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Palindromic products*

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Abstract

A graph G on n vertices is said to be *palindromic* if there is a vertex-labeling bijection $f : V(G) \to \{1, 2, \ldots, n\}$ with the property that for any edge $vw \in E(G)$ there is an edge $xy \in E(G)$ for which $f(x) = n - f(v) + 1$ and $f(y) = n - f(w) + 1$.

This notion was defined and explored in a recent paper [R. Beeler, Palindromic graphs, *Bulletin of the ICA*, **85** (2019) 85–100]. The paper gives sufficient conditions on the factors of a Cartesian product of graphs that ensure the product is palindromic, but states that it is unknown whether the conditions are necessary. We prove that the conditions are indeed necessary. Further, we prove a parallel result for the strong product of graphs.

Keywords: Palindromic graphs, cartesian product of graphs, strong product of graphs.

Math. Subj. Class. (2020): 05C76, 05C78

1 Introduction

A recent article by R. Beeler [1] introduced a new concept. A graph G on n vertices is **palindromic** provided that there is a vertex-labeling bijection $f : V(G) \to \{1, 2, \ldots, n\}$ with the property that to each $vw \in E(G)$ there corresponds an $xy \in E(G)$ for which $f(x) = n + 1 - f(v)$ and $f(y) = n + 1 - f(w)$.

Palindromic graphs, like palindromic words, have a certain symmetry. The mapping $V(G) \to V(G)$ whose effect on labels is $k \mapsto n+1-k$ is an **involution** (an automorphism of order 2). View it as a mirror symmetry, where the vertices are ordered on a line by their labels, as in Figure 1.

This induced involution has no fixed vertex if n is even, and exactly one fixed vertex if n is odd. Indeed, we have the following characterization of palindromic graphs as those graphs admitting an involution that fixes at most one vertex. (The **order** of a graph is its number of vertices. For other standard terms and notations not defined here see West [5].)

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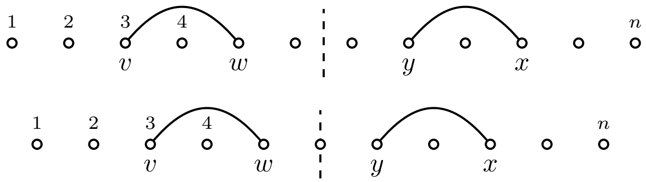


Figure 1: Palindromic graphs of even order admit an involution with no fixed points. Palindromic graphs of odd order admit an involution with exactly one fixed point.

Theorem 1.1 (Beeler [1])**.** *A graph of even order is palindromic if and only if it admits an involution with no fixed vertices. A graph of odd order is palindromic if and only if it admits an involution with exactly one fixed vertex.*

Guided by this theorem, we define a graph to be **even palindromic** if it is palindromic and of even order; it is **odd palindromic** if it is palindromic and of odd order. An involution that fixes at most one vertex is called a **palindromic involution**; one that fixes no vertex is an **even palindromic involution**, and one that fixes exactly one vertex is an **odd palindromic involution**. Thus a graph is even palindromic if and only if admits an even palindromic involution; it is odd palindromic if and only if it admits an odd palindromic involution. A **fixed point** is a fixed vertex.

Beeler [1] characterizes several classes of palindromic graphs, including hypercubes (see Figure 2). More generally he addresses the Cartesian product of graphs, and we will expand upon this in the next section.

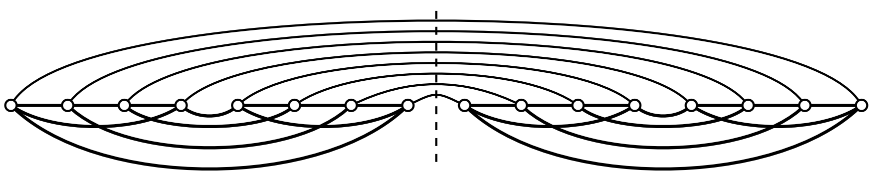


Figure 2: Every hypercube is palindromic. Here is the 4-cube.

2 Cartesian Products

The **Cartesian product** of graphs G and H is the graph $G\Box H$ with vertices $V(G)\times V(H)$ and edges

$$E(G\Box H) = \big\{(x,y)(x',y') \mid xx' \in E(G) \text{ and } y = y', \text{ or } x = x' \text{ and } yy' \in E(H)\big\}.$$

(See Figure 3.) This product is commutative and associative in the sense that the maps $(x, y) \mapsto (y, x)$ and $((x, y), z) \mapsto (x,(y, z))$ are isomorphisms $G\Box H \to H\Box G$ and $(G\Box H)\Box K \to G\Box(H\Box K)$.

Given automorphisms $\alpha : G \to G$ and $\beta : H \to H$, it is straightforward from the definitions that $(x, y) \mapsto (\alpha(x), \beta(y))$ is an automorphism of $G\Box H$. For example, in Figure 3, let $\alpha : G \to G$ be the even palindromic involution of G reflecting G across

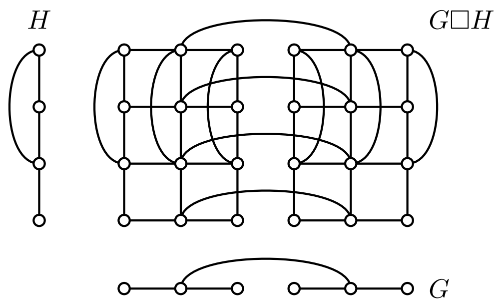


Figure 3: Cartesian product of graphs.

a vertical axis. Let $\beta : H \to H$ be the identity. Then $(x, y) \mapsto (\alpha(x), y)$ is an even palindromic involution of $G \Box H$ reflecting *it* across a vertical axis. This suggests that if one factor of a product is even palindromic, then the product will be even palindromic. Indeed, we have the following result [1, Theorem 4.4].

Lemma 2.1. *If G or H is even palindromic, then $G \Box H$ is even palindromic. If G and H are odd palindromic, then $G \Box H$ is odd palindromic.*

Proof. Let one of G or H (say G) be even palindromic. Theorem 1.1 yields an even palindromic involution $\alpha : G \to G$. Form the even palindromic involution $(x, y) \mapsto (\alpha(x), y)$ of $G \Box H$. Thus the product is even palindromic. For the second statement, say both G and H are odd palindromic. By Theorem 1.1, G has an involution α with exactly one fixed point x_0. (That is, $\alpha(x_0) = x_0$.) For the same reason, H has an involution β with exactly one fixed point y_0. Then $(x, y) \mapsto (\alpha(x), \beta(y))$ is an involution of $G \Box H$ that has exactly one fixed point (x_0, y_0). Therefore $G \Box H$ is odd palindromic. □

Lemma 2.1 spells out conditions on the factors that are sufficient for a palindromic product. Beeler [1] states that it is unknown whether these conditions are also necessary. We will shortly prove that in fact they are, but we first need to review prime factorizations over the Cartesian product.

Observe that $K_1 \Box G \cong G$ for any graph G, so K_1 is the unit for the Cartesian product. A nontrivial graph G is **prime over** □ if for any factoring $G \cong A \Box B$, one of A or B is K_1 and the other is isomorphic to G. Certainly every graph can be factored into prime factors. Sabidussi and Vizing [3, 4] proved that each connected graph has a unique prime factoring up to order and isomorphism of the factors. More precisely, we have the following.

Theorem 2.2 ([2, Theorem 6.8])**.** *Let G and H be isomorphic connected graphs $G = G_1 \Box \cdots \Box G_k$ and $H = H_1 \Box \cdots \Box H_\ell$, where each factor G_i and H_i is prime. Then $k = \ell$, and for any isomorphism $\varphi : G \to H$, there is a permutation π of $\{1, 2, \ldots, k\}$ and isomorphisms $\varphi_i : G_{\pi(i)} \to H_i$ for which*

$$\varphi(x_1, x_2, \ldots, x_k) = \big(\varphi_1(x_{\pi(1)}), \varphi_2(x_{\pi(2)}), \ldots, \varphi_k(x_{\pi(k)})\big).$$

Now we can prove our main result about palindromic Cartesian products.

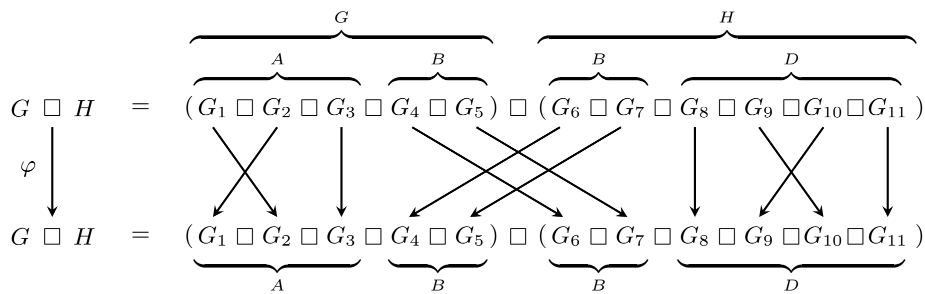
Theorem 2.3. *Suppose G and H are connected graphs. Then:*

(1) *G or H is even palindromic if and only if $G \Box H$ is even palindromic.*

(2) *G and H are odd palindromic if and only if $G \square H$ is odd palindromic.*

Proof. One direction is Lemma 2.1. Conversely, suppose $G \square H$ is palindromic and let φ be a palindromic involution of it. Take prime factorings $G = G_1 \square \cdots \square G_j$ and $H = G_{j+1} \square \cdots \square G_k$, so φ is an involution of $G \square H = (G_1 \square \cdots \square G_j) \square (G_{j+1} \square \cdots \square G_k)$.

The involution φ permutes the prime factors of this product in the sense of Theorem 2.2, where the permutation π satisfies $\pi^2 = \mathrm{id}$. Using commutativity of $\square$, group together the prime factors G_i of G for which $1 < \pi(i) \leq j$, and call their product A. (By convention, $A = K_1$ if there are no such factors G_i. The same applies for the graphs B and D defined below.) Let B be the product of the remaining factors G_i of G. Also group together the prime factors G_i of H for which $j+1 < \pi(i) \leq k$, and call their product D. The Cartesian product of the remaining factors of H is then a graph isomorphic to B. The structure of φ under this scheme is as indicated below, where the arrows represent isomorphisms $\varphi_i : G_{\pi(i)} \to G_i$ between factors.



We have coordinatized G and H as $G = A \square B$ and $H = B \square D$, and φ is an involution of $G \square H = (A \square B) \square (B \square D)$ for which $\varphi\big((a,b),(b',d)\big) = \big((\alpha(a),\beta(b')),(\gamma(b),\delta(d))\big)$, for automorphisms $\alpha : A \to A$, $\beta, \gamma : B \to B$ and $\delta : D \to D$. But because φ^2 is the identity, it must be that $\alpha^2 = \mathrm{id}$, $\gamma = \beta^{-1}$ and $\delta^2 = \mathrm{id}$. Thus we have involutions α and δ of A and D, respectively, and

$$\varphi\big((a,b),(b',d)\big) = \big((\alpha(a),\beta(b')),(\beta^{-1}(b),\delta(d))\big), \tag{2.1}$$

From (2.1) it is evident that the fixed points of φ (if any) are precisely

$$\big((a_0,\beta(b)),(b,d_0)\big) \quad \text{with } \alpha(a_0) = a_0,\ \delta(d_0) = d_0, \text{ and } b \in V(B). \tag{2.2}$$

Thus φ has a fixed point if and only if both α and δ have fixed points. Further, if φ has a fixed point, then it has exactly $|V(B)|$ of them.

Now suppose $G \square H$ is even palindromic. Let φ be an even palindromic involution of $G \square H$ (having no fixed point). From (2.2), at least one of α or δ has no fixed point; say it is α. Then α is an even palindromic involution of A, so A is even palindromic. By the first part of the theorem, $G = A \square B$ is even palindromic. Similarly H is even palindromic if δ has no fixed points.

Suppose $G \square H$ is odd palindromic. Let φ be an odd palindromic involution whose sole fixed point is $\big((a_0,\beta(b_0)),(b_0,d_0)\big)$. The remark following (2.2) implies φ has at least $|V(B)|$ fixed points, so $B = K_1$. Thus we can drop B from our discussion, so $G = A$, $H = D$ and $\varphi(a,d) = \big(\alpha(a),\delta(d)\big)$. We now have involutions $\alpha : G \to G$ and $\delta : H \to H$ with fixed points a_0 and d_0, respectively. Also (a_0,d_0) is a fixed point of φ. If the involution α of G had a second fixed point a_1, then (a_0, d_0) and (a_1, d_0) would be two distinct fixed points of φ. Thus a_0 is the only fixed point of α, so α (hence also G) is odd palindromic. By the same reasoning H is odd palindromic. $\square$

3 Strong Products

The **strong product** of graphs G and H is the graph $G \boxtimes H$ with vertex set $V(G) \times V(H)$, where distinct vertices (x, y) and (x', y') are adjacent whenever

$$\big(xx' \in E(G) \text{ or } y = y'\big) \quad \text{and} \quad \big(x = x' \text{ or } yy' \in E(H)\big).$$

See Figure 3. We quickly review this product's properties; Chapter 7 of [2] proves all assertions made here. The strong product is commutative and associative. If $N_G[x] := N(x) \cup \{x\}$ is the closed neighborhood of a vertex $x \in V(G)$, then

$$N_{G \boxtimes H}[(x, y)] = N_G[x] \times N_H[y]. \tag{3.1}$$

Also $K_1 \boxtimes G \cong G$ for all graphs G. A graph G is **prime** over $\boxtimes$ if for any factoring $G = A \boxtimes B$, one of A or B is K_1 and the other is isomorphic to G.

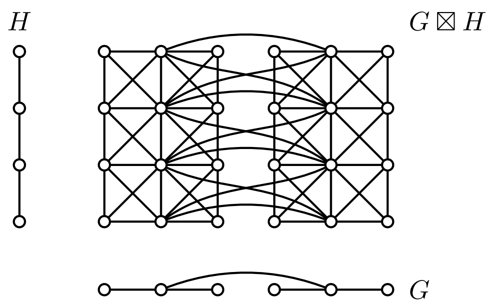


Figure 4: Strong product of graphs.

Given automorphisms $\alpha : G \to G$ and $\beta : H \to H$, it is straightforward from the definitions that $(x, y) \mapsto \big(\alpha(x), \beta(y)\big)$ is an automorphism of $G \boxtimes H$. For instance, in Figure 4, let $\alpha : G \to G$ be the even palindromic involution of G reflecting G across a vertical axis. Say $\beta : H \to H$ is the identity. Then $(x, y) \mapsto \big(\alpha(x), y\big)$ is an even palindromic involution of $G \boxtimes H$ reflecting *it* across a vertical axis (relative to the drawing).

This suggests that we might expect a result for the strong product that is parallel to Theorem 2.3 for the Cartesian product. Indeed, this is exactly the case, but the proof is more involved. The complication is that in general the strong product has no result parallel to Theorem 2.2, unless we impose an additional restriction. A graph is called **S-thin** if no two distinct vertices have the same closed neighborhood. We will need the following analogue of Theorem 2.2 for S-thin graphs.

Theorem 3.1 ([2, Theorem 7.16]). *Let φ be an automorphism of an S-thin connected graph G with prime factorization $G = G_1 \boxtimes G_2 \boxtimes \cdots \boxtimes G_k$. Then there is a permutation π of $\{1, 2, \ldots, k\}$ and isomorphisms $\varphi_i : G_{\pi(i)} \to G_i$ for which $\varphi(x_1, x_2, \ldots, x_k) = \big(\varphi_1(x_{\pi(1)}), \varphi_2(x_{\pi(2)}), \ldots, \varphi_k(x_{\pi(k)})\big)$.*

We say that vertices x and y of a graph are in relation S, written xSy, provided that each has the same closed neighborhood, that is, $N[x] = N[y]$. It is easy to check that S is an equivalence relation of the graph's vertex set. We call an S-equivalence class of $V(G)$ an **S-class** of G. (Note that a graph is S-thin if and only if each S-class consists of a single vertex.) In general, if X is an S-class of graph G, then the subgraph of G induced on X is the complete graph $K_{|X|}$. Also, for any distinct S-classes X and Y, either each vertex of X is adjacent to all vertices of Y, or no vertex of X is adjacent to any vertex of Y.

Given a graph G, we define the quotient G/S to be the graph whose vertices are the S-classes of G, and for which $XY \in E(G/S)$ provided that $X \neq Y$ and G has an edge joining X to Y. Check that G/S is always S-thin.

Because S is defined in terms of the adjaceny structure of a graph, any isomorphism $\varphi : G \to H$ sends S-classes of G bijectively onto S-classes of H. From the discussion above it should be clear that any isomorphism $\varphi : G \to H$ induces an isomorphism $\widetilde{\varphi} : G/S \to H/S$ where $\widetilde{\varphi}(X) = \varphi(X)$, that is, $\widetilde{\varphi}(X)$ is the image of the S-class X under φ.

But the existence of an isomorphism $\widetilde{\varphi} : G/S \to H/S$ does not necessarily mean that there is an isomorphism $\varphi : G \to H$. However, if $|X| = |\widetilde{\varphi}(X)|$ for each $X \in V(G/S)$, then we can lift $\widetilde{\varphi}$ to an isomorphism $\varphi : G \to H$ simply by declaring φ to restrict to a bijection $X \to \widetilde{\varphi}(X)$ for each X.

Using Equation (3.1), one can show that the S-classes of $G \boxtimes H$ are precisely the (set) Cartesian products $X \times Y$, where X is an S-class of G and Y is an S-class of H. In other words, the vertices of $(G \times H)/S$ are $X \times Y$, where $X \in V(G/S)$ and $Y \in V(H/S)$. Further, there is a natural isomorphism

$$\begin{array}{ccc} (G \boxtimes H)/S & \longrightarrow & G/S \boxtimes H/S \\ X \times Y & \longmapsto & (X, Y). \end{array} \tag{3.2}$$

In the proof our main theorem we will switch between $X \times Y$ and (X, Y) when expedient.

The proof also uses all ideas discussed so far in this section.

Theorem 3.2. *Suppose G and H are connected graphs. Then:*

(1) *G or H is even palindromic if and only if $G \boxtimes H$ is even palindromic.*

(2) *G and H are odd palindromic if and only if $G \boxtimes H$ is odd palindromic.*

Proof. If G or H (say G) is even palindromic, then there exists an even palindromic involution α of G, so $(x, y) \mapsto (\alpha(x), y)$ is an even palindromic involution of $G \boxtimes H$. Next suppose G and H are odd palindromic. Then G has an odd palindromic involution α with fixed point x_0, and H has an odd palindromic involution β with fixed point y_0. Then $(x, y) \mapsto (\alpha(x), \beta(y))$ is an odd palindromic involution of $G \boxtimes H$ whose sole fixed point is (x_0, y_0).

It remains to prove the converses of the two statements. We will do this in three parts. The first part codifies the structure of involutions of $G \boxtimes H$.

Part I (Involution structure) Let $\varphi : G \boxtimes H \to G \boxtimes H$ be an involution. By the remarks preceding this theorem, φ induces an automorphism $\widetilde{\varphi}$ of the S-thin graph $(G \boxtimes H)/S \cong G/S \boxtimes H/S$. Because φ is an involution, we have $\widetilde{\varphi}^2 = \mathrm{id}$. (Note that $\widetilde{\varphi}$ could be the identity even if φ is not. This is the case if φ fixes each S-class, i.e., it restricts to a permutation on each S-class.)

Take prime factorings $G/S = G_1 \boxtimes \cdots \boxtimes G_j$ and $H/S = G_{j+1} \boxtimes \cdots \boxtimes G_k$. Then $\widetilde{\varphi}$ is an automorphism (of order 1 or 2) of the graph

$$G/S \boxtimes H/S = (G_1 \boxtimes \cdots \boxtimes G_j) \boxtimes (G_{j+1} \boxtimes \cdots \boxtimes G_k).$$

Now, $\widetilde{\varphi}$ permutes the prime factors of this product in the sense of Theorem 3.1, where the permutation π satisfies $\pi^2 = \mathrm{id}$. As in the proof of Theorem 2.3, group together the prime factors G_i of G/S for which $1 < \pi(i) \leq j$, and call their product A. Let B be the product of the remaining factors of G/S. Also group together the prime factors G_i of H/S for which $j+1 < \pi(i) \leq k$, and call their product D. The product of the remaining factors of H/S is then a graph isomorphic to B. Now we have $G/S = A \boxtimes B$ and $H/S = B \boxtimes D$, and $\widetilde{\varphi}$ is an automorphism of

$$G/S \boxtimes H/S = (A \boxtimes B) \boxtimes (B \boxtimes D)$$

satisfying $\widetilde{\varphi}^2 = \mathrm{id}$, and for which (as in the proof of Theorem 2.3) we have

$$\widetilde{\varphi}\big((a,b),(b',d)\big) = \big((\alpha(a),\beta(b')),(\beta^{-1}(b),\delta(d))\big) \tag{3.3}$$

for automorphisms $\alpha : A \to A$, $\beta : B \to B$ and $\delta : D \to D$, with $\alpha^2 = \mathrm{id}$ and $\delta^2 = \mathrm{id}$.

In (3.3), the ordered pairs (a,b) and $(\alpha(a),\beta(b'))$ are vertices of G/S, which are S-classes of G (subsets of $V(G)$), and hence they have cardinalities $|(a,b)|$ and $|(\alpha(a),\beta(b'))|$. Similarly, (b',d) and $(\beta^{-1}(b),\delta(d))$ are S-classes of H/S.

By the remarks preceding this theorem, the involution φ of $G \boxtimes H$ sends the S-class $(a,b) \times (b',d)$ bijectively to S-class $(\alpha(a),\beta(b')) \times (\beta^{-1}(b),\delta(d))$, so

$$|(a,b)| \cdot |(b',d)| = \big|(\alpha(a),\beta(b'))\big| \cdot \big|(\beta^{-1}(b),\delta(d))\big| \tag{3.4}$$

for all $a \in V(A)$, $b,b' \in V(B)$ and $d \in V(D)$. Putting $b' = \beta^{-1}(b)$ yields

$$|(a,b)| \cdot |(\beta^{-1}(b),d)| = \big|(\alpha(a),b)\big| \cdot \big|(\beta^{-1}(b),\delta(d))\big|. \tag{3.5}$$

In (3.5) replace d with $\delta(d)$ (and use $\delta^2 = \mathrm{id}$) to get

$$|(a,b)| \cdot |(\beta^{-1}(b),\delta(d))| = \big|(\alpha(a),b)\big| \cdot \big|(\beta^{-1}(b),d)\big|. \tag{3.6}$$

Equations (3.5) and (3.6) imply $|(a,b)| = |(\alpha(a),b)|$. Form an automorphism $\widetilde{\alpha} : A \boxtimes B \to A \boxtimes B$ as $\widetilde{\alpha}(a,b) = \big(\alpha(a),b\big)$. Then $\widetilde{\alpha}^2 = \mathrm{id}$, so we have an involution (if it is not the identity map) $\widetilde{\alpha} : G/S \to G/S$ that maps each vertex (S-class) (a,b) to the vertex (S-class) $(\alpha(a),b)$ of the same cardinality.

Also (3.5) and (3.6) yield $\big|\big(\beta^{-1}(b),\delta(d)\big)\big| = \big|\big(\beta^{-1}(b),d\big)\big|$, so $|(b,\delta(d))| = |(b,d)|$ for all $b \in V(B)$ and $d \in V(D)$. Form the automorphism $\widetilde{\delta} : B \boxtimes D \to B \boxtimes D$ where $\widetilde{\delta}(b,d) = \big(b,\delta(d)\big)$. Then $\widetilde{\delta}^2 = \mathrm{id}$, so we have an involution (if not the identity map) $\widetilde{\delta} : H/S \to H/S$ mapping each S-class (b,d) to the S-class $(b,\delta(d))$ of the same cardinality.

In summary, for any involution φ of $G \boxtimes H$, we have constructed automorphisms $\widetilde{\alpha}$ and $\widetilde{\delta}$ of G/S and H/S, respectively, for which $\widetilde{\alpha}^2 = \mathrm{id}$ and $\widetilde{\delta}^2 = \mathrm{id}$. And $|\widetilde{\alpha}((a,b))| = |(a,b)|$ for any S-class (a,b) of G. Thus we can lift $\widetilde{\alpha}$ to an automorphism $\lambda : G \to G$ by declaring that λ restricts to a bijection $(a,b) \to \big(\alpha(a),b\big)$, for each S-class (a,b) of G. Similarly, $|\widetilde{\delta}((b,d))| = |(b,d)|$ for any S-class (b,d) of H, so we can lift $\widetilde{\delta}$ to an automorphism $\mu : H \to H$. In parts II and III of the proof these lifts will be palindromic involutions.

To carry out this plan we will need to consider S-classes of $G \boxtimes H$ that are fixed by φ (i.e., the S-classes whose vertices are permuted by φ.) By Equation (3.3), the fixed points of $\widetilde{\varphi}$ (respectively, the fixed S-classes of φ) are

$$\big((a_0, \beta(b)), (b, d_0)\big) \qquad \text{where } \alpha(a_0) = a_0,\ \delta(d_0) = d_0 \text{ and } b \in V(B) \tag{3.7}$$
$$(a_0, \beta(b)) \times (b, d_0) \qquad \text{where } \alpha(a_0) = a_0,\ \delta(d_0) = d_0 \text{ and } b \in V(B). \tag{3.8}$$

We call an S-class **even** (**odd**) if it has even (odd) cardinality.

Part II (Converse of Statement (1)) Suppose $G \boxtimes H$ is even palindromic. Then there is an even palindromic involution φ of $G \boxtimes H$. We retain the development and notation of Part I of the proof.

Our strategy is to show that one of $\widetilde{\alpha} : G/S \to G/S$ or $\widetilde{\delta} : H/S \to H/S$ has no odd fixed point (S-class). For if this is the case for (say) $\widetilde{\alpha}$, then $\widetilde{\alpha}$ can be lifted to an automorphism $\lambda : G \to G$ sending any S-class (a, b) bijectively to $(\alpha(a), b)$. Whenever $\widetilde{\alpha}$ fixes an S-class (a, b), we can arrange for λ to restrict to an order-2 fixedpoint-free permutation of the even set (a, b). Then λ will be an even palindromic involution of G, so G is even palindromic.

Suppose to the contrary that $\widetilde{\alpha}$ had an odd fixed point (a, b) and $\widetilde{\delta}$ had an odd fixed point (b', d). (So $\alpha(a) = a$ and $\delta(d) = d$.) By (3.4),

$$\underbrace{\big|(a, b)\big|}_{\text{odd}} \cdot \underbrace{\big|(b', d)\big|}_{\text{odd}} = \big|(a, \beta(b'))\big| \cdot \big|(\beta^{-1}(b), d)\big|.$$

Then $(a, \beta(b'))$ is odd, so $(a, \beta(b')) \times (b', d)$ is an odd S-class of $G \boxtimes H$. But the involution φ fixes this odd S-class, by (3.8). Thus φ fixes some point of this S-class, contradicting the fact that φ is even palindromic.

Part III (Converse of Statement (2)) Suppose $G \boxtimes H$ is odd palindromic. Then there is an odd palindromic involution φ of $G \boxtimes H$ with fixed point (x_0, y_0). Then φ fixes the S-class X that contains (x_0, y_0), which necessarily has form $X = (a_0, \beta(b_0)) \times (b_0, d_0)$, where $\alpha(a_0) = a_0$ and $\delta(d_0) = d_0$. (See (3.8) in Part I.) As the involution φ fixes exactly one vertex, which is in X, we know X has odd cardinality. Thus $(a_0, \beta(b_0))$ is an odd S-class of G/S, and (b_0, d_0) is an odd S-class of H/S. Note that $(a_0, \beta(b_0))$ is a fixed point of $\widetilde{\alpha}$ and (b_0, d_0) is a fixed point of $\widetilde{\delta}$. Suppose $\widetilde{\delta}$ had another odd fixed point (b_1, d_1). Then $\delta(d_1) = d_1$ and by Equation (3.4),

$$\underbrace{\big|(a_0, \beta(b_0))\big|}_{\text{odd}} \cdot \underbrace{\big|(b_1, d_1)\big|}_{\text{odd}} = \big|(a_0, \beta(b_1))\big| \cdot \big|(b_0, d_1)\big|.$$

Therefore $\big|(a_0, \beta(b_1))\big|$ and $|(b_0, d_1)|$ are odd. Then $(a_0, \beta(b_1)) \times (b_1, d_1)$ and $(a_0, \beta(b_0)) \times (b_0, d_1)$ are odd S-classes of $G \times H$ that are fixed by φ. But $X = (a_0, \beta(b_0)) \times (b_0, d_0)$ is the only such S-class, hence $\beta(b_1) = \beta(b_0)$ and $d_1 = d_0$. This means $(b_1, d_1) = (b_0, d_0)$. Conclusion: (b_0, d_0) is the only odd S-class of H/S that is fixed by $\widetilde{\delta}$. Therefore we can lift $\widetilde{\delta} : H/S \to H/S$ to an odd palindromic involution $\mu : H \to H$ sending each S-class (b, d) bijectively to $(b, \delta(d))$, having only one fixed vertex on the odd fixed class (b_0, d_0) and no fixed points on any other fixed (even) S-class. Thus H is odd palindromic.

By a symmetric argument, G is also odd palindromic. $\square$

4 Conclusion and Open Questions

Our Theorems 2.3 and 3.2 characterize palindromic Cartesian and strong products in terms of the palindromic properties of their factors. There are four standard associative graph products, the Cartesian, strong, direct and lexicographic products. (See [2].) Here we have only addressed two of these four products. A natural unexplored problem, then, is to establish analogous results for palindromic direct and lexicographic products. However, because the automorphism structure of these products is not as rigid as for the Cartesian and strong products (cf. Theorems 2.2 and 3.1 above), the results and proofs are likely to be substantially different from those presented here.

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A note on a candy sharing game*

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Abstract

Suppose k students sit in a circle and are each distributed some initial amount of candy. Each student begins with an even amount of candy, but their individual amounts may vary. Upon the teacher's signal, each student passes half of their candy to their left and keeps half. After this step, any student with an odd amount of candy receives an extra piece. The game ends if all the students are holding the same amount of candy. We prove, in a generalized setting, that for any initial distribution of n pieces of candy, the game terminates after $O(\log n)$ many iterations and each student ends with $\frac{n}{k} + O(\log n)$ many pieces. Moreover, there exist initial distributions for which the $O(\log n)$ term cannot be improved.

Keywords: Games on graphs, Markov chains.

Math. Subj. Class. (2020): 05C20, 60J10

1 Introduction

In this note, we analyze a game referred to as the *candy sharing game*. Suppose a fixed number k many students sit in a circle and are each distributed some initial amount of candy. Each student begins with an even amount of candy, but their individual amounts may vary. Upon the teacher's signal, each student passes half of their candy to their left and keeps half of their candy. After this step, any student with an odd amount of candy receives an extra piece. The game ends if all the students are holding the same amount of candy. Does the game end after finitely many steps for any initial distribution?

This question originated as a problem in the Beijing Math Olympiad [2]. A web search shows it remains popular as a fun activity for budding mathematicians as well as a challenge

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in computer coding competitions. The answer to the question is "yes" and it boils down to the following three observations: (1) The maximum amount of candy held by any player can never increase. (2) The number of players holding the currently minimum amount of candy will decrease by at least one each step. Thus the minimum amount will increase after at most k steps. (3) Once the maximum and minimum are equal, the game terminates.

In [4], Iba and Tanton consider a generalized version of the game in which each player fixes an integer and at each step shares portions of their candy to some subset of the other players. After receipt of candy from the other players, they round up to the nearest multiple of their chosen integer. They prove that under certain conditions, such games are also *bounded*, i.e., reach a stable state after some finite number of moves. In [1], Cairns considers a version of the game where at each step, each player with more than one piece of candy passes one piece to their left and one to their right. The game is played until it settles into a fixed state or an oscillatory pattern. He completely characterizes the long term behavior given any initial distribution in the case then the number of students and candies are both k.

For the original candy sharing game, given an initial distribution of candy, predicting the length of the game and the final stabilizing amount is an intriguing open question. The main purpose of this note is to provide an upper bound on each of the above parameters which is tight infinitely often (at least in the case when the number of players $k = 3$).

We will consider the candy sharing game played in a generalized setting. We say a directed graph G is *d-regular* if every vertex has in- and out- degree equal to d. The candy sharing game can be played on any d-regular graph G as follows. Each vertex (player) is distributed some amount of candy which is divisible by d. At each step, each player hands $1/d$ proportion of their candy to each of their out-neighbors and is handed candy from each of their in-neighbors. After this, each student is handed $0, 1, \ldots,$ or $d-1$ pieces of candy to ensure they are holding a multiple of d. Thus the original candy sharing game is one played on the directed cycle of length k with a directed loop at each vertex. A directed graph G is *strongly connected* if between any two vertices u and v, there exists a directed u, v-path. G is *aperiodic* if the greatest common divisor of the lengths of its cycles is 1. Our main theorem is as follows.

Theorem 1.1. *Let $k \geq d \geq 2$ be fixed. For any $\boldsymbol{d} = (a_1, \ldots, a_k)$ such that $\sum_{i=1}^{k} a_i = n$ the candy sharing game, played with initial candy distribution $\boldsymbol{d}$, on a strongly connected, aperiodic, d-regular directed graph G ends in $O(\log n)$ turns, and every player will be holding $\frac{n}{k} + O(\log n)$ pieces of candy.*

The next proposition shows that, at least in some specific cases, the order of the $\log n$ term in the final distribution cannot be improved.

Proposition 1.2. *There exist infinitely many values of n such that the candy sharing game with initial distribution $\boldsymbol{d} = (n, 0, 0)$ played on a directed cycle of length 3 with a directed loop on each vertex terminates with after $\Omega(\log n)$ turns with every player holding $\frac{n}{3} + \Omega(\log n)$ pieces of candy.*

In the next section we introduce the definitions and results from the theory of Markov chains necessary for the proof. In Section 3 we prove Theorem 1.1.

2 Notation and background

Throughout the paper, we consider k and d to be fixed, so $O(\cdot)$ and $\Omega(\cdot)$ notation is suppressing constants which may depend on k and d. All logarithms are natural unless otherwise stated. Vectors will be denoted by boldface characters and for a vector $\mathbf{v}$, we use the notation $\mathbf{v}(i)$ to denote the ith entry of $\mathbf{v}$. Suppose the game is played on a directed graph G with vertex set $V(G) = \{1, \ldots, k\}$ and edge set $E(G)$. For $t = 0, 1, 2, \ldots$, let $\mathbf{d}_t = (a_{1,t}, a_{2,t}, \ldots, a_{k,t})$ where $a_{i,t}$ represents the amount of candy held by player i after t steps of the game. One can check that for any $t \geq 1$, if $\mathbf{d}_{t-1} = (b_1, \ldots, b_k)$, then we have the ith entry of $\mathbf{d}_t$ is given by

$$\mathbf{d}_t(i) = d \cdot \left\lceil \frac{1}{d} \sum_{j:ji \in E(G)} \frac{b_j}{d} \right\rceil .$$

At each step of the game, if a player has an amount of candy not divisible by d, then they receive extra pieces. Each piece of candy introduced in this way is referred to as a *draw*. Note that if the game terminates with each player holding s pieces of candy, then the total amount of candy at termination is sk. So the total number of draws in a candy game with n pieces initially distributed is $sk - n$. The number of draws up to and including turn t is denoted Δ_t.

To continue we need some definitions and results regarding Markov chains. All notation and definitions follow those in [5]. A Markov chain with state space Ω and transition matrix P is a sequence of random variables $(X_0, X_1, \ldots)$ on Ω such that for all $i \geq 0$, if X_i has distribution μ, then X_{i+1} has distribution μP. We represent distributions on Ω as row vectors and P as an $|\Omega| \times |\Omega|$ matrix where entry P_{ij} represents the probability of transitioning from state i to state j.

We say a chain is *irreducible* if, for any two states, there is a finite number of steps in which it is possible to transition from one state to the other with positive probability. This number of steps may be dependent on the chosen states. A chain is *aperiodic* if, for each state, the greatest common divisor of the set of times that it is possible to transition from the state back to itself is 1.

Further, a distribution $\boldsymbol{\pi}$ is *stationary for P* if $\boldsymbol{\pi} P = \boldsymbol{\pi}$. Any irreducible chain has a unique stationary distribution, and each starting state will converge to the stationary. The *total variation distance* between two probability distributions, μ and ν is given by $||\mu - \nu||_{TV} = \max_{A \subset \Omega} |\mu(A) - \nu(A)|$. There is a convenient formula for calculating the total variation distance, given by

$$||\mu - \nu||_{TV} = \frac{1}{2} \sum_{x \in \Omega} |\mu(x) - \nu(x)| . \tag{2.1}$$

As we are concerned with how quickly the game stabilizes, we will make use of the following theorem (Thm 4.9 in [5]) which gives a bound on the rate of convergence of an irreducible aperiodic Markov chain.

Theorem 2.1 (Convergence Theorem)**.** *If P is the transition matrix of an irreducible and aperiodic chain, with stationary distribution π, then for any initial probability distribution x on state space Ω, there exist constants $\alpha \in (0, 1)$ and $C > 0$ such that for all $t \geq 0$, $||xP^t - \pi||_{TV} \leq C\alpha^t$.*

3 Proof of Theorem 1.1

Proof of Theorem 1.1. Given a d-regular directed graph G on vertex set $\{1, 2, \dots, k\}$, we may form a $k \times k$ transition matrix P whose ij entry is $1/d$ if ij is an edge, and 0 otherwise. We can view the Markov chain with transition matrix P as a *randomized candy sharing game* where at each stage, vertices no longer draw, but instead individually distribute each piece of candy uniformly at random to one of its out-neighbors. As an example, the transition matrix for the original candy sharing game is given by

$$P = \left.\underbrace{\begin{bmatrix} 1/2 & 1/2 & 0 & 0 & \dots & 0 \\ 0 & 1/2 & 1/2 & 0 & \dots & 0 \\ 0 & 0 & 1/2 & 1/2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1/2 & 0 & 0 & 0 & \dots & 1/2 \end{bmatrix}}_{k}\right\} k.$$

Let $\mathbf{d}_0 = (a_1, \dots, a_k)$ with $n = \sum a_i$, $\tilde{\mathbf{c}}_0 = \mathbf{d}_0$, and for $t \geq 1$, let $\tilde{\mathbf{c}}_t = \tilde{\mathbf{c}}_{t-1}P = \tilde{\mathbf{c}}_0 P^t$. Then $\tilde{\mathbf{c}}_t(i)$ represents the expected amount of candy held by vertex i after t turns of the randomized candy sharing game. Let $\mathbf{c}_t = \frac{1}{n}\tilde{\mathbf{c}}_t$.

Note that since the randomized candy sharing game has no draws, we have the following inequalities:

$$\begin{aligned} \min(\mathbf{d}_t) &\geq \min(\tilde{\mathbf{c}}_t) \\ \max(\mathbf{d}_t) &\leq \max(\tilde{\mathbf{c}}_t) + \Delta_t. \end{aligned} \tag{3.1}$$

Lemma 3.1. *If a candy sharing game is played on a strongly connected, aperiodic, d-regular directed graph G with k vertices, then in at most $k^2 - 2k + 2$ turns either the game will terminate or the minimum amount of candy held by any player will increase.*

Proof. Let t represent the current turn of the game, with $t = 0$ being the initial distribution. Let $\tilde{\mathbf{c}}_t = \mathbf{d}_t = (a_1, a_2, \dots, a_k)$ with $a_i \in \mathbb{Z}_+$ for $i = 1, \dots k$, and further assume that not all entries of this vector are equal (meaning the game has not terminated). In particular, some entry of $\tilde{\mathbf{c}}_t$ is larger than $\min(\tilde{\mathbf{c}}_t)$. Given that G is d-regular the transition matrix, P, for this chain is doubly stochastic, that is, all of its row and column sums are 1. It is easily verifiable that the product of doubly stochastic matrices is also doubly stochastic.

A theorem of Wielandt (see [6]) says that for a $k \times k$, irreducible, non-negative matrix P, there exists an integer $r \leq k^2 - 2k + 2$ such that $P^r_{ij} > 0$ for all i, j. (In fact, a theorem of Dulmage and Mendelsohn [3] says that if P additionally has a non-zero diagonal entry, we have P^r has all positive entries for some $r \leq 2k - 2$.)

Advancing the randomized game, $\tilde{\mathbf{c}}$, by r turns and letting $A_1, \dots A_k$ represent the columns of matrix P^r, we have

$$\tilde{\mathbf{c}}_{t+r} = \tilde{\mathbf{c}}_t P^r = (\tilde{\mathbf{c}}_t \cdot A_1, \tilde{\mathbf{c}}_t \cdot A_2, \dots, \tilde{\mathbf{c}}_t \cdot A_k).$$

Since P^r is doubly stochastic, each entry of $\tilde{\mathbf{c}}_{t+r}$ is a weighted average of the entries of $\tilde{\mathbf{c}}_t$. Since there exist entries of $\tilde{\mathbf{c}}_t$ that are larger than $\min(\tilde{\mathbf{c}}_t)$, each weighted average is larger than $\min(\tilde{\mathbf{c}}_t)$. Thus, using (3.1), we have

$$\min(\mathbf{d}_{t+r}) \geq \min(\tilde{\mathbf{c}}_{t+r}) > \min(\tilde{\mathbf{c}}_t) = \min(\mathbf{d}_t). \qquad \square$$

We can clearly see that if $\max(\mathbf{d}_t) - \min(\mathbf{d}_t) < 1$ for some t, then the discrete game has ended. Let $\boldsymbol{\pi} = \left(\frac{1}{k}, \frac{1}{k}, \ldots, \frac{1}{k}\right)$. Then one can check that $\boldsymbol{\pi} P = \boldsymbol{\pi}$, that is, $\boldsymbol{\pi}$ is the stationary distribution for P. Let α and C be given by Theorem 2.1 for matrix P and initial distribution $\mathbf{c}_0$. Then after t steps we have $||\mathbf{c}_t - \boldsymbol{\pi}||_{TV} < C\alpha^t$. Let $t_0 = \left\lceil \frac{\log(2Cn)}{\log(\frac{1}{\alpha})} \right\rceil$. Then

$$||\mathbf{c}_{t_0} - \boldsymbol{\pi}||_{TV} < \frac{1}{2n}. \tag{3.2}$$

Using the inequalities from (3.1), followed by normalizing the randomized game and utilizing the triangle inequality we have

$$\begin{aligned}
|\max(\mathbf{d}_t) - \min(\mathbf{d}_t)| &\le |\max(\tilde{\mathbf{c}}_t) - \min(\tilde{\mathbf{c}}_t)| + \Delta_t \\
&\le n \left|\max(\mathbf{c}_t) - \min(\mathbf{c}_t)\right| + \Delta_t \\
&\le n \left(\left|\max(\mathbf{c}_t) - \frac{1}{k}\right| + \left|\min(\mathbf{c}_t) - \frac{1}{k}\right| \right) + \Delta_t.
\end{aligned}$$

In the above sum, we compare the distance between two entries of $\mathbf{c}_t$ and two entries of $\boldsymbol{\pi}$. Including the remaining distances leads to the next inequality as $k \ge 2$.

$$n \left(\left|\max(\mathbf{c}_t) - \frac{1}{k}\right| + \left|\min(\mathbf{c}_t) - \frac{1}{k}\right| \right) + \Delta_t \le n \sum_{i=1}^{k} \left|\mathbf{c}_t(i) - \frac{1}{k}\right| + \Delta_t.$$

By (2.1), we have

$$n \sum_{i=1}^{k} \left|\mathbf{c}_t(i) - \frac{1}{k}\right| + \Delta_t = 2n\, ||\mathbf{c}_t - \boldsymbol{\pi}||_{TV} + \Delta_t.$$

Now, Δ_t is bounded above by $(d-1)kt$, since at most every player will have to draw $(d-1)$ pieces every turn. Therefore, after t_0 turns, we have

$$\begin{aligned}
|\max(\mathbf{d}_{t_0}) - \min(\mathbf{d}_{t_0})| &\le 2n\, ||\mathbf{c}_{t_0} - \boldsymbol{\pi}||_{TV} + \Delta_{t_0} \\
&\le 2n \left(\frac{1}{2n}\right) + (d-1)kt_0 \\
&= 1 + (d-1)k \cdot \frac{\log(2Cn)}{\log(\frac{1}{\alpha})}
\end{aligned}$$

where in the second inequality we have used (3.2). Thus we have that there exists a constant $C' = C'(k,d)$ such that after t_0 turns, $|\max(\mathbf{d}_{t_0}) - \min(\mathbf{d}_{t_0})| < C' \log n$.

Recall that $\max(\mathbf{d}_t)$ cannot increase and by Lemma 3.1, $\min(\mathbf{d}_t)$ is guaranteed to increase every $k^2 - 2k + 2 \le k^2$ turns. Therefore, after at most $k^2 C' \log n$ more turns, $|\max(\mathbf{d}_t) - \min(\mathbf{d}_t)|$ will be less than 1, and thus the game will have ended. From this we know the total number of turns the game took is at most $t_0 + k^2 C' \log n = \frac{\log(2Cn)}{\log(\frac{1}{\alpha})} + k^2 C' \log n < C'' \log n$ for some constant $C'' = C''(k,d)$. At worst, each player draws $(d-1)$ pieces of candy every turn. So the total amount of candy at the end of the game is at most $n + (d-1)kC'' \log n$ implying that each player has $\frac{n}{k}$ plus at most $O(\log n)$ pieces. □

4 Proof of Proposition 1.2

Proof of Proposition 1.2. Consider the sequence $(r_i)_{i=1}^{\infty}$ defined recursively by $r_\ell = 4r_{\ell-1} + 2$ and $r_1 = 2$. We examine the game played on a directed cycle of length 3 with loops at each vertex. With initial distribution $\mathbf{d} = (r_\ell, 0, 0)$, the sequence of states is as follows. An arrow indicates advancing a turn, and a number over the arrow represents how many pieces were drawn that turn. For $\ell \geq 2$,

$$\begin{aligned}(r_\ell, 0, 0) = (4r_{\ell-1} + 2, 0, 0) &\xrightarrow{+2} (2r_{\ell-1} + 2, 2r_{\ell-1} + 2, 0) \\ &\xrightarrow{+2} (r_{\ell-1} + 2, 2r_{\ell-1} + 2, r_{\ell-1} + 2).\end{aligned}$$

Notice that each player is holding at least $r_{\ell-1} + 2$ pieces of candy. We can imagine that they each divide their current piles into two: an inner pile consisting of the $r_{\ell-1} + 2$ pieces, and an outer pile containing the remainder. The players would then continue by playing two concurrent games, following the game procedure on the inner pile and outer pile simultaneously. Since the inner pile is the same amount for each player, that game has already terminated and will no longer draw extra pieces. The only draws will then come from the outer game which, after invoking symmetry, is equivalent to the game played with initial distribution $\mathbf{d}' = (r_{\ell-1}, 0, 0)$.

Further, note that the r_ℓ sequence has binary representation

$$10, 1010, 101010, 10101010, \ldots$$

where the ℓ^{th} element is the digits (10) repeated ℓ times. Each time the above recursion is applied to a game of the form $(r_\ell, 0, 0)$, two turns elapse, four pieces of candy are drawn and two digits are removed from the binary representation of the initial candy amount. We finally see that we need to draw 4 times the length of a base 2 expansion of r_ℓ, which is logarithmic. Therefore, letting $n = r_\ell$, the sequence of games played with initial distribution $\mathbf{d} = (n, 0, 0)$ will terminate after $\Theta(\log n)$ turns with each player holding $\frac{n}{3} + \Theta(\log n)$ many pieces of candy. $\square$

5 Conclusion

In this note, we have proved a relationship between the candy sharing game with rounding and a Markov chain without rounding and used the convergence theorem to find a bound on the length of the candy sharing game. The main open problem in candy sharing is to find a closed form expression for the number of rounds and ending amount of candy in terms of the initial candy distribution. It seems this may be quite difficult. Another interesting problem may be to try and prove a result similar to ours for the general games considered in [4].

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Maps and Δ-matroids revisited

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Abstract

Using Tutte's combinatorial definition of a map we define a Δ-matroid purely combinatorially and show that it is identical to Bouchet's topological definition.

Keywords: Matroid, Δ-matroid, cellular map, topological surface.

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1 Matroids and Δ-matroids

A *matroid* M is a finite set E and a non-empty collection $\mathcal{B}$ of subsets of E satisfying the condition that if

(MB) If B_1 and B_2 are in $\mathcal{B}$ and $x \in B_1 \setminus B_2$ then there exists $y \in B_2 \setminus B_1$ such that $(B_1 \cup \{y\}) \setminus \{x\} = B_1 \triangle \{x, y\} \in \mathcal{B}$.

Axiom (MB) is called the *basis exchange axiom*. Sets in $\mathcal{B}$ are called *bases* of M.

Replacing the set difference in Axiom (MB) by the symmetric difference we obtain the symmetric exchange axiom (ΔF) used by Bouchet [1] to define Δ-matroids.

A Δ*-matroid* D is a finite set E and a collection $\mathcal{F}$ of subsets of E satisfying the condition that if

(ΔF) If F_1 and F_2 are in $\mathcal{F}$ and $x \in F_1 \triangle F_2$ then there exists a $y \in F_2 \triangle F_1$ such that $F_1 \triangle \{x, y\} \in \mathcal{F}$.

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Axiom (ΔF) is called the *symmetric exchange axiom* and the sets in $\mathcal{F}$ are called the *feasible sets* of D. It is important to note that y may equal x, so $|F_1 \triangle \{x, y\}| - |F_1| \in \{0, \pm 1, \pm 2\}$.

There are two obvious matroids associated with every Δ-matroid; M_u, the *upper matroid*, whose bases are the feasible sets with largest cardinality, and M_l, the *lower matroid*, whose bases are the feasible sets with least cardinality, [2].

Δ-matroids and maps on surfaces

In [2], Bouchet associates a Δ-matroid to any map. A map is a cellular embedding of a graph G into a compact surface, and, for the Δ-matroid he defined, the lower matroid is the cycle matroid of G, and the upper matroid is the dual of the cycle matroid of the geometric dual, G^*, of G in the surface. For more information about maps see [3, 4, 5]. In this section we would like to reformulate the connection between maps and Δ-matroids in such a way as to clarify both the geometry and the combinatorics.

