BLED WORKSHOPS
IN PHYSICS
VOL. 18, NO. 2

*Proceedings to the 20th Workshop*
***What Comes Beyond . . .*** *(p. 217)*
Bled, Slovenia, July 9–20, 2017

# 13 The Symmetry of 4 × 4 Mass Matrices Predicted by the *Spin-charge-family* Theory — SU(2) × SU(2) × U(1) — Remains in All Loop Corrections

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**Abstract.** The *spin-charge-family* theory [1–11,14–22] predicts the existence of the fourth family to the observed three. The 4 × 4 mass matrices manifest the symmetry SU(2) × SU(2) × U(1), determined on the tree level by the nonzero vacuum expectation values of several scalar fields – the three singlets with the family members quantum numbers (belonging to U(1)) and the two triplets with the family quantum numbers (belonging to SU(2) × SU(2)) with the weak and the hyper charge of the *standard model* higgs field ($\pm\frac{1}{2}, \mp\frac{1}{2}$, respectively). It is demonstrated, using the massless spinor basis, on several cases that (why) the symmetry of 4 × 4 mass matrices remains the same in all loop corrections.

**Povzetek.** Teorija *spinov-nabojev-družin* [1–11,14–22] napove obstoj četrte družine k opazženim trem. Masne matrike 4 × 4 kažejo simetrijo SU(2) × SU(2) × U(1), ki je na drevesnem nivoju določena z neničelnimi vakuumskimi pričakovanimi vrednostmi več skalarnih polj — treh singletov s kvantnimi števili družin (v U(1)) in dveh tripletov s kvantnimi števili družin (v SU(2) × SU(2)), ki imajo šibki in hipernaboj higgsovega polja *standardnega modela*, (enak $\pm\frac{1}{2}$ in $\mp\frac{1}{2}$). Avtorja pokažeta, da (zakaj) se v bazi brezmasnih spinorjev, v več primerih, simetrija masnih matrik 4 × 4 ohranja v vseh redih.

Keywords: Unifying theories, Beyond the standard model, Origin of families, Origin of mass matrices of leptons and quarks, Properties of scalar fields, The fourth family, Origin and properties of gauge bosons, Flavour symmetry, Kaluza-Klein-like theories
PACS:12.15.Ff 12.60.-i 12.90.+b 11.10.Kk 11.30.Hv 12.15.-y 12.10.-g 11.30.-j 14.80.-j

## 13.1 Introduction

The *spin-charge-family* theory [1–11,14–22] predicts before the electroweak break four - rather than the observed three - coupled massless families of quarks and leptons.

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The $4 \times 4$ mass matrices of all the family members demonstrate in this theory the same symmetry [1,5,4,19,20], determined by the scalar fields: the two triplets — the gauge fields of the two family groups $\widetilde{SU}(2)\times\widetilde{SU}(2)$ operating among families — and the three singlets — the gauge fields of the three charges $(Q, Q'$ and $Y')$ distinguishing among family members. All these scalar fields carry the weak and the hyper charge as does the scalar of the *standard model*: ($\pm\frac{1}{2}$ and $\mp\frac{1}{2}$, respectively) [1,4,22].

Although there is no direct observations of the fourth family quarks masses below 1 TeV, while the fourth family quarks with masses above 1 TeV would contribute according to the *standard model* (the *standard model* Yukawa couplings of the quarks with the scalar higgs is proportional to $\frac{m_4^\alpha}{v}$, where $m_4^\alpha$ is the fourth family member ($\alpha = u$, d) mass and $v$ the vacuum expectation value of the scalar) to either the quark-gluon fusion production of the scalar field (the higgs) or to the scalar field decay too much in comparison with the observations, the high energy physicists do not expect the existence of the fourth family members at all [23,24].

One of the authors (N.S.M.B) discusses in Refs. ([1], Sect. 4.2.) that the *standard model* estimation with one higgs scalar might not be the right way to evaluate whether the fourth family, coupled to the observed three, does exist or not. The $u_i$-quarks and $d_i$-quarks of an $i^{th}$ family, namely, if they couple with the opposite sign (with respect to the "$\pm$" degree of freedom) to the scalar fields carrying the family $(\tilde{A}, i)$ quantum numbers and have the same masses, do not contribute to either the quark-gluon fusion production of the scalar fields with the family quantum numbers or to the decay of these scalars into two photons:

The strong influence of the scalar fields carrying the family members quantum numbers to the masses of the lower (observed) three families manifests in the huge differences in the masses of the family members, let say $u_i$ and $d_i$, $i = (1, 2, 3)$, and families $(i)$. For the fourth family quarks, which are more and more decoupled from the observed three families the higher are their masses [20,19], the influence of the scalar fields carrying the family members quantum numbers on their masses is expected to be much weaker. Correspondingly the $u_4$ and $d_4$ masses become closer to each other the higher are their masses and the weaker are their couplings (the mixing matrix elements) to the lower three families. For $u_4$-quarks and $d_4$-quarks with the similar masses the observations might consequently not be in contradiction with the *spin-charge-family* theory prediction that there exists the fourth family coupled to the observed three ([26], which is in preparation).

We demonstrate in the main Sect. 13.2 why the symmetry, which the mass matrices demonstrate on the tree level, keeps the same in all loop corrections.

We present shortly the *spin-charge-family* theory and its achievements so far in Sect. 13.4. All the mathematical support appears in appendices.

Let be here stressed what supports the *spin-charge-family* theory to be the right next step beyond the *standard model*. This theory can not only explain - while starting from the very simple action in $d \geq (13 + 1)$, Eqs. (13.20) in App. 13.4, with the massless fermions (with the spin of the two kinds $\gamma^a$ and $\tilde{\gamma}^a$ , one kind taking care of the spin and of the charges of the family members (Eq. (13.2)), the second kind taking care of the families (Eqs. (13.19, 13.35))) coupled only to the gravity (through the vielbeins and the two kinds of the corresponding spin connections

fields $\omega_{ab\alpha}f^{\alpha}{}_{c}$ and $\tilde{\omega}_{ab\alpha}f^{\alpha}{}_{c}$, the gauge fields of $S^{ab}$ and $\tilde{S}^{ab}$ (Eqs. (13.20)) - all the assumptions of the *standard model*, but also answers several open questions beyond the *standard model*. It offers the explanation for [4–6,1,7–11,14–22]:
**a.** the appearance of all the charges of the left and right handed family members and for their families and their properties,
**b.** the appearance of all the corresponding vector and scalar gauge fields and their properties (explaining the appearance of higgs and Yukawa couplings),
**c.** the appearance and properties of the dark matter,
**d.** the appearance of the matter/antimatter asymmetry in the universe.

The theory predicts for the low energy regime:
**i.** The existence of the fourth family to the observed three.
**ii.** The existence of twice two triplets and three singlets of scalars, all with the properties of the higgs with respect to the weak and hyper charges, what explains the origin of the Yukawa couplings.
**iii.** There are several other predictions, not directly connected with the topic of this paper.

The fact that the fourth family quarks have not yet been observed - directly or indirectly - pushes the fourth family quarks masses to values higher than 1 TeV.

Since the experimental accuracy of the ($3 \times 3$ submatrix of the $4 \times 4$) mixing matrices is not yet high enough [30], it is not possible to calculate the mixing matrix elements among the fourth family and the observed three. Correspondingly it is not possible to estimate masses of the fourth family members by fitting the experimental data to the parameters of mass matrices, determined by the symmetry predicted by the *spin-charge-family* [20,19].

But assuming the masses of the fourth family members the matrix elements can be estimated from the existing $3 \times 3$ subamtrix of the $4 \times 4$ matrix.

The more effort and work is put into the *spin-charge-family* theory, the more explanations of the observed phenomena and the more predictions for the future observations follow out of it. Offering the explanation for so many observed phenomena - keeping in mind that all the explanations for the observed phenomena originate in a simple starting action - qualifies the *spin-charge-family* theory as the candidate for the next step beyond the *standard model*.

The reader is kindly asked to learn more about the *spin-charge-family* theory in Refs. [2–4,1,5,6] and the references there in. We shall point out sections in these references, which might be of particular help, when needed.

## 13.2 The symmetry of family members mass matrices keeps unchanged in all orders of loop corrections

It is demonstrated in this main section that the symmetry $\widetilde{SU}(2)\times \widetilde{SU}(2) \times U(1)$ of the mass term, which manifests in the starting action 13.20 of the *spin-charge-family* theory [4,1,5,6], remains unchanged in all orders of loop corrections. The massless basis will be used for this purpose.

Let us rewrite formally the fermion part of the starting action, Eq. (13.20), in the way that it manifests, Eq. (13.1), the kinetic and the interaction term in

$d = (3+1)$ (the first line, $m = (0,1,2,3)$), the mass term (the second line, $s = (7,8)$) and the rest (the third line, $t = (5,6,9,10,\cdots,14)$).

$$\mathcal{L}_f = \bar{\psi}\gamma^m (p_m - \sum_{A,i} g^{Ai}\tau^{Ai}A^{Ai}_m)\psi + \\ \{\sum_{s=7,8} \bar{\psi}\gamma^s p_{0s}\ \psi\} + \\ \{\sum_{t=5,6,9,\dots,14} \bar{\psi}\gamma^t p_{0t}\ \psi\}, \tag{13.1}$$

where $p_{0s} = p_s - \frac{1}{2}S^{s's"}\omega_{s's"s} - \frac{1}{2}\tilde{S}^{ab}\tilde{\omega}_{abs}$, $p_{0t} = p_t - \frac{1}{2}S^{t't"}\omega_{t't"t} - \frac{1}{2}\tilde{S}^{ab}\tilde{\omega}_{abt}$ [1], with $m \in (0,1,2,3)$, $s \in (7,8)$, $(s',s") \in (5,6,7,8)$, $(a,b)$ (appearing in $\tilde{S}^{ab}$) run within either $(0,1,2,3)$ or $(5,6,7,8)$, $t$ runs $\in (5,\dots,14)$, $(t',t")$ run either $\in (5,6,7,8)$ or $\in (9,10,\dots,14)$. The spinor function $\psi$ represents all family members, presented on Table 13.3 of all the $2^{\frac{7+1}{2}-1} = 8$ families, presented on Table 13.4.

The first line of Eq. (13.1) determines (in $d = (3+1)$) the kinematics and dynamics of spinor (fermion) fields, coupled to the vector gauge fields. The generators $\tau^{Ai}$ of the charge groups are expressible in terms of $S^{ab}$ through the complex coefficients ${c^{Ai}}_{ab}$ (the coefficients ${c^{Ai}}_{ab}$ of $\tau^{Ai}$ can be found in Eqs. (13.23, 13.24) [2],

$$\tau^{Ai} = \sum_{a,b} {c^{Ai}}_{ab}\, S^{ab}\,, \tag{13.2}$$

fulfilling the commutation relations

$$\{\tau^{Ai}, \tau^{Bj}\}_- = i\delta^{AB} f^{Aijk}\tau^{Ak}\,. \tag{13.3}$$

They represent the colour ($\tau^{3i}$), the weak ($\tau^{1i}$) and the hyper (Y) charges, as well as the $SU(2)_{II}$ ($\tau^{2i}$) and $U(1)_{II}$ ($\tau^4$) charges, the gauge fields of these last two groups gain masses interacting with the condensate, Table 13.5. The condensate leaves massless, besides the colour and gravity gauge fields, the weak and the hyper charge vector gauge fields. The corresponding vector gauge fields $A^{Ai}_m$ are expressible with the spin connection fields $\omega_{stm}$ Eq. (13.29)

$$A^{Ai}_m = \sum_{s,t} {c^{Ai}}_{st}\, {\omega^{st}}_m\,. \tag{13.4}$$

The scalar gauge fields of the charges, Eq. (13.30), are expressible with the spin connections and vielbeins [2].

[1] If there are no fermions present, then either $\omega_{abc}$ or $\tilde{\omega}_{abc}$ are expressible by vielbeins ${f^\alpha}_a$ [[2,5], and the references therein]. We assume that there are spinor fields which determine spin connection fields – $\omega_{abc}$ and $\tilde{\omega}_{abc}$. In general one would have [6]: $p_{0a} = {f^\alpha}_a p_{0\alpha} + \frac{1}{2E}\{p_\alpha, E{f^\alpha}_a\}_-$, $p_{0\alpha} = p_\alpha - \frac{1}{2}S^{s's"}\omega_{s's"\alpha} - \frac{1}{2}\tilde{S}^{ab}\tilde{\omega}_{ab\alpha}$. Since the term $\frac{1}{2E}\{p_\alpha, E{f^\alpha}_a\}_-$ does not influece the symmetry of mass matrices, we do not treat it in this paper.

[2] Before the electroweak break there are the conserved charges $\vec{\tau}^1$, $\vec{\tau}^3$ and $Y := \tau^4 + \tau^{23}$, and the non conserved charge $Y' := -\tau^4 \tan^2\vartheta_2 + \tau^{23}$, where $\theta_2$ is the angle of the break of $SU(2)_{II}$ from $SU(2)_I \times SU(2)_{II} \times U(1)_{II}$ to $SU(2)_I \times U(1)_I$. After the electroweak break the conserved charges are $\vec{\tau}^3$ and $Q := Y + \tau^{13}$, the non conserved charge is $Q' := -Y\tan^2\vartheta_1 + \tau^{13}$, where $\theta_1$ is the electroweak angle.

The groups $SO(3,1)$, $SU(3)$, $SU(2)_{I}$, $SU(2)_{II}$ and $U(1)_{II}$ determine spin and charges of fermions, the groups $\widetilde{SO}(3,1)$, $\widetilde{SU}(2)_{I}$, $\widetilde{SU}(2)_{II}$ and $\widetilde{U}(1)_{II}$ determine family quantum numbers [3].

The generators of these groups are expressible by $S^{\tilde{a}b}$

$$\tilde{\tau}^{Ai} = \sum_{a,b} c^{Ai}{}_{ab}\, \tilde{S}^{ab}\,, \tag{13.5}$$

fulfilling again the commutation relations

$$\{\tilde{\tau}^{Ai}, \tilde{\tau}^{Bj}\}_{-} = i\delta^{AB} f^{Aijk} \tilde{\tau}^{Ak}\,, \tag{13.6}$$

while

$$\{\tau^{Ai}, \tilde{\tau}^{Bj}\}_{-} = 0\,. \tag{13.7}$$

The scalar gauge fields of the groups $\widetilde{SU}(2)_{I}$, $\widetilde{SU}(2)_{I}$ and $U(1)$ are presented in Eq. (13.30), the application of the generators of $\vec{\tilde{\tau}}^{1}$, Eq. (13.26), $\vec{\tilde{N}}_{L}$, Eq. (13.25), which distinguish among families and are the same for all the family members, are presented in Eq. (13.12). The application of the family members generators $Q, Y, \tau^4$ and $Y'$ on the family members of any family is presented on Table 13.1.

| $R$ | $Q_{L,R}$ | $Y$ | $\tau^4_{L,R}$ | $Y'$ | $Q'$ | $L$ | $Y$ | $Y'$ | $Q'$ |
|---|---|---|---|---|---|---|---|---|---|
| $u^i_R$ | $\frac{2}{3}$ | $\frac{2}{3}$ | $\frac{1}{6}$ | $\frac{1}{2}\,(1-\frac{1}{3}\tan^2\theta_2)$ | $-\frac{2}{3}\tan^2\theta_1$ | $u^i_L$ | $\frac{1}{6}$ | $-\frac{1}{6}\tan^2\theta_2$ | $\frac{1}{2}(1-\frac{1}{3}\tan^2\theta_1)$ |
| $d^i_R$ | $-\frac{1}{3}$ | $-\frac{1}{3}$ | $\frac{1}{6}$ | $-\frac{1}{2}\,(1+\frac{1}{3}\tan^2\theta_2)$ | $\frac{1}{3}\tan^2\theta_1$ | $d^i_L$ | $\frac{1}{6}$ | $-\frac{1}{6}\tan^2\theta^2$ | $-\frac{1}{2}(1+\tan^2\theta_1)$ |
| $\nu^i_R$ | $0$ | $0$ | $-\frac{1}{2}$ | $\frac{1}{2}\,(1+\tan^2\theta_2)$ | $0$ | $\nu^i_L$ | $-\frac{1}{2}$ | $\frac{1}{2}\tan^2\theta_2$ | $\frac{1}{2}(1+\tan^2\theta_1)$ |
| $e_R$ | $-1$ | $-1$ | $-\frac{1}{2}$ | $\frac{1}{2}\,(-1+\tan^2\theta_2)$ | $\tan^2\theta_1$ | $e_L$ | $-\frac{1}{2}$ | $\frac{1}{2}\tan^2\theta_2$ | $-\frac{1}{2}(1-\tan^2\theta_1)$ |

**Table 13.1.** The quantum numbers $Q, Y, \tau^4, Y', Q'$, Eq. (13.28), of the members of one family (anyone) [6]. Left and right handed members of any family have the same $Q$ and $\tau^4$, the right handed members have $\tau^{13} = 0$ and $\tau^{23} = \frac{1}{2}$, while the left handed members have $\tau^{13} = \frac{1}{2}$ and $\tau^{23} = 0$.

