



11 Small Representation Principle

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Abstract. In a previous article [2] Don Bennett and I looked for, found and proposed a game in which the Standard Model Gauge Group $S(U(2) \times U(3))$ gets singled out as the “winner”. This “game” means that the by Nature chosen gauge group should be just that one, which has the maximal value for a quantity, which is a modification of the ratio of the quadratic Casimir for the adjoint representation and that for a “smallest” faithful representation. In a recent article [1] I proposed to extend this “game” to construct a corresponding game between different potential dimensions for space-time. The idea is to formulate, how the same competition as the one between the potential gauge groups would run out, if restricted to the potential Lorentz or Poincare groups achievable for different dimensions of space-time d . The remarkable point is, that it is the experimental space-time *dimension 4, which wins*.

Our “goal quantity” to be maximized has roughly the favouring meaning that the Lie-group in question can have the “smallest” possible faithful representations. This idea then suggests that the representations of the Standard Model group to be found on the (Weyl)Fermions and the Higgs Boson should be in the detailed way measured by our “goal quantity” be the smallest possible. The Higgs in the Standard Model belongs remarkably enough just to the in such a way “smallest” representation. For the chiral Fermions there are needed restriction so as to avoid anomalies for the gauge symmetries, and in an earlier work[14,12] we have already suggested that the Standard Model Fermion representations could be considered being the smallest possible. We hope in the future to show that also taking smallness in the specific sense suggested here would lead to the correct Standard Model representation system.

So with the suggestion here the whole Standard Model is specified by requiring SMALLEST REPRESENTATIONS! Speculatively we even argue that our principle found suggests the group of gauge transformations and some manifold(suggestive of say general relativity).

Povzetek. V prejšnjem članku [2] sva z Donom Bennettom iskala, našla in predlagala igro, v kateri se umeritvena grupa $S(U(2) \times U(3))$ standardnega modela izkaže kot “zmagovalka”. Ta “igra” pomeni, da je Narava izbrala umeritveno grupo z maksimalno vrednostjo količine, ki je nekoliko spremenjeno razmerje med kvadratom Casimirja za adjungirano upodobitev in le tega za “najmanjšo” zvesto upodobitev. V nedavnem članku [1] sem predlagal razširitev te “igre” s konstrukcijo ustrezne igre med različnimi možnimi razsežnostmi prostor-časa. Tokrat bi tekmovala Lorentzove in Poincarejeve grupe za različne razsežnosti d prostor-časa. Izkaže se, da v tej igri zмага prav opažena *razsežnost 4*. Naša “ciljna količina”, ki jo želimo maksimizirati, v grobem pomeni, da ima iskana Liejeva grupa “najmanjšo” možno zvesto upodobitev. Ali, da ima v primeru upodobitev grup standardnega modela za

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(Weylove) fermione in Higgsov bozon "ciljna količina" najmanjšo vrednost. Higgsov delec v standardnem modelu pripada prav upodobitvi, ki je v zgornjem smislu "najmanjša". Za kiralne fermione moramo zahtevati omejitve, ki poskrbijo, da se izognemo anomalijam umeritvenih simetrij. V prejšnjem delu [14,12] smo že predlagali "ciljno količino", ki zagotovi zmago upodobitvam za fermione standardnega modela. Upati je, da v tem prispevku definirana "ciljna količina" prav tako zagotovi zmago upodobitvam standardnega modela. V skladu z našim predlogom bi torej lahko izpeljali grupe standardnega modela iz zahteve za NAJMANJŠO UPODOBITEV! Postavljamo tedaj spekulativno trditev, da naše načelo predlaga grupo umeritvenih transformacij in mnogoterost (kar namiguje na splošno teorijo relativnosti).

11.1 Introduction

In two earlier articles[2,1] Don Bennett and I proposed a quantity depending on a group - thought of as the gauge *group*(group in the sense of O’Raifeartaigh [3] - which were found to take its largest value on just the Standard Model gauge *group* $S(U(2) \times U(3))$. My article [1] were to tell that the same quantity applied to the Lorentz or by some crude technology to essentially the Poincare group selected as the number of dimensions winning the highest quantity just the experimental number of dimensions 4 for space time. (The prediction of the d=4 dimensions from various reasons have been considered in e.g. [4–7]. N.Brene and I have earlier proposed another quantity to be extremized to select the Standard Model group, namely that it is the most "skew"[8] (i.e. it has the smallest number of automorphisms, appropriately counted). But in this article we shall discuss a "goal quantity" that being maximized as it shall, rather may mean crudely that the group can have as small representations as possible.

To define this wonderful group dependent quantity, which can in this way select as the highest scoring group the by Nature chosen Standard Model *group*, and the by Nature chosen space time dimension 4, let us think of a general Lie group written by means of a cross product of a series of simple Lie groups H_i (take the H_i 's to be the covering groups at first) and a series of real number \mathbf{R} factors in this cross product

$$G_{\text{cover}} = \left(\times_i^I H_i \right) \times \mathbf{R}^J. \quad (11.1)$$

Here the I is the number of, different or identical, as it may be, H_i -groups, which are supposed to be simple Lie groups, while \mathbf{R} denotes the Abelian group of real numbers under addition. The number of Abelian dimensions in the Lie algebra is called J. A very general group is obtained by dividing an invariant discrete subgroup D of the center out of this group G_{cover} . Denoting this general - though assumed connected - group as G we can indeed write it as

$$G = G_{\text{cover}}/D. \quad (11.2)$$

Of course G_{cover} is the covering group of G and the groups H_i ($i = 1, 2, \dots I-1, I$) are its invariant simple Lie groups.

The main ingredient in defining our goal quantity is the ratio of the quadratic Casimirs[10] C_A/C_F of the quadratic Casimir C_A for the adjoint representation

divided by the quadratic Casimir C_F a representation chosen, so as to make the quadratic Casimir C_F of F so small as possible though still requiring the representation F to be faithful or basically to be non-trivial. Here I now ought to remind the reader of the concept of a quadratic Casimir operator:

The easiest may be to remember the concept of quadratic Casimir first for the most well known example of a nonabelian Lie group, namely the group of rotations in 3 dimensions $SO(3)$ (when you do not include reflection in a point but only true rotations) for which the covering group is $SU(2)$. In this case the quadratic Casimir operator is the well known square of the angular momentum operator

$$\vec{J}^2 = J_x^2 + J_y^2 + J_z^2 \tag{11.3}$$

Now our goal quantity, which so nicely points to both the Standard Model group and the dimension of space time, is given as the d_G 'th root of the product with one factor from each invariant simple group H_i , namely $(C_A/C_F)^{d_i}$ (C_F/C_A is related to the Dynkin-index [11]) and some factors e_A^2/e_F^2 for each of the J Abelian factors. (Here the dimension of the simple groups H_i are denoted d_i , while the dimension of the total group G or of G_{cover} is denoted d_G .) Our goal quantity in fact becomes

$$\begin{aligned} & \text{“goal quantity”} \tag{11.4} \\ &= \left(\prod_{\text{simple groups } i} \left(\frac{C_A}{C_F}\right)^{d_i} * \prod_{\text{Abelian factors } j} \left(\frac{e_A^2}{e_F^2}\right)_j \right)^{1/d_G} \tag{11.5} \end{aligned}$$

To fully explain this expression I need to explain what means the “charges” e_F for the “small” representation (essentially F) and e_A for the analogon¹ to the adjoint representation: Of course the reader should have in mind that the Abelian groups, the \mathbf{R} subgroups, have of course no adjoint representations in as far as the basis in the Lie algebra of an Abelian group is only transformed trivially. In stead of defining these “charges” – as we shall do below – by first defining a replacement for the adjoint we shall define these factors $\prod_{\text{Abelian factors } j} \left(\frac{e_A^2}{e_F^2}\right)_j$ from the Abelian factors in the Lie algebra by means of the system of *allowed and not allowed representations* of the group $G = G_{cover}/D = \left(\left(\times_i^{\mathbf{I}} H_i \right) \times \mathbf{R}^{\mathbf{J}} \right) / D$. Each irreducible representation of this G is characterized in addition to its representations under the simple Lie groups H_i also by a “vector” of “charges” representing the phase factors $\exp(i\delta_1 e_{r1} + i\delta_2 e_{r2} + \dots + i\delta_J e_{rJ})$, which multiply the representation vector under an element $(\delta_1, \delta_2, \dots, \delta_J) \in \mathbf{R}^{\mathbf{J}}$, i.e. in the Abelian factor of G . The easiest may be to say that we consider the whole lattice system of allowed “vectors”

¹ We might define an analogon of the adjoint representation for also a set of J properly chosen \mathbf{R} -factors of G by assigning the notion of “analogon to the adjoint representation” to that representation of one of the \mathbf{R} -factors, which has the smallest charge, e_A called, allowed for a representation of $\mathbf{R}/(\mathbf{R} \cap D)$, where \mathbf{R} stands for the \mathbf{R} -factor considered, and $\mathbf{R} \cap D$ for the intersection of the to be divided out discrete group D with this Abelian factor \mathbf{R} .