Bouchet defined a *base* B of a map as a selection of edges from the cellularly embedded graph, $B \subseteq E$, such that, after deleting all the edges of B and all the dual edges of $E \setminus B$, together with their endpoints, the resulting non-compact surface is connected. To perform this operation, it is convenient to use the barycentric subdivision of the map, whose one-skeleton contains both the graph and the dual-graph, with the edges of each subdivided in two, see Figure 1(a) and (b). The *map graph* is the geometric dual of the barycentric

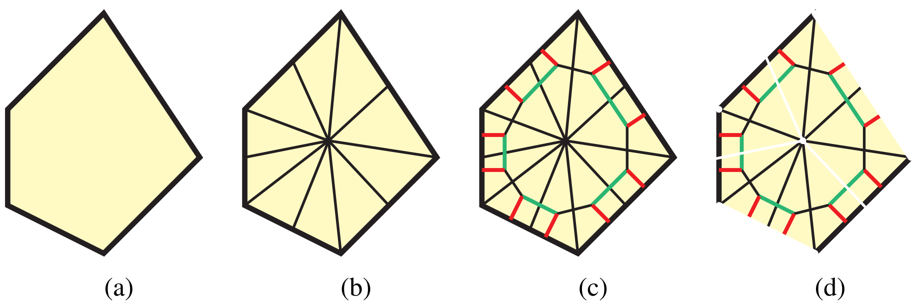
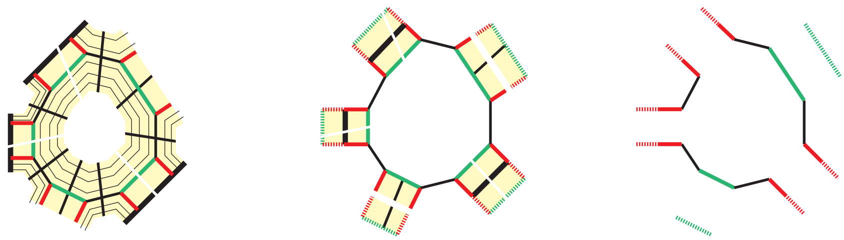


Figure 1: (a) A cell of a map, (b) its barycentric subdivision, (c) the map graph, (d) deleting an edge/dual-edge selection.

subdivision, Figure 1(c), where the edges are colored green, red, and black depending on whether they are parallel to one of the original edges, cross one, or neither. Suppose, as Bouchet did, we delete, for each edge, either the edge or its dual, together with their endpoints, as realized in the barycentric subdivision. If it should happen that some vertex or dual vertex of the map is not deleted, then it is an interior point of the the deleted surface, and we may puncture the surface there without affecting the connectivity. Then, expanding the holes at the vertices and dual vertices, there is a deformation of the punctured surface which respects all the edges and dual edges, so, in particular, respecting the deleted edges. This deformation can continue, expanding the holes until all that is left is the set of black edges of the map graph and the green-red quadrilaterals, each of which has been cut in half, either leaving the green edge pair intact, or the red edge pair. Each of these cut quadrilaterals can be deformed, expanding the cut, onto the surviving color pair, leaving the map graph with one color pair deleted from each quadrilateral, green for those in B, and



(a) (b) (c)

Figure 2: Deforming away from the vertex and dual vertex holes.

red for the others. This is a 2-regular subgraph of the map graph, and contains all the black edges. By the deformation, the surface with the edges and dual edges deleted is connected if and only if the corresponding 2-regular subgraph of the map graph is connected as a topological space, which is true if and only if that 2-regular subgraph is a Hamiltonian cycle.

Bouchet went on to show that the sets B formed the feasible sets of a Δ-matroid on E, using Eulerian splitters. Using the map graph, we may establish this simply and directly.

2 Combinatorial maps and Δ-matroids

Tutte, in the introduction to his paper *What is a map?* [5] remarks

> Maps are usually presented as cellular dissections of topologically defined surfaces. But some combinatorialists, holding that maps are combinatorial in nature, have suggested purely combinatorial axioms for map theory, so that that branch of combinatorics can be developed without appealing to point-set topology.

Tutte's idea is that each edge of a map is associated with four flags, corresponding to the triangles in the barycentric subdivision. Each flag has three vertices: one corresponding to a vertex of the embedded graph (an endpoint of the embedded edge e), one corresponding to an edge (the mid-point of e), and one corresponding to a face (the bary-center of a face incident with e) of the map. The map can be uniquely described in terms of three perfect matchings. Two flags are matched if they share a vertex of the same kind. Faces, Euler characteristic, and orientability can be treated combinatorially without appealing to topology. We now recall Tutte's axiomatic approach as presented in [3, 4].

Let Γ be a connected graph whose edges are partitioned into three classes R, G, and B which we color respectively red, green, and black. Γ is called *map graph* or a *combinatorial map* if the following conditions are satisfied:

1. Each color class is a perfect matching;
2. $R \cup G$ is a union of 4-cycles;
3. Γ is connected.

The graph Γ is 3-regular and edge 2-connected. Γ may have parallel edges, although necessarily not red/green. Γ contains 2-regular subgraphs which use all the black edges of

Γ, which we call *fully black* 2-regular subgraphs; $R \cup B$ and $G \cup B$ are examples, and there always exists a fully black Hamiltonian cycle. To see this, first note that a fully black 2-regular subgraph cannot contain any incident green and red edges, so every red/green quadrilateral intersects a fully black 2-regular subgraph in either two red, or two green edges. Now consider a fully black 2-regular subgraph of Γ with the fewest connected components. If there is not a single component, then there is a green/red quadrilateral which intersects the subgraph in, say, two red edges which belong to two different components, and swapping red and green on that quadrilateral reduces the number of components of the subgraph, violating minimality.

Theorem 2.1. *Given a combinatorial map $\Gamma(R, G, B)$, let E be the set of quadrilaterals of $R \cup G$, and let $\mathcal{F}$ be the collection of subsets of E corresponding to the pairs of green edges in a fully black Hamilton cycle in Γ. Then $(\mathcal{F}, E)$ is a Δ-matroid.*

Proof. We have to show the symmetric exchange property holds. Let F_C and $F_{C'}$ be sets of quadrilaterals corresponding to fully black Hamiltonian cycles C and C'. Let $q \in F_C \triangle F'_C$, so the edges of quadrilateral q are differently colored in C and C', say red and green. There are two cases, either replacing in q the red edges in C with the green of C' results in two components or one. See Figure 3. If it results in just one component, then take $q' = q$, and

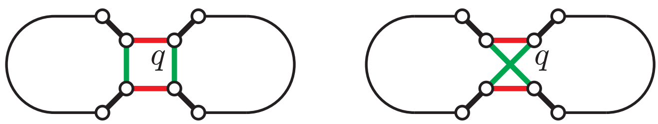


Figure 3

$F_c \triangle \{q, q'\} = F_c \triangle \{q\}$ is the set of green quadrilaterals of a fully black Hamiltonian cycle, and hence feasible, as required.

Otherwise, if there are two components, the Hamiltonian cycle of C' contains a non-black edge, say green, of a quadrilateral q_1, connecting those two components, and necessarily both red edges of q_1 are in C and both green edges of q_1 connect the components, and $q' \in C \triangle C'$. See Figure 4. Regardless of how the green edges of q_1 are placed,

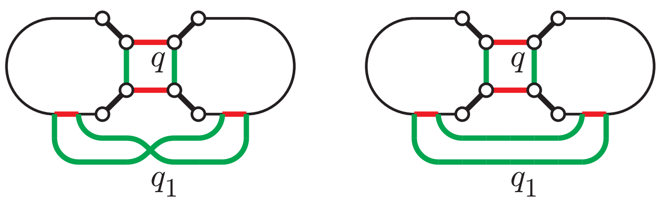


Figure 4

swapping the edges of both q and q' in C yields a new fully black Hamiltonian cycle, so the set $Q \triangle \{q, q_1\}$ is feasible, as required. □

Since R, G and B are perfect matchings, the union of any two them induces a set of disjoint cycles. Let V be the set of cycles of $R \cup B$, E be the set of cycles of $R \cup G$, and V^* be the set of cycles of $G \cup B$. There is a graph (V, E) where incidence is defined between a red-black cycle and a red-green cycle if they share an edge, and, similarly, there is a graph (V^*, E) where incidence is defined between a green-black cycle and a red-green cycle if they share and edge. We say that Γ encodes the graph (V, E) and its geometric dual (V^*, E).

Theorem 2.2. *Let $\Gamma(R, G, B)$ be a combinatorial map and $D_\Gamma = (\mathfrak{F}, E)$ its associated Δ-matroid. Then the lower matroid of D_Γ is the cycle matroid of the graph (V, E) and the upper matroid of D_Γ is the cocycle matroid of the graph (V^*, E).*

Proof. Given $\Gamma(R, G, B)$, recall that the feasible sets of D consist of RG quadrilaterals whose R edges are contained in a fully black Hamilton cycle of Γ. Any fully black Hamilton cycle C of Γ must contain the red edges corresponding to a spanning tree of (V, E) as well as the green edges corresponding to a spanning tree of (V^*, E). So the minimal number of red edges in C is $2(|V| - 1)$, while the maximal number is $2(|E| - |V^*| + 1)$. The edge sets of the spanning trees of (V, E) are the bases of its cycle matroid, while the complements of edge sets of spanning trees in (V^*, E) are the bases of the cocycle matroid of (V^*, E). □

Note that the difference in rank of the upper and lower matroid of $(\mathfrak{F}, E)$ is given by $(|E| - |V^*| + 1) - (|V| - 1) = 2 - \chi$, where χ is the Euler characteristic. Notice also, that if Γ is bipartite, all feasible sets of $D_\Gamma = (\mathfrak{F}, E)$ must have the same parity, since exchanging a red and green pair of edges always disconnects a Hamilton cycle of a bipartite Γ.

Examples of combinatorial maps together with the underlying graph and geometric dual are provided in Figures 5, 6 and 7.

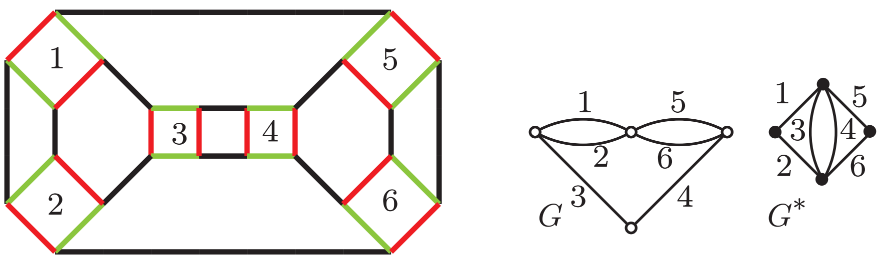


Figure 5

The Δ-matroid associated to the map of Figure 5 has feasible sets

$$\mathfrak{F} = \{\{1,3,4\},\{1,3,5\},\{1,3,6\},\{1,4,5\},\{1,4,6\},\{2,3,4\},\\ \{2,3,5\},\{2,3,6\},\{2,4,5\},\{2,4,6\},\{3,4,5\},\{3,4,6\}\}.$$

Note that $\mathfrak{F}$ is the set of spanning trees of G and at the same time the set of co-trees of G^*, so all feasible sets have the same size and upper and lower matroid are identical.

The Δ-matroid associated to the map of Figure 6 has feasible sets all the sets in $\mathfrak{F}$ together with the two additional sets $\{1,2,3,4,5\}$ and $\{1,2,3,4,6\}$. The lower matroid is again the cycle matroid of G, but the upper matroid is the co-cycle matroid of G^*, which has rank 5 and contains exactly one cycle, namely $\{5,6\}$, which is a minimal cutset of G^* and also a cycle in G.

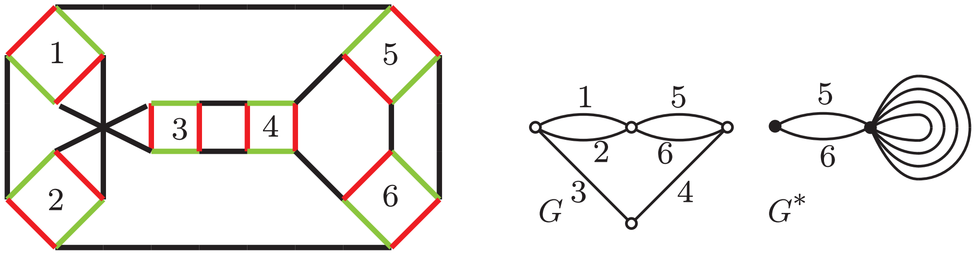


Figure 6

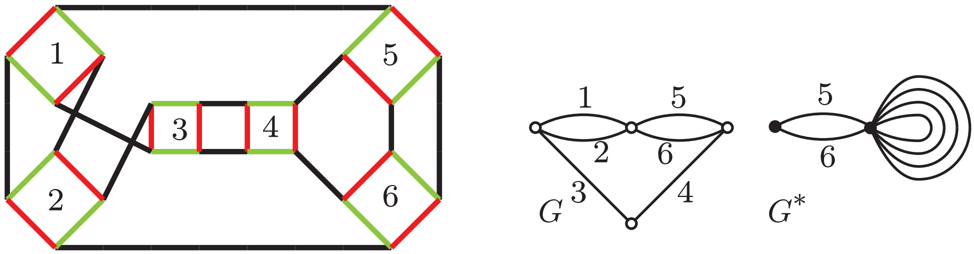


Figure 7

The Δ-matroid associated to the map of Figure 7 has, in addition to the feasible sets of the previous example the feasible set $\{1, 2, 3, 4\}$, whose parity is even, while the parity of all other feasible sets is odd, so this map is not orientable.

As is clear from these examples, the map cannot, in general be recovered from the Δ-matroid information, since the upper or lower matroid do not even determine the graph. Non-isomorphic graphs may have identical cycle-and co-cycle matroids. It is easy to check that $\mathfrak{F}$ is also a list of spanning trees for the graph G', but G is not isomorphic to G'.

However, if both G and G^* are 3-connected, then the map is uniquely recoverable from the Δ-matroid information.

Theorem 2.3. *Let D be the Δ-matroid of a map M with 3-connected upper- and lower matroid. Then M is determined by D.*

Proof. By Whitney's theorem [6], upper and lower matroid uniquely determine G and G^*. To recover M from D, we need to specify a rotation system for each vertex v of G. To determine if two edges e and f with endpoint v follow each other in the rotation about v, it is enough to check if e and f are both incident in G^*, since the vertex co-cycles of G^* correspond to the facial cycles of the embedded G. Now re-construct the map graph. □

For example the lower matroid could be the cycle matroid of K_5, while the upper matroid is the co-cycle matroid of K_5 as well, so this matroid information gives us the graphs G and G^* depicted in Figure 8. By the method in the proof of Theorem 2.3 the map M is easily recovered to be as in Figure 9, which represents the torus map as a doubly periodic tiling. The faces are colored according to the vertex colors in Figure 8.

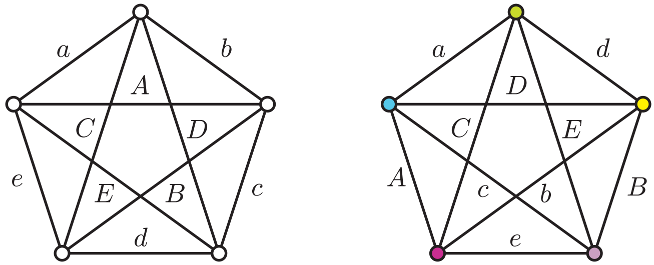


Figure 8

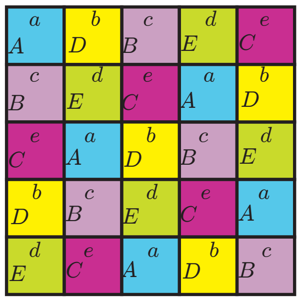


Figure 9

3 Another Δ-matroid from a map

If the objective is to define a natural Δ-matroid from a combinatorial map, the requirement that the subgraph of the map graph be Hamiltonian can be weakened provided that some connectivity is required. Again, let Γ be a map graph with edge set $R \cup G \cup B$, with red edges R, green edges G and black edges B.

Theorem 3.1. *Let K be a fully black 2-valent subgraph of Γ with the property that $K \cup R$ and $K \cup G$ are both connected. Then the set F_K of quadrilaterals in which red is selected in K form the feasible sets of a Δ-matroid.*

Proof. We have to show the symmetric exchange property. Let F_K and $F_{K'}$ be sets of red quadrilaterals corresponding to fully black 2-valent subgraphs K and K', both of which can be connected by adding edges of one color only. Let $q \in F_K \triangle F'_K$, so the edges of quadrilateral q are differently colored in K and K', say red and green respectively. If the red edges of q belong to two different cycles of K, then swapping red for green in q merges the two cycles, then we may take $q' = q$ and the $F_K \triangle \{q, q'\}$ will be connected by the same collections of red, respectively green edges as F_k.

So we may assume that the red edges of q belong to the same component K. If swapping red for green in q does not split the component of K they belong to, see the right side of Figure 3, then just as before, take $q' = q$. So we may assume that the red edges of q belong to the same component of K, and swapping them for green splits that component, see the left side of Figure 3. Let the red edges of q be denoted by q_r and the green edges of q be denoted by q_g. Clearly $(K - q_r + q_g) + R$ is connected since $K + R \subseteq (K - q_r + q_g) + R$ and is connected. The issue is that $(K - q_r + q_g) + G = K + G - q_r$ may have two components. If it has just one, again, take $q' = q$ and we are done. We know that $K' + G$ is connected, so K' must have a red edge of some quadrilateral q' that connects the two components of $(K - q_r + q_g) + G$, so $q' \notin F_K$ and $q' \in F_{K'}$, that is $q' \in F_K \triangle F_{K'}$. $(K - q_r + q'_r) + G$ is connected and we already know $(K - q_r + q'_r) + R$ is connected, so the collections F_K are the feasible sets of a Δ-matroid. □

Let D_Γ be the Δ-matroid as in Theorem 2.3, with feasible sets the pairs of red edges in a fully black Hamilton cycle, and D_K be the Δ-matroid as in Theorem 3.1, with feasible sets the pairs of red edges in a fully black 2-valent subgraph K such that K becomes connected by addition of red edges only as well as by addition of only green edges. D_Γ and D_K are different. For example for the unitary map there are two quadrilaterals, $\{q, q'\}$ and the feasible sets in the first sense are $\{\emptyset, \{q, q'\}\}$, whereas in the second sense are all subsets. For unitary maps the connectivity issue here is void since the $R + B$ and $G + B$ are both connected. The upper and lower matroid for both D_Γ and D_K are clearly the same. However, the Hamiltonian requirement encodes the orientability of the map, by the fact that all feasible sets have the same parity in the orientable case and are of both even and odd cardinality if Γ is not bipartite, while the second approach does not distinguish between the two.

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C_4-face-magic toroidal labelings on $C_m \times C_n$

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Abstract

For a graph $G = (V, E)$ naturally embedded in the torus, let $\mathcal{F}(G)$ denote the set of faces of G. Then, G is called a C_n-face-magic toroidal graph if there exists a bijection $f : V(G) \to \{1, 2, \ldots, |V(G)|\}$ such that for every $F \in \mathcal{F}(G)$ with $F \cong C_n$, the sum of all the vertex labels along C_n is a constant S. Let $x_v = f(v)$ for all $v \in V(G)$. We call $\{x_v : v \in V(G)\}$ a C_n-face-magic toroidal labeling on G. We show that, for all $m, n \geq 2$, $C_m \times C_n$ admits a C_4-face-magic toroidal labeling if and only if either $m = 2$, or $n = 2$, or both m and n are even. We say that a C_4-face-magic toroidal labeling $\{x_{i,j} : (i, j) \in V(C_{2m} \times C_{2n})\}$ on $C_{2m} \times C_{2n}$ is antipodal balanced if $x_{i,j} + x_{i+m,j+n} = \frac{1}{2}S$, for all $(i, j) \in V(C_{2m} \times C_{2n})$. We show that there exists an antipodal balanced C_4-face-magic toroidal labeling on $C_{2m} \times C_{2n}$ if and only if the parity of m and n are the same. Furthermore, when both m and n are even, an antipodal balanced C_4-face-magic toroidal labeling on $C_{2m} \times C_{2n}$ is both row-sum balanced and column-sum balanced. In addition, when $m = n$ is even, an antipodal balanced C_4-face-magic toroidal labeling on $C_{2n} \times C_{2n}$ is diagonal-sum balanced.

Keywords: C_4-face-magic graphs, polyomino, toroidal graphs, Cartesian products of cycles.

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1 Introduction

Kotzig and Rosa [11] formally introduced graph labelings in the 1970s. There are applications of graph labelings to graph decomposition problems, radar pulse code designs, X-ray crystallography and communication network models. The interested reader should read J.A. Gallian's comprehensive dynamic survey on graph labelings [8] for further investigation.

We refer the reader to Bondy and Murty [5] for concepts and notation not explicitly defined in this paper. All graphs in this paper are simple and connected. For a planar graph $G = (V, E)$ embedded in $\mathbb{R}^2$, let $\mathcal{F}(G)$ denote the set of faces of G. Then, G is called a C_n*-face-magic* graph if there exists a bijection $f : V(G) \to \{1, 2, \ldots, |V(G)|\}$ such that for every $F \in \mathcal{F}(G)$ with $F \cong C_n$, the sum of all the vertex labels along C_n is a constant S. Here, the constant S is called a C_n*-face-magic value* of G. A C_n*-face-magic toroidal* (or *cylindrical*) graph G is defined similarly, where G is embedded in the torus (or cylinder), respectively. C_n-face-magic graph labelings are a special case of the more general (a, b, c)-magic labeling introduced by Lih [12]. For assorted values of a, b and c, Baca and others [1, 2, 3, 4, 9, 10, 12] have analyzed the problem for various classes of graphs. Wang [13] showed that the toroidal grid graphs $C_m \times C_n$ are antimagic for all integers $m, n \geq 3$. Butt et al. [6] investigated face antimagic labelings on toroidal and Klein bottle grid graphs.

In this paper, we investigate C_4-face-magic toroidal labelings on $C_m \times C_n$ with its natural embedding in the torus. We show that for all $m, n \geq 2$, there exists a C_4-face-magic toroidal labeling on $C_m \times C_n$ if and only if either $m = 2$, or $n = 2$, or both m and n are even. In the case when $m = n$, we say that a C_4-face-magic toroidal labeling on $C_{2n} \times C_{2n}$ is *torus symmetric* if the labeling is row-sum balanced, column-sum balanced and diagonal-sum balanced. Curran and Low [7] show that, up to symmetries on the torus, there are only three torus symmetric C_4-face-magic toroidal labelings on $C_4 \times C_4$. See Theorem 3.5 in Section 3 for details. In this paper, we search for C_4-face-magic toroidal labelings on $C_{2m} \times C_{2n}$ that are row-sum balanced and column-sum balanced. This investigation leads naturally to the concept of an antipodal balanced labeling. We say that a C_4-face-magic toroidal labeling $\{x_{i,j} : (i, j) \in V(C_{2m} \times C_{2n})\}$ on $C_{2m} \times C_{2n}$ is *antipodal balanced* if $x_{i,j} + x_{i+m,j+n} = \frac{1}{2}S$, for all $(i, j) \in V(C_{2m} \times C_{2n})$. We show that there exists an antipodal balanced C_4-face-magic toroidal labeling on $C_{2m} \times C_{2n}$ if and only if the parity of m and n are the same. Furthermore, when $m = n$ is even, we show that any antipodal balanced C_4-face-magic toroidal labeling on $C_{2n} \times C_{2n}$ is torus symmetric.

2 Preliminaries

Theorem 2.1. *Let $m, n \geq 2$. Then, $P_m \times P_n$ is C_4-face-magic.*

Proof. Label the vertex set of $P_m \times P_n$ as

$$V(P_m \times P_n) = \{(i, j) : 1 \leq i \leq m, 1 \leq j \leq n\}$$

and its edge set as

$$\begin{aligned} E(P_m \times P_n) = &\{\{(i, j), (i + 1, j)\} : 1 \leq i < m, 1 \leq j \leq n\} \\ &\cup \{\{(i, j), (i, j + 1)\} : 1 \leq i \leq m, 1 \leq j < n\}. \end{aligned}$$

We will determine a label $x_{i,j}$ for each vertex $(i, j) \in V(P_m \times P_n)$ and check that this provides a C_4-face-magic labeling.

Case 1. Assume $m-n$ is even. Color the vertex (i,j) white if $i+j$ is even and black if $i+j$ is odd. Note that the vertices $(1,1)$ and (m,n) are white. Let $x_{i,j}=i+m\,(j-1)$ for each white vertex (i,j), and $x_{i,j}=(m-i+1)+m(n-j)$ for each black vertex (i,j). An equivalent definition for $\{x_{i,j}:(i,j)\in V(P_m\times P_n)\}$ would be to write the number $i+m\,(j-1)$ in each cell (i,j), and then rotate the black vertices 180 degrees about the center of the board.

Let $C_{i,j}=\{(i,j),(i+1,j),(i+1,j+1),(i,j+1)\}$. If (i,j) is white, then the two 4-cycles $C_{i,j}$ and $C_{i+1,j}$ have the same face sum, since $x_{i,j}=x_{i+2,j}-2$ and $x_{i,j+1}=x_{i+2,j+1}+2$. If (i,j) is black, then the two 4-cycles $C_{i,j}$ and $C_{i+1,j}$ have the same face sum, since $x_{i,j}=x_{i+2,j}+2$ and $x_{i,j+1}=x_{i+2,j+1}-2$. A similar proof for $C_{i,j}$ and $C_{i,j+1}$ shows that $\{x_{i,j}\}$ is C_4- face-magic. The sum on each face must be the sum on $C_{1,1}$, which is $1+(mn-1)+(m+2)+m\,(n-1)=2mn+2$. This completes Case 1.

Case 2. Without loss of generality, we may assume that m is even and n is odd. Let $m=2m_1$ and $n=2n_1-1$ for some positive integers m_1 and n_1. Again, color vertex (i,j) white if $i+j$ is even and black if $i+j$ is odd.

We first label the white vertices. Let $x_{i,j}=m(j-1)+i$ if both i and j are odd, and $x_{i,j}=m(j-1)+i-1$ if both i and j are even. We observe that $x_{2k-1,2\ell-1}=m(2\ell-2)+2k-1$ for all $1\le k\le m_1$ and $1\le \ell\le n_1$, and $x_{2k,2\ell}=m(2\ell-1)+2k-1$ for all $1\le k\le m_1$ and $1\le \ell< n_1$. Thus each odd label $1,3,5,\ldots,mn-1$ is used exactly once on a white vertex.

Next, we label the black vertices. Let $x_{i,j}=m(n-j+1)-i+1$ if i is odd and j is even, and $x_{i,j}=m(n-j+1)-i+2$ if i is even and j is odd. We observe that whenever vertex (i,j) is white, then vertex $(m-i+1,n-j+1)$ is black and $x_{m-i+1,n-j+1}=x_{i,j}+1$. Thus each even label $2,4,6,\ldots,mn$ is used exactly once on a black vertex.

Let $C_{i,j}=\{(i,j),(i+1,j),(i+1,j+1),(i,j+1)\}$. If (i,j) is white, then $x_{i+2,j}=x_{i,j}+2$ and $x_{i,j+2}=x_{i,j}+2m$. If (i,j) is black, then $x_{i+2,j}=x_{i,j}-2$ and $x_{i,j+2}=x_{i,j}-2m$. An argument similar to that in Case (i) shows that each 4-cycle $C_{i,j}$ has the same face sum as $C_{1,1}$, which is $1+mn+(m+1)+m(n-1)=2mn+2$. This completes Case 2. □

Lemma 2.2. *Let m and n be positive integers. A C_4-face-magic labeling on $P_{2m}\times P_{2n}$ always yields a C_4-face-magic labeling on $C_{2m}\times C_{2n}$ with its natural embedding in the torus. Furthermore, the C_4-face-magic value is $S=2(4mn+1)$.*

Proof. Let $x_{i,j}$ be the C_4-face-magic labeling on vertex (i,j), for $i=1,2,\ldots,2m$ and $j=1,2,\ldots,2n$. Since S is the C_4-face-magic value on $P_{2m}\times P_{2n}$, we have $x_{i,j}+x_{i+1,j}+x_{i,j+1}+x_{i+1,j+1}=S$, for all $i=1,2,\ldots,2m-1$ and $j=1,2,\ldots,2n-1$. We observe that

$$mnS=\sum_{i=1}^{m}\sum_{j=1}^{n}(x_{2i-1,2j-1}+x_{2i,2j-1}+x_{2i-1,2j}+x_{2i,2j})=\sum_{i=1}^{4mn} i=\tfrac{1}{2}(4mn)(4mn+1).$$

Thus, $S=2(4mn+1)$. Since $x_{i,j}+x_{i,j+1}+x_{i+1,j}+x_{i+1,j+1}=S=x_{i+1,j}+x_{i+2,j}+x_{i+1,j+1}+x_{i+2,j+1}$, we have $x_{i,j}+x_{i,j+1}=x_{i+2,j}+x_{i+2,j+1}$. An induction argument shows that $x_{i,j}+x_{i,j+1}=x_{i+2k,j}+x_{i+2k,j+1}$. Since $x_{i+2k,j}+x_{i+2k,j+1}+x_{i+2k+1,j}+x_{i+2k+1,j+1}=S$, we have $x_{i,j}+x_{i,j+1}+x_{i+2k+1,j}+x_{i+2k+1,j+1}=S$.

A similar argument shows that $x_{i,j} + x_{i+1,j} + x_{i,j+2\ell+1} + x_{i+1,j+2\ell+1} = S$. This yields $x_{i,j} + x_{i+2k+1,j} + x_{i,j+2\ell+1} + x_{i+2k+1,j+2\ell+1} = S$. Hence, we have $x_{1,j} + x_{1,j+1} + x_{2m,j} + x_{2m,j+1} = S$, for all $j = 1, 2, \ldots, 2n-1$. Similarly, we have $x_{i,1} + x_{i+1,1} + x_{i,2n} + x_{i+1,2n} = S$, for all $i = 1, 2, \ldots, 2m-1$. Lastly, we have $x_{1,1} + x_{2m,1} + x_{1,2n} + x_{2m,2n} = S$. Therefore, the C_4-face-magic labeling on $P_{2m} \times P_{2n}$ yields a C_4-face-magic labeling on $C_{2m} \times C_{2n}$ with its natural embedding in the torus. □

Lemma 2.3. *Let m and n be integers such that $m \geq 3$ and $n \geq 2$. Suppose $P_m \times C_n$ is a C_4-face-magic cylindrical graph with the natural embedding of $P_m \times C_n$ on the cylinder. Then, n is even.*

Proof. For the purpose of contradiction, suppose n is odd, and let $n = 2n_1 + 1$ for some positive integer n_1. Label the vertex set of $P_m \times C_n$ as

$$V\left(P_m \times C_n\right) = \{(i, j) : 1 \leq i \leq m, 1 \leq j \leq n\}$$

and its edge set as

$$\begin{aligned} E\left(P_m \times C_n\right) = &\left\{\left\{(i,j), (i+1,j)\right\} : 1 \leq i < m, 1 \leq j \leq n\right\} \\ &\cup \left\{\left\{(i,j), (i,j+1)\right\}, \left\{(i,n), (i,1)\right\} : 1 \leq i \leq m, 1 \leq j < n\right\}. \end{aligned}$$

Let $\{x_{i,j} : (i,j) \in V\left(P_m \times C_n\right)\}$ be a C_4-face-magic labeling on $P_m \times C_n$ with C_4-face-magic value S. Let $S_i = x_{i,1} + x_{i+1,1}$, for $i = 1, 2$. Equating the following C_4-face sums to each other:

$$x_{i,j} + x_{i+1,j} + x_{i,j+1} + x_{i+1,j+1} = S = x_{i,j+1} + x_{i+1,j+1} + x_{i,j+2} + x_{i+1,j+2},$$

we obtain $x_{i,j} + x_{i+1,j} = x_{i,j+2} + x_{i+1,j+2}$, where the index j is taken modulo n. Thus,

$$\begin{aligned} &x_{i,1} + x_{i+1,1} = x_{i,2j+1} + x_{i+1,2j+1} \quad \text{and} \\ &x_{i,2} + x_{i+1,2} = x_{i,2j} + x_{i+1,2j}, \text{ for } j = 1, 2, \ldots, n_1. \end{aligned}$$

Also, we have $x_{i,n-1} + x_{i+1,n-1} = x_{i,n+1} + x_{i+1,n+1} = x_{i,1} + x_{i+1,1}$. Hence, $S_i = x_{i,j} + x_{i+1,j}$, for all $i = 1, 2$ and $j = 1, 2, \ldots, n$. From the C_4-face sum

$$S = (x_{i,j} + x_{i+1,j}) + (x_{i,j+1} + x_{i+1,j+1}) = S_i + S_i = 2S_i,$$

we have $S_i = \frac{1}{2}S$. Hence, $x_{1,1} + x_{2,1} = \frac{1}{2}S = x_{2,1} + x_{3,1}$, which in turn, implies that $x_{1,1} = x_{3,1}$. This is a contradiction. Therefore, n is even. □

Proposition 2.4. *Let m be an integer where $m \geq 2$. Then, there is a C_4-face-magic toroidal labeling on $C_m \times C_2$.*

Proof. Let $x_{i,1} = i$ and $x_{i,2} = 2m+1-i$, for $i = 1, 2, \ldots, m$. Then, $x_{i,1} + x_{i,2} = 2m+1$, for $i = 1, 2, \ldots, m$. Thus, $x_{i,1} + x_{i,2} + x_{i+1,1} + x_{i+1,2} = 2(2m+1)$, for $i = 1, 2, \ldots, m$. Hence, $\{x_{i,j} : (i,j) \in V(C_m \times C_2)\}$ is a C_4-face-magic toroidal labeling on $C_m \times C_2$. □

Proposition 2.5. *Let m and n be integers where $m, n \geq 2$. Then, $C_m \times C_n$ has a C_4-face-magic toroidal labeling if and only if either $m = 2$, or $n = 2$, or both m and n are even.*

Proof. ($\Rightarrow$) Suppose $C_m \times C_n$ has a C_4-face-magic toroidal labeling. If either $m = 2$ or $n = 2$, we are done. So assume that $m, n \geq 3$. The C_4-face-magic toroidal labeling on $C_m \times C_n$ is simultaneously a C_4-face-magic cylindrical labeling on both $C_m \times P_n$ and $P_m \times C_n$. By Lemma 2.3, both m and n are even.

($\Leftarrow$) Suppose either $m = 2$, or $n = 2$, or both m and n are even. On the one hand, if $m = 2$ or $n = 2$, by Proposition 2.4, $C_m \times C_n$ has a C_4-face-magic toroidal labeling. On the other hand, if both m and n are even, by Theorem 2.1, $P_m \times P_n$ has a C_4-face-magic labeling. By Lemma 2.2, the C_4-face-magic labeling on $P_m \times P_n$ yields a C_4-face-magic toroidal labeling on $C_m \times C_n$. □

Throughout this paper, if $\{x_{i,j} : (i,j) \in V(C_{2m} \times C_{2n})\}$ is a labeling on $C_{2m} \times C_{2n}$, then for convenience we consider the index i modulo $2m$ and the index j modulo $2n$.

Definition 2.6. We say that the C_4-face-magic torus labeling $\{x_{i,j} : i = 1, 2, \ldots, 2m$ and $j = 1, 2, \ldots, 2n\}$ on $C_{2m} \times C_{2n}$ is *antipodal balanced* if $x_{i,j} + x_{i+m,j+n} = 4mn + 1$, for all integers i and j such that $1 \leq i \leq 2m$ and $1 \leq j \leq 2n$.

Remark 2.7. We give a brief explanation for the term *antipodal balanced*. On the n-sphere $S^n \subsetneqq \mathbb{R}^{n+1}$, the antipodal map $p : S^n \to S^n$ is given by $p(\mathbf{x}) = -\mathbf{x}$. Similarly, on the torus $T^2 = S^1 \times S^1 \subsetneqq \mathbb{C}^2 \cong \mathbb{R}^4$, we define the antipodal map $p : T^2 \to T^2$ by $p(e^{i\theta_1}, e^{i\theta_2}) = -(e^{i\theta_1}, e^{i\theta_2}) = (e^{i(\theta_1+\pi)}, e^{i(\theta_2+\pi)})$. Thus an antipodal balanced C_4-face-magic toroidal labeling on $C_{2m} \times C_{2n}$ is one in which the sum of the labels at a vertex and its antipodal vertex is constant for all vertices in $C_{2m} \times C_{2n}$.

Lemma 2.8. *Let m and n be positive integers. Let $\{x_{i,j} : i = 1, 2, \ldots, 2m$ and $j = 1, 2, \ldots, 2n\}$ be an antipodal balanced C_4-face-magic toroidal labeling on $C_{2m} \times C_{2n}$. For all integers i where $1 \leq i \leq m$, we define*

$$d_i = x_{i(m-1)+1,in+1} - x_{(i-1)(m-1)+1,(i-1)n+1}.$$

Then for all integers i and j where $1 \leq i \leq m$ and $1 \leq j \leq 2n$, we have

$$x_{i(m-1)+1,in+j} = x_{(i-1)(m-1)+1,(i-1)n+j} + (-1)^{j+1} d_i.$$

Proof. By the definition of d_i, we have

$$x_{i(m-1)+1,in+1} = x_{(i-1)(m-1)+1,(i-1)n+1} + d_i.$$

We apply an induction argument on j. Thus we assume that

$$x_{i(m-1)+1,in+j-1} = x_{(i-1)(m-1)+1,(i-1)n+j-1} + (-1)^j d_i. \tag{2.1}$$

Since the labeling is antipodal balanced, we have

$$x_{i(m-1)+2,in+j-1} = \tfrac{1}{2}S - x_{(i-1)(m-1)+1,(i-1)n+j-1}, \text{ and} \tag{2.2}$$

$$x_{i(m-1)+2,in+j} = \tfrac{1}{2}S - x_{(i-1)(m-1)+1,(i-1)n+j}. \tag{2.3}$$

Since $\{x_{i,j}\}$ is a C_4-face-magic toroidal labeling on $C_{2m} \times C_{2n}$, we have

$$x_{i(m-1)+1,in+j-1} + x_{i(m-1)+1,in+j} + x_{i(m-1)+2,in+j-1} + x_{i(m-1)+2,in+j} = S. \tag{2.4}$$

When we substitute the expressions from equations (2.1), (2.2) and (2.3) into equation (2.4), we obtain

$$x_{i(m-1)+1,in+j} = x_{(i-1)(m-1)+1,(i-1)n+j} + (-1)^{j+1} d_i.$$

This completes the proof. $\square$

We next show that if $C_{2m} \times C_{2n}$ has an antipodal balanced C_4-face-magic toroidal labeling, then the parity of m and n are the same.

Lemma 2.9. *Let m and n be positive integers. Let $\{x_{i,j} : i = 1, 2, \ldots, 2m$ and $j = 1, 2, \ldots, 2n\}$ be an antipodal balanced C_4-face-magic toroidal labeling on $C_{2m} \times C_{2n}$. Then, the parity of m and n are the same.*

Proof. We may assume that m is even and n is odd. Let $\{x_{i,j}\}$ be an antipodal balanced C_4-face-magic toroidal labeling on $C_{2m} \times C_{2n}$. We will show that this leads to a contradiction. For all integers i such that $1 \leq i \leq m$, we define

$$d_i = x_{i(m-1)+1,in+1} - x_{(i-1)(m-1)+1,(i-1)n+1}.$$

By Lemma 2.8, for all integers i and j such that $1 \leq i \leq m$ and $1 \leq j \leq 2n$, we have

$$x_{i(m-1)+1,in+j} = x_{(i-1)(m-1)+1,(i-1)n+j} + (-1)^{j+1} d_i.$$

Fix j such that $1 \leq j \leq 2n$. The equations

$$x_{i(m-1)+1,in+j} = x_{(i-1)(m-1)+1,(i-1)n+j} + (-1)^{j+1} d_i \text{ for } i = 1, 2, \ldots, m,$$

yield

$$x_{m+1,j} = x_{m(m-1)+1,mn+j} = x_{1,j} + (-1)^{j+1}\big(d_1 + d_2 + \cdots + d_m\big). \tag{2.5}$$

Setting $j = 1$, equation (2.5) becomes

$$x_{m+1,1} = x_{1,1} + \big(d_1 + d_2 + \cdots + d_m\big). \tag{2.6}$$

Setting $j = n + 1$, equation (2.5) becomes

$$x_{m+1,n+1} = x_{1,n+1} - \big(d_1 + d_2 + \cdots + d_m\big). \tag{2.7}$$

Because $\{x_{i,j}\}$ is antipodal balanced, we have

$$x_{1,1} + x_{m+1,n+1} = \tfrac{1}{2}S = x_{m+1,1} + x_{1,n+1}. \tag{2.8}$$

Substituting equations (2.6) and (2.7) into equation (2.8) and simplifying yields

$$d_1 + d_2 + \cdots + d_m = 0. \tag{2.9}$$

This implies that $x_{m+1,j} = x_{1,j}$, for all $j = 1, 2, \ldots, 2n$. This is a contradiction. $\square$

We next investigate how the conditions of Lemma 2.8 apply to the case $C_{2m} \times C_{2n}$ where both m and n are even.

Lemma 2.10. *Let m and n be positive even integers. Let $\{x_{i,j} : i = 1, 2, \ldots, 2m$ and $j = 1, 2, \ldots, 2n\}$ be an antipodal balanced C_4-face-magic toroidal labeling on $C_{2m} \times C_{2n}$. For all integers i such that $1 \le i \le m$, we define*

$$d_i = x_{i(m-1)+1,in+1} - x_{(i-1)(m-1)+1,(i-1)n+1}.$$

Then by Lemma 2.8, for all integers i and j such that $1 \le i \le m$ and $1 \le j \le 2n$, we have

$$x_{i(m-1)+1,in+j} = x_{(i-1)(m-1)+1,(i-1)n+j} + (-1)^{j+1} d_i.$$

Also, for all integers i and j such that $m+1 \le i \le 2m$ and $1 \le j \le 2n$, we have

$$x_{i(m-1)+1,in+j} = x_{(i-1)(m-1)+1,(i-1)n+j} + (-1)^{j} d_{i-m}.$$

For all integers j such that $1 \le j \le n$, we define

$$d'_j = x_{jm+1,j(n-1)+1} - x_{(j-1)m+1,(j-1)(n-1)+1}.$$

Then for all integers i and j such that $1 \le i \le 2m$ and $1 \le j \le n$, we have

$$x_{jm+i,j(n-1)+1} = x_{(j-1)m+i,(j-1)(n-1)+1} + (-1)^{i+1} d'_j.$$

Also, for all integers i and j such that $1 \le i \le 2m$ and $n+1 \le j \le 2n$, we have

$$x_{jm+i,j(n-1)+1} = x_{(j-1)m+i,(j-1)(n-1)+1} + (-1)^{i} d'_{j-n}.$$

Proof. By Lemma 2.8, for all integers i and j such that $1 \le i \le m$ and $1 \le j \le 2n$, we have

$$x_{i(m-1)+1,in+j} = x_{(i-1)(m-1)+1,(i-1)n+j} + (-1)^{j+1} d_i. \tag{2.10}$$

Since the indices $in + j$ and $(i-1)n + j$ are reduced modulo $2n$, equation (2.10) holds for all integers j. Let i be an integer such that $m+1 \le i \le 2m$. We replace i with $i - m$ and j with $n + j$ in equation (2.10) to obtain

$$x_{(i-m)(m-1)+1,(i-m)n+n+j} = x_{(i-m-1)(m-1)+1,(i-m-1)n+n+j} + (-1)^{n+j+1} d_{i-m}.$$

This reduces to

$$x_{i(m-1)+m+1,(i+1)n+j} = x_{(i-1)(m-1)+m+1,in+j} + (-1)^{j+1} d_{i-m}. \tag{2.11}$$

Since $\{x_{i,j}\}$ is antipodal balanced, we have

$$x_{i(m-1)+m+1,(i+1)n+j} = \tfrac{1}{2}S - x_{i(m-1)+1,in+j} \text{ and} \tag{2.12}$$
$$x_{(i-1)(m-1)+m+1,in+j} = \tfrac{1}{2}S - x_{(i-1)(m-1)+1,(i-1)n+j}. \tag{2.13}$$

When we substitute the expressions in equations (2.12) and (2.13) into equation (2.11), we have, for all integers i and j such that $m+1 \le i \le 2m$ and $1 \le j \le 2n$,

$$x_{i(m-1)+1,in+j} = x_{(i-1)(m-1)+1,(i-1)n+j} + (-1)^{j} d_{i-m}. \tag{2.14}$$

When we interchange the roles of i and j in the previous argument, when $1 \le i \le 2m$ and $1 \le j \le n$, we have

$$x_{jm+i,j(n-1)+1} = x_{(j-1)m+i,(j-1)(n-1)+1} + (-1)^{i+1} d'_j;$$

and when $1 \le i \le 2m$ and $n+1 \le j \le 2n$, we have

$$x_{jm+i,j(n-1)+1} = x_{(j-1)m+i,(j-1)(n-1)+1} + (-1)^{i} d'_{j-n}.$$

□

We next investigate how the conditions of Lemma 2.8 apply to the case $C_{2m} \times C_{2n}$ where both m and n are odd.

Lemma 2.11. *Let m and n be positive odd integers. Let $\{x_{i,j} : i = 1, 2, \dots, 2m$ and $j = 1, 2, \dots, 2n\}$ be an antipodal balanced C_4-face-magic toroidal labeling on $C_{2m} \times C_{2n}$. For all integers i such that $1 \le i \le m$, we define*

$$d_i = x_{i(m-1)+1,in+1} - x_{(i-1)(m-1)+1,(i-1)n+1}.$$

Then, for all integers i and j such that $1 \le i \le m$ and $1 \le j \le 2n$, we have

$$x_{i(m-1)+1,in+j} = x_{(i-1)(m-1)+1,(i-1)n+j} + (-1)^{j+1} d_i \text{ and}$$
$$x_{i(m-1)+m+1,in+j} = x_{(i-1)(m-1)+m+1,(i-1)n+j} + (-1)^{j+1} d_i.$$

For all integers i such that $1 \le j \le n$, we define

$$d'_j = x_{jm+1,j(n-1)+1} - x_{(j-1)m+1,(j-1)(n-1)+1}.$$

Then, for all integers i and j such that $1 \le i \le 2m$ and $1 \le j \le n$, we have

$$x_{jm+i,j(n-1)+1} = x_{(j-1)m+i,(j-1)(n-1)+1} + (-1)^{i+1} d'_j \text{ and}$$
$$x_{jm+i,j(n-1)+n+1} = x_{(j-1)m+i,(j-1)(n-1)+n+1} + (-1)^{i} d'_j.$$

Proof. By Lemma 2.8, for all integers i and j such that $1 \le i \le m$ and $1 \le j \le 2n$, we have

$$x_{i(m-1)+1,in+j} = x_{(i-1)(m-1)+1,(i-1)n+j} + (-1)^{j+1} d_i. \tag{2.15}$$

Because $in + j$ and $(i-1)n + j$ are reduced modulo $2n$, equation (2.15) holds for all integers j. Because $\{x_{i,j}\}$ is antipodal balanced, we have

$$x_{i(m-1)+1,in+j} = \tfrac{1}{2}S - x_{i(m-1)+m+1,(i+1)n+j} \text{ and} \tag{2.16}$$
$$x_{(i-1)(m-1)+1,(i-1)n+j} = \tfrac{1}{2}S - x_{(i-1)(m-1)+m+1,in+j}. \tag{2.17}$$

When we substitute the expressions from equations (2.16) and (2.17) into equation (2.15), we obtain

$$x_{i(m-1)+m+1,(i+1)n+j} = x_{(i-1)(m-1)+m+1,in+j} + (-1)^{j} d_i. \tag{2.18}$$

When we replace j with $j+n$ in equation (2.18), we have, for all integers i and j such that $1 \le i \le m$ and $1 \le j \le 2n$,

$$x_{i(m-1)+m+1,in+j} = x_{(i-1)(m-1)+m+1,(i-1)n+j} + (-1)^{j+1} d_i.$$

When we interchange the roles of i and j in the above argument, we have for all integers i and j such that $1 \le i \le 2m$ and $1 \le j \le n$,

$$x_{jm+i,j(n-1)+1} = x_{(j-1)m+i,(j-1)(n-1)+1} + (-1)^{i+1} d'_j \text{ and}$$
$$x_{jm+i,j(n-1)+n+1} = x_{(j-1)m+i,(j-1)(n-1)+n+1} + (-1)^{i} d'_j.$$

□

3 Results on $C_4 \times C_4$

Curran and Low [7] determine all antipodal balanced C_4-face-magic toroidal labelings on $C_4 \times C_4$ (up to symmetries on a torus). In order to state this result precisely, we must introduce some definitions. This result, as stated in Theorem 3.5, is the basis for the investigation of antipodal balanced C_4-face-magic toroidal labelings on $C_{2m} \times C_{2n}$ in this paper.

Definition 3.1. Let n be a positive integer Let $\{x_{i,j} : i, j = 1, 2, \ldots, 2n\}$ be a C_4-face-magic torus labeling on $C_{2n} \times C_{2n}$. We say that the C_4-face-magic labeling $\{x_{i,j}\}$ on $C_{2n} \times C_{2n}$ is *torus symmetric* if all row sums, column sums, and diagonal sums have a constant value S'. In other words, the sums

$$R_i = \sum_{j=1}^{2n} x_{i,j} = S' \qquad \text{for all } i = 1, 2, \ldots, 2n,$$

$$C_j = \sum_{i=1}^{2n} x_{i,j} = S' \qquad \text{for all } j = 1, 2, \ldots, 2n,$$

$$D_j = \sum_{i=1}^{2n} x_{i,i+j} = S' \qquad \text{for all } j = 1, 2, \ldots, 2n \text{ and}$$

$$D'_j = \sum_{i=1}^{2n} x_{i,j-i} = S' \qquad \text{for all } j = 1, 2, \ldots, 2n,$$

are constant.

According to Lemma 3.2, a torus symmetric C_4-face-magic toroidal labeling on $C_4 \times C_4$ is equivalent to a C_4-face-magic toroidal labeling on $C_4 \times C_4$ in which the four 2×2 block sums given by $B_{i,j} = x_{i,j} + x_{i,j+2} + x_{i+2,j} + x_{i+2,j+2}$, for all $i, j = 1, 2$, also add up to the C_4-face-magic value 34.

