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On the size of maximally non-hamiltonian digraphs*

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Abstract

A graph is called *maximally non-hamiltonian* if it is non-hamiltonian, yet for any two non-adjacent vertices there exists a hamiltonian path between them. In this paper, we naturally extend the concept to directed graphs and bound their size from below and above. Our results on the lower bound constitute our main contribution, while the upper bound can be obtained using a result of Lewin, but we give here a different proof. We describe digraphs attaining the upper bound, but whether our lower bound can be improved remains open.

Keywords: Maximally non-hamiltonian digraphs.

Math. Subj. Class.: 05C45, 05C20

1 Introduction

Throughout this paper all graphs will be simple, finite, connected, and will not admit multiple edges or loops. In a *digraph*, each edge between two adjacent vertices u and v may be oriented from u to v, from v to u, or both ways. We call a digraph an *oriented graph* if no edge has both orientations.

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We were led to the study of the titular subject from a related concept: homogeneous traceability, a notion introduced by Skupień. A digraph is called *homogeneously traceable* if every vertex is the start-vertex of a hamiltonian path. If additionally every vertex is also the end-vertex of some hamiltonian path, the digraph is called *bihomogeneously traceable*. A graph or digraph D is called *hypohamiltonian* if D is non-hamiltonian, but for any vertex v in D, the graph $D - v$ is hamiltonian. Obviously, any digraph that is hamiltonian or hypohamiltonian is bihomogeneously traceable. But not every homogeneously traceable digraph is hamiltonian [4]. Not even bihomogeneous traceability implies hamiltonicity. At a meeting in Kalamazoo (in 1980) Skupień showed that for all $n \geq 7$ there exists a 2-diregular bihomogeneously traceable non-hamiltonian oriented digraph of order n, see [16], which appeared in 1981.

Moreover, Skupień [17] later constructed exponentially many bihomogeneously traceable non-hamiltonian oriented graphs. Independently, in another paper which also appeared in 1981, Hahn and T. Zamfirescu [12] also constructed an infinite sequence of bihomogeneously traceable non-hamiltonian oriented graphs, and gave three special examples: the first is arc-minimal (i.e. with the smallest possible number of arcs for a given number of vertices) and order 7, the second is planar and has 8 vertices (it is proven in [12] that there are no smaller examples), and the third is both arc-minimal and planar, and has 9 vertices. Note that arc-minimality amounts in this context to 2-diregularity.

Hahn and T. Zamfirescu asked in [12] the natural question whether infinitely many planar bihomogeneously traceable non-hamiltonian oriented graphs exist, as very few were known. Infinite families of planar hypohamiltonian digraphs containing opposite arcs have been found by Fouquet and Jolivet [10]. In [20], Thomassen proved that a planar hypohamiltonian digraph with n vertices (and many edges with both orientations—in fact, all but six) exists for each $n \geq 6$. It was shown by the second author [21] that, indeed, there exist infinitely many planar bihomogeneously traceable non-hamiltonian oriented graphs. A stronger result was recently obtained by van Aardt, Burger, and Frick [1], who showed that there exist infinitely many planar hypohamiltonian oriented graphs, thereby solving a problem of Thomassen [20].

If one now asks for an even larger set of spanning paths, one may be led to the case demanding that between any two non-adjacent vertices there exists a hamiltonian path. Such graphs have been studied in the non-oriented case, see for instance [7] and [2], and are called *maximally non-hamiltonian* (which, in the following, will often be abbreviated to MNH). For a digraph D, we write $V(D)$ and $A(D)$ for its set of vertices and arcs, respectively. Mirroring the non-directed definition, a digraph D is *maximally non-hamiltonian* if D is non-hamiltonian, but for every $x, y \in V(D)$ with $yx \notin A(D)$ there is a hamiltonian path from x to y.

A few words concerning the notation. For a set X, we denote with $|X|$ the cardinality of X. In a graph G, for adjacent vertices x and y in G we denote by xy the edge between x and y. If G is a digraph, xy will be the arc from x to y. A digraph D is called *strong* if for any pair $x, y \in V(D)$ there exists a (directed) path from x to y. Denote with $\delta^+(D)$ ($\delta^-(D)$) the minimum out-degree (minimum in-degree) and with $\delta^0(D)$ the *minimum semi-degree* of D, which is the minimum of $\delta^+(D)$ and $\delta^-(D)$. $N^+(x)$ ($N^-(x)$) shall be the set of out-neighbours (in-neighbours) of a vertex x. For a set of vertices S of D we denote the digraph induced by S in D as $D[S]$. Further definitions follow when needed.

