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Configured polytopes and extremal configurations

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Abstract

We examine a class of involutory self-dual convex polytopes with a specified sets of diameters, compare their vertex sets to extremal Lenz configurations, and present some of their realizations.

Keywords: Involutory self-dual polytopes, configured polytopes, Lenz configurations, extremal configurations.

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1 Introduction

We describe points in $\mathbb{R}^d$ by standard coordinates $(x_1, x_2, \ldots, x_d)$. For $3 \leq i \leq d$, let $H_i(b_i)$ denote the hyperplane $x_i = b_i$, and $L_e(b_{e+1}, \ldots, b_d) = \cap_{i=e+1}^{d} H_i(b_i)$, $e = 2, \ldots, d-1$. $L_e(b_{e+1}, \ldots, b_d)$ is an e-flat, and denote the $(e-1)$-sphere with centre c and radius t in $L_e(b_{e+1}, \ldots, b_d)$ by $\mathbb{S}^{e-1}(c, t)$. We denote the origin of $\mathbb{R}^d$ by c_d, and let $(\lambda w, p) := \lambda w + (0, \ldots, 0, p)$, for a point $w \in H_d(0) = L_{d-1}(0)$ and $\{\lambda, p\} \subset \mathbb{R}$.

Let Y be a set of points in $\mathbb{R}^d$. Then $\operatorname{conv}(Y)$ and $\operatorname{aff}(Y)$ denote, respectively, the convex hull and the affine hull of Y. For sets $Y_1, Y_2, \ldots Y_n$, let $[Y_1, Y_2, \ldots Y_n] = \operatorname{conv}(\cup_{i=1}^{n} Y_i)$ and $\langle Y_1, Y_2, \ldots Y_n \rangle = \operatorname{aff}(\cup_{i=1}^{n} Y_i)$. If $Y = \{y_1, y_2, \ldots, y_n\}$ is finite, we let $[y_1, y_2, \ldots, y_n] = \operatorname{conv}(Y)$ and $\langle y_1, y_2, \ldots, y_n \rangle = \operatorname{aff}(Y)$.

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Let $P \subset \mathbb{R}^d$ denote a convex $d-$polytope with $\mathcal{L}(P)$ and $\mathcal{F}_i(P)$, $0 \leq i \leq d-1$, denoting the face lattice and the set of $i-$faces of P. We let $f_i(P) = |\mathcal{F}_i(P)|$, $V(P) = \mathcal{F}_0(P)$ and $\mathcal{F}(\mathcal{P}) = \mathcal{F}_{d-1}(P)$, assume familiarity with the basic notions of convex polytopes, and refer to [3, 6] and [18] for basic terminology and definitions. Specifically, two polytopes P_1 and P_2 are *combinatorially equivalent* ($P_1 \cong P_2$) if there is an isomorphism (inclusion preserving) from $\mathcal{L}(P_1)$ to $\mathcal{L}(P_2)$, and are *dual* if there is an anti-isomorphism (inclusion reversing) from $\mathcal{L}(P_1)$ to $\mathcal{L}(P_2)$. If there is an anti-isomorphism Φ from $\mathcal{L}(P)$ to $\mathcal{L}(P)$ then P is *self-dual*, moreover, if $\Phi^2 = id$ then P is *involutory self-dual*.

Let $P \subset \mathbb{R}^d$ be involutory self-dual via the anti-isomorphism on $\mathcal{L}(P)$ induced by the map $v \to v^*$ with $v \in V(P)$, $v^* \in \mathcal{F}(P)$ and $v \notin v^*$. A segment $[v, w]$, with end-points v and w, both vertices of P and with $w \in v^*$, is called a *principal diagonal* of P and let $\mathcal{D}(P)$ denote the set of principal diagonals of P. Finally, we say that P is *configured* if each principal diagonal in P has length $\operatorname{diam}(P)$, and that P is *strictly configured* if it is configured and only principal diagonals of P have length $\operatorname{diam}(P)$. We note that odd regular polygons are strictly configured.

Let $X_n \subset \mathbb{R}^d$ be a set of $n > d \geq 2$ points and $M_d(X_n)$ be the number of pairs $\{x, y\} \subset X_n$ such that $\operatorname{diam}(X_n) = \|x - y\|$, the distance between x and y. Let $M(d, n)$ be the maximum of $M_d(X_n)$ over all $X_n \subset \mathbb{R}^d$. Then X_n is an *extremal configuration* if $M_d(X_n) = M(d, n)$.

The problem of determining $M(d, n)$ is due to Erdős in [4]. We list contributions to the problem in the References, with specific mention of [11, 12] and [17] and the following results:

(1) $M(2, n) = n$, and $X_n \subset \mathbb{R}^2$ is extremal if and only if $V(P) \subseteq X_n \subseteq \operatorname{bd}(P)$ for some Reuleaux polygon P.

(2) $M(3, n) = 2n - 2$ and $X_n \subset \mathbb{R}^3$ is extremal if and only if X_n is the vertex set of certain types of polytopal (Reuleaux) ball polytopes.

(3) $M(d, n)$, $d \geq 4$, grows quadratically with n, and extremal X_n are attained only by Lenz Constructions.

In this last regard, we note (cf. [17]) that an (even dimensional) *Lenz Configuration* in $\mathbb{R}^d$, $d = 2p \geq 2$, is any translate of a finite subset of $\cup_{i=1}^{p} C_i$ where C_i is a circle with centre at the origin O and radius r_i, so that $r_j^2 + r_k^2 = 1$ for all j, k and the subspaces U_i, spanned by C_i, yield the orthogonal decomposition $\mathbb{R}^d = U_1 \oplus U_2 \oplus ... \oplus U_p$. For odd dimensions $d = 2p + 1$, C_1 is replaced by a 2-sphere with centre O and radius $r = \frac{1}{\sqrt{2}}$.

Theorem 1.1 (K. Swanepoel). *For each $d \geq 4$, there exists a number $N(d)$ such that all extremal configurations X_n, with $n \geq N(d)$, are Lenz Configurations.*

We note that in [17], Swanepoel also determines $M(d, n)$ for sufficiently large n.