Lemma 3.2 ([7], Lemma 8)**.** *Consider the system of linear equations $x_{i,j} + x_{i+1,j} + x_{i,j+1} + x_{i+1,j+1} = 34$ (S1), for all $i = 1, 2, 3$ and $j = 1, 2, 3$ for a C_4-face-magic labeling on $P_4 \times P_4$. Let (S2) be the system S1 together with the equations $B_{i,j} = x_{i,j} + x_{i,j+2} + x_{i+2,j} + x_{i+2,j+2} = 34$, for all $i, j = 1, 2$. If the labeling $\{x_{i,j}\}$ satisfies system (S2), then $\{x_{i,j}\}$ is torus symmetric. Also, let (S3) be the system (S1) together with the equations $R_1 = x_{1,1} + x_{1,2} + x_{1,3} + x_{1,4} = 34$, $C_1 = x_{1,1} + x_{2,1} + x_{3,1} + x_{4,1} = 34$, and $D_4 = x_{1,1} + x_{2,2} + x_{3,3} + x_{4,4} = 34$. Then, (S2) is equivalent to (S3).*

Definition 3.3. Consider the natural embedding of $C_{2n} \times C_{2n}$ in the torus. We say that two torus symmetric C_4-face-magic toroidal labelings on $C_{2n} \times C_{2n}$ are *torus equivalent* if there is a homeomorphism of the torus that maps $C_{2n} \times C_{2n}$ onto itself such that the first C_4-face-magic toroidal labeling on $C_{2n} \times C_{2n}$ is mapped to the second C_4-face-magic toroidal labeling on $C_{2n} \times C_{2n}$.

By Lemma 3.4, a torus symmetric C_4-face-magic toroidal labeling on $C_4 \times C_4$ is antipodal balanced.

Lemma 3.4 ([7], Lemma 13)**.** *Let $\{x_{i,j}\}$ be a torus symmetric C_4-face-magic toroidal labeling on $C_4 \times C_4$. Then, for all i and j, we have $x_{i,j} + x_{i+2,j+2} = 17$ where the indices are taken modulo four. In other words, the labeling $\{x_{i,j}\}$ is antipodal balanced.*

All torus symmetric C_4-face-magic labelings on $C_4 \times C_4$ (up to torus equivalence) are given in the following theorem.

Theorem 3.5 ([7], Theorem 10)**.** *There are three distinct torus nonequivalent torus symmetric C_4-face-magic toroidal labelings on $C_4 \times C_4$. These three distinct torus nonequivalent torus symmetric C_4-face-magic toroidal labelings on $C_4 \times C_4$ are given below:*

1	8	13	12
14	11	2	7
4	5	16	9
15	10	3	6

Table 1: Torus symmetric C_4-face-magic toroidal labeling A on $C_4 \times C_4$.

1	8	11	14
12	13	2	7
6	3	16	9
15	10	5	4

Table 2: Torus symmetric C_4-face-magic toroidal labeling B on $C_4 \times C_4$.

1	12	7	14
8	13	2	11
10	3	16	5
15	6	9	4

Table 3: Torus symmetric C_4-face-magic toroidal labeling C on $C_4 \times C_4$.

An interesting observation about these three labelings is that the row sums and column sums of any two labelings are the 16 C_4-face sums in the third labeling. In the remark below, we indicate how the three torus nonequivalent torus symmetric C_4-face-magic toroidal labelings in Theorem 3.5 can be regarded as coming from one particular labeling on $C_4 \times C_4$.

Remark 3.6 ([7], Remark 24)**.** Label the vertices of $C_4 \times C_4$ with the elements from the set $\{0, 1\}^4$ so that the labelings on each C_4 face adds up to $(2, 2, 2, 2)$. This labeling is given in Table 4. Then the corresponding C_4-face-magic torus labelings on $C_4 \times C_4$ are given by $x_{i,j} = x_{1,1} + a_1 d_1 + a_2 d_2 + a_3 d_3 + a_4 d_4$ where $x_{1,1} = 1$, (a_1, a_2, a_3, a_4) is the labeling on vertex (i, j) in $C_4 \times C_4$ given in Table 4, and (d_1, d_2, d_3, d_4) is one of the three choices of either $(1, 2, 4, 8)$, $(1, 4, 2, 8)$ or $(1, 8, 2, 4)$. The choices of $(1, 2, 4, 8)$, $(1, 4, 2, 8)$ or $(1, 8, 2, 4)$ for (d_1, d_2, d_3, d_4) result in the labelings A, B and C, respectively, in Theorem 3.5.

$(0,0,0,0)$	$(1,1,1,0)$	$(0,0,1,1)$	$(1,1,0,1)$
$(1,0,1,1)$	$(0,1,0,1)$	$(1,0,0,0)$	$(0,1,1,0)$
$(1,1,0,0)$	$(0,0,1,0)$	$(1,1,1,1)$	$(0,0,0,1)$
$(0,1,1,1)$	$(1,0,0,1)$	$(0,1,0,0)$	$(1,0,1,0)$

Table 4: C_4-face-magic toroidal labeling with elements from $\{0,1\}^4$ on $C_4 \times C_4$.

4 Results on $C_{2m} \times C_{2n}$

We first consider antipodal balanced C_4-face-magic toroidal labelings on $C_6 \times C_6$.

Lemma 4.1. *Let $\{x_{i,j}\}$ be an antipodal balanced toroidal C_4-face-magic labeling on $C_6 \times C_6$. Let $d_i = x_{2i+1,3i+1} - x_{2i-1,3i-2}$, for $i = 1,2,3$, and let $d_{j+3} = d'_j = x_{3j+1,2j+1} - x_{3j-2,2j-2}$, for $j = 1,2,3$. Then the value of $x_{i,j}$ is determined by the values of $x_{1,1}$ and d_i, for $i = 1,2,3,4,5,6$ as displayed in Table 5.*

Proof. By Lemma 2.11, for integers i and j, we have

$$x_{2i+1,3i+j} = x_{2i-1,3i-3+j} + (-1)^{j+1} d_i,$$

when $1 \le i \le 3$ and $1 \le j \le 6$ and

$$x_{2i+4,3i+j} = x_{2i+2,3i-3+j} + (-1)^{j+1} d_i,$$

when $1 \le i \le 3$ and $1 \le j \le 6$. Thus $x_{1,4} = x_{1,1} + d_1 + d_2 + d_3$. Furthermore, it suffices to establish the result for the expressions $x_{1,j}$ and $x_{4,j}$, for $j = 1,2,3$. Also, by Lemma 4.1, for integers i and j, we have

$$x_{3j+i,2j+1} = x_{3j-3)+i,2j-1} + (-1)^{i+1} d_{j+3},$$

when $1 \le i \le 6$ and $1 \le j \le 3$ and

$$x_{3j+i,2j+4} = x_{3j-3+i,2j+2} + (-1)^{i} d_{j+3},$$

when $1 \le i \le 6$ and $1 \le j \le 3$. Thus $x_{4,1} = x_{1,1}+d_4+d_5+d_6$ and $x_{4,3} = x_{1,1}+d_4$. From $x_{1,4} = x_{1,1}+d_1+d+2+d_3$ and $x_{1,4} = x_{4,2}-d_6$, we have $x_{4,2} = x_{1,1}+d_1+d_2+d_3+d_6$. From $x_{4,1} = x_{1,1} + d_4 + d_5 + d_6$ and $x_{1,3} = x_{4,1} - d_4$, we have $x_{1,1} = x_{1,1} + d_5 + d_6$. Lastly, from $x_{1,4} = x_{1,1} + d_1 + d + 2 + d_3$, $x_{6,3} = x_{3,1} + d_4$, $x_{3,5} = x_{6,3} + d_5$ and $x_{1,2} = x_{3,5} - d_1$, we have $x_{1,2} = x1,1 + d_1 + d_2 + d_3 + d_4 + d_5$. $\square$

We next consider antipodal balanced C_4-face-magic toroidal labelings on $C_8 \times C_8$.

Lemma 4.2. *Let $\{x_{i,j}\}$ be an antipodal balanced toroidal C_4-face-magic labeling on $C_8 \times C_8$. Let $d_i = x_{3i+1,4i+1} - x_{3i-2,4i-3}$, for $i = 1,2,3,4$, and let $d_{j+4} = d'_j = x_{4j+1,3j+1} - x_{4j-3,3j-2}$, for $j = 1,2,3,4$. Then the value of $x_{i,j}$ is determined by the values of $x_{1,1}$ and d_i, for $i = 1,2,3,4,5,6,7,8$ as displayed in Figure 1.*

$x_{1,1}$	$x_{1,2} = x_{1,1} + d_1 + d_2 + d_3 + d_4 + d_5$	$x_{1,3} = x_{1,1} + d_5 + d_6$	$x_{1,4} = x_{1,1} + d_1 + d_2 + d_3$	$x_{1,5} = x_{1,1} + d_4 + d_5$	$x_{1,6} = x_{1,1} + d_1 + d_2 + d_3 + d_5 + d_6$
$x_{2,1} = x_{1,1} + d_1 + d_2 + d_4 + d_5 + d_6$	$x_{2,2} = x_{1,1} + d_3 + d_6$	$x_{2,3} = x_{1,1} + d_1 + d_2 + d_4$	$x_{2,4} = x_{1,1} + d_3 + d_4 + d_5 + d_6$	$x_{2,5} = x_{1,1} + d_1 + d_2 + d_6$	$x_{2,6} = x_{1,1} + d_3 + d_4$
$x_{3,1} = x_{1,1} + d_2 + d_3$	$x_{3,2} = x_{1,1} + d_1 + d_4 + d_5$	$x_{3,3} = x_{1,1} + d_2 + d_3 + d_5 + d_6$	$x_{3,4} = x_{1,1} + d_1$	$x_{3,5} = x_{1,1} + d_2 + d_3 + d_4 + d_5$	$x_{3,6} = x_{1,1} + d_1 + d_5 + d_6$
$x_{4,1} = x_{1,1} + d_4 + d_5 + d_6$	$x_{4,2} = x_{1,1} + d_1 + d_2 + d_3 + d_6$	$x_{4,3} = x_{1,1} + d_4$	$x_{4,4} = x_{1,1} + d_1 + d_2 + d_3 + d_4 + d_5 + d_6$	$x_{4,5} = x_{1,1} + d_6$	$x_{4,6} = x_{1,1} + d_1 + d_2 + d_3 + d_4$
$x_{5,1} = x_{1,1} + d_1 + d_2$	$x_{5,2} = x_{1,1} + d_3 + d_4 + d_5$	$x_{5,3} = x_{1,1} + d_1 + d_2 + d_5 + d_6$	$x_{5,4} = x_{1,1} + d_3$	$x_{5,5} = x_{1,1} + d_1 + d_2 + d_4 + d_5$	$x_{5,6} = x_{1,1} + d_3 + d_5 + d_6$
$x_{6,1} = x_{1,1} + d_2 + d_3 + d_4 + d_5 + d_6$	$x_{6,2} = x_{1,1} + d_1 + d_6$	$x_{6,3} = x_{1,1} + d_2 + d_3 + d_4$	$x_{6,4} = x_{1,1} + d_1 + d_4 + d_5 + d_6$	$x_{6,5} = x_{1,1} + d_2 + d_3 + d_6$	$x_{6,6} = x_{1,1} + d_1 + d_4$

Table 5: C_4-face-magic toroidal labeling involving the differences d_1, d_2, d_3, d_4, d_5 and d_6 on $C_6 \times C_6$.

Proof. By Lemma 2.10, for integers i and j, we have

$$x_{3i+1,j} = x_{3i-2,j+4} + (-1)^{j+1} d_i,$$

when $1 \leq i \leq 4$ and $1 \leq j \leq 8$ and

$$x_{3i+1,j} = x_{3i-2,j+4} + (-1)^{j} d_{i-4},$$

when $5 \leq i \leq 8$ and $1 \leq j \leq 8$. Thus $x_{5,1} = x_{1,1} + d_1 + d_2 + d_3 + d_4$. Furthermore, we only need to verify the expressions for $x_{1,j}$ for $2 \leq j \leq 8$. Also, by Lemma 2.10, for all integers i and j such that we have

$$x_{i,3j+1} = x_{i+4,3j-2} + (-1)^{i+1} d'_j,$$

when $1 \leq i \leq 8$ and $1 \leq j \leq 4$ and

$$x_{i,3j+1} = x_{i+4,3j-2} + (-1)^{i} d'_{j-4},$$

when $1 \leq i \leq 8$ and $5 \leq j \leq 8$. Hence, $x_{1,7} = x_{1,1} + d_5 + d_6$, $x_{1,5} = x_{1,7} + d_7 + d_8$, $x_{1,3} = x_{1,5} - d_5 - d_6$. Thus $x_{1,5} = x_{1,1} + d_5 + d_6 + d_7 + d_8$ and $x_{1,3} = x_{1,1} + d_7 + d_8$. Also,

$x_{1,4} = x_{5,1} + d_5$, $x_{1,2} = x_{1,4} + d_6 + d_7$, $x_{1,8} = x_{1,2} + d_8 - d_5$ and $x_{1,6} = x_{1,8} - d_6 - d_7$. Since $x_{5,1} = x_{1,1} + d_1 + d_2 + d_3 + d_4$, we have $x_{1,4} = x_{1,1} + d_1 + d_2 + d_3 + d_4 + d_5$ $x_{1,2} = x_{1,1} + d_1 + d_2 + d_3 + d_4 + d_5 + d_6 + d_7$, $x_{1,8} = x_{1,1} + d_1 + d_2 + d_3 + d_4 + d_6 + d_7 + d_8$ and $x_{1,6} = x_{1,1} + d_1 + d_2 + d_3 + d_4 + d_8$. □

Lemma 4.3. *Let $\{x_{i,j}\}$ be an antipodal balanced C_4-face-magic toroidal labeling on $C_{4m} \times C_{4n}$. Then, we have*

$$R_i = \sum_{j=1}^{4n} x_{i,j} = 2n(16mn + 1), \textit{ for all } i = 1, 2, \dots, 4m$$

and

$$C_j = \sum_{i=1}^{4m} x_{i,j} = 2m(16mn + 1), \textit{ for all } j = 1, 2, \dots, 4n.$$

Furthermore, if $m = n$, then we have

$$D_i = \sum_{j=1}^{4n} x_{j,i+j} = 2n(16n^2 + 1), \textit{ for all } i = 1, 2, \dots, 4n$$

and

$$D'_i = \sum_{j=1}^{4n} x_{j,i-j} = 2n(16n^2 + 1), \textit{ for all } i = 1, 2, \dots, 4n.$$

Proof. Let $S = 2(16mn + 1)$ be the C_4-face-magic value of $\{x_{i,j}\}$. We have, for all $i = 1, 2, \dots, 4m$,

$$R_i + R_{i+1} = \sum_{j=1}^{2n} \big(x_{i,2j-1} + x_{i,2j} + x_{i+1,2j-1} + x_{i+1,2j}\big) = 2nS = 4n(16mn + 1).$$

Thus it suffices to show that $R_1 = 2n(16mn + 1)$. We first observe that for all $j = 1, 2, \dots, n$,

$$x_{2m+1,2j-1+2n} = x_{1,2j-1+2n} + (d_1 + d_2 + \cdots + d_{2m}).$$

Thus,

$$\tfrac{1}{2}S = x_{1,2j-1} + x_{2m+1,2j-1+2n} = x_{1,2j-1} + x_{1,2j-1+2n} + (d_1 + d_2 + \cdots + d_{2m}).$$

We next observe that for all $j = 1, 2, \dots, n$,

$$x_{2m+1,2j+2n} = x_{1,2j+2n} - (d_1 + d_2 + \cdots + d_{2m}).$$

Thus,

$$\tfrac{1}{2}S = x_{1,2j} + x_{2m+1,2j+2n} = x_{1,2j} + x_{1,2j+2n} - (d_1 + d_2 + \cdots + d_{2m}).$$

$x_{1,1}$	$x_{1,2} = x_{1,1} + d_1 + d_2 + d_3 + d_4 + d_5 + d_6 + d_7$	$x_{1,3} = x_{1,1} + d_7 + d_8$	$x_{1,4} = x_{1,1} + d_1 + d_2 + d_3 + d_4 + d_5$	$x_{1,5} = x_{1,1} + d_5 + d_6 + d_7 + d_8$	$x_{1,6} = x_{1,1} + d_1 + d_2 + d_3 + d_4 + d_8$	$x_{1,7} = x_{1,1} + d_5 + d_6$	$x_{1,8} = x_{1,1} + d_1 + d_2 + d_3 + d_4 + d_6 + d_7 + d_8$
$x_{2,1} = x_{1,1} + d_1 + d_2 + d_3 + d_5 + d_6 + d_7 + d_8$	$x_{2,2} = x_{1,1} + d_4 + d_8$	$x_{2,3} = x_{1,1} + d_1 + d_2 + d_3 + d_5 + d_6$	$x_{2,4} = x_{1,1} + d_4 + d_6 + d_7 + d_8$	$x_{2,5} = x_{1,1} + d_1 + d_2 + d_3$	$x_{2,6} = x_{1,1} + d_4 + d_5 + d_6 + d_7$	$x_{2,7} = x_{1,1} + d_1 + d_2 + d_3 + d_7 + d_8$	$x_{2,8} = x_{1,1} + d_4 + d_5$
$x_{3,1} = x_{1,1} + d_3 + d_4$	$x_{3,2} = x_{1,1} + d_1 + d_2 + d_5 + d_6 + d_7$	$x_{3,3} = x_{1,1} + d_3 + d_4 + d_7 + d_8$	$x_{3,4} = x_{1,1} + d_1 + d_2 + d_5$	$x_{3,5} = x_{1,1} + d_3 + d_4 + d_5 + d_6 + d_7 + d_8$	$x_{3,6} = x_{1,1} + d_1 + d_2 + d_8$	$x_{3,7} = x_{1,1} + d_3 + d_4 + d_5 + d_6$	$x_{3,8} = x_{1,1} + d_1 + d_2 + d_6 + d_7 + d_8$
$x_{4,1} = x_{1,1} + d_1 + d_5 + d_6 + d_7 + d_8$	$x_{4,2} = x_{1,1} + d_2 + d_3 + d_4 + d_8$	$x_{4,3} = x_{1,1} + d_1 + d_5 + d_6$	$x_{4,4} = x_{1,1} + d_2 + d_3 + d_4 + d_6 + d_7 + d_8$	$x_{4,5} = x_{1,1} + d_1$	$x_{4,6} = x_{1,1} + d_2 + d_3 + d_4 + d_5 + d_6 + d_7$	$x_{4,7} = x_{1,1} + d_1 + d_7 + d_8$	$x_{4,8} = x_{1,1} + d_2 + d_3 + d_4 + d_5$
$x_{5,1} = x_{1,1} + d_1 + d_2 + d_3 + d_4$	$x_{5,2} = x_{1,1} + d_5 + d_6 + d_7$	$x_{5,3} = x_{1,1} + d_1 + d_2 + d_3 + d_4 + d_7 + d_8$	$x_{5,4} = x_{1,1} + d_5$	$x_{5,5} = x_{1,1} + d_1 + d_2 + d_3 + d_4 + d_5 + d_6 + d_7 + d_8$	$x_{5,6} = x_{1,1} + d_8$	$x_{5,7} = x_{1,1} + d_1 + d_2 + d_3 + d_4 + d_5 + d_6$	$x_{5,8} = x_{1,1} + d_6 + d_7 + d_8$
$x_{6,1} = x_{1,1} + d_4 + d_5 + d_6 + d_7 + d_8$	$x_{6,2} = x_{1,1} + d_1 + d_2 + d_3 + d_8$	$x_{6,3} = x_{1,1} + d_4 + d_5 + d_6$	$x_{6,4} = x_{1,1} + d_1 + d_2 + d_3 + d_6 + d_7 + d_8$	$x_{6,5} = x_{1,1} + d_4$	$x_{6,6} = x_{1,1} + d_1 + d_2 + d_3 + d_5 + d_6 + d_7$	$x_{6,7} = x_{1,1} + d_4 + d_7 + d_8$	$x_{6,8} = x_{1,1} + d_1 + d_2 + d_3 + d_5$
$x_{7,1} = x_{1,1} + d_1 + d_2$	$x_{7,2} = x_{1,1} + d_3 + d_4 + d_5 + d_6 + d_7$	$x_{7,3} = x_{1,1} + d_1 + d_2 + d_7 + d_8$	$x_{7,4} = x_{1,1} + d_3 + d_4 + d_5$	$x_{7,5} = x_{1,1} + d_1 + d_2 + d_5 + d_6 + d_7 + d_8$	$x_{7,6} = x_{1,1} + d_3 + d_4 + d_8$	$x_{7,7} = x_{1,1} + d_1 + d_2 + d_5 + d_6$	$x_{7,8} = x_{1,1} + d_3 + d_4 + d_6 + d_7 + d_8$
$x_{8,1} = x_{1,1} + d_2 + d_3 + d_4 + d_5 + d_6 + d_7 + d_8$	$x_{8,2} = x_{1,1} + d_1 + d_8$	$x_{8,3} = x_{1,1} + d_2 + d_3 + d_4 + d_5 + d_6$	$x_{8,4} = x_{1,1} + d_1 + d_6 + d_7 + d_8$	$x_{8,5} = x_{1,1} + d_2 + d_3 + d_4$	$x_{8,6} = x_{1,1} + d_1 + d_5 + d_6 + d_7$	$x_{8,7} = x_{1,1} + d_2 + d_3 + d_4 + d_7 + d_8$	$x_{8,8} = x_{1,1} + d_1 + d_5$

Figure 1: C_4-face-magic toroidal labeling involving the differences d_1, d_2, d_3, d_4, d_5, d_6, d_7 and d_8 on $C_8 \times C_8$.

Hence, we have

$$\begin{aligned} R_1 &= \sum_{j=1}^{4n} x_{1,j} = \sum_{j=1}^{2n} \big(x_{1,2j-1} + x_{1,2j}\big) \\ &= \sum_{j=1}^{n} \big(x_{1,2j-1} + x_{1,2j-1+2n}\big) + \sum_{j=1}^{n} \big(x_{1,2j} + x_{1,2j+2n}\big) \\ &= nS = 2n(16mn+1). \end{aligned}$$

By interchanging the roles of i and j, we have

$$C_j = \sum_{i=1}^{4m} x_{i,j} = 2m(16mn+1), \text{ for all } j = 1, 2, \ldots, 4n.$$

Finally, we assume that $m = n$. Then,

$$D_i = \sum_{j=1}^{4n} x_{j,i+j} = \sum_{j=1}^{2n} \big(x_{j,i+j} + x_{j+2n,i+j+2n}\big) = (2n)\big(\tfrac{1}{2}S\big) = 2n(16n^2+1).$$

A similar argument shows that for all $i = 1, 2, \ldots, 4n$,

$$D'_i = \sum_{j=1}^{4n} x_{j,i-j} = 2n(16n^2+1).$$

□

Proposition 4.4. *Let m and n be integers where $m, n \geq 3$. Let $\{x_{i,j}\}$ be a C_4-face-magic labeling on $P_m \times P_n$ with face-magic value S. Suppose that for all integers i and j such that $1 \leq i \leq m-2$ and $1 \leq j \leq n-2$, we have*

$$x_{i,j} + x_{i,j+2} + x_{i+2,j} + x_{i+2,j+2} = S.$$

Then, $m \leq 4$ and $n \leq 4$.

Proof. For the purpose of contradiction, we assume that $m \geq 5$. We first observe that for all integers i and j such that $1 \leq i \leq m-2$ and $1 \leq j \leq n-2$, we have

$$x_{i,j} + x_{i,j+1} + x_{i+1,j} + x_{i+1,j+1} = S = x_{i,j+1} + x_{i,j+2} + x_{i+1,j+1} + x_{i+1,j+2}.$$

Thus,

$$x_{i,j} + x_{i+1,j} = x_{i,j+2} + x_{i+1,j+2}. \tag{4.1}$$

Replacing i with $i+1$ in equation (4.1) yields

$$x_{i+1,j} + x_{i+2,j} = x_{i+1,j+2} + x_{i+2,j+2}. \tag{4.2}$$

When we subtract equation (4.2) from equation (4.1) and rearrange terms, we obtain

$$x_{i,j} + x_{i+2,j+2} = x_{i+2,j} + x_{i,j+2}. \tag{4.3}$$

Since

$$x_{i,j} + x_{i,j+2} + x_{i+2,j} + x_{i+2,j+2} = S,$$

we have

$$x_{i,j} + x_{i+2,j+2} = \tfrac{1}{2}S = x_{i+2,j} + x_{i,j+2}. \tag{4.4}$$

Let $i = 1$ or 3, and $j = 1$. Then equation (4.4) yields

$$x_{1,1} + x_{3,3} = \tfrac{1}{2}S = x_{5,1} + x_{3,3}.$$

Hence, $x_{1,1} = x_{5,1}$. This is a contradiction. Therefore, $m \leq 4$. A similar argument shows that $n \leq 4$. □

Proposition 4.5. *Let m and n be even positive integers. Let y_j, for $j = 1, 2, \dots, 2n$, be a positive integer, and let $d_1, d_2, \dots, d_m$ be integers. We define a labeling $\{x_{i,j}\}$ on $C_{2m} \times C_{2n}$ by letting, for all integers i and j such that $0 \leq i \leq m$ and $1 \leq j \leq 2n$,*

$$x_{i(m-1)+1,in+j} = y_j + (-1)^{j+1} \sum_{k=1}^{i} d_k \textit{ and}$$

$$x_{i(m-1)+m+1,in+j} = y_j + (-1)^{j+1} \sum_{k=i+1}^{m} d_k.$$

Let $A = \{\sum_{k=1}^{i} d_k, \sum_{k=i}^{m} d_k : 1 \leq i \leq m\} \cup \{0\}$. For all integers j such that $1 \leq j \leq 2n$, let

$$A_j = \{y_j + (-1)^{j+1} a : a \in A\}.$$

Suppose that

1. *for all integers j such that $1 \leq j \leq n$, $y_j + y_{j+n} + (-1)^{j+1} \left(\sum_{k=1}^{m} d_k\right) = 4mn + 1$ and*
2. *the set $\{A_j : 1 \leq j \leq 2n\}$ forms a partition of the set $\{k \in \mathbb{Z} : 1 \leq k \leq 4mn\}$.*

Then, $\{x_{i,j}\}$ is an antipodal balanced C_4-face-magic toroidal labeling on $C_{2m} \times C_{2n}$.

Proof. We first show that the C_4-face sum is preserved for all of the relevant C_4 faces on $C_{2m} \times C_{2n}$. We note that $m - 1$ is relatively prime to $2m$. Thus, $m - 1$ is a generator of $\mathbb{Z}_{2m}$. Hence,

$$\{x_{i(m-1)+1,in+j} : i = 1, 2, \dots, 2m \text{ and } j = 1, 2, \dots, 2n\}$$

is the set $\{x_{i,j} : i = 1, 2, \dots, 2m \text{ and } j = 1, 2, \dots, 2n\}$. We observe that $(i+m)(m-1) + m + 1 = i(m-1) + 1 \pmod{2m}$. We have, for all integers i and j such that $0 \leq i \leq m$ and $1 \leq j \leq 2n$,

$$x_{i(m-1)+1,in+j} = y_j + (-1)^{j+1} \sum_{k=1}^{i} d_k \text{ and} \tag{4.5}$$

$$x_{i(m-1)+m+1,in+j} = y_j + (-1)^{j+1} \sum_{k=i+1}^{m} d_k. \tag{4.6}$$

We replace i with $i-1$ and j with $j+n$ in equation (4.6) to obtain, for all integers i and j such $1 \leq i \leq m+1$ and $1 \leq j \leq 2n$,

$$x_{(i-1)(m-1)+m+1,(i-1)n+j+n} = x_{i(m-1)+2,in+j} = y_{j+n} + (-1)^{j+1} \sum_{k=i}^{m} d_k.$$

Hence, for all integers i and j such $1 \leq i \leq m$ and $1 \leq j \leq 2n$, we have

$$\begin{aligned}
&x_{i(m-1)+1,in+j} + x_{i(m-1)+1,in+j+1} + x_{i(m-1)+2,in+j} + x_{i(m-1)+2,in+j+1} \\
&\qquad = y_j + (-1)^{j+1}\left(\sum_{k=1}^{i} d_k\right) + y_{j+1} + (-1)^{j+2}\left(\sum_{k=1}^{i} d_k\right) \\
&\qquad\qquad + y_{j+n} + (-1)^{j+1}\left(\sum_{k=i}^{m} d_k\right) + y_{j+n+1} + (-1)^{j+2}\left(\sum_{k=i}^{m} d_k\right) \\
&\qquad = \left(y_j + y_{j+n} + (-1)^{j+1}(\sum_{k=1}^{m} d_k)\right) \\
&\qquad\qquad + \left(y_{j+1} + y_{j+n+1} + (-1)^{j+2}(\sum_{k=1}^{m} d_k)\right) \\
&\qquad = \tfrac{1}{2}S + \tfrac{1}{2}S = S.
\end{aligned}$$

Next, we replace i with $i-1$ and j with $j+n$ in equation (4.5) to obtain, for all integers i and j such $1 \leq i \leq m+1$ and $1 \leq j \leq 2n$,

$$x_{(i-1)(m-1)+1,(i-1)n+j+n} = x_{i(m-1)+m+2,in+j} = y_{j+n} + (-1)^{j+1} \sum_{k=1}^{i-1} d_k.$$

Hence, for all integers i and j such $1 \leq i \leq m$ and $1 \leq j \leq 2n$, we have

$$\begin{aligned}
&x_{i(m-1)+m+1,in+j} + x_{i(m-1)+m+1,in+j+1} + x_{i(m-1)+m+2,in+j} + x_{i(m-1)+m+2,in+j+1} \\
&\qquad = y_j + (-1)^{j+1}\left(\sum_{k=i+1}^{m} d_k\right) + y_{j+1} + (-1)^{j+2}\left(\sum_{k=i+1}^{m} d_k\right) \\
&\qquad\qquad + y_{j+n} + (-1)^{j+1}\left(\sum_{k=1}^{i-1} d_k\right) + y_{j+n+1} + (-1)^{j+2}\left(\sum_{k=1}^{i-1} d_k\right) \\
&\qquad = \left(y_j + y_{j+n} + (-1)^{j+1}(\sum_{k=1}^{m} d_k)\right) \\
&\qquad\qquad + \left(y_{j+1} + y_{j+n+1} + (-1)^{j+2}(\sum_{k=1}^{m} d_k)\right) \\
&\qquad = \tfrac{1}{2}S + \tfrac{1}{2}S = S.
\end{aligned}$$

Hence, $\{x_{i,j}\}$ is a C_4-face-magic toroidal labeling on $C_{2m} \times C_{2n}$.

We show that each integer k, where $1 \leq k \leq 4mn$, is used exactly once in the labeling $\{x_{i,j}\}$. For each integer j such that $1 \leq j \leq 2n$, we show that $\{x_{i(m-1)+1,in+j} : 1 \leq i \leq$

$2m\} = A_j$. We have

$$\begin{aligned}\{x_{i(m-1)+1,in+j} : 1 \le i \le 2m\} &= \{x_{i(m-1)+1,in+j}, x_{i(m-1)+m+1,in+j} : 1 \le i \le m\}\\ &= \{y_j, y_j + (-1)^{j+1}(\sum_{k=1}^{i} d_k), y_j + (-1)^{j+1}(\sum_{k=i}^{m} d_k) : 1 \le i \le m\}\\ &= \{y_y + (-1)^{j+1}a : a \in A\} = A_j.\end{aligned}$$

Since $\{A_j : 1 \le j \le 2n\}$ is a partition of the set $\{k \in \mathbb{Z} : 1 \le k \le 4mn\}$, each integer k, where $1 \le k \le 4mn$, is used exactly once in the labeling $\{x_{i,j}\}$.

Finally, we show that $\{x_{i,j}\}$ is an antipodal balanced labeling on $C_{2m} \times C_{2n}$. When we replace j with $j + n$ in equation (4.6), we have

$$x_{(i+m)(m-1)+1,(i+m)n+j+n} = x_{i(m-1)+1+m,in+j+n} = y_{j+n} + (-1)^{j+1}(\sum_{k=i+1}^{m} d_k). \tag{4.7}$$

When we add equations (4.5) and (4.7), we have

$$\begin{aligned}&x_{i(m-1)+1,in+j} + x_{i(m-1)+1+m,in+j+n}\\ &\qquad = y_j + (-1)^{j+1}(\sum_{k=1}^{i} d_k) + y_{j+n} + (-1)^{j+1}(\sum_{k=i+1}^{m} d_k)\\ &\qquad = y_j + y_{j+n} + (-1)^{j+1}(\sum_{k=1}^{m} d_k) = \tfrac{1}{2}S = 4mn + 1.\end{aligned}$$

When we replace i with $i + m$ in this equation, we have

$$\begin{aligned}&x_{(i+m)(m-1)+1+m,(i+m)n+j} + x_{(i+m)(m-1)+1,(i+m)n+j+n}\\ &\qquad = x_{i(m-1)+1,in+j} + x_{i(m-1)+1+m,in+j+n} = 4mn + 1.\end{aligned}$$

This completes the proof. □

Proposition 4.6. *We define a labeling $\{x_{i,j}\}$ on $C_{4m} \times C_{4n}$ in the following manner. For integers i and j such that $1 \le i \le 2m$ and $1 \le j \le 2n$, when j is odd, we let*

$$\begin{aligned}x_{i(2m-1)+1,j+i(2n)} &= 4m(j-1) + 2i,\\ x_{i(2m-1)+1,j+(i+1)(2n)} &= 4m(4n-j) + 2i,\\ x_{i(2m-1)+2m+1,j+i(2n)} &= 4mj - 2i + 1 \text{ and}\\ x_{i(2m-1)+2m+1,j+(i+1)(2n)} &= 4m(4n-j+1) - 2i + 1;\end{aligned}$$

and when j is even, we let

$$\begin{aligned}x_{i(2m-1)+1,j+i(2n)} &= 4mj - 2i + 1,\\ x_{i(2m-1)+1,j+(i+1)(2n)} &= 4m(4n-j+1) - 2i + 1,\\ x_{i(2m-1)+2m+1,j+i(2n)} &= 4m(j-1) + 2i \text{ and}\\ x_{i(2m-1)+2m+1,j+(i+1)(2n)} &= 4m(4n-j) + 2i.\end{aligned}$$

Then, $\{x_{i,j}\}$ is an antipodal balanced C_4-face-magic toroidal labeling on $C_{4m} \times C_{4n}$. Furthermore, by Lemma 4.3, $\{x_{i,j}\}$ is row-sum balanced and column-sum balanced, and $\{x_{i,j}\}$ is torus symmetric whenever $m = n$.

Proof. In the notation of Proposition 4.5, we have $d_1 = 1$ and $d_k = 2$, for $2 \leq k \leq 2m$. Also, for integers j such that $1 \leq j \leq 2n$, we have $y_j = 4m(j-1)+1$ and $y_{j+2n} = 4m(4n-j)+1$ when j is odd, and $y_j = 4mj$ and $y_{j+2n} = 4m(4n-j+1)$ when j is even. Then the labeling $\{x_{i,j}\}$ of $C_{4m} \times C_{4n}$ satisfies, for all integers i and j such that $1 \leq i \leq 2m$ and $1 \leq j \leq 2n$, when j is odd, we have

$$\begin{aligned} x_{i(2m-1)+1,i(2n)+j} &= y_j + (-1)^{j+1}\big(\sum_{k=1}^{i} d_k\big) \\ &= \big(4m(j-1)+1\big) + (2i-1) = 4m(j-1)+2i \quad \text{and} \\ x_{i(2m-1)+1,i(2n)+j+2n} &= y_{j+2n} + (-1)^{j+1}\big(\sum_{k=1}^{i} d_k\big) \\ &= \big(4m(4n-j)+1\big) + (2i-1) = 4m(4n-j)+2i, \end{aligned}$$

and when j is even, we have

$$\begin{aligned} x_{i(2m-1)+1,i(2n)+j} &= y_j + (-1)^{j+1}\big(\sum_{k=1}^{i} d_k\big) \\ &= \big(4mj\big) - (2i-1) = 4mj-2i+1 \quad \text{and} \\ x_{i(2m-1)+1,i(2n)+j+2n} &= y_{j+2n} + (-1)^{j+1}\big(\sum_{k=1}^{i} d_k\big) \\ &= \big(4m(4n-j+1)\big) - (2i-1) = 4m(4n-j+1)-2i+1. \end{aligned}$$

We let $i = 2m$ in each of the previous four equations. Thus, for all integers i and j such that $1 \leq i \leq 2m$ and $1 \leq j \leq 2n$, when j is odd, we have

$$x_{2m+1,j} = 4mj \text{ and } x_{2m+1,j+2n} = 4m(4n-j+1),$$

and when j is even, we have

$$x_{2m+1,j} = 4m(j-1)+1 \text{ and } x_{2m+1,j+2n} = 4m(4n-j)+1.$$

Next, we observe that the labeling $\{x_{i,j}\}$ on $C_{4m} \times C_{4n}$ satisfies, for all integers i and j such that $1 \leq i \leq 2m$ and $1 \leq j \leq 2n$, when j is odd, we have

$$\begin{aligned} x_{i(2m-1)+2m+1,i(2n)+j} &= y_j + (-1)^{j+1}\big(\sum_{k=i+1}^{2m} d_k\big) \\ &= \big(4m(j-1)+1\big) + 2(2m-i) = 4mj-2i \quad \text{and} \\ x_{i(2m-1)+2m+1,i(2n)+j+2n} &= y_{j+2n} + (-1)^{j+1}\big(\sum_{k=i+1}^{2m} d_k\big) \\ &= \big(4m(4n-j)+1\big) + 2(2m-i) = 4m(4n-j+1)-2i, \end{aligned}$$

and when j is even, we have

$$\begin{aligned}
x_{i(2m-1)+2m+1,i(2n)+j} &= y_j + (-1)^{j+1}\big(\sum_{k=i+1}^{2m} d_k\big) \\
&= \big(4mj\big) - 2(2m-i) = 4m(j-1) + 2i \quad \text{and} \\
x_{i(2m-1)+2m+1,i(2n)+j+2n} &= y_{j+2n} + (-1)^{j+1}\big(\sum_{k=i+1}^{2m} d_k\big) \\
&= \big(4m(4n-j+1)\big) - 2(2m-i) = 4m(4n-j) + 2i.
\end{aligned}$$

We show that condition (1) of Proposition 4.5 is satisfied. Let j be an integer such that $1 \leq j \leq 2n$. When j is odd, we have

$$y_j+y_{j+2n}+(-1)^{j+1}\big(\sum_{k=1}^{2m} d_k\big) = \big(4m(j-1)+1\big)+\big(4m(4n-j)+1\big)+(4m-1) = 16mn+1.$$

When j is even, we have

$$y_j + y_{j+2n} + (-1)^{j+1}\big(\sum_{k=1}^{2m} d_k\big) = \big(4mj\big) + \big(4m(4n-j+1)\big) - (4m-1) = 16mn + 1.$$

We show that condition (2) of Proposition 4.5 is satisfied. We first observe that

$$\begin{aligned}
A &= \{\sum_{k=1}^{i} d_k, \sum_{k=i}^{2m} d_k : 1 \leq k \leq 2m\} \cup \{0\} \\
&= \{2i-1 : 1 \leq i \leq 2m\} \cup \{4m-2i+2 : 2 \leq i \leq 2m\} \cup \{0\} \\
&= \{k \in \mathbb{Z} : 0 \leq k \leq 4m-1\}.
\end{aligned}$$

Let j be an integer such that $1 \leq j \leq 2n$. When j is odd, we have

$$\begin{aligned}
A_j &= \{y_j + (-1)^{j+1}a : a \in A\} = \{4m(j-1) + 1 + i : i \in \mathbb{Z}, 0 \leq i \leq 4m-1\} \\
&= \{k \in \mathbb{Z} : 4m(j-1) + 1 \leq j \leq 4mj\}
\end{aligned}$$

and

$$\begin{aligned}
A_{j+2n} &= \{y_{j+2n} + (-1)^{j+1}a : a \in A\} \\
&= \{4m(4n-j) + 1 + i : i \in \mathbb{Z}, 0 \leq i \leq 4m-1\} \\
&= \{k \in \mathbb{Z} : 4m(4n-j) + 1 \leq j \leq 4m(4n-j+1)\}.
\end{aligned}$$

When j is even, we have

$$\begin{aligned}
A_j &= \{y_j + (-1)^{j+1}a : a \in A\} = \{4mj - i : i \in \mathbb{Z}, 0 \leq i \leq 4m-1\} \\
&= \{k \in \mathbb{Z} : 4m(j-1) + 1 \leq j \leq 4mj\}
\end{aligned}$$

and

$$\begin{aligned} A_{j+2n} &= \{y_{j+2n} + (-1)^{j+1}a : a \in A\} \\ &= \{4m(4n-j+1) - i : i \in \mathbb{Z}, 0 \le i \le 4m-1\} \\ &= \{k \in \mathbb{Z} : 4m(4n-j)+1 \le j \le 4m(4n-j+1)\}. \end{aligned}$$

Hence, for all integers j, when $1 \le j \le 2n$, we have

$$A_j = \{k \in \mathbb{Z} : 4m(j-1)+1 \le j \le 4mj\}$$

and when $2n+1 \le j \le 4n$, we have

$$A_j = \{k \in \mathbb{Z} : 4m(4n-j)+1 \le j \le 4m(4n-j+1)\}.$$

Therefore, $\{A_j : 1 \le j \le 4n\}$ is a partition of the set $\{k \in \mathbb{Z} : 1 \le k \le 16mn\}$. This completes the proof. □

Example 4.7. We consider the example of Proposition 4.6 where $m = n = 2$. This example is given in Table 6.

1	16	17	32	57	56	41	40
62	51	46	35	6	11	22	27
5	12	21	28	61	52	45	36
58	55	42	39	2	15	18	31
8	9	24	25	64	49	48	33
59	54	43	38	3	14	19	30
4	13	20	29	60	53	44	37
63	50	47	34	7	10	23	26

Table 6: An antipodal balanced C_4-face-magic toroidal labeling on $C_8 \times C_8$.

We next prove the converse to Proposition 4.5.

Proposition 4.8. *Let m and n be even positive integers. Let $\{x_{i,j} : (i,j) \in V(C_{2m} \times C_{2n})\}$ be an antipodal balanced C_4-face-magic toroidal labeling on $C_{2m} \times C_{2n}$. For all integers j such that $1 \le j \le 2n$, let $y_j = x_{1,j}$. For all integers i such that $1 \le i \le m$, let*

$$d_i = x_{i(m-1)+1,in+1} - x_{(i-1)(m-1)+1,(i-1)n+1}.$$

Then, for all integers i and j such that $0 \le i \le m$ and $1 \le j \le 2n$,

$$x_{i(m-1)+1,in+j} = y_j + (-1)^{j+1} \sum_{k=1}^{i} d_k \text{ and} \tag{4.8}$$

$$x_{i(m-1)+m+1,in+j} = y_j + (-1)^{j+1} \sum_{k=i+1}^{m} d_k. \tag{4.9}$$

Let $A = \{\sum_{k=1}^{i} d_k, \sum_{k=i}^{m} d_k : 1 \le i \le m\} \cup \{0\}$. *For all integers* j *such that* $1 \le j \le 2n$, *let* $A_j = \{y_j + (-1)^{j+1} a : a \in A\}$. *Then*

1. *for all integers* j *such that* $1 \le j \le n$, $y_j + y_{j+n} + (-1)^{j+1}\left(\sum_{k=1}^{m} d_k\right) = 4mn + 1$ *and*
2. *the set* $\{A_j : 1 \le j \le 2n\}$ *forms a partition of the set* $\{k \in \mathbb{Z} : 1 \le k \le 4mn\}$.

Proof. By Lemma 2.10, for all integers i and j such that $1 \le i \le m$ and $1 \le j \le 2n$, we have

$$x_{i(m-1)+1,in+j} = x_{(i-1)(m-1)+1,(i-1)n+j} + (-1)^{j+1} d_i. \tag{4.10}$$

For all integers i and j such that $1 \le i \le m$ and $1 \le j \le 2n$, repeated use of equation (4.10) yields

$$x_{i(m-1)+1,in+j} = x_{1,j} + (-1)^{j+1} \sum_{k=1}^{i} d_k. \tag{4.11}$$

Thus equation (4.8) holds. When we let $i = m$ in equation (4.11), we have

$$x_{m+1,j} = x_{1,j} + (-1)^{j+1} \sum_{k=1}^{m} d_k. \tag{4.12}$$

By Lemma 2.10, for all integers i and j such that $m + 1 \le i \le 2m$ and $1 \le j \le 2n$, we have

$$x_{i(m-1)+1,in+j} = x_{(i-1)(m-1)+1,(i-1)n+j} + (-1)^{j} d_{i-m}. \tag{4.13}$$

Let $1 \le i \le m$. Replacing i with $i + m$ in equation (4.13) yields

$$x_{i(m-1)+m+1,in+j} = x_{(i-1)(m-1)+m+1,(i-1)n+j} + (-1)^{j} d_i. \tag{4.14}$$

For all integers i and j such that $1 \le i \le m$ and $1 \le j \le 2n$, repeated use of equation (4.14) yields

$$x_{i(m-1)+m+1,in+j} = x_{m+1,j} + (-1)^{j} \sum_{k=1}^{i} d_k. \tag{4.15}$$

When we combine equations (4.12) and (4.15), we have

$$x_{i(m-1)+m+1,in+j} = x_{1,j} + (-1)^{j+1} \sum_{k=i+1}^{m} d_k. \tag{4.16}$$

Thus equation (4.9) holds.

Since $\{x_{i,j}\}$ is an antipodal balanced C_4-face-magic toroidal labeling on $C_{2m} \times C_{2n}$, we have $x_{1,n+j} + x_{m+1,j} = 4mn + 1$ for all integers j such that $1 \le j \le n$. By equation (4.12), we have $x_{m+1,j} = y_j + (-1)^{j+1} \sum_{k=1}^{m} d_k$. Since $x_{1,n+j} = y_{j+n}$, we have $y_j + y_{j+n} + (-1)^{j+1}\left(\sum_{k=1}^{m} d_k\right) = x_{1,n+j} + x_{m+1,j} = 4mn + 1$.

By equations (4.8) and (4.9), for integers j such that $1 \le j \le 2n$, we have

$$A_j = \{x_{i(m-1)+1,in+j}, x_{i(m-1)+m+1,in+j} : 0 \le i \le m-1\}.$$

Thus the set $\{A_j : 1 \le j \le 2n\}$ forms a partition of the set $\{x_{i,j} : 1 \le i \le 2m$ and $1 \le j \le 2n\} = \{k \in \mathbb{Z} : 1 \le k \le 4mn\}$. ∎

Proposition 4.9. *Let m and n be odd positive integers. Let y_j and z_j, for $j = 1, 2, \ldots, n$, be a positive integers, and let $d_1, d_2, \ldots, d_m$ be integers. We define a labeling $\{x_{i,j}\}$ on $C_{2m} \times C_{2n}$ by letting, for all integers i and j such $0 \le i \le m$ and $1 \le j \le n$,*

$$x_{i(m-1)+1,in+j} = y_j + (-1)^{j+1} \sum_{k=1}^{i} d_k, \tag{4.17}$$

$$x_{i(m-1)+1,(i+1)n+j} = y_j + (-1)^{j+1} \sum_{k=i+1}^{m} d_k, \tag{4.18}$$

$$x_{i(m-1)+m+1,in+j} = z_j + (-1)^{j+1} \sum_{k=1}^{i} d_k \text{ and} \tag{4.19}$$

$$x_{i(m-1)+m+1,(i+1)n+j} = z_j + (-1)^{j+1} \sum_{k=i+1}^{m} d_k. \tag{4.20}$$

Let $A = \{\sum_{k=1}^{i} d_k, \sum_{k=i}^{m} d_k : 1 \le i \le m\} \cup \{0\}$. For all integers j such that $1 \le j \le n$, let

$$A_j = \{y_j + (-1)^{j+1} a : a \in A\} \text{ and } A_{j+n} = \{z_j + (-1)^{j+1} a : a \in A\}.$$

Suppose

1. *for all integers j such that $1 \le j \le n$, $y_j + z_j + (-1)^{j+1}\left(\sum_{k=1}^{m} d_k\right) = 4mn + 1$, and*
2. *the set $\{A_j : 1 \le j \le 2n\}$ forms a partition of the set $\{k \in \mathbb{Z} : 1 \le k \le 4mn\}$.*

Then, $\{x_{i,j}\}$ is an antipodal balanced C_4-face-magic toroidal labeling on $C_{2m} \times C_{2n}$.

