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5 Understanding the Second Quantization of Fermions in Clifford and in Grassmann Space — New Way of Second Quantization of Fermions — Part II ⋆

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Abstract. We discuss in Part I and Part II of this paper the possibility to present internal part of degrees of freedom of the second quantized fermions in Grassmann space — in Part I — and in Clifford space — Part II [1–3]. They both offer description for second quantized fermions [3]. It is no need in either of these algebras to postulate the second quantization relations as Dirac [13], since both algebras by themselves offer the appropriate anticommutation relations. But while fermions with the internal degrees of freedom described by the Clifford algebras manifest the half integer spins and charges in the fundamental representations — in agreement with the observed properties of quarks and leptons and antiquarks and antileptons — the "Grassmann fermions" manifest integer spins. In Part II we discuss properties of the two kinds of the Clifford algebra objects — both expressible with the Grassmann coordinates, $\gamma^a = (\theta^a + \frac{\partial}{\partial\theta_a})$ and $\tilde{\gamma}^a = i\,(\theta^a i - \frac{\partial}{\partial\theta_a})$ [2,4,5], $\{\gamma^a\,, \tilde{\gamma}^b\}_+ = 0$ — and conditions under which the members of the irreducible representation of the Lorentz algebra carry the family quantum numbers.

Povzetek. Drugi del tega prispevka obravnava obstoj dveh neodvisnih vektorskih prostorov v Cliffordovi algebri, ki sta skupaj ekvivalentna prostoru, ki ga določa Grassmanova algebra. Vsak od vektorskih prostorov v Cliffordovi algebri ponudi kreacijske in anihilacijske operatorje, ki določajo na vakuumskem stanju, ki je vsota produktov anihilacijskih operatorjev na kreacijskih operatotorjih, stanja fermionov s spinom $\frac{1}{2}$ in so rešitve Weylove enačbe. Avtorja postavita zahtevo, da samo ena od obeh Cliffordovih algeber določa vektorski prostor fermionov, druga pa opremi nerazcepne upodobitve Lorentzove grupe v prostoru prve s kantnim številom družine. Zahteva zagotovi, da zadostijo kreacijski in anihilacijski operatorji Diracovim postulatom za fermione v drugi kvantizaciji.

Keywords:Second quantization of fermion fields in Clifford and in Grassmann space, Spinor representations in Clifford and in Grassmann space, Kaluza-Klein-like theories, Higher dimensional spaces, Beyond the standard model

⋆ Talk presented by N.S. Mankoč Borštnik

5.1 Introduction

In Part I of this paper the properties of "Grassmann fermions" of integer spins are presented. Let us repeat: In d-dimensional Grassmann space of anticommuting coordinates θ^a's, $i = (1,..,d)$, there are 2^d "vectors", which are superposition of products of θ^a's. One can arrange them into irreducible representations with respect to the Lorentz group. There are as well derivatives with respect to θ^a's, $\frac{\partial}{\partial\theta_a}$'s, which again form 2^d "vectors", representing Hermitian conjugated partners to the members of the irreducible representations of θ^a's, Eq. (6) of Part I. Grassmann coordinates offer correspondingly $2 \cdot 2^d$ vectors.

Taking superposition of products of θ^a's as creation operators and their Hermitian conjugated partners as annihilation operators, the creation and annihilation operators fulfill, applied on a simple vacuum state $|1>$, the anticommutation relations required for the second quantized fermions, if the unity is not included. The "Grassmann fermions" of an odd products of θ^a's carry integer spins and the charges in adjoint representations. There are no elementary fermions with integer spin observed so far.

In this Part II the properties of the two kinds of the Clifford algebras objects, γ^a's and $\tilde{\gamma}^a$'s, both expressible with θ^a's and $\frac{\partial}{\partial\theta_a}$'s ($\gamma^a = (\theta^a + \frac{\partial}{\partial\theta_a})$, $\tilde{\gamma}^a = i\,(\theta^a - \frac{\partial}{\partial\theta_a})$ [2,4,5]), are presented and the conditions discussed, which limit the space of Clifford "vectors", so that the Clifford algebra "vectors" of each irreducible representation of the corresponding Lorentz algebra of this limited space are equipped by the family quantum numbers. This limited space of the Clifford algebra "vectors", when used to describe the internal degrees of freedom of (the second quantized) fermions, explain the anticommutation relations postulated by Dirac [13].

These *Clifford second quantized fermions* enable the descriptions for not only spins and all the charges of the observed quarks and leptons, but also for their families.

We present in Sect. 5.2 properties of the Clifford algebra "vectors" in the space of d γ^a's and d $\tilde{\gamma}^a$'s and discuss conditions, under which operators of these two kinds of the Clifford algebra objects demonstrate by themselves the anticommutation relations required for the second quantized "fermions", manifesting the half integer spins, offering the explanation for the spin and charges of the observed quarks and leptons and anti-quarks and anti-leptons and also for their families, Refs. [1,2,6–12,3].

In Sect. 5.3 we comment on what we have learned from the second quantized "Grassmann fermions", carrying the integer spins and (from the point of view of $d = (3+1)$) the charges in the adjoint representations and compare these recognitions with the recognitions, which the Clifford algebra is offering for description of the fermions, appearing on families, with half integer spins and charges in the fundamental representations [1,2,6–11,3].

5.2 Second quantized fermions in Clifford space

We learn in Part I that in d-dimensional space of anticommuting Grassmann coordinates (and of their Hermitian conjugated partners — derivatives), Eqs. (2,6) of Part I, there exist two kinds of the Clifford coordinates (operators) — γ^a and $\tilde{\gamma}^a$ — which are expressible in terms of θ^a and their conjugate momentum $p^{\theta a} = i\,\frac{\partial}{\partial\theta_a}$ [2].

$$\gamma^a = (\theta^a + \frac{\partial}{\partial\theta_a})\,, \quad \tilde{\gamma}^a = i\,(\theta^a - \frac{\partial}{\partial\theta_a})\,,$$
$$\theta^a = \frac{1}{2}\,(\gamma^a - i\tilde{\gamma}^a)\,, \quad \frac{\partial}{\partial\theta_a} = \frac{1}{2}\,(\gamma^a + i\tilde{\gamma}^a)\,, \tag{5.1}$$

offering together $2\cdot 2^d$ operators: 2^d of those which are products of γ^a and 2^d of those which are products of $\tilde{\gamma}^a$.

Taking into account Eqs. (1, 2) of Part I, ($\{\theta^a, \theta^b\}_+ = 0$, $\{\frac{\partial}{\partial\theta_a}, \frac{\partial}{\partial\theta_b}\}_+ = 0$, $\{\theta_a, \frac{\partial}{\partial\theta_b}\}_+ = \delta_{ab}$, $\theta^{a\dagger} = \eta^{aa}\,\frac{\partial}{\partial\theta_a}$ and $(\frac{\partial}{\partial\theta_a})^\dagger = \eta^{aa}\theta^a$), one finds

$$\{\gamma^a, \gamma^b\}_+ = 2\eta^{ab} = \{\tilde{\gamma}^a, \tilde{\gamma}^b\}_+\,,$$
$$\{\gamma^a, \tilde{\gamma}^b\}_+ = 0\,, \quad (a, b) = (0, 1, 2, 3, 5, \cdots, d)\,,$$
$$(\gamma^a)^\dagger = \eta^a\,\gamma^a\,, \quad (\tilde{\gamma}^a)^\dagger = \eta^{aa}\,\tilde{\gamma}^a\,, \tag{5.2}$$

with $\eta^{ab} = diag\{1, -1, -1, \cdots, -1\}$.