$\{(e_{r1}, e_{r2}, \dots, e_{rJ}) \mid \text{r allowed by } G\}$ of sets “charges” allowed by the group G , and then compare with corresponding set in which we *only consider those representations* r , which represent the simple non-abelian groups only trivially:

$$\{(e_{r1}, e_{r2}, \dots, e_{rJ}) \mid \text{r allowed by } G, \text{ and with the representation of the } H_i \text{'s being only trivial}\}.$$

In this comparison you ask for a going to an infinitely big region in the J -dimensional lattice after the ratio of the number of “charge vectors” in the first lattice

$$\{(e_{r1}, e_{r2}, \dots, e_{rJ}) \mid \text{r allowed by } G\}$$

relative to that in the second

$$\{(e_{r1}, e_{r2}, \dots, e_{rJ}) \mid \text{r } G\text{-allowed with the } H_i \text{'s represented trivially}\}.$$

Then the whole factor under the d_G 'th root sign is the product of the factor comming from the semisimple part of the group G

$$\text{“Semisimple factor”} = \prod_{\text{simple groups } i} \left(\frac{C_A}{C_F}\right)_i^{d_i} \tag{11.6}$$

and the ratio of the number of charge combinations at all allowed by the group G to the number of charge combinations, when the semisimple groups are restricted to be represented trivially - in the representation of the whole G representing the Abelian part by the charge combination in question :

$$\text{“Abelian factor”} = \left(\frac{\#\{(e_{r1}, e_{r2}, \dots, e_{rJ}) \mid \text{r allowed by } G\}}{\#\{(e_{r1}, e_{r2}, \dots, e_{rJ}) \mid \text{r } G\text{-allowed with the } H_i \text{'s represented trivially}\}} \right)^2 \tag{11.7}$$

Here $\#$ stands for the number of elements in the following set, i.e. the cardinal number; but it must be admitted that the numbers of these charge combinations are infinite, and that to make the finite result, which we shall use, we have to take a cut off and take the limit of the ratio for that cut off going to be a bigger and bigger sphere finally covering the whole J -dimensional space with the charge combinations embedded. So strictly speaking we define rather

$$\text{“Abelian factor”} = \left(\lim_{S \rightarrow \infty} \frac{\#\{(e_{r1}, e_{r2}, \dots, e_{rJ}) \mid \text{r allowed by } G\}_{\text{cut off by } S}}{\#\{(e_{r1}, e_{r2}, \dots, e_{rJ}) \mid \text{r } G\text{-allowed with the } H_i \text{'s represented trivially}\}_{\text{cut off by } S}} \right)^2.$$

where S is some large “sphere say” in the J -dimensional space of charge combinations. The symbol $S \rightarrow \infty$ shall be understood to mean that the region S is taken to be larger and larger in all directions so as to in the limit cover the whole space.

Then our goal quantity to be maximized so as to select the gauge group supposed to be chosen by nature can be written

$$\text{“goal quantity”} = (\text{“Semisimple factor”} * \text{“Abelian factor”})^{1/d_G}. \tag{11.8}$$

Really it is nice to express the quantity “Abelian factor” by means of the representations allowed by the group, because after all the phenomenological determination of the *Lie-group* rather than only the Lie algebra[3] is based on such a system of allowed representations.

11.1.1 Motivation

Before illustrating the calculation of our “goal quantity” with Standard Model as the example, let me stress the motivation or interest in looking for such a function defined on gauge groups or more abstractly somehow on theories and can be used to single out the by Nature chosen model. A major reason making such a singling out especially called for is that the Standard Model and e.g. its group is not in an obvious way anything special! It is a combination of several subgroups like $SU(2)$, $SU(3)$, and $U(1)$ of groups that cannot all be the obvious *one*, since we already use 3. There exist both several groups with lower rank, say than the 4 of the Standard Model group, and of course infinitely many with higher rank. That it truly has been felt, not only by us, but by many physicists that the Standard Model is *a priori* not anything obviously special - except for the fact, that it is the model that agrees with experiment - can be seen from the great interest in - and even belief in - grand unification theories[19] seeking to find e.g. an extended gauge group, of which the Standard Model gauge group is then only the small part, which survived some series of (spontaneous) break downs of part of the larger group. Let me put some of the predictions of the typical grand unification model as $SU(5)$ in the perspective: When they are concerned with representations possible for say the $SU(5)$, there are restrictions for what they can be for the Standard Model “ $SU(2) \times SU(3) \times U(1)$ ” - and they agree with experiment -; but then these restrictions are truly a consequence of that *the subgroup* of $SU(5)$ *having the Lie-algebra (of) $SU(2) \times SU(3) \times U(1)$ is precisely the group $S(U(2) \times U(3))$* . Indeed the condition on the possible representations, when there is an $SU(5)$ GUT theory beyond the Standard Model, is the same condition (11.10) as comes from $S(U(2) \times U(3))$. There is of course more information in specifying the group than only the Lie-algebra; but that of course only implies that an *a priori* not special group is even less special than an algebra, because there are even more *groups* among which to choose than there are *algebras*. (Of course there are truly infinitely many both groups and algebras, but for a given range of ranks, say, there are more *Lie-groups* than *Lie-algebras*).

Another hope of explaining, why the Standard Model including its gauge group is chosen by Nature, is the superstring theories, which predict at the fundamental or string level the gauge groups $E_8 \times E_8$ or $SO(32)$. But from the point of view of our “goal quantity” - as can be seen below from our tables - especially E_8 and consequently also $E_8 \times E_8$ (since our “goal quantity” has the property of being the same for a group G and its cross products with itself any number of times) is the worst group from the point of view of our “goal quantity”: In fact the nature of our “goal quantity” construction is so, that we always must have

$$\text{“goal quantity”} \geq 1. \quad (11.9)$$

But E_8 according to the table below gives just this 1 for its “goal quantity”

$$\text{“goal quantity”}_{E_8 \times E_8} = \text{“goal quantity”}_{E_8} = 1$$

actually because E_8 has no smaller representation than its adjoint representation.

The connection to my personal pet-theory (or dream, or program) of Random Dynamics [13,15,17,16,18,20] is that a priori the present work is ideally phenomenologically - as to be explained in subsection 11.3.3-, i. e. the spirit is to ask nature and just seek to find what is characteristic for the Standard Model group without theoretical guesses behind a priori. However, it(= our phenomenological result) leads to the suggestion that the (gauge)group that wins - gets highest “goal quantity” - is the one that most likely would become approximately a good symmetry by accident. This would then mean, that in a random model, as is the picture in Random Dynamics, the group, that is selected by our game, is just the one most likely to be realized as an approximately good symmetry by accident. So indeed Random Dynamics could be a background theory for the present work. So in this sense random dynamics ends up being favoured by the present article, although we in principle started out purely phenomenologically. (It must be admitted though, that historically the idea appeared as an *extract* from a long Random Dynamics inspired calculation - which has so far not been published - by Don Bennett and myself.) Having approximately gauge symmetries, there is according to some earlier works of ours and others [16,25,29,30] the possibility that the gauge symmetry may become exact by quantum fluctuations; really one first writes it formally as if the remaining small breaking were a Higgsing, and then argue that quantum fluctuations wash away this “Higgs”effect.

11.1.2 Plan of Article

In the next section 11.2 we shall with the Standard Model group as an example tell how to calculate the goal quantity, and we deliver in this section 11.2 also some *tables* to use for such computations. Then in section 11.3 we discuss the attempt to also postdict the dimension of space time; for that several slight modifications are used to in an approximate sense construct a goal quantity like quantity for even the Poincare group in an arbitrary dimension d for space time. Successively in section 11.4 we consider, how we can extend our ideas to measure the *size of a representation* of the Standard Model group, and then the wonderful result is that the representation, under which the *Higgs* fields transform, remarkably enough turns out to be just the *smallest (non- trivial) representation!*

The following sections are about work still under development, and in section 11.5 we review an old work making more precise, what is already rather intuitively obvious: That the fermion representations in the Standard Model are rather “small” and that that together with anomaly conditions settles what they can be *assuming mass protected fermions only*. In the next section 11.6 we point to a way of changing the point of view so as to say, that, what we predict, is rather than the gauge group the group of gauge transformations. This may be the beginning to predict also a manifold structure for the whole gauge theory. Are we on the way to general relativity? We conclude and resume in section 11.7.

11.2 Calculation of “Goal quantity” Illustrated with the Standard Model Group $S(U(2) \times U(3))$

Rather than going into using the structure as a *group* rather than only the *Lie-algebra* structure we just above remarked that we can determine the “Abelian factor” (see 11.8) by studying the system of representation allowed as representations of the *group* rather than being just allowed by the Lie-algebra.

