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ANALOGICAL REASONING IN GEOMETRY PROOFS

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Abstract/Izvleček This study aimed at investigating six high school students' use of analogies while working through geometry proofs in group settings. Along with the analogies used by students and analysis of how they were used, as well as discourse analysis, we investigate students' meta-proof comments to glean some interpretation of students' beliefs about proofs. Overall, this study found that students had different beliefs about the nature and process of proofs and used and understood analogical reasoning in idiosyncratic ways. However, it was also found that students' greater use of analogies did not automatically lead to more success with proofs.

Analogno sklepanje v geometrijskih dokazih
Namen raziskave je bil proučiti uporabo analogije šestih srednješolcev pri razdelavi geometrijskih dokazov v skupinski obliki dela. Poleg analogij, ki so jih uporabljali dijaki, in analize njihove uporabe ter analize diskurza proučujemo komentarje k metadokazom, da bi tako zbrali nekaj interpretacij prepričanj dijakov o dokazih. Če povzamemo, smo s študijo ugotovili, da imajo dijaki različna prepričanja o naravi in procesu dokazovanja ter da so analogno sklepanje uporabljali in razumeli na svojstvene načine. Ugotovili pa smo tudi, da večja uporaba analogij dijakov ne vodi avtomatično v večji uspeh pri dokazih.

Keywords:
Analogical reasoning, cognition, geometry proofs, proving theorems

Ključne besede:
analogno sklepanje, kognicija, geometrijski dokazi, dokazovanje izrekov

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Introduction

While investigating student success in performing proofs in geometry is far from novel, many past studies have investigated whether students are successful rather than how they developed their proofs and which cognitive factors supported or hindered their progress (Bell, 2011; Boesen, Lithner, & Palm, 2010; Lamport, 2012; Magda, 2015; Patkin, 2011; Pfeiffer, 2010; Van Bendegem, 2014; Varghese, 2009).
Research has investigated a number of factors associated with student success in performing geometry proofs. Some of these factors are connected with the cognitive style and cognition of the student (Haavold, 2011; Jonsson, Norqvist, Liljekvist, & Lithner, 2014; Park, Moreno, Seufert, & Brunken, 2010; Schwonke, Renkl, Salden, & Aleven, 2011; Tall, 1998, 2008; Varghese, 2009), the algorithmic nature with which they are taught and learn to perform proofs (Bell, 2011; Jonsson et al., 2014; Park et al., 2010; Tall, 1998, 2008; Varghese, 2009), or the common misconceptions students display in the process of completing proofs (e.g., Bell, 2011; Bem, 2011; Frans & Kosolosky, 2014; Grcar, 2013; Hanna & de Villiers, 2008; Lamport, 2012; Patkin, 2011; Pfeiffer, 2010; Stavrou, 2014; Stylianides & Andreas, 2009; Tall, 1998, 2008; Varghese, 2009). Recently, attention has also been paid to the analogical reasoning students employ in performing geometric proofs (Boesen et al., 2010; Lamport, 2012; Magda, 2015). However, few have investigated the meta-proof conversations and attitudes of students involved in performing proofs. What is needed are studies which simultaneously consider the instances and nature of analogies used by students performing geometry proofs and students' attitudes and beliefs regarding proofs. Presently, the interplay of analogies and students' ideas regarding proofs, proof strategies, and student success with proofs has not been firmly established. This study was developed to address some of these concerns. This study seeks to fill gaps in the literature by investigating some of these simultaneous dimensions by examining procedures and order of proof ideas via the lenses of Anderson, Casey, Thompson, Burrage, Pezaris, and Kosslyn (2008), Boesen et al. (2010), Haavold (2011), Lamport (2012), Magda (2015), Patkin (2011), Schwonke et al. (2011) and Tall (2008). We also examine related work on errors, misconceptions, and degree of reliability in mathematical proofs (Bem, 2011; Frans & Kosolosky, 2014; Grcar, 2013; Stavrou, 2014; Van Bendegem, 2014).

Literature Review

Procedures and order of proof ideas

Recent research has investigated how different cognitive styles and cognition match performance on geometry tasks and proving theorems in Euclidean geometry (Haavold, 2011; Stavrou, 2014). While some have addressed it by investigating how proofs can be learned algorithmically, others have focused on creative reasoning (Bell, 2011; Jonsson et al., 2014; Park et al., 2010; Tall, 2008; Varghese, 2009). Some have demonstrated that different cognitive styles influence student techniques and success on proofs in different contexts (Haavold, 2011; Jonsson et al., 2014; Park et al., 2010; Schwonke et al., 2011; Tall, 2008; Varghese, 2009). Although Schwonke et al. (2011) established a relative sense of association between student cognitive styles and proof procedures and strategies, left unresolved was the selection of assessment tasks and the type of reasoning involved in the ordering of proof ideas. Though incompletely established, Boesen et. al (2010) and Pfeiffer (2010) attempted to resolve this by determining the type of reasoning used by mathematics students.