There are in the *spin-charge-family* theory $2^{\frac{(1+7)}{2}-1} = 8$ families, which split in two groups of four families, due to the break of the symmetry from $\widetilde{SO}(1,7)$ into $\widetilde{SO}(1,3) \times \widetilde{SO}(4)$. Each of these two groups manifests $\widetilde{SU}(2)_{\widetilde{SO}(1,3)} \times \widetilde{SU}(2)_{\widetilde{SO}(4)}$ [6]. These decoupled twice four families are presented in Table 13.4

The lowest of the upper four families, forming neutral clusters with respect to the electromagnetic and colour charges, is the candidate to forms the dark matter [18].

We discuss in this paper symmetry properties of the lower four families, presented in Table 13.4 in the first four lines. We repeat in Table 13.2 the representation and the family quantum numbers of the left and right handed members of the lower four families. Since any of the family members $(u^i_{L,R}, d^i_{L,R}, \nu^i_{L,R}, e^i_{L,R})$

[3] $\widetilde{SU}(3)$ do not contribute to the families at low energies [34].

behave equivalently with respect to all the operators concerning the family groups $\widetilde{SU}(2)_{\widetilde{SO}(1,3)} \times \widetilde{SU}(2)_{\widetilde{SO}(4)}$, we use a common notation $|\psi^i>$.

The interaction, which is responsible for the appearance of masses of fermions, is presented in in Eq. (13.1) in the second line

$$\begin{aligned}
\mathcal{L}_{mass} &= \frac{1}{2} \sum_{+,-} \{\psi^\dagger_L \gamma^0 \stackrel{78}{(\pm)} (-\sum_{A} \tau^A \, A^A_\pm - \sum_{\tilde{A}i} \tilde{\tau}^{Ai} \, A^{Ai}_\pm) \psi_R\} + \text{h.c.}\,, \\
\tau^A &= (Q, Q', Y')\,, \quad \tilde{\tau}^{Ai} = (\vec{\tilde{N}}_L, \vec{\tilde{\tau}}^1, \tilde{\tau}^4)\,, \\
\gamma^0 \stackrel{78}{(\pm)} &= \gamma^0 \frac{1}{2} (\gamma^7 \pm i \gamma^8)\,, \\
A^A_\pm &= \sum_{st} c^A_{st} \, \omega^{st}{}_\pm\,, \quad \omega^{st}{}_\pm = \omega^{st}{}_7 \mp i \, \omega^{st}{}_8\,, \\
\vec{\tilde{A}}^A_\pm &= \sum_{ab} c^A_{ab} \, \tilde{\omega}^{ab}{}_\pm\,, \quad \tilde{\omega}^{ab}{}_\pm = \tilde{\omega}^{ab}{}_7 \mp i \, \tilde{\omega}^{ab}{}_8\,. \qquad (13.8)
\end{aligned}$$

In Eq. (13.8) the $p_s$ is left out since at low energies its contribution is negligible, $A$ determines operators, which distinguish among family members – $(Q, Y, \tau^4)$, the values are presented in Table 13.1 – $(\tilde{A}, i)$ represent the family operators, determined in Eqs. (13.25, 13.26, 13.27). The detailed explanation can be found in Refs. [4,5,1].

Operators $\tau^{Ai}$ are Hermitian, $\gamma^0 \stackrel{78}{(\pm)} = \gamma^0 \stackrel{78}{(\mp)}$. In what follows it is assumed that the scalar fields $A^{Ai}_s$ are Hermitian as well and consequently it follows $(A^{Ai}_\pm)^\dagger = A^{Ai}_\mp$.

While the family operators $\tilde{\tau}^{1i}$ and $\tilde{N}^i_L$ commute with $\gamma^0 \stackrel{78}{(\pm)}$, the family members operators $(Y, Y', Q')$ do not, since $S^{78}$ does not ($S^{78} \gamma^0 \stackrel{78}{(\mp)} = -\gamma^0 \stackrel{78}{(\mp)} S^{78}$). However

$$\begin{aligned}
[\psi^{k\dagger}_L \gamma^0 \stackrel{78}{(\mp)} (Q, Q', Y') A^{(Q,Q',Y')}_\mp \psi^l_R]^\dagger &= \\
&= \psi^{l\dagger}_R \, (Q, Q', Y') \, A^{(Q,Q',Y')\dagger}_\pm \, \gamma^0 \stackrel{78}{(\pm)} \psi^k_L \, \delta_{k,l} = \\
&= \psi^{l\dagger}_R \, (Q^k_R, Q'^k_R, Y'^k_R) \, A^{(Q,Q',Y')}_\pm \, \psi^k_R \, \delta_{k,l}\,, \qquad (13.9)
\end{aligned}$$

where $(Q^k_R, Q'^k_R, Y'^k_R)$ denote the eigenvalues of the corresponding operators on the spinor state $\psi^k_R$. This means that we evaluate in both cases quantum numbers of the right handed partners.

In Table 13.2 four families of spinors belonging to the group with the nonzero values of $\vec{\tilde{N}}_L$ and $\vec{\tilde{\tau}}^1$ are presented in the *technique* 13.5. These are the lower four families, presented in Table 13.4. There are indeed the four families of $\psi^i_{u_R}$ and $\psi^i_{u_L}$. All the $2^{\frac{13+1}{2}-1}$ members of the first family are represented in Table 13.3. The scalar fields $\gamma^0 \stackrel{78}{(\mp)} (Q, Q', Y') \; A^{(Q,Q',Y')}_\mp$ are "diagonal"; They transform a right handed member of one family into the left handed member of the same family, or they transform a left handed member of one family into the right handed member of the same family. These terms are different for different family members but the same for all the families of the same family member.

*We shall prove that the symmetry of mass term keep the same in all the orders of loop corrections in the massless basis.*

Since $Q = (\tau^{13} + \tau^{23} + \tau^4) = (S^{56} + \tau^4)$, $Y' = (-\tau^4 \tan^2\theta_1 + \tau^{23})$ and $Q' = (-(\tau^4 + \tau^{23})\tan^2\theta_1 + \tau^{13})$, we can use as well the operators ($\gamma^0 \overset{78}{(\pm)} \tau^4 A^4_{\pm}$, $\gamma^0 \overset{78}{(\pm)} \tau^{23} A^{23}_{\pm}$, $\gamma^0 \overset{78}{(\pm)} \tau^{13} A^{13}_{\pm}$). In either case we denote the contributions of these terms as $-a^{\alpha}_0$

$$
\begin{aligned}
&-a^{\alpha}_0 = \\
&= -\frac{1}{2}\{\psi^{i\dagger}_L \sum_{+,-} (\gamma^0 \overset{78}{(\pm)} \tau^4 A^4_{\pm} + \gamma^0 \overset{78}{(\pm)} \tau^{23} A^{23}_{\pm} + \gamma^0 \overset{78}{(\pm)} \tau^{13} A^{13}_{\pm})\psi^j_R\}\delta^{ij} + h.c.,
\end{aligned}
\tag{13.10}
$$

where $\alpha$ means that a particular family member ( $\alpha = (u, d, \nu, e)$) is studied. We could make different superposition of these terms. Our proof does not depend on this choice, although each family member has a different value for $a^{\alpha}_0$.

*Transitions among families for any family member are caused by ($\tilde{N}^i_L$ and $\tilde{\tau}^{1i}$), which manifest the symmetry $\widetilde{SU}_{N_L}(2) \times \widetilde{SU}_{\tau^1}(2)$.*

| | | | | $\tilde{\tau}^{13}$ | $\tilde{\tau}^{23}$ | $\tilde{N}^3_L$ | $\tilde{N}^3_R$ | $\tilde{\tau}^4$ |
|---|---|---|---|---|---|---|---|---|
| $\psi^1_{u^{c1}_R}$ | $\overset{03}{(+i)}\,\overset{12}{[+]}\,\vert\,\overset{56}{[+]}\,\overset{78}{(+)}\,\Vert\cdots$ | $\psi^1_{u^{c1}_L}$ | $-\,\overset{03}{[-i]}\,\overset{12}{[+]}\,\vert\,\overset{56}{[+]}\,\overset{78}{[-]}\,\Vert\cdots$ | $-\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | 0 | $-\frac{1}{2}$ |
| $\psi^2_{u^{c1}_R}$ | $\overset{03}{[+i]}\,\overset{12}{(+)}\,\vert\,\overset{56}{[+]}\,\overset{78}{(+)}\,\Vert\ldots$ | $\psi^2_{u^{c1}_L}$ | $-\,\overset{03}{(-i)}\,\overset{12}{(+)}\,\vert\,\overset{56}{[+]}\,\overset{78}{[-]}\,\Vert\cdots$ | $-\frac{1}{2}$ | 0 | $\frac{1}{2}$ | 0 | $-\frac{1}{2}$ |
| $\psi^3_{u^{c1}_R}$ | $\overset{03}{(+i)}\,\overset{12}{[+]}\,\vert\,\overset{56}{(+)}\,\overset{78}{[+]}\,\Vert\cdots$ | $\psi^3_{u^{c1}_L}$ | $-\,\overset{03}{[-i]}\,\overset{12}{[+]}\,\vert\,\overset{56}{(+)}\,\overset{78}{(-)}\,\Vert\cdots$ | $\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | 0 | $-\frac{1}{2}$ |
| $\psi^4_{u^{c1}_R}$ | $\overset{03}{[+i]}\,\overset{12}{(+)}\,\vert\,\overset{56}{(+)}\,\overset{78}{[+]}\,\Vert\cdots$ | $\psi^4_{u^{c1}_L}$ | $-\,\overset{03}{(-i)}\,\overset{12}{(+)}\,\vert\,\overset{56}{(+)}\,\overset{78}{(-)}\,\Vert\cdots$ | $\frac{1}{2}$ | 0 | $\frac{1}{2}$ | 0 | $-\frac{1}{2}$ |

**Table 13.2.** Four families of the right handed $u^{c1}_R$ and of the left handed $u^{c1}_L$ quarks with spin $\frac{1}{2}$ and the colour charge ($\tau^{33} = 1/2$, $\tau^{38} = 1/(2\sqrt{3})$ (the definition of the operators is presented in Eqs. (13.23,13.24) are presented ($1^{st}$ and $7^{th}$ line in Table 13.3). A few examples how to calculate the application of the operators on the states written as products of nilpotents and projectors on the vacuum state can be found in Sect. 13.5. The spin and charges, which distinguish among family members, are not shown in this table, since they commute with $\tilde{N}^i_L$, $\tilde{\tau}^{1i}$ and $\tilde{\tau}^4$, and are correspondingly the same for all the families.

