ARS MATHEMATICA CONTEMPORANEA

ISSN 1855-3966 (printed edn.), ISSN 1855-3974 (electronic edn.)
ARS MATHEMATICA CONTEMPORANEA 22 (2022) #P2.02 / 207–216
https://doi.org/10.26493/1855-3974.2593.1b7
(Also available at http://amc-journal.eu)

Two-distance transitive normal Cayley graphs*

Jun-Jie Huang, Yan-Quan Feng †, Jin-Xin Zhou

School of mathematics and statistics, Beijing Jiaotong University, Beijing 100044, P. R. China

Received 3 April 2021, accepted 10 July 2021, published online 14 April 2022

Abstract

In this paper, we construct an infinite family of normal Cayley graphs, which are 2-distance-transitive but neither distance-transitive nor 2-arc-transitive. This answers a question proposed by Chen, Jin and Li in 2019.

Keywords: Cayley graph, 2*-distance-transitive graph, simple group.*

Math. Subj. Class. (2020): 05C25, 05E18, 20B25

1 Introduction

In this paper, all graphs are finite, simple, and undirected. For a graph Γ, let $V(\Gamma), E(\Gamma)$, $A(\Gamma)$ or $\mathrm{Aut}(\Gamma)$ denote its vertex set, edge set, arc set and its full automorphism group, respectively. The graph Γ is called G*-vertex-transitive*, G*-edge-transitive* or G*-arc-transitive*, with $G \leq \mathrm{Aut}(\Gamma)$, if G is transitive on $V(\Gamma), E(\Gamma)$ or $A(\Gamma)$ respectively, and G*-semisymmetric*, if Γ is G-edge-transitive but not G-vertex-transitive. It is easy to see that a G-semisymmetric graph Γ must be bipartite such that G has two orbits, namely the two parts of Γ, and the stabilizer G_u for any $u \in V(\Gamma)$ is transitive on the neighbourhood of u in Γ. An *s-arc* of Γ is a sequence $v_0, v_1, \ldots, v_s$ of $s+1$ vertices of Γ such that v_{i-1}, v_i are adjacent for $1 \leq i \leq s$ and $v_{i-1} \neq v_{i+1}$ for $1 \leq i \leq s-1$. If Γ has at least one s-arc and $G \leq \mathrm{Aut}(\Gamma)$ is transitive on the set of s-arcs of Γ, then Γ is called (G, s)*-arc-transitive*, and Γ is said to be *s-arc-transitive* if it is $(\mathrm{Aut}(\Gamma), s)$-arc-transitive.

For two vertices u and v in $V(\Gamma)$, the *distance* $d(u, v)$ between u and v in Γ is the smallest length of paths between u and v, and the *diameter* $\mathrm{diam}(\Gamma)$ of Γ is the maximum distance occurring over all pairs of vertices. For $i = 1, 2, \ldots, \mathrm{diam}(\Gamma)$, denote by $\Gamma_i(u)$

*The work was supported by the National Natural Science Foundation of China (11731002, 12011540376, 12011530455, 12071023, 12161141005) and the 111 Project of China (B16002).

†Corresponding author.

E-mail addresses: 20118006@bjtu.edu.cn (Jun-Jie Huang), yqfeng@bjtu.edu.cn (Yan-Quan Feng), jxzhou@bjtu.edu.cn (Jin-Xin Zhou)

the set of vertices at distance i with vertex u in Γ. A graph Γ is called *distance transitive* if, for any vertices u, v, x, y with $d(u, v) = d(x, y)$, there exists $g \in \mathrm{Aut}(\Gamma)$ such that $(u, v)^g = (x, y)$. The graph Γ is called (G, t)*-distance-transitive* with $G \leq \mathrm{Aut}(\Gamma)$ if, for each $1 \leq i \leq t$, the group G is transitive on the ordered pairs of form (u, v) with $d(u, v) = i$, and Γ is said to be t*-distance-transitive* if it is $(\mathrm{Aut}(\Gamma), t)$-distance-transitive.

Distance-transitive graphs were first defined by Biggs and Smith in [2], and they showed that there are only 12 trivalant distance-transitive graphs. Later, distance-transitive graphs of valencies 3, 4, 5, 6 and 7 were classified in [2, 10, 14, 25], and a complete classification of distance-transitive graphs with symmetric or alternating groups of automorphisms was given by Liebeck, Praeger and Saxl [18]. The 2-distance-transitive but not 2-arc-transitive graphs of valency at most 6 were classified in [4, 16], and the 2-distance-primitive graphs (a vertex stabilizer of automorphism group is primitive on both the first step and the second step neighbourhoods of the vertex) with prime valency were classified in [15]. By definition, a 2-arc-transitive graph is 2-distance-transitive, but a 2-distance-transitive graph may not be 2-arc-transitive; an example is the Kneser graph $KG_{6,2}$, see [16]. Furthermore, Corr, Jin and Schneider [5] investigated properties of a connected $(G, 2)$-distance-transitive but not $(G, 2)$-arc-transitive graph of girth 4, and they applied the properties to classify such graphs with prime valency. For more information about 2-distance-transitive graphs, we refer to [6, 7].

For a finite group G and a subset $S \subseteq G \setminus \{1\}$ with $S = S^{-1} := \{s^{-1} \mid s \in S\}$, the *Cayley graph* $\mathrm{Cay}(G, S)$ of the group G with respect to S is the graph with vertex set G and with two vertices g and h adjacent if $hg^{-1} \in S$. For $g \in G$, let $R(g)$ be the permutation of G defined by $x \mapsto xg$ for all $x \in G$. Then $R(G) := \{R(g) \mid g \in G\}$ is a regular group of automorphisms of $\mathrm{Cay}(G, S)$. It is known that a graph Γ is a Cayley graph of G if and only if Γ has a regular group of automorphisms on the vertex set which is isomorphic to G; see [1, Lemma 16.3] and [24]. A Cayley graph $\Gamma = \mathrm{Cay}(G, S)$ is called *normal* if $R(G)$ is a normal subgroup of $\mathrm{Aut}(\Gamma)$. The study of normal Cayley graphs was initiated by Xu [27] and has been investigated under various additional conditions; see [8, 22].