The assumption drawn thus far from Jonsson et al. (2014), Schwonke et al. (2011), and Boesen et al. (2010) is that there is reasonable evidence that the cognitive style employed by students in the context of performing proofs correlates with developing proof competence. What is meant, based on proof ideas and proof competence, is that proofs may have a prescribed order or different directions or multiple solution paths. Proofs may also have definite structures with a specific purpose and end. In addition, some proofs may have paths that are more useful than others and yet formulaic and replicable. As a consequence, it could be conjectured that proof processes are not random but are contextualized natural solution with an existing sequence. Thus, proofs recognize some methodological consistency. On the other hand, proving theorems seeks to emphasize the conversion of verbal problem to pictorial analogies or the reverse. One could also expect to compare pictorial analogy with antagonistic notions, connoting that the picture may be misleading. There are also instances where reasoning becomes subservient to false pictorial analogy.

Magda (2015) advanced this notion by developing a more detailed conceptual understanding of reasoning involved in proving theorems in geometry.

Nevertheless, relatively little has been investigated with respect to reasoning types in the context of proofs (Bem, 2011; Magda, 2015; Patkin, 2011; Schwonke et al., 2011; Tall, 2008; Wiklund-Hörnqvist, Jonsson, & Nyberg, 2014). Instead, the sparse research work examines examples of proofs (Boesen et al., 2010; Frans & Kosolosky, 2014; Lamport, 2012; Stylianides & Andreas, 2009). Nevertheless, it can be argued that procedures employed in proving geometric theorems and the order of ideas provided in a proof may be instrumental in understanding geometric concepts from an analogical perspective.

Numerous authors suggest that students' ability to use analogies and analogical reasoning is valuable, if not necessary, to learning abstract concepts and mathematics new to the student (e.g., Jonsson et al., 2014; Magda, 2015; Patkin, 2011; Stavrou, 2014; Stylianides & Andreas, 2009; Tall, 1998, 2008; Wiklund-Hörnqvist et al., 2014). For instance, while analogies often entail rewriting one mathematical representation (e.g., a verbal statement) into another representation (e.g., a symbolic statement), using analogies differs from performing translations between mathematical representation in that the latter is often prescribed as a task and the former is performed naturally by some students in order to understand, connect, and concretize concepts and solve problems.

In an attempt to investigate how analogical reasoning is applied in geometry, Magda (2015) classified some analogies associated with geometry. First, analogies are recognized as facilitating the comprehension of geometric concepts and the situating of geometric concepts. Magda (2015) defined *analogical reasoning* as thinking that relies upon an analogy, including equivalence, parallelism, similarity, equality, matching, or correspondence. Thus, representational forms of analogical reasoning could be categorized by "…accepted similarities between two systems…" with the view "…to support the conclusion that some further similarity exists" (Magda, 2015, p. 57). Magda (2015) proposes various schemes for solving such analogical reasoning. One such type is shown in Figure 1, which highlights a number of distinct types of analogical reasoning as applied in geometry. The main tenet of Figure 1 suggests that to solve Problem P, one must recognize in it, and distil from it, a problem with more elementary analogues, the Basic Problem (BP), which has in the past been solved. In solving the various steps from 1, 2, …, *n* associated with BP, by using analogy the student can ultimately transform the analogue steps associated with Problem P

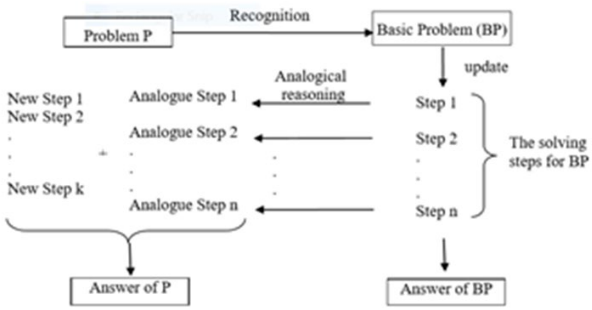


Figure 1. Solving Problem P (Source: Magda, 2015, p. 58)

Second, analogies can build bridges between representations of concepts, theorems, and properties. Verbal representations can be presented with, or through, diagrammatic representations or vice versa. Third, analogies are associated with connecting a particular problem together with the process of solving problems and considering relevant theorems. It is argued that, by observing analogies, it is possible to formulate new mathematical theories. Magda (2015) referred to this form of analogy as a means through which learned/attained concepts, knowledge, and skills are transferred and appropriately used in new states, which may invariably differ from the previous states. Drawing from Magda's (2015) research on types of analogical reasoning, one could deduce that geometry competence can be developed through analogical reasoning. This is attained by considering linkages that exist among concepts, theorems, properties, and similar problems (Anderson et al., 2008; Boesen et al., 2010; Lamport, 2012). Recognizing how to evaluate and synthesize ideas that foster comprehension of analogies existing between various concepts and principles enhances learners' capacity to generalize using analogy. In summary, the procedures for and order of geometric proof are as follows: a well-constructed argument with justifiable reasons contextualized by the problem; certain given information; relevant and auxiliary definitions, theorems, and their properties; and associated statements and reasons. Attention to all of these components, together with available analogies can minimize common errors and misconceptions and increase the degree of reliability of a proof (Grcar, 2013; Stavrou, 2014).

