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Maximal order group actions on Riemann surfaces

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Abstract

A natural problem is to determine, for each value of the integer $g \geq 2$, the largest order of a group that acts on a Riemann surface of genus g. Let $N(g)$ (respectively $M(g)$) be the largest order of a group of automorphisms of a Riemann surface of genus $g \geq 2$ preserving the orientation (respectively possibly reversing the orientation) of the surface.

The basic inequalities comparing $N(g)$ and $M(g)$ are $N(g) \leq M(g) \leq 2N(g)$. There are well-known families of extended Hurwitz groups that provide an infinite number of integers g satisfying $M(g) = 2N(g)$. It is also easy to see that there are solvable groups which provide an infinite number of such examples.

We prove that, perhaps surprisingly, there are an infinite number of integers g such that $N(g) = M(g)$. Specifically, if p is a prime satisfying $p \equiv 1 \pmod 6$ and $g = 3p + 1$ or $g = 2p + 1$, there is a group of order $24(g-1)$ that acts on a surface of genus g preserving the orientation of the surface. For all such values of g larger than a fixed constant, there are no groups with order larger than $24(g-1)$ that act on a surface of genus g.

Keywords: Riemann surface, genus, group action, NEC group, strong symmetric genus.

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1 Introduction

A finite group G can be represented as a group of automorphisms of a compact Riemann surface. In most of the classical work, the group actions were required to preserve the orientation of the Riemann surface. It is also possible to allow the group actions to reverse the orientation of the surfaces.

Among the most interesting group actions for a particular value of the genus g are those such that the orders of the groups are "large" relative to the genus g. A natural problem,

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then, is to determine, for each value of the integer $g \geq 2$, the largest order of a group that acts on a Riemann surface of genus g.

Let $N(g)$ (respectively $M(g)$) be the largest order of a group of automorphisms of a Riemann surface of genus $g \geq 2$ preserving the orientation (respectively possibly reversing the orientation) of the surface. Now suppose the group G acts on the Riemann surface X of genus $g \geq 2$ (possibly reversing the orientation of X). Let G^+ be the subgroup of G consisting of the orientation preserving automorphisms. Then $|G^+| \leq N(g)$ and

$$|G| \leq 2|G^+| \leq 2N(g). \tag{1.1}$$

Consequently, we obtain the basic inequalities comparing $N(g)$ and $M(g)$.

$$N(g) \leq M(g) \leq 2N(g). \tag{1.2}$$

The classical upper bound of Hurwitz shows that, for all $g \geq 2$,

$$N(g) \leq 84(g-1) \text{ and } \mathrm{M(g) \leq 168(g-1)}. \tag{1.3}$$

A group G of order $84(g-1)$ is called a Hurwitz group if it acts on a surface of genus g preserving orientation. If the Hurwitz group has an extension G^* of order $2|G|$ that acts on the same surface, then G^* is an extended Hurwitz group. If g is a genus for which there is an extended Hurwitz group, then $N(g) = 84(g-1)$ and $M(g) = 2N(g)$. These groups have generated considerable interest; see especially [5] but also [3, 4] and [22]. There are known infinite families of extended Hurwitz groups. For example, Conder showed that all symmetric groups Σ_n for $n > 167$ are extended Hurwitz groups [8, p. 75]. Consequently, the bounds in (1.3) and the upper bound for $M(g)$ in (1.2) are attained for infinitely many g.

On the other hand, the general lower bound for $N(g)$ is

$$N(g) \geq 8(g+1) \tag{1.4}$$

for all $g \geq 2$. Further, this lower bound is the best possible, that is, there are infinitely many g such that $N(g) = 8(g+1)$. These results were established independently by Accola [1] and Maclachlan [13].

The corresponding lower bound for $M(g)$ is easy to establish.

Theorem 1.1. *For all integers $g \geq 2$, $M(g) \geq 16(g+1)$. Further, there are infinitely many g such that $M(g) = 16(g+1)$.*

The family of groups used by Accola [1] and Maclachlan [13] to establish the results about the lower bound (1.4) can be extended following the approach of Singerman [22, p. 22]. This yields, for each $g \geq 2$, the construction of a group of order $16(g+1)$ that acts on a Riemann surface of genus g so that $M(g) \geq 16(g+1)$.

These groups are another family of groups such that $M(g) = 2N(g)$ for infinitely many genera g. Indeed, intuitively, one expects $M(g)$ to "often" be equal to $2N(g)$. But it is certainly possible that $M(g) < 2N(g)$. For example, the two smallest values of g for which $M(g) < 2N(g)$ are 17 and 20 with $N(17) = 1344$, $M(17) = 1536$ and $N(20) = 228$, $M(20) = 336$. Also, the classification of orientably regular maps of genus $p+1$ [9] and the Belolipetsky-Jones group of order $12p$ for prime p [2, p. 382] shows that $M(g) < 2N(g)$ for infinitely g.