Proof. We first show that the C_4-face sum is preserved for all relevant C_4 faces on $C_{2m} \times C_{2n}$. Since $\gcd(2m, m-1) = 2$, $m - 1$ generates the subgroup $\langle 2 \rangle$ of $\mathbb{Z}_{2m}$. Hence,

$$\{x_{i(m-1)+1,in+j}, x_{(i+1)(m-1)+1,in+j}, x_{i(m-1)+m+1,in+j}, x_{(i+1)(m-1)+m+1,in+j} : \\ i = 1, 2, \ldots, m \text{ and } j = 1, 2, \ldots, n\}$$

is the set

$$\{x_{i,j} : i = 1, 2, \ldots 2, m \text{ and } j = 1, 2, \ldots, 2n\}.$$

We replace i with $i - 1$ in equations (4.17), (4.18), (4.19) and (4.20) to obtain

$$x_{i(m-1)+m+2,(i+1)n+j} = y_j + (-1)^{j+1} \sum_{k=1}^{i-1} d_k, \tag{4.21}$$

$$x_{i(m-1)+m+2,in+j} = y_j + (-1)^{j+1} \sum_{k=i}^{m} d_k, \tag{4.22}$$

$$x_{i(m-1)+2,(i+1)n+j} = z_j + (-1)^{j+1} \sum_{k=1}^{i-1} d_k \text{ and} \tag{4.23}$$

$$x_{i(m-1)+2,in+j} = z_j + (-1)^{j+1} \sum_{k=i}^{m} d_k. \tag{4.24}$$

We replace j with $j+1$ in equations (4.17), (4.18), (4.19), (4.20), (4.21), (4.22), (4.23) and (4.24) to obtain

$$x_{i(m-1)+1,in+j+1} = y_{j+1} + (-1)^{j+2}\sum_{k=1}^{i} d_k, \tag{4.25}$$

$$x_{i(m-1)+1,(i+1)n+j+1} = y_{j+1} + (-1)^{j+2}\sum_{k=i+1}^{m} d_k, \tag{4.26}$$

$$x_{i(m-1)+m+1,in+j+1} = z_{j+1} + (-1)^{j+2}\sum_{k=1}^{i} d_k, \tag{4.27}$$

$$x_{i(m-1)+m+1,(i+1)n+j+1} = z_{j+1} + (-1)^{j+2}\sum_{k=i+1}^{m} d_k, \tag{4.28}$$

$$x_{i(m-1)+m+2,(i+1)n+j+1} = y_{j+1} + (-1)^{j+2}\sum_{k=1}^{i-1} d_k, \tag{4.29}$$

$$x_{i(m-1)+m+2,in+j+1} = y_{j+1} + (-1)^{j+2}\sum_{k=i}^{m} d_k, \tag{4.30}$$

$$x_{i(m-1)+2,(i+1)n+j+1} = z_{j+1} + (-1)^{j+2}\sum_{k=1}^{i-1} d_k \text{ and} \tag{4.31}$$

$$x_{i(m-1)+2,in+j+1} = z_{j+1} + (-1)^{j+2}\sum_{k=i}^{m} d_k. \tag{4.32}$$

When we add equations (4.17), (4.25), (4.24) and (4.32) together, for $1 \leq i \leq m$ and $1 \leq j \leq n-1$, we obtain

$$\begin{aligned}
&x_{i(m-1)+1,in+j} + x_{i(m-1)+1,in+j+1} + x_{i(m-1)+2,in+j} + x_{i(m-1)+2,in+j+1} \\
&\quad = \Big(y_j + (-1)^{j+1}(\sum_{k=1}^{i} d_k)\Big) + \Big(y_{j+1} + (-1)^{j+2}(\sum_{k=1}^{i} d_k)\Big) \\
&\qquad + \Big(z_j + (-1)^{j+1}(\sum_{k=i}^{m} d_k)\Big) + \Big(z_{j+1} + (-1)^{j+2}(\sum_{k=i}^{m} d_k)\Big) \\
&\quad = \Big(y_j + z_j + (-1)^{j+1}(\sum_{k=1}^{m} d_k)\Big) + \Big(y_{j+1} + z_{j+1} + (-1)^{j+2}(\sum_{k=1}^{m} d_k)\Big) \\
&\quad = S.
\end{aligned}$$

When we add equations (4.18), (4.26), (4.23) and (4.31) together, for $1 \leq i \leq m$ and $1 \leq j \leq n-1$, we obtain

$$\begin{aligned}
&x_{i(m-1)+1,(i+1)n+j} + x_{i(m-1)+1,(i+1)n+j+1} \\
&\qquad + x_{i(m-1)+2,(i+1)n+j} + x_{i(m-1)+2,(i+1)n+j+1} = S.
\end{aligned}$$

Similar C_4-face-magic sums occur when we add equations (4.19), (4.49), (4.22), and (4.52) together; or when we add equations (4.20), (4.50), (4.21) and (4.51) together.

Suppose i is odd and let $j = n$ in equations (4.18), (4.20), (4.21), and (4.23), we have

$$x_{i(m-1)+1,n} = y_n + (-1)^{n+1} \sum_{k=i+1}^{m} d_k, \tag{4.33}$$

$$x_{i(m-1)+m+1,n} = z_n + (-1)^{n+1} \sum_{k=i+1}^{m} d_k, \tag{4.34}$$

$$x_{i(m-1)+m+2,n} = y_n + (-1)^{n+1} \sum_{k=1}^{i-1} d_k \text{ and} \tag{4.35}$$

$$x_{i(m-1)+2,n} = z_n + (-1)^{n+1} \sum_{k=1}^{i-1} d_k. \tag{4.36}$$

Suppose i is odd and let $j = 1$ in equations (4.17), (4.19), (4.22), and (4.24), we have

$$x_{i(m-1)+1,n+1} = y_1 + (-1)^2 \sum_{k=1}^{i} d_k, \tag{4.37}$$

$$x_{i(m-1)+m+1,n+1} = z_1 + (-1)^2 \sum_{k=1}^{i} d_k, \tag{4.38}$$

$$x_{i(m-1)+m+2,n+1} = y_1 + (-1)^2 \sum_{k=i}^{m} d_k \text{ and} \tag{4.39}$$

$$x_{i(m-1)+2,n+1} = z_1 + (-1)^2 \sum_{k=i}^{m} d_k. \tag{4.40}$$

When we add equations (4.33), (4.37), (4.36) and (4.40) together, we have

$$\begin{aligned} x_{i(m-1)+1,n} &+ x_{i(m-1)+1,n+1} + x_{i(m-1)+2,n} + x_{i(m-1)+2,n+1} \\ &= \Big(y_n + (\sum_{k=i+1}^{m} d_k)\Big) + \Big(y_1 + (\sum_{k=1}^{i} d_k)\Big) \\ &\qquad + \Big(z_n + (\sum_{k=1}^{i-1} d_k)\Big) + \Big(z_1 + (\sum_{k=i}^{m} d_k)\Big) \\ &= \Big(y_1 + y_n + (\sum_{k=1}^{m} d_k)\Big) + \Big(z_1 + z_n + (\sum_{k=1}^{m} d_k)\Big) = S. \end{aligned}$$

When we add equations (4.34), (4.38), (4.35) and (4.39) together, we have

$$x_{i(m-1)+m+1,n} + x_{i(m-1)+m+1,n+1} + x_{i(m-1)+m+2,n} + x_{i(m-1)+m+2,n+1} = S.$$

Suppose i is even and let $j = n$ in equations (4.17), (4.19), (4.22) and (4.24), we have

$$x_{i(m-1)+1,n} = y_n + (-1)^{n+1} \sum_{k=1}^{i} d_k, \tag{4.41}$$

$$x_{i(m-1)+m+1,n} = z_n + (-1)^{n+1} \sum_{k=1}^{i} d_k, \tag{4.42}$$

$$x_{i(m-1)+m+2,n} = y_n + (-1)^{n+1} \sum_{k=i}^{m} d_k \text{ and} \tag{4.43}$$

$$x_{i(m-1)+2,n} = z_n + (-1)^{n+1} \sum_{k=i}^{m} d_k. \tag{4.44}$$

Suppose i is even and let $j = 1$ in equations (4.18), (4.20), (4.21) and (4.23), we have

$$x_{i(m-1)+1,n+1} = y_1 + (-1)^2 \sum_{k=i+1}^{m} d_k, \tag{4.45}$$

$$x_{i(m-1)+m+1,n+1} = z_1 + (-1)^2 \sum_{k=i+1}^{m} d_k, \tag{4.46}$$

$$x_{i(m-1)+m+2,n+1} = y_1 + (-1)^2 \sum_{k=1}^{i-1} d_k \text{ and} \tag{4.47}$$

$$x_{i(m-1)+2,n+1} = z_1 + (-1)^2 \sum_{k=1}^{i-1} d_k. \tag{4.48}$$

When we add equations (4.41), (4.45), (4.44) and (4.48) together, we have

$$\begin{aligned}
x_{i(m-1)+1,n} + x_{i(m-1)+1,n+1} + x_{i(m-1)+2,n} + x_{i(m-1)+2,n+1}& \\
= \Big(y_n + (\sum_{k=1}^{i} d_k)\Big) + \Big(y_1 + (\sum_{k=i+1}^{m} d_k)\Big)& \\
+ \Big(z_n + (\sum_{k=i}^{m} d_k)\Big) + \Big(z_1 + (\sum_{k=1}^{i-1} d_k)\Big)& \\
= \Big(y_1 + y_n + (\sum_{k=1}^{m} d_k)\Big) + \Big(z_1 + z_n + (\sum_{k=1}^{m} d_k)\Big) = S.&
\end{aligned}$$

When we add equations (4.42), (4.46), (4.43) and (4.47) together, we have

$$x_{i(m-1)+m+1,n} + x_{i(m-1)+m+1,n+1} + x_{i(m-1)+m+2,n} + x_{i(m-1)+m+2,n+1} = S.$$

For $1 \le j \le n$, by equations (4.17) and (4.18), we have

$$\begin{aligned}
&\{x_{i(m-1)+1,in+j} : i = 0, 1, \ldots, 2m-1\} \\
&\quad = \{x_{i(m-1)+1,in+j}, x_{(i+m)(m-1)+1,(i+m)n+j} : i = 0, 1, \ldots, m-1\} \\
&\quad = \{x_{i(m-1)+1,in+j}, x_{i(m-1)+1,(i+1)n+j} : i = 1, 2, \ldots, m\} \\
&\quad = \{y_j + (-1)^{j+1} \sum_{k=1}^{i} d_k, y_j + (-1)^{j+1} \sum_{k=i+1}^{m} d_k : i = 0, 1, \ldots, m-1\} \\
&\quad = \{y_j + (-1)^{j+1} a : a \in A\} = A_j.
\end{aligned}$$

For $1 \le j \le n$, by equations (4.19) and (4.20), we have

$$\begin{aligned}
&\{x_{i(m-1)+m+1,in+j} : i = 0, 1, \ldots, 2m-1\} \\
&\quad = \{x_{i(m-1)+m+1,in+j}, x_{(i+m)(m-1)+m+1,(i+m)n+j} : i = 0, 1, \ldots, m-1\} \\
&\quad = \{x_{i(m-1)+m+1,in+j}, x_{i(m-1)+m+1,(i+1)n+j} : i = 1, 2, \ldots, m\} \\
&\quad = \{z_j + (-1)^{j+1} \sum_{k=1}^{i} d_k, z_j + (-1)^{j+1} \sum_{k=i+1}^{m} d_k : i = 0, 1, \ldots, m-1\} \\
&\quad = \{z_j + (-1)^{j+1} a : a \in A\} = A_{j+n}.
\end{aligned}$$

Since $\{A_j, A_{j+n} : j = 1, 2, \ldots, n\}$ is a partition of $\{k \in \mathbb{Z} : 1 \le k \le 4mn\}$, $\{x_{i,j} : i = 1, 2, \ldots, 2m \text{ and } j = 1, 2, \ldots, 2n\} = \{k \in \mathbb{Z} : 1 \le k \le 4mn\}$. By Lemma 2.2, $\{x_{i,j}\}$ is a C_4-face-magic labeling on $C_{2m} \times C_{2n}$.

We need to show that $\{x_{i,j}\}$ is antipodal balanced. Let i and j be integers such that $0 \le i \le m$ and $1 \le j \le n$. We add equations (4.17) and (4.20) together to obtain

$$\begin{aligned}
&x_{i(m-1)+1,in+j} + x_{i(m-1)+m+1,(i+1)n+j} = \\
&\qquad = y_j + (-1)^{j+1} \sum_{k=1}^{i} d_k + z_j + (-1)^{j+1} \sum_{k=i+1}^{m} d_k \\
&\qquad = y_j + +z_j + (-1)^{j+1} \sum_{k=1}^{m} d_k = \tfrac{1}{2} S = 4mn + 1.
\end{aligned}$$

We add equations (4.18) and (4.19) together to obtain

$$\begin{aligned}
&x_{i(m-1)+1,(i+1)n+j} + x_{i(m-1)+m+1,in+j} = \\
&\qquad = y_j + (-1)^{j+1} \sum_{k=i+1}^{i} d_k + z_j + (-1)^{j+1} \sum_{k=1}^{i} d_k \\
&\qquad = y_j + +z_j + (-1)^{j+1} \sum_{k=1}^{m} d_k = \tfrac{1}{2} S = 4mn + 1.
\end{aligned}$$

Hence, $\{x_{i,j}\}$ is an antipodal balanced C_4-face-magic toroidal labeling on $C_{2m} \times C_{2n}$. $\square$

Proposition 4.10. *Suppose m and n are odd positive integers. Suppose i and j are integers such that $1 \leq i \leq m$, $1 \leq j \leq n$, and j is odd. Let*

$$\begin{aligned}
x_{i(m-1)+1, in+j} &= 2m(j-1) + 2i, \\
x_{i(m-1)+1, (i+1)n+j} &= 2mj - 2i + 1, \\
x_{i(m-1)+m+1, in+j} &= 2m(2n-j) + 2i \text{ and} \\
x_{i(m-1)+m+1, (i+1)n+j} &= 2m(2n-j+1) - 2i + 1.
\end{aligned}$$

Suppose i and j are integers such that $1 \leq i \leq m$, $1 \leq j \leq n$, and j is even. Let

$$\begin{aligned}
x_{i(m-1)+1, in+j} &= 2mj - 2i + 1, \\
x_{i(m-1)+1, (i+1)n+j} &= 2m(j-1) + 2i, \\
x_{i(m-1)+m+1, in+j} &= 2m(2n-j+1) - 2i + 1 \text{ and} \\
x_{i(m-1)+m+1, (i+1)n+j} &= 2m(2n-j) + 2i.
\end{aligned}$$

Then, $\{x_{i,j}\}$ is an antipodal balanced C_4-face-magic toroidal labeling on $C_{2m} \times C_{2n}$.

Proof. This labeling $\{x_{i,j}\}$ corresponds to the labeling in Proposition 4.9 where for integers j such that $1 \leq j \leq n$ and j is odd, $y_j = 2m(j-1)+1$ and $z_j = 2m(2n-j)+1$; for integers j such that $1 \leq j \leq n$ and j is even, $y_j = 2mj$ and $z_j = 2m(2n-j+1)$; and $d_1 = 1$ and $d_k = 2$ for integers k such that $2 \leq k \leq m$.

Let j be an integer such that $1 \leq j \leq n$ and j is odd. Then,

$$\begin{aligned}
y_j + z_j + (-1)^{j+1}\Big(\sum_{k=1}^{m} d_k\Big) = \big(&2m(j-1)+1\big) \\
&+ \big(2m(2n-j)+1\big) + (2m-1) = 4mn + 1.
\end{aligned}$$

Let j be an integer such that $1 \leq j \leq n$ and j is even. Then,

$$y_j + z_j + (-1)^{j+1}\Big(\sum_{k=1}^{m} d_k\Big) = 2mj + 2m(2n-j+1) - (2m-1) = 4mn + 1.$$

Thus condition (1) of Proposition 4.9 is satisfied.

We have

$$\begin{aligned}
A &= \{\sum_{k=1}^{i} d_k, \sum_{k=i+1}^{m} d_k : i = 1, 2, \ldots, m\} \cup \{0\} \\
&= \{2i-1, 2(m-i) : i = 1, 2, \ldots, m\} \cup \{0\} = \{k \in \mathbb{Z} : 0 \leq k \leq 2m-1\}.
\end{aligned}$$

Let j be an integer such that $1 \leq j \leq n$ and j is odd. Then,

$$A_j = \{y_j + (-1)^{j+1} a : a \in A\} = \{2m(j-1) + 1 + a : a \in A\}$$

and

$$A_{j+n} = \{z_j + (-1)^{j+1} a : a \in A\} = \{2m(2n-j) + 1 + a : a \in A\}.$$

Let j be an integer such that $1 \le j \le n$ and j is even. Then,

$$\begin{aligned} A_j &= \{y_j + (-1)^{j+1}a : a \in A\} = \{2mj - a : a \in A\} \\ &= \{2m(j-1) + 1 + a' : a' \in A\} \end{aligned}$$

and

$$\begin{aligned} A_{j+n} &= \{z_j + (-1)^{j+1}a : a \in A\} = \{2m(2n - j + 1) - a : a \in A\} \\ &= \{2m(2n-j) + 1 + a' : a' \in A\}. \end{aligned}$$

Hence, $\{A_j : 1 \le j \le 2n\}$ forms a partition of $\{k \in \mathbb{Z} : 1 \le k \le 4mn\}$. Thus condition (2) of Proposition 4.9 is satisfied. Therefore by Proposition 4.9, $\{x_{i,j}\}$ is an antipodal balanced C_4-face-magic toroidal labeling on $C_{2m} \times C_{2n}$. □

Example 4.11. We consider the example of Proposition 4.10 where $m = n = 3$. This example is given in Table 7.

1	12	13	6	7	18
34	27	22	33	28	21
5	8	17	2	11	14
31	30	19	36	25	24
4	9	16	3	10	15
35	26	23	32	29	20

Table 7: An antipodal balanced C_4-face-magic toroidal labeling on $C_6 \times C_6$.

We need the following converse to Proposition 4.9 in order to prove our last result in this paper.

Proposition 4.12. *Let m and n be odd positive integers. Let $\{x_{i,j} : (i,j) \in V(C_{2m} \times C_{2n})\}$ be an antipodal balanced C_4-face-magic toroidal labeling on $C_{2m} \times C_{2n}$. For all integers j such that $1 \le j \le n$, let $y_j = x_{1,j}$ and $z_j = x_{m+1,j}$. For all integers i such that $1 \le i \le m$, let*

$$d_i = x_{i(m-1)+1,in+1} - x_{(i-1)(m-1)+1,(i-1)n+1}.$$

Then, for all integers i and j such that $0 \le i \le m$ and $1 \le j \le n$, we have

$$x_{i(m-1)+1,in+j} = y_j + (-1)^{j+1} \sum_{k=1}^{i} d_k, \tag{4.49}$$

$$x_{i(m-1)+1,(i+1)n+j} = y_j + (-1)^{j+1} \sum_{k=i+1}^{m} d_k, \tag{4.50}$$

$$x_{i(m-1)+m+1,in+j} = z_j + (-1)^{j+1} \sum_{k=1}^{i} d_k \text{ and} \tag{4.51}$$

$$x_{i(m-1)+m+1,(i+1)n+j} = z_j + (-1)^{j+1} \sum_{k=i+1}^{m} d_k. \tag{4.52}$$

Let $A = \{\sum_{k=1}^{i} d_k, \sum_{k=i}^{m} d_k : 1 \le i \le m\} \cup \{0\}$*. For all integers* j *such that* $1 \le j \le n$*, let*

$$A_j = \{y_j + (-1)^{j+1}a : a \in A\} \text{ and } A_{j+n} = \{z_j + (-1)^{j+1}a : a \in A\}.$$

Then

1. *for all integers* j *such that* $1 \le j \le n$*,* $y_j + z_j + (-1)^{j+1}\left(\sum_{k=1}^{m} d_k\right) = 4mn + 1$*, and*
2. *the set* $\{A_j : 1 \le j \le 2n\}$ *forms a partition of the set* $\{k \in \mathbb{Z} : 1 \le k \le 4mn\}$*.*

Proof. By Lemma 2.11, for all integers i and j such that $1 \le i \le m$ and $1 \le j \le 2n$, we have

$$x_{i(m-1)+1,in+j} = x_{(i-1)(m-1)+1,(i-1)n+j} + (-1)^{j+1}d_i \text{ and} \tag{4.53}$$
$$x_{i(m-1)+m+1,in+j} = x_{(i-1)(m-1)+m+1,(i-1)n+j} + (-1)^{j+1}d_i. \tag{4.54}$$

For integers i and j such that $1 \le i \le m$ and $1 \le j \le n$, repeated use of equation (4.53) yields

$$x_{i(m-1)+1,in+j} = x_{1,j} + (-1)^{j+1}\sum_{k=1}^{i} d_k. \tag{4.55}$$

Thus equation (4.49) holds. When we let $i = m$ in equation (4.55), we have

$$x_{1,n+j} = x_{1,j} + (-1)^{j+1}\sum_{k=1}^{m} d_k. \tag{4.56}$$

When we replace j with $j + n$ in equation (4.53), for integers i and j such that $1 \le i \le m$ and $1 \le j \le n$, we have

$$x_{i(m-1)+1,(i+1)n+j} = x_{(i-1)(m-1)+1,in+j} + (-1)^{j}d_i. \tag{4.57}$$

For integers i and j such that $1 \le i \le m$ and $1 \le j \le n$, repeated use of equation (4.57) yields

$$x_{i(m-1)+1,(i+1)n+j} = x_{1,n+j} + (-1)^{j}\sum_{k=1}^{i} d_k. \tag{4.58}$$

Combining equation (4.58) with equation (4.56) yields

$$x_{i(m-1)+1,(i+1)n+j} = x_{1,j} + (-1)^{j+1}\sum_{k=i+1}^{m} d_k.$$

Thus equation (4.50) holds. A similar proof using equation (4.54) shows that equations (4.51) and (4.52) hold.

A proof similar to that in Proposition 4.8 shows that, for all integers j such that $1 \le j \le n$, we have $y_j + z_j + (-1)^{j+1}\left(\sum_{k=1}^{m} d_k\right) = 4mn + 1$, and the set $\{A_j : 1 \le j \le 2n\}$ forms a partition of the set $\{k \in \mathbb{Z} : 1 \le k \le 4mn\}$. □

Proposition 4.13. *Let* m *and* n *be positive odd integers. Then,* $C_{2m} \times C_{2n}$ *has no antipodal balanced* C_4*-face-magic toroidal labeling that is both row-sum balanced and column-sum balanced.*

Proof. Let $\{x_{i,j}\}$ be an antipodal balanced C_4-face-magic toroidal labeling on $C_{2m} \times C_{2n}$ that is both row-sum balanced and column-sum balanced. For all integers j such that $1 \leq j \leq n$, let $y_j = x_{1,j}$ and $z_j = x_{m+1,j}$. For all integers i such that $1 \leq i \leq m$, let

$$d_i = x_{i(m-1)+1,in+1} - x_{(i-1)(m-1)+1,(i-1)n+1}.$$

By Proposition 4.12, for all integers i such that $1 \leq i \leq m$, equations (4.49), (4.50), (4.51), and (4.52) hold.

Let

$$T = \sum_{j=1}^{n}\Big(y_j + z_j + (-1)^{j+1}\big(\sum_{k=1}^{m} d_k\big)\Big).$$

By Proposition 4.12, we have $y_j + z_j + (-1)^{j+1}\big(\sum_{k=1}^{m} d_k\big) = 4mn + 1$. Then

$$\begin{aligned} T &= \sum_{j=1}^{n}\Big(y_j + z_j + (-1)^{j+1}\big(\sum_{k=1}^{m} d_k\big)\Big) \\ &= \sum_{j=1}^{n}\big(4mn+1\big) = n(4mn+1) \equiv 1 \pmod 2. \end{aligned}$$

For $i = 1, 2, \ldots, 2m$, let $R_i = \sum_{j=1}^{2n} x_{i,j}$ be the row sum of the labels on the vertices in row i. Also, for $j = 1, 2, \ldots, 2n$, let $C_j = \sum_{i=1}^{2m} x_{i,j}$ be the column sum of the labels on the vertices in column j. Let m_1 be the integer such that $m = 2m_1 + 1$. Then for any integer j such that $1 \leq j \leq n$, we have

$$\begin{aligned} C_j &= \sum_{i=1}^{m}\big(x_{i(m-1)+1,j} + x_{i(m-1)+m+1,j}\big) \\ &= \sum_{i=1}^{m_1+1}\big(x_{(2i-1)(m-1)+1,j} + x_{(2i-1)(m-1)+m+1,j}\big) \\ &\qquad + \sum_{i=1}^{m_1}\big(x_{(2i)(m-1)+1,j} + x_{(2i)(m-1)+m+1,j}\big) \\ &= \sum_{i=1}^{m_1+1}\Big(y_j + (-1)^{j+1}\big(\sum_{k=2i}^{m} d_k\big) + z_j + (-1)^{j+1}\big(\sum_{k=2i}^{m} d_k\big)\Big) \\ &\qquad + \sum_{i=1}^{m_1}\Big(y_j + (-1)^{j+1}\big(\sum_{k=1}^{2i} d_k\big) + z_j + (-1)^{j+1}\big(\sum_{k=1}^{2i} d_k\big)\Big) \\ &= m(y_j + z_j) + (-1)^{j+1}\big(\sum_{k=1}^{m}(m + (-1)^k)d_k\big). \end{aligned}$$

We also have

$$
\begin{aligned}
C_{n+j} &= \sum_{i=1}^{m}\big(x_{i(m-1)+1,n+j} + x_{i(m-1)+m+1,n+j}\big)\\
&= \sum_{i=1}^{m_1+1}\big(x_{(2i-1)(m-1)+1,n+j} + x_{(2i-1)(m-1)+m+1,n+j}\big)\\
&\qquad\qquad + \sum_{i=1}^{m_1}\big(x_{(2i)(m-1)+1,n+j} + x_{(2i)(m-1)+m+1,n+j}\big)\\
&= \sum_{i=1}^{m_1+1}\Big(y_j + (-1)^{j+1}\big(\sum_{k=1}^{2i-1} d_k\big) + z_j + (-1)^{j+1}\big(\sum_{k=1}^{2i-1} d_k\big)\Big)\\
&\qquad\qquad + \sum_{i=1}^{m_1}\Big(y_j + (-1)^{j+1}\big(\sum_{k=2i+1}^{m} d_k\big) + z_j + (-1)^{j+1}\big(\sum_{k=2i+1}^{m} d_k\big)\Big)\\
&= m(y_j + z_j) + (-1)^{j+1}\big(\sum_{k=1}^{m}(m + (-1)^{k+1})d_k\big).
\end{aligned}
$$

Since $C_j = C_{n+j}$, we have $\sum_{k=1}^{m}(-1)^k d_k = 0$.

We have

$$
R_1 = \sum_{j=1}^{n}(x_{1,j} + x_{1,n+j}) = \sum_{j=1}^{n}\Big(y_j + y_j + (-1)^{j+1}\sum_{k=1}^{m} d_k\Big) = 2\sum_{j=1}^{n} y_j + \sum_{k=1}^{m} d_k.
$$

We also have

$$
\begin{aligned}
R_{m+1} &= \sum_{j=1}^{n}(x_{m+1,j} + x_{m+1,n+j}) = \sum_{j=1}^{n}\Big(z_j + z_j + (-1)^{j+1}\sum_{k=1}^{m} d_k\Big)\\
&= 2\sum_{j=1}^{n} z_j + \sum_{k=1}^{m} d_k.
\end{aligned}
$$

Since $R_1 = R_{m+1}$, we have $\sum_{j=1}^{n} y_j = \sum_{j=1}^{n} z_j$. Thus

$$
\begin{aligned}
T &= \sum_{j=1}^{n}\Big(y_j + z_j + (-1)^{j+1}\big(\sum_{k=1}^{m} d_k\big)\Big)\\
&= \sum_{j=1}^{n} y_j + \sum_{j=1}^{n} z_j + \sum_{k=1}^{m} d_k + \sum_{k=1}^{m}(-1)^k d_k\\
&\equiv 2\big(\sum_{j=1}^{n} y_j\big) + \sum_{i=1}^{m_1} 2d_{2i} \equiv 0 \pmod 2.
\end{aligned}
$$

This contradicts $T \equiv 1 \pmod 2$. □

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An alternate proof of the monotonicity of the number of positive entries in nonnegative matrix powers*

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Abstract

Let A be a nonnegative real matrix of order n and $f(A)$ denote the number of positive entries in A. In 2018, Xie proved that if $f(A) \leq 3$ or $f(A) \geq n^2 - 2n + 2$, then the sequence $(f(A^k))_{k=1}^{\infty}$ is monotone for positive integers k. In this note we give an alternate proof of this result by counting walks in a digraph of order n.

Keywords: Digraphs, walks, monotonicity, adjacency matrix.

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1 Introduction

A matrix is *nonnegative* (respectively, *positive*) if all its entries are nonnegative (respectively, positive) real numbers. Nonnegative matrices are widely applied in science, engineering and technology, see [1] and [2]. A nonnegative square matrix A is said to be *primitive* if there exists a positive integer k such that A^k is positive. By $f(A)$ we denote the number of positive entries in A. In [4] Šidák proved that there exists a primitive matrix A of order 9 satisfying $f(A) = 18 > f(A^2) = 16$. Motivated by this observation, in [5] Xi proved that if $f(A) \leq 3$ or $f(A) \geq n^2 - 2n + 2$, then the sequence $(f(A^k))_{k=1}^{\infty}$ is monotone for positive integers k. The proof of this result relies on linear algebra approach considering A as a $0-1$ square matrix, that is, a matrix from the vector space $\mathbb{M}_n(\mathbb{R})$ whose entries are either 0 or 1. Recall, $\mathbb{M}_n(\mathbb{R})$ is the set of all square matrices of size n under the ordinary addition and scalar multiplication of matrices. Clearly, the above restriction on the entries of A is valid since the value of each positive entry in A does not

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effect $f(A^k)$ for all positive integers k. In this note we give an alternate proof of this result using counting method from graph theory.

By a *digraph* we mean a structure $G = (V, A)$, where $V(G)$ is a finite set of *vertices*, and $A(G)$ is a set of ordered pairs (u, v) of vertices $u, v \in V(G)$ called *arcs*. The *order* of the digraph G is the number of vertices in G. An *in-neighbour* of a vertex v in a digraph G is a vertex u such that $(u, v) \in A(G)$. Similarly, an *out-neighbour* of a vertex v is a vertex w such that $(v, w) \in A(G)$. The *in-degree*, respectively *out-degree*, of a vertex $v \in V(G)$ is the number of its in-neighbours, respectively out-neighbours, in G. A *walk* w of length k in G is an alternating sequence $(v_0 a_1 v_1 a_2 \dots a_k v_k)$ of vertices and arcs in G such that $a_i = (v_{i-1}, v_i)$ for each i. If the arcs $a_1, a_2, \dots, a_k$ of a walk w are distinct, w is called a *trail*. A *cycle* C_k of length k is a closed trial of length $k > 0$ with all vertices distinct (except the first and the last).

If a digraph G has n vertices $v_1, v_2, \dots, v_n$, a useful way to represent it is with an $n \times n$ matrix of zeros and ones called its *adjacency matrix*, A_G. The ij-th entry of the adjacency matrix, $(A_G)_{ij}$, is 1 if there is an arc from vertex v_i to vertex v_j and 0 otherwise. That is,

$$(A_G)_{ij} = \begin{cases} 1, & \text{if } (v_i, v_j) \in A(G) \\ 0, & \text{otherwise} \end{cases}$$

The *length-k walk counting matrix* for an n-vertex digraph G is the $n \times n$ matrix C such that

$$C_{uv} := \text{the number of length-}k\text{ walks from } u \text{ to } v.$$

The main result in this note is based on the following well-known result:

Theorem 1.1 ([3]). *The length-k counting matrix of a digraph, G, is $(A_G)^k$, for all $k \in \mathbb{N}$.*

2 Main results

In the following proposition we reprove Theorem 1 and Theorem 2 from [5].

Proposition 2.1. *Let A be a $0-1$ matrix of order n. If $f(A) \leq 3$, then the sequence $(f(A^k))_{k=1}^{\infty}$ is monotone.*

Proof. Let G be a digraph on n vertices $v_1, v_2, \dots, v_n$ corresponding to the adjacency matrix A, that is, there is an arc from vertex v_i to vertex v_j in G ($v_i \to v_j$) if $(A)_{ij} = 1$. We deal with four possible cases.

1. The case when $f(A) = 0$ is trivial. Since $A^k = O_n$, then $f(A^k) = 0$ for any positive integer k.
2. If $f(A) = 1$, then G contains exactly one arc $a = (v_i, v_j)$.
   - If $v_i = v_j$, then for any positive integer k there exists a unique k-walk from v_i to v_i. Therefore $(A^k)_{ii} = 1$. Moreover, since there exists no other k-walk between the vertices of G, the remaining $n^2 - 1$ entries of A^k are zeros. In this case, for any positive integer k we have $f(A^k) = 1$.
   - If $v_i \neq v_j$, then $(A)_{ij} = 1$. It is easy to see that G does not contain a walk of length $k \geq 2$, that is, for any $k \geq 2$ A^k is a zero matrix. Therefore, for any $k \geq 2$ we obtain $1 = f(A) > f(A^k) = 0$.

3. Let $f(A) = 2$, i.e., let $a_1 = (v_i, v_j)$ and $a_2 = (v_r, v_s)$ be two distinct arcs of G.
   If G contains two loops, then we consider one possible case:

   - Let $v_i = v_j \neq v_r = v_s$. For any positive integer $k \geq 1$ there exists exactly one k-walk from vertex v_i to vertex v_j and exactly one k-walk from vertex v_r to vertex v_s. It yields $f(A^k) = 2$.

   If G contains one loop, we consider the following three cases:

   - If $v_i = v_j = v_r \neq v_s$, then $f(A^k) = 2$ for any positive integer $k \geq 1$.
   - If $v_i = v_j = v_s \neq v_r$, then $f(A^k) = 2$ for any positive integer $k \geq 1$.
   - If $v_i = v_j, v_r \neq v_s, v_i \neq v_r$ and $v_i \neq v_s$, then $f(A^k) = 1$ for any positive integer $k \geq 2$.

   If G does not contain loops, then we focus on the cases when at least one of the vertices v_i, v_j, v_r and v_s has positive in-degree and positive out-degree. Otherwise, G does not contain a k-walk for $k \geq 2$.

   - If $v_i \neq v_j = v_r \neq v_s$ and $v_i \neq v_s$, then G contains exactly one 2-walk from v_i to v_s. Moreover, there is no k-walk when $k \geq 3$. Thus $2 = f(A) > 1 = f(A^2) > f(A^k) = 0$ for any positive integer $k \geq 3$.
   - If $v_i \neq v_j = v_r \neq v_s$ and $v_i = v_s$, then $f(A^k) = 2$ for any positive integer k.

4. The proof when $f(A) = 3$ follows the same reasoning as the previous cases.
   Let $a_1 = (v_i, v_j), a_2 = (v_r, v_s)$ and $a_3 = (v_p, v_t)$ be three distinct arcs of G.
   If G contains three loops, then we have:

   - Let $v_i = v_j$, $v_r = v_s$ and $v_p = v_t$. It is easy to see that $f(A^k) = 3$ for any positive integer $k \geq 1$.

   Similarly, if G contains two loops, we treat the following cases.

   - If $v_i = v_j, v_r = v_s, v_p \neq v_t$ and if there is no common vertex between the arcs a_1, a_2 and a_3, then $f(A^k) = 2$ for any positive integer $k \geq 2$.
   - If $v_i = v_j = v_p \neq v_t = v_r = v_s$, then $f(A^k) = 3$ for any positive integer $k \geq 1$.
   - If $v_i = v_j = v_p \neq v_t \neq v_r = v_s$, then $f(A^k) = 3$ for any positive integer $k \geq 1$.
   - If $v_i = v_j = v_t \neq v_p \neq v_r = v_s$, then $f(A^k) = 3$ for any positive integer $k \geq 1$.

   If G contains one loop, we obtain the following cases.

   - If $v_i = v_j, v_r = v_t \neq v_s = v_p$ and $v_i \neq v_r, v_i \neq v_s$, then $f(A^k) = 3$ for any positive integer $k \geq 1$.
   - If $v_i = v_j, v_r \neq v_s, v_p \neq v_t$ and if there is no a common vertex between the arcs a_1, a_2 and a_3, then $f(A^k) = 1$ for any positive integer $k \geq 2$.
   - If $v_i = v_j, v_r \neq v_s = v_p \neq v_t, v_r \neq v_t$ and if there is no common vertex between a_1 and a_2 and a_1 and a_3, then $f(A^2) = 2$ and $f(A^k) = 1$ for any positive integer $k \geq 3$.

- If $v_i = v_j \neq v_r = v_p \neq v_t$, $v_r \neq v_s, v_s \neq v_t, v_i \neq v_s$ and $v_i \neq v_t$, then $f(A^k) = 1$ for any positive integer $k \geq 2$.
- If $v_i = v_j, v_r \neq v_s = v_t \neq v_p, v_r \neq v_p$ and if there is no common vertex between a_1 and a_2 and a_1 and a_3, then $f(A^k) = 1$ for any positive integer $k \geq 2$.
- If $v_i = v_j = v_r \neq v_s, v_p \neq v_t$ and if there is no common vertex between a_1 and a_3 and between a_2 and a_3 , then $f(A^k) = 2$ for any positive integer $k \geq 2$.
- If $v_i = v_j = v_s \neq v_r, v_p \neq v_t$ and if there is no common vertex between a_1 and a_3 and between a_2 and a_3 , then $f(A^k) = 2$ for any positive integer $k \geq 2$.
- If $v_i = v_j = v_r \neq v_s = v_p \neq v_t$ and $v_i \neq v_t$, then $f(A^k) = 3$ for any positive integer $k \geq 1$.
- If $v_i = v_j = v_s \neq v_r = v_p \neq v_t$ and $v_i \neq v_t$, then $f(A^k) = 2$ for any positive integer $k \geq 2$.
- If $v_i = v_j = v_s \neq v_r = v_t \neq v_p$ and $v_i \neq v_p$, then $f(A^k) = 3$ for any positive integer $k \geq 1$.
- If $v_i = v_j = v_r \neq v_s = v_t \neq v_p$ and $v_i \neq v_p$, then $f(A^k) = 2$ for any positive integer $k \geq 2$.
- If $v_i \neq v_j = v_r = v_s = v_p \neq v_t$ and $v_i \neq v_t$, then $f(A^k) = 3$ for any positive integer $k \geq 1$.
- If $v_i \neq v_j = v_r = v_s = v_t \neq v_p$, then $f(A^k) = 3$ for any positive integer $k \geq 1$.
- If $v_j \neq v_i = v_r = v_s = v_p \neq v_t$, then $f(A^k) = 3$ for any positive integer $k \geq 1$.
- If $v_i \neq v_j = v_r = v_s = v_p \neq v_t$ and $v_i = v_t$, then $f(A^k) = 4$ for any positive integer $k \geq 2$.

If G does not contain loops, then each k-walk of G, $k \geq 3$, contains at least two vertices of positive in-degree and positive out-degree. Based on this observation we consider the following cases.

- If $v_i = v_s \neq v_j = v_r, v_p \neq v_t$ and if there is no common vertex between the arcs a_1 and a_3, then $f(A^k) = 2$ for any positive integers $k \geq 2$.
- If $v_i \neq v_j$, $v_r \neq v_s$, $v_p \neq v_t, v_j = v_r, v_s = v_p$ and $v_t = v_i$, then $f(A^k) = 3$ for any positive integer $k \geq 1$.
- If $v_i \neq v_j$, $v_r \neq v_s$, $v_p \neq v_t, v_j = v_r, v_s = v_p$ and $v_i \neq v_t \neq v_j$, then $f(A^2) = 2, f(A^3) = 1$ and $f(A^k) = 0$ for any positive integer $k \geq 4$.
- If $v_t \neq v_p = v_s = v_i \neq v_j = v_r$ and $v_j \neq v_t$, then $f(A^k) = 3$ for any positive integer $k \geq 1$.
- If $v_p \neq v_t = v_s = v_i \neq v_j = v_r$ and $v_j \neq v_p$, then $f(A^k) = 3$ for any positive integer $k \geq 1$. □

The following result is a reproof of Theorem 5 from [5].

Theorem 2.2. *Let A be a $0-1$ matrix of order n. If $f(A) \geq n^2-2n+2$, then the sequence $(f(A^k))_{k=1}^{\infty}$ is non-decreasing.*

Proof. Let G be a digraph on n vertices $v_1, v_2, \ldots, v_n$ which corresponds to the matrix A (A is the adjacency matrix of G consisting of at most $2n-2$ zeros). According to Theorem 1.1, proving $f(A^{k+1}) \geq f(A^k)$ for every positive integer k, is equivalent to proving that the number of pairs of vertices of G for which there exists at least one $(k+1)$-walk is greater or equal than the number of pairs of vertices of G for which there exists at least one k-walk.

Let us suppose that G contains a walk of length k, i.e. let $w = (v_i, v_{i+1}, \ldots, v_j)$ be a k-walk from v_i to $v_j = v_{i+k}$. Thus $(A^k)_{ij} \geq 1$. We prove the following five claims.

Claim 1: If w contains at least four distinct vertices, then there exists at least one $(k+1)$-walk from v_i to v_j. Therefore $(A^{k+1})_{ij} \geq 1$.

Let $w = (v_i, v_{i+1}, \ldots, v_j)$ contain at least four distinct vertices v_i, v_t, v_s and v_j. If w contains a loop, then G contains at least one $(k+1)$-walk from v_i to v_j. Therefore we assume that $(A)_{ii} = (A)_{tt} = (A)_{ss} = (A)_{jj} = 0$. Thus $v_i \neq v_{i+1}$ and $v_{i+1} \neq v_{i+2}$. If there exists no $(k+1)$-walk from v_i to v_j, then for each vertex $v \in V(G) \backslash \{v_i, v_{i+1}\}$, G does not contain 2-walks of type (v_i, v, v_{i+1}). Otherwise we obtain $(k+1)$-walk $(v_i, v, v_{i+1}, v_{i+2}, \ldots, v_j)$. This implies an existence of at least $n-2$ non-connected pairs of vertices among (v_i, v) and (v, v_{i+1}), where $v \in V(G) \backslash \{v_i, v_{i+1}\}$. Similarly, for each vertex $v \in V(G) \backslash \{v_{i+1}, v_{i+2}\}$, G does not contain 2-walks of type (v_{i+1}, v, v_{i+2}). Otherwise we obtain $(k+1)$-walk $(v_i, v_{i+1}, v, v_{i+2}, \ldots, v_j)$.This implies an existence of at least $n-3$ non-connected pairs of vertices among (v_{i+1}, v) and (v, v_{i+2}), where $v \in V(G) \backslash \{v_i, v_{i+1}, v_{i+2}\}$. Since G does not contain at least four loops, we obtain at least $(n-2)+(n-3)+4 = 2n-1$ non-connected pairs of vertices in G, which is not possible.

Claim 2: If $k \geq 3$ and w contains three distinct vertices, then there exists at least one $(k+1)$-walk from v_i to v_j. Therefore $(A^{k+1})_{ij} \geq 1$.

We proceed similarly as in the previous case. Let $w = (v_i, v_{i+1}, \ldots, v_j)$ contain three distinct vertices v_i, v_t and v_j. If w contains a loop, then there exists at least one $(k+1)$-walk from v_i to v_j. Therefore we suppose $(A)_{ii} = (A)_{tt} = (A)_{jj} = 0$. Clearly $v_{i+1} \neq v_i$ and $v_t \neq v_{t+1}$. Without loss of generality let $v_{i+1} = v_t$. If G does not contain a $(k+1)$-walk from v_i to v_j, then for each $v \in V(G) \backslash \{v_i, v_t, v_j\}$ there exist no walks of type (v_i, v, v_{i+1}) and (v_t, v, v_{t+1}). Otherwise we obtain the walks $(v_i, v, v_{i+1}, \ldots, v_j)$ and $(v_i, v_{i+1}, \ldots, v_t, v, v_{t+1}, \ldots, v_j)$, both of length $k+1$. The non-existence of the walks (v_i, v, v_{i+1}) and (v_t, v, v_{t+1}) implies an existence of at least $2(n-3)$ non-connected pairs of vertices among the pairs $(v_i, v), (v, v_{i+1} = v_t), (v_t, v)$ and (v, v_{t+1}).

Let $v_{i+2} = v_i$. We suppose that the walks (v_i, v_j, v_t) and (v_t, v_j, v_i) do not exist. Otherwise we obtain $(k+1)$-walks from v_i to v_j $(v_i, v_j, v_{i+1}, v_{i+2}, \ldots, v_j)$ and $(v_i, v_{i+1}, v_j, v_{i+2}, \ldots, v_j)$, respectively. This yields an existence of at least two non-connected pairs among the pairs $(v_i, v_j), (v_j, v_t), (v_t, v_j)$ and (v_j, v_i). In this case G contains at least $2n-1 = 3+2(n-3)+2$ non-connected pairs of vertices, which is not possible.

Let $v_{i+2} = v_j$. Similarly as in the previous case, we conclude that there exists no a walk (v_i, v_j, v_t). Otherwise we obtain the walk $(v_i, v_j, v_{i+1}, v_{i+2}, \ldots, v_j)$. This yields an existence of at least one non-connected pair among the pairs (v_i, v_j) and (v_j, v_t). In this case G contains at least $2n-2$ non-connected pairs of vertices.

Since A contains at most $2n-2$ zeros, we obtain that v_t and v_j are connected to v_i. For any even $k \geq 4$ we obtain a k-walk $(v_i, v_t, v_i, v_t, \ldots, v_i, v_t, v_j)$. Similarly, if $k = 5$ we obtain the walk $(v_i, v_t, v_j, v_i, v_t, v_j)$. If $k \geq 7$ is an odd number, then $k = 2s+1 = (2s-4)+5$ where $s \geq 3$. In this case we obtain a k-walk from v_i to v_j by connecting the walk $(v_i, v_t, v_i, v_t, \ldots, v_t, v_i)$ of length $2s-4$ and the walk $(v_i, v_t, v_j, v_i, v_t, v_j)$ of length 5.

Claim 3: If $k = 2$ and $w = (v_i, v_t, v_j)$, then $(A^3)_{ij} \geq 1$ or the number of positive entries of A^3 at $(i,i), (i,t), (i,j), (t,i), (t,t), (t,j), (j,i), (j,t)$ and (j,j) position is greater or equal than the number of positive entries of the matrix A^2 at the same positions.

Let G does not contain 3-walk from v_i to v_j and let $v \in V(G) \backslash \{v_i, v_t, v_j\}$. If G contains walks of type (v_i, v, v_t) and (v_t, v, v_j), then there exist 3-walks (v_i, v, v_t, v_j) and (v_i, v_t, v, v_j). In this case $(A^3)_{ij} \geq 1$.

On the other hand, the non-existence of the walks (v_i, v, v_t) and (v_t, v, v_j) implies an existence of at least $2(n-3)$ non-connected pairs among the pairs $(v_i, v), (v, v_t), (v_t, v)$ and (v, v_j). Now, if v_i is connected to v_j, then v_j is not connected to v_i and v_t. Otherwise we obtain the walks (v_i, v_j, v_i, v_j) and (v_i, v_j, v_t, v_j). Since $(A)_{ji} = (A)_{jt} = 0$ the matrix A contains at least $3 + 2(n-3) + 2 = 2n-1$ zeros. This is not possible. If v_i is not connected to v_j, then A contains at least $2n-2$ zeros. Therefore v_j is connected to v_i and v_t, and v_t is connected to v_i. By counting 2-walks between the vertices v_i, v_t and v_j, we find that the matrix A^2 consists of seven positive entries and two zeros at $(i,i), (i,t), (i,j), (t,i), (t,t), (t,j), (j,i), (j,t)$ and (j,j) position. On the other hand, by counting the 3-walks between the vertices v_i, v_t and v_j we conclude that A^3 consists eight positive entries and one zero at the same positions.

Claim 4: Let $w = (v_i, v_{i+1}, \ldots, v_j)$ contain two distinct vertices v_i and v_j. The number of positive entries of A^{k+1} at $(i,i), (i,j), (j,i)$ and (j,j) position is greater or equal than the number of positive entries of the matrix A^k at the same positions.

Let $k \geq 2$. If the walk w contains a loop, then it is easy to conclude that G contains a $(k+1)$-walk from v_i to v_j. In this case $(A^k)_{ij} \geq 1$ implies $(A^{k+1})_{ij} \geq 1$.

If w does not contain loops, then k is an odd number. We observe that G contains a k-walk from vertex v_j to vertex v_i, which implies $(A^k)_{ji} \geq 1$. If there exists no k-walk from v_i to v_i and if there exists no k-walk from v_j to v_j, then $(A^k)_{ii} = (A^k)_{jj} = 0$. Since $k+1$ is an even number, G contains $(k+1)$-walks from v_i to v_i and from v_j to v_j, that is, $(A^{k+1})_{ii} \geq 1$ and $(A^{k+1})_{jj} \geq 1$. Moreover, the digraph G does not contain $(k+1)$-walk from vertex v_i to vertex v_j and from vertex v_j to vertex v_i, that is, $(A^{k+1})_{ij} = (A^{k+1})_{ji} = 0$. Thus, the matrices A^k and A^{k+1} contain two zeros and two positive entries at $(i,i), (i,j), (j,i)$ and (j,j) position.

Similarly, $(A^k)_{ii} \geq 1$ implies $(A^{k+1})_{ij} \geq 1$ and $(A^k)_{jj} \geq 1$ implies $(A^{k+1})_{ji} \geq 1$.

Let $k = 1$. If v_j is connected to v_i, we have the same case as $k \geq 2$. If v_j is not connected to v_i, then there exists at least one 2-walk from v_j to v_i or from v_i to v_j. Otherwise we have at least $2n-1$ non-connected pairs of vertices in G, that is, at least $2n-1$ zeros in A, a contradiction.

Claim 5: If w contains exactly one vertex v_i, then there exists a $(k+1)$-walk from v_i to v_i. Therefore $(A^{k+1})_{ii} \geq 1$.

In this case the walk w is obtained repeating the loop $v_i \to v_i$ k-times. Thus, there exists a $(k+1)$-walk from v_i to v_i.

As a conclusion, in the four cases (whether the k-walk from vertex v_i to vertex v_j contains one, two, three or more distinct vertices), we obtain that the number of positive entries in A^{k+1} is greater or equal than the number of positive entries in A^k, that is, $f(A^{k+1}) \geq f(A^k)$. $\square$

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Semi-perimeter and inner site-perimeter of k-ary words and bargraphs

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Abstract

Given a bargraph B, a *border cell* of B is a cell of B that shares at least one common edge with an outside cell of B. Clearly, the inner site-perimeter of B is the number of border cells of B. A *tangent cell* of B is a cell of B which is not a border cell of B and shares at least one vertex with an outside cell of B. In this paper, we study the generating function for the number of k-ary words, represented as bargraphs, according to the number of horizontal steps, up steps, border cells and tangent cells. This allows us to express some cases via Chebyshev polynomials of the second kind. Moreover, we find an explicit formula for the number of bargraphs according to the number of horizontal steps, up steps, and tangent cells/inner site-perimeter. We also derive asymptotic estimates for the mean number of tangent cells/inner site-perimeter.

Keywords: Bargraphs, Chebyshev polynomials, k-ary words, semi-perimeter, inner site-perimeter.

Math. Subj. Class.: 05A15, 05A16, 60C05

1 Introduction

The solid–on–solid (SOS) model has received a lot of attention. The SOS model arose from the consideration of the boundary between oppositely magnetized phases in the Ising model [9, 23]. The linear SOS model with a magnetic field and wall interaction was solved in [19]. Later, in [20], the restricted SOS (RSOS) has been considered, where the interface takes on a restricted subset of configurations and the interactions of field and single wall interaction in the half-plane. Then, in [21] is presented the solution of the linear RSOS model confined to a slit. Each configuration of the RSOS presented by a *k-bounded bargraph* (see below) with allowing horizontal steps in the x-axis, where the exact solution in [21] is presented by studying the generating function for the number of k-bounded bargraphs according to the semi-perimeter.

