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Counting faces of graphical zonotopes

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Abstract

It is a classical fact that the number of vertices of the graphical zonotope Z_Γ is equal to the number of acyclic orientations of a graph Γ. We show that the f-polynomial of Z_Γ is obtained as the principal specialization of the q-analog of the chromatic symmetric function of Γ.

Keywords: Graphical zonotope, f-vector, graphical matroid, symmetric function.

Math. Subj. Class.: 05E05, 52B05, 16T05

1 Introduction

The f-polynomial of an n-dimensional polytope P is defined by $f(P,q) = \sum_{i=0}^{n} f_i(P)q^i$, where $f_i(P)$ is the number of i-dimensional faces of P. The f-polynomial $f(\mathcal{Z}_\Gamma, q)$ of the graphical zonotope $\mathcal{Z}_\Gamma$ is a combinatorial invariant of a finite, simple graph Γ. The vertices of $\mathcal{Z}_\Gamma$ are in one-to-one correspondence with regions of the graphical hyperplane arrangement $\mathcal{H}_\Gamma$, which are enumerated by acyclic orientations of Γ.

Stanley's chromatic symmetric function $\Psi(\Gamma) = \sum_{f proper} \mathbf{x}_f$ of a graph $\Gamma = (V,E)$, introduced in [7], is the enumerator function of proper colorings $f\colon V \to \mathbb{N}$, where $\mathbf{x}_f = x_{f(1)} \cdots x_{f(n)}$ and f is proper if there are no monochromatic edges. The chromatic polynomial $\chi(\Gamma, d)$ of the graph Γ, which counts proper colorings with a finite number of colors, appears as the principal specialization

$$\chi(\Gamma, d) = \mathbf{ps}(\Psi(\Gamma))(d) = \Psi(\Gamma)\mid_{x_1=\cdots=x_d=1, x_{d+1}=\cdots=0}.$$

The number of acyclic orientations of Γ is determined by the value of the chromatic polynomial $\chi(\Gamma, d)$ at $d = -1$, [6]

$$a(\Gamma) = (-1)^{|V|}\chi(\Gamma, -1). \tag{1.1}$$

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There is a q-analog of the chromatic symmetric function $\Psi_q(\Gamma)$ introduced in a wider context of the combinatorial Hopf algebra of simplicial complexes considered in [2]. It is a symmetric function over the field of rational functions in q. The principal specialization of $\Psi_q(\Gamma)$ is the q-analog of the chromatic polynomial $\chi_q(\Gamma, d)$.

The main result of this paper is the following generalization of formula (1.1):

Theorem 1.1. *Let $\Gamma = (V, E)$ be a simple connected graph and $\mathcal{Z}_\Gamma$ the corresponding graphical zonotope. Then the f-polynomial of $\mathcal{Z}_\Gamma$ is given by*

$$f(\mathcal{Z}_\Gamma, q) = (-1)^{|V|}\chi_{-q}(\Gamma, -1).$$

The cancellation-free formula for the antipode in the Hopf algebra of graphs, obtained by Humpert and Martin in [3], reflects the fact that $f(\mathcal{Z}_\Gamma, q)$ depends only on the graphical matroid $M(\Gamma)$ associated to Γ. For instance, for any tree T_n the graphical matroid is the uniform matroid $M(T_n) = U_n^n$ and the corresponding graphical zonotope is the cube $\mathcal{Z}_{T_n} = I^{n-1}$. Whitney's theorem from 1933 describes how two graphs with the same graphical matroid are related [9]. It can be used to find more interesting nonisomorphic graphs with the same f-polynomials of corresponding graphical zonotopes.

The paper is organized as follows. In Section 2, we review the basic facts about zonotopes. In Section 3, the q-analog of the chromatic symmetric function $\Psi_q(\Gamma)$ of a graph Γ is introduced. Theorem 1.1 is proved in Section 4. We present some examples and calculations in Section 5.