However, for some values of g, $N(g) = M(g)$. The two smallest values of g satisfying $N(g) = M(g)$ are 27 and 28 with $N(27) = M(27) = 624$ and $N(28) = M(28) = 1296$. Surprisingly, this equality holds for infinitely many g. Our main result is the following.

Theorem 1.2. *There are infinitely many g such that $M(g) = N(g)$.*

Specifically, if p is a prime satisfying $p \equiv 1 \pmod 6$ and $g = 3p + 1$, there is a group of order $24(g-1)$ that acts on a surface of genus g preserving the orientation of the surface. For all such values of $g = 3p + 1$ larger than a fixed constant, there are no groups with order larger than $24(g-1)$ that act on a surface of genus g (including those that reverse orientation). Similar results hold if $p \equiv 1 \pmod 6$ and $g = 2p + 1$.

Here we acknowledge our debt to the data on large group actions on surfaces of low genus calculated by Conder [6]. This data was quite helpful in conjecturing Theorem 5.6 and its corollary Theorem 1.2.

We would also like to express our sincere gratitude to the referee for numerous helpful comments. These led to significant improvements in the first three sections.

2 Background results

Much of the following background information is taken from [18]; also see [10, Section 2]. Let the finite group G act on the (compact) Riemann surface X of genus $g \geq 2$. Then represent $X = U/K$, where K is a Fuchsian surface group and obtain an NEC group Γ and a homomorphism $\phi\colon \Gamma \to G$ onto G such that $K = kernel\ \phi$. Associated with the NEC group Γ are its signature and canonical presentation.

Further, the non-euclidean area $\mu(\Gamma)$ of a fundamental region for Γ can be calculated directly from its signature. Here see [21, p.235], where $\mu(\Gamma)$ is given in terms of the topological genus of the quotient surface U/Γ and the periods and link periods of Γ. Then the genus of the surface X on which G acts is given by

$$g = 1 + |G| \cdot \mu(\Gamma)/4\pi. \tag{2.1}$$

The simpler, classical case is that G acts on X preserving orientation. This is the case if and only if Γ is a Fuchsian group and G is generated by elements a_i, b_i for $1 \leq i \leq h$ and x_j of order m_j for $1 \leq j \leq k$ with relation $x_1 \cdots x_k[a_1, b_1] \cdots [a_h, b_h] = 1$. Then the application of (2.1) yields the classical Riemann-Hurwitz equation

$$2g - 2 = |G| \left(2h - 2 + \sum_{j=1}^{k} \left(1 - \frac{1}{m_j} \right) \right). \tag{2.2}$$

The group G acts reversing the orientation of X in case Γ is a proper NEC group. Then it is necessary to check that the surface group K does not contain orientation-reversing elements, or equivalently, the image $\phi(\Gamma^+)$ has index two in G [20, Theorem 1, p. 52]. If this condition holds, then we will say that G has a particular partial presentation *with the Singerman subgroup condition*. The Riemann-Hurwitz equation in this case is more complicated and is in [10, p. 274], for instance. In this case, though, $|G| = 2|G^+|$ and (2.2) can be employed to calculate the relationship between the genus g and $|G|$.

In connection with group actions on surfaces, there are two natural parameters associated with each finite group. The *symmetric genus* $\sigma(G)$ of the group G is the minimum genus of any Riemann surface on which G acts faithfully (possibly reversing orientation). The *strong symmetric genus* $\sigma^0(G)$ of G is the minimum genus of any Riemann surface on which G acts faithfully preserving orientation.

Next we quickly survey the NEC groups with relatively small non-euclidean area. We use the notation of [18]. First, an (ℓ, m, n) triangle group is a Fuchsian group Λ with signature

$$(0; +; [\ell, m, n]; \{\}), \text{ where } 1/\ell + 1/m + 1/n < 1.$$

If the group G is a quotient of Λ by a surface group, then G has a presentation of the form

$$X^\ell = Y^m = (XY)^n = 1. \tag{2.3}$$

We will say that G has partial presentation $T(\ell, m, n)$.

There are two types of NEC groups with a triangle group as canonical Fuchsian subgroup. A full (or extended) (ℓ, m, n) triangle group is an NEC group Γ with signature

$$(0; +; [\,]; \{(\ell, m, n)\}), \text{ where } 1/\ell + 1/m + 1/n < 1.$$

If G is a quotient of Γ (by a surface group), then G has a presentation of the form

$$A^2 = B^2 = C^2 = (AB)^\ell = (BC)^m = (CA)^n = 1, \tag{2.4}$$

and, further, the subgroup generated by AB and BC (the image of Γ^+) has index 2. The partial presentation (2.4) will be denoted $FT(\ell, m, n)$.

