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Commuting graphs and extremal centralizers*

Gregor Dolinar
Faculty of Electrical Engineering, University of Ljubljana
Tržaška cesta 25, SI-1000 Ljubljana, Slovenia

Alexander Guterman †
Faculty of Algebra, Department of Mathematics and Mechanics, Moscow State University
GSP-1, 119991 Moscow, Russia

Bojan Kuzma
University of Primorska, Glagoljaška 8, SI-6000 Koper, Slovenia, and
IMFM, Jadranska 19, SI-1000 Ljubljana, Slovenia

Polona Oblak ‡
Faculty of Computer and Information Science, University of Ljubljana
Tržaška cesta 25, SI-1000 Ljubljana, Slovenia

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Abstract

We determine the conditions for matrix centralizers which can guarantee the connectedness of the commuting graph for the full matrix algebra $M_n(\mathbb{F})$ over an arbitrary field $\mathbb{F}$. It is known that if $\mathbb{F}$ is an algebraically closed field and $n \geq 3$, then the diameter of the commuting graph of $M_n(\mathbb{F})$ is always equal to four. We construct a concrete example showing that if $\mathbb{F}$ is not algebraically closed, then the commuting graph of $M_n(\mathbb{F})$ can be connected with the diameter at least five.

Keywords: Commuting graph, matrix ring, centralizer.

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‡Corresponding author
E-mail addresses: gregor.dolinar@fe.uni-lj.si (Gregor Dolinar), guterman@list.ru (Alexander Guterman), bojan.kuzma@famnit.upr.si (Bojan Kuzma), polona.oblak@fri.uni-lj.si (Polona Oblak)

1 Introduction

Let $n \geq 2$ and let $M_n(\mathbb{F})$ be a ring of all $n \times n$ matrices over a field $\mathbb{F}$. The *commuting graph* $\Gamma(M_n(\mathbb{F}))$ of $M_n(\mathbb{F})$ is a simple graph with the vertex set consisting of all non-scalar matrices from $M_n(\mathbb{F})$, and two vertices form an edge if the corresponding matrices are distinct and commute.

Recently, connections between various algebraic structures and their commuting graphs were investigated, see, e.g. [1, 3, 4, 5, 8, 9, 10, 12, 15]. For example, Mohammadian [15] recently proved that a ring is isomorphic to $M_2(\mathbb{F})$, with $\mathbb{F}$ finite, if and only if the commuting graph of the ring under consideration is isomorphic to $\Gamma(M_2(\mathbb{F}))$. Akbari, Ghandehari, Hadian and Mohammadian conjectured in [3] that this is also true for $M_n(\mathbb{F})$, where $n \geq 2$.

The connectedness and diameter of the commuting graph of a matrix ring $M_n(\mathbb{F})$ have been also studied extensively. If $\mathbb{F}$ is an algebraically closed field and $n \geq 3$, Akbari, Mohammadian, Radjavi, and Raja [4, Corollary 7] proved that the diameter of $\Gamma(M_n(\mathbb{F}))$ is always equal to four. For fields which are not algebraically closed the situation is completely different, e.g. if $\mathbb{F}$ is the field of rational numbers, then the commuting graph is never connected (see [2, Remark 8]). A necessary and sufficient condition for which $\Gamma(M_n(\mathbb{F}))$ is connected was given in [2, Theorem 6]. Namely, it was proven that $\Gamma(M_n(\mathbb{F}))$ is connected if and only if every field extension of $\mathbb{F}$ of degree n contains at least one proper intermediate field. In the case when commuting graph of $M_n(\mathbb{F})$ is connected, its diameter is known to be at most six and it is conjectured that it is at most five, see [4, Theorem 17] and [4, Conjecture 18]. In the present paper, we show that $\Gamma(M_9(\mathbb{Z}_2))$ is connected and has the diameter at least 5, where $\mathbb{Z}_m$ is the ring of integers modulo m. We also characterize connected commuting graphs in the language of centralizers. Observe that this characterization is different from [2, Theorem 6], where the language of field extension was used.