There are many interesting examples of arc-transitive graphs and 2-arc-transitive graphs constructed as normal Cayley graphs. However, the status for 2-distance-transitive graphs is different. Recently, 2-distance-transitive circulants were classified in [3], where the following question was proposed:

Question 1.1 ([3, Question 1.2])**.** Is there a normal Cayley graph which is 2-distance-transitive, but neither distance-transitive nor 2-arc-transitive?

In this paper, we answer the above question by constructing an infinite family of such graphs, which are Cayley graphs of the extraspecial p-groups of exponent p of order p^3.

Theorem 1.2. *For an odd prime p, let* $G = \langle a, b, c \mid a^p = b^p = c^p = 1, [a, b] = c, [c, a] = [c, b] = 1\rangle$ *and* $S = \{a^i, b^i \mid 1 \leq i \leq p-1\}$*. Then* $\mathrm{Cay}(G, S)$ *is a* 2*-distance-transitive normal Cayley graph that is neither distance-transitive nor* 2*-arc-transitive.*

A *clique* of a graph Γ is a maximal complete subgraph, and the *clique graph* Σ of Γ is defined to have the set of all cliques of Γ as its vertex set with two cliques adjacent in Σ if the two cliques have at least one common vertex. Applying Theorem 1.2, we can obtain the following corollary.

Corollary 1.3. *Under the notation given in* Theorem 1.2*, let* $\mathrm{Cos}(G, \langle a\rangle, \langle b\rangle)$ *be the graph with vertex set* $\{\langle a\rangle g \mid g \in G\} \cup \{\langle b\rangle h \mid h \in G\}$ *and with edges all these coset pairs*

$\{\langle a\rangle g, \langle b\rangle h\}$ having non-empty intersection in G. Then $\mathrm{Cos}(G, \langle a\rangle, \langle b\rangle)$ is the clique graph of $\mathrm{Cay}(G, S)$, and $\mathrm{Cay}(G, S)$ is the line graph of $\mathrm{Cos}(G, \langle a\rangle, \langle b\rangle)$. Furthermore, $\mathrm{Cos}(G, \langle a\rangle, \langle b\rangle)$ is 3-arc-transitive.

The graph $\mathrm{Cos}(G, \langle a\rangle, \langle b\rangle)$ was first constructed in [19] as a regular cover of $\mathsf{K}_{p,p}$, where it is said that $\mathrm{Cos}(G, \langle a\rangle, \langle b\rangle)$ is 2-arc-transitive in [19, Theorem 1.1], but not 3-arc-transitive generally for all odd primes p in a remark after [19, Example 4.1]. However, this is not true and Corollary 1.3 implies that $\mathrm{Cos}(G, \langle a\rangle, \langle b\rangle)$ is always 3-arc-transitive for each odd prime p. In fact, $\mathrm{Cos}(G, \langle a\rangle, \langle b\rangle)$ is 3-arc-regular, that is, $\mathrm{Aut}(\mathrm{Cos}(G, \langle a\rangle, \langle b\rangle))$ is regular on the set of 3-arcs of $\mathrm{Cos}(G, \langle a\rangle, \langle b\rangle)$. Some more information about the structure and symmetry properties of $\mathrm{Cos}(G, \langle a\rangle, \langle b\rangle)$ are given in Lemma 3.2.

2 Preliminaries

In this section we list some preliminary results used in this paper. The first one is the well-known orbit-stabilizer theorem (see [9, Theorem 1.4A]).

Proposition 2.1. *Let G be a group with a transitive action on a set Ω and let $\alpha \in \Omega$. Then $|G| = |\Omega||G_\alpha|$.*

The well-known Burnside $p^a q^b$ theorem was given in [12, Theorem 3.3].

Proposition 2.2. *Let p and q be primes and let a and b be positive integers. Then a group of order $p^a q^b$ is soluble.*

The next proposition is an important property of a non-abelian simple group acting transitively on a set with cardinality a prime-power, whose proof depends on the finite simple group classification, and we refer to [13, Corollary 2] or [26, Proposition 2.4].

Proposition 2.3. *Let T be a nonabelian simple group acting transitively on a set Ω with cardinality a p-power for a prime p. If p does not divide the order of a point-stabilizer of T, then T acts 2-transitively on Ω.*

Let $\Gamma = \mathrm{Cay}(G, S)$ be a Cayley graph of a group G with respect to S. Then $R(G)$ is a regular subgroup of $\mathrm{Aut}(\Gamma)$, and $\mathrm{Aut}(G, S) := \{\alpha \in \mathrm{Aut}(G) \mid S^\alpha = S\}$ is also a subgroup of $\mathrm{Aut}(\Gamma)$, which fixes 1. Furthermore, $R(G)$ is normalized by $\mathrm{Aut}(G, S)$, and hence we have a semiproduct $R(G) \rtimes \mathrm{Aut}(G, S)$, where $R(g)^\alpha = R(g^\alpha)$ for any $g \in G$ and $\alpha \in \mathrm{Aut}(G, S)$. Godsil [11] proved that the semiproduct $R(G) \rtimes \mathrm{Aut}(G, S)$ is in fact the normalizer of $R(G)$ in $\mathrm{Aut}(\Gamma)$. By Xu [27], we have the following proposition.