A hybrid $(m; n)$ triangle group is an NEC group Γ with signature

$$(0; +; [m]; \{(n)\}), \text{ where } 2/m + 1/n < 1.$$

The canonical Fuchsian subgroup Γ^+ is a (m, m, n) triangle group. If G is a quotient of Γ, then G has a presentation of the form

$$C^2 = X^m = [C, X]^n = 1, \tag{2.5}$$

and the subgroup generated by X and CXC has index 2. This partial presentation will be denoted $HT(m; n)$.

An (ℓ, m, n, t) quadrilateral group is a Fuchsian group Λ with signature

$$(0; +; [\ell, m, n, t]; \{\ \}), \text{ where } 1/\ell + 1/m + 1/n + 1/t < 2.$$

A quotient group G of Λ has a presentation of the form

$$X^\ell = Y^m = Z^n = (XYZ)^t = 1 \tag{2.6}$$

We will denote this partial presentation $Q(\ell, m, n, t)$. If a group has presentation (2.6) with ℓ, m, n, t all equal to 2, then the group acts on a torus.

Suppose G is a group that acts on a Riemann surface X of genus $g \geq 2$, where X is represented $X = U/K$ and $G = \Gamma/K$. Particularly important here is the case in which $|G| > 24(g-1)$, and we will say G is a *large* group of automorphisms of X. There is, of course, a corresponding restriction on the non-euclidean area of the NEC group Γ and the types of partial presentations that Γ can have. The area restriction is $\mu(\Gamma)/2\pi < 1/12$, which is fairly limiting. A careful check of the signatures gives the following. This result appears in [18, Theorem 2] and also [10, p. 275]. Here we have added the specific Riemann-Hurwitz equation for each case. For example, if G has the partial presentation $FT(2, 4, s)$, then $\mu(\Gamma)/2\pi = (s-4)/8s$. Then using equation (2.2) gives $16(g-1) = |G|(s-4)/s$.

Theorem A. *Let G be a group that acts on a Riemann surface of genus $g \geq 2$. Then $|G| > 24(g-1)$ if and only if G has a partial presentation (with the relations fulfilled) of type $T(2,4,5)$ or $T(2,3,s)$, where $7 \leq s \leq 11$, or one of the following types with the Singerman subgroup condition satisfied. The application of the Riemann-Hurwitz equation is included for each case.*

1. $FT(2,3,s)$, $24(g-1) = |G|(s-6)/s$ *where* $s \geq 7$,
2. $FT(2,4,s)$, $16(g-1) = |G|(s-4)/s$ *where* $5 \leq s \leq 11$,
3. $FT(2,5,s)$, $20(g-1) = |G|(3s-10)/s$ *where* $5 \leq s \leq 7$,
4. $FT(3,3,s)$, $12(g-1) = |G|(s-3)/s$ *where* $4 \leq s \leq 5$,
5. $HT(3;4)$, $48(g-1) = |G|$,
6. $HT(3;5)$, $30(g-1) = |G|$,
7. $HT(5;2)$, $40(g-1) = |G|$.

3 Basic lower bound for $M(g)$

We begin by constructing the family of groups that provides the lower bound in Theorem 1.1.

Fix the integer $m \geq 3$, and let L_m be the group defined by the presentation

$$x^2 = y^4 = z^{2m} = xyz = 1, (z^2)^x = z^{-2}. \tag{3.1}$$

It is easy to see that L_m is an extension of the cyclic group Z_m by the dihedral group D_4 and consequently $|L_m| = 8m$. Then the group L_m has partial presentation $T(2,4,2m)$. Then a calculation using (2.1) shows that L_m acts on a Riemann surface X of genus $g = m-1$ preserving the orientation of the surface. The group L_m has order $8(g+1)$. This family of groups is certainly not new. The family L_m was used, independently, to establish the lower bound $8(g+1)$ by both Accola and Maclachlan; here see [1, p. 400] and [13, Theorem 4, p. 266]. The construction of this family also appears in [2, p. 384].

Next we construct an extension of the group L_m by Z_2, following the approach in [15, p. 128]. To L_m adjoin an element t of order 2 that transforms the elements of L_m according to the automorphism

$$\alpha(x) = x^{-1}, \alpha(y) = y^{-1}. \tag{3.2}$$

Then the extension L_m^* has presentation

$$t^2 = x^2 = y^4 = z^{2m} = xyz = (tx)^2 = (ty)^2 = 1, (z^2)^x = z^{-2}. \tag{3.3}$$

The extension L_m^* of L_m has order $2|L_m|$ and has partial presentation $FT(2,4,2m)$. Thus the group L_m^* is a group of order $16(g+1)$ that acts on the surface X of genus g. This extended family was described by Singerman in [22, p. 24]. Now it is easy to prove Theorem 1.1.