2 Results

An important direction of research on non-directed MNH graphs has been determining the smallest size of an MNH graph of order n, which we shall denote by $f(n)$. This study was initiated by Bondy [6], who showed that for $n \geq 7$ we have $f(n) \geq \lceil \frac{3n}{2} \rceil$. Bollobás [5] conjectured that there exist infinitely many graphs for which this lower bound is in fact attained. This was proven to be correct for all even $n \geq 36$ by Clark and Entringer [7]. In the same paper graphs of order n and size $\lceil \frac{3n}{2} \rceil + 1$ were constructed for all odd $n \geq 55$. Horák and Širáň [13] also found such almost extremal graphs by using a construction of Thomassen [19] (which was originally intended to construct hypohamiltonian graphs). They also proved that for every $n \geq 48$ there exists a triangle-free MNH graph of order n.

Clark and Entringer asked in [7] whether for infinitely many n the MNH graph on n vertices of smallest size is unique or not. Combining the results from [7, 8, 9, 15], one obtains that for infinitely many n there exist two non-isomorphic MNH graphs of order n and smallest size. Still, there were infinitely many orders for which only one MNH graph of smallest size was known. This was investigated by Stacho [18]; he proved that for any $n \geq 88$ there exist three pairwise non-isomorphic MNH graphs of order n and smallest size.

In the following we extend the study of the size of MNH graphs to directed graphs. First, we characterise the non-strong MNH digraphs. A digraph is *symmetric* if for every arc in D the corresponding inverted arc also lies in D. For integers $m \geq 1$ and $p \geq 1$, let $D_{m,p}$ denote the digraph whose vertices are the vertices of two vertex-disjoint complete symmetric digraphs K_m^* and K_p^* and whose arcs are those of the digraphs K_m^*, K_p^*, and additionally the arcs xy with x in K_m^* and y in K_p^*. We claim:

Lemma 2.1. *The non-strong MNH digraphs are the digraphs $D_{m,p}$.*

Proof. Clearly, a digraph $D_{m,p}$ is a non-strong MNH digraph. Conversely, let D be a non-strong MNH digraph. There exists a partition V_1, V_2 of $V(D)$ such that there are no arcs from V_2 to V_1. Let x and y be two distinct vertices of D. It is easy to see that if x, y are both in V_1 or in V_2 there is no hamiltonian path from x to y. Furthermore, if x is in V_2 and y is in V_1, there is also no hamiltonian path from x to y. Then, since D is MNH, it follows that $D[V_1]$ and $D[V_2]$ are complete symmetric digraphs and that every ordered pair xy with $x \in V_1$ and $y \in V_2$ is an arc of D. This means that D is a digraph $D_{m,p}$, and so we are done. □

The upper bound contained in the following theorem can be obtained by using a result of Lewin [14], but we choose to give here a different proof. (In fact, our proof of the upper bound is a new proof of Lewin's [14, Corollary 1].) In upcoming arguments, we require the following.

Theorem 2.2 (Ghouila-Houri [11])**.** *A strong digraph D with $\delta^0(D) \geq |V(D)|/2$ is hamiltonian.*

Theorem 2.3. *For an MNH digraph D of order $n \geq 4$ we have $|A(D)| \leq (n-1)^2$ and this upper bound is attained.*

Proof. There exists a vertex $x \in V(D)$ with $d^+(x) + d^-(x) < n$ (for otherwise, by Theorem 2.2, D would be hamiltonian). Let us put

$$B = N^+(x) \cap N^-(x), \quad A = N^+(x) \setminus B, \quad \text{and} \quad C = N^-(x) \setminus B.$$

Furthermore, let a, b, and c be the respective cardinalities of A, B, and C. Clearly, for a vertex y in A or in C we have $d^+(y) + d^-(y) \leq 2n-3$. For a vertex y in B, we have $d^+(y)+d^-(y) \leq 2n-2$, and for a vertex y not adjacent with x, we have $d^+(y)+d^-(y) \leq 2n-4$. By addition, we get

$$2 \times |A(D)| \leq \\ n-1+a(2n-3)+b(2n-2)+c(2n-3)+(n-1-a-b-c)(2n-4),$$

hence

$$2 \times |A(D)| \leq \\ n-1+(n-1)(2n-4)+a+c+2b = (n-1)(2n-3)+a+c+2b.$$

But $d^+(x)+d^-(x) < n$ means that we have $a+c+2b \leq n-1$. It follows that

$$2 \times |A(D)| \leq (n-1)(2n-3)+n-1 \leq 2(n-1)^2,$$

and the upper bound is proved; it is attained since $D_{1,n-1}$ and $D_{n-1,1}$ are MNH digraphs of size $(n-1)^2$. □