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A *bargraph* is a column-convex polyomino where all the columns are bottom justified. Throughout this paper, we represent a bargraph as a lattice path in $\mathbb{Z}^2$, starting at the origin and after ending upon first return to the x-axis, the progression ends at the origin $(0,0)$. The allowed steps are the up step $u = (0,1)$, the down step $d = (0,-1)$, the horizontal step $h = (1,0)$, and the left horizontal step $(-1,0)$. The first step has to be an up step, the horizontal steps must all lie above the x-axis, the left horizontal steps must all lie on x-axis, and an up step cannot follow a down step and vice versa. Alternatively, we identify a bargraph by the 1×1 squares (called *cells*) that lie inside its lattice path. For instance, Figure 1 represents the bargraph $uhuhhuhuhdhhddd$ (as lattice path) and the word $\sigma_1\sigma_2\cdots\sigma_7 = 1223433$ where the ith column contains exactly σ_i cells. For a given bargraph B, we define the *semi-perimeter* of B to be the number of up steps and horizontal steps in the lattice path, the *site-perimeter* of B to be the number of nearest-neighbour cells outside the boundary of B, and the *inner site-perimeter* to be the number of cells inside B that have at least one edge in common with an outside cell. Figure 1 represents a

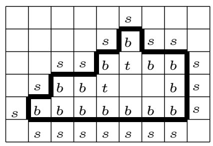

Figure 1: The bargraph of the word 1223433

corresponding bargraph B of the word 1223433 with semi-perimeter 11, site-perimeter 18 (the sum of the cells marked by s), and inner site-perimeter 10 (the sum of the cells marked by b).

In the last decades, the enumeration of bargraphs according to statistics has received a lot of attention (following the interest in Statistical physics as we said at the beginning of the introduction). Earlier authors, such as Prellberg and Brak [22] and Feretić [10] (see also [6, 12]), have found that the generating function that counts all bargraphs (including the empty bargraph) is given by

$$B(x,y) = \frac{1 + x - y - xy - \sqrt{(1 - x - y - xy)^2 - 4x^2y}}{2x}, \tag{1.1}$$

where x counts number of horizontal steps and y counts the number of up steps. Note that the generating function for bargraphs according to the semi-perimeter, often called *the isotropic generating function*, is given by $B(x,x)$. To find the asymptotics of the coefficient of x^n in $B(x,x)$, one computes the dominant singularity ρ which is the positive root of $1 - 4x + 2x^2 + x^4 = 0$. Thus, by singularity analysis (for example, see [11]) we have

$$[x^n]B(x,x) \sim \frac{1}{2}\sqrt{\frac{1-\rho-\rho^3}{\pi\rho n^3}}\rho^{-n} \tag{1.2}$$

with

$$\rho = \frac{1}{3}\left(-1 - \frac{2^{8/3}}{(13+3\sqrt{3})^{1/3}} + 2^{1/3}(13+3\sqrt{33})^{1/3}\right) \approx 0.295598\cdots. \tag{1.3}$$

Later, Blecher et al. refined the generating function $B(x, y)$ by considering a third statistic such as: levels [1], peaks [3], area [2], descents [2] and height [5]. In [6] is studied the number of bargraphs according to the site-perimeter. In [14] is considered the number of bargraphs according to the inner site-perimeter. In [4] it is derived a functional equation for the generating function that counts the number of k-ary words according to the size of the rightmost letter, the number of letters and the site-perimeter. For more details and motivations related to statistical physics and enumerative combinatorics, we refer the reader to [15] and references therein. The aim of this paper, is to refine some of these results.

For a given bargraph B, we define a *border cell* of B as a cell of B that has at least one edge in common with an outside cell of B. Clearly, the inner site-perimeter of B is the number of border cells of B. Further, we define a *tangent cell* of B to be a cell of B which is not a border cell of B and that has at least one vertex in common with an outside cell of B (see Figure 1). In what follows throughout this paper, whenever we refer to k-ary words, we mean their corresponding bargraph representation.

The paper aims to study the generating function for the number of k-ary words (bounded bargraphs that lie below the line $y = k$), according to the number of horizontal steps, up steps, tangent cells and border cells. More precisely: (1) We find an explicit formula for the generating function for the number of bargraphs according to the number of horizontal steps, the number of up steps, inner site-perimeter, and the number of tangent cells. In particular, we present the average number of tangent cells/inner site-perimeter as the semi-perimeter n of the bargraph tends to infinity. (2) We find an explicit formula for the generating function for the number of k-ary words according to the number of horizontal steps and up steps in terms of Chebyshev polynomials of the second kind, then we rederive (1.1). (3) We find an explicit formula for the generating function for the number of k-ary words according to the number of horizontal steps, up steps and tangent cells in terms of Chebyshev polynomials of the second kind. (4) We also study the generating function for the number of k-ary words according to the number of horizontal steps, up steps and inner site-perimeter.

2 Results

Define $[x]_d = \frac{1-x^d}{1-x} = 1 + x + \cdots + x^{d-1}$ for all $d \geq 0$. Let $C_k = C_k(x, y, p, q)$ be the generating function for the number of k-ary words of length n according to the number of horizontal steps (marked by x), up steps (marked by y), border cells (marked by p) and tangent cells (marked by q). We decompose each k-ary word π as

$$\pi = \pi^{(0)}1\pi^{(1)} \cdots 1\pi^{(s)}, s \geq 0$$

where $\pi^{(j)}$ is a word over alphabet $\{2, 3, \ldots, k\}$, for all $j = 0, 1, \ldots, s$. Then

$$\begin{aligned} C_k &= D_k + \sum_{s\geq 1}(xp)^s(y + D_k - 1)(1 + (D_k - 1)/y)^s \\ &= D_k + \frac{xp(y + D_k - 1)(1 + (D_k - 1)/y)}{1 - px(1 + (D_k - 1)/y)}, \end{aligned} \tag{2.1}$$

where $D_k = D_k(x, y, p, q)$ is the generating function for the number of words of length n over alphabet $\{2, 3, \ldots, k\}$, according to the number of horizontal steps, up steps, border cells and tangent cells.

Next we write an equation for the generating function D_k with $k \geq 2$. Clearly, $D_2 = 1 + \frac{xy^2p^2}{1-xp^2}$. In order to write a recurrence relation for D_k, we decompose each word π over alphabet $\{2, 3, \dots, k\}$ as

$$\pi = \pi^{(0)}2\pi^{(1)} \cdots 2\pi^{(s)}, s \geq 0$$

where $\pi^{(j)}$ is a word over alphabet $\{3, 4, \dots, k\}$, for all $j = 0, 1, \dots, s$.

The case $s = 0$ contributes

$$F_0 = 1 + x(yp)^3[yp]_{k-2} + yp^2(D_{k-1} - 1 - x(yp)^2[yp]_{k-2}),$$

where first, second and third term counts the empty word, words with one letter, words with at least two letters, respectively.

Similarly, the case $s = 1$ contributes

$$\begin{aligned} F_1 = xp^2(y^2 + x(yp)^3[yp]_{k-2} + ypq(D_{k-1} - 1 - x(yp)^2[yp]_{k-2}))\cdot \\ (1 + xyp^3[yp]_{k-2} + pq/y(D_{k-1} - 1 - x(yp)^2[yp]_{k-2})), \end{aligned}$$

where $xp^2(y^2 + x(yp)^3[yp]_{k-2} + ypq(D_{k-1} - 1 - x(yp)^2[yp]_{k-2}))$ counts the words of the form $\pi^{(0)}2$, and $1 + xyp^3[yp]_{k-2} + pq/y(D_{k-1} - 1 - x(yp)^2[yp]_{k-2})$ counts either the empty word or the nonempty words of the form $\pi^{(1)}$ without first two up steps.

For $s \geq 2$, we have the contribution

$$F_1(xp^2)^{s-1}(1 + xyp^2q[yp]_{k-2} + pq^2/y(D_{k-1} - 1 - x(yp)^2[yp]_{k-2}))^{s-1}.$$

Therefore, by summing all the contributions, we obtain the following result.

Lemma 2.1. *The generating function $D_k = D_k(x, y, p, q)$ satisfies*

$$\begin{aligned} D_k &= 1 + x(yp)^3[yp]_{k-2} + yp^2(D_{k-1} - 1 - x(yp)^2[yp]_{k-2}) \\ &\quad + xp^2(y^2 + x(yp)^3[yp]_{k-2} + ypq(D_{k-1} - 1 - x(yp)^2[yp]_{k-2}))\cdot \\ &\qquad \cdot (1 + xyp^3[yp]_{k-2} + \frac{pq}{y}(D_{k-1} - 1 - x(yp)^2[yp]_{k-2}))\cdot \\ &\qquad \cdot \left(1 - xp^2(1 + xyp^2q[yp]_{k-2} + \frac{pq^2}{y}(D_{k-1} - 1 - x(yp)^2[yp]_{k-2}))\right)^{-1} \end{aligned}$$

with $D_2 = 1 + \frac{xy^2p^2}{1-xp^2}$.

Hence, by (2.1), we have the following formula for the generating function $C_k(x, p, q)$.

Theorem 2.2. *Let $k \geq 1$. Then*

$$C_k(x, y, p, q) = D_k(x, y, p, q) + \frac{xp(y + D_k(x, y, p, q) - 1)(1 + (D_k(x, y, p, q) - 1)/y)}{1 - px(1 + (D_k(x, y, p, q) - 1)/y)},$$

where $D_k(x, y, p, q)$ is given in Lemma 2.1.

2.1 Bargraphs

Clearly, $C(x, y, p, q) = \lim_{k\to\infty} C_k(x, y, p, q)$ is the generating function for the number of bargraphs according to the number of horizontal steps, number of up steps, inner site-perimeter, and the number of tangent cells. Similarly, $D(x, y, p, q) = \lim_{k\to\infty} D_k(x, y, p, q)$

is the generating function for the number of bargraphs such that each nonempty column contains at least two cells according to the number of horizontal steps, number of up steps, inner site-perimeter, and number of tangent cells. By taking $k \to \infty$, Lemma 2.1 gives

$$\begin{aligned} D(x,y,p,q)-1 &= \frac{-\alpha_1+\sqrt{\alpha_1^2-4\alpha_0\alpha_2}}{2\alpha_2} \\ &= p^2xy^2+p^3xy^3+p^4x^2y^2+p^4xy^4+2p^5x^2y^3+p^5xy^5+\cdots, \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} \alpha_0 &= -p^2y^3(p^2y-1)(py-1)x+p^5y^4(py-1)(p-2q+1)x^2 \\ &\quad +p^7y^5(p^2q^2-2pq^2+pq-p+q)x^3, \\ \alpha_1 &= -y(py-1)^2(p^2y-1)+p^2y(py-1)^2(p^2y-2pqy-1)x \\ &\quad +p^4qy^2(py-1)(2p^3qy-3p^2qy+p^2y-pq+1)x^2, \\ \alpha_2 &= -p^3q^2x(1+py-p^2y)(py-1)^2. \end{aligned}$$

On the other hand, by taking $k \to \infty$, Theorem 2.2 and (2.2) imply the following result.

Theorem 2.3. *The generating function for the number of bargraphs according to the number of horizontal steps, number of up steps, inner site-perimeter, and the number of tangent cells is given by*

$$\begin{aligned} &C(x,y,p,q) \\ &= 1+\frac{-\alpha_1+\sqrt{\alpha_1^2-4\alpha_0\alpha_2}}{2\alpha_2}+\frac{xp(2y\alpha_2-\alpha_1+\sqrt{\alpha_1^2-4\alpha_0\alpha_2})^2}{2\alpha_2(2y\alpha_2-px(2y\alpha_2-\alpha_1+\sqrt{\alpha_1^2-4\alpha_0\alpha_2}))} \\ &= 1+pxy+p^2xy^2+p^2x^2y+p^3xy^3+2p^3x^2y^2+p^3x^3y+\cdots. \end{aligned}$$

We illustrate the above theorem through following 2 examples.

Example 2.4. By Theorem 2.3 we have that the generating function $C(1,1,1,p)$ for the number of bargraphs according to inner site-perimeter is given by

$$\frac{2p^6-2p^5+2p^4-2p^2+1-\sqrt{4p^{11}-4p^9+4p^5+4p^4-4p^3-4p^2+1}}{p(-2p^6+2p^5-4p^4+2p^3+4p^2-1+\sqrt{4p^{11}-4p^9+4p^5+4p^4-4p^3-4p^2+1})}.$$

In the case of counting bargraphs according to site-perimeter, we refer the reader to [6].

Next we define a *strong inner site-perimeter* to be the number of border cells and the number of tangent cells.

Example 2.5. By Theorem 2.3 we have that the generating function $C(1,1,p,p)$ for the number of bargraphs according to strong inner site-perimeter is given by

$$\frac{2p^8-3p^7+p^6+p^5-p^4-2p^2+1-\sqrt{\beta})}{p(-2p^8+3p^7-3p^6+p^5+3p^4+2p^2-1+\sqrt{\beta})},$$

where $\beta = p^{14}-2p^{13}+3p^{12}-5p^{10}+6p^9+p^8-2p^7+2p^6-2p^5+2p^4-4p^2+1$.

In particular, the generating function for the bargraphs according to semi-perimeter and the number of tangent cells is given by $C(x,x,1,q)$. Differentiating $C(x,x,1,q)$ with respect to q and evaluating it at $q=1$ gives

$$\begin{aligned}\frac{\partial}{\partial q}C(x,x,1,q)\mid_{q=1} = &\frac{x^8-x^7+7x^6-10x^5+20x^4-25x^3+24x^2-12x+2}{2x^2\sqrt{x^4+2x^2-4x+1}}\\ &-\frac{x^7-2x^6+7x^5-13x^4+19x^3-18x^2+10x-2}{2x^2(x-1)}.\end{aligned}$$

In what follows ρ is as defined by equation (1.3). By direct calculations and (1.2), we have

$$\lim_{x\mapsto\rho}\frac{\partial}{\partial q}C(x,x,1,q)\mid_{q=1}(1-x/\rho)^{1/2} = 1-\frac{5}{2}\rho-\frac{3}{2}\rho^2.$$

Hence, we have the following result.

Corollary 2.6. *The average number of tangent cells is asymptotic to*

$$\frac{(2-5\rho-3\rho^3)\sqrt{1-\rho-\rho^3}}{2\sqrt{\rho}}n$$

as the semi-perimeter n of the bargraph tends to infinity.

Moreover, by differentiating $C(x,x,p,1)$ with respect to p, evaluating at $p=1$ and using (1.2) gives

$$\lim_{x\mapsto\rho}\frac{\partial}{\partial}C(x,x,p,1)\mid_{p=1}(1-x/\rho)^{1/2} = \frac{25-15\rho-25\rho^2-21\rho^3}{16}.$$

Hence, we have the following result.

Corollary 2.7. *The average inner site-perimeter is asymptotic to*

$$\frac{(25-15\rho-25\rho^2-21\rho^3)\sqrt{1-\rho-\rho^3}}{16\sqrt{\rho}}n$$

as the semi-perimeter n of the bargraph tends to infinity.

2.2 Semi-perimeter and k-ary words

Define $B_k(x,y)=C_k(x,y,1,1)$ and $E_k(x,y)=D_k(x,y,1,1)-1$, for all $k\geq 1$. Then Theorem 2.2 with $p=q=1$ gives

$$B_k(x,y)=\frac{y(1-x+xy)+(xy-x+y)E_k(x,y)}{y(1-x)-xE_k(x,y)}, \tag{2.3}$$

where

$$E_k(x,y)=\frac{xy^3+(1+x)y^2E_{k-1}(x,y)}{y(1-x)-xE_{k-1}(x,y)}.$$

Recall that the Chebyshev polynomials of the second kind $U_m(t)$ satisfy the recurrence relation $U_m(t) = 2tU_{m-1}(t) - U_{m-2}(t)$ with the initial conditions $U_0(t) = 1$ and $U_1(t) = 2t$. By induction ok k, we have

$$E_k(x,y) = \frac{xy\sqrt{y}U_{k-2}\left(\frac{1-x+y+xy}{2\sqrt{y}}\right)}{U_{k-1}\left(\frac{1-x+y+xy}{2\sqrt{y}}\right) - (1+x)\sqrt{y}U_{k-2}\left(\frac{1-x+y+xy}{2\sqrt{y}}\right)}, \tag{2.4}$$

where $U_m(t)$ is the m-th Chebyshev polynomials of the second kind. Substituting into (2.3) gives the following result.

Theorem 2.8. *The generating function for the number of k-ary words, $k \geq 2$, according to the number of horizontal steps and up steps is given by*

$$B_k(x,y) = \frac{(1-x+xy)U_{k-1}\left(\frac{1-x+y+xy}{2\sqrt{y}}\right) - \sqrt{y}U_{k-2}\left(\frac{1-x+y+xy}{2\sqrt{y}}\right)}{(1-x)U_{k-1}\left(\frac{1-x+y+xy}{2\sqrt{y}}\right) - \sqrt{y}U_{k-2}\left(\frac{1-x+y+xy}{2\sqrt{y}}\right)}.$$

Note that $\lim_{k\mapsto\infty} \frac{U_{k-1}(\frac{1}{2\sqrt{t}})}{\sqrt{t}U_k(\frac{1}{2\sqrt{t}})} = C(t)$, where $C(t) = \frac{1-\sqrt{1-4t}}{2t}$ (for example, see [17] and references therein). Thus, Theorem 2.8 shows that

$$\begin{aligned}
\lim_{k\mapsto\infty} B_k(x,y) &= \frac{1-x+xy-\sqrt{y}\lim_{k\mapsto\infty}\frac{U_{k-2}\left(\frac{1-x+y+xy}{2\sqrt{y}}\right)}{U_{k-1}\left(\frac{1-x+y+xy}{2\sqrt{y}}\right)}}{1-x-\sqrt{y}\lim_{k\mapsto\infty}\frac{U_{k-2}\left(\frac{1-x+y+xy}{2\sqrt{y}}\right)}{U_{k-1}\left(\frac{1-x+y+xy}{2\sqrt{y}}\right)}} \\
&= \frac{1-x+xy-\frac{y}{1-x+y+xy}C\left(\frac{y}{(1-x+y+xy)^2}\right)}{1-x-\frac{y}{1-x+y+xy}C\left(\frac{y}{(1-x+y+xy)^2}\right)} \\
&= \frac{(1-x+xy)(1-x+y+xy)-yC\left(\frac{y}{(1-x+y+xy)^2}\right)}{(1-x)(1-x+y+xy)-yC\left(\frac{y}{(1-x+y+xy)^2}\right)} \\
&= \frac{1+x-y-xy-\sqrt{(1-x-y-xy)^2-4x^2y}}{2x},
\end{aligned}$$

which agrees with (1.1).

2.3 Tangent cells

Theorem 2.2 with $p = 1$ gives

$$C_k(x,y,1,q) = \frac{y(1-x+xy)+(xy-x+y)E_k(x,y,1,q)}{y(1-x)-xE_k(x,y,1,q)}, \tag{2.5}$$

with

$$xq^2E_k(x,y,1,q) = -y^2(1-x+2xq+q(1-q)x^2y[y]_{k-2}) + e_k,$$

where $e_k = \frac{y^3(1-x+qx)^2}{y(1-x)+y^2(1+qx)(1-x+qx)-e_{k-1}}$ and $e_2 = \frac{y^2(1-x+qx)^2}{1-x}$.

By induction on k and the definition of Chebyshev polynomials of the second kind, we obtain

$$e_k = \frac{y\sqrt{y}(1-x+qx)(\frac{(1-x)^2-qx^2y(1-x+qx)}{\sqrt{y}(1-x+qx)}U_{k-4}(t)-(1-x)U_{k-5}(t))}{\frac{(1-x)^2-qx^2y(1-x+qx)}{\sqrt{y}(1-x+qx)}U_{k-3}(t)-(1-x)U_{k-4}(t)},$$

where $t = \frac{(1-x)(1+y+qxy)+qxy(1+qx)}{2\sqrt{y}(1-x+qx)}$. By substituting into (2.5), we obtain the following result.

Theorem 2.9. *The generating function for the number of k-ary words, $k \geq 2$, according to the number of horizontal steps, up steps and tangent cells is given by*

$$C_k(x,y,1,q) = 1 + \frac{xy}{1-x} + \frac{y}{1-x} \cdot \frac{E_k(x,y,1,q)}{y(1-x)-xE_k(x,y,1,q)},$$

where

$$\begin{aligned} &xq^2E_k(x,y,1,q) \\ &\quad = -y^2(1-x+2xq+q(1-q)x^2y[y]_{k-2}) \\ &\quad + \frac{y\sqrt{y}(1-x+qx)\left(\frac{(1-x)^2-qx^2y(1-x+qx)}{\sqrt{y}(1-x+qx)}U_{k-4}(t)-(1-x)U_{k-5}(t)\right)}{\frac{(1-x)^2-qx^2y(1-x+qx)}{\sqrt{y}(1-x+qx)}U_{k-3}(t)-(1-x)U_{k-4}(t)} \end{aligned}$$

and $t = \frac{(1-x)(1+y+qxy)+qxy(1+qx)}{2\sqrt{y}(1-x+qx)}$.

Note that by taking $q = 1$ into Theorem 2.9 gives Theorem 2.8, as expected.

Next we turn our attention in finding the generating function for the total number of tangent cells over all k-ary words according to number horizontal steps and up steps. Define $Cq_k(x,y) = \frac{\partial}{\partial q}C_k(x,y,1,q)\mid_{q=1}$ and $Eq_k(x,y) = \frac{\partial}{\partial q}E_k(x,y,1,q)\mid_{q=1}$. Differentiating (2.5) with respect to q and evaluating at $q = 1$ gives

$$Cq_k(x,y) = \frac{y^2}{(y(1-x)-xE_k(x,y))^2}Eq_k(x,y), \tag{2.6}$$

where

$$\begin{aligned} &Eq_k(x,y) \\ &= \frac{y^3}{(y(1-x)-xE_{k-1}(x,y))^2}Eq_{k-1}(x,y) \\ &+ \frac{xy^2(y+E_{k-1}(x,y))}{(y(1-x)-xE_{k-1}(x,y))^2}\left(xy^2(x-2)[y]_{k-2}+(2+x^2y[y]_{k-2})E_{k-1}(x,y)\right). \end{aligned}$$

with $Eq_2(x,y) = 0$.

By induction on k, we have

$$\begin{aligned} &Eq_k(x,y) \\ &= \sum_{j=2}^{k-1}\frac{xy^{3(k-j)-1}(y+E_j(x,y))\left(xy^2(x-2)[y]_{j-1}+(2+x^2y[y]_{j-1})E_j(x,y)\right)}{\prod_{i=j}^{k-1}(y(1-x)-xE_i(x,y))^2}. \end{aligned}$$

Thus, by (2.6), we obtain the following result.

Theorem 2.10. *Let $k \geq 2$. The generating function for the total number of tangent cells over all k-ary words according to the number horizontal steps and up steps is given by*

$$Cq_k(x,y) = \sum_{j=2}^{k-1} \frac{xy^{3(k-j)+1}(y+E_j(x,y))\left(xy^2(x-2)[y]_{j-1}+(2+x^2y[y]_{j-1})E_j(x,y)\right)}{\prod_{i=j}^{k}(y(1-x)-xE_i(x,y))^2}.$$

where $E_k(x,y)$ is given in (2.4).

For instance, Theorem 2.10 gives $Cq_2(x,y)=0$ (as expected, since there are no tangent cells in 2-ary words) and

$$\begin{aligned}Cq_3(x,y) &= \frac{x^3y^3(xy-x+3)}{(x^2y-x^2+xy+2x-1)^2(xy-x+1)}\\ &= \mathbf{3}x^3y^3+14x^4y^3+4x^4y^4+40x^5y^3+90x^6y^3+34x^5y^4+\cdots.\end{aligned}$$

We emphasize in bold, the three 3-ary words with three horizontal steps and three up steps as bargraphs of 232, 233 and 332 with $1+1+1=3$ tangent cells.

2.4 Border cells

Theorem 2.2 with $q=1$ gives

$$C_k(x,y,p,1) = E_k(x,y,p,1)+1+\frac{xp(y+E_k(x,y,p,1))(1+E_k(x,y,p,1)/y)}{1-px(1+E_k(x,y,p,1)/y)},$$

where

$$\begin{aligned}E_k(x,y,p,1) &= xy^3(p^3-p^4)[yp]_{k-2}+yp^2E_{k-1}(x,y,p,1)\\ &\quad+\frac{xp^2(y+pE_{k-1}^2(x,y,p,1))}{1-xp^2(1+xy(p^2-p^3)[yp]_{k-2}+\frac{p}{y}E_{k-1}(x,y,p,1))}\end{aligned}$$

with $E_2(x,y,p,1)=\frac{xy^2p^2}{1-xp^2}$.

Now we find the generating function for the total inner site-perimeter (the number of border cells) over all k-ary words according to the number horizontal steps and up steps. Define $Cp_k(x,y)=\frac{\partial}{\partial p}C_k(x,y,p,1)\mid_{p=1}$ and $Ep_k(x,y)=\frac{\partial}{\partial p}E_k(x,y,p,1)\mid_{p=1}$. Differentiating with respect to p and evaluating at $p=1$ gives

$$Cp_k(x,y)=\frac{y^2}{(y(1-x)-xE_k(x,y))^2}Ep_k(x,y)+\frac{xy(y+E_k(x,y))^2}{(y(1-x)-xE_k(x,y))^2}, \tag{2.7}$$

where

$$Ep_k(x,y)=\frac{y^3}{(y(1-x)-xE_k(x,y))^2}Ep_{k-1}(x,y)+F_k(x,y)$$

with

$$\begin{aligned}F_k(x,y)=&\frac{y^4(3-x-x^2y[y]_{k-2})}{x(y(1-x)-xE_{k-1}(x,y))^2}-\frac{y^3(5-2x^2y[y]_{k-2})}{x(y(1-x)-xE_{k-1}(x,y))}\\ &+\frac{y^2(3-x-2x^2y[y]_{k-2})}{x}-\frac{y}{x}(y(1-x)-xE_{k-1}(x,y)).\end{aligned} \tag{2.8}$$

By induction on k with $Ep_2(x,y)=\frac{2xy^2}{(1-x)^2}$, we obtain

$$Ep_k(x,y)=\sum_{j=2}^{k}\frac{y^{3(k-j)}F_j(x,y)}{\prod_{i=j}^{k-1}(y(1-x)-xE_i(x,y))^2}.$$

Hence, by (2.7), we have the following result.

Theorem 2.11. *Let $k\geq 2$. The generating function for the total inner site-perimeter (the number of border cells) over all k-ary words according to the number of horizontal steps and up steps is given by*

$$Cp_k(x,y)=\sum_{j=2}^{k}\frac{y^{3(k-j)+2}F_j(x,y)}{\prod_{i=j}^{k}(y(1-x)-xE_i(x,y))^2}+\frac{xy(y+E_k(x,y))^2}{(y(1-x)-xE_k(x,y))^2},$$

where $E_k(x,y)$ and $F_k(x,y)$ are given in (2.4) and (2.8), respectively.

For instance, Theorem 2.11 gives

$$\begin{aligned}Cp_2(x,y)&=\frac{xy(x^2(y-1)^2+2x(y-1)+2y+1)}{(x^2(y-1)+2x-1)^2}\\&=xy+2x^2y+2xy^2+\mathbf{10}x^2y^2+3x^3y+28x^3y^2+4x^4y+\cdots.\end{aligned}$$

We emphasize in bold, the three 2-ary words with two horizontal steps and two up steps as bargraphs of 12, 21 and 22 with inner site perimeter $3+3+4=10$.

We end this paper by the following comment on the relation between bargraphs and Chebyshev polynomials. We recall that a *Dyck path* of semi-length n is a lattice path that starts at $(0,0)$, ends at $(2n,0)$, remains weakly above the x-axis, and consists of up steps $(1,1)$ and down steps $(1,-1)$. Apparently, for the first time the relation between restricted permutations and Chebyshev polynomials was discovered by Chow and West in [7], then explored in [18], and characterized as Dyck paths in [13]. Chebyshev polynomials of the second kind also occur in the enumeration of height-restricted Dyck paths, and they are much more natural there (for instance, see [13, 18]). On the other hand, Deutsch and Elizalde [8] established a bijection ρ between Dyck paths and bargraphs, where the semi-length of a Dyck path becomes the semi-perimeter minus the number of peaks of the corresponding bargraph (a peak in a bargraph B is an occurrence of uh^jd for some $j\geq 1$). Besides that, as discussed in [15], due to the geometric nature of bargraphs, we tried to study the statistics tangent cells, semi-perimeter and inner-site perimeter directly on bargraphs, and not to transfer our statistics via the bijection ρ. We followed this approach since sometimes the bargraph statistics can not be transferred to nice statistics in Dyck paths, and sometimes the enumeration of the statistics in Dyck paths requires the same amount of work as working directly in bargraphs. It is the main reason that directs us to choose by our techniques rather then bijection ρ. In our present study, transferring the statistics tangent cells, inner-perimeter in bargraphs to statistics in Dyck paths remains a nice point of exploration for the interested readers.

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A simple construction of exponentially many nonisomorphic orientable triangular embeddings of K_{12s}

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Abstract

Using an index one current graph with the cyclic current group we give a simple construction of 2^{2s-7} nonisomorphic orientable triangular embeddings of the complete graph K_{12s}, $s \geq 4$. These embeddings have no nontrivial automorphisms.

Keywords: Topological embedding, complete graph, nonisomorphic embeddings, triangular embedding.

Math. Subj. Class.: 05C10, 05C15

1 Introduction

In the present paper, by an embedding of a graph we mean a cellular embedding of the graph in an orientable surface. An embedding of a graph is *triangular* if all faces are 3-gonal. Euler's formula allows the possibility for a complete graph K_n to have a triangular embedding if $n \equiv 0, 3, 4$ or $7 \pmod{12}$. Constructing triangular embeddings of complete graphs was a major step in proving the Map Color Theorem [11].

Let K be a graph without loops and multiple edges. An m-gonal face of an embedding of K will be designated as a cyclic sequence $(v_1, v_2, \ldots, v_m)$ of vertices obtained by listing the incident vertices when traversing the boundary walk of the face in some chosen direction. The sequences $(v_1, v_2, \ldots, v_m)$ and $(v_m, \ldots, v_2, v_1)$ designate the same face.

One can differentiate embeddings of graphs as labeled objects (in this case we speak about different labeled embeddings and they have different face sets) and as unlabeled objects (in this case we speak about nonisomorphic embeddings). Two triangular embeddings f_1 and f_2 of K_n are *isomorphic* if there is a bijection ψ between the vertices of K_n such

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that (w_1, w_2, w_3) is a face of f_1 if and only if $(\psi(w_1), \psi(w_2), \psi(w_3))$ is a face of f_2. The bijection ψ is called an *isomorphism* from the embedding f_1 onto the embedding f_2.

During the proof of the Map Color Theorem, one triangular embedding was constructed for every complete graph K_n, $n \equiv 0, 3, 4$ or $7 \pmod{12}$. In this paper we consider the natural question on the rate of growth of the number of nonisomorphic triangular embeddings of complete graphs. At present there are two approaches to construct many such embeddings.

The first approach uses recursive constructions that generate a face 2-colorable triangular embedding of a complete graph from a face 2-colorable triangular embedding of a complete graph of lesser order. First it was shown [1, 3] that there are at least $2^{an^2-o(n^2)}$ (where a is a positive constant) nonisomorphic face 2-colorable triangular embeddings of K_n for some families of values of n such that $n \equiv 3$ or $7 \pmod{12}$, namely, for $n \equiv 7$ or $19 \pmod{36}$, $n \equiv 15 \pmod{60}$, $n \equiv 15$ or $43 \pmod{84}$, etc. Later it was shown [2, 4, 5] that there are at least $n^{bn^2-o(n^2)}$ nonisomorphic face 2-colorable triangular embeddings of K_n for an infinite, but rather sparse set of values of n (where $n \equiv 3$ or $7 \pmod{12}$). This approach having to do with face 2-colorable triangular embeddings does not work in the case of complete graphs of even order.

The second approach [7, 9, 10] uses the current graph technique. Within the limits of the approach, it was shown that there are constants $M, c > 0$, $b \geq 1/12$ such that for every $n \geq M$, $n \equiv 0, 3, 4$ or $7 \pmod{12}$, there are at least $c2^{bn}$ nonisomorphic triangular embeddings of K_n. In the case $n \equiv 0 \pmod{12}$, this approach (see [9]) gives 2^{s-6} nonisomorphic triangular embeddings of K_{12s}, $s \geq 6$, and, up to the present time, this result was the only known result on the number of nonisomorphic triangular embeddings of K_{12s}. This result was obtained by using index four current graphs with the cyclic current group $\mathbb{Z}_{12s}$, and the constructions involved are rather complicated.

In the present paper we give a simple construction of 2^{2s-7} nonisomorphic triangular embeddings of K_{12s}, $s \geq 4$. We use an index one current graph with current group $\mathbb{Z}_{12s-4}$ that was constructed by T. Sun [12] and which generates an embedding of K_{12s-4}, $s \geq 4$, that can be modified into a triangular embedding of K_{12s} (thereby providing a simple construction of a a triangular embedding of K_{12s}, $s \geq 3$). In the present paper, following the approach used in [7, 9, 10], changing rotations of some vertices of the current graph, we obtain 2^{2s-7} different current graphs generating 2^{2s-7} different embeddings of K_{12s-4} that can be modified into 2^{2s-7} different triangular embeddings of K_{12s}. Analyzing faces of the embeddings, we show (Theorem 3.1) that all these 2^{2s-7} different triangular embeddings of K_{12s}, $s \geq 4$, are nonisomorphic, thereby providing a much simpler construction of exponentially many nonisomorphic orientable triangular embeddings of K_{12s}.

2 Index one current graphs

In this section we describe index one current graphs which generate embeddings of K_{12s-4} that can be modified into triangular embeddings of K_{12s}.

First we briefly review some material about index one current graphs in the form used in the paper. The reader is referred to [6, 11] for a more detailed development of the material sketched herein. We assume the reader is familiar with current graphs and embeddings generated by current graphs.

Let G be a connected graph (multiple edges and loops are allowed) with the vertex set $V(G)$ whose edges have been given plus and minus direction. Hence each edge e gives rise to two reverse arcs e^+ and e^- of G. The involutary permutation θ of the arc set $A(G)$ of the graph G that permutes reverse arcs is called the *involution* of G. By a *current assignment* on G we mean a function λ from $A(G)$ into the set of nonzero elements of a group $\mathbb{Z}_n$ such that $\lambda(e^-) = -\lambda(e^+)$ for every edge e. The values of λ are called *currents* and the group $\mathbb{Z}_n$ is called the *current group*. If an edge e is incident with a onevalent vertex w and $\lambda(e^-) = \lambda(e^+)$ (that is, $\lambda(e^+)$ is of order 2 in $\mathbb{Z}_n$), then the arcs e^+ and e^- are identified and this arc is called an *end arc* (and in this case we do not consider w to be a vertex of G).

A *rotation* D of G is a permutation of $A(G)$ whose orbits cyclically permute the arcs directed outwards from each vertex. The rotation D can be represented as $D = \{D_w : w \in V(G)\}$, where D_w, called a rotation of the vertex v, is a cyclic permutation of the arcs directed outwards from v. Consider the permutation $D\theta$ of $A(G)$. It is easy to see that the terminal vertex of an arc a is the initial vertex of the arc $D\theta a$, hence a cycle $(a_1, a_2, \dots, a_m)$ of $D\theta$ can be considered as an oriented path in G called a *circuit* induced by the rotation D of G. By a *one-rotation* of G we mean a rotation of G inducing exactly one circuit.

A triple $\langle G, \lambda, D\rangle$ is called a *current graph*. The *index* of the current graph is the number of circuits induced by D. By the *log* of a circuit $(a_1, a_2, \dots, a_m)$ of the current graph we mean the cyclic sequence $(\lambda(a_1), \lambda(a_2), \dots, \lambda(a_m))$.

A current graph $\langle G, \lambda, D\rangle$ can be represented as a figure of G where the rotations of vertices are indicated. The black vertices denote a clockwise rotation and the white vertices a counterclockwise rotation. Each pair of reverse arcs is represented by one of the arcs with the current indicated. The end arc, as is customary, is depicted as a straight line without an arrow, with a vertex at one end and without a vertex at the other end.

If $(a_1, a_2, \dots, a_t)$ is the rotation of a vertex of a current graph $\langle G, \lambda, D\rangle$, where $\lambda(a_i) = \varepsilon_i$ for $i = 1, 2, \dots, t$, then the cyclic sequence $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_t)$ is called the *current rotation* of the vertex and the element $\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_t$ is the *excess* of the vertex. If the excess of a vertex equals zero, we say that the vertex satisfies Kirchhoff's Current Law (KCL).

Figure 1(a) shows (for now ignore the labels x, y, z, and w, and the boxes connected by lines with edges of the graph) an index one current graph $\langle G, \lambda, D\rangle$ with the current group $\mathbb{Z}_{12s-4}$, $s \geq 4$, having the following properties (A1)-(A6):

(A1) G has two onevalent vertices, one twovalent vertex, and all other vertices are trivalent.

(A2) The log of the circuit contains every nonzero element of $\mathbb{Z}_{12s-4}$ exactly once.

(A3) G has exactly one end arc which has current $6s - 2$.

(A4) Every trivalent vertex satisfies KCL.

(A5) The two onevalent vertices have excess -1 and $6s + 1$, respectively (each of the two excesses has order $12s - 4$ in $\mathbb{Z}_{12s-4}$).

(A6) The twovalent vertex has current rotation $(1, -3)$.

The fragment of the current graph lying inside the dashed box is shown in Figure 1(b). The current graph is slightly different from the current graph given in [12]: we changed the rotations of some vertices for present purposes.

The current graph generates an embedding $f(D)$ of the graph K_{12s-4} whose vertex set is the set $V(s) = \{0, 1, \dots, 12s - 5\}$ of all elements of $\mathbb{Z}_{12s-4}$. There is a mapping from the face set onto the vertex set of the current graph. Given a vertex of the current graph,

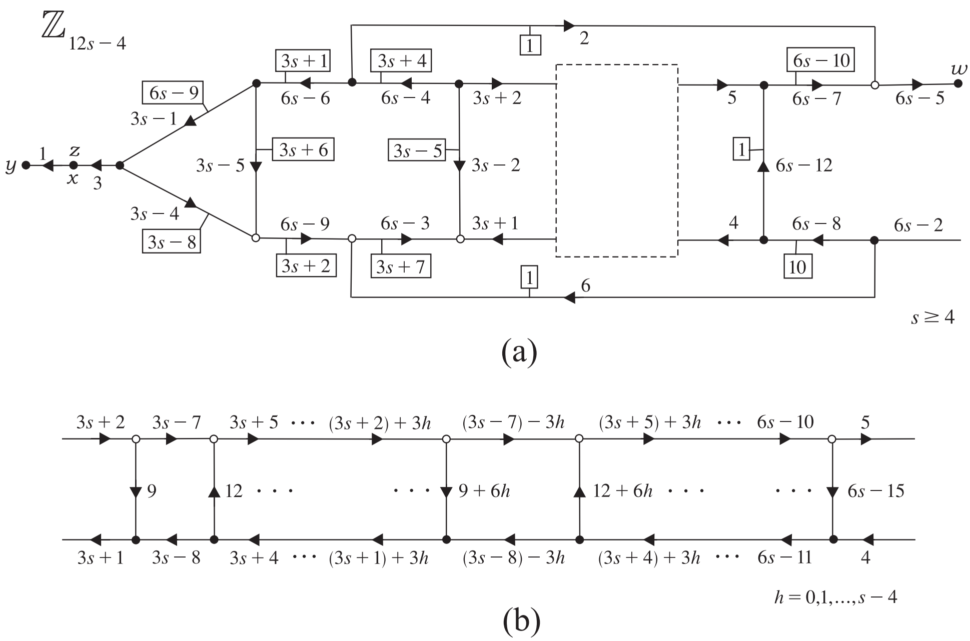

Figure 1: An index one current graph.

the faces mapping onto the vertex are called the faces *induced* by the vertex, and they are determined by Theorem 4.4.1 of [6]. In the case of the current graph $\langle G, \lambda, D\rangle$ satisfying (A1)-(A6) we have the following. A trivalent vertex with current rotation $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ induces $12s-4$ triangular faces $(u, u+\varepsilon_1, u+\varepsilon_1+\varepsilon_2)$, $u \in V(s)$. The onevalent vertex with excess -1 (resp. $6s+1$) induces one $(12s-4)$-gonal face shown in Figure 2(a) (resp. (b)) (now ignore the dashed edges in Figure 2). The twovalent vertex induces two $(12s-4)$-gonal faces shown in Figure 2(c).

The log of the circuit of the current graph $\langle G, \lambda, D\rangle$ (where we ignore the letters x, y, z, and w) determines the cyclic order in which the vertices adjacent to the vertex 0 of G are arranged on the surface around the vertex 0 in $f(D)$.

The fragment of the current graph shown in Figure 1(b) has exactly $2s-7$ vertical edges.

Lemma 2.1 ([8, Lemma 2])**.** *Let a rotation D of a graph G induce exactly one circuit. Let an edge e of G be incident with distinct trivalent vertices v and w. Then there are two ways to choose rotations of v and w without changing the rotations of other vertices, such that the obtained rotation of G induces exactly one circuit.*

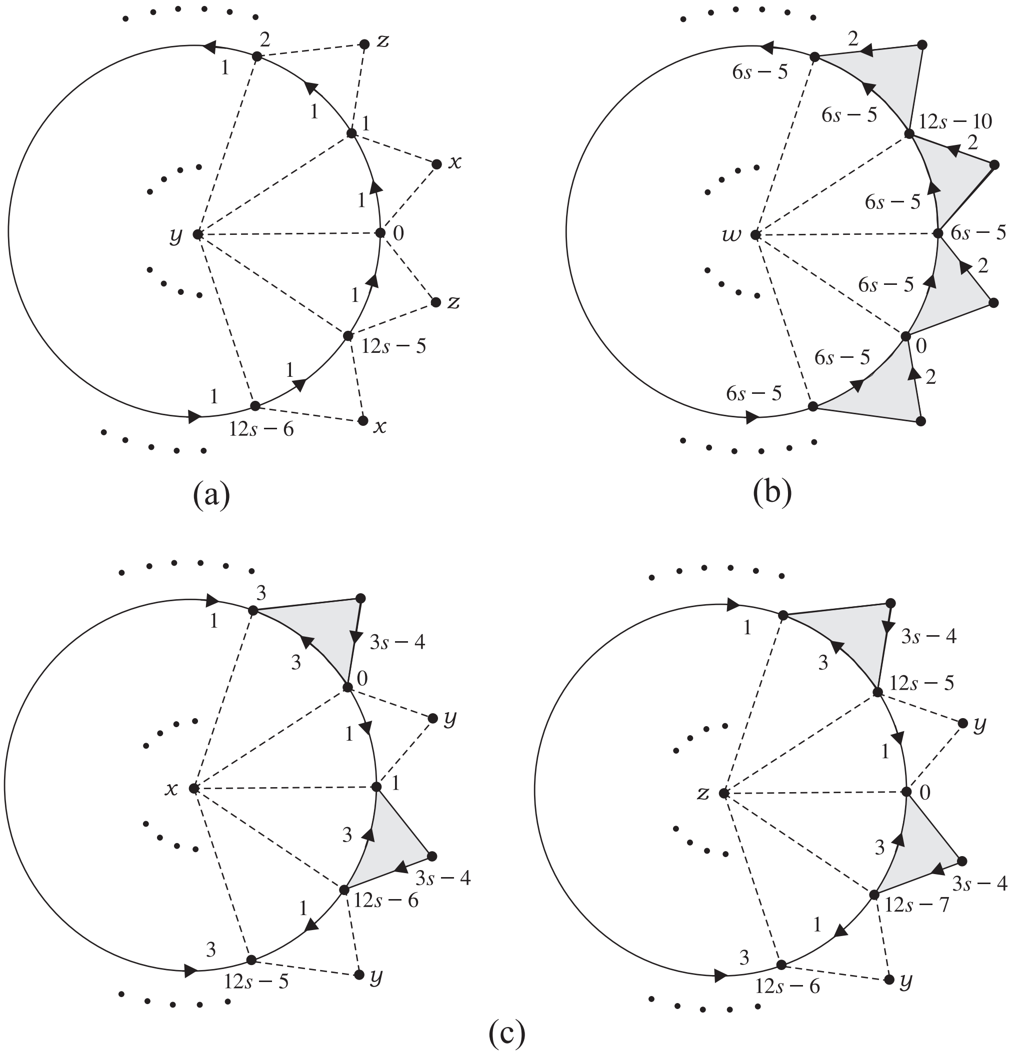

Figure 2: The $(12s-4)$-gonal faces of the embedding $f(Q)$.

Denote by $L(s)$ the set of the vertices of the current graph lying inside the dashed box in Figure 1(a). Now we fix the indicated rotations of the vertices of the current graph in Figure 1(a) that do not lie inside the dashed box, and then, applying Lemma 2.1 to the $2s-7$ vertical edges in Figure 1(b), we can choose the rotations of the vertices of $L(s)$ in 2^{2s-7} different ways such that for the corresponding 2^{2s-7} different one-rotations Q of G, we obtain index one current graphs $\langle G, \lambda, Q\rangle$ satisfying (A1)-(A6). Denote by $\mathcal{D}$ the set of all such 2^{2s-7} different one-rotations Q of G.

The embedding $f(Q)$ of K_{12s-4} generated by $\langle G, \lambda, Q\rangle$, $Q \in \mathcal{D}$, has four $(12s-4)$-gonal faces, and all other faces are triangular. Inserting four new vertices in the four $(12s-4)$-gonal faces, respectively, we obtain a triangular embedding $f'(Q)$ of $K_{12s} - K_4$ which (by attaching one additional handle to gain adjacencies between the new vertices) can be modified into a triangular embedding $\overline{f}(Q)$ of K_{12s}. All embeddings $f'(Q)$ and $\overline{f}(Q)$, $Q \in \mathcal{D}$, have the same vertex set $V(s) \bigcup R$, where $R = \{x, y, z, w\}$ is the set of the four new vertices.

We will show (Theorem 3.1) that all 2^{2s-7} triangular embeddings $\overline{f}(Q)$, $Q \in \mathcal{D}$, are nonisomorphic. Two faces of an embedding are *adjacent* if they share a common edge. To prove Theorem 3.1 we need to know pairs of adjacent faces of the embeddings $\overline{f}(Q)$, $Q \in \mathcal{D}$.

A *link* joining two vertices u and u' of an embedding is every pair (u, u_1, u_2), (u_1, u_2, u') of adjacent triangular faces of the embedding; we say that the vertices u and u' are incident with the link, and that u has the link with u'. By a link $[u, u']$ we mean a link between u and u'.

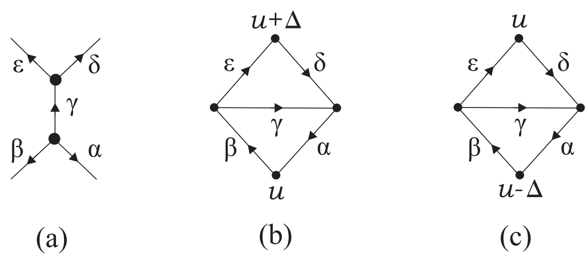

Figure 3: A link of an embedding.

If an edge of $\langle G, \lambda, Q\rangle$, $Q \in \mathcal{D}$, joins two trivalent vertices with current rotations (α, β, γ) and $(\varepsilon, \delta, -\gamma)$, respectively (see Figure 3(a)), then the *type* of the edge is $\Delta = \beta + \varepsilon$. We define the type of an edge up to inversion. Since KCL holds at the vertices, we have $\beta + \varepsilon = -(\alpha + \delta)$, hence the type is well defined. The two adjacent trivalent vertices induce faces of $f(Q)$ that form $12s-4$ links shown in Figure 3(b) where u goes through the values $0, 1, 2, \ldots, 12s-5$; we say that the $12s-4$ links are *induced* by the edge with type $\Delta = \beta + \varepsilon$. Now we have the following.

(B) For any vertex u of $f(Q)$, among the links induced by an edge with type Δ, there are exactly two links incident with u: one of them is a link $[u, u+\Delta]$ shown in Figure 3(b), and another link is a link $[u, u-\Delta]$ shown in Figure 3(c).

Since any two adjacent triangular faces of $f(Q)$ are induced by adjacent trivalent vertices of $\langle G, \lambda, Q\rangle$, every link of $f(Q)$ joining two vertices u and $u+\mu$ is induced by exactly one edge of the current graph and the type of the edge is μ.

3 Links and nonisomorphic embeddings of K_{12s}

To prove Theorem 3.1 we use the fact that in the embeddings $\overline{f}(Q)$, $Q \in \mathcal{D}$, some pairs of vertices have a large number of links joining the vertices, and some pairs of vertices have a small number of links joining the vertices.

Below we describe the modification of $f(Q)$ into $\overline{f}(Q)$, and in so doing we study links of the obtained embeddings.

First we describe links in $f(Q)$. In Figure 1(a) there are 13 edges with their types indicated (the type of an edge is given inside a box connected by a line with the edge). A list of the types of the 13 edges is

$$1, 1, 1, 10, 3s-8, 3s-5, 3s+1, 3s+2, 3s+4, 3s+6, 3s+7, 6s-10, 6s-9.$$

It is easy to check that for $s > 6$, $s = 6$, $s = 5$, and $s = 4$, the list contains, respectively, 11, 10, 9, and 10 different types. Hence $\langle G, \lambda, Q\rangle$ contains at least 9 edges having different types. The current graph $\langle G, \lambda, Q\rangle$ has exactly $6s-2$ edges, and exactly $6s-6$ of them join two trivalent vertices, hence at most $(6s-6)-8$ edges of $\langle G, \lambda, Q\rangle$ have the same type, and, by (B), we obtain the following.

(C) In $f(Q)$, $Q \in \mathcal{D}$, every vertex of $V(s)$ has at most $6s-14$ links with any other vertex of $V(s)$.

In what follows an edge joining vertices u and u' is denoted by (u, u').

Now insert new vertices x, y, z and w in the four $(12s-4)$-gonal faces of $f(Q)$ as shown in Figure 2 as dashed lines. (As is customary, in Figure 1(a), if a onevalent or twovalent vertex is labeled by letters, then the letters denote the new vertices that we insert in the faces induced by the vertex.) We obtain a triangular embedding $f'(Q)$ of $K_{12s} - K_4$. Note that the boundary cycle of the two $(12s-4)$-gonal faces in Figure 2(c) contain all edges $(u, u+1)$ and $(u, u-3)$, $u \in V(s)$, and we insert a new vertex x (resp. z) in the face whose boundary cycle contains all edges $(2i, 2i+1)$ and $(2i+1, 2i-2)$ (resp. $(2i+1, 2i+2)$ and $(2i, 2i-3)$, $i = 0, 1, \ldots, 6s-3$.