Definition 1.1. For a matrix $A \in M_n(\mathbb{F})$, the *centralizer* of A, denoted by $\mathcal{C}(A)$, is the set of all matrices in $M_n(\mathbb{F})$ commuting with A.

Let us remark that the set of non-scalar matrices in $\mathcal{C}(A)$ coincides with the closed neighborhood of vertex $A \in \Gamma(M_n(\mathbb{F}))$.

Centralizer induces a natural equivalence relation on $M_n(\mathbb{F})$:

Definition 1.2. Matrices A and B are *$\mathcal{C}$-equivalent* (abbreviated $A \sim B$) if $\mathcal{C}(A) = \mathcal{C}(B)$.

Definition 1.3. A matrix A is *$\mathcal{C}$-minimal* if for every $X \in M_n(\mathbb{F})$ with $\mathcal{C}(A) \supseteq \mathcal{C}(X)$ it follows that $A \sim X$.

Definition 1.4. A non-scalar matrix A is *$\mathcal{C}$-maximal* if for every non-scalar $X \in M_n(\mathbb{F})$ with $\mathcal{C}(A) \subseteq \mathcal{C}(X)$ it follows that $A \sim X$.

Let us remark that $\mathcal{C}$-minimal and $\mathcal{C}$-maximal matrices in $M_n(\mathbb{F})$ were already classified, see Šemrl [16] and recent paper [7] by the authors. Also, $\mathcal{C}$-minimal and $\mathcal{C}$-maximal matrices were used as a main tool by Dolinar, Kuzma, and Oblak [8] to investigate distances between vertices in the commuting graph $\Gamma(M_n(\mathbb{F}))$ and they will be used to prove the results of this paper as well.

We use the following notations. By E_{ij} we denote the matrices with 1 on (i, j)-th position and 0 elsewhere. By 0_k and I_k we denote the $k \times k$ zero matrix and the $k \times k$

identity matrix, respectively. When it is clear from the context, we omit the subscript. For a given scalar $\lambda \in \mathbb{F}$, define $J_k(\lambda) = \lambda I + \sum_{i=1}^{k-1} E_{i(i+1)}$ to be an elementary upper-triangular Jordan matrix. We denote $J_k = J_k(0)$. A matrix $A \in M_n(\mathbb{F})$ is an *idempotent* if $A^2 = A$, it is a *nilpotent* if there exists an integer $k \geq 1$ such that $A^k = 0$. Non-zero matrix A with $A^2 = 0$ is called a *square-zero matrix*. For a monic polynomial $m \in \mathbb{F}[x]$ we let $C(m) = \sum_{i=1}^{n-1} E_{(i+1)i} - \sum_{i=1}^{n} m_{i-1} E_{in} \in M_n(\mathbb{F})$ be a companion matrix of m, where $m(x) = m_0 + m_1 x + \cdots + m_{n-1} x^{n-1} + x^n$. Given a matrix A, let $\mathbb{F}[A] = \{p(A) |\, p \in \mathbb{F}[x]\}$ be the unital algebra generated by A.

2 Connectedness of commuting graphs

In this section we provide a characterization of matrices for which $\Gamma(M_n(\mathbb{F}))$ is connected in the language of extremal centralizers. We need the following result on $\mathcal{C}$-maximal matrices from our recent paper [7].

Proposition 2.1. [7, Theorem 3.2] *Let $\mathbb{F}$ be an arbitrary field. The following statements are equivalent for a non-scalar matrix $A \in M_n(\mathbb{F})$.*

(i) A is $\mathcal{C}$-maximal.