Proposition 2.4. *Let $\Gamma = \mathrm{Cay}(G, S)$ be a Cayley graph of a finite group G with respect to S, and let $\mathsf{A} = \mathrm{Aut}(\Gamma)$. Then the following hold:*

(1) $N_{\mathsf{A}}(R(G)) = R(G) \rtimes \mathrm{Aut}(G, S)$;

(2) *Γ is a normal Cayley graph if and only if $\mathsf{A}_1 = \mathrm{Aut}(G, S)$, where A_1 is the stabilizer of 1 in A.*

Let Γ be a G-vertex-transitive graph, and let N be a normal subgroup of G. The *normal quotient graph Γ_N* of Γ induced by N is defined to be the graph with vertex set the orbits of N and with two orbits B, C adjacent if some vertex in B is adjacent to some vertex in C in Γ. Furthermore, Γ is called a *normal N-cover* of Γ_N if Γ and Γ_N have the same valency.

Proposition 2.5. *Let Γ be a connected G-vertex-transitive graph and let N be a normal subgroup of G. Suppose that either Γ is an N-cover of Γ_N, or Γ is G-arc-transitive of prime valency and N has at least three orbits on vertices. Then the following statements hold:*

(1) *N is semiregular on $V\Gamma$ and is the kernel of G acting $V(\Gamma_N)$, so $G/N \leq \mathrm{Aut}(\Gamma_N)$;*

(2) *Γ is (G, s)-arc-transitive if and only if Γ_N is $(G/N, s)$-arc-transitive;*

(3) *$G_\alpha \cong (G/N)_\delta$ for any $\alpha \in V\Gamma$ and $\delta \in V(\Gamma_N)$.*

Proposition 2.5 was given in many papers by replacing the condition that Γ is a normal N-cover of Γ_N by one of the following assumptions: (1) N has at least 3-orbits and G is 2-arc-transitive (see [21, Theorem 4.1]); (2) N has at least 3-orbits, G is arc-transitive and Γ has a prime valency (see [20, Theorem 2.5]); (3) N has at least 3-orbits and G is locally primitive (see [17, Lemma 2.5]). The first step for these proofs is to show that for any two vertices $B, C \in V(\Gamma_N)$, the induced subgraph $[B]$ of B in Γ has no edge and if B and C are adjacent in Γ_N then the induced subgraph $[B \cup C]$ in Γ is a matching, which is equivalent to that Γ is a normal N-cover of Γ_N. Then Proposition 2.5(1) - (3) follows from these proofs.

3 Proof Theorem 1.2

For a positive integer n and a prime p, we use $\mathbb{Z}_n$ and $\mathbb{Z}_p^r$ to denote the cyclic group of order n and the elementary abelian group of order p^r, respectively. In this section, we always assume that p is an odd prime, and denote by $\mathbb{Z}_p^*$ the multiplicative group of $\mathbb{Z}_p$ consisting of all non-zero numbers in $\mathbb{Z}_p$. Note that $\mathbb{Z}_p^* \cong \mathbb{Z}_{p-1}$. Furthermore, we also set the following assumptions in this section:

$$G = \langle a, b, c \mid a^p = b^p = c^p = 1, [a, b] = c, [c, a] = [c, b] = 1\rangle,$$
$$S = \{a^i, b^i \mid 1 \leq i \leq p-1\},$$
$$\Gamma = \mathrm{Cay}(G, S),\ \ \mathsf{A} = \mathrm{Aut}(\Gamma),\ \ N = N_{\mathsf{A}}(R(G)) = R(G) \rtimes \mathrm{Aut}(G, S),\ \text{ and } \mathbb{Z}_p^* = \langle t\rangle.$$

By Proposition 2.4, $N_{\mathsf{A}}(R(G)) = R(G) \rtimes \mathrm{Aut}(G, S)$, and $R(g)^\delta = R(g^\delta)$ for any $R(g) \in R(G)$ and $\delta \in \mathrm{Aut}(G, S)$. Since $G = \langle S\rangle$, Γ is a connected Cayley graph of valency $2(p-1)$. Let

$$\begin{array}{llll} \alpha\colon a \longmapsto a^t, & b \longmapsto b, & c \longmapsto c^t; \\ \beta\colon a \longmapsto a, & b \longmapsto b^t, & c \longmapsto c^t; \\ \gamma\colon a \longmapsto b, & b \longmapsto a, & c \longmapsto c^{-1}. \end{array}$$

It is easy to check that a^t, b, c^t satisfy the same relations as a, b, c in G, that is, $[a^t, b] = c^t$, $[c^t, a^t] = [c^t, b] = 1$. By the von Dyck's Theorem (see [23, 2.2.1]), α induces an epimorphism from G to $\langle a^t, b, c^t\rangle$, which must be an automorphism of G because $\langle a^t, b, c^t\rangle = G$. Similarly, β and γ are also automorphisms of G.

Lemma 3.1. $\mathrm{Aut}(G, S) = \langle \alpha, \beta, \gamma\rangle \cong (\mathbb{Z}_{p-1} \times \mathbb{Z}_{p-1}) \rtimes \mathbb{Z}_2$*, and Γ is N-arc-transitive. Furthermore, N has no normal subgroup of order p^2.*

Proof. Since $\mathbb{Z}_p^* = \langle t \rangle$, it is easy to check that $\alpha^{p-1} = \beta^{p-1} = \gamma^2 = 1$, $\alpha\beta = \beta\alpha$ and $\alpha^\gamma = \beta$. Thus $\langle \alpha, \beta, \gamma \rangle \cong (\mathbb{Z}_{p-1} \times \mathbb{Z}_{p-1}) \rtimes \mathbb{Z}_2$. Clearly, $\alpha, \beta, \gamma \in \mathrm{Aut}(G,S)$. To prove $\mathrm{Aut}(G,S) = \langle \alpha, \beta, \gamma \rangle \cong (\mathbb{Z}_{p-1} \times \mathbb{Z}_{p-1}) \rtimes \mathbb{Z}_2$, it suffices to show that $|\mathrm{Aut}(G,S)| \leq 2(p-1)^2$.