Our interests in this paper are realizations (constructions) of strictly configured d-polytopes P, $d \geq 3$, and values of $M_d(P)$ (number of principal diagonals of P). In Section 2, we will show that for strictly configured 4-polytopes there is a formula similar to 1) and 2) that depends on the number of vertices and edges; furthermore we show the convex hull of vertices of an extremal Lenz configuration is never a configured d-polytope. The former raises the question of whether in dimension $d \geq 4$ the situation for $M(d, n)$ may have at least another possible scenario, if the points are not in Lenz configurations. In

Section 3 we will give constructions of configured d-polytopes P for $d \geq 3$ such that for $d = 4$, $M_4(P) \leq 4n$. These constructions consist of two steps: determining self-dual polytopes so that all principal diagonals have length (say 1), and then showing that the diameter of the polytope is 1.

2 Principal diagonals

In this section, we assume that $P \subset \mathbb{R}^d$ is an involutory self-dual d-polytope via the anti-isomorphism on $\mathcal{L}(P)$ induced by $v \in V(P) \to v^* \in \mathcal{F}(P)$, and recall that $\mathcal{D}(P)$ denotes the set of principal diagonals of P.

Theorem 2.1. *Let $P \subset \mathbb{R}^3$ be a configured 3-polytope. Then P is strictly configured and extremal, that is, $|\mathcal{D}(P)| = 2f_0(P) - 2$.*

Proof. Since P is self-dual, we have that $f_0(P) = f_2(P)$ and so, $f_1(P) = 2f_0(P) - 2$ by Euler's Theorem.

Let $v \in V(P)$. Then $v^* \in \mathcal{F}_2(P)$ is a polygon and $f_0(v^*) = f_1(v^*)$. On the one hand, $f_0(v^*) = |\{g \in \mathcal{D}(P) \mid v \in g\}|$ by definition. On the other hand, $v \in e \in \mathcal{F}_1(P)$ iff $e^* \in \mathcal{F}_1(v^*)$, and so, $f_1(v^*) = |\{e \in \mathcal{F}_1(P) \mid v \in e\}|$. Thus $|\{g \in \mathcal{D}(P) \mid v \in g\}| = |\{e \in \mathcal{F}_1(P) \mid v \in e\}|$ and $|\mathcal{D}(P)| = |\mathcal{F}_1(P)|$. □

Theorem 2.2. *Let $P \subset \mathbb{R}^4$ be a strictly configured 4-polytope. Then $|\mathcal{D}(P)| \leq 2f_1(P) - 2f_0(P)$.*

Proof. Let $V(P) = \{v_1, ..., v_n\}$ and $\mathcal{F}_1(P) = \{e_1, ..., e_m\}$. Then $\mathcal{F}_2(P) = \{e_1^*, ..., e_m^*\}$ and $\mathcal{F}(P) = \{v_1^*, ..., v_n^*\}$ by the self-duality of P.

We recall from [1] that $f_{jk}(P)$, $0 \leq j < k \leq 3$, is the number of pairs of j-faces G_j and k-faces G_k such that $G_j \subset G_k$, and that $f_{02}(P) \leq 6f_1(P) - 6f_0(P)$. By the self-duality of P, we have also that

$$\sum_{i=1}^{n} f_1(v_i^*) = f_{13}(P) = f_{02}(P),$$

$$\sum_{i=1}^{n} f_2(v_i^*) = f_{23}(P) = f_{01}(P) \text{ and}$$

$$f_{01}(P) = \sum_{j=1}^{m} f_0(e_j) = 2f_1(P)$$

Finally, let $v \in V(P)$ and $e \in \mathcal{D}(P)$ of a configured $P \subset \mathbb{R}^4$.

Then $v \in e$ if, and only if, $e = [v, w]$ and $w \in \mathcal{F}_0(v^*)$. Thus, $f_0(v^*)$ is the number of principal diagonals of P that contain v, and $\sum_{i=1}^{n} f_0(v_i^*) = 2|\mathcal{D}(P)|$. Then by Euler's Theorem,

$$\begin{aligned} |\mathcal{D}(P)| &= \frac{1}{2}\sum_{i=1}^{n}(2 + f_1(v_i^*) - f_2(v_i^*)) \\ &= n + \frac{1}{2}\sum_{i=1}^{n} f_1(v_i^*) - \frac{1}{2}\sum_{i=1}^{n} f_2(v_i^*) \qquad (2.1) \\ &= f_0(P) + \frac{1}{2}f_{02}(P) - \frac{1}{2}f_{01}(P) \\ &\leq f_0(P) + [3f_1(P) - 3f_0(P)] - f_1(P). \end{aligned}$$

End of Theorem 2.2. □

We let $M_d(Q) = M_d(V(Q))$ for a $d-$polytope Q, and observe that if $P \subset \mathbb{R}^4$ is strictly configured then $M_4(P)$ is linear in $f_1(P)$ and $f_0(P)$. This raises the following question: Is there a set of n vertices of a strictly configured polytope in Lenz Configuration? We show below that the answer is no if $f_0(P) > 5$; in fact, we present in Section 3 a subfamily of such $P \subset \mathbb{R}^4$ with $f_1(P) \leq 3f_0(P)$ and $M_4(P) \leq 4f_0(P)$.

If $n = 5$ and $d = 4$, it is easy to prove that the polytope with vertices $(0, 0, \frac{\sqrt{6}}{12}, \frac{\sqrt{10}}{4})$, $(0, 0, \frac{\sqrt{2}}{3}, 0)$, $\frac{1}{\sqrt{3}}(\cos\frac{\pi}{3}, \sin\frac{\pi}{3}, 0, 0)$, $\frac{1}{\sqrt{3}}(\cos\frac{2\pi}{3}, \sin\frac{2\pi}{3}, 0, 0)$ and $\frac{1}{\sqrt{3}}(1, 0, 0, 0)$ is a Lenz Configuration and that it is strictly configured. This is the only case with $d = 4$ where the vertices of a strictly configured polytope is a Lenz Configuration.

Theorem 2.3. *Let $X \subset \mathbb{R}^4$ be a 4-dimensional extremal Lenz Configuration with $|X| \geq 6$. Then $P = \mathrm{conv}(X)$ is not configured.*

Proof. We assume $X \subset \mathbb{R}^4$ is a 4-dimensional Lenz Configuration with $X \subset C_1 \cup C_2$, $C_i \subset U_i$, where $\mathbb{R}^4 = U_1 \oplus U_2$. It is clear that P is a 4-polytope with $V(P) = X$ and diameter 1. Let $X \cap C_1 = \{w_1, ..., w_a\}$, $X \cap C_2 = \{z_1, ..., z_b\}$ and note that for $i = 1, 2$, $G_i := U_i \cap P \in \mathcal{F}_2(P)$.