| i | | $\vert^a\psi_i>$ | $\Gamma^{(3,1)}$ | $S^{12}$ | $\tau^{13}$ | $\tau^{23}$ | $\tau^{33}$ | $\tau^{38}$ | $\tau^{4}$ | Y | Q |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | (Anti)octet, $\Gamma^{(7,1)} = (-1)\,1\,,\ \Gamma^{(6)} = (1)\,-1$ of (anti)quarks and (anti)leptons | | | | | | | | | |
| 1 | $u^{c1}_R$ | $\stackrel{03}{(+i)}\stackrel{12}{[+]}\mid\stackrel{56}{[+]}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$ | 1 | $\frac{1}{2}$ | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2\sqrt{3}}$ | $\frac{1}{6}$ | $\frac{2}{3}$ | $\frac{2}{3}$ |
| 2 | $u^{c1}_R$ | $\stackrel{03}{[-i]}\stackrel{12}{(-)}\mid\stackrel{56}{[+]}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$ | 1 | $-\frac{1}{2}$ | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2\sqrt{3}}$ | $\frac{1}{6}$ | $\frac{2}{3}$ | $\frac{2}{3}$ |
| 3 | $d^{c1}_R$ | $\stackrel{03}{(+i)}\stackrel{12}{[+]}\mid\stackrel{56}{(-)}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$ | 1 | $\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2\sqrt{3}}$ | $\frac{1}{6}$ | $-\frac{1}{3}$ | $-\frac{1}{3}$ |
| 4 | $d^{c1}_R$ | $\stackrel{03}{[-i]}\stackrel{12}{(-)}\mid\stackrel{56}{(-)}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$ | 1 | $-\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2\sqrt{3}}$ | $\frac{1}{6}$ | $-\frac{1}{3}$ | $-\frac{1}{3}$ |
| 5 | $d^{c1}_L$ | $\stackrel{03}{[-i]}\stackrel{12}{[+]}\mid\stackrel{56}{(-)}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$ | -1 | $\frac{1}{2}$ | $-\frac{1}{2}$ | 0 | $\frac{1}{2}$ | $\frac{1}{2\sqrt{3}}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $-\frac{1}{3}$ |
| 6 | $d^{c1}_L$ | $-\stackrel{03}{(+i)}\stackrel{12}{(-)}\mid\stackrel{56}{(-)}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$ | -1 | $-\frac{1}{2}$ | $-\frac{1}{2}$ | 0 | $\frac{1}{2}$ | $\frac{1}{2\sqrt{3}}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $-\frac{1}{3}$ |
| 7 | $u^{c1}_L$ | $-\stackrel{03}{[-i]}\stackrel{12}{[+]}\mid\stackrel{56}{[+]}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$ | -1 | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | $\frac{1}{2}$ | $\frac{1}{2\sqrt{3}}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{2}{3}$ |
| 8 | $u^{c1}_L$ | $\stackrel{03}{(+i)}\stackrel{12}{(-)}\mid\stackrel{56}{[+]}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$ | -1 | $-\frac{1}{2}$ | $\frac{1}{2}$ | 0 | $\frac{1}{2}$ | $\frac{1}{2\sqrt{3}}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{2}{3}$ |
| 9 | $u^{c2}_R$ | $\stackrel{03}{(+i)}\stackrel{12}{[+]}\mid\stackrel{56}{[+]}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$ | 1 | $\frac{1}{2}$ | 0 | $\frac{1}{2}$ | $-\frac{1}{2}$ | $\frac{1}{2\sqrt{3}}$ | $\frac{1}{6}$ | $\frac{2}{3}$ | $\frac{2}{3}$ |
| 10 | $u^{c2}_R$ | $\stackrel{03}{[-i]}\stackrel{12}{(-)}\mid\stackrel{56}{[+]}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$ | 1 | $-\frac{1}{2}$ | 0 | $\frac{1}{2}$ | $-\frac{1}{2}$ | $\frac{1}{2\sqrt{3}}$ | $\frac{1}{6}$ | $\frac{2}{3}$ | $\frac{2}{3}$ |
| 11 | $d^{c2}_R$ | $\stackrel{03}{(+i)}\stackrel{12}{[+]}\mid\stackrel{56}{(-)}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$ | 1 | $\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | $-\frac{1}{2}$ | $\frac{1}{2\sqrt{3}}$ | $\frac{1}{6}$ | $-\frac{1}{3}$ | $-\frac{1}{3}$ |
| 12 | $d^{c2}_R$ | $\stackrel{03}{[-i]}\stackrel{12}{(-)}\mid\stackrel{56}{(-)}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$ | 1 | $-\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | $-\frac{1}{2}$ | $\frac{1}{2\sqrt{3}}$ | $\frac{1}{6}$ | $-\frac{1}{3}$ | $-\frac{1}{3}$ |
| 13 | $d^{c2}_L$ | $\stackrel{03}{[-i]}\stackrel{12}{[+]}\mid\stackrel{56}{(-)}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$ | -1 | $\frac{1}{2}$ | $-\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | $\frac{1}{2\sqrt{3}}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $-\frac{1}{3}$ |
| 14 | $d^{c2}_L$ | $-\stackrel{03}{(+i)}\stackrel{12}{(-)}\mid\stackrel{56}{(-)}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$ | -1 | $-\frac{1}{2}$ | $-\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | $\frac{1}{2\sqrt{3}}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $-\frac{1}{3}$ |
| 15 | $u^{c2}_L$ | $-\stackrel{03}{[-i]}\stackrel{12}{[+]}\mid\stackrel{56}{[+]}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$ | -1 | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | $\frac{1}{2\sqrt{3}}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{2}{3}$ |
| 16 | $u^{c2}_L$ | $\stackrel{03}{(+i)}\stackrel{12}{(-)}\mid\stackrel{56}{[+]}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$ | -1 | $-\frac{1}{2}$ | $\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | $\frac{1}{2\sqrt{3}}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{2}{3}$ |
| 17 | $u^{c3}_R$ | $\stackrel{03}{(+i)}\stackrel{12}{[+]}\mid\stackrel{56}{[+]}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$ | 1 | $\frac{1}{2}$ | 0 | $\frac{1}{2}$ | 0 | $-\frac{1}{\sqrt{3}}$ | $\frac{1}{6}$ | $\frac{2}{3}$ | $\frac{2}{3}$ |
| 18 | $u^{c3}_R$ | $\stackrel{03}{[-i]}\stackrel{12}{(-)}\mid\stackrel{56}{[+]}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$ | 1 | $-\frac{1}{2}$ | 0 | $\frac{1}{2}$ | 0 | $-\frac{1}{\sqrt{3}}$ | $\frac{1}{6}$ | $\frac{2}{3}$ | $\frac{2}{3}$ |
| 19 | $d^{c3}_R$ | $\stackrel{03}{(+i)}\stackrel{12}{[+]}\mid\stackrel{56}{(-)}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$ | 1 | $\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | 0 | $-\frac{1}{\sqrt{3}}$ | $\frac{1}{6}$ | $-\frac{1}{3}$ | $-\frac{1}{3}$ |
| 20 | $d^{c3}_R$ | $\stackrel{03}{[-i]}\stackrel{12}{(-)}\mid\stackrel{56}{(-)}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$ | 1 | $-\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | 0 | $-\frac{1}{\sqrt{3}}$ | $\frac{1}{6}$ | $-\frac{1}{3}$ | $-\frac{1}{3}$ |
| 21 | $d^{c3}_L$ | $\stackrel{03}{[-i]}\stackrel{12}{[+]}\mid\stackrel{56}{(-)}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$ | -1 | $\frac{1}{2}$ | $-\frac{1}{2}$ | 0 | 0 | $-\frac{1}{\sqrt{3}}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $-\frac{1}{3}$ |
| 22 | $d^{c3}_L$ | $-\stackrel{03}{(+i)}\stackrel{12}{(-)}\mid\stackrel{56}{(-)}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$ | -1 | $-\frac{1}{2}$ | $-\frac{1}{2}$ | 0 | 0 | $-\frac{1}{\sqrt{3}}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $-\frac{1}{3}$ |
| 23 | $u^{c3}_L$ | $-\stackrel{03}{[-i]}\stackrel{12}{[+]}\mid\stackrel{56}{[+]}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$ | -1 | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | 0 | $-\frac{1}{\sqrt{3}}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{2}{3}$ |
| 24 | $u^{c3}_L$ | $\stackrel{03}{(+i)}\stackrel{12}{(-)}\mid\stackrel{56}{[+]}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$ | -1 | $-\frac{1}{2}$ | $\frac{1}{2}$ | 0 | 0 | $-\frac{1}{\sqrt{3}}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{2}{3}$ |
| 25 | $\nu_R$ | $\stackrel{03}{(+i)}\stackrel{12}{[+]}\mid\stackrel{56}{[+]}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$ | 1 | $\frac{1}{2}$ | 0 | $\frac{1}{2}$ | 0 | 0 | $-\frac{1}{2}$ | 0 | 0 |
| 26 | $\nu_R$ | $\stackrel{03}{[-i]}\stackrel{12}{(-)}\mid\stackrel{56}{[+]}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$ | 1 | $-\frac{1}{2}$ | 0 | $\frac{1}{2}$ | 0 | 0 | $-\frac{1}{2}$ | 0 | 0 |
| 27 | $e_R$ | $\stackrel{03}{(+i)}\stackrel{12}{[+]}\mid\stackrel{56}{(-)}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$ | 1 | $\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | 0 | 0 | $-\frac{1}{2}$ | $-1$ | $-1$ |
| 28 | $e_R$ | $\stackrel{03}{[-i]}\stackrel{12}{(-)}\mid\stackrel{56}{(-)}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$ | 1 | $-\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | 0 | 0 | $-\frac{1}{2}$ | $-1$ | $-1$ |
| 29 | $e_L$ | $\stackrel{03}{[-i]}\stackrel{12}{[+]}\mid\stackrel{56}{(-)}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$ | -1 | $\frac{1}{2}$ | $-\frac{1}{2}$ | 0 | 0 | 0 | $-\frac{1}{2}$ | $-\frac{1}{2}$ | $-1$ |
| 30 | $e_L$ | $-\stackrel{03}{(+i)}\stackrel{12}{(-)}\mid\stackrel{56}{(-)}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$ | -1 | $-\frac{1}{2}$ | $-\frac{1}{2}$ | 0 | 0 | 0 | $-\frac{1}{2}$ | $-\frac{1}{2}$ | $-1$ |
| 31 | $\nu_L$ | $-\stackrel{03}{[-i]}\stackrel{12}{[+]}\mid\stackrel{56}{[+]}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$ | -1 | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | 0 | 0 | $-\frac{1}{2}$ | $-\frac{1}{2}$ | 0 |
| 32 | $\nu_L$ | $\stackrel{03}{(+i)}\stackrel{12}{(-)}\mid\stackrel{56}{[+]}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$ | -1 | $-\frac{1}{2}$ | $\frac{1}{2}$ | 0 | 0 | 0 | $-\frac{1}{2}$ | $-\frac{1}{2}$ | 0 |
| 33 | $\bar{d}^{\bar{c}1}_L$ | $\stackrel{03}{[-i]}\stackrel{12}{[+]}\mid\stackrel{56}{[+]}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$ | -1 | $\frac{1}{2}$ | 0 | $\frac{1}{2}$ | $-\frac{1}{2}$ | $-\frac{1}{2\sqrt{3}}$ | $-\frac{1}{6}$ | $\frac{1}{3}$ | $\frac{1}{3}$ |
| 34 | $\bar{d}^{\bar{c}1}_L$ | $\stackrel{03}{(+i)}\stackrel{12}{(-)}\mid\stackrel{56}{[+]}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$ | -1 | $-\frac{1}{2}$ | 0 | $\frac{1}{2}$ | $-\frac{1}{2}$ | $-\frac{1}{2\sqrt{3}}$ | $-\frac{1}{6}$ | $\frac{1}{3}$ | $\frac{1}{3}$ |
| 35 | $\bar{u}^{\bar{c}1}_L$ | $-\stackrel{03}{[-i]}\stackrel{12}{[+]}\mid\stackrel{56}{(-)}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$ | -1 | $\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | $-\frac{1}{2}$ | $-\frac{1}{2\sqrt{3}}$ | $-\frac{1}{6}$ | $-\frac{2}{3}$ | $-\frac{2}{3}$ |
| 36 | $\bar{u}^{\bar{c}1}_L$ | $-\stackrel{03}{(+i)}\stackrel{12}{(-)}\mid\stackrel{56}{(-)}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$ | -1 | $-\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | $-\frac{1}{2}$ | $-\frac{1}{2\sqrt{3}}$ | $-\frac{1}{6}$ | $-\frac{2}{3}$ | $-\frac{2}{3}$ |
| 37 | $\bar{d}^{\bar{c}1}_R$ | $\stackrel{03}{(+i)}\stackrel{12}{[+]}\mid\stackrel{56}{[+]}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$ | 1 | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | $-\frac{1}{2\sqrt{3}}$ | $-\frac{1}{6}$ | $-\frac{1}{6}$ | $\frac{1}{3}$ |
| 38 | $\bar{d}^{\bar{c}1}_R$ | $-\stackrel{03}{[-i]}\stackrel{12}{(-)}\mid\stackrel{56}{[+]}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$ | 1 | $-\frac{1}{2}$ | $\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | $-\frac{1}{2\sqrt{3}}$ | $-\frac{1}{6}$ | $-\frac{1}{6}$ | $\frac{1}{3}$ |
| 39 | $\bar{u}^{\bar{c}1}_R$ | $\stackrel{03}{(+i)}\stackrel{12}{[+]}\mid\stackrel{56}{(-)}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$ | 1 | $\frac{1}{2}$ | $-\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | $-\frac{1}{2\sqrt{3}}$ | $-\frac{1}{6}$ | $-\frac{1}{6}$ | $-\frac{2}{3}$ |

*Continued on next page*

| i | | $\vert^{a}\psi_{i}>$ | $\Gamma^{(3,1)}$ | $S^{12}$ | $\tau^{13}$ | $\tau^{23}$ | $\tau^{33}$ | $\tau^{38}$ | $\tau^{4}$ | Y | Q |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | (Anti)octet, $\Gamma^{(7,1)} = (-1)\,1\,,\ \Gamma^{(6)} = (1)\,-1$ of (anti)quarks and (anti)leptons | | | | | | | | | |
| 40 | $\bar{u}_{R}^{\bar{c}1}$ | $\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$ | 1 | $-\frac{1}{2}$ | $-\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | $-\frac{1}{2\sqrt{3}}$ | $-\frac{1}{6}$ | $-\frac{1}{6}$ | $-\frac{2}{3}$ |
| 41 | $\bar{d}_{L}^{\bar{c}2}$ | $\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$ | -1 | $\frac{1}{2}$ | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ | $-\frac{1}{2\sqrt{3}}$ | $-\frac{1}{6}$ | $\frac{1}{3}$ | $\frac{1}{3}$ |
| 42 | $\bar{d}_{L}^{\bar{c}2}$ | $\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$ | -1 | $-\frac{1}{2}$ | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ | $-\frac{1}{2\sqrt{3}}$ | $-\frac{1}{6}$ | $\frac{1}{3}$ | $\frac{1}{3}$ |
| 43 | $\bar{u}_{L}^{\bar{c}2}$ | $-\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$ | -1 | $\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | $\frac{1}{2}$ | $-\frac{1}{2\sqrt{3}}$ | $-\frac{1}{6}$ | $-\frac{2}{3}$ | $-\frac{2}{3}$ |
| 44 | $\bar{u}_{L}^{\bar{c}2}$ | $-\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$ | -1 | $-\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | $\frac{1}{2}$ | $-\frac{1}{2\sqrt{3}}$ | $-\frac{1}{6}$ | $-\frac{2}{3}$ | $-\frac{2}{3}$ |
| 45 | $\bar{d}_{R}^{\bar{c}2}$ | $\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$ | 1 | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | $\frac{1}{2}$ | $-\frac{1}{2\sqrt{3}}$ | $-\frac{1}{6}$ | $-\frac{1}{6}$ | $\frac{1}{3}$ |
| 46 | $\bar{d}_{R}^{\bar{c}2}$ | $-\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$ | 1 | $-\frac{1}{2}$ | $\frac{1}{2}$ | 0 | $\frac{1}{2}$ | $-\frac{1}{2\sqrt{3}}$ | $-\frac{1}{6}$ | $-\frac{1}{6}$ | $\frac{1}{3}$ |
| 47 | $\bar{u}_{R}^{\bar{c}2}$ | $\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$ | 1 | $\frac{1}{2}$ | $-\frac{1}{2}$ | 0 | $\frac{1}{2}$ | $-\frac{1}{2\sqrt{3}}$ | $-\frac{1}{6}$ | $-\frac{1}{6}$ | $-\frac{2}{3}$ |
| 48 | $\bar{u}_{R}^{\bar{c}2}$ | $\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$ | 1 | $-\frac{1}{2}$ | $-\frac{1}{2}$ | 0 | $\frac{1}{2}$ | $-\frac{1}{2\sqrt{3}}$ | $-\frac{1}{6}$ | $-\frac{1}{6}$ | $-\frac{2}{3}$ |
| 49 | $\bar{d}_{L}^{\bar{c}3}$ | $\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$ | -1 | $\frac{1}{2}$ | 0 | $\frac{1}{2}$ | 0 | $\frac{1}{\sqrt{3}}$ | $-\frac{1}{6}$ | $\frac{1}{3}$ | $\frac{1}{3}$ |
| 50 | $\bar{d}_{L}^{\bar{c}3}$ | $\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$ | -1 | $-\frac{1}{2}$ | 0 | $\frac{1}{2}$ | 0 | $\frac{1}{\sqrt{3}}$ | $-\frac{1}{6}$ | $\frac{1}{3}$ | $\frac{1}{3}$ |
| 51 | $\bar{u}_{L}^{\bar{c}3}$ | $-\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$ | -1 | $\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | 0 | $\frac{1}{\sqrt{3}}$ | $-\frac{1}{6}$ | $-\frac{2}{3}$ | $-\frac{2}{3}$ |
| 52 | $\bar{u}_{L}^{\bar{c}3}$ | $-\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$ | -1 | $-\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | 0 | $\frac{1}{\sqrt{3}}$ | $-\frac{1}{6}$ | $-\frac{2}{3}$ | $-\frac{2}{3}$ |
| 53 | $\bar{d}_{R}^{\bar{c}3}$ | $\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$ | 1 | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | 0 | $\frac{1}{\sqrt{3}}$ | $-\frac{1}{6}$ | $-\frac{1}{6}$ | $\frac{1}{3}$ |
| 54 | $\bar{d}_{R}^{\bar{c}3}$ | $-\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$ | 1 | $-\frac{1}{2}$ | $\frac{1}{2}$ | 0 | 0 | $\frac{1}{\sqrt{3}}$ | $-\frac{1}{6}$ | $-\frac{1}{6}$ | $\frac{1}{3}$ |
| 55 | $\bar{u}_{R}^{\bar{c}3}$ | $\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$ | 1 | $\frac{1}{2}$ | $-\frac{1}{2}$ | 0 | 0 | $\frac{1}{\sqrt{3}}$ | $-\frac{1}{6}$ | $-\frac{1}{6}$ | $-\frac{2}{3}$ |
| 56 | $\bar{u}_{R}^{\bar{c}3}$ | $\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$ | 1 | $-\frac{1}{2}$ | $-\frac{1}{2}$ | 0 | 0 | $\frac{1}{\sqrt{3}}$ | $-\frac{1}{6}$ | $-\frac{1}{6}$ | $-\frac{2}{3}$ |
| 57 | $\bar{e}_{L}$ | $\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$ | -1 | $\frac{1}{2}$ | 0 | $\frac{1}{2}$ | 0 | 0 | $\frac{1}{2}$ | 1 | 1 |
| 58 | $\bar{e}_{L}$ | $\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$ | -1 | $-\frac{1}{2}$ | 0 | $\frac{1}{2}$ | 0 | 0 | $\frac{1}{2}$ | 1 | 1 |
| 59 | $\bar{\nu}_{L}$ | $-\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$ | -1 | $\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | 0 | 0 | $\frac{1}{2}$ | 0 | 0 |
| 60 | $\bar{\nu}_{L}$ | $-\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$ | -1 | $-\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | 0 | 0 | $\frac{1}{2}$ | 0 | 0 |
| 61 | $\bar{\nu}_{R}$ | $\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$ | 1 | $\frac{1}{2}$ | $-\frac{1}{2}$ | 0 | 0 | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 |
| 62 | $\bar{\nu}_{R}$ | $-\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$ | 1 | $-\frac{1}{2}$ | $-\frac{1}{2}$ | 0 | 0 | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 |
| 63 | $\bar{e}_{R}$ | $\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$ | 1 | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | 0 | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ | 1 |
| 64 | $\bar{e}_{R}$ | $\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$ | 1 | $-\frac{1}{2}$ | $\frac{1}{2}$ | 0 | 0 | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ | 1 |

**Table 13.3.** The left handed ($\Gamma^{(13,1)} = -1$, Eq. (13.38)) multiplet of spinors — the members of the fundamental representation of the $SO(13,1)$ group, manifesting the subgroup $SO(7,1)$ of the colour charged quarks and anti-quarks and the colourless leptons and anti-leptons — is presented in the massless basis using the technique presented in App. 13.5. It contains the left handed ($\Gamma^{(3,1)} = -1$) weak ($SU(2)_{I}$) charged ($\tau^{13} = \pm\frac{1}{2}$, Eq. (13.23)), and $SU(2)_{II}$ chargeless ($\tau^{23} = 0$, Eq. (13.23)) quarks and leptons and the right handed ($\Gamma^{(3,1)} = 1$, Sect. 13.5) weak ($SU(2)_{I}$) chargeless and $SU(2)_{II}$ charged ($\tau^{23} = \pm\frac{1}{2}$) quarks and leptons, both with the spin $S^{12}$ up and down ($\pm\frac{1}{2}$, respectively). Quarks distinguish from leptons only in the $SU(3)\times U(1)$ part: Quarks are triplets of three colours ($c^{i} = (\tau^{33}, \tau^{38}) = [(\frac{1}{2}, \frac{1}{2\sqrt{3}}), (-\frac{1}{2}, \frac{1}{2\sqrt{3}}), (0, -\frac{1}{\sqrt{3}})]$, Eq. (13.24)) carrying the "fermion charge" ($\tau^{4} = \frac{1}{6}$, Eq. (13.24)). The colourless leptons carry the "fermion charge" ($\tau^{4} = -\frac{1}{2}$). The same multiplet contains also the left handed weak ($SU(2)_{I}$) chargeless and $SU(2)_{II}$ charged anti-quarks and anti-leptons and the right handed weak ($SU(2)_{I}$) charged and $SU(2)_{II}$ chargeless anti-quarks and anti-leptons. Anti-quarks distinguish from anti-leptons again only in the $SU(3)\times U(1)$ part: Anti-quarks are anti-triplets, carrying the "fermion charge" ($\tau^{4} = -\frac{1}{6}$). The anti-colourless anti-leptons carry the "fermion charge" ($\tau^{4} = \frac{1}{2}$). $Y = (\tau^{23} + \tau^{4})$ is the hyper charge, the electromagnetic charge is $Q = (\tau^{13} + Y)$. The states of opposite charges (anti-particle states) are reachable from the particle states (besides by $S^{ab}$) also by the application of the discrete symmetry operator $\mathcal{C}_{\mathcal{N}}\,\mathcal{P}_{\mathcal{N}}$, presented in Refs. [41,42] and in Sect. 13.5. The vacuum state, on which the nilpotents and projectors operate, is not shown. The reader can find this Weyl representation also in Refs. [5,14,15,4] and in the references therein.