For integers $r \geq 1$ and n with $n \geq 2r+1$ define the digraph $H_{n,r}$ of order n as follows. The vertices of $H_{n,r}$ are those of a complete symmetric digraph K^*_{n-r-1} and $r+1$ additional vertices $y_1, \ldots, y_{r+1}$. Let $x_1, \ldots, x_r$ be r vertices of K^*_{n-r-1}. The arcs of $H_{n,r}$ are the arcs of K^*_{n-r-1}, $y_i x$ where $1 \leq i \leq r+1$, $x \in V(K^*_{n-r-1})$, and $x_i y_j$ where $1 \leq i \leq r$ and $1 \leq j \leq r+1$.

It is easy to see that a digraph $H_{n,r}$, like its converse, is MNH, of minimum semi-degree r and of strong connectivity r.

By Theorem 2.3, the maximum size of a non-strong MNH digraph is at most $(n-1)^2$ and this bound is reached. It was proved in [3] that a digraph D with minimum semi-degree r and with more than

$$a_{n,r} = n^2-(r+2)n+(r+1)^2$$

arcs is hamiltonian. When $r > (n-1)/2$, by Theorem 2.2, D is hamiltonian and therefore cannot be MNH. But when $r \leq (n-1)/2$, the result of [3] shows that the maximum size of an MNH digraph D of order n with $\delta^0(D) = r$ is at most $a_{n,r}$, and since $H_{n,r}$ is MNH, of minimum semi-degree r and of size $a_{n,r}$, this upper bound is reached.

If $1 \leq r \leq s \leq (n-1)/2$, then $a_{n,r} \geq a_{n,s}$, so the maximum size of an MNH digraph D of order n and of strong connectivity $\kappa(D) = r$ is at most $a_{n,r}$. As $\kappa(H_{n,r}) = r$, the bound is sharp.

We now establish a lower bound on the size of an MNH digraph.

Lemma 2.4. *For an MNH digraph D on at least four vertices, either $\delta^0(D) \geq 2$ or $|A(D)| \geq 3n-4$.*

Proof. We proceed by induction on n. It is easy to verify that the assertion is true for $n=4$. Now let $n \geq 5$ and suppose the assertion is true for all $k \leq n-1$. Consider a digraph D of order n having the required property.

Let us put $V(D) = \{x_1, \ldots, x_n\}$. Let there exist a vertex of D which is of out-degree at most 1. W.l.o.g., we may assume that this vertex is x_1. Suppose first that $d^+(x_1) = 0$.

In this case D is a non-strong MNH digraph, and by Lemma 2.1, D is in fact $D_{n-1,1}$. We then have $|A(D)| = (n-1)^2 \geq 3n-4$, and so we are done.

Suppose now that $d^+(x_1) = 1$. W.l.o.g., we may assume that the unique out-neighbour of x_1 is x_2. We claim that $x_2x_1 \in A(D)$. Suppose the opposite. Then there exists a hamiltonian path P from x_1 to x_2. But then x_1 has at least two out-neighbours, a contradiction. We also claim that all of the vertices of $V(D) \setminus \{x_1, x_2\}$ are in-neighbours of x_2. Suppose the opposite. Then there exists a vertex x_i, $i \geq 3$, which is not an in-neighbour of x_2. Thus, there exists a hamiltonian path from x_2 to x_i, whence, x_1 has an out-neighbour in this hamiltonian path distinct from x_2, a contradiction.

We claim that $D' = D - x_1$ is MNH, i.e. that for any two vertices y, z of D' such that $zy \notin A(D)$, there exists a hamiltonian path in D' from y to z. Observe first that $y \neq x_2$. There exists in D a hamiltonian path P from y to z. P contains the arc x_1x_2, and since x_1 is not the first vertex of P it admits an in-neighbour u in P. Then $P' = P - x_1 + ux_2$ (i.e. the path P from which we delete the vertex x_1 and add the arc ux_2) is a hamiltonian path in D' from y to z. So, the claim is proved.