Clearly, $\langle \alpha, \beta, \gamma \rangle$ is transitive on S, and hence Γ is N-arc-transitive. Since $G = \langle S \rangle$, $\mathrm{Aut}(G,S)$ is faithful on S. By Proposition 2.1, $|\mathrm{Aut}(G,S)| = |S||\mathrm{Aut}(G,S)_a|$, where $\mathrm{Aut}(G,S)_a$ is the stabilizer of a in $\mathrm{Aut}(G,S)$. Note that $\mathrm{Aut}(G,S)_a$ fixes a^i for each $1 \leq i \leq p-1$. Again by Proposition 2.1, $|\mathrm{Aut}(G,S)_a| \leq (p-1)|\mathrm{Aut}(G,S)_{a,b}|$, where $\mathrm{Aut}(G,S)_{a,b}$ is the subgroup of $\mathrm{Aut}(G,S)$ fixing a and b. Since $G = \langle a, b \rangle$, we obtain $\mathrm{Aut}(G,S)_{a,b} = 1$, and then $|\mathrm{Aut}(G,S)| \leq 2(p-1)^2$, as required.

Let $H \leq N$ be a subgroup of order p^2. Since $R(G)$ is the unique normal Sylow p-subgroup of $N = R(G) \rtimes \mathrm{Aut}(G,S)$, we have $H \leq R(G)$, and since $|R(G) : H| = p$, we have $H \trianglelefteq R(G)$. Note that the center $C := Z(R(G)) = \langle R(c) \rangle$ and $C \cap H \neq 1$. Thus, $C \cap H = C$ as $|C| = p$, implying $C \leq H$. Since H/C is a subgroup of order p, and $R(G)/C = \langle R(a)C \rangle \times \langle R(b)C \rangle \cong \mathbb{Z}_p^2$, we have $H/C = \langle R(b)C \rangle$ or $\langle R(a)R(b)^iC \rangle$ for some $0 \leq i \leq p-1$. It follows that $H = \langle R(b) \rangle \times C$ or $\langle R(ab^i) \rangle \times C$ for some $0 \leq i \leq p-1$.

Suppose $H \trianglelefteq N$. Since C is characteristic in $R(G)$ and $R(G) \trianglelefteq N$, we have $C \trianglelefteq N$. Recall that $R(a)^\gamma = R(a^\gamma) = R(b)$. Then $(\langle R(a) \rangle \times C)^\gamma = \langle R(b) \rangle \times C$. This implies that both $\langle R(a) \rangle \times C$ and $\langle R(b) \rangle \times C$ are not normal in N. Thus, $H = \langle R(ab^i) \rangle \times C$ for some $1 \leq i \leq p-1$. Since $H \trianglelefteq N$, we have $H^\beta = H$, that is, $\langle R(ab^{ti}) \rangle \times C = H^\beta = H = \langle R(ab^i) \rangle \times C$. It follows that $\langle R(ab^{ti}) \rangle = \langle R(ab^i) \rangle$ and then $R(ab^{ti}) = R(ab^i)$, which further implies $b^{ti} = b^i$. This gives rise to $p \mid i(t-1)$, and since $(i,p) = 1$, we have $t = 1$, contradicting that $\mathbb{Z}_p^* = \langle t \rangle \cong \mathbb{Z}_{p-1}$. Thus, N has no normal subgroup of order p^2. □

For a positive integer n, n_p denotes the largest p-power diving n. By Lemma 3.1, $\Gamma = \mathrm{Cay}(G,S)$ is N-arc-transitive.

Lemma 3.2. *The clique graph Σ of Γ is a connected p-valent bipartite graph of order $2p^2$, A has a faithful natural action on Σ, and Σ is $R(G)$-semisymmetric and N-arc-transitive. Furthermore, $|\mathsf{A}|_p = p^3$.*

Proof. Recall that $G = \langle a, b, c \mid a^p = b^p = c^p = 1, [a,b] = c, [c,a] = [c,b] = 1 \rangle$ and $S = \{a^i, b^i \mid 1 \leq i \leq p-1\}$. Then $\Gamma = \mathrm{Cay}(G,S)$ has exactly two cliques passing through 1, that is, the induced subgraphs of $\langle a \rangle$ and $\langle b \rangle$ in Γ. Since $R(G) \leq \mathrm{Aut}(\Gamma)$ is transitive on vertex set, each clique of Γ is an induced subgraph of the coset $\langle a \rangle x$ or $\langle b \rangle x$ for some $x \in G$. Thus, we may view the vertex set of Σ as $\{\langle a \rangle x, \langle b \rangle x \mid x \in G\}$ with two cosets adjacent in Σ if they have non-empty intersection. It is easy to see that $\langle a \rangle x \cap \langle b \rangle y \neq \emptyset$ if and only if $|\langle a \rangle x \cap \langle b \rangle y| = 1$, and any two distinct cosets, either in $\{\langle a \rangle x \mid x \in G\}$ or in $\{\langle b \rangle x \mid x \in G\}$, have empty intersection. Furthermore, $\langle a \rangle$ has non-empty intersection with exactly p cosets, that is, $\langle b \rangle a^i$ for $0 \leq i \leq p-1$. Thus, Σ is a p-valent bipartite graph of order $2p^2$. The connectedness of Σ follows from that of Γ.

Clearly, A has a natural action on Σ. Let K be the kernel of A on Σ. Then K fixes each coset of $\langle a \rangle x$ and $\langle b \rangle x$ for all $x \in G$. Since $\langle a \rangle x \cap \langle b \rangle x = \{x\}$, K fixes x and hence $K = 1$. Thus, A is faithful on Σ and we may let $\mathsf{A} \leq \mathrm{Aut}(\Sigma)$.