Errors, misconceptions, and degree of reliability in mathematical proofs

Though the relevance, definition, and application of proofs vary from context to context, recent investigation reveals common errors and misconceptions associated with students attempting mathematical proofs (Grcar, 2013; Stavrou, 2014).

Proofs are generally intended to communicate ideas such as "verify, explain, communicate, and systematize statements into deductive systems" (Stavrou, 2014. p. 2). Attempts have been made to understand issues such as "how students learn and solve proofs; teaching techniques of proving; how proofs are validated; how students and teachers perceive proofs; how proofs relate to convincing and refutation; difficulties in the transition of high school to undergraduate mathematics and the extent to which proofs are important in educational settings" (Bell, 2011; Hanna & de Villiers, 2008; Lamport, 2012; Patkin, 2011; Pfeiffer, 2010; Stavrou, 2014, p. 4; Stylianides & Andreas, 2009; Tall, 2008; Varghese, 2009). These issues are closely linked to common errors, misconceptions and to the degree of reliability in mathematical proofs, in that the order is generally associated with techniques, validity, provability, and connectivity with, or transition among, other theorems. Nevertheless, an examination of the types of errors and misconceptions has revealed that there is a tendency to assume a conclusion in order to prove the conclusion of a proof; students use specific examples in proving general statements or abuse and misapply definitions; and in respect to biconditional statements, students often tend to prove only one condition and neglect the other (Stavrou, 2014). Growing bodies of literature regarding the construction of mathematical knowledge suggest rather an ambivalent position. This is because mathematical proofs are epistemically unique and do (not) necessarily correlate with a high level of reliability (Bem, 2011; Frans & Kosolosky, 2014; Grcar, 2013). Since humans are predisposed to errors and subject to various interpretations of theorems, properties and contexts, the opinion on mathematical knowledge and proof is not consistent.

Problem Statement

This study seeks to investigate the instances and nature of analogies used by six students in the context of geometry proofs. Most importantly (and unusual among extant research), we consider the interplay of (a) students' use of analogies and (b) students' ideas regarding proofs, proof strategies, and student success with proofs. Thus, we examine students' order of proof ideas and proving a theorem. In addition (and, again, unusual in the literature), this study recognizes analogies in various forms, such as pictorial (drawn), verbal, relational, physical, gestural, and mental (cognitive, neither physical nor written).

Methodology

Participants and research task

The participants in this study were high school geometry students in North Carolina, U.S. Two distinct groups of students were observed. The group including Student 1 and Student 2 worked through one geometric proof problem and the group including Students 3-6 worked through another geometric proof problem. All six student participants were in the same class, instructed by the same teacher, and had experienced similar learning activities throughout the course. All students were also from the same school system and had all taken the same courses in previous years. The geometric proof problem attempted by each group is identifiable through the transcripts provided below. The geometric proof problems were selected from the geometry curriculum that the students were studying.

Data and analysis

Using a task-based interview design (Goldin, 2000), the two groups of student participants were asked to complete their respective research task. Participants were videotaped as they completed their tasks. Data analysis followed these stages: First, the audio-video recordings were transcribed (Flick, 2009). Second, employing discourse analysis (Wertsch, 1990; Wertsch, Hagstrom & Kikas, 1995), all data (audio-video recordings, participant written work, and transcripts) were reviewed to identify themes (Bogden & Biklen, 2003; Creswell, 2003), and data was sorted into those categories. The researchers particularly sought student understanding of proofs and proof strategies and uses of analogical reasoning while performing geometric proofs. Through the process of check-coding (Miles & Huberman, 1994), researchers' initial coding structures were then compared and contrasted, leading to recognition of similar, different, and missing constructs, and researchers were able to reach consensus (Strauss & Corbin, 1990). The codes developed and employed in this qualitative analysis include those regarding proofs (e.g., students' beliefs regarding proofs, proof strategies, and success with proofs) and analogies (forms, uses, pictorial, relational, verbal, physical, gestural, and mental). In developing this list of observed student analogical techniques we extend and go beyond the majority of the literature regarding student use of analogies. The consensus notes are provided in the transcripts below in order to give the reader more insight into the analysis, coding, and results associated with this study.