(ii) A belongs to one of the following three classes:

(a) A is $\mathcal{C}$-equivalent to an idempotent,

(b) A is $\mathcal{C}$-equivalent to a square-zero matrix,

(c) A is similar to $C \oplus \cdots \oplus C$, where C is a companion matrix of an irreducible polynomial, such that there is no proper intermediate field between $\mathbb{F}$ and $\mathbb{F}[C]$.

Theorem 2.2. *Let $n \geq 2$ and let $\mathbb{F}$ be an arbitrary field. A commuting graph $\Gamma(M_n(\mathbb{F}))$ is not connected if and only if there exists a matrix in $M_n(\mathbb{F})$ which is simultaneously $\mathcal{C}$-minimal and $\mathcal{C}$-maximal.*

Proof. First, let $n = 2$. Then, $\Gamma(M_2(\mathbb{F}))$ is never connected because every non-scalar matrix in $\mathcal{C}(E_{11})$ is diagonal, hence $\mathcal{C}$-equivalent to E_{11} and thus E_{11} and E_{12} are not connected in $\Gamma(M_2(\mathbb{F}))$ (see also Akbari and Raja [5, Remark 8]). Moreover, E_{11} is always a $\mathcal{C}$-minimal and $\mathcal{C}$-maximal matrix, so the theorem is true in the case $n = 2$ for every field $\mathbb{F}$.

Second, let $n \geq 3$. We will prove each direction of the equivalence in the theorem separately.

(i). Suppose A is a $\mathcal{C}$-minimal and $\mathcal{C}$-maximal matrix. According to Proposition 2.1 we have to consider three cases separately.

Case 1. Let A be $\mathcal{C}$-equivalent to an idempotent. Then we may assume that $A = I_r \oplus 0_{n-r}$ for some $r \in \{2, \ldots, n-1\}$. Thus $\mathcal{C}(A) = M_r(\mathbb{F}) \oplus M_{n-r}(\mathbb{F})$. Recall that $J_r \in M_r(\mathbb{F})$ is a nilpotent upper-triangular Jordan cell, so $\mathcal{C}(I_r + J_r) = \{\alpha_0 I_r + \alpha_1 J_r + \cdots + \alpha_{r-1} J^{r-1} |\, \alpha_i \in \mathbb{F}\} = \mathbb{F}[J_r]$. It easily follows that

$$\mathcal{C}((I_r + J_r) \oplus 0_{n-r}) = \mathbb{F}[J_r] \oplus M_{n-r}(\mathbb{F}).$$

Hence $\mathcal{C}((I_r + J_r) \oplus 0_{n-r}) \subsetneq \mathcal{C}(A)$, so A is not $\mathcal{C}$-minimal, a contradiction.

Case 2. Let A be $\mathcal{C}$-equivalent to a non-scalar square-zero matrix. Then we may assume $A = \begin{pmatrix} 0_r & 0 & I_r \\ 0 & 0_{n-2r} & 0 \\ 0 & 0 & 0_r \end{pmatrix} = J_n^{n-r}$ for some integer r, $1 \leq r \leq \frac{n}{2}$. However, $\mathcal{C}(J_n) \subsetneq \mathcal{C}(A)$, hence A is not $\mathcal{C}$-minimal, a contradiction.

Case 3. Let $A = C \oplus \cdots \oplus C$, where C is a companion matrix of some irreducible monic polynomial $m \in \mathbb{F}[x]$, such that there is no proper intermediate field between $\mathbb{F}$ and $\mathbb{F}[C]$. We will prove that actually $A = C$.

Suppose $\mathbb{F}$ is an infinite field. Then A being $\mathcal{C}$-minimal implies that A is non-derogatory (see [7, Lemma 2.7]). Therefore $A = C$, i.e. A contains only one summand. Hence $\mathbb{F}[C]$ is a field extension of $\mathbb{F}$ of degree n. Recall that C is a companion matrix of an irreducible polynomial, such that there is no proper intermediate field between $\mathbb{F}$ and $\mathbb{F}[C]$ thus it follows by [2, Theorem 6] that $\Gamma(M_n(\mathbb{F}))$ is not connected.