For example the phenomenological feature of the Standard Model, that gives rise to, that the Standard Model Group indeed must be taken as $S(U(2) \times U(3))$ [3], is the restriction on the weak hypercharge y quantization (or rather we prefer to use the half weak hypercharge $y/2$) realizing the usual assumption in the Standard Model about electric charge quantization (Milikan quantization extended with the well known rules for quarks). This rule become written for the Standard Model:

$$y/2 + I_W + \text{“trianality”}/3 = 0(\text{mod}1). \tag{11.10}$$

According to the rule to calculate the Abelian factor we shall in the limit of a going to infinity big range of $y/2$ -values ask for what fraction of the number of values possible with the rule (11.10) imposed and the same but only including representations with the simple groups $SU(2)$ and $SU(3)$ in the Lie algebra of the Standard Model represented trivially. If we only allowed the adjoint or the trivial representations of these simple groups, so that $I_W = 0(\text{mod}1)$ and “trianality” = 0, it is quite obvious in our Standard Model example, that the Standard Model rule (11.10) allows, when the simple representations can be adjusted, all $y/2$ being an integer multiplum of $1/6$. If we, however, limit the simple groups to have trivial (or adjoint) representations only, then we can only have $y/2$ being integer. It is clear that this means in the limit of the large range S that there are 6 times as many $y/2$ values allowed, when the representations of the simple groups are free, as when it is restricted to be trivial (or adjoint). We therefore immediately find for the Standard Model Group

$$\text{“Abelian factor”}_{S(U(2) \times U(3))} = 6^2 = 36. \tag{11.11}$$

In order to calculate the factor “Semisimple factor” (11.6) we must look up the table for the C_A/C_F for the simple groups involved, then raise these factors to the power of the dimension of the Lie-algebras in question, and very finally after having multiplied also by the “Abelian factor” we must take the root of the total dimension of the whole group.

11.2.1 Useful Table

Here we give the table to use, our (essentially inverse Dynkin index [11]) ratios for the *simple* Lie groups, with the representation F selected so as to provide the biggest possible ratio C_A/C_F still keeping F non trivial, or let us say faithful (in a few cases the choice of this F is not clear at the outset and the user of the table has to choose the largest number among “vector” and “spinor” after he has provided the rank n he wants to use):

Our Ratio of Adjoint to “Simplest” (or smallest) Quadratic Casimirs C_A/C_F

$$\frac{C_A}{C_F}|_{A_n} = \frac{2(n+1)^2}{n(n+2)} = \frac{2(n+1)^2}{(n+1)^2-1} = \frac{2}{1-\frac{1}{(n+1)^2}} \quad (11.12)$$

$$\frac{C_A}{C_{F \text{ vector}}}|_{B_n} = \frac{2n-1}{n} = 2 - \frac{1}{n} \quad (11.13)$$

$$\frac{C_A}{C_{F \text{ spinor}}}|_{B_n} = \frac{2n-1}{\frac{2n^2+n}{8}} = \frac{16n-8}{n(2n+1)} \quad (11.14)$$

$$\frac{C_A}{C_F}|_{C_n} = \frac{n+1}{n/2+1/4} = \frac{4(n+1)}{2n+1} \quad (11.15)$$

$$\frac{C_A}{C_{F \text{ vector}}}|_{D_n} = \frac{2(n-1)}{n-1/2} = \frac{4(n-1)}{2n-1} \quad (11.16)$$

$$\frac{C_A}{C_{F \text{ spinor}}}|_{D_n} = \frac{2(n-1)}{\frac{2n^2-n}{8}} = \frac{16(n-1)}{n(2n-1)} \quad (11.17)$$

$$\frac{C_A}{C_F}|_{G_2} = \frac{4}{2} = 2 \quad (11.18)$$

$$\frac{C_A}{C_F}|_{F_4} = \frac{9}{6} = \frac{3}{2} \quad (11.19)$$

$$\frac{C_A}{C_F}|_{E_6} = \frac{12}{\frac{26}{3}} = \frac{18}{13} \quad (11.20)$$

$$\frac{C_A}{C_F}|_{E_7} = \frac{18}{\frac{57}{4}} = \frac{72}{57} = \frac{24}{19} \quad (11.21)$$

$$\frac{C_A}{C_F}|_{E_8} = \frac{30}{30} = 1 \quad (11.22)$$

For calculation of this table seek help in[27,26].

In the just above table we have of course used the conventional notation for the classification of Lie algebras, wherein the index n on the capital letter denotes the rank (the rank n is the maximal number of mutually commuting basis-vectors in the Lie algebra) of the Lie algebra, and:

- A_n is $SU(n+1)$,
- B_n is the odd dimension orthogonal group Lie algebra i.e. for $SO(2n+1)$ or for its covering group $Spin(2n+1)$,
- C_n are the symplectic Lie algebras.
- D_n is the even dimension orthogonal Lie algebra i.e. for $SO(2n)$ or its covering group $Spin(2n)$,
- while F_4 , G_2 , and E_n for $n=6,7,8$ are the exceptional Lie algebras.

The words spinor or vector following in the index the letter F , which itself denotes the “small” representation - i.e. most promising for giving a small quadratic Casimir C_F - means that we have used for F respectively the smallest spinor and the smallest vector representation.

11.2.2 End of calculation of the “goal quantity” for the Standard Model Group

Since the Lie-algebra in addition to the Abelian part ($U(1)$ usually called) consists of $SU(2)$ and $SU(3)$ we must look these two simple Lie algebras up in the table above, finding respectively for the C_A/C_F ratios $8/3$ and $9/4$, which must be taken to respectively the powers 3 and 8, since the dimensions of the $A_n = SU(n + 1)$ Lie-groups are “dimension” = $(n + 1)^2 - 1$, leading to

$$\begin{aligned} \text{“Semisimple factor”}_{S(U(2) \times U(3))} &= \left(\frac{8}{3}\right)^3 \cdot \left(\frac{9}{4}\right)^8 = 3^{13} \cdot 2^{-7} = 1594323/128 \\ &= 12455.6484375. \end{aligned} \tag{11.23}$$

Remembering that we got $6 = 3 \cdot 2$ for the ratio of numbers of $y/2$ -values, when all representation obeying (11.10) were counted relative to this number for only the representations with trivial representations of $SU(2)$ and $SU(3)$, the “Abelian factor” = $6^2 = 3^2 \cdot 2^2$. Then the whole factor, of which to next take the 12th root (since the total dimensionality of the Standard Model group is 12) becomes

$$\begin{aligned} \text{“Semisimplefactor”} \cdot \text{“Abelianfactor”} &= \left(\frac{8}{3}\right)^3 \left(\frac{9}{4}\right)^8 \cdot 36 = 2^{-5} \cdot 3^{21} \\ &= 448403.34375. \end{aligned} \tag{11.24}$$

Thus we just have to take the 12th root of this quantity to obtain the score or “goal quantity” for the Standard Model group $S(U(2) \times U(3))$

$$\begin{aligned} \text{“goalquantity”}_{S(U(2) \times U(3))} &= (2^{-5} \cdot 3^{21})^{1/12} = 3 \cdot \left(\frac{27}{32}\right)^{1/12} = 3 \cdot 0.985941504 \\ &= \mathbf{2.957824511}. \end{aligned} \tag{11.25}$$

Similar calculations give the “goal quantity” for other groups. But it requires of course either a lot of work or some rules and experiences with calculating such goal quantities in order to see, which alternative groups are the severe competitors of the Standard Model group $S(U(2) \times U(3))$ that have to have their “goal quantities” computed in order to establish that the by Nature selected Standard Model group $S(U(2) \times U(3))$ is indeed the winner in obtaining the highest “goal quantity” (except for groups being higher powers of the Standard Model group itself).

For example a very near competitor is the group $U(2)$, for which one easily calculates

$$\text{“Semisimplefactor”}_{U(2)} = \left(\frac{8}{3}\right)^3 = 2^9/3^3 = 18.962962963 \tag{11.26}$$

$$\text{“Abelianfactor”}_{U(2)} = 2^2 = 4 \tag{11.27}$$

$$\begin{aligned} \text{“goalquantity”}_{U(2)} &= (2^{11}/3^3)^{1/4} = 2^3/3 \cdot \left(\frac{3}{2}\right)^{1/4} \\ &= 2^3/3 \cdot 1.10668192 = \mathbf{2.951151786} \end{aligned} \tag{11.28}$$

On the Fig. 11.1 we illustrate the three groups getting the three highest “goal quantities”. The third group winning so to speak the bronze medal in this competition is $Spin(5) \times SU(3) \times U(1)/“Z_6”$ (where “ Z_6 ” stands for a certain with the

integers modulo 6 isomorphic subgroup of the center of the cross product group; it arranges a quantization rule for the allowed representations quite analogous to that of the Standard Model group except, that the weak Lie algebra $SU(2)$ has been replaced by $Spin(5)$ (which is the covering group of $SO(5)$), which is very analogous to the Standard Model group just with $SO(5)$ or rather $Spin(5)$ which is its covering group replacing the $SU(2)$ in the Standard Model:

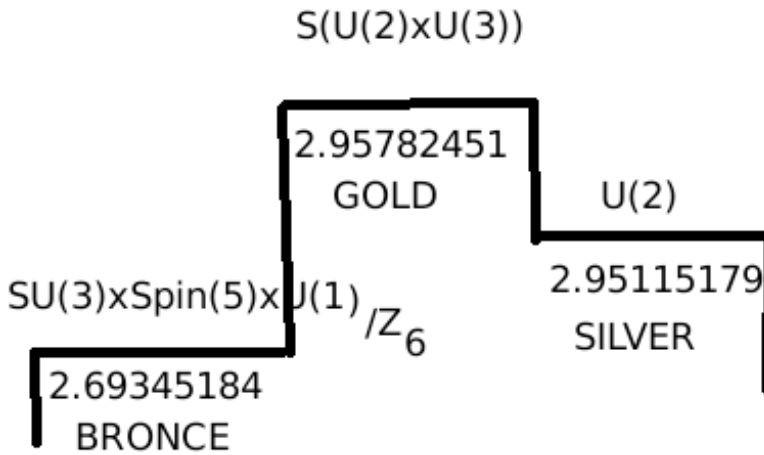


Fig. 11.1. This figure illustrates the three Lie groups getting in our game the highest scores for our “goal quantity” as were the sportsmen winning gold silver and bronze medals.