From [17], we have that $M_4(X) = M(4, n)$ with $|X \cap C_1| = \lceil \frac{n}{2} \rceil$ and $|X \cap C_2| = \lfloor \frac{n}{2} \rfloor$, say. Furthermore, $M(4, 6) = t_2(6) + 4$, $M(4, 7) = t_2(7) + 4$ and $M(4, n) \leq t_2(n) + \lceil \frac{n}{2} \rceil + 1$ for $n \geq 8$ where $t_2(n)$ is the number of pairs $\{w_j, z_k\}$ such that $\|w_j - z_k\| = 1$. Accordingly, there are $M(4, n) - t_2(n)$ diameters of X that have end points in either C_1 or C_2.

We suppose that P is configured via the anti-isomorphism induced by $v \to v^*$, $v \notin v^*$, and seek a contradiction. Then $a \geq 3$, $b \geq 3$, $v \notin v^*$ and $\mathcal{F}(P) = \{w_1^*, ..., w_a^*, z_1^*, ..., z_b^*\}$ yield that $v^* \cap C_1 \neq \emptyset \neq v^* \cap C_2$ for $v \in X \cap C_1$, and $G_1 = z_1^* \cap z_2^*$ and $G_2 = w_1^* \cap w_2^*$ say: Thus, $w_j^* \cap G_1 \in \mathcal{F}_1(w_j^*)$ and $z_k^* \cap G_2 \in \mathcal{F}_1(z_k^*)$ for $3 \leq j \leq a$ and $3 \leq k \leq b$.

It now follows that the number of principal diagonals of P in G_1 and G_2 is:

- two through each w_j and z_k with $j \geq 3$, $k \geq 3$ and
- at least one through each of w_1, w_2, z_1 and z_2;

that is, at least $\frac{1}{2}(2(a-2)+2(b-2)+4) = a+b-2 = n-2$ and $n-2 \leq \lceil \frac{n}{2} \rceil \neq 1$. Then $n = 6$, $w_3^* \cap G_1 = [w_1, w_2]$ and so, $w_3 \in w_1^* \cap w_2^*$, $[G_1, w_3] \subset w_1^* \cap w_2^*$, and $w_1^* = w_2^*$; a contradiction. □

We note that the arguments and the result in Theorem 2.3 extend to $d \geq 5$ for extremal Lenz Configuration X with sufficiently large $|X|$. This raises the issue of how to realize configured polytopes with a large number of vertices in higher dimensions.

3 Constructions of strictly configured polytopes

In this section, we present realizations of strictly configured polytopes that are $(d-2)$-fold d-pyramids or "stratified" d-polytopes. We note that configured polytopes play an important part in the study of, among others, graphs, hypergraphs, and bodies of constant width.

3.1 Prismoids

Let $m \geq d \geq 3$ and $\mathcal{Q} \subset H_d(0)$ be a $(d-1)$-polytope with $V(\mathcal{Q}) = \{w_1, w_2, \ldots, w_m\}$ and c_d as a relative interior point.

We consider translated homothetic copies (homotheties) $\mathcal{Q}_{jm}$ of $\mathcal{Q}$. For $k \geq 2$ and $1 \leq j \leq k$, let $\mathcal{Q}_{jm} = [y_{j1}, y_{j2}, \ldots, y_{jm}]$ with $y_{jr} = (\lambda_{jr} w_r, p_j)$, $p_k < p_{k-1} < \cdots < p_1$ and $\lambda_j > 0$. We let $R_{km} = [\mathcal{Q}_{1m}, \mathcal{Q}_{2m}, \ldots, \mathcal{Q}_{km}]$, and say that R_{km} is a *k-layered d-prismoid* if $|V(R_{km})| = km$ and for $r = 1, \ldots, m$, $[y_{(j-1)r}, y_{jr}]$ are the edges of R_{km} that intersect $\mathcal{Q}_{(j-1)m}$ and $\mathcal{Q}_{jm}$.

Then $[\mathcal{Q}_{im}, \mathcal{Q}_{jm}]$ is a d-prismoid for $1 \leq i \leq j \leq m$, $\{\mathcal{Q}_{1m}, \mathcal{Q}_{km}\} \subset \mathcal{F}(R_{km})$ and we let $P_{km} = [y_{00}, R_{km}]$ for some point $y_{00} = (0, \ldots, 0, q) \in \mathbb{R}^d$. We say that P_{km} is a *stratified* d-polytope if y_{00} is beyond either $\mathcal{Q}_{1m}$ or $\mathcal{Q}_{km}$, and beneath all other facets of R_{km} (cf. [6] p. 78), and hence, $|V(P_{km})| = km + 1$.

In what follows, we assume that $P_{km} = [y_{00}, R_{km}] \subset \mathbb{R}^d$ is stratified with R_{km} as above and y_{00} beyond exactly $\mathcal{Q}_{1m}$. It is clear that P_{km} is dependent upon the $(d-1)$-polytope $\mathcal{Q} = [w_1, w_2, \ldots, w_m] \subset H_d(0)$, and we examine properties of P_{km} that are inherited from $\mathcal{Q}$.

As a point of reference, $P_{2m} \subset \mathbb{R}^3$ is called an apexed 3-prism in [11].

3.1.1

Let $\mathcal{Q} = [w_1, w_2, \ldots, w_m] \subset H_d(0)$ be involutary self-dual via the anti-isomorphism on $\mathcal{L}(\mathcal{Q})$ induced by $w_r \to \tilde{w}_r \in \mathcal{F}(\mathcal{Q})$. Then $\mathcal{F}(\mathcal{Q}) = \{\tilde{w}_1, \tilde{w}_2, \ldots, \tilde{w}_m\}$ and we have that

- $\mathcal{Q}_{jm}$ is involutary self-dual via the anti-isomorphism of $\mathcal{L}(\mathcal{Q}_{jm})$ that sends $y_{jr} \to \tilde{y}_{jr}$, and $y_{js} \in \tilde{y}_{jr}$ if, and only if, $w_s \in \tilde{w}_r$,
- $\mathcal{F}(\mathcal{Q}_{jm}) = \{\tilde{y}_{j1}, \tilde{y}_{j2}, \ldots, \tilde{y}_{jm}\}$,
- $\mathcal{F}(R_{km}) = \{\mathcal{Q}_{1m}, \mathcal{Q}_{km}\} \cup \{\tilde{y}_{(j-1)r}, \tilde{y}_{jr}\} | 2 \leq j \leq k, 1 \leq r \leq m\}$ and
- $\mathcal{F}(P_{km}) = (\mathcal{F}(R_{km}) \setminus \{\mathcal{Q}_{1m}\}) \cup \{[y_{00}, \tilde{y}_{1r}] | 1 \leq r \leq m\}$.