Suppose $\mathbb{F} = GF(p^k)$ is a finite field and suppose that $A = C \oplus \cdots \oplus C$ contains more than one summand. Since $\mathbb{F}[C]$ is a field extension of $\mathbb{F}$ with degree $d = \deg m$, it is isomorphic to $\mathbb{K} = GF(p^{kd})$. Let $\gamma_C \in \mathbb{K}$ correspond to matrix C under this isomorphism. Since A contains more than one summand, d is a proper divisor of n. So, $\mathbb{K} = GF(p^{kd})$ is a proper intermediate field between $\mathbb{F}$ and $GF(p^{kn})$. It is known that the multiplicative group of $GF(p^{kn})$ is cyclic, so let $\xi \in GF(p^{kn})$ be its generator. Then $\mathbb{F}[\xi] = GF(p^{kn})$ and minimal polynomial $f \in \mathbb{F}[x]$ for ξ is irreducible over $\mathbb{F}$ of degree n. For matrix $X = C(f) \in M_n(\mathbb{F})$, $\mathbb{F}[X]$ is a field isomorphic to $GF(p^{kn})$. Since $GF(p^{kd}) \subseteq GF(p^{kn})$, some polynomial in X is isomorphic to $\gamma_C \in GF(p^{kd})$, and hence also to C. Consequently, by Skolem-Noether theorem, $p(X)$ is similar to a matrix $A = C \oplus \cdots \oplus C$ for some $p \in \mathbb{F}[x]$. By applying a suitable similarity to X we can assume that $p(X) = A$, hence $\mathcal{C}(X) \subseteq \mathcal{C}(A)$. Since A is $\mathcal{C}$-minimal, $\mathcal{C}(X) = \mathcal{C}(A)$, so X is a polynomial in A by the centralizer Theorem (see [13, p. 113, Corollary 1] and also [17, p. 106, Theorem 2]). Therefore the rational form of X (see [11, Chapter 3] for details) has at least as many cells as the rational form of A. A contradiction to the fact that X is similar to $C(f)$. It follows that $A = C$ and we conclude as in the infinite case that $\Gamma(M_n(\mathbb{F}))$ is not connected.

(ii). Suppose the commuting graph $\Gamma(M_n(\mathbb{F}))$ is not connected. By [4, Theorem 11] any two non-scalar idempotents are connected and thus there exists a non-scalar matrix A which is not connected to any non-scalar idempotent. We may assume that A is already in its rational form. Then A consists of a single cell because otherwise a matrix $A_1 \oplus A_2$ would be connected to an idempotent $I \oplus 0$. Hence $A = C(m^\alpha)$ for some irreducible polynomial m and positive integer α. If $\alpha \geq 2$, then A would be connected to a non-scalar square-zero matrix $B = m(A)^{\alpha-1}$. It is easy to see that B commutes with a rank-one matrix which further commutes with an idempotent, a contradiction. So, $\alpha = 1$ and thus A is non-derogatory, hence $\mathcal{C}$-minimal as it was proved in [7, Theorem 2.6]. If $A = C(m)$ is not $\mathcal{C}$-maximal, then there exists a proper intermediate field $\mathbb{K}$ between $\mathbb{F}$ and $\mathbb{F}[A]$ by Proposition 2.1. We can assume $\mathbb{K} = \mathbb{F}[X]$ is a simple extension for some $X \in \mathbb{F}[A]$. The minimal polynomial of X has smaller degree than the minimal polynomial of A, otherwise $\mathbb{F}[X] = \mathbb{F}[A]$. Hence the rational form of X contains more than one cell and therefore X, and thus also A is connected to a non-scalar idempotent, a contradiction. □

3 Commuting graph with diameter greater than four

Recall that the diameter of a commuting graph $\Gamma(M_n(\mathbb{F}))$, where $\mathbb{F}$ is algebraically closed and $n \geq 3$, is equal to four [4]. Below we provide an example showing that if $\mathbb{F}$ is not algebraically closed, then the diameter of $\Gamma(M_n(\mathbb{F}))$ can be indeed greater than 4.