Note that $R(G)$ is not transitive on $\{\langle a \rangle x, \langle b \rangle x \mid x \in G\}$, but transitive on $\{\langle a \rangle x \mid x \in G\}$ and $\{\langle b \rangle x \mid x \in G\}$. Furthermore, $R(\langle a \rangle)$ fixes $\langle a \rangle$ and is transitive on $\{\langle b \rangle a^i \mid 0 \leq i \leq p-1\}$, the neighbourhood of $\langle a \rangle$ in Σ, and similarly, $R(\langle b \rangle)$ fixes $\langle b \rangle$

and is transitive on the neighbourhood $\{\langle a\rangle b^i \mid 0 \le i \le p-1\}$ of $\langle b\rangle$ in Σ. It follows that Σ is $R(G)$-semisymmetric. Recall that $N = R(G) \rtimes \mathrm{Aut}(G,S)$ and $\mathrm{Aut}(G,S) = \langle \alpha, \beta, \gamma\rangle$. Since $a^\gamma = b$ and $b^\gamma = a$, γ interchanges $\{\langle a\rangle x \mid x \in G\}$ and $\{\langle b\rangle x \mid x \in G\}$. This yields that Σ is $R(G) \rtimes \langle\gamma\rangle$-arc-transitive and hence N-arc-transitive.

Since Σ is a connected graph with prime valency p, we have $p^2 \nmid |\mathrm{Aut}(\Sigma)_u|$ for any $u \in V(\Sigma)$, and in particular, $p^2 \nmid |\mathsf{A}_u|$. Note that $p \mid |\mathsf{A}_u|$. By Proposition 2.1, $|\mathsf{A}| = |\Sigma||\mathsf{A}_u| = 2p^2|\mathsf{A}_u|$. This implies that $|\mathsf{A}|_p = p^3$. □

Lemma 3.3. $\mathsf{A} = \mathrm{Aut}(\Gamma) = R(G) \rtimes \mathrm{Aut}(G,S)$.

Proof. By Lemma 3.2, $|\mathsf{A}|_p = p^3$, and since $|V(\Gamma)| = p^3$ and A is vertex-transitive on $V(\Gamma)$, the vertex stabilizer A_1 is a p'-group, that is, $p \nmid |\mathsf{A}_1|$. To prove the lemma, by Proposition 2.4 we only need to show that $R(G) \trianglelefteq \mathsf{A}$, and since $R(G)$ is a Sylow p-subgroup of A, it suffices to show that A has a normal Sylow p-subgroup.

Let M be a minimal normal subgroup of A. Then $M = T_1 \times T_2 \cdots \times T_d$, where $T_i \cong T$ for each $1 \le i \le d$ with a simple group T. Since $|V(\Gamma)| = p^3$, each orbit of M has length a p-power and hence each orbit of T_i has length a p-power. It follows that $p \mid |T|$. Assume that $|T|_p = p^\ell$. Then $|M|_p = p^{d\ell}$ and $d\ell = 1, 2$ or 3 as $|\mathsf{A}|_p = p^3$.

We process the proof by considering the two cases: M is insoluble or soluble.

Case 1: M is insoluble.

In this case, T is a non-abelian simple group. We prove that this case cannot happen by deriving contradictions. Recall that $d\ell = 1, 2$ or 3.

Assume that $d\ell = 1$. Then $|M|_p = p$. By Lemma 3.2, $M \trianglelefteq \mathsf{A} \le \mathrm{Aut}(\Sigma)$, and since $|V(\Sigma)| = 2p^2$, M has at least three orbits. Since Σ has valency p, Proposition 2.5 implies that M is semiregular on $V(\Sigma)$ and hence $|M| \mid 2p^2$. By Proposition 2.2, M is soluble, a contradiction.

Assume that $d\ell = 2$. Since $R(G)$ is a Sylow p-subgroup of A and $M \trianglelefteq \mathsf{A}$, $R(G) \cap M$ is a Sylow p-subgroup of M and hence $|R(G) \cap M| = |M|_p = p^2$. Since $R(G) \trianglelefteq N$ and $M \trianglelefteq \mathsf{A}$, $M \cap R(G)$ is a normal subgroup of order p^2 in N, contradicting to Lemma 3.1.

Assume that $d\ell = 3$. Then $(d, \ell) = (1, 3)$ or $(3, 1)$. Since $|M|_p = p^3 = |\mathsf{A}|_p$, we deduce $R(G) \le M$ and hence M is transitive on Γ.

For $(d, \ell) = (1, 3)$, M is a non-abelian simple group. Since $M_1 \le \mathsf{A}_1$ is a p'-group, Proposition 2.3 implies that M is 2-transitive on Γ, forcing that Γ is the complete graph of order p^3, a contradiction.

For $(d, \ell) = (3, 1)$, we have $M = T_1 \times T_2 \times T_3$. Then $|M|_p = p^3$, and since $M \trianglelefteq \mathsf{A}$, we derive $R(G) \le M$. By Lemma 3.2 $M \le \mathrm{Aut}(\Sigma)$, and Σ is $R(G)$-semisymmetric. Since M has no subgroup of index 2, M fixes the two parts of Σ setwise, and hence Σ is M-semisymmetric. Noting that γ interchanges the two parts of Σ, we have that Σ is $M\langle\gamma\rangle$-arc-transitive. Since γ is an involution, under conjugacy it fixes T_i for some $1 \le i \le 3$, say T_1. Then $T_1 \trianglelefteq \langle M, \gamma\rangle$ and by Proposition 2.5, T_1 is semiregular on Σ. This gives rise to $|T_1| \mid 2p^2$, contrary to the simplicity of T_1.

Case 2: M is soluble.