Proof. Fix $g \geq 2$, and set $m = g+1$. Then $M(g) \geq |L_m^*| = 16(g+1)$. Also, there are infinitely many values of g such that $N(g) = 8(g+1)$; here see [1, Theorem 4, p. 407] or [13, Theorem 5, p. 272]. Then for such a value of g, $M(g) \leq 2N(g) = 16(g+1)$ using the basic inequality (1.2). Hence there are infinitely many values of g such that $M(g) = 16(g+1)$. □

Before proving Theorem 1.2, we establish an interesting result about the family of groups L_m. We have seen that L_m acts preserving orientation on a Riemann surface of genus $g = m - 1$. In fact, this value is the strong symmetric genus of the group L_m.

First, we get rid of a redundant generator in the definition of L_m and obtain the presentation

$$x^2 = z^{2m} = (zx)^4 = 1, (z^2)^x = z^{-2}. \tag{3.4}$$

Theorem 3.1. $\sigma^0(L_m) = m - 1$.

Proof. Let L_m have generators x and z and relations (3.4) and be generated by u and v. Define $N = \langle z^2, (zx)^2\rangle$. The element $(zx)^2$ is in the center of L_m and since conjugation by x inverts z^2, N is a normal subgroup of L_m. Since $L_m/N \cong Z_2 \times Z_2$, uN, vN and uvN are the same as the cosets xN, zN and zxN in some order. All elements of L_m in the set xN have order 2 and all elements in the set zxN have order 4. The elements of the set zN are of the form z^k or $z^k(zx)^2$ and have order $2m/d$, where $d = gcd(k, 2m)$.

Let u be an element from xN and v an element from zxN. The product is contained in zN. So $uv = (xz)^2z^t = ((xz)^2z)^t$ or $uv = z^t$, where t is odd. Next, suppose the product uv has order smaller than $2m$. So $gcd(t, m) = d > 2$. Let $M = \langle (xz)^2, z^t\rangle$. Since $xzx = (xz)^2z^{-1}$, M is a normal subgroup of L_m of order $4m/d$. It follows that $\langle u, v\rangle = \langle u, uv\rangle \subseteq \langle u, M\rangle$ and since $|\langle u, M\rangle| = 8m/d \neq 8m$, the elements u and v do not generate L_m. Therefore, the product of two generators, u of order 2 and v of order 4, must have order $2m$.

Hence L_m has presentation $T(2, 4, 2m)$, and the corresponding triangle group is the only one that maps faithfully onto L_m.

Suppose that L_m has partial presentation $Q(2, 2, 2, 2)$ and acts on a torus. Let L_m be generated by involutions s, t, u and v satisfying $stuv = 1$. Let $N = \langle (zx)^2, z^2\rangle$. All elements of order 2 are either contained in the coset xN or are in the normal subgroup $V = \langle (zx)^2, z^m\rangle$ of L_m. Since there are no elements of order 2 in the coset zxN, an even number of s, t, u or v must be from xN. If none of the generators are from xN, then $\langle s, t, u, v\rangle \subseteq \langle z, N\rangle \neq L_m$. If all four generators are in xN, then $\langle s, t, u, v\rangle \subseteq \langle x, N\rangle \neq L_m$. Suppose that only two of the generators are from xN, say u and v. Then $\langle s, t, u, v\rangle \subseteq \langle x, V\rangle \neq L_m$ and again we get a contradiction. It also follows that $\sigma^0(L_m) \neq 1$ for all $m > 2$.

Suppose that L_m has partial presentation $Q(2, 2, 2, 3)$. Let L_m be generated by involutions s, t, u and the element v of order 3. The element v must be contained in $\langle z^2\rangle \subseteq N$. Since there are no elements of order 2 in the coset zxN and $(sN)(tN)(uN) = (1N)$, we can't have one of the cosets be xN and another be zN, since then the third would be in zxN. If one or more of s, t and u are in xN, then $\langle s, t, u, v\rangle \subseteq \langle x, N\rangle \neq L_m$. If one or more of s, t and u are in zN, then $\langle s, t, u, v\rangle \subseteq \langle z, N\rangle \neq L_m$. So L_m does not have presentation $\Gamma(2, 2, 2, 3)$. No other Fuchsian group has small enough non-euclidean area and the proof is complete. □

Theorem 3.1 shows that there is at least one group with strong symmetric genus g for all g, which is the main result of [16]. One interesting thing here is that the well-known groups of Theorem 3.1 provide an alternate proof of [16, Theorem 1], which was established using groups of the form $Z_k \times D_n$.

Theorem 3.1 also has a consequence for the function that counts the number of groups of each genus. Using direct products and dicyclic groups, it was shown that there are at least four groups of strong symmetric genus g for all $g \geq 0$ [14, Theorem 1]. It is not hard to see that the group L_{g+1} is another group of genus g, and we have the following.