By induction hypothesis, either every vertex of D' is of in-degree at least 2 in D', or $|A(D')| \geq 3(n-1) - 4 = 3n - 7$. In the first case, since x_2 is of in-degree $n-2$ in D', we have $|A(D')| \geq n - 2 + 2(n-2)$, hence $|A(D')| \geq 3n - 6$. Since x_1x_2 and x_2x_1 are arcs of D but not of D', we get $|A(D)| \geq 3n - 4$, and the theorem is proved in this case.

Suppose now that $|A(D')| \geq 3n - 7$. Assume first that there are no distinct vertices x_i, x_j, where $3 \leq i, j \leq n$, such that x_jx_i is not an arc of D'. Then we have $|A(D')| \geq (n-2)^2 \geq 3n-6$. Thus $|A(D)| \geq 3n-4$, and again we are done. Suppose now that there exist distinct vertices x_i, x_j, where $3 \leq i, j \leq n$, such that $x_jx_i \notin A(D')$. Then there exists a hamiltonian path of D from x_i to x_j, and then necessarily x_1 has an in-neighbour x_k with $3 \leq k \leq n$. It follows that $|A(D)| \geq 3n - 7 + 3 = 3n - 4$, and we are done. □

As a corollary, we can state:

Theorem 2.5. *For an MNH digraph D of order $n \geq 4$ we have $|A(D)| \geq 2n$.*

Proof. If $|A(D)| \geq 3n - 4$, we are done. If $|A(D)| < 3n - 4$, by Lemma 2.4, each vertex of D is of out-degree at least 2, and we get $|A(D)| \geq 2n$. □

Observe that for $n = 3$, this lower bound inequality is untrue: consider the digraph $D = (\{x, y, z\}, \{xy, yx, xz, yz\})$. D is an MNH digraph of order 3 and size 4. Also note that for $n = 4$, the lower bound is tight due to the digraph $D_{2,2}$, which is not regular.

From [6] we know that for $n \geq 7$ the size of an MNH graph of order n is at least $\lceil \frac{3n}{2} \rceil$. For every $n \geq 19$, this lower bound is reached; consult [18] and [15] for details. Thus, for $n \geq 19$, there exists an MNH graph G of order n and size $\lceil \frac{3n}{2} \rceil$. It is easy to prove that the symmetric digraph obtained by doubly orienting each edge of G is an MNH digraph of order n and size $2 \times \lceil \frac{3n}{2} \rceil$. So the minimum size of an MNH digraph of order $n \geq 19$ is at least $2n$ and at most $2 \times \lceil \frac{3n}{2} \rceil$.

We now give a lower bound on the size of an MNH non-strong digraph:

Theorem 2.6. *Let D be an MNH non-strong digraph of order $n \geq 2$. Then*

$$|A(D)| \geq \left\lceil \frac{3}{4}n^2 \right\rceil - n.$$

Proof. We know that D is of the form $D_{\alpha n,(1-\alpha)n}$, where $0 < \alpha < 1$ and αn is an integer. Then we have

$$\begin{aligned}|A(D)| &= \alpha n(\alpha n - 1) + (1-\alpha)n((1-\alpha)n - 1) + \alpha n(1-\alpha)n \\ &= n^2(\alpha^2 - \alpha + 1) - n \geq \tfrac{3}{4}n^2 - n,\end{aligned}$$

since $\alpha^2 - \alpha + 1 \geq \frac{3}{4}$. □

If the strong connectivity is 1, we have the following result.

Theorem 2.7. *Let D be an MNH digraph of order $n \geq 4$ and of strong connectivity 1. Then*

$$|A(D)| \geq \min\left\{3n - 4, \left\lceil \frac{5}{2}n \right\rceil - 1\right\}.$$

Proof. If $\delta^+(D) \leq 1$ or $\delta^-(D) \leq 1$, by Lemma 2.4, we have $|A(D)| \geq 3n - 4$, and the result is proved.

Suppose now that $\delta^+(D) \geq 2$ and $\delta^-(D) \geq 2$, i.e. each vertex of D is of out-degree at least 2 and of in-degree at least 2. There is a vertex x such that the digraph $D - x$ is not strong, so there exists a partition of $V(D) \setminus \{x\}$ into two non-empty sets A and B such that there are no arcs from B to A. W.l.o.g., we suppose that $|B| \leq |A|$, so $|A| \geq 2$. We claim that all the vertices of B are out-neighbours of x. Let us suppose this not to be the case. Then there exists a vertex y of B such that $xy \notin A(D)$. Since D is MNH, there exists a hamiltonian path P from y to x. Necessarily, P contains an arc uv with $u \in B$ and $v \in A$, which is impossible. Similarly, all the vertices of A are in-neighbours of x. Since x has at least one out-neighbour in A and at least one in-neighbour in B, we get $d^+(x) + d^-(x) \geq n + 1$.