Theorem 3.1. *The graph* $\Gamma(M_9(\mathbb{Z}_2))$ *is connected with diameter at least* 5.

Proof. Note that $\mathbb{Z}_2$ permits only one field extension of degree $n = 9$, the Galois field $GF(2^9)$ which contains $GF(2^3)$ as the only proper intermediate field. So, by [2, Theorem 6] the commuting graph of $M_9(\mathbb{Z}_2)$ is connected. To see that its diameter is at least 5, consider a polynomial $m(\lambda) = \lambda^9 + \lambda^8 + \lambda^4 + \lambda^2 + 1 \in \mathbb{Z}_2[\lambda]$. It is easy to see that this polynomial is irreducible. Let $\widehat{A} = C(m) \in M_9(\mathbb{Z}_2)$ be its companion matrix. Since $\widehat{A}$ has a cyclic vector, $\mathcal{C}(\widehat{A}) = \mathbb{Z}_2[\widehat{A}]$ by a well known Frobenius result on dimension of centralizer (see for example [2, Corollary 1]), and this is a field extension of $\mathbb{Z}_2$ [14, Theorem 4.14, p. 472] of index $n = 9$. Actually, $\mathcal{C}(\widehat{A})$ is isomorphic to $GF(2^9)$ by the uniqueness of field extensions for finite fields. In the sequel we identify these two fields.

Since the field extension $\mathbb{Z}_2 \subset GF(2^9)$ contains only $GF(2^3)$ as a proper intermediate field, we see that each $X \in \mathcal{C}(\widehat{A}) \setminus GF(2^3)$ satisfies $\mathbb{Z}_2[X] = \mathbb{Z}_2[\widehat{A}] = \mathcal{C}(\widehat{A})$ and in particular X and $\widehat{A}$ are polynomials in each other so they are $\mathcal{C}$-equivalent. Moreover, each non-scalar $\widehat{Y} \in GF(2^3)$ satisfies $\mathbb{Z}_2[\widehat{Y}] = GF(2^3)$, because no proper intermediate fields exist between $\mathbb{Z}_2$ and its overfield $GF(2^3)$, and in particular, $\mathcal{C}(\widehat{Y_1}) = \mathcal{C}(\widehat{Y_2})$ for any two non-scalar $\widehat{Y_1}, \widehat{Y_2} \in GF(2^3) \subset GF(2^9) = \mathcal{C}(\widehat{A})$.

There exists a polynomial p so that $\widehat{Y} = p(\widehat{A}) \in GF(2^3) \setminus \{0, 1\}$. As the field $GF(2^3)$ contains no idempotents other than 0 and 1 we see that the rational canonical form of $\widehat{Y}$ consists only of cells which correspond to some powers of the same irreducible polynomial. Likewise, the field contains no non-zero nilpotents, so each cell of $\widehat{Y}$ corresponds to the same irreducible polynomial. Moreover, $GF(2^3)$ has no subfields other than $\mathbb{Z}_2$, so $\mathbb{Z}_2[\widehat{Y}] = GF(2^3)$ and hence the minimal polynomial of $\widehat{Y} \in GF(2^3)$ has degree $[GF(2^3) : \mathbb{Z}_2] = 3$. This polynomial is relatively prime to its derivative, so in a splitting field, $\widehat{Y}$ has three distinct eigenvalues. It easily follows that $\widehat{Y}$ is similar to a matrix $C \oplus C \oplus C$, with C being a 3×3 companion matrix of some irreducible polynomial of degree 3. Let S_1 be an invertible matrix such that $\widehat{Y} = S_1^{-1}(C \oplus C \oplus C)S_1$ and define