Then (cf. [2], Theorem 2.1) P_{km} is involutary self-dual via the anti-isomorphism on $\mathcal{L}(P_{km})$ induced by the map $y_{jr} \to Y_{jr}$ with $Y_{00} = \mathcal{Q}_{km}$, $Y_{kr} = [y_{00}, \tilde{y}_{1r}]$ and $Y_{jr} = [\tilde{y}_{(k-j)r}, \tilde{y}_{(k-j+1)r}]$ for $j = 1, \ldots, k-1$ and $r = 1, \ldots, m$. □

3.1.2

With $\mathcal{Q}$ as in 3.1.1, let $V(\mathcal{Q}) \subset \mathbb{S}^{d-2}(c_d, t) \subset H_d(0)$ and $\|w_r - w_s\| = 1$ for each $w_r \in V(\mathcal{Q})$ and $w_s \in \tilde{w}_r$. We say that P_{km} is metrically embedded in $\mathbb{R}^d$ if $\|y - y'\| = 1$ for every $\{y, y'\} \subset V(P_{km})$ such that $[y, y']$ is a principal diagonal of P_{km}. Thus, a metrically embedded P_{km} of diameter 1 is configured.

From Theorem 4.1 in [2]; if $y_{00} = (0, 0, \ldots, 0, q)$, then there are $0 < \lambda_k \leq \lambda_1 < \cdots < \lambda_j \leq \lambda_{k-j} < \cdots < \lambda_{[\frac{k+1}{2}]} = 1$ that yield $0 = p_k < p_{k-1} < \cdots < p_1 < q$ so that for every $y_{jr} \in V(P_{km})$: if $y_{is} \in Y_{jr}$ then $\|y_{jr} - y_{is}\| = 1$. Specifically, we note that $q^2 = 1 - \lambda_k^2 t^2$, $p_1^2 = 1 - \|\lambda_k w_r - \lambda_1 w_s\|^2$ and $p_{k-1} = p_1 - \sqrt{\beta}$ with $\beta = 1 - \|\lambda_{k-1} w_r - \lambda_1 w_s\|^2$. □

Our present interest is to determine involutary self-dual $P_{km} \subset \mathbb{R}^d$ of, say, diameter 1 and then to characterize its diameters. To that end, we seek involutary self-dual $\mathcal{Q} \subset H_d(0)$ of diameter 1 and with vertices on a $(d-2)$-sphere.

3.2 Pyramids with polygonal bases

With the a_i's to be specified, let $d \geq 3$ and $\mathcal{Q} \subset L_2(-a_3, \ldots, -a_d)$ be a regular m-gon with cyclically labeled vertices $w_1, w_2, \ldots, w_m$, the circumradius t, the diameter 1 and $m = 2u + 1 \geq 3$. Then it is well known that $1 = \|w_r - w_{r+u}\| = \|w_r - w_{r+u+1}\|$ for each w_r, and that $\mathcal{Q}$ has $2m$ diameters.

As a simplification, we write $w_r = (x_1, x_2, -a_3, \ldots, -a_d)$ as $w_r = (x_1, x_2)$ in relation to the plane $L_2(-a_3, \ldots, -a_d)$.

3.2.1

With $\theta = \frac{2\pi}{m}$ and $w_r = t(\cos(r\theta), \sin(r\theta))$ for $r = 1, \ldots, m$, we note that $w_m = (t, 0), w_{m+u} = w_u$ and $1 = \|w_m - w_u\|^2 = 2t^2(1 - \cos(u\theta)) = 2t^2(1 + \cos(\frac{\pi}{m}))$ from $m = 2u + 1$.

3.2.2

With $m = 2u + 1 \geq 5$ and $\lambda > 0$, we claim that $\|\lambda w_r - w_j\| < \|\lambda w_r - w_{r+u}\|$ for $w_j \in V(\mathcal{Q}) \setminus \{w_r, w_{r+u}, w_{r+u+1}\}$.

With coordinates as in 3.2.1, we may assume that $w_r = w_m$ and that w_j is in the upper half-plane. Then $0 < j\theta < u\theta < \pi$ and $\cos(u\theta) < \cos(j\theta)$ and $\|\lambda w_m - w_u\|^2 - \|\lambda w_m - w_j\|^2 = 2\lambda t^2(\cos(j\theta) - \cos(u\theta))$. □

3.2.3

For $\lambda > \mu > 0$ and $w_s \in \{w_{r+u}, w_{r+u+1}\}$, we have that $[\lambda w_r, \mu w_r, \mu w_s, \lambda w_s]$ is an isosceles trapezoid of side lengths λ, μ and $(\lambda - \mu)t$ and $\|\lambda w_r - \mu w_s\|^2 = \lambda\mu + (\lambda - \mu)^2 t^2 = \|\lambda w_s - \mu w_r\|^2$. □

3.2.4

From $1 = \|w_m - w_u\|^2 = 2t^2(1 + \cos(\frac{\pi}{m}))$ and $m \geq 3$, we obtain that $\frac{1}{4} < t^2 \leq \frac{1}{3}$ and $\frac{1}{3} < \frac{1}{4(1-t^2)} \leq \frac{3}{8}$. We let $t_2 = t$, $t_d^2 = \frac{1}{4(1-t_{d-1}^2)}$ for $d \geq 3$ and note that $\frac{1}{3} < t_3^2 \leq \frac{3}{8} < t_4^2 \leq \frac{2}{5} < t_5^2 \leq \frac{5}{12} < t_6^3 \leq \frac{3}{7} < t_7^2 \leq \frac{7}{16} < t_d^2 < \frac{1}{2}$ with $d \geq 8$. □