2 Zonotopes

A *zonotope* $\mathcal{Z} = \mathcal{Z}(v_1, \dots, v_m)$ is a convex polytope determined by a collection of vectors $\{v_1, \dots, v_m\}$ in $\mathbb{R}^n$ as the Minkowski sum of line segments

$$\mathcal{Z} = [-v_1, v_1] + \cdots + [-v_m, v_m].$$

It is a projection of the m-cube $[-1, 1]^m$ under the linear map $\mathbf{t} \mapsto A\mathbf{t}, \mathbf{t} \in [-1, 1]^m$, where $A = [v_1 \cdots v_m]$ is an $n \times m$-matrix whose columns are vectors $v_1, \dots, v_m$. The zonotope $\mathcal{Z}$ is symmetric about the origin and all its faces are translations of zonotopes.

To a collection of vectors $\{v_1, \dots, v_m\}$ is associated a central arrangement of hyperplanes $\mathcal{H} = \{H_{v_1}, \dots, H_{v_m}\}$, where H_v denotes the hyperplane perpendicular to a vector $v \in \mathbb{R}^n$. The zonotope $\mathcal{Z}$ and the corresponding arrangement of hyperplanes $\mathcal{H}$ are closely related. In fact the associated fan $\mathcal{F}_\mathcal{H}$ of the arrangement $\mathcal{H}$ is the normal fan $\mathcal{N}(\mathcal{Z})$ of the zonotope $\mathcal{Z}$ (see [10, Theorem 7.16]). It follows that the face lattice of $\mathcal{F}_\mathcal{H}$ and the reverse face lattice of $\mathcal{Z}$ are isomorphic. In particular, vertices of $\mathcal{Z}$ correspond to regions of $\mathcal{H}$ and their total numbers coincide

$$f_0(\mathcal{Z}) = r(\mathcal{H}). \tag{2.1}$$

The faces of the zonotope $\mathcal{Z}$ are encoded by covectors of the oriented matroid $\mathcal{M}$ associated to the collection of vectors $\{v_1, \dots, v_m\}$. The covectors are sign vectors

$$\mathcal{V}^* = \{\mathrm{sign}(v) \in \{+, -, 0\}^m \mid v \in \mathbb{R}^n\},$$

where $\mathrm{sign}(v)_i = \begin{cases} +, & \langle v, v_i\rangle > 0 \\ 0, & \langle v, v_i\rangle = 0 \\ -, & \langle v, v_i\rangle < 0 \end{cases}$, $i = 1, \dots, m$. The face lattice of the zonotope $\mathcal{Z}$ is isomorphic to the lattice of covectors componentwise induced by $+, - < 0$ on $\mathcal{V}^*$.

A special class of zonotopes is determined by simple graphs. To a connected graph $\Gamma = (V, E)$, whose vertices are enumerated by integers $V = \{1, \ldots, n\}$, are associated the *graphical zonotope*

$$\mathcal{Z}_\Gamma = \mathcal{Z}(e_i - e_j \mid i < j, \{i, j\} \in E)$$

and the *graphical arrangement* in $\mathbb{R}^n$

$$\mathcal{H}_\Gamma = \{H_{e_i - e_j} \mid i < j, \{i, j\} \in E\}.$$

There is a bijective correspondence between regions of $\mathcal{H}_\Gamma$ and acyclic orientations of Γ, [8, Proposition 2.5], which by (2.1) implies

$$f_0(\mathcal{Z}_\Gamma) = r(\mathcal{H}_\Gamma) = a(\Gamma). \tag{2.2}$$

The arrangement $\mathcal{H}_\Gamma$ is refined by the braid arrangement $\mathcal{A}_{n-1}$ consisting of all hyperplanes $H_{e_i - e_j}, 1 \leq i < j \leq n$. Thus $\mathcal{Z}_\Gamma$ belongs to a wider class of convex polytopes called generalized permutohedra introduced in [4]. Since arrangements $\mathcal{H}_\Gamma$ and $\mathcal{A}_{n-1}$ are not essential we take their quotients by the line $l : x_1 = \cdots = x_n$ and without confusing retain the same notation. Consequently $\dim \mathcal{Z}_\Gamma = n - 1$.