It follows for the generators of the Lorentz algebra of each of the two kinds of the Clifford algebra operators, S^{ab} and $\tilde{S}^{ab}$, that:

$$S^{ab} = \frac{i}{4}(\gamma^a\gamma^b - \gamma^b\gamma^a)\,, \quad \tilde{S}^{ab} = \frac{i}{4}(\tilde{\gamma}^a\tilde{\gamma}^b - \tilde{\gamma}^b\tilde{\gamma}^a)\,,$$
$$\mathbf{S}^{ab} = S^{ab} + \tilde{S}^{ab}\,, \quad \{S^{ab}, \tilde{S}^{ab}\}_- = 0\,,$$
$$\{S^{ab}, \gamma^c\}_- = i(\eta^{bc}\gamma^a - \eta^{ac}\gamma^b)\,, \quad \{\tilde{S}^{ab}, \tilde{\gamma}^c\}_- = i(\eta^{bc}\tilde{\gamma}^a - \eta^{ac}\tilde{\gamma}^b)\,,$$
$$\{S^{ab}, \tilde{\gamma}^c\}_- = 0\,, \quad \{\tilde{S}^{ab}, \gamma^c\}_- = 0\,, \tag{5.3}$$

where $\mathbf{S}^{ab} = i\,(\theta^a\frac{\partial}{\partial\theta_b} - \theta^b\frac{\partial}{\partial\theta_a})$, Eq. (3) of Part I.

Let us make a choice of the Cartan subalgebra of the commuting operators of the Lorentz algebra for each of the two kinds of the operators of the Clifford algebra, S^{ab} and $\tilde{S}^{ab}$,

$$S^{03}, S^{12}, S^{56}, \cdots, S^{d-1\;d}\,,$$
$$\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56}, \cdots, \tilde{S}^{d-1\;d}\,. \tag{5.4}$$

The two kinds of the Lorentz algebras, the one generated by γ^a and the other by $\tilde{\gamma}^a$, are obviously completely independent. We make a choice of the irreducible representations of the two Lorentz groups to be the "eigenvectors" of the corresponding Cartan subalgebras of Eq. (5.4), and take into account Eq. (5.2),

$$
\begin{aligned}
S^{ab}\frac{1}{2}(\gamma^a+\frac{\eta^{aa}}{ik}\gamma^b) &= \frac{k}{2}\,\frac{1}{2}(\gamma^a+\frac{\eta^{aa}}{ik}\gamma^b)\,,\\
S^{ab}\frac{1}{2}(1+\frac{i}{k}\gamma^a\gamma^b) &= \frac{k}{2}\,\frac{1}{2}(1+\frac{i}{k}\gamma^a\gamma^b)\\
\tilde{S}^{ab}\frac{1}{2}(\tilde{\gamma}^a+\frac{\eta^{aa}}{ik}\tilde{\gamma}^b) &= \frac{k}{2}\,\frac{1}{2}(\tilde{\gamma}^a+\frac{\eta^{aa}}{ik}\tilde{\gamma}^b)\,,\\
\tilde{S}^{ab}\frac{1}{2}(1+\frac{i}{k}\tilde{\gamma}^a\tilde{\gamma}^b) &= \frac{k}{2}\,\frac{1}{2}(1+\frac{i}{k}\tilde{\gamma}^a\tilde{\gamma}^b)\,.
\end{aligned}
\tag{5.5}
$$

The Clifford "vectors" of both kinds are normalized, up to a phase, with respect to Eq. (4.21) of App. 4.4. Both have half integer spin. The "eigenvalues" of the operator S^{03}, for example, for the "vector" $\frac{1}{2}(\gamma^0 \mp \gamma^3)$ are equal to $\pm\,\frac{i}{2}$, respectively, for the "vector" $\frac{1}{2}(1\pm\gamma^0\gamma^3)$ are $\pm\,\frac{i}{2}$, respectively, while all the rest "vectors" have "eigenvalues" $\pm\,\frac{1}{2}$. One finds equivalently for the "eigenvectors" of the operator $\tilde{S}^{03}$: for $\frac{1}{2}\,(\tilde{\gamma}^0\mp\tilde{\gamma}^3)$ the "eigenvalues" $\pm\,\frac{i}{2}$, respectively, and for the "eigenvectors" $\frac{1}{2}(1\pm\tilde{\gamma}^0\tilde{\gamma}^3)$ the "eigenvalues" $k=\pm\,\frac{i}{2}$, respectively, while all the rest "vectors" have $k=\pm\,\frac{1}{2}$.

To make discussions easier let us introduce the notation for the "eigenvectors" of the two Cartan subalgebras, Eq. (5.4), Ref. [4,2].

$$
\begin{aligned}
&\stackrel{ab}{(k)}:=\frac{1}{2}(\gamma^a+\frac{\eta^{aa}}{ik}\gamma^b)\,, &&\stackrel{ab}{(k)}^{\dagger}=\eta^{aa}\stackrel{ab}{(-k)}\,, &&(\stackrel{ab}{(k)})^2=0\,,\\
&\stackrel{ab}{[k]}:=\frac{1}{2}(1+\frac{i}{k}\gamma^a\gamma^b)\,, &&\stackrel{ab}{[k]}^{\dagger}=\stackrel{ab}{[k]}\,, &&(\stackrel{ab}{[k]})^2=\stackrel{ab}{[k]}\,,\\
&\stackrel{ab}{(\tilde{k})}:=\frac{1}{2}(\tilde{\gamma}^a+\frac{\eta^{aa}}{ik}\tilde{\gamma}^b)\,, &&\stackrel{ab}{(\tilde{k})}^{\dagger}=\eta^{aa}\stackrel{ab}{(-\tilde{k})}\,, &&(\stackrel{ab}{(\tilde{k})})^2=0\,,\\
&\stackrel{ab}{[\tilde{k}]}:=\frac{1}{2}(1+\frac{i}{k}\tilde{\gamma}^a\tilde{\gamma}^b)\,, &&\stackrel{ab}{[\tilde{k}]}^{\dagger}=\stackrel{ab}{[\tilde{k}]}\,, &&(\stackrel{ab}{[\tilde{k}]})^2=\stackrel{ab}{[\tilde{k}]}\,,
\end{aligned}
\tag{5.6}
$$

with $k^2=\eta^{aa}\eta^{bb}$. Let us point out that the eigenvectors of the Cartan subalgebras are either the nilpotents — $(\stackrel{ab}{(k)})^2=0$ and $(\stackrel{ab}{(\tilde{k})})^2=0$ — or projectors — $(\stackrel{ab}{[k]})^2=\stackrel{ab}{[k]}$ and $(\stackrel{ab}{[\tilde{k}]})^2=\stackrel{ab}{[\tilde{k}]}$.

Representations of γ^a and representations of $\tilde{\gamma}^a$ are completely independent, each with $2^{\frac{d}{2}-1}$ members in $2\cdot 2\cdot 2^{\frac{d}{2}-1}$ representations.

5.2.1 Properties of Clifford vectors

2^{d-1} odd and 2^{d-1} even Grassmann operators, which are superposition of odd and even products of θ^a's, are well distinguishable from their 2^{d-1} odd and 2^{d-1} even Hermitian conjugated operators, which are superposition of odd and even products of $\frac{\partial}{\partial\theta_a}$'s, Eq. (6) in Part I.