Since $p \mid |M|$, we have $M = \mathbb{Z}_p^d$ with $1 \le d \le 3$. If $d = 3$ then A has a normal Sylow p-subgroup, as required. If $d = 2$ then $M \le R(G) \le N$ and N has a normal subgroup of order p^2, contrary to Lemma 3.1. Thus, we may let $d = 1$, and since $M \le R(G)$ and $R(G)$ has a unique normal subgroup of order p that is the center of $R(G)$, we derive that $M = \langle R(c)\rangle$.

Now it is easy to see that the quotient graph $\Gamma_M = \mathrm{Cay}(G/M, S/M)$ with $S/M = \{a^iM, b^iM \mid 1 \le i \le p-1\}$. Note that $G/M = \langle aM\rangle \times \langle bM\rangle \cong \mathbb{Z}_p^2$. Then Γ_M is a connected Cayley graph of order p^2 with valency $2(p-1)$, so Γ is a normal M-cover of Γ_M. By Proposition 2.5, we may let $\mathsf{A}/M \le \mathrm{Aut}(\Gamma_M)$ and Γ_M is A/M-arc-transitive.

Let H/M be a minimal normal subgroup of A/M. Then $H \trianglelefteq \mathsf{A}$ and $H/M = L_1/M \times \cdots \times L_r/M$, where $L_i \trianglelefteq H$ and L_i/M $(1 \le i \le r)$ are isomorphic simple groups. Since $|\Gamma_M| = p^2$, we infer $p \mid |H/M|$ and similarly, $p \mid |L_i/M|$. Let $|L_i/M|_p = p^s$. Then $|H/M|_p = p^{rs}$, and since $|\mathsf{A}/M|_p = p^2$, we obtain that $sr = 1$ or 2.

We finish the proof by considering the two subcases: H/M is insoluble or soluble.

Subcase 2.1: H/M is insoluble.

In this subcase, L_i/M are isomorphic non-abelain simple groups. We prove this subcase cannot happen by deriving contradictions. Recall that $sr = 1$ or 2.

Let $sr = 1$. Then $|H/M|_p = p$, and therefore $|H|_p = p^2$. Since $H \trianglelefteq \mathsf{A}$, $H \cap R(G)$ is a Sylow p-subgroup of H, implying $|H \cap R(G)| = p^2$, and then $R(G) \trianglelefteq N$ yields that $H \cap R(G)$ is a normal subgroup of order p^2 in N, contrary to Lemma 3.1.

Let $rs = 2$. Then $|H/M|_p = p^2$ and $|H|_p = p^3$. This yields $R(G) \le H$ and H is transitive on Γ, so H/M is transitive on $V(\Gamma_M)$. Note that $(r, s) = (1, 2)$ or $(2, 1)$.

For $(r, s) = (1, 2)$, H/M is a nonabelian simple group. By Propostion 2.5, $(H/M)_u$ for $u \in V(\Gamma_M)$ is a p'-group because $H_1 \le \mathsf{A}_1$ is a p'-group, and by Proposition 2.3, H/M is 2-transitive on $V(\Gamma_M)$, forcing that Γ_M is a complete group of order p^2, a contradiction.

For $(r, s) = (2, 1)$, $H/M \cong L_1/M \times L_2/M$, where L_1/M and L_2/M are isomorphic nonabelain simple groups and $|L_i/M|_p = p$. It follows that $|H|_p = p^3$ and $|L_i|_p = p^2$ for $1 \le i \le 2$. Since $H \trianglelefteq \mathsf{A}$, we derive $R(G) \le H$. Note that H has no subgroup of index 2. Since Σ is bipartite, it is H-semisymmetric. Let Δ_1 and Δ_2 be the two parts of Σ. Then $|\Delta_1| = |\Delta_2| = p^2$, and H is transitive on both Δ_1 and Δ_2.

Suppose $(L_1)_u = 1$ for some $u \in V(\Sigma) = \Delta_1 \cup \Delta_2$. By Proposition 2.1, $|L_1| = |u^{L_1}|$, and since $L_1 \trianglelefteq H$ and $|\Delta_1| = |\Delta_2| = p^2$, we derive $|L_1| = p$ or p^2, contrary to the insolubleness of L_1. Thus $(L_1)_u \ne 1$. Since Σ has prime valency p, H_u is primitive on the neighbourhood $\Sigma(u)$ of u in Σ, and since $(L_1)_u \trianglelefteq H_u$, $(L_1)_u$ is transitive on $\Sigma(u)$, which implies that $|(L_1)_u|_p = p$. Since $|L_1|_p = p^2$, each orbit of L_1 on Δ_1 or Δ_2 has length p.

Let $x \in \Delta_1$ and $y \in \Delta_2$ be adjacent in Σ, and let Δ_{11} and Δ_{21} be the orbits of L_1 containing x and y, respectively. Then $|\Delta_{11}| = |\Delta_{21}| = p$. Since $(L_1)_x$ is transitive on $\Sigma(x)$, x is adjacent to each vertex in Δ_{21}, and therefore, each vertex in Δ_{11} is adjacent to each vertex in Δ_{21}, that is, the induced subgroup $[\Delta_{11} \cup \Delta_{21}]$ is the complete bipartite graph $\mathsf{K}_{p,p}$. It follows that $\Sigma \cong p\mathsf{K}_{p,p}$, contrary to the connectedness of Σ.

Subcase 2.2: H/M is soluble.

In this case, $|H| = p^2$ or p^3. Recall that $H \trianglelefteq \mathsf{A}$. If $|H| = p^2$ then $H \le R(G)$ and N has normal subgroup of order p^2, contradicts Lemma 3.1. Thus, $|H| = p^3$ and A has a normal Sylow p-subgroup, as required. This completes the proof. □

Now we are ready to finish the proof.

Proof of Theorem 1.2. By Lemmas 3.1 and 3.3, Γ is a arc-transitive normal Cayley graph. In particular, Γ is 1-distance transitive. Since $S = \{a^i, b^i \mid 1 \le i \le p-1\}$, Γ has girth 3, so it is not 2-arc-transitive.