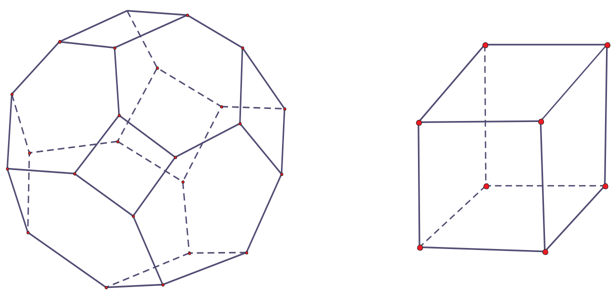

Figure 1: Permutohedron Pe^3 and cube I^3.

Example 2.1. (i) The permutohedron Pe^{n-1} is represented as the graphical zonotope $\mathcal{Z}_{K_n}$ corresponding to the complete graph K_n on n vertices (Figure 1).

(ii) The cube I^{n-1} is represented as the graphical zonotope $\mathcal{Z}_{T_n}$ corresponding to an arbitrary tree T_n on n vertices. This shows that the graph Γ is not determined by the combinatorial type of the zonotope $\mathcal{Z}_\Gamma$.

3 q-analog of chromatic symmetric function of graph

Stanley's chromatic symmetric function $\Psi(\Gamma)$ can be obtained in a purely algebraic way. A combinatorial Hopf algebra $\mathcal{H}$ is a graded, connected Hopf algebra equipped with the multiplicative linear functional $\zeta \colon \mathcal{H} \to \mathbf{k}$ to the ground field $\mathbf{k}$. For the theory of combinatorial Hopf algebras see [1]. Consider the combinatorial Hopf algebra of graphs $\mathcal{G}$ which is linearly generated over a field $\mathbf{k}$ by simple finite graphs with the product defined by disjoint union $\Gamma_1 \cdot \Gamma_2 = \Gamma_1 \sqcup \Gamma_2$ and the coproduct

$$\Delta(\Gamma) = \sum_{I \subset V} \Gamma\mid_I \otimes \Gamma\mid_{V \setminus I},$$

where $\Gamma\mid_I$ denotes the induced subgraph on $I \subset V$. The structure of $\mathcal{G}$ is completed by the character $\zeta\colon \mathcal{G} \to \mathbf{k}$ defined to be $\zeta(\Gamma) = 1$ for Γ with no edges and $\zeta(\Gamma) = 0$ otherwise. Then it turns out that $\Psi(\Gamma)$ is the image of the unique morphism of combinatorial Hopf algebras to symmetric functions $\Psi : \mathcal{G} \to Sym$, ([1, Example 4.5]).

An important part of the structure of the Hopf algebra $\mathcal{G}$ is the antipode $S\colon \mathcal{G} \to \mathcal{G}$. The cancellation-free formula for the antipode in terms of acyclic orientations of a graph Γ is obtained in [3]. We recall some basic definitions. Terminology comes from matroid theory. Given a graph $\Gamma = (V, E)$, for a collection of edges $F \subset E$ denote by $\Gamma_{V,F}$ the graph on V with the edge set F. A *flat* F of the graph Γ is a collection of its edges such that components of $\Gamma_{V,F}$ are induced subgraphs. The rank $\mathrm{rk}(F)$ is the size of spanning forests of $\Gamma_{V,F}$. We have that $|V| = \mathrm{rk}(F) + c(F)$, where $c(F)$ is the number of components of $\Gamma_{V,F}$. By contracting edges from a flat F we obtain the graph Γ/F. Finally, let $a(\Gamma)$ be the number of acyclic orientations of Γ. The formula of Humpert and Martin is as follows

$$S(\Gamma) = \sum_{F\in\mathcal{F}(\Gamma)} (-1)^{c(F)} a(\Gamma/F)\Gamma_{V,F}, \tag{3.1}$$

where the sum is over the set of flats $\mathcal{F}(\Gamma)$.