Results

Throughout the transcripts provided below, the researchers' consensus notes are included. These notes take two distinct forms. First, in brackets and italics are comments regarding student actions or other aspects to give the reader better understanding of the transcripts. For example: [*Student draws in the segment and marks the perpendicular angle and the segments AD and BD as congruent.*] Second, in braces and italics are observations of *students' beliefs regarding proofs and proof techniques*, and *students' use of analogies* are coded using red and blue fonts. For example: {*Converts verbal problem to a pictorial analogy. An automatic act toward a proof*}. These notes have been retained in the transcripts to assist the reader to see how the researchers interpreted the transcripts and to build reliability and replicability in future similar research. The following discussions begin by initially considering some aspects associated with proofs in respect to the student participants and then considering the domain of analogical reasoning. Thus, we examine students order of proof ideas and proving a theorem via proofs, analogies, connecting proofs with analogies.
It is important to note that the transcripts are replete with errors and misstatements. These have been left intact to more accurately represent the communication of the students. For instance, very often students denoted segments by naming only the endpoints. They would say "*AB*" to mean segment *AB* or $\overline{AB}$. To maintain the simplicity of mathematical reading, when students stated "is congruent to" the notes transcribe "≅". Conjoining these two situations, a student may state that "*AB* is congruent to *BC*" and mean "segment *AB* is congruent to segment *BC*." In other cases, when students name vertices in the wrong order, they are accurately reported and represented in the transcript.

Proofs

The following transcripts are in two parts. The first part comprises a group of two students considering the nature of proof in the context of proving two triangles as congruent. The second transcripts represent a group of four students attempting to prove the isosceles triangle theorem.

On order of proof ideas

S1: I don't know the right order [*of steps for this proof*]. {*Proof has a prescribed order.*}

S2: You can use any order you want that will get you there. {*Numerous directions to proofs.*}

S1: But [*the teacher*] always says that my ideas are out of order. {*Proof has a prescribed order.*}

S2: That can be true. There are some things that can be out of order. {*Proof has structure.*}

S1: Then I can't use ANY order.

S2: Yes and no. You need to keep your goal in mind. {*Proof has a purpose.*}

S1: Prove the last statement in the theorem. {*Proof has a specific end.*}

S2: Right. But you've got to get there. {*Proof has multiple solution paths.*}

S1: That's where I get confused. She says that either I put too much in or I don't put in enough. I wish that she would show us the right way to do it every time. {*Proof is formulaic and to be replicated.*}

S2: It doesn't work that way. You can prove it almost any way you want. {*Proof has multiple solution paths.*} Here's my trick. {*Some solution paths are more useful than others.*} If I need parts or angles to be congruent for the theorem, I know that almost every time I need to show that triangles are congruent. And to show that the triangles are congruent, I need to use either SSS, SAS, ASA, or AAS. {*Analogy made with triangle congruence theorems.*}

S1: I know that those are the ways for the triangles. But I never know which ones to use. {*Proof is formulaic and to be replicated.*}

S2: I use the one that I see based on the information that I know. {*Proof processes are not random; a contextualized natural solution sequence exists.*} If I know that I have two sides congruent, I start looking for either another side for SSS or an angle to get SAS. If I have two angles congruent, I start looking for a side to get either ASA or AAS. If I have an angle and a side congruent, then I will aim for SAS, ASA, or AAS. {*A mental (cognitive, neither physical nor written) analogical connection between geometric elements and theorems.*}

S1: I get that. I really do. But when I get into doing a proof on my own, I get lost. {*Proof has a linear ordering.*}

S2: I think that you forget to prove that the triangles are congruent and look for the information that will do that first. Then you prove that last little piece of the theorem. {*Proofs recognize some methodological consistency. Mental analogical connection of geometric elements with theorems.*}

S1: I can figure out problems, but doing proofs is so tough. For problems, there is usually one way to do it. For proofs, I wish there was only one way. {*Problems are formulaic and mechanical. Proofs should be also.*}
S2: Proofs take a little bit of creativity. {*A subjective component is involved in proofs.*}
S1: I don't want to be creative. I want to get them done the way [the teacher] does. {*Proof is a task to be accomplished, not an experience in which to participate.*}

It is immediately apparent that some students articulated different beliefs regarding the nature of proofs than others. Interestingly, and to be remembered, all student participants were from the same classroom, were taught by the same teacher, and had learning experiences in geometry which were more similar than different. Before considering commonalities and differences between student beliefs, we summarize the beliefs discerned through each student's articulations.

> Student 1: Proofs are formulaic and mechanical, with a prescribed, linear ordering and a specific end.
>
> Student 2: While there is a subjective component is involved in proofs, allowing for multiple solution paths, some paths are more useful than others. Although variation is possible, a proof has structure and some methodological consistency; proof heuristics are not random and a contextualized natural solution sequence exists.
>
> Student 3: While there are multiple possible proof heuristics, they are limited in number. Particular facts must be recognized or deduced in order to lead to a direction for a heuristic.
>
> Student 4: While there may be multiple solutions to a proof, proofs possess a common methodology and some heuristics may be ineffective or advantageous.
>
> Student 5: Proofs have multiple possible heuristics. Different recognized or deduced information leads to different proof directions.
>
> Student 6: Different known or deduced facts lead to different proof heuristics, of which there are many.