3.2.5

With $d \geq 4$ and $\mathcal{Q} \subset L_2(-a_3, \ldots, -a_d) \subset L_3(-a_4, \ldots, -a_d)$ as above, we write $w_r = (t_2 \cos(r\theta), t_2 \sin(r\theta), -a_3)$ in relation to $L_3(-a_4, \ldots, -a_d)$. We consider the 2-sphere $\mathbb{S}^2 := \mathbb{S}^2((0,0,0), t_3) \subset L_3(-a_4, \ldots, -a_d)$ with $t_3^2 = \frac{1}{4(1-t_2^2)}$, and let $a_3 = \sqrt{t_3^2 - t_2^2}$. Then $V(\mathcal{Q}) \subset \mathbb{S}^2$ and with $w_{m+1} = (0, 0, t_3)$, we claim that $\|w_{m+1} - w_r\| = 1$ for $r = 1, 2, \ldots, m$.

As $\mathcal{Q}$ is symmetric about the x_3-axis, we verify the claim with $w_r = w_m = (t_2, 0, -a_3)$.

From $t_3^2 = \|w_m\|^2 = t_2^2 + a_3^2$ and $t_2^2 = \frac{4t_3^2 - 1}{4t_3^2}$, it follows that $\|w_{m+1} - w_m\|^2 =$

$$t_2^2 + (t_3 + a_3)^2 = 2t_3^2 + 2t_3\sqrt{t_3^2 - t_2^2} = 2t_3^2 + 2t_3\left(\frac{(1-2t_3^2)^2}{4t_3^2}\right)^{\frac{1}{2}} = 1.$$

Theorem 3.1. *Let $d \geq 3$ and $\mathcal{Q}^2 = [w_1, \dots, w_m] \subset L_2(-a_3, \dots, -a_d)$ be a regular m-gon of diameter 1 and circumradius t_2; $m = 2u+1 \geq 3$. Then for $e = 3, \dots, d$, $t_e^2 = \frac{1}{4(1-t_{e-1}^2)}$, $a_e^2 = t_e^2 - t_{e-1}^2$ and $c_e = (0, \dots, -a_{e+1}, \dots, -a_d)$ if $e \neq d$, there is an involutary self-dual $(e-2)$-fold e-pyramid $\mathcal{Q}^e = [w_1, \dots, w_m, \dots, w_{m+e-2}]$ of diameter 1 and basis $\mathcal{Q}^2$ such that*

(i) *$\mathcal{Q}^e \subset L_e(-a_{e+1}, \dots, -a_d)$ if $e \neq d$,*

(ii) *$V(\mathcal{Q}^e) \subset \mathbb{S}^{e-1}(c_e, t_e)$ and*

(iii) *$\mathcal{Q}^e$ is strictly configured.*

Proof. With reference to Subsections 3.2.1, 3.2.2, 3.2.3, 3.2.4 and 3.2.5, we let:

- $w_i = (t_2 \cos(i\theta), t_2 \sin(i\theta), -a_3, \dots, -a_d)$ for $i = 1, \dots, m$
- $w_{m+i} = (0, \dots, 0, t_{i+2}, -a_{i+3}, \dots, -a_d)$ for $i = 1, \dots, d-3$ and
- $w_{m+d-2} = (0, \dots, 0, t_d)$.

We observe first that for $2 \leq i < j \leq d$, $t_i^2 + a_{i+1}^2 = t_{i+1}^2$ and so, $t_i^2 + a_{a+1}^2 + \cdots + a_j^2 = t_j^2$. From this it follows that $\|w_i - c_e\|^2 = t_2^2 + a_3^2 + \cdots + a_e^2 = t_e^2$ for $w_i \in V(\mathcal{Q}^2)$, $3 \leq e \leq d$ $\|w_{m+i} - c_e\|^2 = t_{i+2}^2 + a_{i+3}^2 + \cdots + a_e^2 = t_e^2$ for $i+2 \leq e \leq d-1$ and $\|w_j - c_d\|^2 = \|w_j\|^2 = t_d^2$ for $w_j \in V(\mathcal{Q}^d)$.

Next, with $w_r = (t_2 \cos(r\theta), t_2 \sin(r\theta), -a_3, \dots, -a_d)$ and $w_r' = (t_2 \cos(r+u)\theta, t_2 \sin(r+u)\theta, -a_3, \cdots - a_d)$, we note that $\mathcal{Q}^2$ is involutary self-dual via the anti-isomorphism of $\mathcal{L}(\mathcal{Q}^2)$ induced by $w_r \to \bar{w}_r = [w_r', w_{r+1}']$. Then for $e = 3, \dots, d$,

$\mathcal{F}(\mathcal{Q}^e) = \{[\bar{w}_r, w_{m+1}, \dots w_{m+e-2}] | r = 1, \dots m\} \cup \{[V(\mathcal{Q}^e) \setminus \{w_r\}] | r = m+1, \dots, m+e-2\}$

and $\mathcal{Q}^e$ is involutary self-dual via the anti-isomorphism on $\mathcal{L}(\mathcal{Q}^e)$ induced by $w_r \to \tilde{w}_r$ where

$$\tilde{w}_r = \begin{cases} [\bar{w}_r, w_{m+1}, \dots w_{m+e-2}], & r = 1, \dots, m; \\ [V(\mathcal{Q}^e) \setminus \{w_r\}], & r = m+1, \dots, m+e-2. \end{cases}$$

Finally, we observe that for $1 \leq j \leq m+i$, $\|w_{m+i} - w_j\|^2 = t_{i+1}^2 + (t_{i+2} + a_{i+2})^2$. Then, as in 3.2.5, $t_{i+1}^2 = \frac{4t_{i+2}^2 - 1}{4t^2}$ yields that $\|w_{m+i} - w_j\| = 1$. From this and $t_2^2 = \frac{1}{2(1+\cos(\frac{\pi}{m}))}$, we obtain that $\|w_r - w_s\| = 1$ for $w_s \in \tilde{w}_r$; furthermore, if $\{w_r, w_z\} \subset V(\mathcal{Q}^2)$ and $w_z \notin \tilde{w}_r$ then $\|w_r - w_z\| < 1$. □

We note that $M_e(\mathcal{Q}^e) = 2M_2(\mathcal{Q}^2) + \sum_{m+1}^{m+e-3} j = (e-1)m + \binom{e-2}{2}$ and that $\mathcal{Q}^3$ is extremal.