The following modification of the character ζ is considered in [2] in a wider context of the combinatorial Hopf algebra of simplicial complexes. Define $\zeta_q(\Gamma) = q^{\mathrm{rk}(\Gamma)}$, which determines the algebra morphism $\zeta_q\colon \mathcal{G} \to \mathbf{k}(q)$, where $\mathbf{k}(q)$ is the field of rational functions in q. This character produces the unique morphism $\Psi_q\colon \mathcal{G} \to QSym$ to quasisymmetric functions over $\mathbf{k}(q)$. The expansion of $\Psi_q(\Gamma)$ in the monomial basis of quasisymmetric functions is determined by the universal formula [1, Theorem 4.1]

$$\Psi_q(\Gamma) = \sum_{\alpha\models n} (\zeta_q)_\alpha(\Gamma) M_\alpha.$$

The sum above is over all compositions of the integer $n = |V|$ and the coefficient of the expansion corresponding to the composition $\alpha = (a_1, \ldots, a_k) \models n$ is given by

$$(\zeta_q)_\alpha(\Gamma) = \sum_{I_1\sqcup\ldots\sqcup I_k = V} q^{\mathrm{rk}(\Gamma|_{I_1})+\cdots+\mathrm{rk}(\Gamma|_{I_k})},$$

where the sum is over all set compositions of V of the type α. The coefficients $(\zeta_q)_\alpha(\Gamma)$ depend only on the partition corresponding to a composition α, so the function $\Psi_q(\Gamma)$ is actually symmetric and it can be expressed in the monomial basis of symmetric functions.

The invariant $\Psi_q(\Gamma)$ is more subtle than $\Psi(\Gamma)$. Obviously $\Psi_0(\Gamma)$ is the chromatic symmetric function of a graph Γ. It remains open to find two nonisomorphic graphs Γ_1 and Γ_2 with the same q-chromatic symmetric functions $\Psi_q(\Gamma_1) = \Psi_q(\Gamma_2)$. Let

$$\chi_q(\Gamma, d) = \mathbf{ps}(\Psi_q(\Gamma))(d)$$

be the q-analog of the chromatic polynomial $\chi(\Gamma, d)$. It is a consequence of a general fact for combinatorial Hopf algebras (see [1]) that

$$\chi_q(\Gamma, -1) = (\zeta_q \circ S)(\Gamma). \tag{3.2}$$

Example 3.1. Consider the graph Γ on four vertices with the edge set $E=\{12,13,23,34\}$. We find that

$$\Psi_q(\Gamma) = 24m_{1,1,1,1} + (8q+4)m_{2,1,1} + (2q^2+4q)m_{2,2} + (3q^2+q)m_{3,1} + q^3m_4.$$

By principal specialization and taking into account that

$$\mathbf{ps}(m_{\lambda_1^{i_1},\dots,\lambda_k^{i_k}})(d) = \frac{(i_1+\cdots+i_k)!}{i_1!\cdots i_k!}\binom{d}{i_1+\cdots+i_k},$$

we obtain

$$\chi_q(\Gamma,d) = d(d-1)^2(d-2) + qd(d-1)(4d-5) + 4q^2d(d-1) + q^3d,$$

which by Theorem 1.1 gives

$$f(\mathcal{Z}_\Gamma,q) = 12 + 18q + 8q^2 + q^3.$$

4 Proof of Theorem 1.1

Proof. By applying (3.2) and the formula for antipode (3.1) we obtain

$$(-1)^{|V|}\chi_{-q}(\Gamma,-1) = (-1)^{|V|}\sum_{F\in\mathcal{F}(\Gamma)}(-1)^{c(\Gamma)}a(\Gamma/F)(-q)^{\mathrm{rk}(F)}.$$

It follows that the statement of the theorem is equivalent to the following expression of the f-polynomial

$$f(\mathcal{Z}_\Gamma,q) = \sum_{F\in\mathcal{F}(\Gamma)} a(\Gamma/F)q^{\mathrm{rk}(F)}. \tag{4.1}$$

Therefore it should be shown that components of f-vectors are determined by

$$f_k(\mathcal{Z}_\Gamma) = \sum_{\substack{F\in\mathcal{F}(\Gamma)\\ \mathrm{rk}(F)=k}} a(\Gamma/F),\ \ 0\le k\le n-1. \tag{4.2}$$