In the Clifford case (of either γ^a's or $\tilde{\gamma}^a$'s) the "vectors", made of products of nilpotents ($\stackrel{ab}{(k)}$ or $\stackrel{ab}{(\tilde{k})}$) and projectors ($\stackrel{ab}{[k]}$ or $\stackrel{ab}{[\tilde{k}]}$), Eq. (5.6), which each of them are "eigenvectors" of one of the member of the Cartan subalgebra of one of the two kinds, Eq. (5.4), the relations among "vectors" and their Hermitian conjugated partners are less transparent (although easy to be evaluated). This can be noticed in Eq. (5.6), since $\frac{1}{\sqrt{2}}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b)^\dagger$ is $\eta^{aa}\ \frac{1}{\sqrt{2}}(\gamma^a + \frac{\eta^{aa}}{i(-k)}\gamma^b)$, while $\frac{1}{\sqrt{2}}(1 + \frac{i}{k}\gamma^a\gamma^b)$ are self adjoint. This is the case also for representations in the sector of $\tilde{\gamma}^a$'s.

Let us recognize the properties of the nilpotents and projectors. The relations are taken from Ref. [6].

$$
\begin{aligned}
&\overset{ab}{(k)}\overset{ab}{(k)} = 0\,, \quad &&\overset{ab}{(k)}\overset{ab}{(-k)} = \eta^{aa}\ \overset{ab}{[k]}\,, \quad &&\overset{ab}{[k]}\overset{ab}{[k]} = \overset{ab}{[k]}\,, \quad &&\overset{ab}{[k]}\overset{ab}{[-k]} = 0\,,\\
&\overset{ab}{(k)}\overset{ab}{[k]} = 0\,, \quad &&\overset{ab}{[k]}\overset{ab}{(k)} = \overset{ab}{(k)}\,, \quad &&\overset{ab}{(k)}\overset{ab}{[-k]} = \overset{ab}{(k)}\,, \quad &&\overset{ab}{[k]}\overset{ab}{(-k)} = 0\,.
\end{aligned}
\tag{5.7}
$$

The same relations are valid also if one replaces $\overset{ab}{(k)}$ with $\overset{ab}{(\tilde{k})}$ and $\overset{ab}{[k]}$ with $\overset{ab}{[\tilde{k}]}$.

We illustrate properties of "vectors" of the Clifford algebra of γ^a's on irreducible representations of the Lorentz group $SO(5,1)$ and their subgroups $SO(3,1)$ and $SO(1,1)$, presented in Table 5.1, for the case of $\tilde{\gamma}^a$'s all $\overset{ab}{(k)}$'s have to be replaced by $\overset{ab}{(\tilde{k})}$'s and all $\overset{ab}{[k]}$ by $\overset{ab}{[\tilde{k}]}$'s.

odd I	i	quadruplet a 03 12 56	quadruplet b 03 12 56	quadruplet c 03 12 56	quadruplet d 03 12 56	S^{03}	S^{12}	S^{56}	$\Gamma^{(5+1)}$	$\Gamma^{(3+1)}$
		03 12 56	03 12 56	03 12 56	03 12 56					
	1	(+i)(+)(+)	[+i][+](+)	[+i](+)[+]	(+i)[+][+]	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1
	2	[−i][−](+)	(−i)(−)(+)	(−i)[−][+]	[−i](−)[+]	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	1	1
	3	[−i](+)[−]	(−i)[+][−]	(−i)(+)(−)	[−i][+](−)	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	1	−1
	4	(+i)[−][−]	[+i](−)[−]	[+i][−](−)	(+i)(−)(−)	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	1	−1
odd II	i	quadruplet a 03 12 56	quadruplet b 03 12 56	quadruplet c 03 12 56	quadruplet d 03 12 56	S^{03}	S^{12}	S^{56}	$\Gamma^{(5+1)}$	$\Gamma^{(3+1)}$
	1	(−i)(+)(+)	[−i][+](+)	[−i](+)[+]	(−i)[+][+]	$-\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	−1	−1
	2	[+i][−](+)	(+i)(−)(+)	(+i)[−][+]	[+i](−)[+]	$\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	−1	−1
	3	[+i](+)[−]	(+i)[+][−]	(+i)(+)(−)	[+i][+](−)	$\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	−1	1
	4	(−i)[−][−]	[−i](−)[−]	[−i][−](−)	(−i)(−)(−)	$-\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	−1	1
even I	i	quadruplet a 03 12 56	quadruplet b 03 12 56	quadruplet c 03 12 56	quadruplet d 03 12 56	S^{03}	S^{12}	S^{56}	$\Gamma^{(5+1)}$	$\Gamma^{(3+1)}$
	1	[−i](+)(+)	(−i)[+](+)	[−i][+][+]	(−i)(+)[+]	$-\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	−1	−1
	2	(+i)[−](+)	[+i](−)(+)	(+i)(−)[+]	[+i][−][+]	$\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	−1	−1
	3	(+i)(+)[−]	[+i][+][−]	(+i)[+](−)	[+i](+)(−)	$\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	−1	1
	4	[−i][−][−]	(−i)(−)[−]	[−i](−)(−)	(−i)[−](−)	$-\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	−1	1
even II	i	quadruplet a 03 12 56	quadruplet b 03 12 56	quadruplet c 03 12 56	quadruplet d 03 12 56	S^{03}	S^{12}	S^{56}	$\Gamma^{(5+1)}$	$\Gamma^{(3+1)}$
	1	[+i](+)(+)	(+i)[+](+)	[+i][+][+]	(+i)(+)[+]	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1
	2	(−i)[−](+)	[−i](−)(+)	(−i)(−)[+]	[−i][−][+]	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	1	1
	3	(−i)(+)[−]	[−i][+][−]	(−i)[+](−)	[−i](+)(−)	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	1	−1
	4	[+i][−][−]	(+i)(−)[−]	[+i](−)(−)	(+i)[−](−)	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	1	−1

Table 5.1. $2^d = 64$ "eigenvectors" of the Cartan subalgebra, Eq. (5.4), of the Clifford γ^a algebra in $d = (5+1)$ are presented, divided into four groups of four irreducible representations. Two of four groups have an odd number of γ^a's. "Vectors" in the *odd I* part have Hermitian conjugated partners among "vectors" of the *odd II* part, and the opposite. The two groups with the even number of γ^a's, *even I* and *even II*, have their Hermitian conjugated partners within their own group each. Numbers — 03 12 56 — explain the indexes of the corresponding Cartan subalgebra. Equivalent table for $\tilde{\gamma}^a$'s follow by replacing all $\overset{ab}{(k)}$ by $\overset{ab}{(\tilde{k})}$ and $\overset{ab}{[k]}$ by $\overset{ab}{[\tilde{k}]}$.

There are in the γ^a part of the Clifford algebra "vectors" twice $2^{\frac{6}{2}-1} = 4$ odd irreducible representations, each representation with $2^{\frac{6}{2}-1} = 4$ members and twice 4 even irreducible representations with 4 members, as presented in Table 5.1. The representations for the $\tilde{\gamma}^a$ sector follow from Table 5.1, if one replaces $\overset{ab}{(k)}$ with $\overset{ab}{(\tilde{k})}$ and $\overset{ab}{[k]}$ with $\overset{ab}{[\tilde{k}]}$.