Recall that $G = \langle a, b, c \mid a^p = b^p = c^p = 1, [a, b] = c, [c, a] = [c, b] = 1\rangle$. Clearly,

$$\begin{aligned}\Gamma_1(1) &= S = \{a^i, b^i \mid 1 \le i \le p-1\},\\ \Gamma_2(1) &= \{b^j a^i, a^j b^i \mid 1 \le i, j \le p-1\}.\end{aligned}$$

Note that $\mathrm{Aut}(G, S) = \langle \alpha, \beta, \gamma \mid \alpha^{p-1} = \beta^{p-1} = \gamma^2 = 1, \alpha^\beta = \alpha, \alpha^\gamma = \beta\rangle$, where $a^\alpha = a^t, b^\alpha = b, c^\alpha = c^t$, $a^\beta = a, b^\beta = b^t, c^\beta = c^t$, $a^\gamma = b, b^\gamma = a$ and $c^\gamma = c^{-1}$. Then $(ba)^{\alpha^i\beta^j} = b^{t^i}a^{t^j}$, and since $\mathbb{Z}_p^* = \langle t\rangle$, we obtain that $\langle\alpha, \beta\rangle$ is transitive on the set $\{b^j a^i \mid 1 \le i, j \le p-1\}$. Similarly, $\langle\alpha, \beta\rangle$ is transitive on $\{a^j b^i \mid 1 \le i, j \le p-1\}$. Furthermore, γ interchanges the two sets $\{b^j a^i \mid 1 \le i, j \le p-1\}$ and $\{a^j b^i \mid 1 \le i, j \le p-1\}$. It follows that $\mathrm{Aut}(G, S)$ is transitive on $\Gamma_2(1)$ and hence Γ is 2-distance transitive.

Noting that $ab = bac$, we have that $b^{-1}ab = ac \in \Gamma_3(1)$ and $aba = ba^2c \in \Gamma_3(1)$. Also it is easy to see that $(ac)^{\mathrm{Aut}(G,S)} = (ac)^{\langle\alpha,\beta,\gamma\rangle} = \{a^i c^j, b^i c^j \mid 1 \le i, j \le p-1\}$. Now it is easy to see that $ba^2c \notin (ac)^{\mathrm{Aut}(G,S)}$, and since $\mathsf{A}_1 = \mathrm{Aut}(G, S)$ by Proposition 2.4, Γ is not distance-transitive. □

Proof of Corollary 1.3. Recall that Σ is the clique graph of Γ. By the first paragraph in the proof of Lemma 3.2 and the definition of $\mathrm{Cos}(G, \langle a\rangle, \langle b\rangle)$ in Corollary 1.3, we have $\Sigma = \mathrm{Cos}(G, \langle a\rangle, \langle b\rangle)$. Again by Lemma 3.2, Σ is $R(G)$-semisymmetric, and since $|E(\Sigma)| = (2p^2 \cdot p)/2 = p^3 = |R(G)|$, $R(G)$ is regular on the edge set $E(\Sigma)$ of Σ. Thus, the line graph of Σ is a Cayley graph on G.

For a given edge $\{\langle a\rangle x, \langle b\rangle y\} \in E(\Sigma)$, we have $|\langle a\rangle x \cap \langle b\rangle y| = 1$, and then we may identify this edge with the unique element in $\langle a\rangle x \cap \langle b\rangle y$. Note that Σ has valency $2(p-1)$. Then the edge $1 = \langle a\rangle \cap \langle b\rangle$ in Σ is exactly incident to all edges in $S = \{a^i, b^i \mid 1 \le i \le p-1\}$, because $\{a^i\} = \langle a\rangle \cap \langle b\rangle a^i$ and $\{b^i\} = \langle b\rangle \cap \langle a\rangle b^i$. It follows that $\Gamma = \mathrm{Cay}(G, S)$ is exactly the line graph of Σ.

If $\alpha \in \mathrm{Aut}(\Sigma)$ fixes each edge in Σ then α fixes all vertices of Σ, that is, $\mathrm{Aut}(\Sigma)$ acts faithfully on Γ. Thus, we may view $\mathrm{Aut}(\Sigma)$ as a subgroup of $\mathrm{Aut}(\Gamma)$. By Lemmas 3.2 and 3.3, we have $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(\Sigma) = R(G) \rtimes \mathrm{Aut}(G, S)$.

Recall that $\mathrm{Aut}(G, S) = \langle\alpha, \beta, \gamma\rangle$ and Σ is arc-transitive. Since $a^\beta = a$, $b^\beta = b^t$ and $c^\beta = c^t$, where $\mathbb{Z}_p^* = \langle t\rangle$, $\langle\beta\rangle$ fixes the arc $(\langle a\rangle, \langle b\rangle)$ in Σ and is transitive on the vertex set $\{\langle a\rangle b^i \mid 1 \le i \le p-1\}$, where $\{\langle a\rangle\} \cup \{\langle a\rangle b^i \mid 1 \le i \le p-1\}$ is the neighbourhood of $\langle b\rangle$ in Σ. Thus, Σ is 2-arc-transitive. Since $a^\alpha = a^t$, $b^\alpha = b$ and $c^\alpha = c^t$, $\langle\alpha\rangle$ fixes the 2-arc $(\langle a\rangle, \langle b\rangle, \langle a\rangle b)$ and is transitive on the vertex set $\{\langle b\rangle a^i b \mid 1 \le i \le p-1\}$, where $\{\langle b\rangle\} \cup \{\langle b\rangle a^i b \mid 1 \le i \le p-1\}$ is the neighbourhood of $\langle a\rangle b$ in Σ. It follows that Σ is 3-arc-transitive. It is easy to see that the number of 3-arcs in Σ equals to $|A| = 2p^3(p-1)^2$, A is regular on the set of 3-arcs of Σ. □

ORCID iDs

Jun-Jie Huang https://orcid.org/0000-0002-9250-0672
Yan-Quan Feng https://orcid.org/0000-0003-3214-0609
Jin-Xin Zhou https://orcid.org/0000-0002-8353-896X

References

[1] N. Biggs, *Algebraic Graph Theory*, Camb. Math. Libr., Cambridge University Press, Cambridge, 1993, doi:10.1017/cbo9780511608704.