By duality between the face lattice of $\mathcal{Z}_\Gamma$ and the face lattice of the fan $\mathcal{F}_{\mathcal{H}_\Gamma}$ we have

$$f_k(\mathcal{Z}_\Gamma) = f_{n-k-1}(\mathcal{F}_{\mathcal{H}_\Gamma}).$$

Let $L(\mathcal{H}_\Gamma)$ be the intersection lattice of the graphical arrangement $\mathcal{H}_\Gamma$. For a subspace $X\in L(\mathcal{H}_\Gamma)$ there is an arrangement of hyperplanes

$$\mathcal{H}_\Gamma^X = \{X\cap H \mid X\nsubseteq H, H\in\mathcal{H}_\Gamma\}$$

whose intersection lattice $L(\mathcal{H}_\Gamma^X)$ is isomorphic to the upper cone of X in $L(\mathcal{H}_\Gamma)$. Since $\mathcal{H}_\Gamma$ is central and essential we have

$$f_{n-k-1}(\mathcal{F}_{\mathcal{H}_\Gamma}) = \sum_{\substack{X\in L(\mathcal{H}_\Gamma)\\ \dim(X)=n-k-1}} r(\mathcal{H}_\Gamma^X), \tag{4.3}$$

where $r(\mathcal{H}_\Gamma^X)$ is the number of regions of the arrangement $\mathcal{H}_\Gamma^X$, see [8, Theorem 2.6].

The intersection lattice $L(\mathcal{H}_\Gamma)$ is isomorphic to the lattice of flats of the graphical matroid $M(\Gamma)$. By this isomorphism to a flat F of rank k corresponds the intersection subspace $X^F = \cap_{\{i,j\}\in F}H_{e_i-e_j}$ of dimension $n-k-1$. It is easy to see that arrangements $\mathcal{H}_\Gamma^{X^F}$ and $\mathcal{H}_{\Gamma/F}$ coincide, which by (2.2) and comparing formulas (4.2) and (4.3) proves theorem. ☐

5 Examples

By applying Theorem 1.1 we obtain the following interpretation of identities elaborated in [2, Propositions 17, 19].

Example 5.1. (i) For the permutohedron $Pe^{n-1} = \mathcal{Z}_{K_n}$, the f-polynomial is given by

$$f(\mathcal{Z}_{K_n}, q) = A_n(q + 1),$$

where $A_n(q) = \sum_{\pi \in S_n} q^{\mathrm{des}(\pi)}$ is the Euler polynomial. Recall that $\mathrm{des}(\pi)$ is the number of descents of a permutation $\pi \in S_n$. It recovers the fact that the h-polynomial of the permutohedron Pe^{n-1} is the Euler polynomial $A_n(q)$.

(ii) For the cube $I^{n-1} = \mathcal{Z}_{T_n}$, where T_n is a tree on n vertices, the f-polynomial is given by

$$f(\mathcal{Z}_{T_n}, q) = (q + 2)^{n-1}.$$

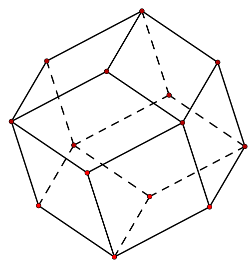

Figure 2: Rhombic dodecahedron $\mathcal{Z}_{C_4}$.

Proposition 5.2. *The f-polynomial of the graphical zonotope $\mathcal{Z}_{C_n}$ associated to the cycle graph C_n on n vertices is given by*

$$f(\mathcal{Z}_{C_n}, q) = q^n + q^{n-1} + (q + 2)^n - 2(q + 1)^n.$$

Proof. A flat $F \in \mathcal{F}(C_n)$ is determined by the complementary set of edges. If $\mathrm{rk}(F) = n - k, k > 1$ then the complementary set has k edges and $C_n/F = C_k$. Since $a(C_k) = 2^k - 2, k > 1$, by formula (4.2), we obtain

$$f_{n-k}(\mathcal{Z}_{C_n}) = (2^k - 2)\binom{n}{k}, \;\; 2 \le k \le n,$$

which leads to the required formula. □

Specially, for $n = 4$ the resulting zonotope is the rhombic dodecahedron (see Figure 2). We have

$$f(\mathcal{Z}_{C_4}, q) = 14 + 24q + 12q^2 + q^3.$$

Proposition 5.3. *Let $\Gamma = \Gamma_1 \vee_v \Gamma_2$ be the wedge of two connected graphs Γ_1 and Γ_2 at the common vertex v. Then*

$$f(\mathcal{Z}_\Gamma, q) = f(\mathcal{Z}_{\Gamma_1}, q) f(\mathcal{Z}_{\Gamma_2}, q).$$