Every link of $f(Q)$ is a link of $f'(Q)$. After we insert a new vertex in a $(12s-4)$-gonal face, every vertex of $V(s)$ lying on the boundary cycle of the face gains a new link with two different vertices of $V(s)$ lying on the cycle. Now, considering Figure 2 where we depict all triangular faces incident with the edges of the boundary cycles of the $(12s-4)$-gonal faces, we obtain the following:

(D) The links of $f'(Q)$ which are not links of $f(Q)$ are as follows: the vertex y has $6s-2$ links with each of x and z; the vertex w has exactly one link with every vertex of $V(s)$; the vertex x (resp. z) has exactly one link with each even (resp. odd) vertex of $V(s)$; every vertex $u \in V(s)$ has three new links $[u, u+2]$, three new links $[u, u-2]$, one new link $[u, u+6]$ and one new link $[u, u-6]$.

The triangular embedding $\overline{f}(Q)$, $Q \in \mathcal{D}$, of K_{12s} is obtained from the embedding $f'(Q)$ of $K_{12s} - K_4$ in the following way. The log of the circuit of $\langle G, \lambda, Q\rangle$ (following [11], the letters x, y, z, w enter the log) determines the cyclic order in which the vertices adjacent to the vertex 0 are arranged on the surface around the vertex 0 in $f'(Q)$. As easily

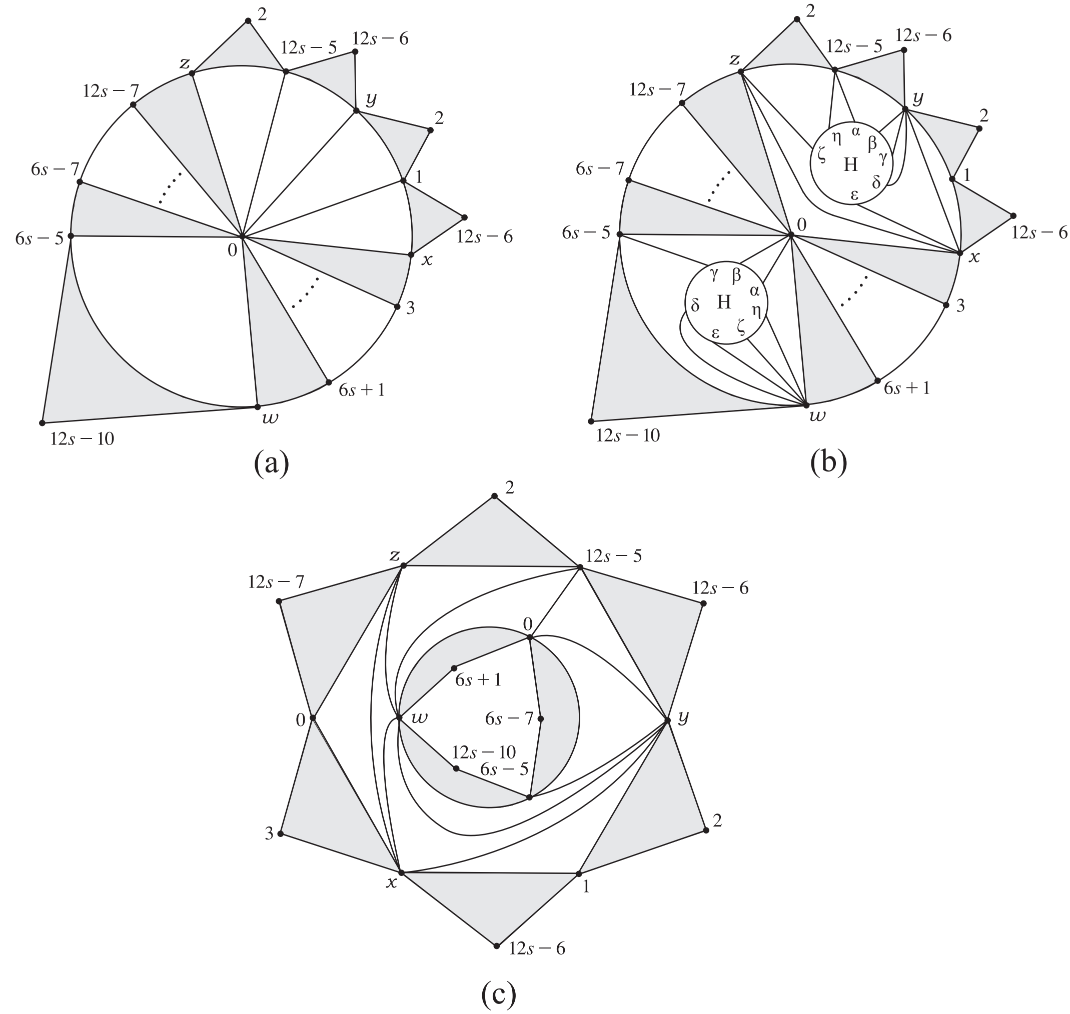

Figure 4: Attaching a handle.

seen, the log is of the form

$$(\ldots, 9s, 3, x, 1, y, 12s-5, z, 12s-7, 9s-3, \ldots, 6s-7, 6s-5, w, 6s+1, 12s-6, \ldots)$$

so that the faces of $f'(Q)$ incident with the vertex 0 are arranged as shown in Figure 4(a). Now, in Figure 4(a), we delete edges $(0, 12s-5), (0, y), (0, 1)$, and then, as shown in Figure 4(b), using a handle (depicted as two blank cycles with the letter H inside; the cycles are to be identified, and the edge ends labeled by the same Greek letter $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta$ are to be identified as well) we gain adjacencies

$$(x, y), (x, w), (x, z), (y, w), (y, 0), (y, 6s-5), (z, w), (0, 12s-5), (w, 12s-5).$$

In Figure 4 (and as in what follows in Figure 5) the shaded faces are faces of $f'(Q)$ that remain unchanged when modifying $f'(Q)$ into $\overline{f}(Q)$. As a result, we obtain a triangular embedding of the graph K_{12s} without two edges (y, z) and $(0, 1)$, but with two extra edges $(w, 12s-5)$ and $(y, 6s-5)$. Note that the faces shown in Figure 4(b) are the same for all $Q \in \mathcal{D}$.

The embedding $f'(Q)$ contains five pairs of adjacent faces shown as non-shaded faces in Figures 5(a) – (e), respectively. For each of the five pairs, we show a fragment of $\langle G, \lambda, Q\rangle$ whose vertices induce the faces of the pair and all the other (shaded) faces adjacent to the faces of the pair (the vertices of the fragment are not vertices of $L(s)$, so that the faces shown in Figures 5(a) – (e) are the same for all $Q \in \mathcal{D}$). The reader can consult Figures 2 and 3 when checking pairs of adjacent faces in Figures 5(a) – (e).

The diagonal flips in the pairs of adjacent non-shaded faces shown in Figures 5(a), (b), and (c), replace the edges $(w, 12s-5)$, $(6s, 6s-6)$, and $(12s-8, 12s-9)$ by the edges $(6s, 6s-6)$, $(12s-8, 12s-9)$, and (y, z), respectively, depicted in dashed line. As a result, we lose an extra edge $(w, 12s-5)$ and gain a missing edge (y, z). The diagonal flips in the pairs of adjacent non-shaded faces shown in Figures 5(d) and (e) replace the edges $(y, 6s-5)$ and $(6s-6, 6s-4)$ by the edges $(6s-6, 6s-4)$ and $(0, 1)$, respectively. As a result, we lose an extra edge $(y, 6s-5)$ and gain a missing edge $(0, 1)$. We obtain the triangular embedding $\overline{f}(Q)$ of K_{12s}. Note that the diagonal flips do not affect the faces shown in Figure 4(b), hence all faces shown in Figure 4(b) are faces of $\overline{f}(Q)$.

Now we need to know what new links we gain and what links incident with vertices of R we lose when modifying $f'(Q)$ into $\overline{f}(Q)$.

In Figure 4(b), a new additional handle is attached to the 6-gonal face $(0, z, 12s-5, y, 1, x)$ and the triangular face $(w, 0, 6s-5)$, and then some new edges are embedded. If we actually identify the two cycles with the letter H inside, then the faces of $\overline{f}(Q)$ incident with the new edges shown in Figure 4(b) can be redrawn as shown in Figure 4(c). Every lost link contains a face that we lose during the modification, hence all lost links are incident with vertices incident with lost faces. Every new link contains a new face, hence all new links are incident with vertices incident with new faces. The reader can consult Figure 2 when checking faces in Figure 4.

When considering the five diagonal flips shown in Figures 5(a) – (e), it is easy to see that if in Figure 5(f) we replace the edge (a, b) by (c, d), then we lose links $[c, d], [a, h], [a, f], [b, g], [b, e]$ and gain new links $[a, b], [c, e], [c, f], [d, g], [d, h]$.

By inspection of Figures 4(c) and 5(a) – (e), the reader can check that during the modification of $f'(Q)$ into $\overline{f}(Q)$: the vertex y lost two links with each of x and z; each of x and z gained at most one link with any vertex of $V(s)$; the vertex w gained at most three new

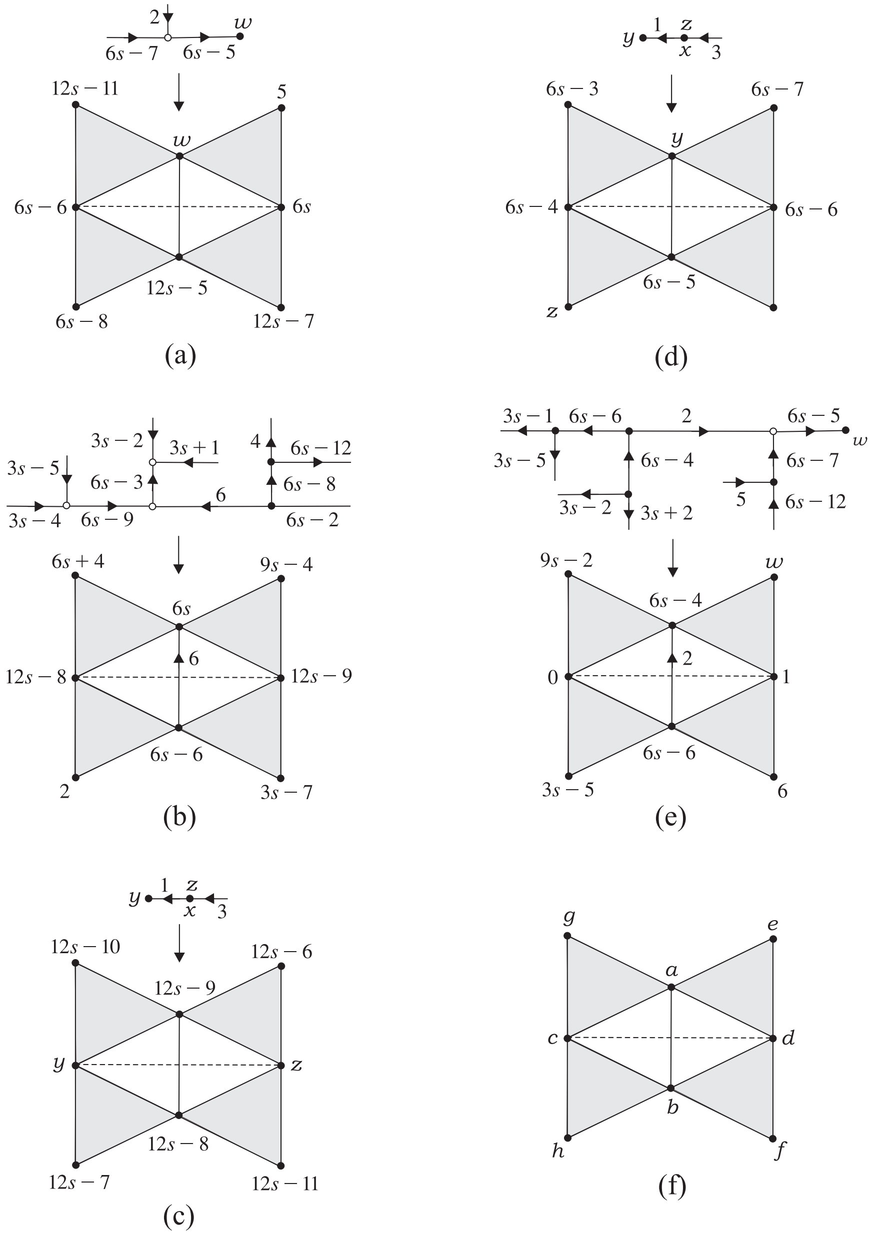

Figure 5: Diagonal flips.

links with any vertex of $V(s)$, one new link with y, and no links with each of x and z; any two vertices of $V(s)$ gained at most one new link; any vertex of $V(s)$ gained at most four new links with vertices of R.

Now, taking into account (C) and (D), we obtain the following.

(E) For any $Q \in \mathcal{D}$, in $\overline{f}(Q)$, we have the following: the vertex y has $6s-4$ links with each of x and z; each of x and z has $6s-4$ links with y only; the vertex w has at most 4 links with any vertex of $V(s) \bigcup R$, and has no links with x and z; every vertex of $V(s)$ has a link either with x or z, and has less than $6s-4$ links with every other vertex of $V(s) \bigcup R$.

By an *automorphism* of $\overline{f}(Q)$ we mean any isomorphism from $\overline{f}(Q)$ onto $\overline{f}(Q)$.

Theorem 3.1. *All 2^{2s-7} embeddings $\overline{f}(Q)$, $Q \in \mathcal{D}$, of K_{12s}, $s \geq 4$, are nonisomorphic and each of them has no nontrivial automorphisms.*

Proof. Suppose there is an isomorphism ψ of $\overline{f}(Q_1)$ onto $\overline{f}(Q_2)$, where $Q_1, Q_2 \in \mathcal{D}$. If two adjacent faces (u_1, u_2, u_3) and (u_2, u_3, u_4) are a link in $\overline{f}(Q_1)$, then the two adjacent faces $(\psi(u_1), \psi(u_2), \psi(u_3))$ and $(\psi(u_2), \psi(u_3), \psi(u_4))$ are a link in $\overline{f}(Q_2)$. Since $\overline{f}(Q_1)$ and $\overline{f}(Q_2)$ have the same number of links, namely, the number of edges of K_{12s}, it follows that the number of links between any two vertices u and u' in $\overline{f}(Q_1)$ equals the number of links between any two vertices $\psi(u)$ and $\psi(u')$ in $\overline{f}(Q_2)$.

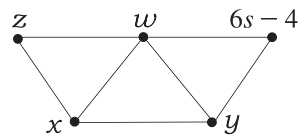

Figure 6: Common faces of all $\overline{f}(Q)$, $Q \in \mathcal{D}$.

By (E), the vertex y is the only vertex in each of $\overline{f}(Q_1)$ and $\overline{f}(Q_2)$ that has $6s-4$ links with each of two vertices, hence $\psi(y) = y$. Since x and z are the only vertices in each of $\overline{f}(Q_1)$ and $\overline{f}(Q_2)$ such that each of the vertices has $6s-4$ links with exactly one other vertex, we have $\{\psi(x), \psi(z)\} = \{x, z\}$. Since w is the only vertex of $V(s) \bigcup \{w\}$ in each of $\overline{f}(Q_1)$ and $\overline{f}(Q_2)$ that has no links with x and z, we have $\psi(w) = w$. Considering Figure 4(c), we see that $\overline{f}(Q_1)$ and $\overline{f}(Q_2)$ have the same faces shown in Figure 6. Since $(\psi(w), \psi(x), \psi(y)) = (w, \psi(x), y)$ is a face of $\overline{f}(Q_2)$, and $\psi(x) \in \{x, z\}$, we obtain (see Figure 6) that $\psi(x) = x$, and then $\psi(z) = z$.

If $\overline{f}(Q_1)$ and $\overline{f}(Q_2)$ have common adjacent faces $(\psi(u_1), \psi(u_2), \psi(u_3))$ and $(\psi(u_2), \psi(u_3), \psi(u_4))$, where $\psi(u_j) = u_j$ for $j = 1, 2, 3$, then $\psi(u_4) = u_4$. The faces incident with w (the faces are the same for all $\overline{f}(Q)$, $Q \in \mathcal{D}$), form a sequence $F_1, F_2, \ldots, F_{12s-1}$ where:

(i) $F_1 = (w, x, y)$ and $\psi(w) = w$, $\psi(x) = x$, $\psi(y) = y$;

(ii) for $j = 1, 2, \ldots, 12s-1$, the faces F_j and F_{j+1} (here $F_{12s} = F_1$) share a common edge (w, b_j), where $\{b_1, b_2, \ldots, b_{12s-1}\} = (V(s) \bigcup R) \setminus \{w\}$.

It follows that $\psi(u) = u$ for every $u \in V(s) \bigcup R$, hence $\overline{f}(Q_1)$ and $\overline{f}(Q_2)$ have the same faces. If $Q_1 = Q_2$, then we obtain that ψ is a trivial automorphism, hence $\overline{f}(Q_1)$ does not have nontrivial automorphisms.

Suppose, for a contradiction, that $Q_1 \neq Q_2$. Since $\overline{f}(Q_1)$ and $\overline{f}(Q_2)$ have the same faces, considering the modification of $f(Q)$ into $\overline{f}(Q)$, we see that $f(Q_1)$ and $f(Q_2)$ have the same faces as well, hence we have:

(a) The cyclic order in which the vertices adjacent to the vertex 0 are arranged on the surface around the vertex 0 in $f(Q_1)$ is (up to reversal) the cyclic order in which the vertices adjacent to the vertex 0 are arranged on the surface around the vertex 0 in $f(Q_2)$.

The embeddings $f(Q_1)$ and $f(Q_2)$ are generated by the current graphs $\langle G, \lambda, Q_1 \rangle$ and $\langle G, \lambda, Q_2 \rangle$, respectively. Since $Q_1 \neq Q_2$, a trivalent vertex v (resp. w) of G has the same rotation (resp. different rotations) in Q_1 and Q_2. Then the circuit of $\langle G, \lambda, Q_1 \rangle$ (resp. $\langle G, \lambda, Q_2 \rangle$) is of the form $(a_1, a_2, \ldots, b_1, b_2, \ldots)$ (resp. $(a_1, a_2, \ldots, b_1, b_3, \ldots)$) where a_1 and a_2 are arcs incident with v, and b_1, b_2, b_3 are arcs incident with w, where $b_2 \neq b_3$. Hence the two cascades have different logs of their circuits, namely, $(\lambda(a_1), \lambda(a_2), \ldots, \lambda(b_1), \lambda(b_2), \ldots)$ and $(\lambda(a_1), \lambda(a_2), \ldots, \lambda(b_1), \lambda(b_3), \ldots)$ where $\lambda(b_2) \neq \lambda(b_3)$, contrary to (a) (note that in the cascades, $\lambda(a) \neq \lambda(a')$ for different arcs a and a'). $\square$

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Two new families of non-CCA groups*

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Abstract

We determine two new infinite families of Cayley graphs that admit colour-preserving automorphisms that do not come from the group action. By definition, this means that these Cayley graphs fail to have the CCA (Cayley Colour Automorphism) property, and the corresponding infinite families of groups also fail to have the CCA property. The families of groups consist of the direct product of any dihedral group of order $2n$ where $n \geq 3$ is odd, with either itself, or the cyclic group of order n. In particular, this family of examples includes the smallest non-CCA group that does not fit into any previous family of known non-CCA groups.

Keywords: Cayley graphs, automorphisms, colour preserving, CCA

Math. Subj. Class.: 05C25

1 Introduction

All groups and graphs in this paper are finite. All of our graphs are simple, undirected, and have no loops.

A Cayley graph of G with respect to C (a subset of $G \backslash \{e\}$) is the graph $Cay(G, C)$ whose vertices are the elements of G, with an edge from g to gc if and only if $g \in G, c \in C$. The set C is known as the connection set of $Cay(G, C)$. This connection set gives a natural colouring of the edges where we colour the edge from g to gc (which is the same as the edge from gc to g) with a colour associated to $\{c, c^{-1}\}$. A colour-preserving automorphism of $Cay(G, C)$ is a permutation of the vertices that preserves edges and non-edges as well as edge colour. A Cayley graph $Cay(G, C)$ is said to have the **Cayley Colour**

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Automorphism (CCA) property if every colour-preserving automorphism of the graph is an affine function on G. The group G is said to be CCA if every connected Cayley graph of G is CCA.

The study of this property has only come up recently in history. In 2012, M. Conder, T. Pizanski and A. Žitnik [1] proposed a question about the permutations of circulant graphs that preserve a certain edge colouring that the second author [7] answered. The second author showed that for any connected Cayley graph on the cyclic group C_n, all colour-preserving automorphisms that fix the identity are automorphisms of C_n. In 2014, A. Hujdurović, K. Kutnar, D. W. Morris, and J. Morris [3] extended the original question by looking at Cayley graphs, using the natural edge colouring described. In early 2017, L. Morgan, J. Morris and G. Verret [5, 6] gave new results for finite simple groups and Sylow cyclic groups that generalized results produced by E. Dobson, A. Hujdurović, K. Kutnar, and J. Morris in [2]. The problem of determining colour-preserving and colour-permuting automorphisms for directed Cayley graphs has already been studied and is well understood: see for example [10], where the authors showed that for a connected Cayley digraph, every colour-preserving automorphism is a left-translation by some element of the group.

In his M.Sc. thesis, the first author produced code using GAP [9] and Sage [8] that determines whether or not a group or graph has the CCA property, and ran this code on all groups of order up to 200 (excluding orders 128 and 192). With this data in hand, a logical step was to look for theoretical methods to explain some of the small non-CCA groups that were not previously understood, and if possible to find new infinite families of non-CCA groups using this method.

In this paper, we use results from [5] to show that whenever $n \geq 3$, the groups $C_n \times D_{2n}$ and $D_{2n} \times D_{2n}$ are non-CCA groups. Section 2 contains some basic background, definitions, and notation, along with the statements of the results we need from [5]. Section 3 provides proofs of our main results.

2 Background

The following notation is used for the remainder of this paper. We use C_n to represent the cyclic group of order n, and D_{2n} (for $n \geq 3$) to represent the dihedral group of order $2n$. We also have Q_8 as the quaternion group of order 8.

The notation $\Gamma = (V(\Gamma), E(\Gamma))$ represents a graph of finite order, consisting of a set $V = V(\Gamma)$ of vertices and a set $E = E(\Gamma) \subseteq \{\{u, v\} \mid u, v \in V\}$ of edges. The set of vertices that are adjacent to a vertex v, denoted $\Gamma(v)$, is called the neighbourhood of v. We use $\mathcal{L}(\Gamma)$ to indicate the line graph of the graph Γ, and $\mathcal{S}(\Gamma)$ is the subdivision graph of the graph Γ.

If G acts on a graph Γ and $S \subseteq V(\Gamma)$ is fixed setwise under the action of G, then G^S is the restriction of the action of G to S. We use G_v to denote the stabiliser subgroup (elements of G that fix v).

Definition 2.1 ([3, Definition. 2.6])**.** For an abelian group A of even order and an involution $y \in A$, the corresponding **generalized dicyclic group** is

$$\mathrm{Dic}(A, y) = \langle x, A \mid x^2 = y, x^{-1}ax = a^{-1}, \forall a \in A\rangle.$$

Definition 2.2 ([3, Definition. 5.1])**.** The **generalized dihedral group** over an abelian group A is the group

$$\mathrm{Dih}(A) = \langle \sigma, A \mid \sigma^2 = e, \sigma a \sigma = a^{-1}, \forall a \in A \rangle$$

Definition 2.3 ([5, Definition 4.5])**.** Let B be a permutation group and G a regular subgroup of B. Let $\mathcal{A}^0$ be the colour-preserving automorphism group of the complete Cayley colour graph $K_G = \mathrm{Cay}(G, G \setminus \{e\})$, and let $\widehat{G}$ be the subgroup of $\mathcal{A}^0$ consisting of all left translations by elements of G. We say that (G, B) is a **complete colour pair** if B is a subgroup of $\mathcal{A}^0$ and G is one of the following:

- G is abelian but not an elementary abelian 2-group, and $\mathcal{A}^0 \cong \mathrm{Dih}(G)$.
- $G \cong \mathrm{Dic}(A, y)$ but not of the form $Q_8 \times C_2^n$ and $\mathcal{A}^0 = \widehat{G} \rtimes \langle \sigma \rangle$, where σ is the permutation that fixes A pointwise and maps every element of the coset Ax to its inverse.
- $G \cong Q_8 \times C_2^n$ and $\mathcal{A}^0 = \langle \widehat{G}, \sigma_i, \sigma_j, \sigma_k \rangle$, where σ_α is the permutation of $Q_8 \times C_2^n$ that inverts every element of $\{\pm\alpha\} \times C_2^n$ and fixes every other element.

The importance of Definition 2.3 comes from the fact that if (G, B) is a complete colour pair, then in each case we have a colour-preserving automorphism of K_G that is not an element of $\widehat{G}$.

An *arc* is an orientation for an edge in a graph. So the edge $\{u, v\}$ admits two possible orientations: (u, v), or (v, u).

Definition 2.4. Let Γ be a graph and G a permutation group acting on the vertices of Γ. We say that Γ is a ***G*-arc-regular graph** if for each pair of arcs $e_1 = (u, v)$ and $e_2 = (w, x)$ (each an oriented edge from $E(\Gamma)$), there exists a unique element of G that maps u to w and v to x, so that it maps the chosen orientation for e_1 to the chosen orientation for e_2.

Notation 2.5. For the remainder of this paper we use the following notation. Consider the complete bipartite graph $K_{n,n}$. We define ρ_1 to be a cyclic permutation on one of the bipartition sets, and ρ_2 be a cyclic permutation on the other bipartition set, with τ an involution that commutes with $\rho_1\rho_2$ and switches the bipartition sets. Let σ_1 be an involution acting on the first bipartition set that inverts ρ_1, and σ_2 an involution acting on the second bipartition set that inverts ρ_2.

We label the edges of $\mathcal{S}(K_{n,n})$ as follows. Use v to denote the unique vertex in the second bipartition set of $K_{n,n}$ that is fixed under the action of σ_2. Now in $\mathcal{S}(K_{n,n})$ label the edge from $\tau(v)$ to the vertex subdividing $\{v, \tau(v)\}$ with the identity element e of G, and label each other edge by the unique element of G that maps the edge e to that edge. This produces a labeling that shows us that $\mathcal{L}(\mathcal{S}(K_{n,n}))$ is a Cayley graph on G. From this it is straightforward to observe that the connection set C (which consists of all neighbours of e) is $\{\tau\} \cup \{\rho_2^i : 1 \le i \le n-1\}$.

Corollary 2.6 ([5, Corollary 4.10])**.** *Let Γ be a connected G-arc-regular graph. If H is a group of automorphisms of Γ such that:*

- *$G \le H$, and*
- *$(G_v^{\Gamma(v)}, H_v^{\Gamma(v)})$ is a complete colour pair for every vertex v of Γ,*

then H is a colour-preserving group of automorphisms of $\mathcal{L}(\mathcal{S}(\Gamma))$ viewed as a Cayley graph on G.

The real point of this corollary is that if we show that some element of H is not an affine function, then this implies that $\mathcal{L}(\mathcal{S}(\Gamma))$ is a non-CCA graph, and so G is a non-CCA group. The fact that $(G_v^{\Gamma(v)}, H_v^{\Gamma(v)})$ is a complete colour pair is what allows us to produce the desired non-affine element of H.

3 Main results

In our main result, we show that $K_{n,n}$ is a (connected) $C_n \times D_{2n}$-arc-regular graph and therefore if we take $\Gamma = K_{n,n}$, $G = C_n \times D_{2n}$, and $H = D_{2n} \wr C_2$ then all of the conditions of Corollary 2.6 are satisfied. For clarity, we are using $D_{2n} \wr C_2$ to denote the semidirect product $(D_{2n} \times D_{2n}) \rtimes C_2$, where the C_2 is acting on the coordinates in the direct product. Hence $D_{2n} \wr C_2$ is a colour-preserving group of automorphisms of $\mathcal{L}(\mathcal{S}(K_{n,n}))$. With this we find an element in $D_{2n} \wr C_2$, a colour-preserving automorphism, that is a non-affine function to show that $\mathcal{L}(\mathcal{S}(K_{n,n}))$ is non-CCA. The proof is not particularly difficult; the difficulty of this result lies in finding an arc-regular graph and corresponding permutation groups to which we can apply Corollary 2.6.

Theorem 3.1. *The graph $\mathcal{L}(\mathcal{S}(K_{n,n}))$ viewed as a Cayley graph on $C_n \times D_{2n}$ is non-CCA whenever $n \geq 3$ is odd.*

Specifically, if $G = \langle \rho_1, \rho_2, \tau \rangle$ and $C = \{\tau\} \cup \{\rho_2^i : 1 \leq i \leq n-1\}$, then σ_2 is a non-affine colour-preserving automorphism of $\mathrm{Cay}(G, C)$.

Proof. We use Notation 2.5 and the labelling that is given in the paragraph following that notation to view $\mathcal{L}(\mathcal{S}(\Gamma))$ as a Cayley graph on G. Observe that $G = \langle \rho_1, \rho_2, \tau \rangle = \langle \rho_1\rho_2, \rho_1\rho_2^{-1}, \tau \rangle \cong C_n \times D_{2n}$ since n is odd so that $\langle \rho_2^2 \rangle = \langle \rho_2 \rangle$. Notice that G acts regularly on the arcs of $K_{n,n}$, so that $K_{n,n}$ is G-arc-regular.

Consider now the group $H = \langle \rho_1, \rho_2, \tau, \sigma_1, \sigma_2 \rangle \cong D_{2n} \wr C_2$ where each copy of D_{2n} acts independently on one of the bipartition sets of $K_{n,n}$, and the C_2 (generated by τ) exchanges the coordinates. The first copy of D_{2n} is generated by ρ_1 and σ_1. The second copy is generated by ρ_2 and σ_2. It is clear that $G \leq H$ since $\rho_1, \rho_2, \tau \in H$.

Let v be an arbitrary vertex of the second bipartition set. The neighbours of v are all the elements of the first bipartition set. We notice that $G_v^{\Gamma(v)}$ is the subgroup of G that fixes v and its action is restricted to the bipartition set that v is not in. We see that ρ_1 is the cyclic permutation of $\Gamma(v)$. Since $G = \langle \rho_1, \rho_2, \tau \rangle$, it is not hard to observe that $G_v^{\Gamma(v)} = \langle \rho_1 \rangle \cong C_n$. Similarly since $H = \langle \rho_1, \rho_2, \tau, \sigma_1, \sigma_2 \rangle$ we have that $H_v^{\Gamma(v)} = \langle \rho_1, \sigma_1 \rangle \cong D_{2n}$. Thus we only need to show that (C_n, D_{2n}) is a complete colour pair.

We can see (C_n, D_{2n}) is a complete colour pair using Definition 2.3. Let $\mathcal{A}^0$ be the colour-preserving automorphism group for the Cayley graph $K_{G_v^{\Gamma(v)}}$. We know that $\mathcal{A}^0 = D_{2n} = \mathrm{Dih}(C_n)$ and thus since C_n is abelian and is not an elementary abelian 2-group ($n \geq 3$), all the properties of the first possibility for a complete colour pair are met. (In this case, $B = D_{2n} = \mathcal{A}^0$.) We thus conclude (using Corollary 2.6) that every element of H is a colour-preserving automorphism of $\mathcal{L}(\mathcal{S}(K_{n,n}))$ viewed as a Cayley graph on G.

It remains to show that some element of H is not affine. We claim that σ_2 (acting on G as an automorphism of the Cayley graph) is such an element. In order to prove this, we show that $\sigma_2^{-1}\tau\sigma_2$ is not an element of G. Let v be the unique vertex in the second bipartition set that is fixed by σ_2. Clearly, $\sigma_2^{-1}\tau\sigma_2 = \sigma_2\tau\sigma_2$ maps the arc $(v, \tau(v))$ to the arc $(\tau(v), v)$, since σ_2 fixes both v and $\tau(v)$. Since G is acting arc-regularly, it has a unique element that maps $(v, \tau(v))$ to the arc $(\tau(v), v)$, and we know that this element is τ. So if σ_2 normalises G, we must have $\sigma_2\tau\sigma_2 = \tau$. It is straightforward to verify that this is not the case. For example, $\tau\rho_2(v) = \tau\rho_1\rho_2(v) = \rho_1\rho_2\tau(v) = \rho_1\tau(v)$ (the first equality follows from the fact that ρ_1 fixes the bipartition set that contains v; the second equality from the fact that τ and $\rho_1\rho_2$ commute, and the third from the fact that ρ_2 fixes the bipartition set that does not contain v). However, $\sigma_2\tau\sigma_2\rho_2(v) = \tau\sigma_2\rho_2(v) = \tau\rho_2^{-1}\sigma_2(v) = \tau\rho_2^{-1}(v)$ (the first equality follows because σ_2 fixes the bipartition set that does not contain v; the second because $\langle\sigma_2, \rho_2\rangle \cong D_{2n}$, so σ_2 inverts ρ_2; and the third because σ_2 fixes v). However, since $n \geq 3$, $\tau\rho_2^{-1}(v)$ is not the same as $\tau\rho_2(v)$, because the order of ρ_2 is n. Thus, $\sigma_2 \in H$ does not normalise G, as claimed. □

Corollary 3.2. *The group $C_n \times D_{2n}$ is non-CCA whenever $n \geq 3$ is odd.*

We use the above result to show that $D_{2n} \times D_{2n}$ is not CCA whenever $n \geq 3$ is odd.

Proposition 3.3. *The group $D_{2n} \times D_{2n}$ is non-CCA whenever $n \geq 3$ is odd.*

Proof. Let $G = \langle\rho_1, \rho_2, \tau\rangle$ where these permutations are as defined in Notation 2.5. Define $H = \langle G, \gamma\rangle$, where γ is an involution that commutes with τ and with $\rho_1^{-1}\rho_2$, and inverts $\rho_1\rho_2$. Notice that this implies $H \cong D_{2n} \times D_{2n}$.

By Theorem 3.1, if $G = \langle\rho_1, \rho_2, \tau\rangle$ and $C = \{\tau\} \cup \{\rho_2^i : 1 \leq i \leq n-1\}$, then σ_2 is a non-affine automorphism of $\mathrm{Cay}(G, C)$ (in its action on G as an automorphism of this Cayley graph). We use this to produce a non-affine colour-preserving automorphism φ on $\Gamma = \mathrm{Cay}(H, C \cup \{\gamma\})$.

Define φ by $\varphi(g) = \sigma_2(g)$, and $\varphi(g\gamma) = \sigma_2(g)\gamma$ for every $g \in G$. We first show that φ is colour-preserving on Γ.

Consider any edge e of Γ. If both endpoints of e are in G then $\varphi(e) = \sigma_2(e)$ and since σ_2 preserves colours, so does φ.

If one endpoint of e is in G and the other is not, then it must be the case that e is coloured γ, and its endpoints are g and $g\gamma$ for some $g \in G$. Furthermore, by definition of φ we have $\varphi(g\gamma) = \varphi(g)\gamma$, so there is an edge between $\varphi(g)$ and $\varphi(g\gamma)$, and its colour is γ. Thus φ also preserves the colour of any such edge.

The final case to consider is if both endpoints of e are in $G\gamma$. Suppose the endpoints of e are $\rho_1^{i_1}\rho_2^{i_2}\tau^{f_1}\gamma$ and $\rho_1^{j_1}\rho_2^{j_2}\tau^{f_2}\gamma$, where $0 \leq i_1, i_2, j_1, j_2 \leq n-1$, and $0 \leq f_1, f_2 \leq 1$. Since there is an edge between these vertices, we must have $\gamma\tau^{f_1}\rho_1^{j_1-i_i}\rho_2^{j_2-i_2}\tau^{f_2}\gamma \in C$ (recall that γ and τ are both involutions). Note that $\rho_1^a\rho_2^b = (\rho_1\rho_2)^{(a+b)/2}(\rho_1^{-1}\rho_2)^{(b-a)/2}$; we want to use this because we know that γ commutes with τ and with $\rho_1^{-1}\rho_2$ but inverts $\rho_1\rho_2$. So we have

$$\begin{aligned}
&\gamma\tau^{f_1}(\rho_1\rho_2)^{(j_1+j_2-i_1-i_2)/2}(\rho_1^{-1}\rho_2)^{(j_2+i_1-i_2-j_1)/2}\tau^{f_2}\gamma \\
&\qquad = \tau^{f_1}(\rho_1\rho_2)^{(i_1+i_2-j_1-j_2)/2}(\rho_1^{-1}\rho_2)^{(j_2+i_1-i_2-j_1)/2}\tau^{f_2} \\
&\qquad = \tau^{f_1}\rho_1^{i_2-j_2}\rho_2^{i_1-j_1}\tau^{f_2} \in C.
\end{aligned}$$

Since we know the elements of C, this implies one of three possibilities:

- the element is τ, so that $i_2 = j_2$ and $i_1 = j_1$, and $\{f_1, f_2\} = \{0, 1\}$;

- $f_1 = f_2 = 0$ and the element is ρ_2^j for some $1 \le j \le n-1$, so $i_2 = j_2$, and $j = i_1 - j_1$); or
- $f_1 = f_2 = 1$ and the element is ρ_2^j for some $1 \le j \le n-1$, so (using the above equation and the fact that τ commutes with $\rho_1\rho_2$ and inverts $\rho_1^{-1}\rho_2$) $i_1 = j_1$, and $j = j_2 - i_2$.

We now need to understand the images of the endpoints of e under φ. Recall from the labelling established immediately following Notation 2.5 that we choose the vertex v to be the unique vertex in the second bipartition set that is fixed by σ_2, and in $\mathcal{S}(K_{n,n})$ we label the edge from $\tau(v)$ to the vertex subdividing $\{v, \tau(v)\}$ with the identity element of G. This means that the edge from $\rho_1^{i_1}\tau(v)$ to the vertex subdividing $\{\rho_1^{i_1}\tau(v), \rho_2^{i_2}(v)\}$ will be the image of the edge labelled with the identity under the action of $\rho_1^{i_1}\rho_2^{i_2}$, so is labelled $\rho_1^{i_1}\rho_2^{i_2}$. Similarly, since the edge from v to the vertex subdividing $\{v, \tau(v)\}$ has the label τ, the edge from $\rho_2^{i_2}(v)$ to the vertex subdividing $\{\rho_1^{i_1}\tau(v), \rho_2^{i_2}(v)\}$ will be labelled $\rho_1^{i_1}\rho_2^{i_2}\tau$. This is the other "half" of the same subdivided edge from $K_{n,n}$. It should now be apparent that $\sigma_2(\rho_1^{i_1}\rho_2^{i_2}) = \rho_1^{i_1}\rho_2^{-i_2}$ and therefore $\sigma_2(\rho_1^{i_1}\rho_2^{i_2}\tau) = \rho_1^{i_1}\rho_2^{-i_2}\tau$ (the other half of the same subdivided edge from $K_{n,n}$). Thus, the images of the endpoints of e under φ are $\rho_1^{i_1}\rho_2^{-i_2}\tau^{f_1}\gamma$ and $\rho_1^{j_1}\rho_2^{-j_2}\tau^{f_2}\gamma$.

Now using similar calculations to those above, the colour of the edge between these images is

$$\tau^{f_1}\rho_1^{j_2-i_2}\rho_2^{i_1-j_1}\tau^{f_2}$$

(together with its inverse). Taking the three possibilities identified above in turn, if $i_1 = j_1$, $i_2 = j_2$, and $\{f_1, f_2\} = \{0, 1\}$ then this colour is τ as before, so φ has preserved the colour. If $f_1 = f_2 = 0$, $i_2 = j_2$, and the colour of e was $\{\rho_2^j, \rho_2^{-j}\}$ where $j = i_1 - j_1$, then the colour of this edge is also $\{\rho_2^j, \rho_2^{-j}\}$. Finally, if $f_1 = f_2 = 1$, $i_1 = j_1$, and the colour of e was $\{\rho_2^j, \rho_2^{-j}\}$ where $j = j_2 - i_2$, then the colour of this edge is $\{\rho_2^j, \rho_2^{-j}\}$. So in all cases the colour of e is preserved under the action of φ. This completes the proof that φ is colour-preserving.

Since $\mathrm{Cay}(H, C)$ has two connected components (on G and $G\gamma$), any colour-preserving automorphism of Γ must preserve these components. Therefore, if φ is affine then its restriction to G (which is σ_2) would have to be affine on G. By Theorem 3.1 this is not the case. Thus, φ is a colour-preserving automorphism of Γ that is not affine, and therefore Γ and H are not CCA. □

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Transit sets of two-point crossover*

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Abstract

Genetic Algorithms typically invoke crossover operators to produce offsprings that are a "mixture" of two parents x and y. On strings, k-point crossover breaks parental genotypes at at most k corresponding positions and concatenates alternating fragments for the

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two parents. The transit set $R_k(x, y)$ comprises all offsprings of this form. It forms the tope set of an uniform oriented matroid with Vapnik-Chervonenkis dimension $k + 1$. The Topological Representation Theorem for oriented matroids thus implies a representation in terms of pseudosphere arrangements. This makes it possible to study 2-point crossover in detail and to characterize the partial cubes defined by the transit sets of two-point crossover.

Keywords: Genetic algorithms, recombination, transit functions, oriented matroids, Vapnik-Chervonenkis dimension.

Math. Subj. Class.: 05C62, 05C75

1 Introduction

Genetic Algorithms, Evolutionary Algorithms, and Genetic Programming are heuristics commonly employed to solve complex optimization problems. A key component are crossover operators, which generate offsprings that are a mixture of two parents [16, 18, 22, 25]. Here we consider crossover operators on the set $X = \mathcal{A}^n$ strings with a fixed length n over some alphabet $\mathcal{A}$. A *k-mask* m is a binary string of length n with a most k break points between consecutive runs of 0s and 1s. That is, there are $0 \le h \le k < n$ "break points" $0 < i_1 < i_2 < \cdots < i_h < n$, such that (with $i_0 := 0$ and $i_{h+1} = n$) m satisfies $m_i = \mathtt{0}$ for $i_j < i \le i_{j+1}$ for even j and $m_i = \mathtt{1}$ for $i_j < i \le i_{j+1}$ for odd j. By definition, every k-mask starts with 0. For example, for $n = 15$ and $i_1 = 3$, $i_2 = 5$, $i_3 = 8$, $i_4 = 12$, we have the 4-mask

$$m = \mathtt{000110001111000}.$$

Note that m is also a k-mask for $4 \le k \le 15$. A k-mask thus is a binary string with *at most* $k + 1$ alternating runs of 0s and 1.

Definition 1.1. A string $z \in X$ is a k-point crossover offspring of $x, y \in X$ if there is k-mask m such that either $z_i = x_i$ if $m_i = \mathtt{0}$ and $z_i = y_i$ if $m_i = \mathtt{1}$ for $1 \le i \le n$, or $z_i = y_i$ if $m_i = \mathtt{0}$ and $z_i = x_i$ if $m_i = \mathtt{1}$ for $1 \le i \le n$.

For instance, given two parents x and y, as well as the 4-mask m, we obtain the two offsprings z_1 and z_2 as follows:

$$\begin{aligned} x &= \mathtt{++-++-++-++-+++} \\ y &= \mathtt{-+--++--++--+--} \\ m &= \mathtt{000110001111000} \\ z_1 &= \mathtt{++--+-++++--+++} \end{aligned} \qquad \begin{aligned} x &= \mathtt{++-++-++-++-+++} \\ y &= \mathtt{-+--++--++--+--} \\ m &= \mathtt{000110001111000} \\ z_2 &= \mathtt{-+-+++---++-+--} \end{aligned}$$

Intuitively, k-point crossover subdivides the parents x and y into *at most* $k + 1$ consecutive fragments that alternate in the offspring z. There is a rich literature on various aspects of k-point crossover operators. Algebraic properties are the focus of [7, 21, 24], disruption analysis is studied in [5], the relation between search spaces of crossover and mutation is discussed in [4, 23], coordinate transformation are explored in [8, 15]. The *recombination*

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sets $R_k(x,y)$ of possible crossover offsprings z of two parents x and y under k-point crossover. The function $R_k : X \times X \to 2^X$ satisfies, for all $x, y \in X$, (T1) $x, y \in R_k(x,y)$, (T2) $R_k(x,y) = R_k(y,x)$, and (T3) $R_k(x,x) = \{x\}$ [14]. These three axioms define *transit functions* [19], forming a common framework to describe intervals, convexities, and betweenness. In [3], we studied properties of the transit functions R_k deriving from k-point crossover. Convexity as a property of crossover operators is studied e.g. in [11, 12].

Here, we focus on the transit sets $R_k(x,y)$ themselves. Since $R_k(x,y)$ depends only on the positions in which x and y differ, it suffices to consider a two-letter alphabet $\mathcal{A} = \{+,-\}$ and thus $X = \{+,-\}^n$. We therefore interpret X as the vertex set of the n-dimensional Boolean Hypercube, and $R_k(x,y)$ as an induced subgraph of X. It is shown in [3, Cor. 4.2] that $R_k(x,y)$ is a *partial cube*, that is, an isometric subgraph of n-dimensional Boolean Hypercube [6].

The Hamming distance on X is the number $d(x,y)$ of positions in which x and y differ. Any two vertices x and y span a sub-hypercube $Q(x,y)$ of X with dimension $d(x,y)$, which coincides with the set of all crossover offsprings $R_k(x,y)$ whenever $d(x,y) \le k$. Otherwise, $R_k(x,y)$ is an induced subgraph of $Q(x,y)$. Its cardinality

$$|R_k(x,y)| = \begin{cases} 2^t & \text{if} \quad t \le k \\ 2\Phi_k(t-1) & \text{if} \quad t > k \end{cases} \tag{1.1}$$

depends only on the Hamming distance $t := d(x,y)$ and the parameter k [3, 14], where $\Phi_h(n) := \sum_{i=0}^{h} \binom{n}{i}$. In fact, the graphs $R_k(x,y)$ depend only on k and the Hamming distance $d(x,y)$:

Lemma 1.2. *Let $x, y \in \{+,-\}^n$ and $x', y' \in \{+,-\}^{n'}$. Then $R_k(x,y)$ and $R_k(x',y')$ are isomorphic if and only if $d(x,y) = d(x',y')$.*

Proof. Since every coordinate i for which $x_i = y_i$ is constant in $R_k(x,y)$ we know that $R_k(x,y)$ is an isometric subgraph of the subcube spanned by the $d := d(x,y)$ coordinates i with $x_i \neq y_i$. Relabeling the coordinates on $\{+,-\}^d$ is an isomorphism, hence $R_k(x,y)$ is isomorphic to $R_k(-^d, +^d)$, where $-^d$ and $+^d$ are the strings of length d with all coordinates being $-$ and $+$, respectively. Thus $R_k(x,y)$ and $R_k(x',y')$ are isomorphic if $d(x,y) = d(x',y')$. On the other hand, $R_k(-^d, +^d)$ and $R_k(-^{d'}, +^{d'})$ cannot be isomorphic if $d \neq d'$ since the diameter of the graphs differs. □

In this contribution, we show that the transit set of k-point forms the tope set of an uniform oriented matroid, which provides a means of gaining further insight into their structure and allows a characterization of the transit sets of two-point crossover.

2 VC-Dimension of Recombination Sets $R_k(x,y)$

The Vapnik-Chervonenkis dimension (VC-dimension) quantifies the complexity of set systems [26, 27]. Given some base set Y of cardinality $n := |Y|$, a family $\mathcal{H} \subseteq 2^Y$ forms an induced subgraph G of the Boolean hypercube $\{+,-\}^n$: for $A \in \mathcal{H}$, we identify $y \in A \subseteq Y$ with the y-coordinate of the corresponding point being $+$, while $y \notin A$ corresponds to $-$. A set $C \subseteq Y$ is said to be *shattered by* $\mathcal{H}$ if $\{Q \cap C | Q \in \mathcal{H}\} = 2^C$. The VC-dimension of $\mathcal{H}$ is the largest integer d_{VC} such that there is a set $C \subseteq Y$ of cardinality d_{VC} shattered by $\mathcal{H}$. By convention, $d_{VC} = -1$ for $\mathcal{H} = \emptyset$. Clearly, Y is always shattered by $\mathcal{H} = 2^Y$. Thus the VC-dimension of the Boolean hypercube $\{+,-\}^n$ itself is

n. Analogously, every subset $Y' \subseteq Y$ is shattered by $2^{Y'}$ and thus the VC-dimension of a sub-hypercube of dimensions $|Y| = n'$ is $d_{VC} = n'$.

As noted in [14], the 1-point crossover recombination set $R_1(x,y)$ is an isometrically embedded cycle C_{2t} for $t \geq 2$. It is not hard to check that $d_{VC} = 2$ in this case. For a partial cube G with d cuts the VC-dimension equals the dimension of the largest cube-minor in G, i.e., the largest cardinality of a set of coordinates shattered by the set of all d cuts of G. Here, a partial cube minor is either a contraction of cuts or the restriction to one of its sides, i.e., a specialization of the standard notion of graph minors [17]. Moreover, the cube-minor of a partial cube G is a graph isomorphic to a hypercube that can be obtained from G by a series of contractions and restrictions. Note that contractions can be seen as simply ignoring a coordinate.

Proposition 2.1. $d_{VC}(R_k(x,y)) = \begin{cases} k+1 & \textit{if} \quad d(x,y) > k \\ d(x,y) & \textit{if} \quad d(x,y) \leq k \end{cases}$

Proof. By Lemma 1.2 it suffices to consider $R_k(-^n, +^n)$. From the definition of k-point crossover it straightforwardly follows that $R_k(x,y) = \{+,-\}^n$, when $k = n-1$, since there is a break point between any two coordinates. Now suppose $k < n-1$. If the break points are consecutive, i.e., $i_j = j$ for $1 \leq j \leq k$, then $R_k(x,y)$ induces $\{+,-\}^{k+1}$ on the first $k+1$ coordinates. The same holds if the break points are not consecutive and we contract consecutive coordinates j and $j+1$ that do not have a break point between them. On the other hand, with k break points we can only "crossover" at most $k+1$ coordinates, whence $d_{VC}(R_k(x,y)) \leq k+1$. □

3 Oriented matroids and 2-point recombination sets

Oriented matroids [1] are an axiomatic abstraction of geometric and topological structures including convex polytopes, vector configurations, (pseudo)hyperplane arrangements, point configurations in the Euclidean space, directed graphs, and linear programs. They reflect properties such as linear dependencies, facial relationship, convexity, duality, and have bearing on solutions of associated optimization problems. Among several equivalent axiomatizations of oriented matroids, the face or covector axioms best captures the geometric flavour and thus is the most convenient one for our purposes.