[2] N. Biggs and D. Smith, On trivalent graphs, *Bull. Lond. Math. Soc.* **3** (1971), 155–158, doi: 10.1112/blms/3.2.155.

[3] J. Chen, W. Jin and C. H. Li, On 2-distance-transitive circulants, *J. Algebr. Comb.* **49** (2019), 179–191, doi:10.1007/s10801-018-0825-3.

[4] B. P. Corr, W. Jin and C. Schneider, Two-distance-transitive but not two-arc-transitive graphs, in press.

[5] B. P. Corr, W. Jin and C. Schneider, Finite 2-distance transitive graphs, *J. Graph Theory* **86** (2017), 78–91, doi:10.1002/jgt.22112.

[6] A. Devillers, M. Giudici, C. H. Li and C. E. Praeger, Locally s-distance transitive graphs, *J. Graph Theory* **69** (2012), 176–197, doi:10.1002/jgt.20574.

[7] A. Devillers, M. Giudici, C. H. Li and C. E. Praeger, Locally s-distance transitive graphs and pairwise transitive designs, *J. Comb. Theory, Ser. A* **120** (2013), 1855–1870, doi:10.1016/j.jcta.2013.07.003.

[8] A. Devillers, W. Jin, C. H. Li and C. E. Praeger, On normal 2-geodesic transitive Cayley graphs, *J. Algebr. Comb.* **39** (2014), 903–918, doi:10.1007/s10801-013-0472-7.

[9] J. D. Dixon and B. Mortimer, *Permutation Groups*, Springer-Verlag, Berlin, 1996, doi:10.1007/978-1-4612-0731-3.

[10] A. Gardiner and C. E. Praeger, Distance-transitive graphs of valency five, *Proc. Edinb. Math. Soc., II. Ser.* **30** (1987), 73–81, doi:10.1017/s0013091500017983.

[11] C. D. Godsil, On the full automorphism group of a graph, *Combinatorica* **1** (1981), 243–256, doi:10.1007/bf02579330.

[12] D. Gorenstein, *Finite Groups*, Chelsea Publishing Company, New York, 1980, `https://books.google.si/books?id=bxRrwQEACAAJ`.

[13] R. M. Guralnick, Subgroups of prime power index in a simple group, *J. Algebra* **81** (1983), 304–311, doi:10.1016/0021-8693(83)90190-4.

[14] A. A. Ivanov, A. V. Ivanov and I. A. Faradzhev, Distance-transitive graphs of valency 5, 6 and 7, *Eur. J. Comb.* **7** (1986), 303–319, doi:10.1016/0041-5553(84)90010-7.

[15] W. Jin, Y. Huang and W. J. Liu, Two-distance-primitive graphs with prime valency, *Appl. Math. Comput.* **357** (2019), 310–316, doi:10.1016/j.amc.2019.03.052.

[16] W. Jin and L. Tan, Finite two-distance-transitive graphs of valency 6, *Ars Math. Contemp.* **11** (2016), 49–58, doi:10.26493/1855-3974.781.d31.

[17] C. H. Li and J. Pan, Finite 2-arc-transitive abelian Cayley graphs, *Eur. J. Comb.* **29** (2008), 148–158, doi:10.1016/j.ejc.2006.12.001.

[18] M. W. Liebeck, C. E. Praeger and J. Saxl, Distance transitive graphs with symmetric or alternating automorphism group, *Bull. Aust. Math. Soc.* **35** (1987), 1–25, doi:10.1017/s0004972700012995.

[19] J. Pan, Z. Huang and Z. Liu, Arc-transitive regular cyclic covers of the complete bipartite graph $\mathsf{K}_{p,p}$, *J. Algebr. Comb.* **42** (2015), 619–633, doi:10.1007/s10801-015-0594-1.

[20] J. M. Pan and F. G. Yin, Symmetric graphs of order four times a prime power and valency seven, *J. Algebra Appl.* **17** (2018), 12, doi:10.1142/s0219498818500937, id/No 1850093.

[21] C. E. Praeger, An O'Nan-Scott theorem for finite quasiprimitive permutation groups and an application to 2-arc transitive graphs, *J. Lond. Math. Soc., II. Ser.* **47** (1992), 227–239, doi:10.1112/jlms/s2-47.2.227.

[22] C. E. Praeger, Finite normal edge-transitive Cayley graphs, *Bull. Aust. Math. Soc.* **60** (1999), 207–220, doi:10.1017/s0004972700036340.

[23] D. Robinson, *A Course in the Theory of Groups*, Springer-Verlag, New York, 1995, doi:10.1007/978-1-4419-8594-1.

[24] G. Sabidussi, Vertex-transitive graphs, *Monatsh. Math.* **68** (1964), 426–438, doi:10.1007/bf01304186.

[25] D. H. Smith, Distance-transitive graphs of valency four, *J. Lond. Math. Soc., II. Ser.* **8** (1974), 377–384, doi:10.1112/jlms/s2-8.2.377.

[26] Y. Wang, Y.-Q. Feng and J.-X. Zhou, Cayley digraphs of 2-genetic groups of odd prime-power order, *J. Comb. Theory, Ser. A* **143** (2016), 88–106, doi:10.1016/j.jcta.2016.05.001.

[27] M. Xu, Automorphism groups and isomorphisms of Cayley digraphs, *Discrete Math.* **182** (1998), 309–319, doi:10.1016/s0012-365x(97)00152-0.