Proof. The graphical matroids of involving graphs are related by $M(\Gamma) = M(\Gamma_1) \oplus M(\Gamma_2)$. For the sets of flats it holds $\mathcal{F}(\Gamma) = \{F_1 \cup F_2 \mid F_i \in \mathcal{F}(\Gamma_i), i = 1,2\}$. For $F = F_1 \cup F_2$ we have $\Gamma/F = \Gamma_1/F_1 \vee_{[v]} \Gamma_2/F_2$, where $[v]$ is the component of the vertex v in $\Gamma_{V,F}$. Obviously $a(\Gamma/F) = a(\Gamma_1/F_1)a(\Gamma_2/F_2)$ and $\mathrm{rk}(F) = \mathrm{rk}(F_1) + \mathrm{rk}(F_2)$. The proposition follows from formula (4.1). □

The formula for cubes in Example 5.1 (ii) follows from Proposition 5.3 since any tree is a consecutive wedge of edges and $f(I^1, q) = q+2$. It also allows us to restrict ourselves only to biconnected graphs. For a biconnected graph Γ with a disconnecting pair of vertices $\{u, v\}$ Whitney introduced the transformation called the *twist* around the pair $\{u, v\}$. This transformation does not have an affect on the graphical matroid $M(\Gamma)$ [9].

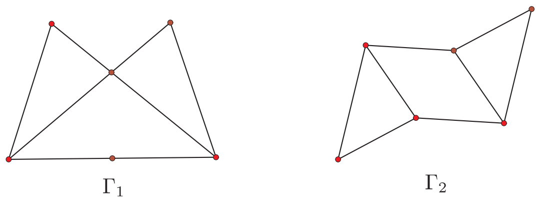

Figure 3: Biconnected graphs related by twist transformation.

Example 5.4. Figure 3 shows the pair of biconnected graphs on six vertices obtained one from another by the twist transformation. The corresponding zonotopes have the same f-polynomial

$$f(\mathcal{Z}_{\Gamma_1}, q) = f(\mathcal{Z}_{\Gamma_2}, q) = 126 + 348q + 358q^2 + 164q^3 + 30q^4 + q^5.$$

On the other hand their q-chromatic symmetric functions are different. One can check that corresponding coefficients by $m_{3,1^3}$ are different

$$[m_{3,1^3}]\Psi_q(\Gamma_1) = (11q^2 + 8q + 1) \cdot 3!,$$
$$[m_{3,1^3}]\Psi_q(\Gamma_2) = (10q^2 + 10q) \cdot 3!.$$

This shows that the q-analog of the chromatic symmetric function of a graph is not determined by the corresponding graphical matroid. By taking $q = 0$ we obtain that even the chromatic symmetric functions are different since $[m_{3,1^3}]\Psi(\Gamma_1) = 6$ and $[m_{3,1^3}]\Psi(\Gamma_2) = 0$.

Let us now consider Stanley's example of nonisomorphic graphs with the same chromatic symmetric functions, see [7]. We find that the f-polynomials of the corresponding graphical zonotopes differ for those graphs. From these examples we conclude that chromatic properties of a graph and the f-vector of the corresponding graphical zonotope are not related.

We have already noted that graphical zonotopes are generalized permutohedra. The h-polynomials of simple generalized permutohedra are determined in [5, Theorem 4.2]. The only simple graphical zonotopes are products of permutohedra [5, Proposition 5.2]. They are characterized by graphs whose biconnected components are complete subgraphs. Therefore Proposition 5.3 together with Example 5.1 (i) prove that the h-polynomial of a simple graphical zonotope is the product of Eulerian polynomials, the fact obtained in [5, Corollary 5.4]. Example 3.1 is of this sort and represents the hexagonal prism which is the product $\mathcal{Z}_{K_3} \times \mathcal{Z}_{K_2}$.

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