Taking into account Table 13.3 and Eqs. (13.34, 13.43) one easily finds what do operators $\gamma^{0}\stackrel{78}{(\pm)}$ do on the left handed and the right handed members of any family $i = (1, 2, 3, 4)$.

$$
\begin{aligned}
\gamma^0 \stackrel{78}{(-)} |\psi^i_{u_R,\nu_R} > &= -|\psi^i_{u_L,\nu_L} >\,,\\
\gamma^0 \stackrel{78}{(+)} |\psi^i_{u_L,\nu_L} > &= |\psi^i_{u_R,\nu_R} >\,,\\
\gamma^0 \stackrel{78}{(+)} |\psi^i_{d_R,e_R} > &= |\psi^i_{d_L,e_L} >\,,\\
\gamma^0 \stackrel{78}{(-)} |\psi^i_{d_L,e_L} > &= |\psi^i_{d_R,e_R} >\,. \qquad (13.11)
\end{aligned}
$$

We need to know also what do operators ($\tilde{\tau}^{1\pm} = \tilde{\tau}^{11} \pm i\,\tilde{\tau}^{12}$, $\tilde{\tau}^{13}$) and ($\tilde{N}^{\pm}_L = \tilde{N}^1_L \pm i\,\tilde{N}^2_L$, $\tilde{N}^3_L$) do when operating on any member ($u_{L,R}$, $\nu_{L,R}$, $d_{L,R}$, $e_{L,R}$) of a particular family $\psi^i$, $i = (1, 2, 3, 4)$.

Taking into account, Eqs. (13.32, 13.33, 13.43, 13.45, 13.36, 13.25, 13.26),

$$
\begin{aligned}
&\tilde{N}^{\pm}_L = -\stackrel{03}{\widetilde{(\mp i)}}\stackrel{12}{\widetilde{(\pm)}}\,, \qquad \tilde{\tau}^{1\pm} = (\mp)\stackrel{56}{\widetilde{(\pm)}}\stackrel{78}{\widetilde{(\mp)}}\,,\\
&\tilde{N}^3_L = \frac{1}{2}\,(\tilde{S}^{12} + i\,\tilde{S}^{03})\,, \quad \tilde{\tau}^{13} = \frac{1}{2}\,(\tilde{S}^{56} - \tilde{S}^{78})\,,\\
&\stackrel{ab}{\widetilde{(-k)}}\stackrel{ab}{(k)} = -i\eta^{aa}\stackrel{ab}{[k]}\,, \qquad \stackrel{ab}{\widetilde{(k)}}\stackrel{ab}{(k)} = 0\,,\\
&\stackrel{ab}{\widetilde{(k)}}\stackrel{ab}{[k]} = i\stackrel{ab}{(k)}\,, \qquad \stackrel{ab}{\widetilde{(k)}}\stackrel{ab}{[-k]} = 0\,,\\
&\stackrel{ab}{\widetilde{(k)}} = \frac{1}{2}\,(\tilde{\gamma}^a + \frac{\eta^{aa}}{ik}\,\tilde{\gamma}^b)\,, \qquad \stackrel{ab}{\widetilde{[k]}} = \frac{1}{2}\,(1 + \frac{i}{k}\,\tilde{\gamma}^a\tilde{\gamma}^b)\,, \qquad (13.12)
\end{aligned}
$$

one finds

$$
\begin{aligned}
&\tilde{N}^{+}_L\,|\psi^1> = |\psi^2>\,, \quad \tilde{N}^{+}_L\,|\psi^2> = 0\,,\\
&\tilde{N}^{-}_L\,|\psi^2> = |\psi^1>\,, \quad \tilde{N}^{-}_L\,|\psi^1> = 0\,,\\
&\tilde{N}^{+}_L\,|\psi^3> = |\psi^4>\,, \quad \tilde{N}^{+}_L\,|\psi^4> = 0\,,\\
&\tilde{N}^{-}_L\,|\psi^4> = |\psi^3>\,, \quad \tilde{N}^{-}_L\,|\psi^3> = 0\,,\\
&\tilde{\tau}^{1+}\,|\psi^1> = |\psi^3>\,, \quad \tilde{\tau}^{1+}\,|\psi^3> = 0\,,\\
&\tilde{\tau}^{1-}\,|\psi^3> = |\psi^1>\,, \quad \tilde{\tau}^{1-}\,|\psi^1> = 0\,,\\
&\tilde{\tau}^{1-}\,|\psi^4> = |\psi^2>\,, \quad \tilde{\tau}^{1-}\,|\psi^2> = 0\,,\\
&\tilde{\tau}^{1+}\,|\psi^2> = |\psi^4>\,, \quad \tilde{\tau}^{1+}\,|\psi^4> = 0\,,\\
&\tilde{N}^3_L\,|\psi^1> = -\frac{1}{2}\,|\psi^1>\,, \quad \tilde{N}^3_L\,|\psi^2> = +\frac{1}{2}\,|\psi^2>\,,\\
&\tilde{N}^3_L\,|\psi^3> = -\frac{1}{2}\,|\psi^3>\,, \quad \tilde{N}^3_L\,|\psi^4> = +\frac{1}{2}\,|\psi^4>\,,\\
&\tilde{\tau}^{13}\,|\psi^1> = -\frac{1}{2}\,|\psi^1>\,, \quad \tilde{\tau}^{13}\,|\psi^2> = -\frac{1}{2}\,|\psi^2>\,,\\
&\tilde{\tau}^{13}\,|\psi^3> = +\frac{1}{2}\,|\psi^3>\,, \quad \tilde{\tau}^{13}\,|\psi^4> = +\frac{1}{2}\,|\psi^4>\,. \qquad (13.13)
\end{aligned}
$$

Let the scalars ($\tilde{A}^{N_L\boxplus}_{(\pm)}$, $\tilde{A}^{N_L 3}_{(\pm)}$, $\tilde{A}^{1\boxplus}_{(\pm)}$, $\tilde{A}^{13}_{(\pm)}$) be the scalar gauge fields of the operators ($\tilde{N}^{\pm}_L$, $\tilde{N}^3_L$, $\tilde{\tau}^{1\pm}$, $\tilde{\tau}^{13}$), respectively. Here $\tilde{A}_{(\pm)} = \tilde{A}_7 \mp i\,\tilde{A}_8$ for all the scalar gauge fields, while $\tilde{A}^{N_L \boxplus}_{(\pm)} = \frac{1}{2}\,(\tilde{A}^{N_L 1}_{(\pm)} \mp i\,\tilde{A}^{N_L 2}_{(\pm)})$, respectively, and $\tilde{A}^{1\boxplus}_{(\pm)} = \frac{1}{2}\,(\tilde{A}^{11}_{(\pm)} \mp i\,\tilde{A}^{1}_{(\pm)})$, respectively. All these fields can be expressed by $\tilde{\omega}_{abc}$, as presented in Eq. (13.30).

We are prepared now to calculate the mass matrix elements for any of the family members. Let us notice that the operators $\gamma^0 \stackrel{78}{(\mp)}$, as well as the operators of spin and charges, distinguish between $|\psi^i_L>$ and $|\psi^i_R>$. Correspondingly all the diagrams must have an odd number of contribution.

*We use the massless basis* $|\psi^i_{L,R}>$. We shall simplify the calculation by making a choice of the $\frac{1}{\sqrt{2}}\,(|\psi^i_L> + |\psi^i_R>)$, keeping in mind that we must have an odd number of contributions

We can calculate the mass matrix for any family member using Eqs. (13.13). Below we present the mass matrix on the tree level, where $(\tilde{a}_1, \tilde{a}_2, a_\alpha)$ represent the vacuum expectation values of $\frac{1}{2}\,\frac{1}{\sqrt{2}}\,(\tilde{A}^{\tilde{1}3}_{(+)} + \tilde{A}^{\tilde{1}3}_{(-)})$, $\frac{1}{2}\,\frac{1}{\sqrt{2}}\,(\tilde{A}^{\tilde{N}_L 3}_{(+)} + \tilde{A}^{\tilde{N}_L 3}_{(-)})$, $\frac{1}{\sqrt{2}}\,(A^{\alpha}_{(+)} + A^{\alpha}_{(-)})$, respectively and where to $A^{\alpha}_{(\pm)}$ the sum of $\tau^{4\alpha}A^{4}_{(\pm)}$, $\tau^{13\alpha}A^{13}_{(\pm)}$ and $\tau^{23\alpha}A^{23}_{(\pm)}$, Eq. (13.10), is contributing.

We use the notation $<\tilde{A}^{\tilde{N}_L \boxminus}> = \frac{1}{\sqrt{2}}\,(<\tilde{A}^{\tilde{N}_L \boxminus}_{(+)}> + <\tilde{A}^{\tilde{N}_L \boxminus}_{(-)}>)$ and $<\tilde{A}^{\tilde{1}\boxminus}> = \frac{1}{\sqrt{2}}\,(<\tilde{A}^{\tilde{1}\boxminus}_{(+)}> + <\tilde{A}^{\tilde{1}\boxminus}_{(-)}>)$, since we use the basis $\frac{1}{\sqrt{2}}\,(|\psi^i_L> + |\psi^i_R>)$.

On the tree level is the contribution to the matrix elements $<\psi^1|..|\psi^4>$, $<\psi^2|..|\psi^3>$, $<\psi^3|..|\psi^2>$ and $<\psi^4|..|\psi^1>$ equal to zero. One can come, however, from $<\psi^1|..|\psi^4>$ in three steps (not two, due to the left right jumps in each step): $<\psi^4|\sum_{+,-}\tilde{\tau}^{\tilde{1}\boxminus}\tilde{A}^{\tilde{1}\boxminus}\sum_k|\psi^k><\psi^k|\sum_{+,-}\tilde{N}^{\boxminus}_L\tilde{A}^{\tilde{N}_L\boxminus}|\psi^4><\psi^4|(\tilde{a}_1+\tilde{a}_2+a^\alpha)|\psi^4>$, there are all together six such terms, since the diagonal term appears also at the beginning as $(-\tilde{a}_1-\tilde{a}_2+a^\alpha)$ and in the middle as $(\tilde{a}_1-\tilde{a}_2+a^\alpha)$, and since the operators $\sum_{+,-}\tilde{\tau}^{\tilde{1}\boxminus}\tilde{A}^{\tilde{1}\boxminus}$ and $\sum_{+,-}\tilde{N}^{\boxminus}_L\tilde{A}^{\tilde{N}_L\boxminus}$ appear in the opposite order as well. Summing all this six terms for each of four matrix elements ($<1|..|4>$, $<2|..|3>$, $<3|..|2>$, $<4|..|1>$) we find:

$$
\begin{aligned}
<1|..|4> &= 6a^\alpha\ <\tilde{A}^{\tilde{1}\boxminus}><\tilde{A}^{\tilde{N}_L\boxminus}>\,,\\
<2|..|3> &= 6a^\alpha\ <\tilde{A}^{\tilde{1}\boxminus}><\tilde{A}^{\tilde{N}_L\boxplus}>\,,\\
<3|..|2> &= 6a^\alpha\ <\tilde{A}^{\tilde{1}\boxplus}><\tilde{A}^{\tilde{N}_L\boxminus}>\,,\\
<4|..|1> &= 6a^\alpha\ <\tilde{A}^{\tilde{1}\boxplus}><\tilde{A}^{\tilde{N}_L\boxplus}>\ .
\end{aligned}
\tag{13.14}
$$

These matrix elements are presented in Eq. (13.15).

$$
{}^{\alpha}\mathcal{M}_{(o)} =
\begin{pmatrix}
-\tilde{a}_1-\tilde{a}_2+a^\alpha & <\tilde{A}^{\tilde{N}_L\boxminus}> & <\tilde{A}^{\tilde{1}\boxminus}> & 6a^\alpha<\tilde{A}^{\tilde{1}\boxminus}\tilde{A}^{\tilde{N}_L\boxminus}> \\
<\tilde{A}^{\tilde{N}_L\boxplus}> & -\tilde{a}_1+\tilde{a}_2+a^\alpha & 6a^\alpha<\tilde{A}^{\tilde{1}\boxminus}\tilde{A}^{\tilde{N}_L\boxplus}> & <\tilde{A}^{\tilde{1}\boxminus}> \\
<\tilde{A}^{\tilde{1}\boxplus}> & 6a^\alpha<\tilde{A}^{\tilde{1}\boxplus}\tilde{A}^{\tilde{N}_L\boxminus}>b & \tilde{a}_1-\tilde{a}_2+a^\alpha & <\tilde{A}^{\tilde{N}_L\boxminus}> \\
6a^\alpha<\tilde{A}^{\tilde{1}\boxplus}\tilde{A}^{\tilde{N}_L\boxplus}> & <\tilde{A}^{\tilde{1}\boxplus}> & <\tilde{A}^{\tilde{N}_L\boxplus}> & \tilde{a}_1+\tilde{a}_2+a^\alpha
\end{pmatrix}
\tag{13.15}
$$

One notices that the diagonal terms have on the tree level the symmetry $<\psi^1|..|\psi^1> + <\psi^4|..|\psi^4> = a^\alpha = <\psi^2|..|\psi^2> + <\psi^3|..|\psi^3>$ and that in the off diagonal elements in next order to zero the contribution of the fields, which depend on particular family member $\alpha = (u, d, \nu, e)$ enter. We also notice that $< \psi^i|..|\psi^j >^{\dagger} = < \psi^j|..|\psi^i >$. In the case that $< \tilde{A}^{\tilde{1}\boxminus} > = < \tilde{A}^{\tilde{1}\boxplus} > = e$ and $< \tilde{A}^{\tilde{N}_L\boxminus} > = < \tilde{A}^{\tilde{N}_L\boxplus} > = d$, which would mean that all the matrix elements are real, the mass matrix simplifies to

$$\mathcal{M}^{\alpha}_{(o)} = \begin{pmatrix} -\tilde{a}_1 - \tilde{a}_2 + a^{\alpha} & d & e & 6a^{\alpha}ed \\ d & -\tilde{a}_1 + \tilde{a}_2 + a^{\alpha} & 6a^{\alpha}ed & e \\ e & 6a^{\alpha}ed & \tilde{a}_1 - \tilde{a}_2 + a^{\alpha} & d \\ 6a^{\alpha}ed & e & d & \tilde{a}_1 + \tilde{a}_2 + a^{\alpha} \end{pmatrix}. \tag{13.16}$$

### 13.2.1 Mass matrices beyond the tree level

To make a proof that the symmetry $\widetilde{SU}(2) \times \widetilde{SU}(2) \times U(1)$ of the mass matrix, presented in Eq. (13.15), is kept in all orders of loop corrections, we need to proof only that at each order the matrix element, let say, $< 1|..|2 >$ (in Eq. (13.15) this matrix element is equal to $< \tilde{A}^{\tilde{N}_L\boxminus} >$) remains equal to $< 3|..|4 >$ in all orders, while $< 2|..|1 >$ remains to be equal to $< 1|..|2 >^{\dagger} = < 4|..|3 >$ $(= < \tilde{A}^{\tilde{N}_L\boxminus} >)$. These should be done for all the matrix elements appearing in Eq. (13.15.