Common themes emerge from these student perspectives: a proof has structure and some methodological consistency (S2, S4); proofs have multiple solution paths (S2, S4-6), but some paths are better than others (S2, S4); and proof directions are based on recognized or constructed relationships among elements (S2, S3, S5, S6).

Divergent notions from the students' responses include the following: proofs are formulaic and mechanical, with a prescribed, linear ordering (S1), and there are a limited number of possible proof heuristics for a given theorem (S3).

In proving theories in mathematics, mathematicians and geometers have considered the use of various forms of mechanisms, procedures, and ideas regarding the ordering of proofs (Boesen et al., 2010; Gathercole & Pickering, 2000; Haavold, 2011; Magda, 2015; Park, 2010; Stavrou, 2014; Tall, 2008; van den Broek, Takashima, Segers, Fernández, & Verhoeven, 2013; Wiklund-Hörnqvist et al., 2014). The students in this study recognized that there were necessary components to ordering proofs. Most came to understand that the relational information deduced in the problem led to the direction and ordering of the proof. Student 1 believed that this ordering was defined by the teacher and to be replicated. While most students recognized that there should exist proper methodological orderings within proofs, not all were able to discern these.

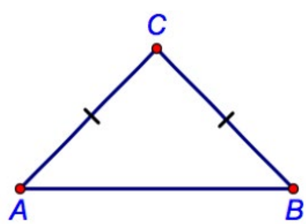
For Students 3-6, the discussion of whether segment *CD* could be an altitude, median, angle bisector, or perpendicular bisector is somewhat revealing. Student 4 created a perpendicular bisector of segment *AB* and assumed that it passed through *C*. He seemed to follow this tack because he did not have a particular direction for the proof. Students who chose the other constructions did so purposely, either because they recognized the direction in which the proof would progress or as an investigatory tool to discern a direction.

Analogies

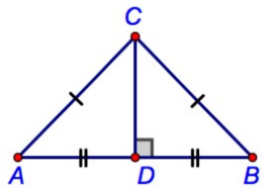
It is apparent through the transcripts that student participants employed analogies in different ways, and often seemed to have different relationships regarding analogies.

On proving a theorem

S3: We're trying to prove that base angles on an isosceles triangle are congruent. So, we can start with this picture. {*Converts verbal problem to a pictorial analogy. An automatic act toward a proof.*}



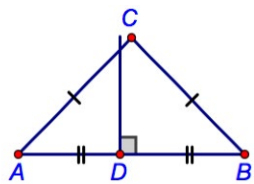
S4: Now we can make the perpendicular bisector. [*Student draws in the segment and marks the perpendicular angle and the segments AD and BD as congruent.*] {*Extends the pictorial analogy. May lack planning regarding the proof.*}



S5: Wait. How do you know that the perpendicular bisector of *AB* goes through *C*? {*Compares pictorial analogy with antagonistic notions, connoting that the picture may be misleading.*}

S4: What do you mean? It's right there. {*Reasoning becomes subservient to false pictorial analogy.*}

S5: We can construct a perpendicular bisector, but it might not go through *C*. Like this. {*Employs a pictorial analogy counterexample.*}



S4: But for that to happen, *D* is not at the midpoint. {*Constructs analogy between the picture and known geometric ideas.*}

S6: How do you know? It's marked as it is. {*Didactically challenges others to reconsider pictorial analogies.*}

S5: Yours could be really close to going through *C*, but not exactly. {*Introduces analogy regarding proximity.*}

S4: But we always use perpendicular bisectors. {*Returns to a previous analogical idea. Believes proofs possess a common methodology.*}

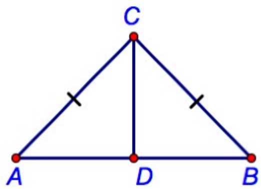
S5: From the midpoint on the line, not to a point off the line. {*Introduces a mental analogy of a line and point.*}

S4: I don't get it. {*Does not connect with previous student's analogy.*}

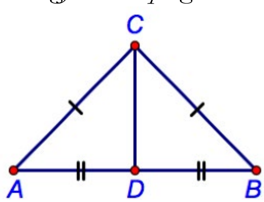
S3: Ok. [*Using hands and gesturing from down to up.* {*Introduces a gestural analogy.*}] We can't raise a perpendicular bisector from *AB* through *C*. But can we drop a perpendicular bisector from *C* to *AB*? {*Returns to a mental analogy regarding a point and a line.*}