Suppose now that x has exactly one out-neighbour y in A. It is easy to see that $A \setminus \{y\}$, $B \cup \{x\}$ is a partition of $V(D) \setminus \{y\}$ into non-empty sets such that there are no arcs from $B \cup \{x\}$ to $A \setminus \{y\}$. So, $D - y$ is not strong, and from the preceding arguments, we have $d^+(y) + d^-(y) \geq n + 1$. Since $d^+(z) + d^-(z) \geq 4$ for z distinct from x and y, by addition we get $2 \times |A(D)| \geq 2(n + 1) + 4(n - 2)$, hence $2 \times |A(D)| \geq 6n - 6$, and $|A(D)| \geq 3n - 3 > 3n - 4$, and the result is proved.

Suppose now that x has at least two out-neighbours in A. We then get $d^+(x)+d^-(x) \geq n + 2$, and for $z \neq x$ we have $d^+(z) + d^-(z) \geq 4$. By addition this yields $2 \times |A(D)| \geq n + 2 + 4(n - 1)$, hence $|A(D)| \geq \lceil \frac{5}{2}n \rceil - 1$, as stated. □

With above conclusions in mind, an MNH digraph D with

$$|A(D)| < \min\left\{3n - 4, \left\lceil \frac{5}{2}n \right\rceil - 1, \left\lceil \frac{3}{4}n^2 \right\rceil - n\right\}$$

is necessarily of strong connectivity at least 2, and thus, of semi-degree at least 2. Now it is easy to see that for $n \geq 5$, if the lower bound $2n$ is attained by an MNH digraph D of order n, then necessarily D is a 2-diregular digraph of order n and of strong connectivity 2. We were not able to show the existence of such digraphs, but an advance in this direction is perhaps the following:

Theorem 2.8. *Let D be a 2-diregular MNH digraph of order n. Then D is hypohamiltonian.*

Proof. We know that D is non-hamiltonian. Suppose for the sake of a contradiction that there exists a vertex z such that $D - z$ is non-hamiltonian. Let x be an in-neighbour of z, and let y be the other out-neighbour of x. We claim that $zy \in A(D)$. Suppose the opposite. Then there exists a hamiltonian path from y to z, and it is easy to see that the in-neighbour of z in P is x. But then $P - z + xy$ is a hamiltonian cycle of $D - z$, a contradiction.

Assume that y is an in-neighbour of z. Then y and z are of in-degree 2 in the induced sub-digraph $D[\{x, y, z\}]$. Since D is 2-strong and $n \geq 5$, this is not possible. Therefore y is not an in-neighbour of z. Suppose that x is an out-neighbour of z. Then x and z are of out-degree 2 in the induced sub-digraph $D[\{x, y, z\}]$. Since D is 2-strong and $n \geq 5$, this is not possible.

So x is not an out-neighbour of z. Then z has an in-neighbour u distinct from x, y, and z. The vertex y is not an out-neighbour of u (for otherwise y would be of in-degree at least 3, which is impossible), and x is not an out-neighbour of u (for otherwise, with a previous argument, zx would be an arc of D, which is false). Thus u has an out-neighbour v distinct from x, y, and z. Then, by previous arguments, zv is an arc of D, and v is not an in-neighbour of z. The last statement implies that there exists a hamiltonian path P' from z to v. The second vertex of P' is necessarily y, and then it is easy to see that x has an out-neighbour w distinct from y and z. Since x is of out-degree 2, this is impossible. Thus $D - z$ cannot be non-hamiltonian, and so the theorem is proved. $\square$

The above discussions concerning the size of MNH digraphs led us to the following question.

Problem. *Does every non-hamiltonian oriented graph of order at least* 3 *contain an arc* xy *for which there is no hamiltonian path from* x *to* y*?*

Lastly, we would like to point out the connection between maximally non-hamiltonian graphs and so-called *platypus*, which are non-hamiltonian graphs in which every vertex-deleted subgraph is traceable. They contain the family of all hypohamiltonian and hypotraceable graphs. The second author showed [22] that a maximally non-hamiltonian graph G is a platypus if and only if $\Delta(G) < |V(G)| - 1$, where $\Delta(G)$ denotes the maximum degree of G. Directed platypus have not yet been investigated, but the above results may provide a good starting point.

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