$$A = S_1 \widehat{A} S_1^{-1}.$$

Clearly, $p(A) = S_1 \widehat{Y} S_1^{-1} = C \oplus C \oplus C$ and it follows that

$$\mathcal{C}(p(A)) = \begin{bmatrix} \mathbb{Z}_2[C] & \mathbb{Z}_2[C] & \mathbb{Z}_2[C] \\ \mathbb{Z}_2[C] & \mathbb{Z}_2[C] & \mathbb{Z}_2[C] \\ \mathbb{Z}_2[C] & \mathbb{Z}_2[C] & \mathbb{Z}_2[C] \end{bmatrix}. \tag{3.1}$$

Since $\mathbb{Z}_2[\widehat{Y}] = GF(2^3)$ we obtain $\mathbb{Z}_2[C] = GF(2^3)$.

Consider a 3×3 block matrix

$$N = \begin{bmatrix} E_{13} & 0 & 0 \\ 0 & 0 & E_{13} \\ E_{32} & 0 & 0 \end{bmatrix}, \quad E_{13}, E_{13}, E_{32} \in M_3(\mathbb{Z}_2).$$

It is immediate that $N^3 = 0$, so $I + N$ is invertible. Define

$$B = (I + N)A(I + N)^{-1}.$$

We will show that $d(A, B) \geq 5$.

Suppose there exists a path $A — V — Z — W — B$ of length 4. Note that $V \in GF(2^3) \subset \mathcal{C}(A)$. Otherwise, if $V \in \mathcal{C}(A) \setminus GF(2^3)$, then $\mathcal{C}(V) = \mathcal{C}(A)$ and such V has exactly the same neighbours as A. Since $B = (I + N)A(I + N)^{-1}$, it follows $W = (I + N)U(I + N)^{-1}$ for some $U \in GF(2^3) \subset \mathcal{C}(A) = (I + N)^{-1}\mathcal{C}(B)(I + N)$. Recall that any two non-scalar elements in $GF(2^3)$ have the same centralizer. So in particular we might take $U = V = p(A) = C \oplus C \oplus C$ where polynomial p was defined before. For any $Z \in \mathcal{C}(V) \cap \mathcal{C}((I + N)V(I + N)^{-1})$ we have

$$Z = (I + N)\widehat{Z}(I + N)^{-1}, \qquad Z, \widehat{Z} \in \mathcal{C}(V)$$

and hence, by postmultiplying with $(I + N)$ and rearranging,

$$Z - \widehat{Z} = N\widehat{Z} - ZN. \tag{3.2}$$

Let us write $Z = \left[Z_{ij}\right]_{1\leq i,j\leq 3}$ and $\widehat{Z} = \left[\widehat{Z}_{ij}\right]_{1\leq i,j\leq 3}$ as 3×3 block matrices and by (3.1) we have that $Z_{ij}, \widehat{Z}_{ij} \in \mathbb{Z}_2[C] = GF(2^3) \subseteq M_3(\mathbb{Z}_2)$, hence each of them is either zero or invertible. Then (3.2) implies

$$\left[Z_{ij} - \widehat{Z}_{ij}\right]_{ij} = \begin{bmatrix} -Z_{11}E_{13} - Z_{13}E_{32} + E_{13}\widehat{Z}_{11} & E_{13}\widehat{Z}_{12} & E_{13}\widehat{Z}_{13} - Z_{12}E_{13} \\ -Z_{21}E_{13} - Z_{23}E_{32} + E_{13}\widehat{Z}_{31} & E_{13}\widehat{Z}_{32} & E_{13}\widehat{Z}_{33} - Z_{22}E_{13} \\ -Z_{31}E_{13} - Z_{33}E_{32} + E_{32}\widehat{Z}_{11} & E_{32}\widehat{Z}_{12} & E_{32}\widehat{Z}_{13} - Z_{32}E_{13} \end{bmatrix}.$$