Hermitian conjugation transforms $2^{\frac{d}{2}-1}$ Clifford *odd* representations with $2^{\frac{d}{2}-1}$ members, into $2^{\frac{d}{2}-1} \cdot 2^{\frac{d}{2}-1}$ Hermitian conjugated partners for each kind of the two kinds of the Clifford algebra operators — γ^a and $\tilde{\gamma}^a$. Hermitian conjugated partners of one Lorentz irreducible representation with $2^{\frac{d}{2}-1}$ members, however, belong to $2^{\frac{d}{2}-1}$ Lorentz irreducible representations: The first column of the four representations in the *odd I* part has the corresponding Hermitian conjugated partners in the fourth line of the *odd II*, for example.

In Table 5.2 only one *quadruplet* is presented, the *quadruplet a* from Table 5.1, together with the corresponding Hermitian conjugated partner. All the "vectors" of the quadruplet are orthogonal among themselves and so are also the "vectors" of the Hermitian conjugated partners. The product of each of the Hermitian conjugated partner with its "vector" gives $\overset{03}{[-i]}\overset{12}{[-1]}(\overset{56}{[-1]}$. For the first "vector" one finds: $\overset{03}{(-i)}\overset{12}{(-)}\overset{56}{(-)} \cdot \overset{03}{(+i)}\overset{12}{(+)}\overset{56}{(+)}=\overset{03}{[-i]}\overset{12}{[-1]}\overset{56}{[-1]}$. This follows by taking into account Eq. (5.7).

If we denote by $\hat{b}^{m\dagger}_f$, with $f = 1$ and $m = (1, 2, 3, 4)$, the first four "vectors" of Table 5.2, and their Hermitian conjugated partners by $(\hat{b}^{m\dagger}_f)^\dagger = \hat{b}^m_f$, with $f = 1$ and $m = (1, 2, 3, 4)$, we can write

$$\begin{aligned}\hat{b}^{m'}_f \cdot \hat{b}^{m\dagger}_f &= \delta^{mm'} \; \overset{03}{[-i]}\overset{12}{[-1]}(\overset{56}{[-1]}\,,\\ &\text{for } f = 1 \text{ and all } (m, m')\,. \end{aligned} \tag{5.8}$$

One easily checks, taking into account Eq. (5.7), that *quadruplets (a,b,c,d)* of the irreducible representation *odd I* fulfill the equivalent relations, only the products of Hermitian conjugated partner m with its "vector" m change: It follows that $\hat{b}^{m'}_f \cdot \hat{b}^{m\dagger}_f = \delta^{mm'}\, (\overset{03}{[-i]}\overset{12}{[-1]}\overset{56}{[-1]}\,, \overset{03}{[+i]}\overset{12}{[+1]}\overset{56}{[-1]}\,, \overset{03}{[+i]}\overset{12}{[-1]}\overset{56}{[+1]}\,, \overset{03}{[-i]}\overset{12}{[+1]}\overset{56}{[+1]}\,)$ for $f = (1, 2, 3, 4)$, respectively. All these "vectors", which are products of $\hat{b}^m_f \cdot \hat{b}^{m\dagger}_f$, are products of selfadjoint projectors only, having an even Clifford character.

One can check for $d = (5 + 1)$, using Eq. (5.7), that it follows.

$$\begin{aligned}
&\hat{b}^m_f \cdot \hat{b}^{m'}_{f'} = 0\,,\\
&\hat{b}^{m\dagger}_f \cdot \hat{b}^{m'\dagger}_{f'} = 0\,,\\
&\hat{b}^m_f \cdot \hat{b}^{m'\dagger}_f = \delta^{mm'}\, |\psi_{oc}>\,, \quad \text{for a chosen } f\,,\\
&\hat{b}^{m\dagger}_f\, |\psi_{oc}> = |\psi^m_f>\,,\\
&\hat{b}^m_f\, |\psi_{oc}> = 0\,,
\end{aligned} \tag{5.9}$$

for all (f, f') and all (m, m') of Clifford odd Lorentz irreducible representations, with the normalized vacuum state $|\psi_{oc}>= \frac{1}{\sqrt{2^{\frac{6}{2}-1}}} (\stackrel{03}{[-i]}\stackrel{12}{[-1]}\stackrel{56}{[-1]} +(\stackrel{03}{[+i]}\stackrel{12}{[+1]}\stackrel{56}{[-1]}$ $+(\stackrel{03}{[+i]}\stackrel{12}{[-1]}\stackrel{56}{[+1]} +(\stackrel{03}{[-i]}\stackrel{12}{[+1]}\stackrel{56}{[+1]})$.

The generalization of these recognitions to any even d, if d is either $d = 2(2n+1)$ or $d = 4n$, n is a positive integer, is straightforward. We shall do this in Subsect. 5.2.3).

i	quadruplet a	Her. con. quadruplet a
1	$\stackrel{03}{(+i)}\stackrel{12}{(+)}\stackrel{56}{(+)}$	$\stackrel{03}{(-i)}\stackrel{12}{(-)}\stackrel{56}{(-)}$
2	$\stackrel{03}{[-i]}\stackrel{12}{[-]}\stackrel{56}{(+)}$	$\stackrel{03}{[-i]}\stackrel{12}{[-]}\stackrel{56}{(-)}$
3	$\stackrel{03}{[-i]}\stackrel{12}{(+)}\stackrel{56}{[-]}$	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\stackrel{56}{[-]}$
4	$\stackrel{03}{(+i)}\stackrel{12}{[-]}\stackrel{56}{[-]}$	$\stackrel{03}{(-i)}\stackrel{12}{[-]}\stackrel{56}{[-]}$

Table 5.2. The *quadruplet a* of the irreducible representation *odd I*, from Table 5.1, $d = (5+1)$, together with the Hermitian conjugated partner is presented. Each member of the *quadruplet a* is a product of nilpotents and projectors, which are the "eigenvectors" of the Cartan subalgebra, Eq. (5.4), of the Clifford γ^a algebra.

Let us noticed that all the vectors of the first column, *odd I*, when applied on the selfadjoint "vector" of the *quadruplet a* of *even I*, give the vectors of the first column, *odd I*, back, Eq. (5.7). The vectors of the second column, *quadruplet b*, *odd I*, when applied on the selfadjoint "vector" of the *quadruplet b*, *even I*, give the vectors of the second column back. This also happens to the third column, *quadruplet c*, *odd I*, when applied on the selfadjoint "vector" of the *quadruplet c*, *even I*, and to the fourth column, *quadruplet d*, *odd I*, when applied on the self adjoint vector of the *quadruplet d even I*. Similar properties follow when the columns of *odd II* apply on the corresponding selfadjoint operators of *even II*.

Let us notice also that all the annihilation operators anticommute among themselves, $\{\hat{b}^{m'}_{f}, \hat{b}^{m}_{f'}\}_{+} = 0$, the same is true for creation operators, $\{\hat{b}^{m'\dagger}_{f}, \hat{b}^{m\dagger}_{f}\}_{+} = 0$, while $\{\hat{b}^{m'}_{f'}, \hat{b}^{m\dagger}_{f}\}_{+}|_{f'=f} = \delta^{mm'}|\psi_{oc}>$ is valid only for $f' = f$ and not for the rest members of particular family to which $\hat{b}^{m'}_{f}$ belong [1].