**a.** It is not difficult to see that each of the diagonal terms ($\tilde{\tau}^{\tilde{1}3} < \tilde{A}^{\tilde{1}3} >$, $\tilde{N}^3_L < \tilde{A}^{\tilde{N}_L 3} >$, $\tau^A < A^A >$, with $\tau^A = \tau^4, \tau^{13}, \tau^{23}$) have the property that the sum of the contributions $x + xxx + xxxxx + ...$ (in all orders) keeps the symmetry of the tree level. Let us check for $\tilde{\tau}^{\tilde{1}3} < \tilde{A}^{\tilde{1}3} >$. One obtains for each of the four families $i = [1,2,3,4]$ the values $[-\tilde{a}^1(1+(-\tilde{a}^1)^2+(-\tilde{a}^1)^4+\dots), -\tilde{a}^1(1+(-\tilde{a}^1)^2+(-\tilde{a}^1)^4+\dots), \tilde{a}^1(1+(\tilde{a}^1)^2+(\tilde{a}^1)^4+\dots), \tilde{a}^1(1+(\tilde{a}^1)^2+(\tilde{a}^1)^4+\dots)]$, which we call $[-\mathbf{\tilde{a}^1}, -\mathbf{\tilde{a}^1}, \mathbf{\tilde{a}^1}, \mathbf{\tilde{a}^1}]$ for the four families $i = [1,2,3,4]$, respectively. Correspondingly one finds for the same kind of diagrams for $\tilde{N}^3_L < \tilde{A}^{\tilde{N}_L 3} >$ the four values $[-\mathbf{\tilde{a}^2}$, $\mathbf{\tilde{a}^2}$, $-\mathbf{\tilde{a}^2}$, $\mathbf{\tilde{a}^2}]$ for the four families $i = [1,2,3,4]$, respectively. While for $\tau^A A^A$ we obtain, when summing over the diagrams $x + xxx + xxxxx + ...$, the same value $\mathbf{a}^{\alpha}$ for a particular family member $\alpha = (u, d, \nu, e)$ all four families. Family members properties enter in the left/right basis $\frac{1}{\sqrt{2}}(|\psi^i_L > +|\psi^i_R >$ into the mass matrix only through $a^{\alpha}$.

One reproduces that the sum of $< 1|..|1 > + < 4|..|4 > = < 2|..|2 > + < 3|..|3 >$

Correspondingly it is not difficult to see that all the matrix elements, not only diagonal but also off diagonal, keep the symmetry of the mass matrix of Eq. (13.15) in all orders of corrections, provided that the matrix elements of the kind $\alpha\tilde{a}_1 + \beta\tilde{a}_2 + a^{\alpha}$ — or of the kind in the $\alpha\mathbf{\tilde{a}_1} + \beta\mathbf{\tilde{a}_2} + \mathbf{a}^{\alpha}$ — appears in the diagrams in first power only. Here $(\alpha, \beta)$ are $\pm 1$, they are determined by the eigenvalues of the operators $\tilde{\tau}^{\tilde{1}3}$ (for $\tilde{a}_1$) and $\tilde{N}^3_L$ (for $\tilde{a}_2$), respectively, on a particular family, Eq. (13.13).

**b.** Let us add to the diagonal terms the loop corrections. Let us evaluate, using the massless basis $|\psi^i>=\frac{1}{\sqrt{2}}(|\psi^i_L>+|\psi^i_R>)$, the contribution:

$$
\begin{aligned}
&<\psi^i|\sum_{-,+,\boxplus,\boxminus,j}\gamma^0 \overset{78}{(\pm)}\;[\tilde{N}_L^{\boxplus}\tilde{A}^{\tilde{N}_L\boxplus}+\tilde{\tau}^{1\boxplus}\tilde{A}^{\tilde{1}\boxplus}]|\psi^j>\\
&\qquad <\psi^j|\sum_{-,+}\gamma^0 \overset{78}{(\pm)}\;[\tilde{N}_L^{3}\tilde{A}^{\tilde{N}_L 3}+\tilde{\tau}^{13}\tilde{A}^{\tilde{1}3}+\sum_{A}\tau^A A^A]|\psi^j>\\
&\qquad\qquad <\psi^j|\sum_{-,+,\boxplus,\boxminus}\gamma^0 \overset{78}{(\pm)}\;[\tilde{N}_L^{\boxplus}\tilde{A}^{\tilde{N}_L\boxplus}+\tilde{\tau}^{1\boxplus}\tilde{A}^{\tilde{1}\boxplus}]|\psi^i>\,. \qquad (13.17)
\end{aligned}
$$

One finds for $i=[1,2,3,4]$ the values $[\tilde{A}^{\tilde{1}\boxplus}\tilde{A}^{\tilde{1}\boxminus}(\tilde{a}^1-\tilde{a}^2+a^\alpha)+\tilde{A}^{\tilde{N}_L\boxplus}\tilde{A}^{\tilde{N}_L\boxminus}(-\tilde{a}^1+\tilde{a}^2+a^\alpha)$, $\tilde{A}^{\tilde{1}\boxplus}\tilde{A}^{\tilde{1}\boxminus}(\tilde{a}^1+\tilde{a}^2+a^\alpha)+\tilde{A}^{\tilde{N}_L\boxminus}\tilde{A}^{\tilde{N}_L\boxplus}(-\tilde{a}^1-\tilde{a}^2+a^\alpha)$, $\tilde{A}^{\tilde{1}\boxminus}\tilde{A}^{\tilde{1}\boxplus}(-\tilde{a}^1-\tilde{a}^2+a^\alpha)+\tilde{A}^{\tilde{N}_L\boxplus}\tilde{A}^{\tilde{N}_L\boxminus}(+\tilde{a}^1+\tilde{a}^2+a^\alpha)$, $\tilde{A}^{\tilde{1}\boxminus}\tilde{A}^{\tilde{1}\boxplus}(-\tilde{a}^1+\tilde{a}^2+a^\alpha)+\tilde{A}^{\tilde{N}_L\boxminus}\tilde{A}^{\tilde{N}_L\boxplus}(+\tilde{a}^1-\tilde{a}^2+a^\alpha)]$, respectively, which again has the symmetry of the tree level state $<1|\ldots|1>+<4|\ldots|4>=<2|\ldots|2>+<3|\ldots|3>$.

One can make three such loops, or any kind of loops in any order of loop corrections with one $(\alpha\tilde{a}^1+\beta\tilde{a}^2+a^\alpha)$ and the symmetry of tree level state $<1|\ldots|1>+<4|\ldots|4>=<2|\ldots|2>+<3|\ldots|3>$ is manifested.

**c.** Let us look at the loop corrections to the off diagonal terms $<1|\ldots|2>, <1|\ldots|3>, <2|\ldots|4>, <3|\ldots|4>$, as well as their complex conjugate values.

Let us evaluate, using the massless basis $|\psi^i>=\frac{1}{\sqrt{2}}(|\psi^i_L>+|\psi^i_R>)$, the contribution:

$$
\begin{aligned}
&<\psi^4|\sum_{-,+,\boxplus,\boxminus,j,k}\gamma^0 \overset{78}{(\pm)}\;[\tilde{N}_L^{\boxplus}\tilde{A}^{\tilde{N}_L\boxplus}+\tilde{\tau}^{1\boxplus}\tilde{A}^{\tilde{1}\boxplus}]|\psi^j>\\
&\qquad <\psi^j|\sum_{-,+}\gamma^0 \overset{78}{(\pm)}\;[\tilde{N}_L^{\boxplus}\tilde{A}^{\tilde{N}_L\boxplus}+\tilde{\tau}^{1\boxplus}\tilde{A}^{\tilde{1}\boxplus}]|\psi^k>\\
&\qquad <\psi^k|\sum_{-,+,\boxplus,\boxminus}\gamma^0 \overset{78}{(\pm)}\;[\tilde{N}_L^{\boxplus}\tilde{A}^{\tilde{N}_L\boxplus}+\tilde{\tau}^{1\boxplus}\tilde{A}^{\tilde{1}\boxplus}]|\psi^2>\\
&+<\psi^4|\sum_{-,+,\boxplus,\boxminus,j}\gamma^0 \overset{78}{(\pm)}\;[\tilde{N}_L^{3}\tilde{A}^{\tilde{N}_L 3}+\tilde{\tau}^{13}\tilde{A}^{\tilde{1}3}+\sum_{A}\tau^A A^A]|\psi^4>\\
&\qquad <\psi^4|\sum_{-,+,\boxplus,\boxminus}\gamma^0 \overset{78}{(\pm)}\;[\tilde{N}_L^{\boxplus}\tilde{A}^{\tilde{N}_L\boxplus}+\tilde{\tau}^{1\boxplus}\tilde{A}^{\tilde{1}\boxplus}]|\psi^j>\\
&\qquad <\psi^j|\sum_{-,+,\boxplus,\boxminus}\gamma^0 \overset{78}{(\pm)}\;[\tilde{N}_L^{3}\tilde{A}^{\tilde{N}_L 3}+\tilde{\tau}^{13}\tilde{A}^{\tilde{1}3}+\sum_{A}\tau^A A^A]|\psi^2>\,. \qquad (13.18)
\end{aligned}
$$

One obtains for this term $<4|...|2>=<\tilde{A}^{\tilde{1}\boxplus}>\{\tilde{A}^{\tilde{N}_L\boxplus}\tilde{A}^{\tilde{N}_L\boxminus}+|\tilde{A}^{\tilde{N}_L 3}|^2+|\tilde{A}^{\tilde{1}3}|^2+|\tau^A A^A|^2\}$, which is equal to the equivalent loop correction term for the matrix element $<3|...|1>$.

Checking the loop corrections for the off diagonal elements $<1|\ldots|2>$, $<1|\ldots|3>, <2|\ldots|4>, <3|\ldots|4>$ in all loop corrections one finds that the symmetry of these off diagonal terms is kept in all orders.

**d.** There are still the terms $< 1|\ldots|4 >, < 2|\ldots|3 >, < 3|\ldots|2 >$ and $< 4|\ldots|1 >$ to be checked in loop corrections. Adding loop corrections in the way we did in **c.** we find that also these matrix elements keep the symmetry of Eq. (13.15).

## 13.3 Conclusions

We demonstrate in this contribution on several cases that the matrix elements of mass matrices $4 \times 4$, predicted by the *spin-charge-family* theory for each family member $\alpha = (u, d, \nu, e)$ to have the symmetry $\widetilde{SU}(2)_{\tilde{SO}(4)_{1+3}} \times \widetilde{SU}(2)_{\tilde{SO}(4)_{\text{"weak"}}} \times U(1)$ on the tree level, keeps this symmetry in all loop corrections. The first to groups concern the family groups, the last one concern the family members group.

The only dependence of the mass matrix on the family member ($\alpha = (u, d, \nu, e)$) quantum numbers is on the tree level through the vacuum expectation values of the operators $\gamma^0 \stackrel{78}{(\pm)} QA^Q_{\pm}$, $\gamma^0 \stackrel{78}{(\pm)} Q'A^{Q'}_{\pm}$ and $\gamma^0 \stackrel{78}{(\pm)} \tau^4 A^4_{\pm}$, appearing on a tree level in the diagonal terms of the mass matrix only and are the same for each of four families — $I_{4\times4}a^{\alpha}$, I is the unite matrix. In the loop corrections these operators enter into all the off diagonal matrix elements, causing the difference in the masses of the family members. The right handed neutrino, which is the regular member of the four families, Table 13.3, has the nonzero value of the operator $\tau^4 A^4_{\pm}$ only (while the family part of the mass matrix is on the tree level the same for all the members).

We demonstrate on several cases, why does the symmetry of the mass matrix, which shows up on the tree level, remain in the loop corrections in all orders.

Although we are not (yet) able to calculate these matrix elements, the predicted symmetry will enable to predict masses of the fourth family (to the observed three), since the $3 \times 3$ submatrix of the $4 \times 4$ matrix determines $4 \times 4$ matrix uniquely [19,4]. We only must wait for accurate enough data for mixing matrices of quarks and leptons to predict, using the symmetry of mass matrices predicted by the *spin-charge-family*, the masses of the fourth family quarks and leptons.

## 13.4 APPENDIX: Short presentation of the *spin-charge-family* theory

This subsection follows similar sections in Refs. [1,4–7].

The *spin-charge-family* theory [1–11,14–22] assumes:

**A.** A simple action (Eq. (13.20)) in an even dimensional space ($d = 2n$, $d > 5$), d is chosen to be $(13 + 1)$. This choice makes that the action manifests in $d = (3 + 1)$ in the low energy regime all the observed degrees of freedom, explaining all the assumptions of the *standard model*, as well as other observed phenomena.

There are two kinds of the Clifford algebra objects, $\gamma^a$'s and $\tilde{\gamma}^a$'s in this theory with the properties.

$$\{\gamma^a, \gamma^b\}_+ = 2\eta^{ab}\,, \qquad \{\tilde{\gamma}^a, \tilde{\gamma}^b\}_+ = 2\eta^{ab}\,, \quad , \quad \{\gamma^a, \tilde{\gamma}^b\}_+ = 0\,. \quad (13.19)$$

Fermions interact with the vielbeins $f^{\alpha}{}_{a}$ and the two kinds of the spin-connection fields - $\omega_{ab\alpha}$ and $\tilde{\omega}_{ab\alpha}$ - the gauge fields of $S^{ab} = \frac{i}{4}(\gamma^a \gamma^b - \gamma^b \gamma^a)$ and $\tilde{S}^{ab} = \frac{i}{4}(\tilde{\gamma}^a \tilde{\gamma}^b - \tilde{\gamma}^b \tilde{\gamma}^a)$, respectively.

The action

$$\mathcal{A} = \int d^dx \; E \; \frac{1}{2} \, (\bar{\psi} \, \gamma^a p_{0a} \psi) + \text{h.c.} + \int d^dx \; E \; (\alpha \, R + \tilde{\alpha} \, \tilde{R}) \,, \tag{13.20}$$

in which $p_{0a} = f^{\alpha}{}_{a} \, p_{0\alpha} + \frac{1}{2E} \{p_{\alpha}, E f^{\alpha}{}_{a}\}_{-}$, $p_{0\alpha} = p_{\alpha} - \frac{1}{2} \, S^{ab} \, \omega_{ab\alpha} - \frac{1}{2} \, \tilde{S}^{ab} \, \tilde{\omega}_{ab\alpha}$, and

$$R = \frac{1}{2} \{f^{\alpha[a} f^{\beta b]} \; (\omega_{ab\alpha,\beta} - \omega_{ca\alpha} \, \omega^{c}{}_{b\beta})\} + \text{h.c.},$$

$$\tilde{R} = \frac{1}{2} \{f^{\alpha[a} f^{\beta b]} \; (\tilde{\omega}_{ab\alpha,\beta} - \tilde{\omega}_{ca\alpha} \, \tilde{\omega}^{c}{}_{b\beta})\} + \text{h.c.}$$

[4], introduces two kinds of the Clifford algebra objects, $\gamma^a$ and $\tilde{\gamma}^a$, $\{\gamma^a, \gamma^b\}_{+} = 2\eta^{ab} = \{\tilde{\gamma}^a, \tilde{\gamma}^b\}_{+}$. $f^{\alpha}{}_{a}$ are vielbeins inverted to $e^{a}{}_{\alpha}$, Latin letters $(a, b, ..)$ denote flat indices, Greek letters $(\alpha, \beta, ..)$ are Einstein indices, $(m, n, ..)$ and $(\mu, \nu, ..)$ denote the corresponding indices in $(0, 1, 2, 3)$, while $(s, t, ..)$ and $(\sigma, \tau, ..)$ denote the corresponding indices in $d \geq 5$:

$$e^{a}{}_{\alpha} f^{\beta}{}_{a} = \delta^{\beta}_{\alpha} \,, \quad e^{a}{}_{\alpha} f^{\alpha}{}_{b} = \delta^{a}_{b} \,, \tag{13.21}$$

$E = \det(e^{a}{}_{\alpha})$.

**B.** The *spin-charge-family* theory assumes in addition that the manifold $M^{(13+1)}$ breaks first into $M^{(7+1)} \times M^{(6)}$ (which manifests as $SO(7,1) \times SU(3) \times U(1)$), affecting both internal degrees of freedom - the one represented by $\gamma^a$ and the one represented by $\tilde{\gamma}^a$. Since the left handed (with respect to $M^{(7+1)}$) spinors couple differently to scalar (with respect to $M^{(7+1)}$) fields than the right handed ones, the break can leave massless and mass protected $2^{((7+1)/2-1)}$ families [34]. The rest of families get heavy masses [5].

**C.** There is additional breaking of symmetry: The manifold $M^{(7+1)}$ breaks further into $M^{(3+1)} \times M^{(4)}$.

**D.** There is a scalar condensate (Table 13.5) of two right handed neutrinos with the family quantum numbers of the upper four families, bringing masses of the scale $\propto 10^{16}$ GeV or higher to all the vector and scalar gauge fields, which interact with the condensate [5].

**E.** There are the scalar fields with the space index $(7, 8)$ carrying the weak $(\tau^{1i})$ and the hyper charges ($Y = \tau^{23} + \tau^4$, $\tau^{1i}$ and $\tau^{2i}$ are generators of the subgroups of

[4] Whenever two indexes are equal the summation over these two is meant.

[5] A toy model [34,35] was studied in $d = (5 + 1)$ with the same action as in Eq. (13.20). The break from $d = (5 + 1)$ to $d = (3 + 1) \times$ an almost $S^2$ was studied. For a particular choice of vielbeins and for a class of spin connection fields the manifold $M^{(5+1)}$ breaks into $M^{(3+1)}$ times an almost $S^2$, while $2^{((3+1)/2-1)}$ families remain massless and mass protected. Equivalent assumption, although not yet proved how does it really work, is made in the $d = (13 + 1)$ case. This study is in progress.