Theorem 3.2. *If g is a non-negative integer, then there are at least 5 groups of strong symmetric genus g.*

We remark here that these families L_m and L_m^* are groups that act on the torus. These groups are in in Proulx classes (g) and (k) respectively; for the associated partial presentations, see [12, pp. 291,292]. The orientation preserving subgroup of the action of L_m^* on the torus is not L_m, even though L_m is the orientation preserving subgroup of the action of L_m^* on the surface of genus $m-1$. Consequently, the two families L_m and L_m^* are of no help in filling the symmetric genus spectrum. The groups $Z_k \times D_n$ used in [16] to fill all the gaps in the strong symmetric genus spectrum are also groups that act on the torus and have symmetric genus one.

4 A family of $24(g-1)$ automorphisms

Our main task here is to show there are infinitely many values of g such that $M(g) = N(g)$. This result was something of a surprise, to us at least, and it is not easy to prove.

We start with the construction of another family of groups. Let p be a prime satisfying $p \equiv 1 \pmod 6$ and m an integer satisfying $m^3 \equiv 1 \pmod p$ and not congruent to 1 $\pmod p$. Define the groups J_p by the presentation

$$x^3 = u^3 = v^2 = z^p = (uv)^4 = [x,u] = [x,v] = [z,u] = 1, \qquad (4.1)$$
$$z^x = z^m, z^v = z^{-1}.$$

It is easy to see that J_p is the semidirect product of the cyclic group Z_p by the group $Z_3 \times \Sigma_4$, namely $Z_p \times_\phi (Z_3 \times \Sigma_4)$ where ϕ is a homomorphism mapping x into $z \to z^m$, u into $z \to z$ and v into $z \to z^{-1}$.

Theorem 4.1. *The group J_p has partial presentation $T(2,3,12)$ and hence acts on a surface of genus $1+3p$.*

Proof. We will use the presentation (4.1). First, $o(v) = 2$, $o(ux) = 3$ and $o(vux) = 12$. In addition, $\langle v, ux\rangle = \langle x, u, v\rangle \cong Z_3 \times \Sigma_4$.

Define $r = vz$ and $w = z^{-1}ux$. Clearly, $o(r) = 2$. It is easy to verify that $z^{-k}x = xz^{-km}$. Therefore, $w^3 = (z^{-1}x)^3 = z^{-(m^2+m+1)} = 1$, since $m^2+m+1 \equiv 0 \pmod p$. It follows that $o(w) = 3$ and $o(rw) = 12$.

Next, we need to show that $J_p = \langle r, w\rangle$. First, $[r,w] = z^{-(m+1)}[v,u]$. Next, we show that $[r,w]^3 = z^{-3(m+1)}$. If $m \equiv -1 \pmod p$, then $m^3 \equiv -1 \pmod p$ and this is false. Therefore, $z \in \langle [r,w]\rangle$ and it follows that $\langle r,w\rangle = \langle z,x,u,v\rangle = J_p$. Thus J_p has partial presentation $T(2,3,12)$.

□

It is not difficult to see that, in fact, $\sigma^0(J_p) = 1+3p$. An obvious consequence of Theorem 4.1 is the following.

Theorem 4.2. *Let p be a prime such that $p \equiv 1 \pmod 6$, and let $g = 3p+1$. Then the group J_p is a group of order $24(g-1)$ that acts on a surface of genus g preserving the orientation of the surface. Consequently, for any such g,*

$$M(g) \geq N(g) \geq 24(g-1). \qquad (4.2)$$

There are, of course, infinitely many such g. We will show that, for most of these values of g, $M(g) = N(g) = 24(g-1)$, establishing Theorem 1.2.

5 Large groups of automorphisms

Assume p is a prime such that $p \equiv 1 \pmod 6$, and let $g = 3p + 1$. Then Theorem 4.2 shows that there is a group of order $24(g-1)$ that acts on a surface of genus g preserving the orientation of the surface, and inequality (4.2) holds. The hard part of the proof of Theorem 1.2 is to show that, for most of these values of g, there are no large groups of automorphisms, that is, no groups with order larger than $24(g-1)$. We use Theorem A. In this section we do not assume that $p \equiv 1 \pmod 6$. However, in the proof it is necessary to assume that that the prime p is not small. This will enable us to apply the following useful result of Accola [1, Lemma 5, p. 402].

Accola's Lemma. *Let G be a non-abelian image of the triangle group $T(2,3,\lambda)$ of order $\mu\lambda$. Then $\lambda \le \mu^2$.*

Let X be a Riemann surface of genus g, and suppose that G were a large group of automorphisms of X. Then $|G| > 24(g-1) = 72p$, and G has one of the partial presentations in Theorem A. We show that, in fact, G cannot have any of these partial presentations. While it is necessary to consider each presentation, we describe the overall outline of the argument but omit some details. In addition, to apply Accola's Lemma, it is necessary to assume that the prime p is not small, and we assume that $p > (36)^2$.