In any even dimensional space there is in any Clifford even irreducible representation of the corresponding Lorentz algebra of the two kinds of Clifford "vectors" (defined by either γ^a's or $\tilde{\gamma}^a$'s) one member, which is the product of $\frac{d}{2}$ selfadjoint projectors $(1 + \frac{i}{k}\gamma^a\gamma^b)$. Correspondingly the whole "vector" is self-

[1] Anticommutator $\{\stackrel{03}{(+i)}\stackrel{12}{(+)}\stackrel{56}{(+)}, \stackrel{03}{[+i]}\stackrel{12}{[+]}\stackrel{56}{(-)}\}_{+} = -\stackrel{03}{(+i)}\stackrel{12}{(+)}\stackrel{56}{[-]}$, for example, and applied on the first summand of $|\psi_{oc}>$ gives this Clifford even creation operator $-\stackrel{03}{(+i)}\stackrel{12}{(+)}\stackrel{56}{[-]}$ back, which can be found in Table 5.1 among *even I* in the third line of the column *quadruplet a*, while $\stackrel{03}{[+i]}\stackrel{12}{[+]}\stackrel{56}{(-)}$ appears in the third line of *quadruplet d* in *odd II* and $\stackrel{03}{(+i)}\stackrel{12}{(+)}\stackrel{56}{(+)}$ appears in the first line of *quadruplet a* in *odd I* of the same table.

adjoint. In Table 5.1 there are in *even I* representations of Clifford even "vectors" four "vectors" ($m = (4, 3, 1, 2)$ of *quadruplets (a,b,c,d)*, respectively), which can be obtained as well from the application of the annihilation operator $\hat{b}^{m'}_{f}$ (*odd II*) on its creation partner $\hat{b}^{m\dagger}_{f}$ (*odd I*), for each irreducible representation f separately.

The selfadjoint even "vectors" appear also in *even II* sector, belonging as well to different irreducible representations of the Lorentz group (in the *quadruplets (a,b,c,d)* they carry the family member number $m = (4, 3, 1, 2)$, respectively). All the Clifford even "vectors" of the same irreducible Lorentz representation, applied on their selfadjoint "vector", gives these "vectors" back.

All the Clifford even representations follow from the products of the Clifford odd "vectors",

Equivalent Clifford even representations as in the space of γ^a's appear also in the space of $\tilde{\gamma}^a$'s.

5.2.2 Second quantized "Clifford fermions"

We learned in Subsect. 5.2.1 that:

a. The two vector spaces, the one spanned by γ^a's and the second one spanned by $\tilde{\gamma}^a$'s, are completely independent vector spaces, each with 2^d "vectors". The Clifford odd "vectors" (the superposition of products of odd numbers of γ^a's or $\tilde{\gamma}^a$'s, respectively) can be arranged for each kind of the Clifford algebras as twice $2^{\frac{d}{2}-1} \cdot 2^{\frac{d}{2}-1}$ irreducible representations of the Lorentz group.

The Clifford even part (made of superposition of products of even numbers of γ^a's and $\tilde{\gamma}^a$'s, respectively) splits again into twice $2^{\frac{d}{2}-1} \cdot 2^{\frac{d}{2}-1}$ irreducible representations of the Lorentz group. **b.** The two groups of the Clifford odd parts (of each of the two kinds) of "vectors", each with $2^{\frac{d}{2}-1}$ irreducible representations of $2^{\frac{d}{2}-1}$ members, are Hermitian conjugated to each other.

b.i. The members of one irreducible representation share all the quantum numbers (determined by the members of the Cartan sublagebra (of either S^{ab} or $\tilde{S}^{ab}$) with the corresponding members of another irreducible representations. The same is true also for their Hermitian conjugated partners.

b.ii. The $2^{\frac{d}{2}-1}$ members of each of the $2^{\frac{d}{2}-1}$ irreducible representations are orthogonal and so are orthogonal their corresponding Hermitian conjugated partners.

b.iii. Making a choice of "vectors" and denoting them by $\hat{b}^{m\dagger}_{f}$, (where f denotes different irreducible representations and m a member in the representation f), and their Hermitian conjugate partners by $\hat{b}^{m}_{f} = (\hat{b}^{m\dagger}_{f})^{\dagger}$, while choosing the vacuum state $|\psi_{oc}>$ as the sum of all the products of $\hat{b}^{m}_{f} \cdot \hat{b}^{m\dagger}_{f}$ for all $f = (1, 2, \cdots, 2^{\frac{d}{2}-1})$, we end up with Eq. (5.9), valid for superposition of odd products of either γ^a's or $\tilde{\gamma}^a$'s, each in its own "vector space".

b.iv. The Clifford odd creation and annihilation operators of any irreducible representation f obey the anticommutation relations, postulated by Dirac for fermions. However (as we learn in Subsect. 5.2.1), there exist among annihilation operators $2^{\frac{d}{2}-1} - 1$ members of the same irreducible representation of annihilation operators, to which the particular Hermitian conjugated partner $\hat{b}^{m}_{f}$ (of a particular creation operator $\hat{b}^{m\dagger}_{f}$) belong (obviously obtainable by the generators of the

Lorentz transformations, S^{ab} or $\tilde{S}^{ab}$, respectively), the anticommutators of which with the creation operator $\hat{b}_f^{m\dagger}$ gives one of the $2^{\frac{d}{2}-1}$ members (In Table 5.1 one gets *quadruplets (a,b,c,d)* of *even I*, if one chooses $\hat{b}_f^{m\dagger}$ from *odd I* — otherwise one would get one member of *even II* — which does not belong to self adjoint operators).

c. There are the same number of the Clifford even irreducible representations — twice $2^{\frac{d}{2}-1}$, each with $2^{\frac{d}{2}-1}$ number of members — as in the case of the odd irreducible representations. While in the case of the odd irreducible representations the two groups of $2^{\frac{d}{2}-1}$ representations, each with $2^{\frac{d}{2}-1}$ members, are Hermitian conjugated to each other, the Hermitian conjugated partners appear in the even case within each of the two groups separately.

c.i. The members of one irreducible representation share all the quantum numbers (determined by the members of the Cartan sublagebra (of either S^{ab} or $\tilde{S}^{ab}$) with the corresponding members of another irreducible representations.

c.ii. Only $2^{\frac{d}{2}-1}-1$ members of each of the $2^{\frac{d}{2}-1}$ irreducible representations of each of the two groups are orthogonal to each other, while their application on the member which is the product of the projectors only, gives the same member back. All the members of one irreducible representation are orthogonal to all the members of another representation and to all the members of all the representations of another group.

c.iii. All the Clifford even "vectors" can be expressed as the products of the Clifford odd "vectors".

The creation and annihilation operators of an odd Clifford algebras of both kinds, of either γ^a's or $\tilde{\gamma}^a$'s, would obviously obey the anticommutation relations for the second quantized fermions, postulated by Dirac, provided that each of the irreducible representations would carry a different quantum number.

But we know that a particular member m of all the irreducible representations have the same quantum numbers, that is the same "eigenvalues" of the Cartan subalgebra (for the vector space of either γ^a's or $\tilde{\gamma}^a$'s) Eq. (5.6).