S4: Isn't that the same thing? {*Mental analogy from S3 is ineffective.*}

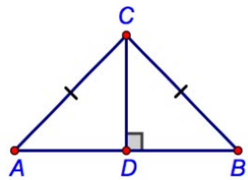
S6: [*Pointing at the diagram.*] You can drop a perpendicular to *AB*, but you don't know that it will hit at a midpoint… {*Returns to a previous pictorial analogy. Refining possible proof heuristics.*}
S5: Or you can drop a median from *C*, but then you won't know if it is perpendicular. {*Providing additional proof heuristics.*}
S4. What if *AB* is above *C*? {*Introduces a mental analogy.*}
S5: What do you mean?
S4: If *AB* is above *C*, then we drop the line from *AB* to *C*. {*Employs the mental analogy.*}
S6: No. 'Drop' and 'raise' don't really mean going down or going up. It means going from and going to. So, the first thing is where we start and the second thing is where we end. {*Employs and extends a verbal analogy.*}
S3: [*Dismissing the previous discussion of raising and dropping as inconsequential.*] But we still need the line, like this. But what is it? {*Develops a new pictorial analogy. Recognizes that the segment is a component of the proof.*}



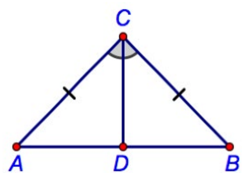
S5: I think it can be a couple of the things already said. It can be a perpendicular from *C*, or a median from *C*. {*Proofs have more than one possible heuristic.*}
S4: Is one better than the other? {*Some possible proof heuristics may be more advantageous.*}
S6: I like the median. It gets me to the easiest proof. Because we need to prove the base angles [$\angle CAB$ *and* $\angle CBA$] congruent, we need to first make the triangles congruent. The median gets me to SSS. {*Introduces analogy of triangle congruence. Recognizes the value of the median toward triangle congruence and the necessity of the latter in respect to this proof.*}
S3: What do you mean?
S6: [*Pointing to and editing the previous picture.*] Look. With the median from *C*, *AD* is congruent to *BD*. And I already had AC is congruent with *BC*. Now, *CD* is congruent to itself. {*Enhancing pictorial analogy. Developing and verifying a proof heuristic.*}



S4: That's great. Because the triangles are congruent, then CPCTC [*congruent parts of congruent triangles are congruent*] and $\angle CAB \cong \angle CBA$. We have a proof.
S3: [*Speaking to S6*] But you said that we can use other lines. Can we do the perpendicular from *C* to *AB*? {*Considers a mental analogy. Considers another proof heuristic.*}
S5: I think so. Let's see. [*Draws the accompanying picture.*] That gives us $\angle ADC \cong \angle BDC$, $AC \cong BC$, and $CD \cong CD$. That gives us SSA. Oh oh. That's ASS, which is not one of our choices. {*Constructs pictorial analogy and makes analogy with known theorems. Recognizes a false proof heuristic.*}



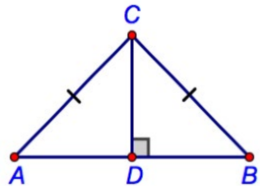
S4: So, we can't use the perpendicular from *C*. {*Recognizes that some proof heuristics are ineffective.*}
S3: I guess that's it. {*Assumes that all potential proof heuristics have been considered.*}
S6: What about an angle bisector at *C*? {*Introduces a mental analogy. Investigates another proof heuristic.*}



S3: Let's try. [*Draws the accompanying figure.*] That gives us… Hey, we have SAS. {*Employs pictorial analogy and connects with analogy using an additional triangle congruence theorem. Investigates and uses another proof heuristic.*}
S4: Hey. Too fast. How do you know that the angle bisector goes through *D*. {*Finds S3's analogy too quickly developed and used.*}
S5: No, it is not that the angle bisector goes through some point *D* already there. The angle bisector goes through *AB*. We make that point *D*, no matter where it is. So, the angle bisector goes through *AB* and that point is *D*. {*Employs verbal analogy.*}
S3: We have $AC \cong BC$, $\angle ACD \cong \angle BCD$, and $CD \cong CD$. {*Employs analogies regarding congruence. Considers the components of congruence in respect to the proof.*}
S4: Cool. {*Accepts analogies from others. Recognizes completion of the proof.*}
S5: Now I think that those are all the options: a median and an angle bisector. {*Verification of multiple proof heuristics.*}

S6: And which one we picked led us to another choice of SSS or SAS. {*Different proof heuristics lead to different results.*}

S3: We couldn't do ASA or AAS. But can we look at the perpendicular again? [*Constructs the accompanying figure.*] We have $AC \cong BC$, $\angle ADC \cong \angle BDC$, $\angle ACD \cong \angle BCD$, and $CD \cong CD$. That seems like a lot of stuff. {*Constructs pictorial analogy using another construction analogy. Considers congruences and how these lead to a possible proof heuristic.*}



S5: How do you know that $\angle ACD \cong \angle BCD$? {*Reconsiders pictorial analogy through congruence. Questions how this construct will lead to a proof.*}