$SO(4)$, $\tau^4$ and $\tau^{3i}$ are the generators of $U(1)_{II}$ and $SU(3)$, respectively, which are subgroups of $SO(6)$), which with their nonzero vacuum expectation values change the properties of the vacuum and break the weak charge and the hyper charge. Interacting with fermions and with the weak and hyper bosons, they bring masses to heavy bosons and to twice four groups of families. Carrying no electromagnetic ($Q = \tau^{13} + Y$) and colour ($\tau^{3i}$) charges and no $SO(3,1)$ spin, the scalar fields leave the electromagnetic, colour and gravity fields in $d = (3+1)$ massless.

The assumed action $\mathcal{A}$ and the assumpions offer the explanation for the origin and all the properties **o.** of the observed fermions:

**o.i.** of the family members, on Table 13.3 the family members, belonging to one Weyl (fundamental) representation of massless spinors of the group $SO(13,1)$ are presented in the "technique" [9–11,14–16,12,13] and analyzed with respect to the subgroups $SO(3,1)$, $SU(2)_I$, $SU(2)_{II}$, $SU(3)$, $U(1)_{II}$), Eqs. (13.22, 13.23, 13.24), with the generators $\tau^{Ai} = \sum_{s,t} c^{Ai}{}_{st}\, S^{st}$,

**o.ii.**of the families analyzed with respect to the subgroups $\widetilde{SO}(3,1)$, $\widetilde{SU}(2)_I$, $\widetilde{SU}(2)_{II}$, $\widetilde{U}(1)_{II}$), with the generators $\tilde{\tau}^{Ai} = \sum_{ab} c^{Ai}{}_{ab}\, \tilde{S}^{st}$, Eqs. (13.25, 13.26, 13.27), are presented on Table 13.4, all the families are singlets with respect to $\widetilde{SU}(3)$,

**oo.i.** of the observed vector gauge fields of the charges

$$SU(2)_I, SU(2)_{II}, SU(3), U(1)_{II})$$

discussed in Refs. ([1,4,2], and the references therein), all the vector gauge fields are the superposition of the $\omega_{stm}$, $A^{Ai}_m = \sum_{s,t} c^{Ai}{}_{st}\, \omega_{stm}$, Eq. vect

**oo.ii.** of the Higgs's scalar and of the Yukawa couplings, explainable with the scalar fields with the space index $(7,8)$, there are two groups of two triplets, which are scalar gauge fields of the charges $\tilde{\tau}^{Ai}$, expressible with the superposition of the $\tilde{\omega}_{abs}$, $A^{Ai}_m = \sum_{a,b} c^{Ai}{}_{ab}\, \omega_{abs}$ and three singlets, the gauge fields of $Q, Q', S'$, Eqs. (13.28), all with the weak and the hyper charges as assumed by the *standard model* for the Higgs's scalars,

**oo.iii.** of the scalar fields explaining the origin of the matter-antimatter asymmetry, Ref. [5],

**oo.iv.** of the appearance of the dark matter, there are two decoupled groups of four families, carrying family charges ($\vec{\tilde{N}}_L$, $\vec{\tilde{\tau}}^1$) and ($\vec{\tilde{N}}_R$, $\vec{\tilde{\tau}}^2$), Eqs. (13.25, 13.26), both groups carry also the family members charges ($Q, Q', Y'$), Eq. (13.28).

The *standard model* groups of spins and charges are the subgroups of the $SO(13,1)$ group with the generator of the infinitesimal transformations expressible with $S^{ab}$ ($= \frac{i}{2}(\gamma^a\gamma^b - \gamma^b\gamma^a)$, $\{S^{ab}, S^{cd}\}_- = -i(\eta^{ad}S^{bc} + \eta^{bc}S^{ad} - \eta^{ac}S^{bd} - \eta^{bd}S^{ac})$) for the spin

$$\vec{N}_{\pm}(= \vec{N}_{(L,R)}) := \frac{1}{2}(S^{23} \pm iS^{01}, S^{31} \pm iS^{02}, S^{12} \pm iS^{03}), \tag{13.22}$$

for the weak charge, $SU(2)_{I}$, and the second $SU(2)_{II}$, these two groups are the invariant subgroups of $SO(4)$,

$$\vec{\tau}^{1} := \frac{1}{2}(S^{58} - S^{67}, S^{57} + S^{68}, S^{56} - S^{78}),$$
$$\vec{\tau}^{2} := \frac{1}{2}(S^{58} + S^{67}, S^{57} - S^{68}, S^{56} + S^{78}), \qquad (13.23)$$

for the colour charge $SU(3)$ and for the "fermion charge" $U(1)_{II}$, these two groups are subgroups of $SO(6)$,

$$\begin{aligned}\vec{\tau}^{3} := & \frac{1}{2}\{S^{9\,12} - S^{10\,11}, S^{9\,11} + S^{10\,12}, S^{9\,10} - S^{11\,12}, \\ & S^{9\,14} - S^{10\,13}, S^{9\,13} + S^{10\,14}, S^{11\,14} - S^{12\,13}, \\ & S^{11\,13} + S^{12\,14}, \frac{1}{\sqrt{3}}(S^{9\,10} + S^{11\,12} - 2S^{13\,14})\}, \\ \tau^{4} := & -\frac{1}{3}(S^{9\,10} + S^{11\,12} + S^{13\,14}),\end{aligned} \qquad (13.24)$$

$\tau^{4}$ is the "fermion charge", while the hyper charge $Y = \tau^{23} + \tau^{4}$.

The generators of the family quantum numbers are the superposition of the generators $\tilde{S}^{ab}$ ($\tilde{S}^{ab} = \frac{i}{4}\{\tilde{\gamma}^{a}, \tilde{\gamma}^{b}\}_{-}$, $\{\tilde{S}^{ab}, \tilde{S}^{cd}\}_{-} = -i(\eta^{ad}\tilde{S}^{bc} + \eta^{bc}\tilde{S}^{ad} - \eta^{ac}\tilde{S}^{bd} - \eta^{bd}\tilde{S}^{ac})$, $\{\tilde{S}^{ab}, S^{cd}\}_{-} = 0$. One correspondingly finds the generators of the subgroups of $\widetilde{SO}(7,1)$,

$$\vec{\tilde{N}}_{L,R} := \frac{1}{2}(\tilde{S}^{23} \pm i\tilde{S}^{01}, \tilde{S}^{31} \pm i\tilde{S}^{02}, \tilde{S}^{12} \pm i\tilde{S}^{03}), \qquad (13.25)$$

which determine representations of the two $\widetilde{SU}(2)$ invariant subgroups of $\widetilde{SO}(3,1)$, while

$$\vec{\tilde{\tau}}^{1} := \frac{1}{2}(\tilde{S}^{58} - \tilde{S}^{67}, \tilde{S}^{57} + \tilde{S}^{68}, \tilde{S}^{56} - \tilde{S}^{78}),$$
$$\vec{\tilde{\tau}}^{2} := \frac{1}{2}(\tilde{S}^{58} + \tilde{S}^{67}, \tilde{S}^{57} - \tilde{S}^{68}, \tilde{S}^{56} + \tilde{S}^{78}), \qquad (13.26)$$

determine representations of $\widetilde{SU}(2)_{I} \times \widetilde{SU}(2)_{II}$ of $\widetilde{SO}(4)$. Both, $\widetilde{SO}(3,1)$ and $\widetilde{SO}(4)$, are the subgroups of $\widetilde{SO}(7,1)$. One finds for the infinitesimal generator $\tilde{\tau}^{4}$ of $\tilde{U}(1)$ originating in $\widetilde{SO}(6)$ the expression

$$\tilde{\tau}^{4} := -\frac{1}{3}(\tilde{S}^{9\,10} + \tilde{S}^{11\,12} + \tilde{S}^{13\,14}). \qquad (13.27)$$

The operators for the charges $Y$ and $Q$ of the *standard model*, together with $Q'$ and $Y'$, and the corresponding operators of the family charges $\tilde{Y}$, $\tilde{Y}'$, $\tilde{Q}$, $\tilde{Q}'$ are defined as follows:

$$\begin{aligned}&Y := \tau^{4} + \tau^{23}, \quad Y' := -\tau^{4}\tan^{2}\vartheta_{2} + \tau^{23}, \quad Q := \tau^{13} + Y, \quad Q' := -Y\tan^{2}\vartheta_{1} + \tau^{13}, \\ &\tilde{Y} := \tilde{\tau}^{4} + \tilde{\tau}^{23}, \quad \tilde{Y}' := -\tilde{\tau}^{4}\tan^{2}\vartheta_{2} + \tilde{\tau}^{23}, \quad \tilde{Q} := \tilde{Y} + \tilde{\tau}^{13}, \quad \tilde{Q}' = -\tilde{Y}\tan^{2}\vartheta_{1} + \tilde{\tau}^{13}.\end{aligned} \qquad (13.28)$$

The families split into two groups of four families, each manifesting the

$$\widetilde{SU}(2) \times \widetilde{SU}(2) \times U(1),$$

with the generators of of the infinitesimal transformations ($\vec{\tilde{N}}_L, \vec{\tilde{\tau}}^1, Q, Q', Y'$) and ($\vec{\tilde{N}}_R, \vec{\tilde{\tau}}^2, Q, Q', Y'$), respectively. The generators of $U(1)$ group $(Q, Q', Y')$, Eq. 13.28, distinguish among family members and are the same for both groups of four families, presented on Table 13.4, taken from Ref. [4].

The vector gauge fields of the charges $\vec{\tau}^1$, $\vec{\tau}^2$, $\vec{\tau}^3$ and $\tau^4$ follow from the requirement $\sum_{Ai} \tau^{Ai} A^{Ai}_m = \sum_{s,t} \frac{1}{2} S^{st}\, \omega_{stm}$ and the requirement that $\tau^{Ai} = \sum_{a,b}\ c^{Ai}{}_{ab}\ S^{ab}$, Eq. (13.2), fulfilling the commutation relations $\{\tau^{Ai}, \tau^{Bj}\}_- = i\delta^{AB} f^{Aijk} \tau^{Ak}$, Eq. (13.3). Correspondingly we find $A^{Ai}_m = \sum_{s,t}\ c^{Ai}{}_{st}\ \omega^{st}{}_m$, Eq. (13.4), with $(s, t)$ either in $(5, 6, 7, 8)$ or in $(9, \ldots, 14)$.

The explicit expressions for these vector gauge fields in terms of $\omega_{stm}$ are as follows

$$\begin{aligned}
\vec{A}^1_m &= (\omega_{58m} - \omega_{67m}, \omega_{57m} + \omega_{68m}, \omega_{56m} - \omega_{78m}),\\
\vec{A}^2_m &= (\omega_{58m} + \omega_{67m}, \omega_{57m} - \omega_{68m}, \omega_{56m} + \omega_{78m}),\\
A^Q_m &= \omega_{56m} - (\omega_{9\,10m} + \omega_{11\,12m} + \omega_{13\,14m}),\\
A^Y_m &= (\omega_{56m} + \omega_{78m}) - (\omega_{9\,10m} + \omega_{11\,12m} + \omega_{13\,14m}),\\
\vec{A}^3_m &= (\omega_{9\,12m} - \omega_{10\,11m}, \omega_{9\,11m} + \omega_{10\,12m}, \omega_{9\,10m} - \omega_{11\,12m},\\
&\quad \omega_{9\,14m} - \omega_{10\,13m}, \omega_{9\,13m} + \omega_{10\,14m}, \omega_{11\,14m} - \omega_{12\,13m},\\
&\quad \omega_{11\,13m} + \omega_{12\,14m}, \frac{1}{\sqrt{3}}(\omega_{9\,10m} + \omega_{11\,12m} - 2\omega_{13\,14m})),\\
A^4_m &= (\omega_{9\,10m} + \omega_{11\,12m} + \omega_{13\,14m}). \qquad (13.29)
\end{aligned}$$

All $\omega_{stm}$ vector gauge fields are real fields. Here the fields contain the coupling constants which are not necessarily the same for all of them. In the case that the coupling constants would be the same, than the angles $\theta^2_2$ and $\theta^2_1$ would be equal to one, which is not the case (at least $\sin^2_1 \approx 0.22$.)

One obtains in a similar way the scalar gauge fields, which determine mass matrices of family members. They carry the space index $s = (7, 8)$.

$$\begin{aligned}
\vec{\tilde{A}}^1_s &= (\tilde{\omega}_{58s} - \tilde{\omega}_{67s}, \tilde{\omega}_{57s} + \tilde{\omega}_{68s}, \tilde{\omega}_{56s} - \tilde{\omega}_{78s}),\\
\vec{\tilde{A}}^2_s &= (\tilde{\omega}_{58s} + \tilde{\omega}_{67s}, \tilde{\omega}_{57s} - \tilde{\omega}_{68s}, \tilde{\omega}_{56s} + \tilde{\omega}_{78s}),\\
\vec{\tilde{A}}^N_{Ls} &= (\tilde{\omega}_{23s} + i\tilde{\omega}_{01s}, \tilde{\omega}_{31s} + i\tilde{\omega}_{02s}, \tilde{\omega}_{12s} + \tilde{\omega}_{03s}),\\
\vec{\tilde{A}}^N_{Rs} &= (\tilde{\omega}_{23s} - i\tilde{\omega}_{01s}, \tilde{\omega}_{31s} - i\tilde{\omega}_{02s}, \tilde{\omega}_{12s} - i\tilde{\omega}_{03s}),\\
A^Q_s &= \omega_{56s} - (\omega_{9\,10s} + \omega_{11\,12s} + \omega_{13\,14s}),\\
A^Y_s &= (\omega_{56s} + \omega_{78s}) - (\omega_{9\,10s} + \omega_{11\,12s} + \omega_{13\,14s})\\
A^4_s &= -(\omega_{9\,10s} + \omega_{11\,12s} + \omega_{13\,14s}). \qquad (13.30)
\end{aligned}$$

All $\omega_{sts'}$, $\tilde{\omega}_{sts'}$, $(s, t, s') = (5, \cdot, 14)$, $\tilde{\omega}_{i,j,s'}$ and $i\,\tilde{\omega}_{0,s'}$, $(i, j) = (1, 2, 3)$ scalar gauge fields are real fields.

| | | | | | $\tilde{\tau}^{13}$ | $\tilde{\tau}^{23}$ | $\tilde{N}^3_L$ | $\tilde{N}^3_R$ | $\tilde{\tau}^4$ |
|---|---|---|---|---|---|---|---|---|---|
| I | $u^{c1}_{R1}$ | $\stackrel{03}{(+i)}\stackrel{12}{[+]} \mid \stackrel{56}{[+]}\stackrel{78}{(+)} \parallel \stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$ | $\nu_{R1}$ | $\stackrel{03}{(+i)}\stackrel{12}{[+]} \mid \stackrel{56}{[+]}\stackrel{78}{(+)} \parallel \stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$ | $-\frac{1}{2}$ | $0$ | $-\frac{1}{2}$ | $0$ | $-\frac{1}{2}$ |
| I | $u^{c1}_{R2}$ | $\stackrel{03}{[+i]}\stackrel{12}{(+)} \mid \stackrel{56}{[+]}\stackrel{78}{(+)} \parallel \stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$ | $\nu_{R2}$ | $\stackrel{03}{[+i]}\stackrel{12}{(+)} \mid \stackrel{56}{[+]}\stackrel{78}{(+)} \parallel \stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$ | $-\frac{1}{2}$ | $0$ | $\frac{1}{2}$ | $0$ | $-\frac{1}{2}$ |
| I | $u^{c1}_{R3}$ | $\stackrel{03}{(+i)}\stackrel{12}{[+]} \mid \stackrel{56}{(+)}\stackrel{78}{[+]} \parallel \stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$ | $\nu_{R3}$ | $\stackrel{03}{(+i)}\stackrel{12}{[+]} \mid \stackrel{56}{(+)}\stackrel{78}{[+]} \parallel \stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$ | $\frac{1}{2}$ | $0$ | $-\frac{1}{2}$ | $0$ | $-\frac{1}{2}$ |
| I | $u^{c1}_{R4}$ | $\stackrel{03}{[+i]}\stackrel{12}{(+)} \mid \stackrel{56}{(+)}\stackrel{78}{[+]} \parallel \stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$ | $\nu_{R4}$ | $\stackrel{03}{[+i]}\stackrel{12}{(+)} \mid \stackrel{56}{(+)}\stackrel{78}{[+]} \parallel \stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$ | $\frac{1}{2}$ | $0$ | $\frac{1}{2}$ | $0$ | $-\frac{1}{2}$ |
| II | $u^{c1}_{R5}$ | $\stackrel{03}{[+i]}\stackrel{12}{[+]} \mid \stackrel{56}{[+]}\stackrel{78}{[+]} \parallel \stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$ | $\nu_{R5}$ | $\stackrel{03}{[+i]}\stackrel{12}{[+]} \mid \stackrel{56}{[+]}\stackrel{78}{[+]} \parallel \stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$ | $0$ | $-\frac{1}{2}$ | $0$ | $-\frac{1}{2}$ | $-\frac{1}{2}$ |
| II | $u^{c1}_{R6}$ | $\stackrel{03}{(+i)}\stackrel{12}{(+)} \mid \stackrel{56}{[+]}\stackrel{78}{[+]} \parallel \stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$ | $\nu_{R6}$ | $\stackrel{03}{(+i)}\stackrel{12}{(+)} \mid \stackrel{56}{[+]}\stackrel{78}{[+]} \parallel \stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$ | $0$ | $-\frac{1}{2}$ | $0$ | $\frac{1}{2}$ | $-\frac{1}{2}$ |
| II | $u^{c1}_{R7}$ | $\stackrel{03}{[+i]}\stackrel{12}{[+]} \mid \stackrel{56}{(+)}\stackrel{78}{(+)} \parallel \stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$ | $\nu_{R7}$ | $\stackrel{03}{[+i]}\stackrel{12}{[+]} \mid \stackrel{56}{(+)}\stackrel{78}{(+)} \parallel \stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$ | $0$ | $\frac{1}{2}$ | $0$ | $-\frac{1}{2}$ | $-\frac{1}{2}$ |
| II | $u^{c1}_{R8}$ | $\stackrel{03}{(+i)}\stackrel{12}{(+)} \mid \stackrel{56}{(+)}\stackrel{78}{(+)} \parallel \stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$ | $\nu_{R8}$ | $\stackrel{03}{(+i)}\stackrel{12}{(+)} \mid \stackrel{56}{(+)}\stackrel{78}{(+)} \parallel \stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$ | $0$ | $\frac{1}{2}$ | $0$ | $\frac{1}{2}$ | $-\frac{1}{2}$ |