The only possibility to "dress" each irreducible representation of one kind of the two independent vector spaces with a new, let us say "family" quantum number, is that we "sacrifice" one of the two vector spaces, let us make a choice of $\tilde{\gamma}^a$'s, and use these operators to define the "family" quantum number for the irreducible representation of the vector space of γ^a's, keeping the relations of Eq. (5.2) unchanged: $\{\gamma^a, \gamma^b\}_+ = 2\eta^{ab} = \{\tilde{\gamma}^a, \tilde{\gamma}^b\}_+, \{\gamma^a, \tilde{\gamma}^b\}_+ = 0, (\gamma^a)^\dagger = \eta^{aa}\gamma^a$, $(\tilde{\gamma}^a)^\dagger = \eta^{aa}\tilde{\gamma}^a$, $(a, b) = (0, 1, 2, 3, 5, \cdots, d)$.

We therefore *postulate*:

Let $\tilde{\gamma}^a$'s operate on γ^a's as follows [5,2,10,11,5,3]

$$\tilde{\gamma}^a B(\gamma^a) = (-)^B \, i \, B\gamma^a \,, \tag{5.10}$$

with $(-)^B = -1$, if B is an odd product of γ^a's, otherwise $(-)^B = 1$ [5].

The vector space of $\tilde{\gamma}^a$'s have correspondingly no meaning any longer, it is "frozen out". (No vector space of $\tilde{\gamma}^a$'s can be taken into account any longer).

Taking into account Eq. (5.10) we can check that

a. Relations of Eq. (5.2) remain unchanged.

b. Relations of Eq. (5.6) remain unchanged.

c. The eigenvalues besides of the operators S^{ab} also of $\tilde{S}^{ab}$ on nilpotents and projectors of γ^{a}'s can be calculated, leading to

$$
\begin{aligned}
S^{ab} \overset{ab}{(k)} &= \frac{k}{2} \overset{ab}{(k)}\,, &\qquad \tilde{S}^{ab} \overset{ab}{(k)} &= \frac{k}{2} \overset{ab}{(k)}\,,\\
S^{ab} \overset{ab}{[k]} &= \frac{k}{2} \overset{ab}{[k]}\,, &\qquad \tilde{S}^{ab} \overset{ab}{[k]} &= -\frac{k}{2} \overset{ab}{[k]}\,,
\end{aligned}
\tag{5.11}
$$

demonstrating that the eigenvalues of S^{ab} on nilpotents and projectors of γ^{a}'s differ from the eigenvalues of $\tilde{S}^{ab}$, so that $\tilde{S}^{ab}$ can be used to denote irreducible representations of S^{ab} with the "family" quantum number.

d. We further recognize that γ^{a} transform $\overset{ab}{(k)}$ into $\overset{ab}{[-k]}$, never to $\overset{ab}{[k]}$, while $\tilde{\gamma}^{a}$ transform $\overset{ab}{(k)}$ into $\overset{ab}{[k]}$, never to $\overset{ab}{[-k]}$

$$
\begin{aligned}
&\gamma^{a} \overset{ab}{(k)} = \eta^{aa} \overset{ab}{[-k]},\; \gamma^{b} \overset{ab}{(k)} = -ik \overset{ab}{[-k]},\; \gamma^{a} \overset{ab}{[k]} = \overset{ab}{(-k)},\; \gamma^{b} \overset{ab}{[k]} = -ik\eta^{aa} \overset{ab}{(-k)}\,,\\
&\tilde{\gamma}^{a} \overset{ab}{(k)} = -i\eta^{aa} \overset{ab}{[k]},\; \tilde{\gamma}^{b} \overset{ab}{(k)} = -k \overset{ab}{[k]},\; \tilde{\gamma}^{a} \overset{ab}{[k]} = \;i \overset{ab}{(k)},\; \tilde{\gamma}^{b} \overset{ab}{[k]} = -k\eta^{aa} \overset{ab}{(k)}
\end{aligned}
\tag{5.12}
$$

e. One finds, using Eq. (5.10),

$$
\begin{aligned}
&\overset{ab}{(\tilde{k})} \overset{ab}{(k)} = 0\,, \quad \overset{ab}{(\widetilde{-k})} \overset{ab}{(k)} = -i\eta^{aa} \overset{ab}{[k]}\,, \quad \overset{ab}{(\tilde{k})} \overset{ab}{[k]} = i \overset{ab}{(k)}\,, \quad \overset{ab}{(\tilde{k})} \overset{ab}{[-k]} = 0\,,\\
&\overset{ab}{[\tilde{k}]} \overset{ab}{(k)} = \overset{ab}{(k)}\,, \quad \overset{ab}{[\widetilde{-k}]} \overset{ab}{(k)} = 0\,, \quad \overset{ab}{[\tilde{k}]} \overset{ab}{[k]} = 0\,, \quad \overset{ab}{[\widetilde{-k}]} \overset{ab}{[k]} = \overset{ab}{[k]}\;.
\end{aligned}
\tag{5.13}
$$

f. From Eq. (5.12) it follows

$$
\begin{aligned}
S^{ac} \overset{ab}{(k)}\overset{cd}{(k)} &= -\frac{i}{2}\eta^{aa}\eta^{cc} \overset{ab}{[-k]}\overset{cd}{[-k]}\,, &\quad \tilde{S}^{ac} \overset{ab}{(k)}\overset{cd}{(k)} &= \frac{i}{2}\eta^{aa}\eta^{cc} \overset{ab}{[k]}\overset{cd}{[k]}\,,\\
S^{ac} \overset{ab}{[k]}\overset{cd}{[k]} &= \frac{i}{2} \overset{ab}{(-k)}\overset{cd}{(-k)}\,, &\quad \tilde{S}^{ac} \overset{ab}{[k]}\overset{cd}{[k]} &= -\frac{i}{2} \overset{ab}{(k)}\overset{cd}{(k)}\,,\\
S^{ac} \overset{ab}{(k)}\overset{cd}{[k]} &= -\frac{i}{2}\eta^{aa} \overset{ab}{[-k]}\overset{cd}{(-k)}\,, &\quad \tilde{S}^{ac} \overset{ab}{(k)}\overset{cd}{[k]} &= -\frac{i}{2}\eta^{aa} \overset{ab}{[k]}\overset{cd}{(k)}\,,\\
S^{ac} \overset{ab}{[k]}\overset{cd}{(k)} &= \frac{i}{2}\eta^{cc} \overset{ab}{(-k)}\overset{cd}{[-k]}\,, &\quad \tilde{S}^{ac} \overset{ab}{[k]}\overset{cd}{(k)} &= \frac{i}{2}\eta^{cc} \overset{ab}{(k)}\overset{cd}{[k]}\;.
\end{aligned}
\tag{5.14}
$$

g. Each irreducible representation of the *odd I* has now the "family" quantum number, determined by $\tilde{S}^{ab}$ of the Cartan subalgebra of Eq. (5.4). Correspondingly the creation and annihilation operators fulfill the anticommutation relations of Dirac fermions, without postulating them.

$$
\begin{aligned}
\{\hat{b}^{m}_{f}, \hat{b}^{m'\dagger}_{f'}\}_{+} \,|\psi_{oc}> &= \delta^{mm'}\,\delta_{ff'}\,|\psi_{oc}>\,,\\
\{\hat{b}^{m}_{f}, \hat{b}^{m'}_{f'}\}_{+} \,|\psi_{oc}> &= 0\,|\psi_{oc}>\,,\\
\{\hat{b}^{m\dagger}_{f}, \hat{b}^{m'\dagger}_{f'}\}_{+} \,|\psi_{oc}> &= 0\,|\psi_{oc}>\,,\\
\hat{b}^{m\dagger}_{f} \,|\psi_{oc}> &= |\psi^{m}_{f}>\,,\\
\hat{b}^{m}_{f} \,|\psi_{oc}> &= 0\,|\psi_{oc}>\,,
\end{aligned}
\tag{5.15}
$$

with (m, m') denoting the "family" member and (f, f') denoting "families".