S2: Doesn't it? {*Angle congruence analogy may supersede reasoning.*}

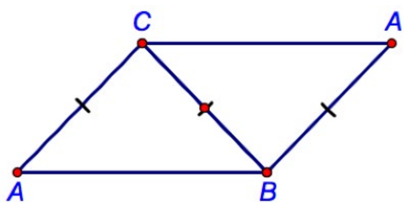
S5: We know it does, but not yet. After we get congruent triangles, then we can CPCTC it. {*Recognizes direction in the proof.*}

S3: Ok. We don't have $\angle ACD \cong \angle BCD$. But we still have $\angle ADC \cong \angle BDC$, $AC \cong BC$, and $CD \cong CD$. That seems like an awful lot to not be able to do anything with. {*Considers necessary information for a proof.*}

S5: Let's work through the list again. For SSS, we would need $AD \cong BD$. We don't have that. For SAS, we would again need $AD \cong BD$. For ASA, we would need $\angle ACD \cong \angle BCD$; but we don't have that either. For AAS, we would again need $\angle ACD \cong \angle BCD$. Those are all of our options and none of them work. {*Makes analogies across required element congruences and triangle congruence theorems. Recognizes information regarding elements leads to different proof directions.*}

S3: So, we got all the proofs. {*Believes that proof directions are limited.*}

S6: All the proofs using that kind of a segment. We could rotate the triangle on one side, get a parallelogram, and then use alternate interior angles. But that would be a lot harder. {*Introduces a new analogy involving a parallelogram. Considers new proof heuristic.*}



Before considering commonalities and differences among student beliefs in the next section, we summarize each student's use and beliefs regarding analogies.

Student 1: Made no discernible use of, nor comment about, analogies.

Student 2: Made analogies with known theorems and employed a mental (cognitive, neither physical nor written) analogical connection between geometric elements and theorems. He encountered an angle congruence analogy that seemingly took pre-eminence and may have hindered further reasoning.

Student 3: Considers a mental (cognitive, neither physical nor written) analogy forwarded by another. He constructs pictorial analogy using construction analogies and employs analogies regarding congruence. He introduces a gestural analogy and uses a mental analogy regarding a point and a line.

Student 4: Accepts analogies from others, but does not always understand them. He constructs and extends analogies between a picture and known geometric ideas and employs mental (cognitive, neither physical nor written) analogies of his own making and from others. His reasoning becomes subservient to false pictorial analogy and he returns to a previous analogical idea.

Student 5: Compares a pictorial analogy with antagonistic notions, connoting that the picture may be misleading. He constructs pictorial analogy, makes an analogy with known theorems, and employs a pictorial analogy as a counterexample. He employs a verbal analogy, introduces a mental (cognitive, neither physical nor written) analogy of a line and point, and introduces an analogy regarding proximity. He makes analogies across required element congruences and triangle congruence theorems and reconsiders a pictorial analogy through congruence.

Student 6: Didactically challenges others to reconsider pictorial analogies. He employs and extends a verbal analogy, enhances a pictorial analogy, and introduces a mental (cognitive, neither physical nor written) analogy. He introduces a new analogy involving a parallelogram and an analogy of triangle congruence.

Discussion

Geometry problem solving has often been characterized by illustration of diagrams as a central heuristic (Bell, 2011; Hanna & de Villiers, 2008; Lamport, 2012; Lin & Lin, 2011; Patkin, 2011; Pfeiffer, 2010; Stavrou, 2014; Stylianides & Andreas, 2009; Tall, 2008; Varghese, 2009). The argument is that most diagrams provide images that are sufficiently obtainable and visualized for problem solving. However, in the context of geometry, it is readily noticed that the student participants in this study employed analogies in numerous ways: making analogies with known theorems; employing mental (cognitive, neither physical nor written) analogies; constructing pictorial analogies; employing analogies regarding geometric relationships; using gestural analogies; comparing pictorial analogies with antagonistic notions; and employing verbal analogies; using an analogy regarding proximity. These varied uses of analogy are consistent with Magda (2015), who recognizes the use of analogies as a creative act. Notably, analogies seem rarely singular; rather, they seem to be interconnected and multimodal.

Analogies are used both as a tool toward geometric proof and as a mechanism of communication. The transcripts demonstrate that a number of these students employed analogies to personally interpret and solve the respective problems. However, in the natural flow of conversation, students also used analogies to communicate their ideas to others. The student transcripts seem to reveal that, while some analogies effectively communicate ideas to others, occasionally this communication is ineffective, since the listener does not seem to grasp the analogy (e.g., Student 4). This may imply that analogies are more idiosyncratic – both in the presenting and in the receiving of such – than previously noted.

While analogies are employed to solve geometric problems and complete proofs, they may also have inhibiting characteristics. For instance, Student 4 seemed to become stymied by some analogies. First, he becomes unable to consider that his analogy could be incorrect and then balks because Student 3's analogy is too quickly constructed and employed without Student 4 fully digesting it. Second, his reasoning seems to become subservient to even his own analogy; once he draws an analogical diagram, he cannot see beyond the fact that it may be incorrect.