11.3 Dimension of Space-time Also

The main point of my progress since last year [2] is to say:

The choice of dimensionality of space time, that nature have made, - at least 3+1 for practical purpose - can be considered also a choice of a group, - and even a gauge group, if we invoke general relativity -namely say the Lorentz group or the Poincare group. So if we have a “game” or a “goal quantity” selecting by letting it be maximal the gauge group of the Standard Model, it is in principle possible to ask:

Which among the as Lorentz or Poincare group applicable groups get the highest “goal quantity” score? Which dimension wins the competition among Lorentz or Poincare groups?.

We would of course by extrapolation from the gauge group story (= previous work(with Don)[2]) expect that Nature should again have chosen the “winner”.

It is my point now that - with only very little “cheat” - I can claim that indeed **Nature has chosen that dimension $d = 4$** (presumably meant to be the practical one, we see, and not necessarily the fundamental dimension, since our quantity could represent some stability against collapsing the dimension) **that gives the biggest score for the Poincare group!** (for the Lorentz groups $d = 4$ and $d = 3$ share the winner place !)

11.3.1 Development of Goal Quantities for dimension fitting.

In the present article we shall ignore anthropic principle arguments for what space time dimension should be, and seek to get a statement, that the experimental number of dimensions (4 if you count the truly observed one and take the convention to include time as one dimension) just maximizes some quantity, that is a relatively simple function of the group structure of, say, the Lorentz group, and which we then call a “goal quantity”.

Making a “goal quantity” for Dimension is a Two step Procedure:

- 1) We first use the proposals in my work with Don Bennett to give a number - a goal quantity - for any Lie group.
- 2) We have to specify on which group we shall take and use the procedure of the previous work; shall it be the Lorentz group?, its covering group ? or somehow an attempt with the Poincare group ? :

Developing a “Goal quantity” for “predicting”(fitting) the Space Time dimension

A series of four proposals:

- a. Just take the Lorentz group and calculate for that the inverse Dynkin index or rather the quantity which we already used as “goal quantity” in the previous work and above (11.5) C_A/C_F . (Semi-simple Lorentz groups except for dimension $d = 2$ or smaller and in fact simple for 3, 5 and higher).
- b. We supplement in a somewhat *ad hoc way* the above *a.*, i.e. C_A/C_F by taking its $\frac{d+1}{d-1}$ th power. The idea behind this proposal is that we think of the *Poincare group* instead of as under *a.* only on the Lorentz group part, though still *in a crude way*. This means we think of a group, which is the Poincare group, except that we for simplicity *ignore that the translation generators do not commute with the Lorentz group part*. Then we assign in accordance with the *ad hoc rule* used for the gauge group the Abelian sub-Lie-algebra a formal replacement 1 for the ratio of the quadratic Casimirs C_A/C_f - because there is no limit to how small momenta can be quantized and no natural way to obtain the charges e_r for restricted representations, since we have essentially \mathbf{R} as the Abelian group rather than $U(1)$ or complicated discrete subgroups D being divided out -: I.e. we put “ e_A^2/e_F^2 ” = “ $C_A/C_F|_{\text{Abelian formal}}'' = 1$. Next we construct an “average” averaged *in a logarithmic way* (meaning that we average the logarithms and then exponentiate again) weighted with the dimension of the Lie groups over all the dimensions of the Poincare Lie group. Since

the dimension of the Lorentz group for d dimensional space-time is $\frac{d(d-1)}{2}$ while the Poincare group has dimension $\frac{d(d-1)}{2} + d = \frac{d(d+1)}{2}$ the logarithmic averaging means that we get

$$\begin{aligned} \exp\left(\frac{\frac{d(d-1)}{2} \ln(C_A/C_F)|_{\text{Lorentz}} + \ln(1) * d}{d(d+1)/2}\right) &= (C_A/C_F)|_{\text{Lorentz}}^{\frac{d(d-1)/2}{d(d+1)/2}} \\ &= (C_A/C_F)|_{\text{Lorentz}}^{\frac{d-1}{d+1}} \end{aligned} \quad (11.29)$$

That is to say we shall make a certain ad hoc partial inclusion of the Abelian dimensions in the Poincare groups.

To be concrete we here propose to say crudely: Let the Poincare group have of course d "Abelian" generators or dimensions. Let the dimension of the Lorentz group be $d_{\text{Lor}} = d(d-1)/2$; then the total dimension of the Poincare group is $d_{\text{Poi}} = d + d_{\text{Lor}} = d(d+1)/2$. If we crudely followed the idea of weighting proposed in the previous article [2] or above (11.5 as if the d "abelian" generators were just simple cross product factors - and not as they really are: not quite usual, because they do not commute with the Lorentz generators - then since we formally are from this previous article suggested to use the *as if number 1 for the Abelian groups*, we should use the quantity

$$(C_A/C_F)|_{\text{Lor}}^{\frac{d_{\text{Lor}}}{d_{\text{Poi}}}} = (C_A/C_F)|_{\text{Lor}}^{\frac{d-1}{d+1}} \quad (11.30)$$

as goal quantity.

Really you can simply say: we put the "Abelian factor" =1, but still take the $d_{\text{Poi}} = d(d+1)/2$ th root at the end, by using the total dimension of the Poincare group d_{Poi} . The crux of taking this "1" is that we do not have anything corresponding to the division out of a discrete group giving the restriction like (11.10 in the Poincare case.

- c. We could improve the above proposals for goal quantities a . or b . by including into the quadratic Casimir C_A for the adjoint representation also contributions from the translation generating generators, so as to define a quadratic Casimir for the whole Poincare group. This would mean, that we for calculating our goal quantity would do as above but

$$\text{Replace } :C_A \rightarrow C_A + C_V, \quad (11.31)$$

where C_V is the vector representation quadratic Casimir, meaning the representation under which the translation generators transform under the Lorentz group. Since in the below table we in the lines denoted "no fermions" have taken the "small representation" F to be this vector representation V , this replacement means, that we replace the goal quantity ratio C_A/C_F like this:

(S)O(d), "no spinors":

$$C_A/C_F = C_A/C_V \rightarrow (C_A + C_V)/C_F = C_A/C_F + 1 \quad (11.32)$$

Spin(d), "with spinors":

$$\begin{aligned} C_A/C_F &\rightarrow (C_A + C_V)/C_F \\ &= C_A/C_F + (C_A/C_V)^{-1}(C_A/C_F) \\ &= (1 + (C_A/C_F)|_{\text{no spinors}}^{-1})C_A/C_F. \end{aligned} \quad (11.33)$$

Let me stress though that this proposal c. is not quite “fair” in as far as it is based on the Poincare group, while the representations considered are not faithful w.r.t. to the whole Poincare group, but only w.r.t. the Lorentz group

- d. To make the proposal c. a bit more “fair” we should at least say: Since we in c. considered a representation which were only faithful w.r.t. the Lorentz subgroup of the Poincare group we should at least correct the quadratic Casimir - expected crudely to be “proportional” to the number of dimensions of the (Lie)group - by a factor $\frac{d+1}{d-1}$ being the ratio of the dimension of the Poincare (Lie)group, $d + d(d-1)/2$ to that of actually faithfully represented Lorentz group $d(d-1)/2$. That is to say we should before forming the ratio of the improved C_A meaning $C_A + C_V$ (as calculated under c.) to C_F replace this C_F by $\frac{d+1}{d-1} * C_F$, i.e. we perform the replacement:

$$C_F \rightarrow C_F * \frac{d(d-1)/2 + d}{d(d-2)/2} = C_F * \frac{d+1}{d-1}. \quad (11.34)$$

Inserted into $(C_A + C_V)/C_F$ from c. we obtain for the in this way made more “fair” approximate “goal quantity”

$$\text{“goal quantity”}|_{\text{no spinor}} = (C_A/C_F + 1) * \frac{d-1}{d+1} \quad (11.35)$$

$$\text{“goal quantity”}|_{\text{w. spinor}} = (1 + (C_A/C_F)|_{\text{no spinor}}^{-1}) * C_A/C_F * \frac{d-1}{d+1} \quad (11.36)$$

This proposal d. should then at least be crudely balanced with respect to how many dimensions that are represented faithfully.