h. The vacuum state for the vector space determined by γ^a's remains unchanged $|\psi_{oc}>$, Eq. (80) of Ref. [3].

$$
\begin{aligned}
|\psi_{oc}> &= \stackrel{03}{[-i]}\stackrel{12}{[-]}\stackrel{56}{[-]}\cdots\stackrel{d-1\,d}{[-]} + \stackrel{03}{[+i]}\stackrel{12}{[+]}\stackrel{56}{[-]}\cdots\stackrel{d-1\,d}{[-]} + \stackrel{03}{[+i]}\stackrel{12}{[-]}\stackrel{56}{[+]}\cdots\stackrel{d-1\,d}{[-]} + \cdots |1>\,,\\
&\text{for } d = 2(2n+1)\,,\\
|\psi_{oc}> &= \stackrel{03}{[-i]}\stackrel{12}{[-]}\stackrel{35}{[-]}\cdots\stackrel{d-3\,d-2}{[-]}\stackrel{d-1\,d}{[+]} + \stackrel{03}{[+i]}\stackrel{12}{[+]}\stackrel{56}{[-]}\cdots\stackrel{d-3\,d-2}{[-]}\stackrel{d-1\,d}{[+]} + \cdots |1>\,,\\
&\text{for } d = 4n\,,
\end{aligned}
\tag{5.16}
$$

n is a positive integer.

i. Taking into account relation among θ^a in Eq. (5.1) it follows from Eq. (5.10), since $\tilde{\gamma}^a \cdot 1 = i\gamma^a$

$$
\theta^a = \gamma^a\,, \quad \frac{\partial}{\partial \theta_a} = 0\,. \tag{5.17}
$$

The Hermitian conjugated part of the space in the Grassmann case "freezed out" together with the "vector" space of $\tilde{\gamma}^a$'s.

5.2.3 Second quantization of "Clifford fermions" with families in any d

Let us generalize what we learned in Subsect. 5.2.2 to any dimension d, with the vector space determined by γ^a's, while $\tilde{\gamma}^a$'s define the family quantum numbers of each creation operator $\hat{b}^{m\dagger}_f$, which is the product of nilpotents and projectors, Eq. (5.6).

Let us make a choice of the starting creation operator $\hat{b}^{1\dagger}_1$ of an odd Clifford character and their Hermitian conjugated partner in $d = 2(2n+1)$ as follows

$$
\begin{aligned}
\hat{b}^{1\dagger}_1 &:= \stackrel{03}{(+i)}\stackrel{12}{(+)}\stackrel{56}{(+)}\cdots\stackrel{d-3\,d-2}{(+)}\stackrel{d-1\,d}{(+)}\,,\\
\hat{b}^{1}_1 &= (\hat{b}^{1\dagger}_1)^\dagger = \stackrel{d-1\,d}{(-)}\stackrel{d-3\,d-2}{(-)}\cdots\stackrel{56}{(-)}\stackrel{12}{(-)}\stackrel{01}{(-i)}\,.
\end{aligned}
\tag{5.18}
$$

All the rest "vectors", belonging to the same Lorentz representation, follow by the application of the Lorentz generators S^{ab}'s.

The representations with different "family" quantum numbers are reachable by $\tilde{S}^{ab}$, since, according to Eq. (5.14), we recognize that $\tilde{S}^{ac}$ transforms two nilpotents $\stackrel{ab}{(k)}\stackrel{cd}{(k)}$ into two projectors $\stackrel{ab}{[k]}\stackrel{cd}{[k]}$, without changing k ($\tilde{S}^{ac}$ transforms $\stackrel{ab}{[k]}\stackrel{cd}{[k]}$ into $\stackrel{ab}{(k)}\stackrel{cd}{(k)}$, as well as $\stackrel{ab}{[k]}\stackrel{cd}{(k)}$ into $\stackrel{ab}{(k)}\stackrel{cd}{[k]}$). All the "family" members are reachable from one member of a new family also by the application of S^{ab}'s from any of the family members of a particular family.

In this way, by starting with the creation operator $\hat{b}^{1\dagger}_1$, Eq. (5.18), $2^{\frac{d}{2}-1}$ "families" each with $2^{\frac{d}{2}-1}$ "family" members follow. (In the *odd I* part of Table 5.1 we correspondingly recognize four representations with the "family" quantum numbers $(\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56}) = [(\frac{i}{2}, \frac{1}{2}, \frac{1}{2}), (-\frac{i}{2}, -\frac{1}{2}, \frac{1}{2}), (-\frac{i}{2}, \frac{1}{2}, -\frac{1}{2}), (\frac{i}{2}, -\frac{1}{2}, -\frac{1}{2})]$, respectively, for $d = (5+1)$.)

The corresponding annihilation operators, that is the Hermitian conjugated partners of $2^{\frac{d}{2}-1}$ "families", each with $2^{\frac{d}{2}-1}$ "family" members, following from the starting creation operator $\hat{b}^{1\dagger}_{1}$, can be obtained besides with the Hermitian conjugation also by the application of $\tilde{\gamma}^a \gamma^a$ on any member of any "family" of the Clifford odd creation operators. (The application of $\tilde{\gamma}^0 \gamma^0$ on $\hat{b}^{1\dagger}_{1}$ leads to $\hat{b}^{1}_{1}$), all the rest $2^{\frac{d}{2}-1} \cdot 2^{\frac{d}{2}-1}$ annihilation operators follow by the application of S^{ab} and $\tilde{S}^{ab}$ on $\hat{b}^{1}_{1}$). (Table 5.1 represents in the *odd II* part the annihilation operators to the creation partners of the *odd I* part.)

The creation and annihilation operators of an odd Clifford character, expressed by nilpotents and projectors of γ^a's, obey the anticommutation relations of Eq. (5.15), *without postulating the second quantized anticommutation relations*.

The even partners of the Clifford odd creation and annihilation operators follow by either the application of γ^a on the creation operators, leading to $2^{\frac{d}{2}-1}$ "families", each with $2^{\frac{d}{2}-1}$ members, or with the application of $\tilde{\gamma}^a$ on the creation operators, leading to another group of the Clifford even operators, again with the $2^{\frac{d}{2}-1}$ "families", each with $2^{\frac{d}{2}-1}$ members.

It is not difficult to recognize, that each of the Clifford even "families", obtained by the application of γ^a on the creation operators contains one selfadjoint operator, which is the product of projectors only, determining the vacuum state, Eq. (5.16). (Table 5.1 represents in the *even I* part these four selfadjoint operators, together with the rest of $(2^{\frac{6}{2}-1} - 1) \cdot 2^{\frac{6}{2}-1}$ Clifford even operators.)

The second Clifford even group of $2^{\frac{d}{2}-1}$ "families" with $2^{\frac{d}{2}-1}$ members, which follows by the application of γ^a on the annihilation operators, has again $2^{\frac{d}{2}-1}$ selfadjoint operators, which would determine the vacuum state, if the annihilation and the creation operators would exchange their roles. (Table 5.1 represents in the *even II* part the second group of even operators, with $\cdot 2^{\frac{6}{2}-1}$ selfadjoint operators, together with the rest of $(2^{\frac{d}{2}-1} - 1) \cdot 2^{\frac{d}{2}-1}$ Clifford even operators.)