**Table 13.4.** Eight families of the right handed $u^{c1}_R$ (13.3) quark with spin $\frac{1}{2}$, the colour charge ($\tau^{33} = 1/2, \tau^{38} = 1/(2\sqrt{3})$ (the definition of the operators is presented in Eqs. (13.23,13.24), a few examples how to calculate the application of these operators on the states can be found in Subsect. 13.5. The definition of the operators, expressible with $\tilde{S}^{ab}$ is: $\vec{\tilde{N}}_{L,R} = \frac{1}{2}(\tilde{S}^{23} \pm i\tilde{S}^{01}, \tilde{S}^{31} \pm i\tilde{S}^{02}, \tilde{S}^{12} \pm i\tilde{S}^{03})$, $\vec{\tilde{\tau}}^1 = \frac{1}{2}(\tilde{S}^{58} - \tilde{S}^{67}, \tilde{S}^{57} + \tilde{S}^{68}, \tilde{S}^{56} - \tilde{S}^{78})$, $\vec{\tilde{\tau}}^2 = \frac{1}{2}(\tilde{S}^{58} + \tilde{S}^{67}, \tilde{S}^{57} - \tilde{S}^{68}, \tilde{S}^{56} + \tilde{S}^{78})$ and $\tilde{\tau}^4 = -\frac{1}{3}(\tilde{S}^{9\,10} + \tilde{S}^{11\,12} + \tilde{S}^{13\,14})$), and of the colourless right handed neutrino $\nu_R$ of spin $\frac{1}{2}$ are presented in the left and in the right column, respectively. They belong to two groups of four families, one (II) is a doublet with respect to ($\vec{\tilde{N}}_L$ and $\vec{\tilde{\tau}}^1$) and a singlet with respect to ($\vec{\tilde{N}}_R$ and $\vec{\tilde{\tau}}^2$), the other (I) is a singlet with respect to ($\vec{\tilde{N}}_L$ and $\vec{\tilde{\tau}}^1$) and a doublet with with respect to ($\vec{\tilde{N}}_R$ and $\vec{\tilde{\tau}}^2$). All the families follow from the starting one by the application of the operators ($\tilde{N}^{\pm}_{R,L}$, $\tilde{\tau}^{(2,1)\pm}$), Eq. (13.45). The generators ($N^{\pm}_{R,L}$, $\tau^{(2,1)\pm}$) (Eq. (13.45)) transform $u_{Ri}, i = (1, \cdots, 8)$, to all the members of the same colour of thefamily. The same generators transform equivalently the right handed neutrino $\nu_{Ri}, i = (1, \cdots, 8)$, to all the colourless members of the $i^{th}$ family.

The theory predicts, due to commutation relations of generators of the infinitesimal transformations of the family groups, $\widetilde{SU}(2)_{I} \times\widetilde{SU}(2)_{I}$ and $\widetilde{SU}(2)_{II}$ $\times\widetilde{SU}(2)_{II}$, the first one with the generators $\vec{\tilde{N}}_L$ and $\vec{\tilde{\tau}}^1$, and the second one with the generators $\vec{\tilde{N}}_R$ and $\vec{\tilde{\tau}}^2$, Eqs. (13.25,13.26), two groups of four families.

The theory offers (so far) several predictions:

**i.** several new scalars, those coupled to the lower group of four families — two triplets and three singlets, the superposition of ($\vec{\tilde{A}}^1_s$, $\vec{\tilde{A}}^N_{Ls}$ and $A^Q_s, A^Y_s, A^4_s$, Eq. (13.30) — some of them to be observed at the LHC ([1,5,4],

**ii.** the fourth family to the observed three to be observed at the LHC ([1,5,4] and the references therein),

**iii.** new nuclear force among nucleons built from the quarks of the upper four families.

The theory offers also the explanation for several phenomena, like it is the "miraculous" cancellation of the*standard model* triangle anomalies [3].

The breaks of the symmetries, manifesting in Eqs. (13.22, 13.25, 13.23, 13.26, 13.24, 13.27), are in the *spin-charge-family* theory caused by the scalar condensate of the two right handed neutrinos belonging to one group of four families, Table 13.5, and by the nonzero vacuum expectation values of the scalar fields carrying the space index $(7,8)$ (Refs. [4,1] and the references therein). The space breaks first to $SO(7,1) \times SU(3) \times U(1)_{II}$ and then further to $SO(3,1) \times SU(2)_{I} \times U(1)_{I} \times SU(3) \times U(1)_{II}$, what explains the connections between the weak and the hyper charges and the handedness of spinors [3].

| state | $S^{03}$ | $S^{12}$ | $\tau^{13}$ | $\tau^{23}$ | $\tau^4$ | $Y$ | $Q$ | $\tilde{\tau}^{13}$ | $\tilde{\tau}^{23}$ | $\tilde{\tau}^4$ | $\tilde{Y}$ | $\tilde{Q}$ | $\tilde{N}^3_L$ | $\tilde{N}^3_R$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $(\vert\nu^{\mathbf{VIII}}_{\mathbf{1R}} >_{\mathbf{1}} \vert\nu^{\mathbf{VIII}}_{\mathbf{2R}} >_{\mathbf{2}})$ | 0 | 0 | 0 | 1 | −1 | 0 | 0 | 0 | 1 | −1 | 0 | 0 | 0 | 1 |
| $(\vert\nu^{VIII}_{1R} >_1 \vert e^{VIII}_{2R} >_2)$ | 0 | 0 | 0 | 0 | −1 | −1 | −1 | 0 | 1 | −1 | 0 | 0 | 0 | 1 |
| $(\vert e^{VIII}_{1R} >_1 \vert e^{VIII}_{2R} >_2)$ | 0 | 0 | 0 | −1 | −1 | −2 | −2 | 0 | 1 | −1 | 0 | 0 | 0 | 1 |

**Table 13.5.** This table is taken from [5]. The condensate of the two right handed neutrinos $\nu_R$, with the $VIII^{th}$ family quantum numbers, coupled to spin zero and belonging to a triplet with respect to the generators $\tau^{2i}$, is presented together with its two partners. The right handed neutrino has $Q = 0 = Y$. The triplet carries $\tau^4 = -1$, $\tilde{\tau}^{23} = 1$, $\tilde{\tau}^4 = -1$, $\tilde{N}^3_R = 1$, $\tilde{N}^3_L = 0$, $\tilde{Y} = 0$, $\tilde{Q} = 0$. The $\tilde{\tau}^{31} = 0$. The family quantum numbers are presented in Table 13.4.

The stable of the upper four families is the candidate for the dark matter, the fourth of the lower four families is predicted to be measured at the LHC.

## 13.5 APPENDIX: Short presentation of spinor technique [1,4,10,12,13]

This appendix is a short review (taken from [4]) of the technique [10,40,12,13], initiated and developed in Ref. [10] by one of the authors (N.S.M.B.), while proposing the *spin-charge-family* theory [2,4,5,7,8,1,14,15,9–11,16–22]. All the internal degrees of freedom of spinors, with family quantum numbers included, are describable with two kinds of the Clifford algebra objects, besides with $\gamma^a$'s, used in this theory to describe spins and all the charges of fermions, also with $\tilde{\gamma}^a$'s, used in this theory to describe families of spinors:

$$\{\gamma^a, \gamma^b\}_+ = 2\eta^{ab}\,, \qquad \{\tilde{\gamma}^a, \tilde{\gamma}^b\}_+ = 2\eta^{ab}\,, \qquad \{\gamma^a, \tilde{\gamma}^b\}_+ = 0\,. \quad (13.31)$$

We assume the "Hermiticity" property for $\gamma^a$'s (and $\tilde{\gamma}^a$'s) $\gamma^{a\dagger} = \eta^{aa}\gamma^a$ (and $\tilde{\gamma}^{a\dagger} = \eta^{aa}\tilde{\gamma}^a$), in order that $\gamma^a$ (and $\tilde{\gamma}^a$) are compatible with (13.31) and formally unitary, i.e. $\gamma^{a\,\dagger}\,\gamma^a = I$ (and $\tilde{\gamma}^{a\,\dagger}\tilde{\gamma}^a = I$). One correspondingly finds that $(S^{ab})^\dagger = \eta^{aa}\eta^{bb}S^{ab}$ (and $(\tilde{S}^{ab})^\dagger = \eta^{aa}\eta^{bb}\tilde{S}^{ab}$).

Spinor states are represented as products of nilpotents and projectors, formed as odd and even objects of $\gamma^a$'s, respectively, chosen to be the eigenstates of a Cartan subalgebra of the Lorentz groups defined by $\gamma^a$'s

$$\stackrel{ab}{(k)}:= \frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b)\,, \qquad \stackrel{ab}{[k]}:= \frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b)\,, \quad (13.32)$$

where $k^2 = \eta^{aa}\eta^{bb}$. We further have [4]

$$\begin{aligned}
\gamma^a \stackrel{ab}{(k)}: &= \frac{1}{2}(\gamma^a\gamma^a + \frac{\eta^{aa}}{ik}\gamma^a\gamma^b) = \eta^{aa} \stackrel{ab}{[-k]}\,,\\
\gamma^a \stackrel{ab}{[k]}: &= \frac{1}{2}(\gamma^a + \frac{i}{k}\gamma^a\gamma^a\gamma^b) = \stackrel{ab}{(-k)}\,,\\
\tilde{\gamma}^a \stackrel{ab}{(k)}: &= -i\frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b)\gamma^a = -i\eta^{aa} \stackrel{ab}{[k]}\,,\\
\tilde{\gamma}^a \stackrel{ab}{[k]}: &= i\frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b)\gamma^a = -i \stackrel{ab}{(k)}\,,
\end{aligned} \quad (13.33)$$

where we assume that all the operators apply on the vacuum state $|\psi_0\rangle$. We define a vacuum state $|\psi_0>$ so that one finds $< \stackrel{ab}{(k)}^\dagger \stackrel{ab}{(k)} >= 1\,, \quad < \stackrel{ab}{[k]}^\dagger \stackrel{ab}{[k]} >= 1$.

We recognize that $\gamma^a$ transform $\stackrel{ab}{(k)}$ into $\stackrel{ab}{[-k]}$, never to $\stackrel{ab}{[k]}$, while $\tilde{\gamma}^a$ transform $\stackrel{ab}{(k)}$ into $\stackrel{ab}{[k]}$, never to $\stackrel{ab}{[-k]}$

$$\begin{aligned}
&\gamma^a \stackrel{ab}{(k)}= \eta^{aa} \stackrel{ab}{[-k]},\; \gamma^b \stackrel{ab}{(k)}= -ik \stackrel{ab}{[-k]},\; \gamma^a \stackrel{ab}{[k]}=\stackrel{ab}{(-k)},\; \gamma^b \stackrel{ab}{[k]}= -ik\eta^{aa} \stackrel{ab}{(-k)}\,,\\
&\tilde{\gamma^a} \stackrel{ab}{(k)}= -i\eta^{aa} \stackrel{ab}{[k]},\; \tilde{\gamma^b} \stackrel{ab}{(k)}= -k \stackrel{ab}{[k]},\; \tilde{\gamma^a} \stackrel{ab}{[k]}= \;i \stackrel{ab}{(k)},\; \tilde{\gamma^b} \stackrel{ab}{[k]}= -k\eta^{aa} \stackrel{ab}{(k)}\;.
\end{aligned} \quad (13.34)$$

The Clifford algebra objects $S^{ab}$ and $\tilde{S}^{ab}$ close the algebra of the Lorentz group

$$\begin{aligned}
S^{ab} &:= (i/4)(\gamma^a\gamma^b - \gamma^b\gamma^a)\,,\\
\tilde{S}^{ab} &:= (i/4)(\tilde{\gamma}^a\tilde{\gamma}^b - \tilde{\gamma}^b\tilde{\gamma}^a)\,,
\end{aligned} \quad (13.35)$$

$\{S^{ab}, \tilde{S}^{cd}\}_- = 0\,, \{S^{ab}, S^{cd}\}_- = i(\eta^{ad}S^{bc} + \eta^{bc}S^{ad} - \eta^{ac}S^{bd} - \eta^{bd}S^{ac})\,, \{\tilde{S}^{ab}, \tilde{S}^{cd}\}_- = i(\eta^{ad}\tilde{S}^{bc} + \eta^{bc}\tilde{S}^{ad} - \eta^{ac}\tilde{S}^{bd} - \eta^{bd}\tilde{S}^{ac})\,.$

One can easily check that the nilpotent $\overset{ab}{(k)}$ and the projector $\overset{ab}{[k]}$ are "eigenstates" of $S^{ab}$ and $\tilde{S}^{ab}$

$$\begin{aligned} S^{ab}\ \overset{ab}{(k)} &= \frac{1}{2} k\ \overset{ab}{(k)}, \qquad & S^{ab}\ \overset{ab}{[k]} &= \ \frac{1}{2} k\ \overset{ab}{[k]}, \\ \tilde{S}^{ab}\ \overset{ab}{(k)} &= \frac{1}{2} k\ \overset{ab}{(k)}, \qquad & \tilde{S}^{ab}\ \overset{ab}{[k]} &= -\frac{1}{2} k\ \overset{ab}{[k]}, \end{aligned} \tag{13.36}$$

where the vacuum state $|\psi_0\rangle$ is meant to stay on the right hand sides of projectors and nilpotents. This means that multiplication of nilpotents $\overset{ab}{(k)}$ and projectors $\overset{ab}{[k]}$ by $S^{ab}$ get the same objects back multiplied by the constant $\frac{1}{2}k$, while $\tilde{S}^{ab}$ multiply $\overset{ab}{(k)}$ by $\frac{k}{2}$ and $\overset{ab}{[k]}$ by $(-\frac{k}{2})$ (rather than by $\frac{k}{2}$). This also means that when $\overset{ab}{(k)}$ and $\overset{ab}{[k]}$ act from the left hand side on a vacuum state $|\psi_0\rangle$ the obtained states are the eigenvectors of $S^{ab}$.

The technique can be used to construct a spinor basis for any dimension $d$ and any signature in an easy and transparent way. Equipped with nilpotents and projectors of Eq. (13.32), the technique offers an elegant way to see all the quantum numbers of states with respect to the two Lorentz groups, as well as transformation properties of the states under the application of any Clifford algebra object.