Let E be a finite set. A *sign vector* X on E is a vector $(X_e : e \in E)$ with coordinates $X_e \in \{+,0,-\}$. The support of a sign vector X is the set $\underline{X} = \{e \in E \,|\, X_e \neq 0\}$. The *composition* $X \circ Y$ of two sign vectors X and Y is defined coordinate-wisely as $(X \circ Y)_e = X_e$, if $X_e \neq 0$, and $(X \circ Y)_e = Y_e$ otherwise. Their difference set is $D(X,Y) = \{e \in E \,|\, X_e = -Y_e\}$. We denote by $\leq$ the product (partial) order on $\{-,0,+\}^E$ implied by the standard ordering $- < 0 < +$ of signs.

An *oriented matroid* M is ordered pair $(E, \mathcal{F})$ of a finite set E and a set of *covectors* $\mathcal{F} \subseteq \{+,-,0\}^E$ satisfying, for all $X, Y \in \mathcal{F}$, the following (face or covector) axioms:

(F0) $\mathbf{0} = (0,0,\ldots,0) \in \mathcal{F}$.

(F1) $-X \in \mathcal{F}$.

(F2) $X \circ Y \in \mathcal{F}$.

(F3) There is $Z \in \mathcal{F}$ with $Z_e = 0$ for $e \in D(X,Y)$ and $Z_f = (X \circ Y)_f$ for $f \in E \setminus D(X,Y)$.

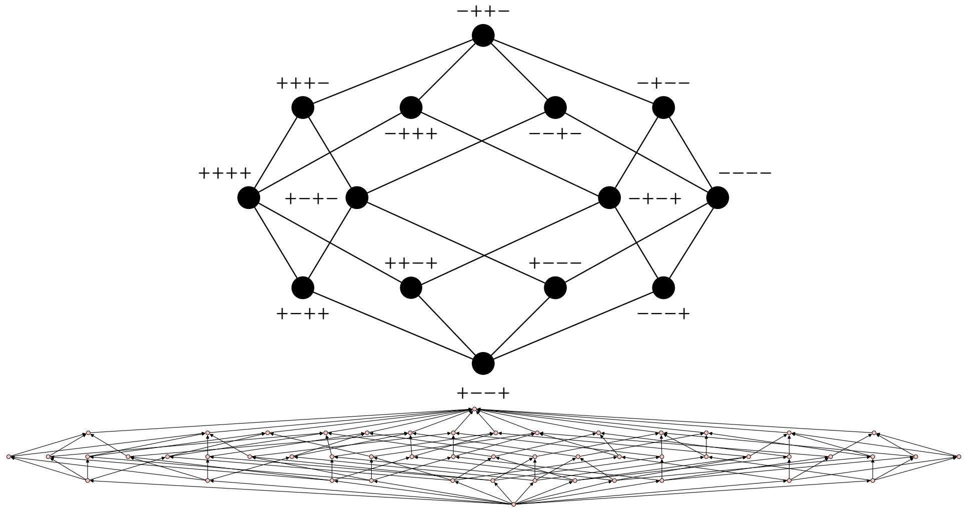

Figure 1: The rhombododecahedral graph $R_2(----, ++++)$ (top) with the binary labeling corresponding to the isometric embedding into 4-dimensional hypercube. Below we show its big face lattice generated using SageMath (`www.sagemath.org`).

Consider a subspace $V \subseteq \mathbb{R}^{|E|}$, define, for every $v \in V$, its sign vector $s(v)$ coordinatewise by $s_e(v) = \mathrm{sgn}(v_e)$ for all $e \in E$, and denote by $\mathcal{F}$ the set of all sign vectors of V. Oriented matroids obtained from a vector space in this manner are called *representable* or *linear*.

The set $\mathcal{C} \subset \mathcal{F}$ of *cocircuits* or *vertices* of M consists of the non-zero covectors that are minimal with respect to the partial order $\leq$. The set $\mathcal{T} \subset \mathcal{F}$ of *topes* of M comprises the covectors that are maximal with respect to $\leq$. The cocircuits determine the set of covectors: every covector $X \in \mathcal{F} \setminus \{0\}$ has a representation of the form $X = V_1 \circ V_2 \circ \ldots \circ V_k$, where $V_1, V_2, \ldots V_k$ are cocircuits, and $V_1, V_2, \ldots V_k \leq X$. Similarly, the topes determine the oriented matroid: $\mathcal{F} = \{X \in \{+, -, 0\}^E \mid \forall T \in \mathcal{T} : X \circ T \in \mathcal{T}\}$.

$M = (E, \mathcal{F})$ is *uniform* of rank r if $|\underline{X}| = r + 1$ for all cocircuits. The *big face lattice* $\widehat{\mathcal{F}}$ is a lattice obtained by adding the unique maximal element $\widehat{1}$ to the partial order $\leq$ on $\mathcal{F}$. The rank of a covector X is defined as its height in $\widehat{\mathcal{F}}$. The rank $\mathrm{rk}(M)$ of M is the maximal rank of its covectors. The corank of M is $|E| - \mathrm{rk}(M)$.

As an example consider $R_2(x, y)$ with $d(x, y) = 4$. It can be verified that the elements of $R_2(----, ++++)$ are exactly the topes of the oriented matroid corresponding to the Rhombododecahedron. It is shown together with its big face lattice in Figure 1. This observation can be generalized with the help of the following result:

Proposition 3.1 ([13]). *A set $T \subseteq \{+, -\}^X$ of VC-dimension d is the set of topes of a uniform oriented matroid M on X if and only if $T = -T$ and $|T| = 2\Phi_{d-1}(|X| - 1)$.*

By Proposition 2.1, Equ.(1.1), and Theorem 3.1, this immediately implies

Theorem 3.2. *For $x, y \in \{+, -\}^X$, with $d(x, y) = |X| = n$ the elements of $R_k(x, y)$ form the set of topes of a uniform oriented matroid M on X with VC-dimension $d_{VC} = \mathrm{rk}(M) = k + 1$.*

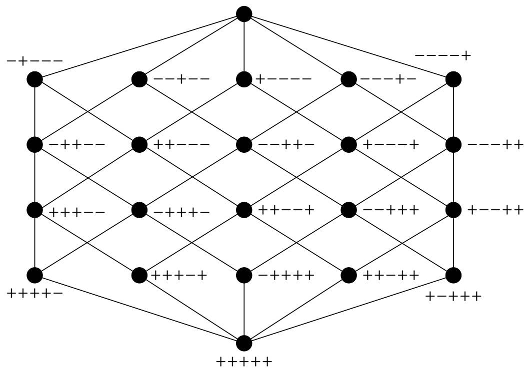

Figure 2: The transit graph $R_2(-----,+++++)$.

Since many of the known results on oriented matroids depend on the corank, we note that $R_k(x,y)$ has corank $n-k-1$.

One of the cornerstones of the theory of oriented matroids is the Topological Representation Theorem, which connects oriented matroids with pseudosphere arrangements, see Appendix A for detailed definitions. Together with Theorem 3.2, it immediately implies the following topological characterization of the recombination sets of k-point crossover:

Theorem 3.3. *For $x,y \in \{+,-\}^X$, with $d(x,y) = |X| = n$, the recombination set $R_k(x,y)$ can be topologically represented by a pseudosphere arrangement of dimension k, where the minimal elements in the big face lattice correspond to the intersections of exactly k pseudospheres, and there are $2\binom{n}{k-1}$ such intersections.*

The significance of this result is that it provides a representation of crossover operators in terms of topological objects. As an illustration of the usefulness of Theorem 3.3, we now turn to a full characterization of the transit graphs of 2-point crossover operators. The smallest non-trivial examples are the graphs $R_2(----,++++)$ in Figure 1 and $R_2(-----,+++++)$ in Figure 2.

Theorem 3.4. *$R_2(a,b)$ with $d(a,b) = t > 3$ induces antipodal planar quandrangulation, that is, a partial cube of diameter t with t^2-t+2 vertices, $2t^2-2t$ edges, t^2-t quadrangles, and all cuts of size $2t-2$.*

Proof. Let $|V|$, $|E|$, $|Q|$ and $|C|$ denote the number of vertices, edges, 4-faces, and edges of a cut, respectively. From the definition of crossover operator, we can arbitrarily permute coordinates, hence it follows that each cut has the same number of edges, this justifies that we study $|C|$. From Theorem 3.2 it follows that vertices of $R_2(a,b)$ form the set of topes of a uniform oriented matroid of rank 3 and corank $t-3$. As shown by [10] and in the book by [1], rank 3 oriented matroids can be represented by pseudocircle arrangement on $\mathbb{S}^2$. The corresponding tope graph is therefore planar. Hence $R_2(a,b)$ induces in particular a planar antipodal partial cube. Corank $t-3$ implies that each intersection of pseudocurves is the intersection of exactly two of them. Hence all faces of the dual – the tope graph – are 4-cycles, therefore $R_2(a,b)$ induces planar quadrangulation. Moreover, each intersection

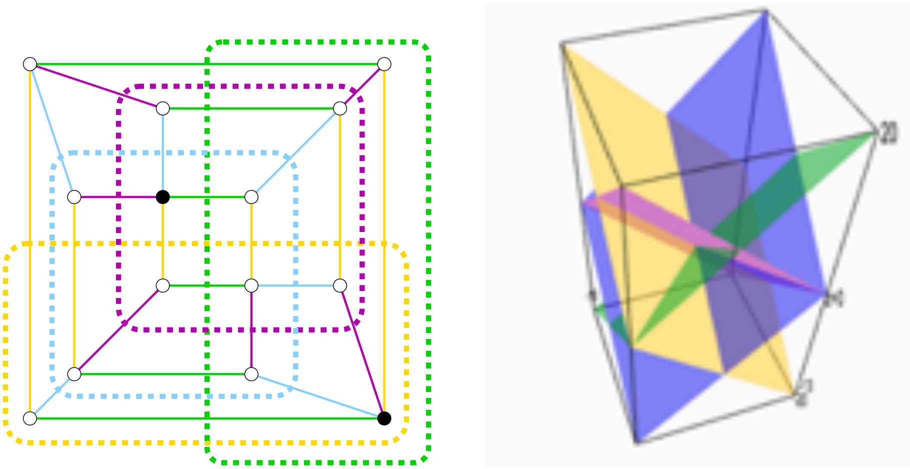

Figure 3: Topological representation of rhombododecahedron (l.h.s.) in terms of its pseudocircle arrangement (doted curves) and the corresponding hyperplane arrangement (r.h.s.).

of two pseudocircles corresponds to cocircuit. In uniform oriented matroid of corank $t-3$ there are exactly $2\binom{t}{t-2}$ cocircuits, which correspond to the 4-cycles in the dual graph.

Quadrangulations are maximal planar bipartite graphs – no edge can be added so that graph remains planar and bipartite. Using Euler formula for planar graphs [20], we obtain $|E| = 2|V| - 4$. Equ.(1.1) furthermore, implies $|E| = 2t^2 - 2t$ and thus $|C| = |E|/t = 2t - 2$. □

As an example, Figure 3 shows the pseudocircle arrangement and the equivalent hyperplane arrangement of transit graph $R_2(----, ++++)$ of Figure 1.

In order to get a better intuition on the structure of the 2-point crossover graphs we derive their degree sequence.

Theorem 3.5. *The degree sequence of* $R_2(a, b)$ *with* $t := d(a, b) > 3$ *equals* $(t, t, 4, \dots, 4, 3, \dots, 3)$ *with* $t^2 - 3t$ *vertices of degree* 4 *and* $2t$ *vertices of degree* 3.

Proof. W.l.o.g., let $a = 0\dots0$ and $b = 1\dots1$. For any vertex $c = x\dots xyx\dots x$, $x, y \in \{0, 1\}$ we have that $c \in R_2(a, b)$, hence $deg(a) = deg(b) = t$. Let $c \in R_2(a, b) \setminus \{a, b\}$. Then we have two cases:

Case 1. $c = xx\dots xxyy\dots yy$ and $\{x, y\} = \{0, 1\}$. Then c has at most four neighbors in $R_2(a, b)$: $c_1 = yx\dots xxyy\dots yy$, $c_2 = xx\dots xxyy\dots yx$, $c_3 = xx\dots xyyy\dots yy$ and $c_4 = xx\dots xxxy\dots yy$. Since $t > 3$ it follows that c also has at least three neighbors in $R_2(a, b)$.

Case 2. $c = x\dots xxyy\dots yyxx\dots x$ and $\{x, y\} = \{0, 1\}$. Then c has at most four neighbors in $R_2(a, b)$: $c_1 = x\dots xxxy\dots yyxx\dots x$, $c_2 = x\dots xyyy\dots yyxx\dots x$, $c_3 = x\dots xxyy\dots yxxx\dots x$, and $c_4 = x\dots xxyy\dots yyyx\dots x$. Since $t > 3$ it follows that c also has at least three neighbors in $R_2(a, b)$.

Let x_3 and x_4 denote the number of vertices of degree 3 and 4 respectively. By the handshaking lemma $2|E| = \sum_{v \in V(G)} deg(v)$. Therefore, it follows from arguments above and Theorem 3.4 that

$$4t^2 - 4t = 2t + \sum_{v \in V(G)\setminus\{a,b\}} deg(v)$$
$$4t^2 - 6t = 3x_3 + 4x_4$$

Theorem 3.4 also implies that $t^2 - t = x_3 + x_4$. Solving this system of linear equations yields $x_3 = 2t$ and $x_4 = t^2 - 3t$. □

4 Concluding remarks

The recombination sets of 1-point crossover operators form isometric cycles in hypercube. The partial cubes corresponding to k-point crossover operators have a VC-dimension of $k+1$ unless they are smaller sub-hypercubes. We have considered here the uniform oriented matroids that correspond to the k-point crossover operators and used this connection to characterize the partial cubes of 2-point recombination sets. It remains an open question for future research whether the connection with oriented matroids and their topological representations can be utilized to better understand the structure of k-point recombination graphs.

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Appendix A: Pseudosphere arrangements

Consider the d-dimensional sphere $\mathbb{S}^d$ in $\mathbb{R}^{d+1}$ and the corresponding $(d+1)$-dimensional ball $\mathbb{B}^{d+1} = \{(x_1, \ldots, x_{d+1}) \in \mathbb{R}^{d+1} \,|\, x_1^2 + \ldots + x_{d+1}^2 \leq 1\}$, whose boundary surface is $\mathbb{S}^d$.

A pseudosphere $S \subset \mathbb{S}^d$ is a tame embedded $(d-1)$-dimensional sphere. Its complement in B^d consist of exactly two regions, hence S can be oriented, by labeling one region by S_e^+ and the other by S_e^-. A pseudosphere arrangement $\mathcal{S} = \{S_e \,|\, e \in E\}$ in the Euclidean space $\mathbb{R}^d$ is a collection of $(d-1)$-dimensional pseudospheres on the d-dimensional unit sphere $\mathbb{S}^d$, where the intersection of any number of spheres is again a sphere and the intersection of an arbitrary collection of closed sides is either a sphere or a ball, i.e., for all $R \subset E$ holds

(i) $S_R = \mathbb{S}^d \cap_{i \in R} S_i$ is empty or homeomorphic to a sphere.

(ii) If $e \in E$ and $S_R \not\subset S_e$ then $S_R \cap S_e$ is a pseudosphere in S_R, $S_R \cap S_e^+ \neq \emptyset$ and $S_R \cap S_e^- \neq \emptyset$.

For a pseudosphere arrangement $\mathcal{S}$, the position vector $\sigma(x)$ of a point $x \in \mathbb{S}^d$ is defined as $\sigma(x)_e = 0$ for $x \in S_e$, $\sigma(x)_e = +$ for $x \in S_e^+$ and $\sigma(x)_e = -$ for $x \in S_e^-$. The set of all position vectors of $\mathcal{S}$ is denoted by $\sigma(\mathcal{S})$. A famous theorem due to [9] establishes an correspondence between oriented matroids and pseudosphere arrangement.

Topological Representation Theorem ([2, 9]). *Let $M = (E, \mathcal{F})$ be an oriented matroid of rank d. Then there exists a pseudosphere arrangement $\mathcal{S}$ in $\mathbb{S}^d$ such that $\sigma(\mathcal{S}) = \mathcal{F}$. Conversely, if $\mathcal{S}$ is a pseudosphere arrangement in $\mathbb{S}^d$, then $(E, \sigma(\mathcal{S}))$ is an oriented matroid of rank d.*

A pseudosphere arrangement naturally induces a cell complex on $\mathbb{S}^d$, whose partial order of faces corresponds precisely to the partial order $\leq$ on covectors of the corresponding oriented matroid. This fact served as motivation for the concept of covectors in the theory of oriented matroids.

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On the equivalence between a conjecture of Babai-Godsil and a conjecture of Xu concerning the enumeration of Cayley graphs

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Abstract

In this paper we show that two distinct conjectures, the first proposed by Babai and Godsil in 1982 and the second proposed by Xu in 1998, concerning the asymptotic enumeration of Cayley graphs are in fact equivalent. This result follows from a more general theorem concerning the asymptotic enumeration of a certain family of Cayley graphs.

Keywords: Regular representation, Cayley graph, automorphism group, asymptotic enumeration, graphical regular representation, GRR, normal Cayley graph, Babai-Godsil conjecture, Xu conjecture.

Math. Subj. Class.: 05C25, 05C30, 20B25, 20B15

1 Introduction

All digraphs and groups considered in this paper are finite. A ***digraph*** Γ is an ordered pair (V, A) where the ***vertex-set*** V is a finite non-empty set and the ***arc-set*** $A \subseteq V \times V$ is a binary relation on V. The elements of V and A are called ***vertices*** and ***arcs*** of Γ, respectively. An ***automorphism*** of Γ is a permutation σ of V with $A^\sigma = A$, that is, $(x^\sigma, y^\sigma) \in A$ for every $(x, y) \in A$. Let R be a group and let S be a subset of R. The ***Cayley digraph*** on R with connection set S (which we denote by $\Gamma(R, S)$) is the digraph with vertex-set R and with (g, h) being an arc if and only if $hg^{-1} \in S$. The group R acts regularly as a group of automorphisms of $\Gamma(R, S)$ by right multiplication and hence $R \leq \mathrm{Aut}(\Gamma(R, S))$. When $R = \mathrm{Aut}(\Gamma(R, S))$, the digraph Γ is called a ***DRR*** (for digraphical regular representation). Babai and Godsil made the following conjecture.

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Conjecture 1.1 ([6, Conjecture 3.13], [2])**.** *Let R be a group of order r. The proportion of subsets S of R such that $\Gamma(R,S)$ is a* DRR *goes to* 1 *as $r \to \infty$. More precisely,*

$$\lim_{r\to\infty} \min\left\{\frac{|\{S \subseteq R : \mathrm{Aut}(\Gamma(R,S)) = R\}|}{2^r} : |R| = r\right\} = 1.$$

This conjecture has been recently proved in [12].

This paper is the first step for proving yet another conjecture of Babai and Godsil concerning the enumeration of ***Cayley graphs***. A Cayley graph over R is a Cayley digraph $\Gamma(R,S)$ whose binary relation $\{(g,h) \in R\times R \mid gh^{-1} \in S\}$ defining the arc-set of $\Gamma(R,S)$ is symmetric. (Incidentally, the binary relation $\{(g,h) \in R\times R \mid gh^{-1} \in S\}$ is riflexive if and only if S contains the identity element of G.) In terms of the connection set $S \subseteq R$, $\Gamma(R,S)$ is a Cayley graph if and only if $S = S^{-1}$, where $S^{-1} := \{s^{-1} \mid s \in S\}$. Given a subset S of R, we say that S is ***inverse-closed*** if $S = S^{-1}$, that is, $\Gamma(R,S)$ is undirected, which in turn means that $\Gamma(R,S)$ is a Cayley graph. When $R = \mathrm{Aut}(\Gamma(R,S))$ and S is inverse-closed, the graph Γ is called a ***GRR*** (for graphical regular representation).

While the number of Cayley digraphs on R is $2^{|R|}$, which is a number that depends on the cardinality of R only, the number of undirected Cayley graphs on R is $2^{\frac{|R|+|\mathbf{I}(R)|}{2}}$ (see Lemma 2.2), where $\mathbf{I}(R) := \{\iota \in R \mid \iota^2 = 1\}$, and hence depends on the algebraic structure of R.

Although the difference between Cayley digraphs and Cayley graphs seems only minor and to some extent only aesthetic, the behaviour between these two classes of combinatorial objects with respect to their automorphisms can be dramatically different. For instance, it was proved by Babai [1, Theorem 2.1] that, except for

$$Q_8,\ C_2\times C_2,\ C_2\times C_2\times C_2,\ C_2\times C_2\times C_2\times C_2 \text{ and } C_3\times C_3,$$

every finite group R admits a DRR. Borrowing a phrase which I once heard from Tom Tucker: "Besides some low level noise, every finite group admits a DRR". The analogue for GRRs is not the same. Indeed, it turns out that there are two (and only two) infinite families of groups that do no admit GRRs. The first family consists of abelian groups of exponent greater than two. If R is such a group and ι is the automorphism of R mapping every element to its inverse, then every Cayley graph on R admits $R \rtimes \langle\iota\rangle$ as a group of automorphisms. Since R has exponent greater than 2, $\iota \neq 1$ and hence no Cayley graph on R is a GRR. The other family of groups that do not admit GRRs are the generalised dicyclic groups, see [14, Definition 1.1] for a definition and also Definition 2.4 below. These two families were discovered by Mark Watkins [19].

It was proved by Godsil [5] that abelian groups of exponent greater than 2 and generalised dicyclic groups are the only two infinite families of groups that do not admit GRRs. (A lot of papers have been published for determining those groups admitting a GRR, and some of the most influential works along the way appeared in [7, 8, 9, 15, 16, 20].) The stronger Conjecture 1.2 was made (at various times) by Babai, Godsil, Imrich and Lovász.

Conjecture 1.2 (see [2, Conjecture 2.1] and [6, Conjecture 3.13])**.** *Let R be a group of order r which is neither generalized dicyclic nor abelian of exponent greater than* 2*. The proportion of inverse-closed subsets S of R such that $\Gamma(R,S)$ is a* GRR *goes to* 1 *as $r \to \infty$. More precisely,*

$$\lim_{r\to\infty} \min\left\{\frac{|\{S \subseteq R : \mathrm{Aut}(\Gamma(R,S)) = R\}|}{2^{\mathbf{c}(R)}} : R \text{ admits a GRR and } |R| = r\right\} = 1.$$

This conjecture is open at the moment and some of the techniques developed in [12] for dealing with digraphs are not suited for dealing with undirected graphs.

The scope of this paper is twofold. Broadly speaking, we aim to start a long process where we try to generalize and adapt the results obtained in [12] for eventually dealing with undirected graphs and proving Conjecture 1.2. Given an inverse-closed subset S of R, we let $A := \mathrm{Aut}(\Gamma(R,S))$. Now, the set S fails to give rise to a GRR essentially for two different reasons.

1. There are non-identity group automorphisms of R leaving the set S invariant. This case arises when $\mathbf{N}_A(R) > R$ (this is the typical obstruction and we have encountered this obstruction already when we briefly discussed abelian groups of exponent greater than 2).
2. The only group automorphism of R leaving the set S invariant is the identity and there are some automorphisms of $\Gamma(R,S)$ not lying in R. This case arises when $\mathbf{N}_A(R) = R$ and $A > R$: this obstruction is somehow mysterious and much harder to analyze.

These two obstructions are clear (if not obvious) to readers familiar with the enumeration problem of Cayley graphs [12] and in particular to readers familiar with [2]. Actually the same obstructions arise in the enumeration problem of other types of Cayley graphs, for instance in the asymptotic enumeration of DFRs [17] and GFRs [4, 18] and in the recent solution of the GFR conjecture [18]. We start this process by dealing with the ***first natural obstruction*** for the existence of GRRs.[1]

Theorem 1.3. *Let R be a group of order r which is neither generalized dicyclic nor abelian of exponent greater than* 2. *The proportion of inverse-closed subsets S of R such that $\mathbf{N}_{\mathrm{Aut}(\Gamma(R,S))}(R) > R$ goes to* 0 *as $r \to \infty$.*

We observe that in Proposition 2.9 we have a quantified version of Theorem 1.3. Moreover, in Lemma 2.8 we have a more technical version of Theorem 1.3 which includes also generalized dicyclic groups and abelian groups of exponent greater than 2. These two more technical results are in our opinion needed to follow the footsteps of the argument in [12] for the asymptotic enumeration of Cayley digraphs.

The second scope of this paper is to prove that a famous conjecture of Xu on the asymptotic enumeration of normal Cayley graphs is actually equivalent to Conjecture 1.2. A Cayley (di)graph Γ on R is said to be a ***normal Cayley (di)graph on*** R if the regular representation of R is normal in $\mathrm{Aut}(\Gamma)$, that is, $R \trianglelefteq \mathrm{Aut}(\Gamma)$. Clearly, every DRR and every GRR Γ on R is a normal Cayley (di)graph because $R = \mathrm{Aut}(\Gamma)$. Xu has conjectured that almost all Cayley (di)graphs on R are normal Cayley (di)graphs on R. Indeed, Xu in [21] has posed the following two conjectures.

Conjecture 1.4 (see [21])**.** *The minimum, over all groups R of order r, of the proportion of subsets S of R such that $\Gamma(R,S)$ is a normal Cayley graph tends to* 1 *as $r \to \infty$. More precisely,*

$$\lim_{r\to\infty} \min \left\{ \frac{|\{S \subseteq R : R \trianglelefteq \mathrm{Aut}(\Gamma(R,S))\}|}{2^r} : |R| = r \right\} = 1.$$

[1] During the refereeing process of this paper a substantial step towards the second obstruction has been obtained in [13]

Conjecture 1.5 (see [21]). *The minimum, over all groups R of order r, of the proportion of inverse-closed subsets S of R such that $\Gamma(R,S)$ is a normal Cayley graph tends to 1 as $r \to \infty$. More precisely,*

$$\lim_{r\to\infty} \min\left\{\frac{|\{S \subseteq R : R \trianglelefteq \mathrm{Aut}(\Gamma(R,S))\}|}{2^{\mathbf{c}(R)}} : |R| = r\right\} = 1.$$

Conjecture 1.4 was shown to be true in [12] by proving the stronger Conjecture 1.1. The veracity of Conjecture 1.5 when R is an abelian group and when R is a dicyclic group was proved in [3, 14]. In this paper we show that Conjecture 1.2 and Conjecture 1.5 are actually equivalent.

Theorem 1.6. *Conjecture 1.2 holds true if and only if Conjecture 1.5 holds true.*

2 Group automorphisms

Definition 2.1. Given a finite group R and $x \in R$, we let $o(x)$ denote the order of the element x and we let $\mathbf{I}(R) := \{x \in R \mid o(x) \leq 2\}$ be the set of elements of R having order at most 2. We let $\mathbf{c}(R)$ denote the fraction $(|R| + |\mathbf{I}(R)|)/2$, that is,

$$\mathbf{c}(R) = \frac{|R| + |\mathbf{I}(R)|}{2}.$$

Given a subset X of R, we write $\mathbf{I}(X) := X \cap \mathbf{I}(R)$. Finally, we denote by $\mathbf{Z}(R)$ the centre of R.

Lemma 2.2. *Let R be a finite group. The number of inverse-closed subsets S of R is $2^{\mathbf{c}(R)}$.*

Proof. Given an arbitrary inverse-closed subset S of R, $S \cap \mathbf{I}(R)$ is an arbitrary subset of $\mathbf{I}(R)$ whereas in $S \cap (R \setminus \mathbf{I}(R))$ the elements come in pairs, where each element is paired up to its inverse. Thus the number of inverse-closed subsets of R is

$$2^{|\mathbf{I}(R)|} \cdot 2^{\frac{|R\setminus \mathbf{I}(R)|}{2}} = 2^{\mathbf{c}(R)}. \qquad \square$$

Definition 2.3. Let R be a finite group. Given an automorphism φ of R, we set

$$\mathbf{C}_R(\varphi) := \{x \in R \mid x^\varphi = x\},$$
$$\mathbf{C}_R(\varphi)^{\mathrm{inv}} := \{x \in R \mid x^\varphi = x^{-1}\}.$$

Observe that, when $\varphi = id_R$ is the identity automorphism of R, $\mathbf{C}_R(\varphi)^{\mathrm{inv}} = \mathbf{I}(R)$.

Given $x \in R$, we denote by $\iota_x : R \to R$ the inner automorphism of R induced by x, that is, $m^{\iota_x} = xmx^{-1}$, for every $m \in R$. (Usually, the automorphism ι_x is defined by $m \mapsto m^{\iota_x} = x^{-1}mx$, however for our application it is more convenient to define ι_x by $m \mapsto m^{\iota_x} = xmx^{-1}$.) When $A \trianglelefteq R$, we still denote by ι_x the restriction to A of the automorphism ι_x, this makes the notation not too cumbersome to use and hopefully will cause no confusion.

Finally, we let $\iota : R \to R$ be the permutation defined by $x^\iota = x^{-1}$, for every $x \in R$. In particular, when R is abelian, ι is an automorphism of R. Furthermore, $\iota = id_R$ if and only if R is an abelian group of exponent at most 2.

Definition 2.4. Let A be an abelian group of even order and of exponent greater than 2, and let y be an involution of A. The *generalised dicyclic group* $\mathrm{Dic}(A, y, x)$ is the group $\langle A, x \mid x^2 = y, a^x = a^{-1}, \forall a \in A\rangle$. A group is called *generalised dicyclic* if it is isomorphic to some $\mathrm{Dic}(A, y, x)$. When A is cyclic, $\mathrm{Dic}(A, y, x)$ is called a *dicyclic* or *generalised quaternion group*.

We let $\bar{\iota}_A : \mathrm{Dic}(A, y, x) \to \mathrm{Dic}(A, y, x)$ be the mapping defined by $(ax)^{\bar{\iota}_A} = ax^{-1}$ and $a^{\bar{\iota}_A} = a$, for every $a \in A$. In particular, $\bar{\iota}_A$ is an automorphism of $\mathrm{Dic}(A, y, x)$. The role of the label "A" in $\bar{\iota}_A$ seems unnecessary, however we use this label to stress one important fact. An abstract group R might be isomorphic to $\mathrm{Dic}(A, y, x)$, for various choices of A. Therefore, since the automorphism $\bar{\iota}_A$ depends on A and since we might have more than one choice of A, we prefer a notation that emphasizes this fact.

Lemma 2.5. *Let R be a finite group and let φ be an automorphism of R with $|R : \mathbf{C}_R(\varphi)| = 2$. Then one of the following holds:*

1. $\frac{1}{4}(|R| + |\mathbf{I}(R)| + |\mathbf{C}_R(\varphi)| + |\mathbf{C}_R(\varphi)^{\mathrm{inv}}|) \le \mathbf{c}(R) - \frac{|R|}{32}$,
2. *R is generalized dicyclic over the abelian group $\mathbf{C}_R(\varphi)$ and $\varphi = \bar{\iota}_{\mathbf{C}_R(\varphi)}$,*
3. *R is abelian of exponent greater than 2 and $\varphi = \iota$.*

Proof. For simplicity, we let $A := \mathbf{C}_R(\varphi)$ and we let o denote the left-hand side in (1).

Suppose that $\mathbf{C}_R(\varphi)^{\mathrm{inv}} \subseteq A$. Then

$$\mathbf{C}_R(\varphi)^{\mathrm{inv}} = \mathbf{C}_R(\varphi)^{\mathrm{inv}} \cap A = \{a \in A \mid a^\varphi = a^{-1}\} = \mathbf{C}_A(\varphi)^{\mathrm{inv}}.$$

Since $A = \mathbf{C}_R(\varphi)$, we have $a^\varphi = a$ for every $a \in A$ and hence

$$\mathbf{C}_R(\varphi)^{\mathrm{inv}} = \mathbf{C}_A(\varphi)^{\mathrm{inv}} = \mathbf{C}_A(id_A)^{\mathrm{inv}}.$$

Clearly, $a \in \mathbf{C}_A(id_A)^{\mathrm{inv}}$ if and only if $a = a^{-1}$, that is, $a \in \mathbf{I}(A)$. Therefore

$$\mathbf{C}_R(\varphi)^{\mathrm{inv}} = \mathbf{C}_A(\varphi)^{\mathrm{inv}} = \mathbf{C}_A(id_A)^{\mathrm{inv}} = \mathbf{I}(A).$$

Thus

$$\begin{aligned}
o &= \frac{1}{4}\left(|R| + |\mathbf{I}(R)| + \frac{|R|}{2} + |\mathbf{I}(A)|\right)\\
&\le \frac{1}{4}\left(\frac{3}{2}|R| + 2|\mathbf{I}(R)|\right)\\
&= \frac{|R| + |\mathbf{I}(R)|}{2} - \frac{|R|}{8}\\
&= \mathbf{c}(R) - \frac{|R|}{8},
\end{aligned}$$

and (1) holds in this case.

Suppose that $\mathbf{C}_R(\varphi)^{\mathrm{inv}} \not\subseteq A$. In particular, there exists $x \in R \setminus A$ with $x^\varphi = x^{-1}$. As $|R : A| = 2$, we have $R = A \cup Ax$. For every $a \in A$, since φ is an automorphism of R fixing point-wise A and since $xax^{-1} \in A$, we deduce

$$xax^{-1} = (xax^{-1})^\varphi = x^\varphi a^\varphi (x^{-1})^\varphi = x^{-1}ax$$

and hence $x^2a = ax^2$, that is, $x^2 \in \mathbf{Z}(\langle A, x\rangle) = \mathbf{Z}(R)$. As $x^2 \in A$, we have $x^2 = (x^2)^\varphi = (x^\varphi)^2 = (x^{-1})^2 = x^{-2}$, that is, $x^4 = 1$. Summing up,

$$x^2 \in \mathbf{Z}(R), \quad x^4 = 1. \tag{2.1}$$

Now, let $y \in \mathbf{C}_R(\varphi)^{\mathrm{inv}} \setminus A$. Then, $y = ax$, for some $a \in A$. Moreover, $y^\varphi = y^{-1} = (ax)^{-1} = x^{-1}a^{-1}$ and $y^\varphi = (ax)^\varphi = a^\varphi x^\varphi = ax^{-1}$. Thus

$$x^{-1}a^{-1} = ax^{-1},$$

that is, $xax^{-1} = a^{-1}$. Recall that $\iota_x : A \to A$ is the restriction to the normal subgroup A of the inner automorphism of R determined by x, that is, $a^{\iota_x} = xax^{-1}$, for every $a \in A$. We have shown that $\mathbf{C}_R(\varphi)^{\mathrm{inv}} \setminus A = \mathbf{C}_A(\iota_x)^{\mathrm{inv}}x$. As $\mathbf{C}_R(\varphi)^{\mathrm{inv}} \cap A = \mathbf{I}(A)$, we get

$$\mathbf{C}_R(\varphi)^{\mathrm{inv}} = \mathbf{I}(A) \cup \mathbf{C}_A(\iota_x)^{\mathrm{inv}}x \tag{2.2}$$

and $|\mathbf{C}_R(\varphi)^{\mathrm{inv}}| = |\mathbf{I}(A)| + |\mathbf{C}_A(\iota_x)^{\mathrm{inv}}|$.

Suppose that $|\mathbf{C}_A(\iota_x)^{\mathrm{inv}}| \le 3|A|/4$. Thus, by (2.2), we have

$$\begin{aligned}
o &= \frac{1}{4}\left(\frac{3}{2}|R| + |\mathbf{I}(R)| + |\mathbf{I}(A)| + |\mathbf{C}_A(\iota_x)^{\mathrm{inv}}|\right) \\
&\le \frac{1}{4}\left(\frac{3}{2}|R| + 2|\mathbf{I}(R)| + \frac{3|A|}{4}\right) \\
&= \frac{1}{4}\left(\frac{3}{2}|R| + 2|\mathbf{I}(R)| + \frac{3|R|}{8}\right) \\
&= \frac{|R| + |\mathbf{I}(R)|}{2} - \frac{|R|}{32} \\
&= \mathbf{c}(R) - \frac{|R|}{32},
\end{aligned}$$

and (1) holds in this case.

Suppose that $|\mathbf{C}_A(\iota_x)^{\mathrm{inv}}| > 3|A|/4$, that is, the automorphism ι_x of A inverts more than $3/4$ of its elements. By a result of Miller [11], A is abelian. Since A is abelian, it is easy to verify that $\mathbf{C}_A(\iota_x)^{\mathrm{inv}}$ is a subgroup of A. As $|\mathbf{C}_A(\iota_x)^{\mathrm{inv}}| > 3|A|/4$, we get $\mathbf{C}_A(\iota_x)^{\mathrm{inv}} = A$ and ι_x acts on A inverting each of its elements. From (2.2), we have

$$\mathbf{C}_R(\varphi)^{\mathrm{inv}} = \mathbf{I}(A) \cup Ax. \tag{2.3}$$

If $\mathbf{I}(R) \subseteq \mathbf{I}(A)$, then no element in Ax is an involution and hence x has order 4 from (2.1). When A has exponent greater than 2, we deduce $R \cong \mathrm{Dic}(A, x^2, x)$ is a generalized dicyclic group over A, $\varphi = \bar{\iota}_A$ and (2) holds in this case. When A has exponent at most 2, we have $\mathbf{I}(A) = A$ and $\varphi = \iota$. Hence $\mathbf{I}(R) = A$, R is an abelian group of exponent greater than 2 and (3) holds in this case. Therefore, we may suppose $\mathbf{I}(R) \nsubseteq \mathbf{I}(A)$.

Let $x' \in \mathbf{I}(R) \setminus A$. Then, $x' = ax$, for some $a \in A$. Then $1 = x'^2 = (ax)^2 = axax = a(xax^{-1})x^2 = aa^{-1}x^2 = x^2$ and hence $x^2 = 1$. Now, for every $b \in A$, we have $(bx)^2 = bxbx = b(xbx^{-1}) = bb^{-1} = 1$. This shows $\mathbf{I}(R) \setminus A = Ax$. Therefore,

$\mathbf{I}(R) = \mathbf{I}(A) \cup Ax$ and hence $\mathbf{I}(R) = \mathbf{C}_R(\varphi)^{\mathrm{inv}}$ from (2.3). We deduce

$$\begin{aligned} o &= \frac{1}{4}\left(\frac{3|R|}{4} + |\mathbf{I}(R)| + |\mathbf{C}_R(\varphi)^{\mathrm{inv}}|\right) \\ &= \frac{1}{4}\left(\frac{3}{2}|R| + 2|\mathbf{I}(R)|\right) \\ &= \frac{|R| + |\mathbf{I}(R)|}{2} - \frac{|R|}{8} = \mathbf{c}(R) - \frac{|R|}{8}, \end{aligned}$$

and (1) holds in this case. $\square$

Lemma 2.6. *Let R be a finite group and let φ be an automorphism of R with $|R : \mathbf{C}_R(\varphi)| = 3$. Then one of the following holds:*

1. $\frac{1}{4}(|R| + |\mathbf{I}(R)| + |\mathbf{C}_R(\varphi)| + |\mathbf{C}_R(\varphi)^{\mathrm{inv}}|) \leq \mathbf{c}(R) - \frac{|R|}{96}$,
2. *R is abelian of exponent greater than 2 and $\varphi = \iota$.*

Proof. For simplicity, we let $A := \mathbf{C}_R(\varphi)$ and we let o denote the left-hand side in (1). As $|R : A| = 3$, we may write $R = A \cup Ax \cup Ax'$, for some $x, x' \in R$.

Suppose that $\mathbf{C}_R(\varphi)^{\mathrm{inv}} \subseteq A \cup Ay$, for some $y \in \{x, x'\}$. Then

$$\mathbf{C}_R(\varphi)^{\mathrm{inv}} = (\mathbf{C}_R(\varphi)^{\mathrm{inv}} \cap A) \cup (\mathbf{C}_R(\varphi)^{\mathrm{inv}} \cap Ay) = \mathbf{I}(A) \cup (\mathbf{C}_R(\varphi)^{\mathrm{inv}} \cap Ay) \subseteq \mathbf{I}(A) \cup Ay,$$

because φ fixes each element of A. Thus $|\mathbf{C}_R(\varphi)^{\mathrm{inv}}| \leq |\mathbf{I}(R)| + |A|$ and

$$\begin{aligned} o &\leq \frac{1}{4}\left(\frac{4}{3}|R| + |\mathbf{I}(R)| + |\mathbf{I}(R)| + |A|\right) \\ &\leq \frac{1}{4}\left(\frac{4}{3}|R| + 2|\mathbf{I}(R)| + \frac{|R|}{3}\right) \\ &= \frac{|R| + |\mathbf{I}(R)|}{2} - \frac{|R|}{12} \\ &= \mathbf{c}(R) - \frac{|R|}{12}, \end{aligned}$$

and (1) holds in this case.

Therefore we may suppose that $\mathbf{C}_R(\varphi)^{\mathrm{inv}} \cap Ax \neq \emptyset$ and $\mathbf{C}_R(\varphi)^{\mathrm{inv}} \cap Ax' \neq \emptyset$. In particular, replacing x and x' if necessary, we may suppose that $x, x' \in \mathbf{C}_R(\varphi)^{\mathrm{inv}}$, that is, $x^\varphi = x^{-1}$ and $x'^\varphi = x'^{-1}$.

CASE: $A \trianglelefteq R$.

As R/A is cyclic of order 3, we may assume that $x' = x^{-1}$ and that x has odd order. For every $a \in A$, we have $xax^{-1} \in A$ and hence

$$xax^{-1} = (xax^{-1})^\varphi = x^\varphi a^\varphi (x^{-1})^\varphi = x^{-1}ax,$$

that is, $x^2a = ax^2$. Therefore $x^2 \in \mathbf{Z}(\langle x, A\rangle) = \mathbf{Z}(R)$. As x has odd order, we deduce $x \in \mathbf{Z}(R)$. From this it is easy to deduce that

$$\mathbf{C}_R(\varphi)^{\mathrm{inv}} = \mathbf{I}(A) \cup \mathbf{I}(A)x \cup \mathbf{I}(A)x^{-1}. \tag{2.4}$$

Assume that $|\mathbf{I}(A)| \le 3|A|/4$. Thus, by (2.4), we have

$$\begin{aligned} o &= \frac{1}{4}\left(\frac{4}{3}|R| + |\mathbf{I}(R)| + |\mathbf{I}(A)| + |\mathbf{I}(A)| + |\mathbf{I}(A)|\right) \\ &\le \frac{1}{4}\left(\frac{4}{3}|R| + 2|\mathbf{I}(R)| + 2|\mathbf{I}(A)|\right) \\ &\le \frac{1}{4}\left(\frac{4}{3}|R| + 2|\mathbf{I}(R)| + 2\frac{3|A|}{4}\right) = \frac{1}{4}\left(\frac{4}{3}|R| + 2|\mathbf{I}(R)| + \frac{|R|}{2}\right) \\ &= \frac{1}{4}\left(\frac{11}{6}|R| + 2|\mathbf{I}(R)|\right) \\ &= \frac{|R| + |\mathbf{I}(R)|}{2} - \frac{|R|}{24} = \mathbf{c}(R) - \frac{|R|}{24}, \end{aligned}$$

and (1) holds in this case.

Assume that $|\mathbf{I}(A)| > 3|A|/4$. By [11], A is abelian. Thus $\mathbf{I}(A)$ is a subgroup of A with $|\mathbf{I}(A)| > 3|A|/4$. It follows that A is an elementary abelian 2-group. As $x \in \mathbf{Z}(R)$, we deduce that R is abelian and $\varphi = \iota$; thus (2) holds in this case.

Case: A is not normal in R.

Let K be the core of A in R. Observe that the group R acts on the right cosets of A in R. As $|R : A| = 3$, this action gives rise to a transitive permutation representation of R inside the symmetric group of degree 3. The kernel of this permutation representation is, by definition, K and hence R/K is isomorphic to a subgroup of the symmetric group of degree 3. Therefore $|R : K| \le 3! = 6$. Since by hypothesis A is not normal in R, we deduce that K is a proper subgroup of A. As $|R : A| = 3$ and $|R : K| \le 6$, we get that $|R : K| = 6$ and that R/K is isomorphic to the dihedral group of order 6.

Suppose that $\mathbf{C}_R(\varphi)^{\mathrm{inv}} \cap Ky = \emptyset$, for some $y \in R \setminus A$. As $R \setminus A$ is the union of four K-cosets and as $\mathbf{C}_R(\varphi)^{\mathrm{inv}} \cap Ky = \emptyset$, we deduce $|\mathbf{C}_R(\varphi)^{\mathrm{inv}} \cap (R \setminus A)| \le 3|K|$. As $\mathbf{C}_R(\varphi)^{\mathrm{inv}} \cap A = \mathbf{I}(A)$, we get $|\mathbf{C}_R(\varphi)^{\mathrm{inv}}| = |\mathbf{C}_R(\varphi)^{\mathrm{inv}} \cap A| + |\mathbf{C}_R(\varphi)^{\mathrm{inv}} \cap (R \setminus A)| \le |\mathbf{I}(A)| + 3|K|$ and hence

$$\begin{aligned} o &\le \frac{1}{4}\left(\frac{4}{3}|R| + |\mathbf{I}(R)| + |\mathbf{I}(A)| + 3|K|\right) \le \frac{1}{4}\left(\frac{4}{3}|R| + 2|\mathbf{I}(R)| + 3\frac{|R|}{6}\right) \\ &= \frac{1}{4}\left(\frac{11}{6}|R| + 2|\mathbf{I}(R)|\right) = \frac{|R| + |\mathbf{I}(R)|}{2} - \frac{|R|}{24} = \mathbf{c}(R) - \frac{|R|}{24}, \end{aligned}$$

and (1) holds in this case. Thus we may suppose $\mathbf{C}_R(\varphi)^{\mathrm{inv}} \cap Ky \ne \emptyset$, for every $y \in R \setminus A$.

Let $x_1, x_2, x_3, x_4 \in R \setminus A$ with $R = A \cup Kx_1 \cup Kx_2 \cup Kx_3 \cup Kx_4$ and with $x_1, x_2, x_3, x_4 \in \mathbf{C}_R(\varphi)^{\mathrm{inv}}$. As usual we denote by $\iota_{x_i} : K \to K$ the automorphism of K defined by $k^{\iota_{x_i}} = x_i k x_i^{-1}$, for every $k \in K$. For each $i \in \{1, \ldots, 4\}$, let $y \in \mathbf{C}_R(\varphi)^{\mathrm{inv}} \cap Kx_i$. Then $y = kx_i$, for some $k \in K$ and hence $x_i^{-1}k^{-1} = (kx_i)^{-1} = y^{-1} = y^{\varphi} = (kx_i)^{\varphi} = k^{\varphi}x_i^{\varphi} = kx_i^{-1}$, that is, $x_i k x_i^{-1} = k^{-1}$ and $k \in \mathbf{C}_K(\iota_{x_i})^{\mathrm{inv}}$. This shows

$$\mathbf{C}_R(\varphi)^{\mathrm{inv}} = \mathbf{I}(A) \cup \mathbf{C}_K(\iota_{x_1})^{\mathrm{inv}}x_1 \cup \mathbf{C}_K(\iota_{x_2})^{\mathrm{inv}}x_2 \cup \mathbf{C}_K(\iota_{x_3})^{\mathrm{inv}}x_3 \cup \mathbf{C}_K(\iota_{x_4})^{\mathrm{inv}}x_4. \tag{2.5}$$

Suppose that $|\mathbf{C}_K(\iota_{x_i})^{\mathrm{inv}}| \le 3|K|/4$, for some $i \in \{1, 2, 3, 4\}$. Then

$$|\mathbf{C}_R(\varphi)^{\mathrm{inv}}| \le |\mathbf{I}(A)| + 3|K| + \frac{3|K|}{4} = |\mathbf{I}(A)| + \frac{15|K|}{4} = |\mathbf{I}(A)| + \frac{5|R|}{8}.$$

Thus

$$\begin{aligned}
o &\leq \frac{1}{4}\left(\frac{4}{3}|R|+|\mathbf{I}(R)|+|\mathbf{I}(A)|+\frac{5|R|}{8}\right)\leq\frac{1}{4}\left(\frac{47}{24}|R|+2|\mathbf{I}(R)|\right)\\
&= \frac{|R|+|\mathbf{I}(R)|}{2}-\frac{|R|}{96}=\mathbf{c}(R)-\frac{|R|}{96},
\end{aligned}$$

and (1) holds in this case. Therefore, we may suppose that $|\mathbf{C}_K(\iota_{x_i})^{\mathrm{inv}}|>3|K|/4$, for each $i\in\{1,2,3,4\}$. The work of Miller [11] shows that K is abelian and that, for every $i\in\{1,2,3,4\}$, x_i acts by conjugation on K by inverting each of its elements. In particular, (2.5) becomes

$$\mathbf{C}_R(\varphi)^{\mathrm{inv}}=\mathbf{I}(A)\cup(R\setminus A). \tag{2.6}$$

As R/K is isomorphic to the dihedral group of order 6, we deduce that there exist $i,j,k\in\{1,2,3,4\}$ with $x_ix_j\in Kx_k$. From the previous paragraph, x_i,x_j and x_k act by conjugation on K by inverting each of its elements. Therefore, for every $y\in K$, we have

$$y^{-1}=y^{x_k}=y^{x_ix_j}=(y^{x_i})^{x_j}=(y^{-1})^{x_j}=y,$$

that is, $y^2=1$. This yields that K is an elementary abelian 2-group and hence $K\subseteq\mathbf{I}(A)$. Eq (2.6) gives $\mathbf{C}_R(\varphi)^{\mathrm{inv}}\supseteq K\cup(R\setminus A)$ and hence $|\mathbf{C}_R(\varphi)^{\mathrm{inv}}|\geq|K|+|R\setminus A|=5|R|/6>3|R|/4$. Again, from the work of Miller [11], we deduce that R is abelian and $\varphi=\iota$, and (2) holds in this case. □

Before proving the main step towards the proof of Theorem 1.3, we need a preliminary observation.

Lemma 2.7. *Let φ be an automorphism of a finite group R and let*

$$\kappa:=\frac{|\mathbf{C}_R(\varphi)^{\mathit{inv}}|}{|R|}.$$

If $\frac{1}{2}<\kappa<1$, then there exists a positive integer $q\geq 2$ with $\kappa=\frac{q+1}{2q}$. In particular, if $\frac{2}{3}<\kappa$, then $\kappa=\frac{3}{4}$ and there exists an abelian subgroup A of R such that $|R:A|=|A:\mathbf{C}_A(x)|=2$ for every $x\in R\setminus A$.

Proof. The first assertion follows at once from the classification of Liebeck and MacHale [10, Structure Theorem, page 61] of the finite groups admitting an automorphism inverting more than half of the elements. (Actually, the first statement of this lemma can also be found in the third paragraph of the introductory section in [10].)