11.3.2 Philosophy of the goal quantity construction/development

The reader should consider these different proposals for a quantity to maximize (= use as goal quantity) as rather closely related versions of a quantity suggested by a perhaps a bit vague ideas being improved successively by treating the from our point of view a bit more difficult to treat Abelian part (=the translation part of the Poincare group) at least in an approximate way.

One should have in mind, that this somewhat vague basic idea behind is: The group selected by nature is the one that counted in a “normalization determined from the Lie algebra of the group” can be said to have a faithful representation (F) the matrices of which move as little as possible, when the group element being represented move around in the group.

Let me at least clarify a bit, what is meant by this statement:

We think by representations as usual on linear representations, and thus it really means representation of the group by means of a homomorphism of the group into a group of matrices. The requirement of the representation being faithful then means, that this group of matrices shall actually be an isomorphic image of the original group. Now on a system of matrices we have a natural metric,

namely the metric in which the distance between two matrices \mathbf{A} and \mathbf{B} is given by the square root of the trace of the numerical square of the difference

$$\text{dist} = \sqrt{\text{tr}((\mathbf{A} - \mathbf{B})(\mathbf{A} - \mathbf{B})^+)}. \quad (11.37)$$

To make a comparison of one group and some representation of it with another group and its representation w.r.t. to, how fast the representation matrices move for a given motion of the group elements, we need a normalization giving us a well-defined metric on the groups, w.r.t. which we can ask for the rate of variation of the representations. In my short statement I suggested that this "normalization should be determined from the Lie algebra of the group". This is to be taken to mean more precisely, that one shall consider the *adjoint* representation, which is in fact completely given by the Lie algebra, and then use the same distance concept as we just proposed for the matrix representation $\sqrt{\text{tr}((\mathbf{A} - \mathbf{B})(\mathbf{A} - \mathbf{B})^+)}$. In this way the quantity to minimize would be the ratio of the motion-distance in the representation - F say - and in the Lie algebra representation - i.e. the adjoint representation. But that ratio is just for infinitesimal motions $\sqrt{C_F/C_A}$. So if we instead of talking about what to minimize, inverted it and claimed we should maximize we would get $\sqrt{C_A/C_F}$ to be *maximized*. Of course the square root does not matter, and we thus obtain in this way a means to look at the ratio C_A/C_F as a measure for the motion of an element in the group compared to the same element motion on the representation.

It might not really be so wild to think that a group which can be represented in a way so that the representation varies little when the group element moves around would be easier to get realized in nature than one that varies more. If one imagine that the potential groups become good symmetries by accident, then at least it would be less of an accident required the less the degrees of freedom moves around under the to the group corresponding symmetry (approximately). It is really such a philosophy of it being easier to get some groups approximately being good symmetries than other, and those with biggest C_A/C_F should be the easiest to become good symmetries by accident, we argue for. That is indeed the speculation behind the present article as well as the previous one [2] that symmetries may appear by accident (then perhaps being strengthened to be exact by some means [16,25]).

11.3.3 Phenomenological Philosophy

But let us stress that you can also look at the present work and the previous one in the following phenomenological philosophy:

We wonder, why Nature has chosen just 4 (=3+1) dimensions and why Nature - at the present experimentally accessible scale at least - has chosen just the Standard Model group $S(U(2) \times U(3))$? Then we speculate that there might be some quantity characterizing groups, which measures how well they "are suited" to be the groups for Nature. And then we begin to *seek* that quantity as being some function defined on the class of abstract groups - i.e. giving a number for each abstract (Lie?) group - of course by proposing for ourselves at least various versions or ideas for what such a *relatively simple* function defined on the abstract Lie groups

could be. Then the present works - this paper and the previous ones[2] and [1] - represents the present status of the search: We found that with small variations the types of such functions representing the spirit of the *little motion of the "best" faithful representation*, i.e. essentially the largest C_A/C_F , turned out truly to bring Nature's choices to be (essentially) the winners.

In this sense we may then claim that we have found by phenomenology, that at least the "direction" of a quantity like C_A/C_F or light modifications of it is a very good quantity to make up a "theory" for, why we have got the groups we got!

Here we bring the table in which we present the calculations of our for the space-time dimension relevant various "goal quantities":

Di- men- sion	Lorentz group, covering	Ratio C_A/C_F for spinor	Ratio C_A/C_F as no spinor	c.-quan- tity max c)	$\frac{d-1}{d+1}$	d.-quan- tity max d)
2^2	U(1)	-(for- mally 2)	-(for- mally 1)	4	1/3	4/3 =1.33
3	spin(3)	$\frac{8}{3} = 2.67$	1	$\frac{16}{3} = 5.3$	$\frac{2}{4}$	$\frac{8}{3} = 2.67$
4	Spin(4) SU(2) × SU(2)	$\frac{8}{3}$ =2.67	$\frac{4}{3}$	$\frac{14}{3}$ =4.67	$\frac{3}{5}$	$\frac{14}{5}$ =2.8
5	Spin(5)	$\frac{12}{5} = 2.4$	$\frac{3}{2} = 1.5$	4	$\frac{4}{6}$	$\frac{8}{3} = 2.67$
6	Spin(6)	$\frac{32}{15}$	$\frac{8}{5} = 1.6$	$\frac{52}{15} = 3.5$	$\frac{5}{7}$	$\frac{52}{21} = 2.5$
d odd	Spin(d)	$\frac{8(2n-1)}{n(2n+1)}$ $\frac{16(d-2)}{d(d-1)}$	$2 - 1/n =$ $2 - \frac{2}{d-1}$	$\frac{8(3d-5)}{d(d-1)}$	$\frac{d-1}{d+1}$	$\frac{8(3d-5)}{d(d+1)}$
d even	Spin(d)	$\frac{16(d-2)}{d(d-1)}$	$\frac{4(n-1)}{2n-1}$ $= \frac{2d-4}{d-1}$	$\frac{8(3d-5)}{d(d-1)}$	$\frac{d-1}{d+1}$	$\frac{8(3d-5)}{d(d+1)}$
d ∞	Spin(d)	$\approx 16/d$	$\rightarrow 2$	$\approx 24/d$	$\rightarrow 1$	$\approx 24/d$ $\rightarrow 0$
d ∞	Spin(d)	$\approx 16/d$	$\rightarrow 2$	$\approx 24/d$	$\rightarrow 1$	$\approx 24/d$ $\rightarrow 0$

Caption: We have put the goal-numbers for the third proposal c in which I (a bit more in detail) seek to make an analogon to the number used in the reference [2] in which we studied the gauge group of the Standard Model. The purpose of c is to approximate using the *Poincare group* a bit more detailed, but still not by making a true representation of the Poincare group. I.e. it is still not truly the Poincare group we represent faithfully, but only the Lorentz group, or here in the table only the covering group Spin(d) of the Lorentz group. However, I include in the column marked "c., max c)" in the quadratic Casimir C_A of the Lorentz group an extra term coming from the structure constants describing the non-commutativity of the Lorentz group generators with the translation generators C_V so as to replace C_A in the starting expression of ours C_A/C_F by $C_A + C_V$. In the column marked "d., max d)" we correct the ratio to be more "fair" by counting at least that because of truly faithfully represented part of the Poincare group in the representations, I use, has only dimension $d(d-1)/2$ (it is namely only the Lorentz group) while the

full Poincare group - which were already in c . but also in d . used in the improved C_A being $C_A + C_V$ - is $d(d-1)/2 + d = d(d+1)/2$. The correction is crudely made by the dimension ratio $\dim(\text{Lorentz})/\dim(\text{Poincare}) = (d-1)/(d+1)$ given in the next to last column.