Lemma 5.1. *If the prime $p > (36)^2$, then p divides $|G|$ but p^2 does not.*

Proof. Suppose first that G has any of the partial presentations in Theorem A except $FT(2,3,s)$. In these cases, the Riemann-Hurwitz formulas in Theorem A give $|G|$ in terms of the parameter s, and for the values of s that can occur, $|G|$ is a multiple of p but p^2 does not divide $|G|$ (for large p). For example, suppose G has partial presentation $FT(2,5,s)$, where s is 5, 6 or 7. Then if s is 5 or 6, then G is $120p$ or $90p$, respectively, and p^2 does not divide $|G|$ if $p > 5$. If $s = 7$, then $|G|$ is not an integer.

Suppose now that G has partial presentation $FT(2,3,s)$ where $s \ge 7$. In this case, $|G| = 72ps/(s-6)$ so that $72ps = |G|(s-6)$. First, for small s, $7 \le s \le 12, s \ne 11$, $|G|$ is a multiple of p but p^2 does not divide $|G|$ (for large p). For example, if $s = 8$, $|G| = 288p$. If $s = 11$, $|G|$ is not an integer.

Assume then that G has partial presentation $FT(2,3,s)$ where $s > 12$, the hard case. Now by Euclid's Lemma, either p divides $|G|$ or p divides $(s-6)$.

Assume that p divides $(s-6)$ and write $s - 6 = mp$ for some integer $m \ge 1$. Now $s = mp + 6 > p > (36)^2$ (by assumption). But on the other hand, $|G| = 72ps/mp = 72s/m$. Then $|G^+| = 36s/m$. The group of orientation preserving automorphisms G^+ is a $T(2,3,s)$ group of order cs, where $c = 36/m \le 36$. Now by Accola's Lemma, $p < s \le c^2 \le (36)^2$, an obvious contradiction. Thus, if G is a $FT(2,3,s)$ group (and $p > (36)^2$), then p divides $|G|$.

Finally, we have $|G|/p = 72s/(s-6)$. With $s > 12$, $s/(s-6) < 2$ so that $|G|/p = 72s/(s-6) < 144$. Hence, p^2 does not divide $|G|$ for large p. □

Lemma 5.2. *The Sylow p-subgroup $S_p \cong Z_p$ of G is normal in G.*

Proof. A review of the calculations in the previous proof shows that in each case that is arithmetically possible, $|G| = cp$ for some constant $c < p$ for large p. The constant c depends on the presentation, of course. Now, obviously, $|S_p| = p$. Also, the number n_p of Sylow p-subgroups of G is $\equiv 1 \pmod p$ and is a divisor of $|G|$. Then $n_p \geq (p+1)$ is clearly not possible. Hence S_p is normal in G. □

Now let S_p act on X with $Y = X/S_p$ the quotient space, γ the genus of Y and $\pi\colon X \to Y$ the quotient map.

Lemma 5.3. *The quotient map π is unramified, and the quotient space $Y = X/S_p$ has genus $\gamma = 4$. Further, the quotient group $Q = G/S_p$ is a large group of automorphisms of Y.*

Proof. Let τ be the number of branch points of π. Then the Riemann-Hurwitz formula gives

$$2(g-1)/p = 2(\gamma - 1) + \tau(p-1)/p. \tag{5.1}$$

Then $2(g-1) = 2p(\gamma-1) + \tau(p-1)$ and we have $g - 1 = 3p$. Now $\tau(p-1) = 6p - 2p(\gamma - 1) = 2p(4-\gamma)$. Since $\tau(p-1) \geq 0$, $4 \geq \gamma$. If $\gamma = 4$, then $\tau = 0$. Assume $\gamma < 4$. Then $p-1$ divides $p(8-2\gamma)$. Since $p-1$ and p are relatively prime, $p-1$ divides $(8-2\gamma)$ so that $p - 1 \leq 8 - 2\gamma \leq 8$. Now $p \leq 9$ contradicting the assumption that p is large. Thus $\gamma = 4$ and the number of branch points $\tau = 0$, that is, the quotient map π is unramified.

Now the quotient group Q acts on the surface Y of genus 4. Since G is a large group of automorphisms of X, $|G| > 24(g-1) = 72p$. Then $|Q| = |G|/p > 72 = 24(4-1)$ and Q is a large group of automorphisms of Y. □

The large group actions on Riemann surfaces of genus 4 have been classified, and these are presented in Table 1. These group actions were considered in determining the groups of symmetric genus 4; here see [18, pp. 4089,4090] and [10, p. 285]. With a single exception, these actions correspond to groups of reflexible regular maps. A description of the connection between groups of regular maps and large groups of automorphisms of Riemann surfaces is in [17, p. 24]. The regular maps of genus 4 were first classified by Garbe [11, p. 53]. These maps also appear in [7, Table 1]. In Table 1, we give the group number in the MAGMA small groups library. Map symbols are from [7].

Table 1: Large Group Actions on Surfaces of Genus 4.