Suppose now that $\frac{2}{3}<\kappa$. Then, from the first statement, there exists $q\in\mathbb{N}$ with $q\geq 2$ and $\kappa=\frac{q+1}{2q}$. Now, $\frac{2}{3}<\frac{q+1}{2q}$ only when $q=2$; hence $\kappa=\frac{3}{4}$. We now invoke once again the work of Liebeck and MacHale. In [10, Structure Theorem, page 61], the finite groups admitting an automorphism inverting more than half of the elements are partitioned into three types. Namely, **Type I***, **Type II*** and **Type III***. It is readily seen that none of the groups in **Type II*** or **Type III*** admits an automorphism φ with $\frac{|\mathbf{C}_R(\varphi)^{\mathrm{inv}}|}{|R|}=\frac{3}{4}$. Therefore, R is of **Type I***, which means that there exists an abelian subgroup A with $|R:A|=|A:\mathbf{C}_A(x)|$, for every $x\in R\setminus A$. □

Lemma 2.8. *Let R be a finite group and let φ be a non-identity automorphism of R. Then, one of the following holds*

1. *the number of φ-invariant inverse-closed subsets of R is at most $2^{\mathbf{c}(R)-\frac{|R|}{96}}$,*
2. *$\mathbf{C}_R(\varphi)$ is abelian of exponent greater than 2 and has index 2 in R, R is a generalized dicyclic group over $\mathbf{C}_R(\varphi)$ and $\varphi = \bar{\iota}_{\mathbf{C}_R(\varphi)}$,*
3. *R is abelian of exponent greater than 2 and $\varphi = \iota$.*

Proof. In the first part of the proof, we establish the result when φ has order p, where p is a prime number.

Recall that $\iota : R \to R$ is the permutation of R defined by $x^\iota = x^{-1}$, for every $x \in R$. Let $H := \langle \iota, \varphi\rangle \leq \mathrm{Sym}(R)$. Clearly, the number of φ-invariant inverse-closed subsets of R is 2^o, where o is the number of H-orbits, that is, o is the number of orbits of H in its action on R. From the orbit-counting lemma, we have

$$o = \frac{1}{|H|}\sum_{h\in H} |\mathrm{Fix}_R(h)|, \tag{2.7}$$

where $\mathrm{Fix}_R(h) := \{x \in R \mid x^h = x\}$ is the fixed-point set of h in its action on R.

For every $x \in R$, we have $x^{\iota\varphi} = (x^{-1})^\varphi = (x^\varphi)^{-1} = x^{\varphi\iota}$ and hence $\iota\varphi = \varphi\iota$. Therefore H is an abelian group. Moreover, $\mathrm{Fix}_R(\iota) = \mathbf{I}(R)$ and $\mathrm{Fix}_R(\varphi^\ell) = \mathbf{C}_R(\varphi)$ for every $\ell \in \{1, \dots, p-1\}$.

Suppose R is abelian of exponent at most 2. As R has exponent at most 2, ι is the identity permutation and hence $H = \langle\varphi\rangle$ is cyclic of prime order p. From (2.7) and from the fact that $|\mathbf{C}_R(\varphi)| \leq |R|/2$, we obtain

$$o = \frac{1}{p}(|R|+(p-1)|\mathbf{C}_R(\varphi)|) \leq \frac{1}{p}\left(|R| + (p-1)\frac{|R|}{2}\right) = \frac{(p+1)|R|}{2p} \leq \frac{3|R|}{4} = |R| - \frac{|R|}{4}$$

and part (1) of the lemma holds in this case because $\mathbf{c}(R) = (|R| + |\mathbf{I}(R)|)/2 = |R|$ and thus

$$o = |R| - \frac{|R|}{4} = \mathbf{c}(R) - \frac{|R|}{4}.$$

In particular, for the rest of the argument we suppose that R has exponent greater than 2. Thus H has order $2p$.

CASE 1: p is odd.

As H is abelian of order $2p$, we deduce that H is cyclic and $\mathrm{Fix}_R(\iota\varphi^\ell) = \mathbf{C}_R(\varphi^\ell) \cap \mathrm{Fix}_R(\iota) = \mathbf{C}_R(\varphi) \cap \mathbf{I}(R)$, for every $\ell \in \{1, \dots, p-1\}$. Now, (2.7) yields (in the second inequality we are using the fact that φ is not the identity automorphism and hence $|\mathbf{C}_R(\varphi)| \leq |R|/2$)

$$\begin{aligned}
o &= \frac{1}{2p}\left(|R| + |\mathbf{I}(R)| + (p-1)|\mathbf{C}_R(\varphi)| + (p-1)|\mathbf{C}_R(\varphi) \cap \mathbf{I}(R)|\right)\\
&\leq \frac{1}{2p}\left(|R| + |\mathbf{I}(R)| + (p-1)|\mathbf{C}_R(\varphi)| + (p-1)|\mathbf{I}(R)|\right)\\
&= \frac{|R| + |\mathbf{I}(R)|}{2} - \frac{|R|}{2} + \frac{1}{2p}\left(|R| + (p-1)|\mathbf{C}_R(\varphi)|\right)\\
&\leq \mathbf{c}(R) - \frac{|R|}{2} + \frac{1}{2p}\left(|R| + (p-1)\frac{|R|}{2}\right) = \mathbf{c}(R) - |R|\left(\frac{1}{2} - \frac{p+1}{4p}\right)\\
&= \mathbf{c}(R) - |R|\frac{p-1}{4p} \leq \mathbf{c}(R) - \frac{|R|}{6}.
\end{aligned}$$

CASE 2: $p = 2$.

If $\varphi = \iota$, then R is an abelian group of exponent greater than 2 and we obtain that part (3) holds in this case. Therefore, we may suppose that $\varphi \neq \iota$. As $H = \langle \varphi, \iota \rangle$ is abelian of order $2p$, we deduce that $H = \{id_R, \iota, \varphi, \iota\varphi\}$ is elementary abelian of order 4. Moreover, $\mathrm{Fix}_R(\iota) = \mathbf{I}(R)$, $\mathrm{Fix}_R(\varphi) = \mathbf{C}_R(\varphi)$ and $\mathrm{Fix}_R(\iota\varphi) := \mathbf{C}_R(\varphi)^{\mathrm{inv}}$. Thus

$$o = \frac{1}{4}\left(|R| + |\mathbf{I}(R)| + |\mathbf{C}_R(\varphi)| + |\mathbf{C}_R(\varphi)^{\mathrm{inv}}|\right).$$

From Lemmas 2.5 and 2.6, we may suppose that $|R : \mathbf{C}_R(\varphi)| \geq 4$.

Miller [11] has shown that a non-identity automorphism of a non-abelian group inverts at most $3|R|/4$ elements. Therefore, $|\mathbf{C}_R(\varphi)^{\mathrm{inv}}| \leq 3|R|/4$. Observe that the same inequality holds when R is abelian because $\mathbf{C}_R(\varphi)^{\mathrm{inv}}$ is a proper subgroup of R and hence $|\mathbf{C}_R(\varphi)^{\mathrm{inv}}| \leq |R|/2 \leq 3|R|/4$. In particular, if $|R : \mathbf{C}_R(\varphi)| \geq 5$, then we deduce

$$\begin{aligned} o &= \frac{1}{4}\left(|R| + |\mathbf{I}(R)| + \frac{|R|}{5} + \frac{3|R|}{4}\right) \\ &= \frac{1}{4}\left(\frac{39}{20}|R| + |\mathbf{I}(R)|\right) \\ &\leq \frac{|R| + |\mathbf{I}(R)|}{2} - \frac{|R|}{80} = \mathbf{c}(R) - \frac{|R|}{80}. \end{aligned}$$

For the rest of the argument we may suppose that $|R : \mathbf{C}_R(\varphi)| \leq 4$ and hence $|R : \mathbf{C}_R(\varphi)| = 4$. Therefore

$$o = \frac{1}{4}\left(\frac{5|R|}{4} + |\mathbf{I}(R)| + |\mathbf{C}_R(\varphi)^{\mathrm{inv}}|\right). \tag{2.8}$$

Assume $|\mathbf{C}_R(\varphi)^{\mathrm{inv}}| \leq 2|R|/3$. Then, from (2.8), we get

$$\begin{aligned} o &= \frac{1}{4}\left(\frac{5|R|}{4} + |\mathbf{I}(R)| + \frac{2|R|}{3}\right) \\ &= \frac{1}{4}\left(\frac{23}{12}|R| + |\mathbf{I}(R)|\right) \leq \frac{|R| + |\mathbf{I}(R)|}{2} - \frac{|R|}{48} = \mathbf{c}(R) - \frac{|R|}{48}. \end{aligned}$$

Therefore, we may assume that $|\mathbf{C}_R(\varphi)^{\mathrm{inv}}| > 2|R|/3$.

As $2|R|/3 < |\mathbf{C}_R(\varphi)^{\mathrm{inv}}| \leq 3|R|/4$, from Lemma 2.7 we deduce that

$$|\mathbf{C}_R(\varphi)^{\mathrm{inv}}| = \frac{3|R|}{4}$$

and that R contains an abelian subgroup A with $|R : A| = |A : \mathbf{C}_A(x)| = 2$, for every $x \in R \setminus A$.

Suppose that A is not φ-invariant. Since φ has order $p = 2$, $A \cap A^\varphi$ has index 4 in R and is φ-invariant. Observe that $R/(A \cap A^\varphi)$ is an elementary abelian 2-group of order 4. Let T be the index 2 subgroup of R containing $A \cap A^\varphi$ and with $A \neq T \neq A^\varphi$. We have

$$\mathbf{C}_R(\varphi)^{\mathrm{inv}} = (\mathbf{C}_R(\varphi)^{\mathrm{inv}} \cap A) \cup (\mathbf{C}_R(\varphi)^{\mathrm{inv}} \cap A^\varphi) \cup (\mathbf{C}_R(\varphi)^{\mathrm{inv}} \cap T).$$

Let $a \in \mathbf{C}_R(\varphi)^{\mathrm{inv}} \cap A$. Then $a^{-1} = a^\varphi \in A^\varphi \cap A$ and hence $\mathbf{C}_R(\varphi)^{\mathrm{inv}} \cap A = \mathbf{C}_{A\cap A^\varphi}(\varphi)^{\mathrm{inv}}$ and (similarly) $\mathbf{C}_R(\varphi)^{\mathrm{inv}} \cap A^\varphi = \mathbf{C}_{A\cap A^\varphi}(\varphi)^{\mathrm{inv}}$. Therefore

$$\mathbf{C}_R(\varphi)^{\mathrm{inv}} = \mathbf{C}_{A\cap A^\varphi}(\varphi)^{\mathrm{inv}} \cup (\mathbf{C}_R(\varphi)^{\mathrm{inv}} \cap T).$$

We deduce

$$\begin{aligned}|\mathbf{C}_R(\varphi)^{\mathrm{inv}}| &= |\mathbf{C}_{A\cap A^\varphi}(\varphi)^{\mathrm{inv}}| + |\mathbf{C}_R(\varphi)^{\mathrm{inv}} \cap (T \setminus (A\cap A^\varphi))|\\ &\le |A\cap A^\varphi| + (|T| - |A\cap A^\varphi|)\\ &= |T| = \frac{|R|}{2};\end{aligned}$$

however this contradicts $|\mathbf{C}_R(\varphi)^{\mathrm{inv}}| = 3|R|/4$. Thus A is φ-invariant.

CASE 2.1: φ inverts each element in A, that is, $a^\varphi = a^{-1}$, for every $a \in A$.

As $|\mathbf{C}_R(\varphi)^{\mathrm{inv}}| = 3|R|/4 > |R|/2 = |A|$, there exists $x \in R \setminus A$ with $x^\varphi = x^{-1}$. It follows that $\mathbf{C}_R(\varphi)^{\mathrm{inv}} = A \cup \mathbf{C}_A(x)x$ and hence

$$|\mathbf{C}_R(\varphi)^{\mathrm{inv}}| = |A| + \frac{|A|}{2}. \tag{2.9}$$

A computation gives $\mathbf{C}_R(\varphi) = \mathbf{I}(A) \cup \{ax \mid a \in A, a^2 = x^{-2}\}$. Let $a, b \in A$ with the property that $a^2 = x^{-2} = b^2$. Then $(ab^{-1})^2 = a^2b^{-2} = x^{-2}x^2 = 1$. This shows that either $\{ax \mid a \in A, a^2 = x^{-2}\}$ is the empty set or $\{ax \mid a \in A, a^2 = x^{-2}\} = \{b\bar{a}x \mid b \in \mathbf{I}(A)\}$, where $\bar{a} \in A$ is a fixed element with $\bar{a}^2 = x^{-2}$. In particular, $|\mathbf{C}_R(\varphi)| \in \{|\mathbf{I}(A)|, 2|\mathbf{I}(A)|\}$. As $|R : \mathbf{C}_R(\varphi)| = 4$, we deduce that either $|A : \mathbf{I}(A)| = 2$ and $\{ax \mid a \in A, a^2 = x^{-2}\} = \emptyset$, or $|A : \mathbf{I}(A)| = 4$ and $\{ax \mid a \in A, a^2 = x^{-2}\} \neq \emptyset$. In the first case, from (2.9), we have

$$|\mathbf{C}_R(\varphi)^{\mathrm{inv}}| = |A| + |A|/2 = |A| + |\mathbf{I}(A)| \le \frac{|R|}{2} + |\mathbf{I}(R)|.$$

Thus

$$\begin{aligned}o &\le \frac{1}{4}\left(\frac{5}{4}|R| + |\mathbf{I}(R)| + \frac{|R|}{2} + |\mathbf{I}(R)|\right) \le \frac{1}{4}\left(\frac{7}{4}|R| + 2|\mathbf{I}(R)|\right)\\ &= \frac{|R| + |\mathbf{I}(R)|}{2} - \frac{|R|}{16} = \mathbf{c}(R) - \frac{|R|}{16}.\end{aligned}$$

In the second case, from (2.9), we have

$$\begin{aligned}|\mathbf{C}_R(\varphi)^{\mathrm{inv}}| &= |A| + |A|/2 = |A| + 2|\mathbf{I}(A)|\\ &= \frac{|R|}{2} + |\mathbf{I}(A)| + |\mathbf{I}(A)| = \frac{|R|}{2} + \frac{|R|}{8} + |\mathbf{I}(A)|\\ &\le \frac{5|R|}{8} + |\mathbf{I}(R)|.\end{aligned}$$

Thus

$$\begin{aligned}o &\le \frac{1}{4}\left(\frac{5}{4}|R| + |\mathbf{I}(R)| + \frac{5|R|}{8} + |\mathbf{I}(R)|\right) = \frac{1}{4}\left(\frac{15}{8}|R| + 2|\mathbf{I}(R)|\right)\\ &= \frac{|R| + |\mathbf{I}(R)|}{2} - \frac{|R|}{32} = \mathbf{c}(R) - \frac{|R|}{32}.\end{aligned}$$

CASE 2.2: φ does not invert each element in $R \setminus A$.

Observe that $\mathbf{C}_A(\varphi)^{\mathrm{inv}}$ is a subgroup of A because A is abelian. In particular, $|\mathbf{C}_R(\varphi)^{\mathrm{inv}} \cap A| \le |A|/2 = |R|/4$. As $|\mathbf{C}_R(\varphi)^{\mathrm{inv}}| = 3|R|/4$, we deduce that

- φ inverts each element in $R \setminus A$ and
- $|\mathbf{C}_R(\varphi)^{\mathrm{inv}} \cap A| = |R|/4$.

Fix $x \in R \setminus A$. In particular, for every $a \in A$, we have $x^{-1}a^{-1} = (ax)^{-1} = (ax)^{\varphi} = a^{\varphi}x^{\varphi} = a^{\varphi}x^{-1}$ and hence $a^{\varphi} = x^{-1}a^{-1}x$. From this it follows

$$\mathbf{C}_R(\varphi)^{\mathrm{inv}} = \mathbf{C}_A(x) \cup Ax \text{ and } \mathbf{C}_R(\varphi) = \mathbf{C}_A(\iota_x)^{\mathrm{inv}} \cup \mathbf{I}(R \setminus A),$$

where $\mathbf{I}(R \setminus A) := \{m \in R \setminus A \mid m^2 = 1\}$.

Suppose that $\mathbf{I}(R \setminus A) = \emptyset$. Then $\mathbf{C}_R(\varphi) = \mathbf{C}_A(\iota_x)^{\mathrm{inv}}$ and hence $|A : \mathbf{C}_A(\iota_x)^{\mathrm{inv}}| = 2$ because $|R : \mathbf{C}_R(\varphi)| = 4$. As $|A : \mathbf{C}_A(x)| = 2$, we deduce that $|A : \mathbf{C}_A(x) \cap \mathbf{C}_A(\iota_x)^{\mathrm{inv}}| \le 4$. Clearly, $\mathbf{C}_A(x) \cap \mathbf{C}_A(\iota_x)^{\mathrm{inv}} \subseteq \mathbf{I}(A)$ and hence $|A : \mathbf{I}(A)| \le 4$. We deduce

$$\begin{aligned}|\mathbf{C}_R(\varphi)| = |\mathbf{C}_A(\iota_x)^{\mathrm{inv}}| &= |\mathbf{C}_A(\iota_x)^{\mathrm{inv}} \cap (A \setminus \mathbf{C}_A(x))| + |\mathbf{C}_A(\iota_x)^{\mathrm{inv}} \cap \mathbf{C}_A(x)| \\ &\le \frac{|A|}{4} + |\mathbf{I}(A)| \le \frac{|R|}{8} + |\mathbf{I}(R)|.\end{aligned}$$

Thus

$$\begin{aligned} o &= \frac{1}{4}\left(|R| + |\mathbf{I}(R)| + |\mathbf{C}_R(\varphi)| + |\mathbf{C}_R(\varphi)^{\mathrm{inv}}|\right) \\ &= \frac{1}{4}\left(\frac{7|R|}{4} + |\mathbf{I}(R)| + |\mathbf{C}_R(\varphi)|\right) \le \frac{1}{4}\left(\frac{7|R|}{4} + |\mathbf{I}(R)| + \frac{|R|}{8} + |\mathbf{I}(R)|\right) \\ &= \frac{1}{4}\left(\frac{15|R|}{8} + 2|\mathbf{I}(R)|\right) \le \frac{|R| + |\mathbf{I}(R)|}{2} - \frac{|R|}{32} = \mathbf{c}(R) - \frac{|R|}{32}.\end{aligned}$$

Suppose that $\mathbf{I}(R \setminus A) \neq \emptyset$. In particular, we may suppose that $x \in \mathbf{I}(R \setminus A)$, that is, $x^2 = 1$. From this it follows that

$$\begin{aligned}\mathbf{C}_R(\varphi) &= \mathbf{C}_A(\iota_x)^{\mathrm{inv}} \cup \mathbf{C}_A(\iota_x)^{\mathrm{inv}}x, \\ \mathbf{C}_R(\varphi)^{\mathrm{inv}} &= \mathbf{C}_A(x) \cup Ax, \\ \mathbf{I}(R) &= \mathbf{I}(A) \cup \mathbf{C}_A(\iota_x)^{\mathrm{inv}}x.\end{aligned}$$

As $|\mathbf{C}_R(\varphi)| = |R|/4$, we deduce $|\mathbf{C}_A(\iota_x)^{\mathrm{inv}}| = |A|/4$. Assume that $|\mathbf{I}(R)| \ge |\mathbf{C}_R(\varphi)|$, that is, $|\mathbf{I}(A)| \ge |\mathbf{C}_A(\iota_x)^{\mathrm{inv}}|$. Thus

$$\begin{aligned} o &= \frac{1}{4}\left(|R| + |\mathbf{I}(R)| + |\mathbf{C}_R(\varphi)| + |\mathbf{C}_R(\varphi)^{\mathrm{inv}}|\right) \\ &= \frac{1}{4}\left(\frac{7|R|}{4} + 2|\mathbf{I}(R)|\right) \le \frac{|R| + |\mathbf{I}(R)|}{2} - \frac{|R|}{16} = \mathbf{c}(R) - \frac{|R|}{16}.\end{aligned}$$

Assume that $|\mathbf{I}(R)| < |\mathbf{C}_R(\varphi)|$, that is, $|\mathbf{I}(A)| < |\mathbf{C}_A(\iota_x)^{\mathrm{inv}}|$. Observe now

$$\mathbf{C}_A(x) \cap \mathbf{I}(A) = \mathbf{C}_A(\iota_x)^{\mathrm{inv}} \cap \mathbf{I}(A) = \mathbf{C}_A(x) \cap \mathbf{C}_A(\iota_x)^{\mathrm{inv}}.$$

As $|\mathbf{I}(R)| < |\mathbf{C}_R(\varphi)|$, from these equalities we deduce $\mathbf{I}(A) = \mathbf{C}_A(x) \cap \mathbf{C}_A(\iota_x)^{\mathrm{inv}}$ and that $\mathbf{C}_A(x) \neq \mathbf{C}_A(\iota_x)^{\mathrm{inv}}$. Moreover,

$$|\mathbf{I}(A)| = \frac{|A|}{8},\ |\mathbf{C}_A(x)| = \frac{|A|}{2},\ |\mathbf{C}_A(\iota_x)^{\mathrm{inv}}| = \frac{|A|}{4}.$$

In particular, $\mathbf{c}(R) = (|R| + |\mathbf{I}(R)|)/2 = 19|R|/32$. Thus

$$o \quad = \quad \frac{1}{4}|R|\left(1 + \frac{3}{16} + \frac{1}{4} + \frac{3}{4}\right) = \frac{35|R|}{64} = \frac{19|R|}{32} - \frac{3|R|}{64} = \mathbf{c}(R) - \frac{3|R|}{64}.$$

The proof of the lemma is now completed when φ has prime order.

Suppose now that $o(\varphi)$ is not a prime number. Let p be the largest prime divisor of $o(\varphi)$ and let $\psi := \varphi^{o(\varphi)/p}$. As ψ is a non-identity automorphism of R of prime order, we are in the position to apply Lemma 2.8 to the group R and to the automorphism ψ. Let 2^o be the number of orbits of $\langle\varphi\rangle$ on R.

If part (1) of Lemma 2.8 holds for ψ, then part (1) of Lemma 2.8 holds for φ because every φ-invariant subset of R is also ψ-invariant.

Assume then that part (3) of Lemma 2.8 holds for ψ. Then R is abelian of exponent greater than 2 and $\psi = \iota$. Hence $p = o(\psi) = 2$. As p is the largest prime divisor of d, we deduce that d is a power of 2. As $o(\varphi) \geq 4$ and $\varphi^{d/2} = \iota$, the action of $\langle\varphi\rangle$ on R has orbits of cardinality 1 on $\mathbf{C}_R(\varphi)$, of cardinality at least 2 on $\mathbf{C}_R(\iota) \setminus \mathbf{C}_R(\varphi)$, and of cardinality at least 4 on $R \setminus \mathbf{C}_R(\iota)$. It follows that the number of subsets of R which are φ-invariant is at most

$$2^{|\mathbf{C}_R(\varphi)|} \cdot 2^{\frac{|\mathbf{C}_R(\iota)\setminus\mathbf{C}_R(\varphi)|}{2}} \cdot 2^{\frac{|R\setminus\mathbf{C}_R(\iota)|}{4}} = 2^{\frac{|R|+|\mathbf{C}_R(\iota)|}{4} + \frac{|\mathbf{C}_R(\varphi)|}{2}}.$$

Observe that every φ-invariant subset of R is also inverse-closed because $\varphi^{d/2} = \iota$. Thus

$$2^o \leq 2^{\frac{|R|+|\mathbf{C}_R(\iota)|}{4} + \frac{|\mathbf{C}_R(\varphi)|}{2}}.$$

Observe, also, that $\mathbf{C}_R(\iota) = \mathbf{I}(R)$. As $\mathbf{c}(R) = (|R| + |\mathbf{I}(R)|)/2$, by rewriting the previous equation, we deduce

$$2^o \leq 2^{\mathbf{c}(R) - \frac{|R|+|\mathbf{I}(R)|-2|\mathbf{C}_R(\varphi)|}{4}}. \tag{2.10}$$

If $|\mathbf{I}(R)| - 2|\mathbf{C}_R(\varphi)| \geq 0$, then (2.10) yields

$$o \leq 2^{\mathbf{c}(R) - \frac{|R|}{4}}.$$

Thus part (1) holds for R and φ. If $|\mathbf{I}(R)| - 2|\mathbf{C}_R(\varphi)| < 0$, then $|\mathbf{I}(R)| < 2|\mathbf{C}_R(\varphi)|$. However, as $\mathbf{C}_R(\varphi) \leq \mathbf{C}_R(\psi) = \mathbf{I}(R)$, we deduce $\mathbf{I}(R) = \mathbf{C}_R(\varphi)$ and hence (2.10) yields

$$o \leq 2^{\mathbf{c}(R) - \frac{|R|-|\mathbf{I}(R)|}{4}}.$$

Since R has exponent greater than 2, we have $|R| - |\mathbf{I}(R)| \leq |R|/2$ and hence

$$o \leq 2^{\mathbf{c}(R) - \frac{|R|}{8}}.$$

Thus part (1) holds for R and φ.

Assume then that part (2) of Lemma 2.8 holds for ψ. Then $\mathbf{C}_R(\psi)$ is abelian of exponent greater than 2 and has index 2 in R, R is a generalized dicyclic group over $\mathbf{C}_R(\psi)$

and $\psi = \bar{\iota}_{\mathbf{C}_R(\psi)}$. Hence $p = o(\psi) = 2$. As p is the largest prime divisor of d, we deduce that d is a power of 2. As $o(\varphi) \geq 4$ and $\varphi^{d/2} = \psi = \bar{\iota}_{\mathbf{C}_R(\psi)}$, the action of $\langle\varphi\rangle$ on R has orbits of cardinality 1 on $\mathbf{C}_R(\varphi)$, of cardinality at least 2 on $\mathbf{C}_R(\bar{\iota}_{\mathbf{C}_R(\psi)}) \setminus \mathbf{C}_R(\varphi)$, and of cardinality at least 4 on $R \setminus \mathbf{C}_R(\bar{\iota}_{\mathbf{C}_R(\psi)})$. As $\mathbf{C}_R(\psi) = \mathbf{C}_R(\bar{\iota}_{\mathbf{C}_R(\psi)})$, the action of $\langle\varphi\rangle$ on R has orbits of cardinality 1 on $\mathbf{C}_R(\varphi)$, of cardinality at least 2 on $\mathbf{C}_R(\psi) \setminus \mathbf{C}_R(\varphi)$, and of cardinality at least 4 on $R \setminus \mathbf{C}_R(\psi)$. Since the number of inverse-closed subsets of $\mathbf{C}_R(\varphi)$ is $\mathbf{c}(\mathbf{C}_R(\varphi)) = (|\mathbf{C}_R(\varphi)| + |\mathbf{I}(\mathbf{C}_R(\varphi))|)/2$, it follows that the number of inverse-closed subsets of R which are φ-invariant is at most

$$2^{\frac{|\mathbf{C}_R(\varphi)|+|\mathbf{I}(\mathbf{C}_R(\varphi))|}{2}} \cdot 2^{\frac{|\mathbf{C}_R(\psi)\setminus\mathbf{C}_R(\varphi)|}{2}} \cdot 2^{\frac{|R\setminus\mathbf{C}_R(\psi)|}{4}} = 2^{\frac{|R|+|\mathbf{C}_R(\psi)|}{4}+\frac{|\mathbf{I}(\mathbf{C}_R(\varphi))|}{2}}.$$

As $|\mathbf{C}_R(\psi)| = |R|/2$, we have

$$2^o \leq 2^{\frac{3|R|}{8}+\frac{|\mathbf{I}(\mathbf{C}_R(\varphi))|}{2}}.$$

As $\mathbf{c}(R) = (|R| + |\mathbf{I}(R)|)/2$, by rewriting the previous equation, we deduce

$$2^o \leq 2^{\mathbf{c}(R)-\frac{|R|+4|\mathbf{I}(R)|-4|\mathbf{I}(\mathbf{C}_R(\varphi))|}{8}}. \tag{2.11}$$

As $|\mathbf{I}(R)| - |\mathbf{I}(\mathbf{C}_R(\varphi))| \geq 0$, from (2.11) e deduce

$$2^o \leq 2^{\mathbf{c}(R)-\frac{|R|}{8}}.$$

In particular, part (1) holds for R and φ. □

Proposition 2.9. *Let R be a finite group and suppose that R is not an abelian group of exponent greater than 2 and that R is not a generalized dicyclic group. Then the set*

$$\{S \subseteq R \mid S = S^{-1}, R \neq \mathbf{N}_{\mathrm{Aut}(\Gamma(R,S))}(R)\}$$

has cardinality at most $2^{\mathbf{c}(R)-|R|/96+(\log_2|R|)^2}$.

As $R \leq \mathbf{N}_{\mathrm{Aut}(\Gamma(R,S))}(R)$, the condition $R \neq \mathbf{N}_{\mathrm{Aut}(\Gamma(R,S))}(R)$ is equivalent to the fact that R is a proper subgroup of $\mathbf{N}_{\mathrm{Aut}(\Gamma(R,S))}(R)$.

Proof. Let, for the time being, R be any finite group. For every $\varphi \in \mathrm{Aut}(R)$ with $\varphi \neq id_R$ and for every $S \subseteq R$ with $S = S^{-1}$ and $S^\varphi = S$, we have $\varphi \in \mathbf{N}_{\mathrm{Aut}(\Gamma(R,S))}(R) \setminus R$ and φ fixes the identity vertex of $\Gamma(R,S)$. Conversely, for every $S \subseteq R$ with $S = S^{-1}$ and for every $\varphi \in \mathbf{N}_{\mathrm{Aut}(\Gamma(R,S))}(R) \setminus R$ with φ fixing the identity vertex of $\Gamma(R,S)$, we have $\varphi \in \mathrm{Aut}(R)$ with $\varphi \neq id_R$. Therefore,

$$\{S \subseteq R \mid S = S^{-1}, R \neq \mathbf{N}_{\mathrm{Aut}(\Gamma(R,S))}(R)\} = \bigcup_{\substack{\varphi\in\mathrm{Aut}(R)\\ \varphi\neq id_R}} \{S \subseteq R \mid S = S^{-1}, S^\varphi = S\}$$

and

$$|\{S \subseteq R \mid S = S^{-1}, R \neq \mathbf{N}_{\mathrm{Aut}(\Gamma(R,S))}(R)\}| = \sum_{\substack{\varphi\in\mathrm{Aut}(R)\\ \varphi\neq id_R}} |\{S \subseteq R \mid S = S^{-1}, S^\varphi = S\}|. \tag{2.12}$$

Suppose now that R is not an abelian group of exponent greater than 2 and that R is not a generalized dicyclic group.

Since a chain of subgroups of R has length at most $\log_2(|R|)$, R has a generating set of cardinality at most $\lfloor \log_2(|R|)\rfloor \leq \log_2(|R|)$. Any automorphism of R is uniquely determined by its action on the elements of a generating set for R. Therefore

$$|\operatorname{Aut}(R)| \leq |R|^{\lfloor \log_2(|R|)\rfloor} \leq 2^{(\log_2(|R|))^2}. \tag{2.13}$$

Let $\varphi \in \operatorname{Aut}(R)$ with $\varphi \neq id_R$. We now apply Lemma 2.8 to the group R and to the non-identity automorphism φ of R. As R is neither abelian of exponent greater than 2 nor generalized dicyclic, parts (2) and (3) of Lemma 2.8 do not hold. Hence, part (1) of Lemma 2.8 holds, that is,

$$|\{S \subseteq R \mid S = S^{-1}, S^{\varphi} = S\}| \leq 2^{\mathbf{c}(R)-\frac{|R|}{96}}. \tag{2.14}$$

Now the proof follows from (2.12), (2.13) and (2.14). $\square$

3 Proofs of Theorems 1.3 and 1.6

Proof of Theorem 1.3. Let R be a finite group of order r which is neither generalized dicyclic nor abelian of exponent greater than 2. By Lemma 2.2 and Proposition 2.9, we have

$$\lim_{r\to\infty} \frac{|\{S \subseteq R \mid S = S^{-1}, R \neq \mathbf{N}_{\operatorname{Aut}(\Gamma(R,S))}(R)\}|}{|\{S \subseteq R \mid S = S^{-1}\}|} \leq \lim_{r\to\infty} 2^{-\frac{r}{96}+(\log_2(r))^2} = 0. \quad \square$$

Proof of Theorem 1.6. Let R be a finite group. It was shown in [3, 14] that Xu conjecture holds true when R is a generalized dicyclic group or when R is an abelian group of exponent greater than 2. In particular, for the rest of the proof we may assume that R is neither a generalized dicyclic group nor an abelian group of exponent greater than 2.

Let us denote by $\mathcal{N}(R) := \{S \subseteq R \mid S = S^{-1}, R \trianglelefteq \operatorname{Aut}(\Gamma(R,S))\}$, $\mathcal{C}(R) := \{S \subseteq R \mid S = S^{-1}, R = \operatorname{Aut}(\Gamma(R,S))\}$ and $\mathcal{T}(R) := \{S \subseteq R \mid S = S^{-1}\}$. If Conjecture 1.2 holds true, then

$$\lim_{|R|\to\infty} \frac{|\mathcal{C}(R)|}{|\mathcal{T}(R)|} = 1$$

and hence

$$\lim_{|R|\to\infty} \frac{|\mathcal{N}(R)|}{|\mathcal{T}(R)|} = 1,$$

because $\mathcal{C}(R) \subseteq \mathcal{N}(R)$, that is, Conjecture 1.5 holds true. Conversely, suppose that Conjecture 1.5 holds true, that is, $\lim_{|R|\to\infty} |\mathcal{N}(R)|/|\mathcal{T}(R)| = 1$. Now,

$$\begin{aligned}
\mathcal{N}(R) &= \mathcal{C}(R) \cup \{S \subseteq R \mid S = S^{-1}, R \trianglelefteq \operatorname{Aut}(\Gamma(R,S)), R < \operatorname{Aut}(\Gamma(R,S))\} \\
&\subseteq \mathcal{C}(R) \cup \{S \subseteq R \mid S = S^{-1}, R < \mathbf{N}_{\operatorname{Aut}(\Gamma(R,S))}(R)\}
\end{aligned}$$

and hence, by Theorem 1.3, we have

$$\begin{aligned}
1 &= \lim_{|R|\to\infty} \frac{|\mathcal{N}(R)|}{|\mathcal{T}(R)|} \\
&\le \lim_{|R|\to\infty} \frac{|\mathcal{C}(R)|}{|\mathcal{T}(R)|} + \lim_{|R|\to\infty} \frac{|\{S \subseteq R \mid S = S^{-1}, R < \mathbf{N}_{\mathrm{Aut}(\Gamma(R,S))}(R)\}|}{|\mathcal{T}(R)|} \\
&= \lim_{|R|\to\infty} \frac{|\mathcal{C}(R)|}{|\mathcal{T}(R)|},
\end{aligned}$$

that is, Theorem 1.2 holds true.

□

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Noncommutative Lattices
by Jonathan E. Leech

J. E. Leech, *Noncommutative Lattices: Skew Lattices, Skew Boolean Algebras and Beyond*, volume 4 of *Famnit Lectures*, Slovenian Discrete and Applied Mathematics Society and University of Primorska Press, Koper, 2021.

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Weakenings of lattices, where the meet and join operations may fail to be commutative, attracted from time to time the attention of various communities of scholars, including ordered algebra theorists (for their connection with preordered sets) and semigroup theorists (who viewed them as structurally enriched bands possessing a dual multiplication). Recently, noncommutative generalisations of lattices and related structures have seen a growth in interest, with new ideas and applications emerging. The adjective "noncommutative" is used here in the inclusive sense of "not-necessarily-commutative". Much of this recent activity derives in some way from the initiation, thirty years ago, by Jonathan Leech, of a research program into structures based on Pascual Jordan's notion of a noncommutative lattice. Indeed, the research began by studying multiplicative bands of idempotents in rings, and realising that under certain conditions such bands would also be closed under an "upward multiplication". In particular, for multiplicative bands that were left regular, any maximal such band in a ring was also closed under the circle operation (or quadratic join) $x \circ y = x + y - xy$. And any band closed under both operations satisfied certain absorption identities, e.g., $e(e \circ f) = e = e \circ (ef)$. These observations indicated the presence of structurally strengthened bands with a roughly lattice-like structure. These algebras are called *skew lattices* and are defined as algebras $(S; \wedge, \vee)$ of type $(2, 2)$, where both operations $\wedge$ and $\vee$ are associative and satisfy the four absorption identities $x \wedge (x \vee y) = x = (y \vee x) \wedge x$ and their dual. Absorption causes both operations to be idempotent. In the case of maximal left regular bands in rings, $\wedge$ and $\vee$ are given as $e \wedge f = ef$ and $e \vee f = e + f - ef$.

Parallel to this was an expanding role of results related to structures that were weakened or modified forms of (generalised) Boolean algebras. This was especially important in the study of a second class of motivating examples, algebras of partial functions between pairs of sets, A and B. Here, the skew operations are defined as follows:

$$f \wedge g = g_{|G \cap F}\,; \quad f \vee g = f \cup g_{|G-F}\,,$$

where $F, G \subseteq A$ denote the actual functional domains of the partial functions f and g respectively. A relative complement, defined by $f \setminus g = f_{|F-G}$, can be added to the algebraic structure, making definable the complement in the Boolean interval $\{g : g \subseteq f\}$ of all functions approximating a given f. These algebras of partial functions provided examples of so-called *skew Boolean algebras* and related structures, much as subsets of a given set led to basic examples of Boolean algebras and distributive lattices. The pioneering papers by Leech on skew lattices and skew Boolean algebras have attracted the attention of mathematicians from around the world, and in the last thirty years many interesting papers have been published on the subject. As a result of these developments, skew lattices have grown into a theory worth studying for its own sake. The 2020/21 monograph *Noncommutative Lattices: Skew Lattices, Skew Boolean Algebras and Beyond* by Jonathan E. Leech provides an excellent, organised and comprehensive account of much that has been published on the subject up through 2017. The book is mainly written for algebraists and mathematicians, but readers interested in applications to logic and computer science may also find it useful. The core of this monograph is the first four chapters. More specialised topics are studied in the last three chapters. The content of the monograph will be explained in more detail in the remaining part of this review.

In the first chapter of the book the author recalls various basic facts about bands (idempotent semigroups) that are pertinent to the rest of the monograph. In particular, he emphasises that a knowledge of regular bands, and their left and right-sided cases, is crucial to understanding much that will be said about skew lattices. The author of this review has particularly appreciated Section 1.3, where a noncommutative lattice is defined as a double band satisfying a specified set among eight possible absorption identities. The comparison of these absorption laws naturally brings the reader into the definitions of quasilattices, paralattices, antilattices and skew lattices.

The basic theory for skew lattices is developed in Chapter 2. Of particular importance in Section 2.1 are the two core structural results for skew lattices, analogues of the Clifford-McLean Theorem and the Kimura Factorization Theorem, given originally for bands and regular bands, respectively. There are two basic subvarieties of skew lattices: lattices (full commutation) and anti-lattices (no non trivial commutation). The Clifford-McLean Theorem for skew lattices thus states that every skew lattice is a lattice of anti-lattices. More precisely, Green's relation $\mathcal{D}$ on a skew lattice S, defined by $x \,\mathcal{D}\, y$ iff $x \wedge y \wedge x = x$ and $y \wedge x \wedge y = y$, is a congruence making $S/\mathcal{D}$ the maximal lattice image of S, and all congruence classes of $\mathcal{D}$ are maximal anti-lattices in S (Theorem 2.1.3). The Kimura Factorisation Theorem for skew lattices interestingly states that every skew lattice S factorises as the fibered product of its maximal right-handed image and its maximal left-handed image (Theorem 2.1.5).

An element that join commutes with all elements in a skew lattice also meet commutes with all elements (and conversely). In general two elements commuting under one operation need not commute under the other operation. A skew lattice S is called *symmetric* if this does not happen. All skew lattices in rings (using multiplication and the circle operation) are symmetric. In Section 2.2 many results on symmetric skew lattices are presented. We mention here that, if S is a symmetric skew lattice for which $S/\mathcal{D}$ is countable, then S has a lattice section (Theorem 2.2.7), i.e., a sublattice T of S having nonempty intersection with each $\mathcal{D}$-class of S, in which case, $T \cong S/\mathcal{D}$. A characterisation of having a left-handed section and a right-handed section is also given in Theorem 2.2.8.

In Section 2.3 *normal* skew lattices are studied. In a normal skew lattice the lower set $\{x : x \leq e\}$ is a lattice, for every e. Of special interest are distributive, symmetric, normal skew lattices characterised in Theorem 2.3.4 by the identities $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ and $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$. This strengthened form of distributivity is called *strong distributivity*. Thanks to Theorem 2.3.6, every normal skew lattice of idempotents in a ring is strongly distributive. In this case the operations are given by $e \wedge f = ef$ again, but $e \vee f = (e + f - ef)^2 = e + f + fe - efe - fef$, the cubic join. Of course when $e + f - ef$ is idempotent, both outcomes agree. Strongly distributive skew lattices are also of interest due to their connections to skew Boolean algebras, the subject of Chapter 4. A skew lattice can be embedded into (the skew lattice reduct of) a skew Boolean algebra precisely when it is strongly distributive.

In Section 2.4 and 2.5 a detailed study of the natural partial order $\leq$ on a skew lattice is provided. This study is based on the behaviour of *primitive* skew lattices, consisting of exactly two $\mathcal{D}$-classes. Primitive skew lattices have a simple description given in terms of cosets (Theorem 2.4.1). In Section 2.6 the decompositions of (mostly symmetric) normal skew lattices are studied. For instance, the Reduction Theorem 2.6.9 implies that every symmetric normal skew lattice can be embedded in the product of its maximal lattice image and its maximal distributive image. In Theorem 2.6.12 and its corollaries the variety of strongly distributive skew lattices is shown to be generated by a special primitive skew lattice $\mathbf{5}$, a noncommutative 5-element variant of the lattice $\mathbf{2}$ for which the latter is its maximal lattice image. A similar result holds for the variety of symmetric, normal skew lattices.

Chapter 3 is devoted to the study of quasilattices, paralattices, and especially refined quasilattices. The variety of refined quasilattices contains the variety of skew lattices and it is defined as intersection of the variety of quasilattices and of the variety of paralattices. Particular attention is given to their congruence lattices and to related topics such as Green's equivalences and simple algebras. Since all skew lattices are refined quasilattices, the study in this chapter has implications for skew lattices.

Skew Boolean algebras are studied in Chapters 4 and 7. In Section 4.1 skew Boolean algebras are formally defined as structural enhancements of strongly distributive skew lattices. Skew Boolean algebras are shown to be subdirect products of primitive skew Boolean algebras; moreover, every skew Boolean algebra can be embedded into a power of $\mathbf{5}$, a 5-element primitive algebra (Corollaries 4.1.6 and 4.1.7). In Section 4.2, special attention is given to classifying finite algebras, and in particular, to classifying finitely generated (and thus finite) free skew Boolean algebras. The main results are Theorems 4.2.2 and 4.2.6, with the latter describing the structure of finitely generated free algebras. In Section 4.4 skew Boolean algebras with finite intersections are introduced, that is, algebras for which the natural partial order has meets that are called intersections. All skew Boolean algebras S for which $S/\mathcal{D}$ is finite have intersections as do, more generally, all complete skew Boolean algebras. In Chapter 7 the role of skew Boolean algebras in universal algebra is examined, in particular in the study of what might be termed "generalised Boolean phenomena", a topic of interest in universal algebra, with connections to discriminator varieties, iBCK-algebras and more recently, Church algebras. The reviewer thinks that the characterisation of one-pointed discriminator varieties in terms of right-handed skew Boolean algebras with intersections is one of the most beautiful results of the theory. Chapter 6 (Skew Lattices in Rings) is also devoted to the skew Boolean algebras of idempotents in rings, and in particular, the case where the idempotents in a ring are closed under multiplication and thus naturally form a skew Boolean algebra.

We conclude the review of this excellent monograph with the belief that it will be the main reference on the subject of noncommutative lattices for many years.

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THE ART OF DISCRETE AND APPLIED MATHEMATICS

Minisymposium Announcement and Call for Papers – Chemical Graph Theory

This is a call for submission of papers for a special issue of the journal *The Art of Discrete and Applied Mathematics* (ADAM), on topics in Chemical Graph Theory.

Additionally, we are announcing a related 15-speaker minisymposium on Chemical Graph Theory at CanaDAM 2021 (Canadian Discrete and Applied Mathematics) Conference. The CanaDAM 2021 Conference is taking place online from May 25 – May 28, 2021. Further information on CanaDAM 2021 can be found at

https://2021.canadam.math.ca/

About the minisymposium: This minisymposium in chemical graph theory explores various applications of graph theory to chemistry. A molecule can be described as a graph, where vertices represent atoms and edges represent chemical bonds: benzenoids and fullerenes are two examples of such graph classes. Properties of those graphs, such as perfect matchings and graph spectra, can be used to model characteristics of molecules, including stability, reactivity, and electronic structure. Other related topics in chemical graph theory include enumeration of graphs classes and algorithms for their enumeration.

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Papers should be submitted by 31 December 2021 via the ADAM website. When submitting a paper, please choose the option "Chemical Graph Theory Issue of ADAM" so that it is directed to the correct editors. Papers that are accepted will appear online soon after acceptance, and papers that are not processed in time for the special issue may still be accepted and published in a subsequent regular issue of ADAM.

Nino Bašić and Elizabeth Hartung

Guest Editors

Jonathan E. Leech: Noncommutative Lattices: Skew Lattices, Skew Boolean Algebras and Beyond

About the book: The extended study of non-commutative lattices was begun in 1949 by Ernst Pascual Jordan, a theoretical and mathematical physicist and co-worker of Max Born and Werner Karl Heisenberg. Jordan introduced noncommutative lattices as algebraic structures potentially suitable to encompass the logic of the quantum world. The modern theory of noncommutative lattices began forty years later with Jonathan Leech's 1989 paper "Skew lattices in rings." Recently, noncommutative generalizations of lattices and related structures have seen an upsurge in interest, with new ideas and applications emerging, from quasilattices to skew Heyting algebras. Much of this activity is derived in some way from the initiation of Jonathan Leech's program of research in this area. The present book consists of seven chapters, mainly covering skew lattices, quasilattices and paralattices, skew lattices of idempotents in rings and skew Boolean algebras. As such, it is the first research monograph covering major results due to this renewed study of noncommutative lattices. It will serve as a valuable graduate textbook on the subject, as well as a handy reference to researchers of noncommutative algebras.

About the author: Jonathan Leech graduated from the University of Hawaii and earned a PhD at the University of California, Los Angeles. He has taught mathematics at the University of Tennessee, later at Missouri Western State University and finally at Westmont College in Santa Barbara, California. He has been a Visiting Professor at Case Western Reserve University, the Universidad de Granada in Spain and Universidade Mackenzie in Brazil, and a scholar in residence at both the University of Sidney and the University of Tasmania in Australia. Throughout his academic career Professor Leech has studied algebraic structures related to semigroups, with much of his emphasis being on the theory of noncommutative lattices, and of skew lattices in particular. He laid the foundations of the modern theory of noncommutative lattices in a number of (co)authored seminal publications. His work has inspired many mathematicians around the world to pursue research in this area.

J. E. Leech, *Noncommutative Lattices: Skew Lattices, Skew Boolean Algebras and Beyond*, volume 4 of *Famnit Lectures*, Slovenian Discrete and Applied Mathematics Society and University of Primorska Press, Koper, 2021, 284 pp., ISBN 978-961-95273-0-6.

The paperback edition of the book was published on March 5, 2021 by SDAMS, the Slovenian Discrete and Applied Mathematics Society. The cost of the book is 20.00 EUR + shipping. Society members have discount of 5.00 EUR. Orders should be sent to info@sdams.si. An invoice will be sent upon receipt of the order. The book will be shipped after payment is received.

THE ART OF DISCRETE AND APPLIED MATHEMATICS

Call for papers for the Wilfried Imrich 80 issue of ADAM

On May 25, 2021, Wilfried Imrich turned 80. Wilfried's impact on the development of the Slovenian graph theory school is utmost important and lasting. As a small token of appreciation, we are opening a special issue of ADAM dedicated to Wilfried to be edited by Tanja Dravec, Marko Jakovac, Sandi Klavžar, and Janez Žerovnik. You are invited to submit a paper related to Wilfried's work by October 1, 2021. All accepted papers will be published on-line as soon as possible, the issue will be completed in 2022. When submitting a paper, please choose the option "The Wilfried Imrich 80 Issue of ADAM" so that it is directed to the correct editors.

Tanja Dravec, Marko Jakovac, Sandi Klavžar, Iztok Peterin, Janez Žerovnik
Guest editors

Petra Šparl Award 2022: Call for Nominations

The Petra Šparl Award was established in 2017 to recognise in each even-numbered year the best paper published in the previous five years by a young woman mathematician in one of the two journals *Ars Mathematica Contemporanea* (*AMC*) and *The Art of Discrete and Applied Mathematics* (*ADAM*). It was named after Dr Petra Šparl, a talented woman mathematician who died mid-career in 2016.

The award consists of a certificate with the recipient's name, and invitations to give a lecture at the Mathematics Colloquium at the University of Primorska, and lectures at the University of Maribor and University of Ljubljana. The first award was made in 2018 to Dr Monika Pilśniak (AGH University, Poland) for a paper on the distinguishing index of graphs, and then two awards were made for 2020, to Dr Simona Bonvicini (Università di Modena e Reggio Emilia, Italy) for her contributions to a paper giving solutions to some Hamilton-Waterloo problems, and Dr Klavdija Kutnar (University of Primorska, Slovenia), for her contributions to a paper on odd automorphisms in vertex-transitive graphs.

The Petra Šparl Award Committee is now calling for nominations for the next award.

Eligibility: Each nominee must be a woman author or co-author of a paper published in either *AMC* or *ADAM* in the calendar years 2017 to 2021, who was at most 40 years old at the time of the paper's first submission.

Nomination Format: Each nomination should specify the following:

(a) the name, birth-date and affiliation of the candidate;
(b) the title and other bibliographic details of the paper for which the award is recommended;
(c) reasons why the candidate's contribution to the paper is worthy of the award, in at most 500 words; and
(d) names and email addresses of one or two referees who could be consulted with regard to the quality of the paper.

Procedure: Nominations should be submitted by email to any one of the three members of the Petra Šparl Award Committee (see below), **by 31 October 2021**.

Award Committee:

- Marston Conder, m.conder@auckland.ac.nz
- Asia Ivić Weiss, weiss@yorku.ca
- Aleksander Malnič, aleksander.malnic@guest.arnes.si

Marston Conder, Asia Ivić Weiss and Aleksander Malnič
Members of the *2022 Petra Šparl Award Committee*