Di- men- sion	Lorentz group (covering)	Ratio C_F/C_A for spinor	Ratio C_A/C_F "no spinor"	c - quantity max c)	d - quantity max d)
2^3	U(1)	-(f.: 2)	-(f.: 1)	4	$4/3=1.33$
3	spin(3)	$\frac{8}{3} = 2.67$	1	$\frac{16}{3} = 5.33$	$\frac{8}{3} = 2.67$
4	Spin(4) = SU(2) \times SU(2)	$\frac{8}{3} = 2.67$	$\frac{4}{3}$	$\frac{14}{3} = 4.67$	$\frac{14}{5} = 2.8$
5	Spin(5)	$\frac{12}{5} = 2.4$	$\frac{3}{2} = 1.5$	4	$\frac{8}{3} = 2.6667$
6	Spin(6)	$\frac{32}{15}$	$\frac{8}{5} = 1.6$	$\frac{52}{15} = 3.47$	$\frac{52}{21} = 2.4762$
d odd	Spin(d)	$\frac{8(2n-1)}{n(2n+1)}$ $= \frac{16(d-2)}{d(d-1)}$	$2 - 1/n =$ $2 - 2/(d-1)$	$\frac{8(3d-5)}{d(d-1)}$	$\frac{8(3d-5)}{d(d+1)}$
d even	Spin(d)	$\frac{16(d-2)}{d(d-1)}$	$\frac{4(n-1)}{2n-1} = \frac{2d-4}{d-1}$	$\frac{8(3d-5)}{d(d-1)}$	$\frac{8(3d-5)}{d(d+1)}$
d odd $\rightarrow \infty$	Spin(d)	$\approx 16/d$	$\rightarrow 2$	$\approx 24/d$	$\approx 24/d \rightarrow 0$
d even $\rightarrow \infty$	Spin(d)	$\approx 16/d$	$\rightarrow 2$	$\approx 24/d$	$\approx 24/d \rightarrow 0$

11.4 The Higgs Representation

A rather simple and successful application of our ideas is to seek the answer to the question: Why has the Higgs field just got the representation $(2, 1, y/2 = 1/2)$ under the Standard Model group with the Lie algebra factors written in the order $SU(2) \times SU(3) \times U(1)$?

Note that the selection of the gauge group by our "goal quantity" had the character of being obtained as a ratio - of the quadratic Casimirs C_A for adjoint and C_F for another faithful representation or some "replacements" for them in the Abelian cases - of an adjoint representation parameter to one for *another representation* F . Also this other representation F gets basically selected by the same principle as the selection of the whole gauge group by maximizing our "goal quantity", because we also *select the representation F from the requirement that our "goal quantity" be maximized.*

Thus in reality we have hit on a quantity that tends to select both a *group* and a smallest C_F representation.

Now strictly speaking most irreducible representations of say the Standard Model group $S(U(2) \times U(3))$ will not usually be completely faithful. It is rather so that the various representations F appearing as representations of the simple subgroups will not be truly faithful, but rather *only be faithful for some subgroup* of the $S(U(2) \times U(3))$ group say. If we therefore now shall make some numbers assigned to the various *not completely faithful* representations which are allowed

as representations of the Standard Model group $S(U(2) \times U(3))$, it would be most “fair” to count the ratio of the quadratic Casimir in the “Adjoint” representation - or better in the group itself - *by not using the full say Standard Model Group, but rather only that part of the group $S(U(2) \times U(3))$ that is indeed faithfully represented on the representation, which is up to be tested, with a number to specify which representation should be favoured.*

So let us say we have some representation R of say the Standard Model group, i.e. an allowed one, which of course then also obeys the quantization rule (such as) (11.10).

Now there is always a kernel K consisting of the elements in the group $S(U(2) \times U(3))$ or more generally the Lie group G, with which we work, for which the elements in R are transformed trivially, it means not shifting to another element, but only to itself. This kernel K is of course an invariant subgroup of the full group G. This means that G/K is a well defined factor group in G. Then we should naturally suggest the “fair” rule that we construct the number according to which the representation R should be selected *as the number we would get by calculating the “goal quantity” of ours for the group G/R with though the restriction that the F should correspond to R.*

Let us illustrate this rule proposed by looking at a couple of examples:

If we want to consider one of the representations F giving the maximal G_A/G_F for one of the simple subgroups, which in Standard Model can only be $SU(2)$ or $SU(3)$, then for these two groups the F-representations are respectively **2** and **3** (or one could take the equivalent $\bar{3}$ for the $SU(3)$). But of course say **2** alone without any $y/2$ charge would not be allowed by the Standard Model group $S(U(2) \times U(3))$. Thus we are forced to include an appropriate $y/2$. Doing that you can easily find that the relevant factor groups $S(U(2) \times U(3))/K$ becomes in the two cases respectively $U(2)$ and $U(3)$. Actually with smallest $y/2$ values allowed in the two cases $y/2 = 1/2$ (or $-1/2$) for $SU(2)$ and $y/2 = -1/3$ for $SU(3)$ with **3** we get just the same F as is used in our calculations of our “goal quantity”. This means that quantities to select the representation happens to be in our two cases just the “goal quantities” for the two groups $U(2)$ and $U(3)$, namely just the factor groups. We already know that $U(2)$ were the “silver medal winner” and thus that it should be trivially $U(2)$ related to measuring the size of the representation **2, 1, $y/2 = 1/2$** which gets selected. This means the winning - and that means “smallest” representation of the Standard Model (measured by using the associated factor group for which it is faithful) - representation of the Standard Model group became this **2, 1, $y/2 = 1/2$** . This is just the representation of the Higgs. So the Higgs representation is predicted this way (as the “smallest” in our way of counting, closely related to the game we used to tell the gauge group with) !

11.5 The (chiral) Fermion Representations

It is now the idea to use the very same “goal quantity” as the one, with which we exercised in deriving the Higgs representation above, to argue for the Fermion rep-

representations in the Standard Model - or rather what we in the present philosophy expect for the choice of Nature - as to what they should be.

Here the situation is somewhat more complicated because the requirement that there be no gauge- nor gauge gravity anomalies imposes restrictions on the whole system of representations for the chiral fermions. Assuming that we work with 3+1 dimensions we can take it as our convention to work with only left handed spin 1/2 Fermions, because we can let the right handed ones simply be represented by their CP-analogue left handed ones.

We must therefore first write down the non-anomaly conditions for having various thinkable numbers of families for the various representations of the Standard Model.

Now the use of anomaly conditions together with the assumption of "small representations" (in some meaning or the other) we already used in some articles years ago. For instance in "Why do we have parity violation?" [12] Colin Froggatt and I sought to answer this question by using the principle of small representations to derive the representations that the Standard Model should have and thus why they would give parity violation the well known way. Also in [14] we allude to the principle of small representations (here in the last section). In fact in the section XIII, called "Hahn-Nambu-like Charges" we sought to derive the system of the representations of Weyl Fermions (we use a notation there of only counting the left handed spin 1/2 fermions, letting it be understood that the right handed components achieved by CP i.e. of the anti particles of course exists but are just not listed in the way we keep track of the particles in this notation; that is to say that normally considered right handed Weyl particles are just counted by their CP-antiparticle, which if left handed). We sought to derive it from the no-anomaly-conditions and a principle of "small representations". The latter were not exactly the same as we seek to develop in the present more recent but in some approximate sense it were very close to the present idea of a small representation principle, as we claim the choice of nature of the Standard Model group $S(U(2) \times U(3))$ indicates. Nevertheless the two ideas of a "small representation principle" are so close that at least I give/gave them the same name "small representations".

In the section XIII of the Puerto-Rico conference proceedings [14] we use somewhat special technology to argue that *imposing the conditions for:*

- 1. no chiral anomalies and no mixed anomalies for the gauge charge conservations,
- 2. together with a *small representation principle* (formulated using the concept to be explained of "Hahn Nambu charges")
- 3. and that the fermions shall be *mass protected* (i.e. get zero mass due to gauge (charge) conservation, were it not for the "Higgsing"),

lead to the Standard Model representations spectrum basically (i.e. we get that there should be a number of families of the type we know, but how many we do not predict from these assumptions).

The technology used in [14] was to consider only a certain subset of charges - called there "Hahn-Nambu charges" - of the Cartan algebra of the Standard Model Lie algebra, or of the Lie algebra for any other gauge group being discussed.

Since the rank of the Standard model (gauge)group is 4, there are of course 4 linearly independent Cartan algebra charges. But now we used in the reference [14] not linearly independent charges, but rather linear combinations of the Cartan algebra charges selected to have the special property, that for representations allowed for the Standard Model *group* these specially selected Cartan algebra charges had only the integer values and even in the usual Standard Model system of representations took only the values $-1, 0, \text{ or } 1$.

Let me explain the technique of our Costa Rica proceedings paper [14] a bit more:

Starting from assuming a gauge group with rank four say (but we really have in mind using a similar discussion on any potential gauge group, so that we also with those considerations could hope for approaching a derivation of an answer to why just the Standard Model) and deciding to consider only the Cartan algebra part, we would basically have assumed effectively an \mathbf{R}^4 gauge Lie algebra. But as a rudiment of as well the explicite charge quantization rule resulting from the *group* structure as from the charge quantizations caused by the non-abelian Lie algebra structure present before we threw the non-abelian parts away - only keeping the Cartan algebra - we would have quantization rules for the Cartan algebra charges. Indeed we would rather obtain an effective gauge *group* after this keeping nothing but the Cartan algebra being $U(1)^4$ than the here first mentioned \mathbf{R}^4 . This would mean that in the appropriate basis choice for these Cartan algebra charges they would all be restricted by the group structure to be integers. Making sums and/or differences of such "basis" charges restricted to be integers one can easily write down combinations which again would be restricted to have only integer charges.

But now the main question of interest in our earlier quantization of certain Cartan algebra charges were to implement the requirement/assumption of "small representations" or for Abelian equivalently "small charges".