Theorem 3.2. *Let $d \geq 3$, $m = 2u+1$, $n = m+d-3$ and $k \in \{2, 3\}$. Then there is an involutary self-dual stratified $P_{kn} = [y_{00}, R_{kn}] \subset \mathbb{R}^d$ of diameter 1 that is strictly configured.*

Proof. With reference to Subsection 3.1 and Theorem 3.1 with $e = d-1$ and $a_d = 0$, we consider P_{kn} with the property that:

- y_{00} is beyond exactly $\mathcal{Q}_{1n}$.

- $\mathcal{Q} = [w_1, ..., w_n] \subset L_{d-1}(-a_d) = H_d(0)$,
- $\mathcal{Q}^{d-1}$ is a involutary self-dual $(d-3)$-fold $(d-1)$-pyramid with diameter 1 and basis $\mathcal{Q}^2$, and
- $\mathcal{Q}^2 = [w_1, ..., w_m] \subset L_2(-a_3, ..., -a_d)$ is a regular m-gon of diameter 1.

Then $c_{d-1} = (0, \dots, 0, -a_d) = c_d$ and with $t_2, \dots t_{d-1}$ as in 3.2.4, we simplify notation and let $t = t_{d-1}$.

We now apply 3.1.2 with $y_{00} = (0, \dots, 0, q)$ and $p_k < p_{k-1} < \cdots < p_1 < q$.

Case 1: $k = 2$ and hence, $\lambda_1 = 1$ and $p_2 = 0$.

With $0 < \lambda_2 < 1$: P_{2n} is stratified, $Y_{00} = Q_{2n}$, $Y_{1r} = [\tilde{y}_{1r}, \tilde{y}_{2r}]$ and $Y_{2r} = [y_{00}, \tilde{y}_{1r}]$. With $q^2 = 1 - \lambda_2 t^2$ and $p_1^2 = 1 - \|\lambda_2 w_r - w_s\|^2 = 1 - (\lambda_2 + (1-\lambda_2)^2 t^2)$ (cf. 3.2.3), we have that $\|y_{jr} - y_{is}\| = 1$ for $y_{is} \in Y_{jr}$.

With $\lambda_2 = \frac{1}{2}$; we have $q^2 = \frac{4-t^2}{4}$, $p_1^2 = \frac{2-t^2}{4}$ and claim that $\|y_{jr} - y_{iz}\| < 1$ for $y_{iz} \notin Y_{jr}$. From $\frac{1}{3} < t^2 < \frac{1}{2}$, we obtain that

$$\begin{aligned} \|y_{00} - y_{1r}\|^2 &= \|(0,q) - (w_r, p_1)\|^2 = \|w_r\|^2 + (q - p_1)^2 \\ &= t^2 + q^2 + p_1^2 - 2qp_1 \\ &= \frac{1}{4}(6 - 2t^2 - 2\sqrt{4-t^2}\sqrt{2-t^2}) \\ &\leq \frac{1}{4}\left(6 + 2\left(\frac{1}{2}\right) - 2\sqrt{4 - \frac{1}{3}}\sqrt{2 - \frac{1}{3}}\right) < 1 \end{aligned} \tag{3.1}$$

Let $y_{iz} \neq y_{00} \neq y_{jr}$ and $y_{iz} \notin Y_{jr}$. Then $y_{iz} = (\lambda_i w_z, p_i)$, $y_{jr} = (\lambda_j w_r, p_j)$ and $w_z \notin \tilde{w}_r$ (cf. 3.1.1). Since Q_{1n} and Q_{2n} are homothets of Q, we may assume by Theorem 3.1(iii) that $j = 1$ and $i = 2$, say. Since $w_z \notin \tilde{w}_r$, it follows as in the proof of Theorem 3.1 that $w_z = w_r$ or $\{w_z, w_r\} \subset V(Q^2)$. If $w_z = w_r$, then $\|y_{1r} - y_{2r}\|^2 = \frac{t^2}{4} + p_1^2 = \frac{1}{2}$. If $\{w_z, w_r\} \subset V(Q^2)$, then it follows from 3.2.2 that $\|w_r - \frac{1}{2}w_z\| < \|w_r - \frac{1}{2}w_s\|$ with $w_s \in \tilde{w}_r \cap V(Q^2)$. Hence, $\|y_{1r} - y_{2z}\| < \|y_{1r} - y_{2s}\| = 1$.

Case 2: $k = 3$ and hence, $\lambda_2 = 1$ and $p_3 = 0$.

Let $Y_{00} = Q_{3n}$, $Y_{1r} = [\tilde{y}_{2r}, \tilde{y}_{3r}]$, $Y_{2r} = [\tilde{y}_{1r}, \tilde{y}_{2r}]$ and $Y_{3r} = [y_{00}, \tilde{y}_{1r}]$. With $\lambda = \lambda_1 = \lambda_3 = \frac{1}{2}$ and $q^2 = 1 - \lambda t^2 = \frac{4-t}{4}$, $p_1^2 = 1 - \|\lambda w_r - \lambda w_s\|^2 = 1 - \lambda^2 = \frac{3}{4}$ (cf. 3.1.2 and 3.2.3), $\beta = 1 - \|\lambda_2 w_r - \lambda_1 w_s\|^2 = 1 - \|w_r - \lambda w_s\|^2 = 1 - \lambda + (1-\lambda)^2 t^2 = \frac{2-t^2}{4}$ and $p_2 = p_1 - \sqrt{\beta}$, we obtain that $\|y_{jr} - y_{is}\| = 1$ for $y_{is} \in Y_{jr}$.

Let $y_{iz} \notin Y_{jr}$. We claim that $\|y_{jr} - y_{iz}\| < 1$ and then it follows that each Y_{jr} is a facet of P_{3n}; that is, R_{3n} is a 3-layered prismoid and P_{3n} is stratified.