Recognizing from Eq.(13.35) that the two Clifford algebra objects $(S^{ab}, S^{cd})$ with all indexes different commute (and equivalently for $(\tilde{S}^{ab}, \tilde{S}^{cd})$), we select the Cartan subalgebra of the algebra of the two groups, which form equivalent representations with respect to one another

$$\begin{aligned} &S^{03}, S^{12}, S^{56}, \cdots, S^{d-1\ d}, \quad \text{if} \quad d = 2n \geq 4, \\ &\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56}, \cdots, \tilde{S}^{d-1\ d}, \quad \text{if} \quad d = 2n \geq 4\,. \end{aligned} \tag{13.37}$$

The choice of the Cartan subalgebra in $d < 4$ is straightforward. It is useful to define one of the Casimirs of the Lorentz group — the handedness $\Gamma$ ($\{\Gamma, S^{ab}\}_- = 0$) (as well as $\tilde{\Gamma}$) in any $d = 2n$

$$\begin{aligned} \Gamma^{(d)} &:= (i)^{d/2} \qquad \prod_a \quad (\sqrt{\eta^{aa}} \gamma^a), \quad \text{if} \quad d = 2n, \\ \tilde{\Gamma}^{(d)} &:= (i)^{(d-1)/2} \prod_a \quad (\sqrt{\eta^{aa}} \tilde{\gamma}^a), \quad \text{if} \quad d = 2n\,. \end{aligned} \tag{13.38}$$

We understand the product of $\gamma^a$'s in the ascending order with respect to the index $a$: $\gamma^0 \gamma^1 \cdots \gamma^d$. It follows from the Hermiticity properties of $\gamma^a$ for any choice of the signature $\eta^{aa}$ that $\Gamma^\dagger = \Gamma$, $\Gamma^2 = I$.( Equivalent relations are valid for $\tilde{\Gamma}$.) We also find that for $d$ even the handedness anticommutes with the Clifford algebra objects $\gamma^a$ ($\{\gamma^a, \Gamma\}_+ = 0$) (while for $d$ odd it commutes with $\gamma^a$ ($\{\gamma^a, \Gamma\}_- = 0$)).

Taking into account the above equations it is easy to find a Weyl spinor irreducible representation for $d$-dimensional space, with $d$ even or odd [6]. For $d$ even we simply make a starting state as a product of $d/2$, let us say, only

[6] For $d$ odd the basic states are products of $(d-1)/2$ nilpotents and a factor $(1 \pm \Gamma)$.

nilpotents $\stackrel{ab}{(k)}$, one for each $S^{ab}$ of the Cartan subalgebra elements (Eqs.(13.37, 13.35)), applying it on an (unimportant) vacuum state. Then the generators $S^{ab}$, which do not belong to the Cartan subalgebra, being applied on the starting state from the left hand side, generate all the members of one Weyl spinor.

$$
\begin{array}{l}
\stackrel{0d}{(k_{0d})}\stackrel{12}{(k_{12})}\stackrel{35}{(k_{35})}\cdots\stackrel{d-1\;d-2}{(k_{d-1\;d-2})}\;|\psi_0>\\
\stackrel{0d}{[-k_{0d}]}\stackrel{12}{[-k_{12}]}\stackrel{35}{(k_{35})}\cdots\stackrel{d-1\;d-2}{(k_{d-1\;d-2})}\;|\psi_0>\\
\stackrel{0d}{[-k_{0d}]}\stackrel{12}{(k_{12})}\stackrel{35}{[-k_{35}]}\cdots\stackrel{d-1\;d-2}{(k_{d-1\;d-2})}\;|\psi_0>\\
\vdots\\
\stackrel{0d}{[-k_{0d}]}\stackrel{12}{(k_{12})}\stackrel{35}{(k_{35})}\cdots\stackrel{d-1\;d-2}{[-k_{d-1\;d-2}]}\;|\psi_0>\\
\stackrel{od}{(k_{0d})}\stackrel{12}{[-k_{12}]}\stackrel{35}{[-k_{35}]}\cdots\stackrel{d-1\;d-2}{(k_{d-1\;d-2})}\;|\psi_0>\\
\vdots
\end{array}
\tag{13.39}
$$

All the states have the same handedness $\Gamma$, since $\{\Gamma, S^{ab}\}_- = 0$. States, belonging to one multiplet with respect to the group $SO(q, d-q)$, that is to one irreducible representation of spinors (one Weyl spinor), can have any phase. We could make a choice of the simplest one, taking all phases equal to one. (In order to have the usual transformation properties for spinors under the rotation of spin and under $\mathcal{C}_\mathcal{N}\;\mathcal{P}_\mathcal{N}$,some of the states must be multiplied by $(-1)$.)

The above representation demonstrates that for $d$ even all the states of one irreducible Weyl representation of a definite handedness follow from a starting state, which is, for example, a product of nilpotents $\stackrel{ab}{(k_{ab})}$, by transforming all possible pairs of $\stackrel{ab}{(k_{ab})}\stackrel{mn}{(k_{mn})}$ into $\stackrel{ab}{[-k_{ab}]}\stackrel{mn}{[-k_{mn}]}$. There are $S^{am}, S^{an}, S^{bm}, S^{bn}$, which do this. The procedure gives $2^{(d/2-1)}$ states. A Clifford algebra object $\gamma^a$ being applied from the left hand side, transforms a Weyl spinor of one handedness into a Weyl spinor of the opposite handedness.

We shall speak about left handedness when $\Gamma = -1$ and about right handedness when $\Gamma = 1$.

While $S^{ab}$, which do not belong to the Cartan subalgebra (Eq. (13.37)), generate all the states of one representation, $\tilde{S}^{ab}$, which do not belong to the Cartan subalgebra (Eq. (13.37)), generate the states of $2^{d/2-1}$ equivalent representations.

Making a choice of the Cartan subalgebra set (Eq. (13.37)) of the algebra $S^{ab}$ and $\tilde{S}^{ab}$: ($S^{03}$, $S^{12}$, $S^{56}$, $S^{78}$, $S^{9\;10}$, $S^{11\;12}$, $S^{13\;14}$), ($\tilde{S}^{03}$, $\tilde{S}^{12}$, $\tilde{S}^{56}$, $\tilde{S}^{78}$, $\tilde{S}^{9\;10}$, $\tilde{S}^{11\;12}$, $\tilde{S}^{13\;14}$ ), a left handed ($\Gamma^{(13,1)} = -1$) eigenstate of all the members of the Cartan subalgebra, representing a weak chargeless $u_R$-quark with spin up, hyper charge (2/3) and colour $(1/2\,, 1/(2\sqrt{3}))$, for example, can be written as

$$
\begin{array}{l}
\stackrel{03}{(+i)}\stackrel{12}{(+)}\,|\,\stackrel{56}{(+)}\stackrel{78}{(+)}\,||\,\stackrel{9\,10}{(+)}\;\stackrel{11\,12}{(-)}\;\stackrel{13\,14}{(-)}\;|\psi_0\rangle =\\
\frac{1}{2^7}(\gamma^0-\gamma^3)(\gamma^1+i\gamma^2)|(\gamma^5+i\gamma^6)(\gamma^7+i\gamma^8)||\\
(\gamma^9+i\gamma^{10})(\gamma^{11}-i\gamma^{12})(\gamma^{13}-i\gamma^{14})|\psi_0\rangle\,.
\end{array}
\tag{13.40}
$$

This state is an eigenstate of all $S^{ab}$ and $\tilde{S}^{ab}$ which are members of the Cartan subalgebra (Eq. (13.37)).

The operators $\tilde{S}^{ab}$, which do not belong to the Cartan subalgebra (Eq. (13.37)), generate families from the starting $u_R$ quark, transforming the $u_R$ quark from Eq. (13.40) to the $u_R$ of another family, keeping all of the properties with respect to $S^{ab}$ unchanged. In particular, $\tilde{S}^{01}$ applied on a right handed $u_R$-quark from Eq. (13.40) generates a state which is again a right handed $u_R$-quark, weak chargeless, with spin up, hyper charge (2/3) and the colour charge $(1/2\,, 1/(2\sqrt{3}))$

$$\tilde{S}^{01}\;\stackrel{03}{(+i)}\stackrel{12}{(+)}\,|\,\stackrel{56}{(+)}\stackrel{78}{(+)}\,||\,\stackrel{9\,10}{(+)}\,\stackrel{11\,12}{(-)}\,\stackrel{13\,14}{(-)} = -\frac{i}{2}\;\stackrel{03}{[+i]}\stackrel{12}{[+]}\,|\,\stackrel{56}{(+)}\stackrel{78}{(+)}\,||\,\stackrel{9\,10}{(+)}\,\stackrel{11\,12}{(-)}\,\stackrel{13\,14}{(-)}\;. \tag{13.41}$$

One can find both states in Table 13.4, the first $u_R$ as $u_{R8}$ in the eighth line of this table, the second one as $u_{R7}$ in the seventh line of this table.

Below some useful relations follow. From Eq.(13.34) one has

$$\begin{aligned}
S^{ac}\,\stackrel{ab}{(k)}\stackrel{cd}{(k)} &= -\frac{i}{2}\eta^{aa}\eta^{cc}\,\stackrel{ab}{[-k]}\stackrel{cd}{[-k]}\,, &\quad \tilde{S}^{ac}\,\stackrel{ab}{(k)}\stackrel{cd}{(k)} &= \frac{i}{2}\eta^{aa}\eta^{cc}\,\stackrel{ab}{[k]}\stackrel{cd}{[k]}\,,\\
S^{ac}\,\stackrel{ab}{[k]}\stackrel{cd}{[k]} &= \frac{i}{2}\,\stackrel{ab}{(-k)}\stackrel{cd}{(-k)}\,, & \tilde{S}^{ac}\,\stackrel{ab}{[k]}\stackrel{cd}{[k]} &= -\frac{i}{2}\,\stackrel{ab}{(k)}\stackrel{cd}{(k)}\,,\\
S^{ac}\,\stackrel{ab}{(k)}\stackrel{cd}{[k]} &= -\frac{i}{2}\eta^{aa}\,\stackrel{ab}{[-k]}\stackrel{cd}{(-k)}\,, & \tilde{S}^{ac}\,\stackrel{ab}{(k)}\stackrel{cd}{[k]} &= -\frac{i}{2}\eta^{aa}\,\stackrel{ab}{[k]}\stackrel{cd}{(k)}\,,\\
S^{ac}\,\stackrel{ab}{[k]}\stackrel{cd}{(k)} &= \frac{i}{2}\eta^{cc}\,\stackrel{ab}{(-k)}\stackrel{cd}{[-k]}\,, & \tilde{S}^{ac}\,\stackrel{ab}{[k]}\stackrel{cd}{(k)} &= \frac{i}{2}\eta^{cc}\,\stackrel{ab}{(k)}\stackrel{cd}{[k]}\;.
\end{aligned} \tag{13.42}$$

We conclude from the above equation that $\tilde{S}^{ab}$ generate the equivalent representations with respect to $S^{ab}$ and opposite.

We recognize in Eq. (13.43) the demonstration of the nilpotent and the projector character of the Clifford algebra objects $\stackrel{ab}{(k)}$ and $\stackrel{ab}{[k]}$, respectively.

$$\begin{aligned}
&\stackrel{ab}{(k)}\stackrel{ab}{(k)} = 0\,, && \stackrel{ab}{(k)}\stackrel{ab}{(-k)} = \eta^{aa}\,\stackrel{ab}{[k]}\,, && \stackrel{ab}{(-k)}\stackrel{ab}{(k)} = \eta^{aa}\,\stackrel{ab}{[-k]}\,, && \stackrel{ab}{(-k)}\stackrel{ab}{(-k)} = 0\,,\\
&\stackrel{ab}{[k]}\stackrel{ab}{[k]} = \stackrel{ab}{[k]}\,, && \stackrel{ab}{[k]}\stackrel{ab}{[-k]} = 0\,, && \stackrel{ab}{[-k]}\stackrel{ab}{[k]} = 0\,, && \stackrel{ab}{[-k]}\stackrel{ab}{[-k]} = \stackrel{ab}{[-k]}\,,\\
&\stackrel{ab}{(k)}\stackrel{ab}{[k]} = 0\,, && \stackrel{ab}{[k]}\stackrel{ab}{(k)} = \stackrel{ab}{(k)}\,, && \stackrel{ab}{(-k)}\stackrel{ab}{[k]} = \stackrel{ab}{(-k)}\,, && \stackrel{ab}{(-k)}\stackrel{ab}{[-k]} = 0\,,\\
&\stackrel{ab}{(k)}\stackrel{ab}{[-k]} = \stackrel{ab}{(k)}\,, && \stackrel{ab}{[k]}\stackrel{ab}{(-k)} = 0, && \stackrel{ab}{[-k]}\stackrel{ab}{(k)} = 0\,, && \stackrel{ab}{[-k]}\stackrel{ab}{(-k)} = \stackrel{ab}{(-k)}\;.
\end{aligned} \tag{13.43}$$

Defining

$$\begin{aligned}
\stackrel{ab}{(\tilde{\pm} i)} &= \frac{1}{2}\,(\tilde{\gamma}^a \mp \tilde{\gamma}^b)\,, &\quad \stackrel{ab}{(\tilde{\pm} 1)} &= \frac{1}{2}\,(\tilde{\gamma}^a \pm i\tilde{\gamma}^b)\,,\\
\stackrel{ab}{[\tilde{\pm} i]} &= \frac{1}{2}(1 \pm \tilde{\gamma}^a\tilde{\gamma}^b), & \stackrel{ab}{[\tilde{\pm} 1]} &= \frac{1}{2}(1 \pm i\tilde{\gamma}^a\tilde{\gamma}^b).
\end{aligned}$$

one recognizes that

$$\stackrel{ab}{\tilde{(k)}}\stackrel{ab}{(k)} = 0\,, \quad \stackrel{ab}{\tilde{(-k)}}\stackrel{ab}{(k)} = -i\eta^{aa}\stackrel{ab}{[k]}\,, \quad \stackrel{ab}{\tilde{(k)}}\stackrel{ab}{[k]} = i\stackrel{ab}{(k)}\,, \quad \stackrel{ab}{\tilde{(k)}}\stackrel{ab}{[-k]} = 0\,. \tag{13.44}$$

Below some more useful relations [14] are presented:

$$\begin{aligned}
N^{\pm}_{+} &= N^{1}_{+} \pm i\,N^{2}_{+} = -\stackrel{03}{(\mp i)}\stackrel{12}{(\pm)}\,, \quad N^{\pm}_{-} = N^{1}_{-} \pm i\,N^{2}_{-} = \stackrel{03}{(\pm i)}\stackrel{12}{(\pm)}\,,\\
\tilde{N}^{\pm}_{+} &= -\stackrel{03}{\tilde{(\mp i)}}\stackrel{12}{\tilde{(\pm)}}\,, \quad \tilde{N}^{\pm}_{-} = \stackrel{03}{\tilde{(\pm i)}}\stackrel{12}{\tilde{(\pm)}}\,,\\
\tau^{1\pm} &= (\mp)\stackrel{56}{(\pm)}\stackrel{78}{(\mp)}\,, \quad \tau^{2\mp} = (\mp)\stackrel{56}{(\mp)}\stackrel{78}{(\mp)}\,,\\
\tilde{\tau}^{1\pm} &= (\mp)\stackrel{56}{\tilde{(\pm)}}\stackrel{78}{\tilde{(\mp)}}\,, \quad \tilde{\tau}^{2\mp} = (\mp)\stackrel{56}{\tilde{(\mp)}}\stackrel{78}{\tilde{(\mp)}}\,.
\end{aligned} \tag{13.45}$$

In Table 13.4 [4] the eight families of the first member in Table 13.3 (member number 1) of the eight-plet of quarks and the $25^{th}$ member in Table 13.3 of the eight-plet of leptons are presented as an example. The eight families of the right handed $u_{1R}$ quark are presented in the left column of Table 13.4 [4]. In the right column of the same table the equivalent eight-plet of the right handed neutrinos $\nu_{1R}$ are presented. All the other members of any of the eight families of quarks or leptons follow from any member of a particular family by the application of the operators $N^{\pm}_{R,L}$ and $\tau^{(2,1)\pm}$, Eq. (13.45) on this particular member.

The eight-plets separate into two group of four families: One group contains doublets with respect to $\vec{\tilde{N}}_R$ and $\vec{\tilde{\tau}}^2$, these families are singlets with respect to $\vec{\tilde{N}}_L$ and $\vec{\tilde{\tau}}^1$. Another group of families contains doublets with respect to $\vec{\tilde{N}}_L$ and $\vec{\tilde{\tau}}^1$, these families are singlets with respect to $\vec{\tilde{N}}_R$ and $\vec{\tilde{\tau}}^2$.

The scalar fields which are the gauge scalars of $\vec{\tilde{N}}_R$ and $\vec{\tilde{\tau}}^2$ couple only to the four families which are doublets with respect to these two groups. The scalar fields which are the gauge scalars of $\vec{\tilde{N}}_L$ and $\vec{\tilde{\tau}}^1$ couple only to the four families which are doublets with respect to these last two groups.

After the electroweak phase transition, caused by the scalar fields with the space index $(7, 8)$, the two groups of four families become massive. The lowest of the two groups of four families contains the observed three, while the fourth remains to be measured. The lowest of the upper four families is the candidate for the dark matter [1].

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