We formulated the requirement of such "small charges" via defining a concept of a "Hahn Nambu charge". Such a type of charge, which we would denote as "Hahn Nambu charge", were by our definition assumed to obey:

- A "Hahn Nambu charge" should be one of the combinations of the Cartan algebra charges, which precisely were allowed to take on integers - no more no less - (due to the group structure of the $U(1)^4$ say for rank 4).
- But in the actual detailed model one should for the Hahn Nambu charge *only find the charge eigenvalues* $-1, 0, \text{ or } 1$. (This assumption is, one may say, an assumption of small charge values -for the Hahn Nambu charge type - in as far as the charge value numerically less than or equal to 1 is "small" compared to the quantization interval assumed just above to be 1).

Then instead of assuming in some other way, that we seek a model with the smallest possible charge values, we used then in [14] and [12] to say in stead - and crudely equivalently - that we should arrange so many "Hahn Nambu charges" to exist in the model to be sought as possible.

In order that the reader shall get an idea what type of charges these "Hahn Nambu charges" are, let me mention the Hahn Nambu charges of the Standard

Model:

$$\text{“HNred”} = y/2 + I_{W3} + \sqrt{3}\lambda_{8\text{red}} \quad (11.38)$$

$$\text{“HNblue”} = y/2 + I_{W3} + \sqrt{3}\lambda_{8\text{blue}} \quad (11.39)$$

$$\text{“HNyellow”} = y/2 + I_{W3} + \sqrt{3}\lambda_{8\text{yellow}} \quad (11.40)$$

$$\lambda_{2\text{yellow}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (11.41)$$

$$\lambda_{2\text{yellow}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (11.42)$$

$$\lambda_{2\text{red}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad (11.43)$$

$$\lambda_{2\text{blue}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (11.44)$$

$$\text{“Twice weak isospin } I_{W3}\text{”} = 2I_{W3} \quad (11.45)$$

Here we have used a notation, wherein the colors are listed in the series (“red”, “blue”, “yellow”) in columns and rows and defined the variously color-defined λ_8 -matrices:

$$\sqrt{3}\lambda_{8\text{red}} = \begin{pmatrix} -2/3 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/3 \end{pmatrix} \quad (11.46)$$

$$\sqrt{3}\lambda_{8\text{blue}} = \begin{pmatrix} 1/3 & 0 & 0 \\ 0 & -2/3 & 0 \\ 0 & 0 & 1/3 \end{pmatrix} \quad (11.47)$$

$$\sqrt{3}\lambda_{8\text{yellow}} = \begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & -2/3 \end{pmatrix} \quad (11.48)$$

It is easy to check that these 7 “Hahn Nambu charges” are related to each other by being sums or differences of each other, and also that they are indeed according to our definition indeed “Hahn Nambu charges” in the wellknown Standard Model. Indeed you should also see that the first three of them, “HNred”, “HNblue”, and “HNyellow” are indeed three color choices for what historically Hahn and Nambu proposed as the electric charge to be used in a QCD including model. Nowadays we know, that quarks only have electric charges $2/3$ or $-1/3$ fundamental charges, but the original Hahn Nambu charge were precisely constructed to have only the integer Millikan charge *even for the quarks*.

The crux of the calculation, we want to extract from the study of Hahn Nambu charges in our old works [14,12], is that imposing the no gauge anomaly conditions for the Cartan subalgebra, using the the assumption that we have as

many “Hahn Nambu charges” as possible still having a mass protected system of (Weyl)fermions, we are led to a system of representations which *is* indeed the usual one when extended to get the non-abelian charges too.

The technique we used in the old paper(s) [14] were in fact to study the no-anomaly constraint equations modulo 2, which for Hahn Nambu charges, that never take by assumption/definition charge values bigger than 1 numerically, close to be enough.

Actually it turned out that we could first find a system of mass-protected Weyl-fermions, when the dimension of the Cartan algebra (= the number of linearly independent Hahn Nambu charges) became at least 4. In that case then we had indeed to have a system of Weyl fermions, which modulo some trivial symmetries, had to be the one found experimentally w.r.t. these “Hahn Nambu charges”.

This should be interpreted to say, that requiring maximal numbers of “Hahn Nambu Charges” in our sense, which is a slightly special way of requiring small representations together with the assumptions of mass protection and no anomalies, leads to the Standard Model fermion system.

That is to say we should consider the structure of a family in the Standard Model to essentially come out of such requirements. In this way we can count the fermion system/spectrum as largely being a successful result coming out from a “Small representation principle”!

11.6 Speculations on the Full Group of Gauge Transformations and Diffeomorphism Symmetry

In the above discussion and in the previous articles in the present series of papers [2,1] we sought to find a game leading to the “gauge group”. But now we want to have in mind that the “gauge group” is not truly the most physical and simple concept in as far as the true symmetry in a gauge theory with “gauge group” G is really not truly G , but rather a cross product of one copy of G , say $G(x)$ for every point x in space time. That is to say the true symmetry group of the gauge theory having the “gauge group” G is rather $\times_x G(x) = G \times G \times \dots \times G$, where in the cross product it is supposed that we have one factor for every space time point x .

Above we saw that the goal quantity for a group were suggested to be of a type, that is balanced in such a way, that the score or goal quantity is the *same for a group G and for the cross product of this group with itself $G \times G \times \dots \times G \times G$ any number of times.*

This means of course, if as we found the Standard Model group $S(U(2) \times U(3))$ wins our game, then in fact any product of this group with itself any number of times can also be said to get just the same score, and thus it will also win! That it to say that we might reinterpret our work by saying: It is not truly the gauge group for the realized gauge theory we predict to be the winner. Rather we could say that the group that wins is the whole symmetry group of the full quantum field theory supposed to be realized. The concept of the full gauge symmetry (or we could say reformulation symmetry) is – we would say – a simpler concept than the concept of the “gauge group” for which it would have to be specified how this

gauge group would have to be applied, namely one should construct a group of all gauge transformations $\prod_x G(x) = G \times G \times \dots \times G$.

But since this full group gets just the same score as the more complicatedly defined “gauge group” we could claim that our prediction is, that it is this group of all the gauge transformations that gets the maximal score.

This would mean in some sense a slight simplification of our assumption.

11.6.1 Could we even predict the manifold?

Very speculatively - and with the success of predicting the dimension in mind - we could seek to argue that the group of gauge transformations $\prod_x G(x) = G \times G \times \dots \times G$ in some way could be claimed to represent a somewhat larger group than just this $\prod_x G(x) = G \times G \times \dots \times G$ in as far as we even on the same representation space of a direct sum of the representations F for the different points in space time could claim to represent *also a diffeomorphism group*. Since this diffeomorphism group shuffles around the direct sum of the F -type representations we could claim, that we managed to represent a group which is really the combination of the diffeomorphism group and the group of gauge transformations on just the same space of linear representations as the group of gauge transformations alone gets represented on as its “record (in our game) representation”. Intuitively this means that we have got an even bigger group relative to the representation than if we just represent the Standard Model group $S(U(2) \times U(3))$ on its F 's. Thus including such a diffeomorphism extension sounds like providing a superwinner superseding the formal winner itself the $S(U(2) \times U(3))$ (or its cross products with itself). So there is the hope that formulating the details appropriately we could arrange to get our true prediction become *the group of gauge transformations with the gauge group $S(U(2) \times U(3))$ extended with a diffeomorphism group*. If indeed in addition the dimension $d = 4$ for space time favoured by our game, because of its gauge group for general relativity, and thus hopefully the group of diffeomorphisms for just a four dimensional manifold would get exceptionally high score, it becomes very reasonable to expect that our game could predict just the right dimension of the manifold, on which the cross product of the standard model group with itself gets extended by the diffeomorphism symmetry.

This means that we are very close to have an argument that the most favoured symmetry group would precisely be the group of Standard Model gauge transformations extended by just a four dimensional diffeomorphism symmetry.

But if so, it would mean, that we had found a principle, a game, favouring precisely the group of gauge transformations found empirically.

Well, it must here be admitted a little caveat: The groups we considered to derive the dimension were the group of Lorentz transformation or Poincare transformations, *and not the full group of linear d -dimensional maps as would locally correspond to the diffeomorphism symmetry*. Thus one should presumably rather hope for our scheme to lead not to the full diffeomorphism symmetry as part of the winning symmetry group, but rather only that part of the diffeomorphism group, which does not shift the metric tensor $g_{\mu\nu}$. It would namely rather be this

subgroup of the diffeomorphism group that would locally be like the Spin(4) or SO(4) as we discussed in the dimension fitting.

But somehow this is presumably also rather what we should hope for to have a successful theory of ours.

“Going for” the Standard Model as were our starting point means that we really concentrated on only looking for the long wave length or practically accessible part of whatever the true theory for physics might be. This long wave length practical section should presumably be defined as what we can learn from few particle collisions with energies only up to about a few TeV. But in such few particle practical experiments we should not discover gravity and general relativity. We should only “see” the flat Minkowski space time and the Standard Model. But that should then mean that we should not truly “see” diffeomorphism group, but only some rudiments associated with the metric tensor leaving part of this group.