We observe that if $a < t^2 \leq b$ then

$$\begin{aligned} \|y_{00} - y_{2r}\|^2 &= \|(0,q) - (w_r, p_2)\|^2 = \|w_r\|^2 + (q - p_2)^2 \\ &= t^2 + q^2 + p_1^2 + \beta + 2q\sqrt{\beta} - 2p_1\left(q + \sqrt{\beta}\right) \\ &= \frac{1}{4}\left(9 + 2t^2 + 2\sqrt{(4-t^2)(2-t^2)} - 2\sqrt{3}(\sqrt{4-t^2} + \sqrt{2-t^2}\right) \\ &< \frac{1}{4}\left(9 + 2b + 2\sqrt{(4-a)(2-a)} - 2\sqrt{3}(\sqrt{4-b} + \sqrt{2-b}\right) \end{aligned} \tag{3.2}$$

and $\|y_{00} - y_{2r}\| < 1$ for $(a, b) \in \{(\frac{1}{3}, \frac{3}{8}), (\frac{3}{8}, \frac{2}{5}), (\frac{2}{5}, \frac{5}{12}), (\frac{5}{12}, \frac{3}{7}), (\frac{3}{7}, \frac{7}{16}), (\frac{7}{16}, \frac{1}{2})\}$, that is, for each $d \geq 3$ (cf. 3.2.4).

It is clear that $\|y_{00} - y_{1r}\| < \|y_{00} - y_{2r}\|$, and hence, we may assume that $y_{iz} = (\lambda_i w_z, p_i)$, $y_{jr} = (\lambda_j w_r, p_j)$ and $w_z \notin \tilde{w}_r$. Then $\|w_r - w_z\| < \|w_r - w_s\|$ for $w_s \notin \tilde{w}_r$, and $\|y_{1r} - y_{3z}\| < \|y_{1r} - y_{3s}\| = 1$ for $y_{3s} \in \tilde{y}_{1r} \subset Y_{1r}$.

From $t^2 < \frac{1}{2}$, we obtain that $\beta > \frac{3}{16} = \frac{p_1^2}{4}$, $p_2 = p_1 - \sqrt{\beta} < \frac{p_1}{2}$ and $p_2 < p_1 - p_2$. Thus, $\|y_{3r} - y_{2z}\| < \|y_{1r} - y_{2z}\|$ and we argue as above that $\|y_{1r} - y_{2z}\| < 1$.

In summary; $\|y_{jr} - y'\| < 1$ for $\{y_{jr}, y'\} \subset \{y_{00}\} \cup \{y_{jr} | j = 1, \ldots, k$ and $r = 1, \ldots, n\}$, and with equality if and only if $y' \in Y_{jr}$. Thus

$$\mathcal{F}(P_{kn}) = \{Y_{00}\} \cup \{Y_{jr} | j = 1, \ldots, k, r = 1, \ldots, n\},$$
$$V(P_{kn}) = \{y_{00}\} \cup \{y_{jr} | j = 1, \ldots, k, r = 1, \ldots, n\}$$

and P_{kn} is involutary self-dual under the anti-isomorphism on $\mathcal{L}(P_{kn})$ induced by $y_{jr} \to Y_{jr}$. □

Theorem 3.3. *Let $P_{km} \subset \mathbb{R}^3$ be an involutary self-dual stratified 3-polytope that is configured with diameter 1; $k \geq 2$ and $m = 2u + 1 \geq 3$. Then there is an involutary self-dual stratified $P_{(k+1)m} \subset \mathbb{R}^3$ that is configured with diameter 1.*

Proof. We let $l = k + 1$ and denote P_{km} as in 3.1.1 and 3.1.2 with $d = 3$. Specifically,

- $Q = [w_1, \ldots, w_m] \subset H_3(0)$ is a regular m-gon of diameter 1 and circumcentre $c_3 = (0, 0, 0)$ as in 3.2.1,
- $Q_{jm} = [y_{j1}, \ldots, y_{jm}]$ with $y_{jr} = (\lambda_j w_r, p_j)$ and $0 < \lambda_k \leq \lambda_1 < \cdots < \lambda_j \leq \lambda_{k-j} < \cdots < \lambda_{[\frac{l}{2}]} = 1$, $0 < p_k < p_{k-1} < \cdots < p_1 < q \leq 1$ and $y_{00} = (0, 0, q)$,
- the anti-isomorphism on $\mathcal{L}(P_{km})$ is induced by $y_{jr} \to Y_{jr}$ with $Y_{00} = Q_{km}$, $Y_{km} = [y_{00}, \tilde{y}_{1r}]$, $Y_{jr} = [\tilde{y}_{(k-j)r}, \tilde{y}_{(l-j)r}]$, $1 \leq j \leq k - 1$, and $\tilde{y}_{jr} = [y_{j(r+u)}, y_{j(r+u+1)}]$, and
- $\|y_{jr} - y_{is}\| = 1$ if, and only if, $y_{is} \in Y_{jr}$.

Let $\mathbb{S}(y) := \mathbb{S}^2(y, 1)$ for $y \in \mathbb{R}^3$, and consider the homothets $Q_{0m} = [y_{01}, \ldots, y_{om}]$ of Q with $y_{0r} = (\lambda_0 w_r, p_0)$, $0 < \lambda_0 < \lambda_1$ and $p_1 < p_0 < q$. From $[y_{k(r+u)}, y_{k(r+u+1)}] = \tilde{y}_{kr} = Y_{00} \cap Y_{1r}$, it follows that $\|y_{00} - y_{ks}\| = 1 = \|y_{1r} - y_{rs}\|$ for $s \in \{r+u, r+u+1\}$, and so,

$$\{y_{00}, y_{1r}\} \subset C_{kr} := \mathbb{S}(y_{k(r+u)}) \cap \mathbb{S}(y_{k(r+u+1)}),$$

a circle with centre $\frac{1}{2}(y_{k(r+u)} + y_{k(r+u+1)})$. It is now clear that

(i) for each $p_1 < p_0 < q$, there is $0 < \lambda_0 < \lambda_1$ such that $y_{0r} \in C_{kr}$.

In fact, $y_{0r} \in \alpha_{kr}$, the shorter arc of C_{kr} with end points y_{00} and y_{1r}. We note also that $V(Q_{0m}) \cap V(P_{km}) = \emptyset$ for each such p_0. Let $V = V(P_{km})$, $B(y) = [\mathbb{S}(y)]$ and $B(V) = \cap_{y \in V} B(y)$. Since $\operatorname{diam}(P_{km}) = 1$, it follows that

(ii) $\alpha_{kr} \subset \operatorname{bd}(B(V))$ for $r = 1, \ldots m$.