Connecting proofs with analogies

While students' use of analogies – particularly diagrammatic analogies – may be helpful in constructing geometric proofs (Bell, 2011; Hanna & de Villiers, 2008; Lamport, 2012; Lin & Lin, 2011; Patkin, 2011; Pfeiffer, 2010; Stavrou, 2014; Stylianides & Andreas, 2009; Tall, 2008; Varghese, 2009), their use seems far from guaranteeing the successful completion of proofs. Indeed, some analogies seem to have the occasional effect of impeding some students and hindering them from progressing beyond the analogy (e.g., Students 2 and 4); some students hear analogies posed by others but do not necessarily grasp them (e.g., Student 4); and some analogies may be misleading (e.g., Students 5 and 6). The fact that analogies may not necessarily lead to success in geometric proofs may be for a number of interconnected ideas: First, since the use of analogies is idiosyncratic, no particular analogy seems to be a panacea for any situation. Second, not all analogies are understood in the same way among individual students. Third, not all analogies effectively communicate ideas. Fourth, in fluid conversation, analogies seem to be posed, used or discarded, and extended or replaced with amazing quickness, with little time for the interlocutors to adequately digest each subsequent analogy. Thus, it is difficult to ascertain the full effect of any analogy.

There is much agreement in the literature that that most mechanisms used in proofs rest on the following: (1) visual (enactive) proof of geometric statements; (2) graphic proof of numeric and algebraic statements; (3) proof in arithmetic by specific and generic calculation; and (4) algebraic proof by algebraic manipulation (Tall, 1998). Some have argued that geometric proofs are often reduced to using generalized arithmetic or algebra to justify conjectures (Bell, 2011; Jonsson et al., 2014; Park et al., 2010; Tall, 1998; Varghese, 2009). In the act of performing proofs, Magda (2015) has also proposed the use of algorithmic and creative reasoning. Summarily, analogical reasoning in proving theorems in geometry incorporates the list provided by Tall (1998, along with algorithmic and creative reasoning as commonly typified (see Fig 2) by visuals, numeric and algebraic symbolism, specific and generic concepts and theorems, as well as algebraic manipulations.

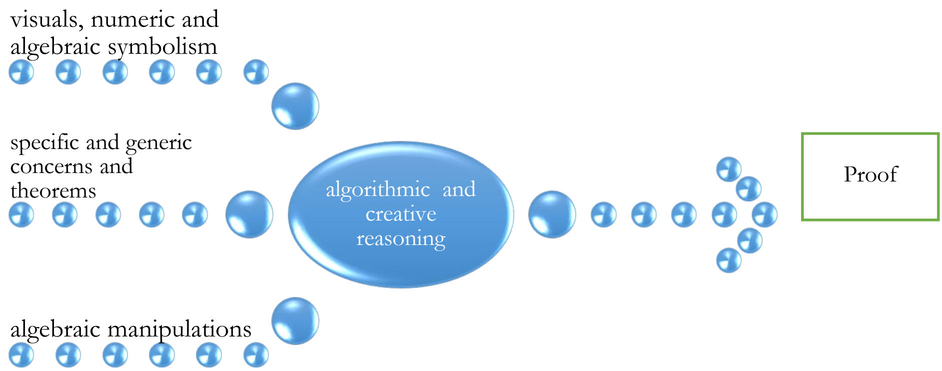


Figure 2. Analogical reasoning in proving theorems

Implications

While this study may have yielded valuable findings, the implications may be of even greater value. Some of these implications are discussed below. The use of analogies has value in the completion of geometric proofs. Therefore, it may seem reasonable that educators should seek opportunities to teach students how to use analogies. However, teaching students to employ analogies may be very complex, particularly when analogies can be recognized to take a myriad of verbal, pictorial, theoretical, mental (cognitive, neither physical nor written), or even gestural forms, and each person finds some forms and uses to be more personally useful and comprehensible than others. It may not be feasible to teach students such a great variety of analogical types, particularly when some types would not be found meaningful or valuable to individual students. As previously discussed, the use of analogies seems to be idiosyncratic to the student. This again would make the teaching of analogies equally idiosyncratic and possibly quite difficult in the classroom. The nature and natural uses of analogies seems to further imply that instruction regarding such would be problematic. The nature and use of analogies seem to be constructed upon one another, be logically nonlinear, and not be equally understood among students. Unfortunately, these implications bode poorly for teaching students about, and regarding the application of, analogies. It may be that the use of analogies is too idiosyncratic to be useful fodder in the classroom. Even if students could be taught to more freely employ analogical reasoning, there is no current understanding that this reasoning will be employed effectively by each student.

Concluding Remarks

In short, this study reveals how very far we need to go to gain a more comprehensible and useable understanding of students' use of analogies, particularly in the context of proofs, before we can make more definitive claims regarding how to increase their use of analogical reasoning. However, we hope that we have cleared a path for others to continue this avenue of research.

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