Observe that each block on the left side belongs to $\mathbb{Z}_2[C] = GF(2^3) \subseteq M_3(\mathbb{Z}_2)$, and so is either zero or invertible. On the other hand, on the right side, each block in the last two columns has rank at most two. We deduce that the last two columns on both sides are zero. In particular, comparing the second columns we see that $\widehat{Z}_{12} = Z_{12} = 0$ and $\widehat{Z}_{32} = 0$, so $Z_{22} = \widehat{Z}_{22}$, and $Z_{32} = \widehat{Z}_{32} = 0$. Putting this in the above equation and simplifying, the last column gives $\widehat{Z}_{13} = 0$, so $Z_{13} = \widehat{Z}_{13} = 0$, $Z_{23} = \widehat{Z}_{23}$, and $Z_{33} = \widehat{Z}_{33}$. Also, comparing the $(2, 3)$ positions, we obtain

$$0 = Z_{23} - \widehat{Z}_{23} = E_{13}\widehat{Z}_{33} - \widehat{Z}_{22}E_{13} = e_1(\widehat{Z}_{33}^{\mathrm{T}}e_3)^{\mathrm{T}} - \widehat{Z}_{22}e_1e_3^{\mathrm{T}}.$$

Moreover, $\widehat{Z}_{33}^{\mathrm{T}}e_3 = \lambda e_3$ and $\widehat{Z}_{22}e_1 = \lambda e_1$, $\lambda \in \mathbb{Z}_2$. Since $\widehat{Z}_{33}, \widehat{Z}_{22} \in \mathbb{Z}_2[C]$ and every vector is cyclic for C we see that $\widehat{Z}_{33} = \widehat{Z}_{22} = \lambda I_3$. The matrix equation therefore simplifies to

$$\begin{bmatrix} Z_{11} - \widehat{Z}_{11} & 0 & 0 \\ Z_{21} - \widehat{Z}_{21} & 0 & 0 \\ Z_{31} - \widehat{Z}_{31} & 0 & 0 \end{bmatrix} = \begin{bmatrix} -Z_{11}E_{13} + E_{13}\widehat{Z}_{11} & 0 & 0 \\ -Z_{21}E_{13} - \widehat{Z}_{23}E_{32} + E_{13}\widehat{Z}_{31} & 0 & 0 \\ -Z_{31}E_{13} - \lambda E_{32} + E_{32}\widehat{Z}_{11} & 0 & 0 \end{bmatrix}.$$

Comparing the position $(1, 1)$ gives by similar arguments as above that $\widehat{Z}_{11} = Z_{11} = \mu I_3$. Inserting this into the equation we see after rearrangement that the rank of the block at position $(3, 1)$ is equal to $\mathrm{rk}((\mu - \lambda)E_{32} - Z_{31}E_{13}) \leq 2$, which forces the two blocks at position $(3,1)$ to be zero, i.e. $Z_{31} - \widehat{Z}_{31} = 0 = (\mu - \lambda)E_{32} - Z_{31}E_{13} = (\mu - \lambda)e_3 e_2^{\mathrm{T}} - Z_{31}e_1 e_3^{\mathrm{T}}$. We immediately get $Z_{31} = \widehat{Z}_{31} = 0 = (\mu - \lambda)$. Therefore, $Z_{11} = Z_{22} = Z_{33} = \lambda I_3$. Finally, comparing the $(2,1)$ positions gives

$$Z_{21} - \widehat{Z}_{21} = -Z_{21}E_{13} - \widehat{Z}_{23}E_{32},$$

and arguing as above, $Z_{21} = \widehat{Z}_{21} = 0$. Hence, Z is a scalar matrix. So, $\mathcal{C}(V) \cap \mathcal{C}(W)$ contains only scalar matrices, which gives that $d(A,B) \geq 5$. $\square$

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