Since P_{km} is involutary self-dual with no fixed points, it follows from Theorem 3.2 of [13] that $B(V)$ is polytopal and the face polyhedral structure of $B(V)$ is a lattice isomorphic to $\mathcal{L}(P_{km})$. Accordingly, $B(V)$ is similarly self-dual and from Theorem 4.1 of [13], any surface $\Phi \subset \mathbb{R}^3$ obtained from $\mathrm{bd}(B(V))$ (by performing their surgery on one edge-arc of each pair of dual edge-arcs of $\mathrm{bd}(B(V))$) is the boundary of a body of constant width. In this case, $V \subset \Phi$ and $\mathrm{diam}(V) = 1$ yield Φ is of constant width 1.

We note that dual edge-arcs of $\mathrm{bd}(B(V))$ correspond to dual edges of $\mathcal{L}(P_{km})$. Thus, the duality $[y_{00}, y_{1r}] \longleftrightarrow Y_{00} \cap Y_{1r} = \tilde{y}_{kr}$
yields that α_{kr} is dual to the shorter edge-arc in $\mathbb{S}(y_{00}) \cap \mathbb{S}(y_{1r})$ with end point $y_{k(r+u)}$ and $y_{k(r+u+1)}$. We consider those Φ that contain each of $\alpha_{k1}, \alpha_{k2}, \ldots, \alpha_{km}$. Then the symmetry of P_{km} about the x_3–axis and $i)$ yield that

(iii) $V' = V \cup V(Q_{0m}) \subset \Phi$ and $\mathrm{diam}(V') = 1$,

(iv) $\mathbb{S}(y_{00}) \cap V' = V(Q_{km})$ and the spherical region $\mathbb{S}(y_{00}) \cap \Phi$ is not empty and bounded in $H_3(0)$ by the circumcircle of Q_{km}, and

(v) $y'_{00} = (0, 0, q-1) \in \mathbb{S}(y_{00}) \cap \Phi$.

From $\mathrm{diam}(V) = 1$, $|V| = km+1$, $M(3, km+1) = 2km$ and Theorem 2.1, we have that $M_3(V) = 2km+1$. From $\mathrm{diam}(V') = 1$, $|V'| = lm+1$ and $i)$, we have that $M_3(V') \geq M_3(V) + 2m = 2lm$. Thus, $M_3(V') = 2lm$ and

(vi) $\|y_{0r} - y\| < 1$ for $y_{0r} \in V(Q_{0m})$ and $y \in V \setminus \{y_{k(r+u)}, y_{k(r+u+1)}\}$.

Let $V'' = V' \cup \{y'_{00}\}$. Then $\mathrm{diam}(V'') = 1$, $|V''| = lm+2$, $\|y_{00} - y'_{00}\| = 1$ and $2|V''| - 2 = 2lm + 2 \geq M_3(V'') \geq 2lm+1$. From the rotational symmetry of V'' and $\mathbb{S}(y'_{00})$ about the x_3–axis, it follows that

(vii) $\|y'_{00} - y\| < 1$ for $y \in V' \setminus \{y_0\}$, and

(viii) $\|y_\epsilon - y\| < 1$ for $y \in V' \setminus \{y_0\}$ for sufficiently small $\epsilon > 0$ and $y_\epsilon = (0, 0, q-1-\epsilon)$.

Let $p_0 = q - \epsilon$ and μ be the radius of the circle $H_3(p_0) \cap \mathbb{S}(y'_{00})$. Then $\{(0, 0, p_0)\} = H_3(p_0) \cap \mathbb{S}(y_\epsilon) \subset Q_{0m} \subset [H_3(p_0) \cap \mathbb{S}(y'_{00})]$ and with λ_0 chosen so that $0 < \lambda_0 < \lambda_1$ and $y_{0r} \in \alpha_{kr}$, we have that $0 < \lambda_0 t \leq \mu$. Accordingly, there is a point $z_{00} \in [y'_{00}, y_\epsilon]$ such that $\lambda_0 t$ is the radius of $H_3(p_0) \cap \mathbb{S}(z_{00})$; that is,

(ix) $\|z_{00} - y_{0r}\| = 1$ for $r = 1, 2, \ldots, m$.

Finally, let $z_{jr} = y_{(l-j)r}, \tilde{z}_{jr} = \tilde{y}_{(l-j)r}$ and $Q'_{jm} = Q_{(l-j)m}$ for $j = 1, 2, \ldots, l$ and $r = 1, 2, \ldots, m$. In addition, let $Z_{00} = Q'_{lm} = Q_{0m}$, $Z_{lr} = [z_{00}, \tilde{z}_{1r}] = [z_{00}, \tilde{y}_{kr}]$ and $Z_{jr} = [\tilde{z}_{(l-j)r}, \tilde{z}_{(l-j+1)r}] = [\tilde{y}_{jr}, \tilde{y}_{(j-1)r}]$. From the preceding, we have that $P_{lm} = [z_{00}, Q'_{1m}, \ldots, Q'_{lm}]$ is involutary self-dual via $z_{jr} \to Z_{jr}$, stratified and configured with diameter 1. □

Finally, we show that if a set of n points are the vertices of a configured 4-polytope P such as in Theorem 3.2 then $M_4(P) \leq 4n$.

Theorem 3.4. *Let $P_{km} = [y_{00}, R_{km}] \subset \mathbb{R}^4$ be a configured stratified 4-polytope, with $n = km+1$ vertices. Then number of principal diagonals of P_{km} is at most $4n$.*

Proof. By Theorem 2.2, it is sufficient to prove that $f_1(P) \leq 3n$ for every configured stratified 4-polytope. By construction, $R_{km} = [\mathcal{Q}_{1m}, \mathcal{Q}_{2m}, \ldots \mathcal{Q}_{km}]$ where each copy

$\mathcal{Q}_{im}$ is self-dual and contains m vertices, and thus, $f_1(\mathcal{Q}_{im}) = 2m-2$ by Euler's Theorem and self-duality.

Finally, there are m edges through y_{00} and $m(k-1)$ edges connecting the k homothets $\mathcal{Q}_{im}$, and so, $f_1(P_{km}) = k(2m-2)+m(k-1)+m = 3km-2k \leq 3km+3 = 3n$. $\square$

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