5.2.4 *Action for free massless Clifford "fermions" with half integer spin*

The Lorentz invariant action for a free massless fermion in Clifford space is well known

$$\mathcal{A} = \int d^d x \, \frac{1}{2} \, (\psi^{\dagger} \gamma^0 \, \gamma^a p_a \psi) + h.c. \,, \tag{5.19}$$

$p_a = i \, \frac{\partial}{\partial x^a}$, leading to the equations of motion

$$\gamma^a p_a |\psi> = 0 \,, \tag{5.20}$$

which fulfill also the Klein-Gordon equation

$$\gamma^a p_a \gamma^b p_b |\psi> = p^a p_a |\psi> = 0 \,, \tag{5.21}$$

for each of the basic states $|\psi^m_f>$. γ^0 appears in the action to take care of the Lorentz invariance of the action.

Solutions of Eq. (5.20) are for free massless "fermions" superposition of $\hat{b}^{m\dagger}_{f}$, for a chosen "family" f, describing internal degrees of freedom, with coefficients depending on momentum p^a, $a = (0,1,2,3,5,\dots,d)$ of the plane wave solution $e^{-ip_a x^a}$

$$|\phi^s_{fp}> = \sum_m c^{ms}{}_{fp}\, \hat{b}^{m\dagger}_{f}\, e^{-ip_a x^a}\, |\psi_{oc}>,$$
$$\hat{b}^{s\dagger}_{fp} = \sum_m c^{ms}{}_{fp}\, \hat{b}^{m\dagger}_{f}\, e^{-ip_a x^a}\,, \tag{5.22}$$

s represents different solutions of the equations of motion, and, since they are orthonormalized, they fulfill the relation $< \phi^s_{fp}|\phi^{s'}_{f'p'} >= \delta_{ss'}\, \delta_{ff'}\, \delta^{pp'}$, where we assumed the discretization of momenta p^a.

5.2.5 *Solutions for n free massless Clifford "fermions" with half integer spin with the family quantum number*

The number of creation operators $\hat{b}^{s\dagger}_{fp}$ in d-dimensional space is

$$2^{\frac{d}{2}-1} \cdot 2^{\frac{d}{2}-1} \tag{5.23}$$

for a chosen momentum p^a, due to the number of families and number of members in each family, respectively. They all anticommute, fulfilling with the annihilation operators Eq. (5.15) ([3] andreferences therein).

When we discus more then one "fermion", we must keep in mind that the number of creation operators for a particular momentum is

$$2^{2^{\frac{d}{2}-1}\cdot 2^{\frac{d}{2}-1}}, \tag{5.24}$$

since each state can be either fulfilled by a fermion or empty. Since the momentum can be any and the solutions of different momentum are, in the discretized case, orthogonal, the number of states is correspondingly infinite.

Since the states are for different momentum orthogonal, the creation and annihilation operators fulfill the anticommutation relations of Eq. (5.15) for each momentum p^a.

$$\{\hat{b}^s_{fp}, \hat{b}^{s'\dagger}_{f'p'}\}_+ |\psi_{oc}> = \delta^{ss'}\, \delta_{ff'}\, \delta_{pp'}\, |\psi_{oc}>,$$
$$\{\hat{b}^s_{fp}, \hat{b}^{s'}_{f'p'}\}_+ |\psi_{oc}> = 0\, |\psi_{oc}>,$$
$$\{\hat{b}^{s\dagger}_{fp}, \hat{b}^{s'\dagger}_{f'p'}\}_+ |\psi_{oc}> = 0\, |\psi_{oc}>,$$
$$\hat{b}^{s\dagger}_{fp}\, |\psi_{oc}> = |\psi^s_{fp}>,$$
$$\hat{b}^s_{fp}\, |\psi_{oc}> = 0\, |\psi_{oc}>\,. \tag{5.25}$$

In Ref. [3], Eqs. (47, 65, 87), discuss properties of the n fermion states.

5.3 Conclusions

We learn in Part I of this paper, that odd products of superposition of θ^a's, Eqs. (7,6) in Part I, exist, which together with their Hermitian conjugated partners, fulfill all the requirements for the anticommutation relations for the Dirac fermions. There is no need to postulate the anticommutation relations. However, *these "fermions" carry the integer spin and the corresponding charges originating in* $d \geq 5$ *belong to adjoint representations*. No families appear in this case, that means that there is no available operators, which would connect different irreducible representations of the Lorentz group.

In Part II we learn that the Grassmann space offers two kinds of the Clifford operators — γ^a's and $\tilde{\gamma}^a$'s. Both kinds of the Clifford objects define two kinds of independent Clifford spaces. "Vectors" of an odd products of γ^a's or $\tilde{\gamma}^a$'s, respectively, carry the half integer spins and charges, originating in $d \geq 5$, in fundamental representations. Both kinds of odd Clifford "vectors" together offer two times $2^{\frac{d}{2}-1} \cdot 2^{\frac{d}{2}-1}$ creation operators and two times $2^{\frac{d}{2}-1} \cdot 2^{\frac{d}{2}-1}$ annihilation operators. The Clifford odd creation and annihilation operators of both kinds of the Clifford spaces for each of the corresponding irreducible Lorentz representations separately fulfill the anticommutation relations for the Dirac fermions – without postulating them.

To achieve that at least in one of the two groups of the Clifford odd creation and annihilation operators fulfill all the requirements for the Dirac fermions also when different irreducible representations are taken into account, the "family" quantum number must be introduced for any of the irreducible representation.

To achieve this we "sacrifice" one of the two kinds of the Clifford vector spaces — the one determined by $\tilde{\gamma}^a$'s — and use the corresponding $\tilde{S}^{ab}$'s to define the "family" quantum number for each irreducible representation of S^{ab}. The creation operators $\hat{b}^{m\dagger}_f$ and the annihilation operators $\hat{b}^{m'}_{f'}$ — (f, f') determine now family quantum numbers and (m, m') determine family members quantum numbers — fulfill the anticommutation relations of Eq. (5.15). The solutions of equations of motion for free massless fermions, Eq. (5.20), for a particular momentum p^a fulfill correspondingly the anticommutation relations of Eq. (5.25).

Solutions of equation of motion of different moments p^a obviously anticommute, due to the fact that the creation and annihillation operators fullfil the anticommutation relations of of Eq. (5.15). *There is no need to postulate anticommutation relations as Dirac did for the second quantized fermions.*

The Clifford algebra by itself, including "families", explains the Dirac assumption for second quantized fermions with the half integer spins and the charges in the fundamental representations, if charges origin in $d \geq 5$.

The reduction of the Clifford space, defined with two completely independent operators γ^a's and $\tilde{\gamma}^a$'s, into the space spanned by γ^a's only has as the consequencethat θ^a"s become γ^a's, while their Hermitian conjugated partners do not exist any longer.

While in Grassmann space the Grassmann odd "vectors" fulfill the anticommutation relations for "fermions" with integer spins and charges in the adjoint representations (originating in $d \geq 5$), and the Grassmann even "vectors" commute, with the vacuum state in both cases, which is just the identity, the Clifford even "vectors" are used to determine the (rather complicated) vacuum state.

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