Group	Order	Library Number	Partial Presentation	Map Symbol	G/G′
$\Sigma_3 \times \Sigma_4$	144	183	$FT(2,3,12)$	R4.1	$(Z_2)^2$
$Z_2 \times \Sigma_5$	240	189	$FT(2,4,5)$	R4.2	$(Z_2)^2$
Σ_5	120	34	$T(2,4,5)$	R4.2	Z_2
	144	186	FT(2,4,6)	R4.3	$(Z_2)^3$
$D_4 \times D_5$	80	39	FT(2,4,10)	R4.4	$(Z_2)^3$
$Z_2 \times A_5$	120	35	FT(2,5,5)	R4.6	Z_2
Σ_5	120	34	$HT(5;2)$		Z_2

The group G is an extension of $S_p \cong Z_p$ by Q. Since $|Q|$ is relatively prime to p, the group G is a semidirect product, by the Schur-Zassenhaus Lemma.

Lemma 5.4. $G \cong Z_p \times_\phi Q$.

The following is important here. The proof is an exercise using the definition of semidirect product.

Lemma 5.5. *Let H be the semidirect product $K \times_\theta Q$, and let $L = kernel(\theta)$. Then L is normal in the big group H.*

Theorem 5.6. *Let p be a prime such that $p > (36)^2$. There are no large groups of automorphisms that act on a surface of genus $g = 3p + 1$.*

Proof. For each of the possibilities for Q, we show that G cannot have the relevant partial presentation.

First suppose there is a group G of order $144p$ with partial presentation $FT(2, 3, 12)$. In particular, G is generated by involutions. Then $G \cong Z_p \times_\phi Q$, where $Q \cong \Sigma_3 \times \Sigma_4$. Let $L = kernel(\phi)$. Since $\phi\colon Q \to Aut(Z_p) \cong Z_{p-1}$, Q/L is cyclic. It follows that $Q' \subset L \subset Q$. Now a calculation shows that the commutator quotient group $Q/Q' \cong (Z_2)^2$. Hence L must have index 1 or 2 in Q, and L is normal in G by Lemma 5.5. If $L = Q$, then $G \cong Z_p \times Q$. Then G is obviously not generated by involutions, since Z_p is not. Hence $[Q : L] = 2$ and the quotient group G/L has order $2p$ so that G/L is isomorphic to either Z_{2p} or the dihedral group D_p. Since Z_{2p} is not generated by involutions, we must have $G/L \cong D_p$. But D_p is not a quotient of a $FT(2, 3, 12)$ group (the product of reflections in D_p has order p or 1). Thus there is no group of order $144p$ with partial presentation $FT(2, 3, 12)$.

Essentially the same proof (using the same notation) shows that there are no groups of order $80p$ with presentation $FT(2, 4, 10)$ and also none of order $144p$ with presentation $FT(2, 4, 6)$. The only difference in each of the cases is that the commutator quotient group $Q/Q' \cong (Z_2)^3$. But it still follows that L has index 1 or 2 in Q.

The proof is very similar but even easier in case there were a group G of order $120p$ with presentation $FT(2, 5, 5)$. Then $G \cong Z_p \times_\phi Q$, where $Q \cong Z_2 \times A_5$. Now $Q' \cong A_5$ so that either $L = Q$ or $L = Q'$. Again, as in the previous cases, L has index 1 or 2 in Q, and it follows in the same way that this case is not possible either.

Now suppose there were such a group G with order $120p$ with partial presentation $T(2, 4, 5)$. Then G is generated by two elements of orders 2 and 5. Then $G \cong Z_p \times_\phi Q$, where $Q \cong \Sigma_5$. Then $Q' \cong A_5$, and thus either $L = Q$ or $L = Q'$. If $L = Q$, then $G \cong Z_p \times Q$. Then G is obviously not a quotient of a $T(2, 4, 5)$ group, since Z_p is not (Z_p has no elements of order 2). Hence $L = Q'$ and the quotient group G/L has order $2p$ so that G/L is isomorphic to either Z_{2p} or the dihedral group D_p. Neither group is a quotient of $T(2, 4, 5)$ groups; neither group has an element of order 5. Thus there is no group of order $120p$ with partial presentation $T(2, 4, 5)$.

Further, there is no group G of order $240p$ with partial presentation $FT(2, 4, 5)$. If there were such a group G acting on a surface X of genus $3p + 1$, then the group G^+ of orientation preserving automorphisms would be a $T(2, 4, 5)$ group acting on X. But we have just seen that this is not possible. Hence there is no $FT(2, 4, 5)$ group of order $240p$.