The ideal picture which we should hope to become the prediction in this low energy section philosophy should rather be that the geometrical symmetries are only the flat Poincare group combined with the full gauge group for the Standard Model.

11.7 Conclusion

The main point of the present article is the suggestion that in a way - that may have to be made a bit precise in the future/coming further work - a principle of “small representations” should be sufficient to imply a significant part of the details of the Standard Model. The real recently most important progress in the work with Don Bennett [2] is that it seems that even for the selection of the gauge group itself this selection of “small representations” is so important *that the very group is selected so as to in the appropriate way of counting have the smallest faithful representations*. That is to say the Standard Model gauge group should have been selected to be the model of Nature precisely, because it could cope with smaller representations, measured in our slightly specific way, than any other proposal for the gauge group (except for cross products of the Standard Model group with itself a number of times). This so successful specific way of measuring the “smallness” of the representations takes its outset from the (inverted) Dynkin index in the case of simple Lie groups: C_A/C_F . This is then averaged actually in the way that the logarithms of it is averaged weighted with the dimensions of the various simple groups in the cross product (and then we may of course reexponentiate if we want) and extended to the most natural analogue for the Abelian Lie-algebra parts, essentially replacing the C_A/C_F by e_A^2/e_F^2 meaning the charge square ratio for two representations analogous to the adjoint and the F ones.

The philosophy that taking outset in C_A/C_F with F, as we did, being chosen so as to maximize this ratio C_A/C_F can be considered assuming a principle of “small representations” is obvious. If we consider the adjoint representation quadratic Casimir C_A for the simple group under investigation as just a normalization - to have something to compare quadratic Casimirs of other representations to - maximizing our starting quantity C_A/C_F means really selecting a (simple) group according to how small faithful representations F one can find for it. So it is really

selecting the group with the smallest representations. Here of course then the concept of the size of the representation has been identified with the size of the quadratic Casimir, but that is at first a very natural identification and secondly, that were the one with which we had the success. It is also the quadratic Casimir, which is connected with natural metric on the space of unitary matrices in the representations. In fact our outset quantity C_A/C_F becomes the square of the ratio of the distance the unitary representation matrix moves for an infinitesimal motion of the group element in the adjoint and in the representation F , wherein by choice of F this distance is minimal. So our "goal quantity" which is the appropriate average of the ratio C_A/C_F and its extension to the Abelian parts becomes (essentially) the square of the volume of the representation space - in representations of the F 's - and the corresponding representation space using the adjoint representation or an analogue of adjoint space representation, if Abelian parts are present. But the crux of the matter is *a surprisingly large amount of details of the Standard Model including its Gauge group is determined from a requirement of essentially minimizing the quadratic Casimirs of the representations:*

- First the **gauge group** - and here we stress *group* - $S(U(2) \times U(3))$ of the Standard Model is *selected* by for our "goal quantity" (11.5) obtaining the highest score 2.95782451 which is rather tinnily, 0.0067, above the next (silver medal) (not being just a trivial cross product including the Standard Model itself), namely $U(2)$ (= standard model missing the strong interactions QCD) 2.95115179.
- **The dimension 4 for space time is also selected** by the Poincare group getting the highest score for approximately the same "goal quantity", which we used for the gauge group. It must be admitted though that we did not treat the Poincare group exactly - because it does not have the nice finite dimensional representations we would like to keep to have as strong similarity with the gauge group as possible - but instead made the trick of making some crude corrections starting from the Lorentz group. When using the Lorentz group dimension $d=3$ and $d=4$ stand equal. When we correct in reasonably "fair" ways the dimension $d=4$ (the experimental one for practical purposes in our notation that include the time) wins by having the highest corrected "goal quantity" for the Lorentz group, corrected to simulate the Poincare group. In this sense our principle, which is at the end a principle of small representations, point to the experimentally observed number of dimensions $d=4$.
- **The representation of the Higgs field** is when we use our "goal quantity" inspired way of defining in a very precise way numerically the smallest of the possible various irreducible representations to be the inverse of the this "goal quantity" for the factor group $G/K = S(U(2) \times U(3))/K$, for which the thought upon representation R is faithful. By this we just mean that we define K as the (invariant) subgroup, the elements of which are represented just by the unit matrix in the representation R . This we then in principle go through for all irreducible representations R for the Standard Model and ask for each possible R : what is the "goal quantity" for the corresponding $S(U(2) \times U(3))/K$ (here K depends on R of course) group. For $R = (2, 1, y/2 = 1/2)$ this factor group $S(U(2) \times U(3))/K$ turns out to be just $U(2)$ and score "goal quantity"

for the representation $R = (2, 1, y/2 = 1/2)$ is just that of the group $U(2)$ because it happens that the F for the $SU(2)$ inside $U(2)$ is just the 2 . Thus the quantity to determine to decide on the representation $R = (2, 1, y/2 = 1/2)$ becomes exactly the “goal quantity” _{$U(2)$} which we knew already were unbeatable (except if there should have been an irreducible representation faithful for the whole Standard Model group, but there is not). Thus assuming that the representation is smallest meaning, since “size” = $1/\text{“goal quantity”}$ for representations using our scheme, for predict representations the Higgs which is scalar and has no anomaly problems should be that representation that won $R = (2, 1, y/2 = 1/2)$, and that is precisely the representation of the Higgs!

- The Fermion representations all for mass protected Fermions (meaning that gauge symmetry would have to be broken, spontaneously by a Higgs presumably) in order for the Fermions to obtain nonzero masses. This makes them easily make anomalies in the gauge symmetries (charge conservations). In order that no anomalies really occur relations between the number of species of Fermions in various representations get severely restricted. Together with some requirement of “small representations” it looks rather suggestive, that the Standard Model system of particles in a family comes out just intuitively. In our article [14] we did an attempt to make the requirement of small representations precise in a quite different way than in the present article- but it were an attempt to assume small representations in some way at least -and we mainly worked with the Cartan algebra only. But the result was, that the Standard Model representations came out/were postdicted for the Cartan algebra at least.
- At the end we sought to change the point of view as to what group should be the one, that shall win the game of getting the largest “goal quantity” from being *the gauge group to be the group of all the gauge transformations*. Since it happens that we had balanced our “goal quantity” so much in order to avoid making the dimension of the group of much influence the value of this “goal quantity” had turned out to be exactly the same for a group and its cross product with itself, ever so many times. Since now the group of all gauge transformations is basically an infinite cross product of what we usually call the gauge group, it means that w.r.t. our competition selecting the gauge group or the group of gauge transformations makes no difference. So if we e.g. should think that the group of all the gauge transformations is a more fundamental and well defined concept, we are free to choose our scheme to select that group of gauge transformations rather than the gauge group. But if we are very speculatively optimistic we might find some argument that many cross product factors would occur and hope in the long run to get a kind of understanding of the gauge symmetry on a whole manifold to optimistically come out of our game. Perhaps extension of this point of view to the Lorentz (or crudely Poincare) group as gauge symmetry should in later work give a better way of arguing for the dimension of space time $d=4$, at the same time getting close to general relativity.

This series of ideas for points resulting from some principle or another, but presumably best by using our “goal quantity” (11.5), shows that such a type of principle is close to deriving a lot of the structure of the Standard Model: The gauge group, in the “group” included some quantization rule (11.10), the space time dimension, the Higgs representation, the fermion representations, and more doubtfully some argument that we have gauge symmetry at all.

In conclusion I think that this kind of principle - a precise making of a principle of “small representations” - could have a very good chance to explain a lot of the structure of the Standard Model and thereby of the physics structure, we see today!

11.7.1 Outlook and speculation on finestructure constants

If we take the above results of having success with “goal quantity” related to the representations F being in fact the representations of the Standard Model group ($y/2 = 1/2, \underline{2}, \underline{1}$) and ($y/2 = -1/3, \underline{1}, \underline{3}$) to mean that these two representations represent the dominant fields (for the gauge field on a lattice say), then it *happens* that we got an “important representation” being the direct sum of these two representations. This sum corresponds to the $\underline{5}$ of the $SU(5)$ in grand unification [19]. If we also took it, that the involvement of the natural measure on the representation space of unitary matrices in the definition of our successful goal quantity to mean, that we should use this distance measure on the representations to suggest the strength of the gauge couplings, we would end up with a simulated $SU(5)$ -unification prediction!

We hope that our scheme might suggest an *approximate* $SU(5)$ -relation between the couplings only, because we presumably even would if this should work at all for our kind of thinking rather at some fundamental/Planck scale than at an adjustable scale like in conventional Grand Unification. (We hope to return to our hopes of obtaining approximate $SU(5)$ coupling relations at the Planck scale in later works in which we should then take into account that there are also secondary representations in the series of our smallness and that how much they should contribute might be something we at least at first could start fitting and playing with).

Acknowledgement

I want to thank the Niels Bohr Institute for allowing me to stay as emeritus and for support of the travel to Bled, where this proceedings contribution were presented and the participants at the workshop there and in addition Svend Erik Rugh for helpful discussions. (Hope Svend might participate in next work on these ideas).

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