Finally, assume that there were a group G of order $120p$ with partial presentation $HT(5; 2)$. Then G is generated by two elements of orders 2 and 5, and $G \cong Z_p \times_\phi Q$, where $Q \cong \Sigma_5$. Now $Q' \cong A_5$, and either $L = Q$ or $L = Q'$. If $L = Q$, then $G \cong Z_p \times Q$. Then G is obviously not a quotient of a $HT(5; 2)$ group, since Z_p is not (Z_p has no elements of order 2). Hence $L = Q'$ and the quotient group G/L has order $2p$ so that G/L is isomorphic to either Z_{2p} or the dihedral group D_p. Neither of these groups is a quotient of a $HT(5;2)$ group; neither group has an element of order 5. Thus there is no group of order $120p$ with partial presentation $HT(5;2)$.

In summary, none of the partial presentations listed in Table 1 are possible, and there is no large group action on a surface of genus $g = 3p + 1$ for large $p > (36)^2$. □

Combining Theorems 4.2 and 5.6 gives the following.

Theorem 5.7. *Let p be a prime such that $p \equiv 1 \pmod 6$ and $p > (36)^2$, and let $g = 3p + 1$. Then for any such g,*

$$M(g) = N(g) = 24(g - 1). \tag{5.2}$$

Applying Dirichlet's Theorem about the number of primes in an arithmetic sequence establishes Theorem 1.2.

6 Another family of $24(g - 1)$ automorphisms

There is another interesting family of groups that can be used to determine an infinite sequence of odd values of g such that $M(g) = N(g)$. This provides an alternate proof of Theorem 1.2, and it may be established using arguments similar to those in the two previous sections. But there is no improvement to Theorem 1.2, of course, and technically the proof is somewhat harder. We describe this family of groups but only comment very briefly on the arguments in this case.

Let p be a prime satisfying $p \equiv 1 \pmod 6$ and m an integer satisfying $m^3 \equiv 1 \pmod p$ and not congruent to $1 \pmod p$. Define the groups K_p by the presentation

$$u^3 = v^2 = (uv)^3(u^{-1}v)^3 = z^p = 1, z^u = z^m, z^v = z^{-1}. \tag{6.1}$$

It is easy to see that K_p is the semidirect product of the cyclic group Z_p by the group P_{48}. The group P_{48} has order 48 and contains $SL(2,3)$ as a subgroup; a presentation is in [15, p. 116]. It is one of the groups of symmetric genus 2 [15, Theorem 4]. The group K_p has partial presentation $T(2,3,12)$ and acts on a surface of genus $1 + 2p$. This gives the following analog of Theorem 4.2.

Theorem 6.1. *Let p be a prime such that $p \equiv 1 \pmod 6$, and let $g = 2p + 1$. Then the group K_p is a group of order $24(g - 1)$ that acts on a surface of genus g preserving the orientation of the surface. Consequently, for any such g,*

$$M(g) \geq N(g) \geq 24(g - 1). \tag{6.2}$$

Using the approach (and notation) of Section 6, it is possible to show that there are no large groups of automorphisms for most of these values of g. The analog of Lemma 5.1 holds; it is necessary to assume that $p > (24)^2$ to apply Accola's result. Then the analog of Lemma 5.2 is easy to establish. The result corresponding to Lemma 5.3 holds with a similar proof. There is an important difference here, though. The quotient space Y has genus $\gamma = 3$. It is necessary, then, to consider the large group actions on Riemann surfaces of genus 3. These actions have been classified; see [18, p. 4089] and [10, p. 285]. The regular maps of genus 3 were classified by Sherk [19]; also see [7, Table 1]. There are 10 large group actions in all. Eight of these are map groups, but there are also two groups of 96 to consider, a $FT(3,3,4)$ group and a $HT(3;4)$ group. The analogs of Lemmas 5.4 and 5.5 continue to hold, as does the following companion to Theorem 5.6.

Theorem 6.2. *Let p be a prime such that $p > (24)^2$. There are no large groups of automorphisms that act on a surface of genus $g = 2p + 1$.*

It is necessary to consider the ten possibilities for the quotient group Q. In nine of the cases, as in the proof of Theorem 5.6, an argument using the commutator quotient group suffices; in these cases, Q/Q' is isomorphic to 1, Z_2, $(Z_2)^2$, or $(Z_2)^3$. The exceptional case is the one in which G is a group of order $96p$ with partial presentation $HT(3; 4)$. In this case, $Q/Q' \cong Z_6$; this case can be handled by considering the group G^+. Then it is not hard to show that there is no $T(3, 3, 4)$ group of order $48p$ and hence no $HT(3; 4)$ group of order $96p$.

Combining Theorems 6.1 and 6.2 gives the following.

Theorem 6.3. *Let p be a prime such that $p \equiv 1 \pmod 6$ and $p > (24)^2$, and let $g = 2p + 1$. Then for any such g,*

$$M(g) = N(g) = 24(g - 1). \tag{6.3}$$

Finally, it is worth noting that there are genera in which $N(g) = M(g)$ but the genus g does not have either the form $2p + 1$ or the form $3p + 1$ for a prime p. Two examples are genus 28 and genus 37.

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