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Few-body problems inspired by hadron spectroscopy

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Abstract. I discuss some results derived in very simplified models of hadron spectroscopy, where a static potential is associated with non-relativistic kinematics. Several regularity patterns of the experimental spectrum are explained in such simple models. It is underlined that some methods developed for hadronic physics have applications in other fields, in particular atomic physics. A few results can be extended to cases involving spin-dependent forces or relativistic kinematics.

1 Introduction

As discussed in several contributions at this nice workshop, the dynamics of light quarks is far from being simple, with non-perturbative effects even at short distances, and highly-relativistic motion of the constituents inside hadrons. Nevertheless, it is interesting to consider a fictitious world, with the hadron spectrum governed by a simple Hamiltonian where a non-relativistic kinematics is supplemented by a static, flavour-independent potential. The regularities derived from the properties of the Schrödinger equation are similar to these observed in the actual spectrum. This suggests that the actual QCD theory of quark confinement should exhibit similar regularities.

One should also notice that several results derived in the context of quark models of hadrons have been successfully applied to other few-body problems, in particular in atomic physics.

Another challenge consists of extending theorems on level order, convexity, etc., to less naive Hamiltonians with spin-dependent forces and relativistic kinematics. Some of the first results will be mentioned.

2 Results on mesons

The discovery of Ψ and Υ resonances and their excitations has stimulated many studies in the quark model. In particular, the successful description of these spectra by the same potential has motivated investigations on the consequences of flavour independence. The rigorous results have been summarized in the reviews by Quigg and Rosner and by Martin and Grosse. A few examples are given below, dealing with energy levels.

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All potentials models reproduce the observed pattern of quarkonium that $E(1P) < E(2S)$. Note the notation adopted here, (n, ℓ), in terms of which the principal quantum number of atomic physics is $n + \ell$. It has been proved that $E(n+1, \ell) > E(n, \ell+1)$ if $\Delta V > 0$, and the reverse if $\Delta V < 0$. The Coulomb degeneracy is recovered as a limiting case. The sign of Δ reflects whether the charge $Q(r)$ seen at distance r grows (asymptotic freedom), decreases or remains constant (Gauss theorem).

This "Coulomb theorem" can be applied successfully to muonic atoms, which are sensitive to the size of the nucleus ($Q(r) \nearrow$), and to alkaline atoms whose last electron penetrates the inner electron shells ($Q(r) \searrow$).

Another theorem describes how the harmonic oscillator (h.o.) degeneracy $E(n+1, \ell) = E(n, \ell+2)$ is broken. A strict inequality is obtained if the sign of V'' is constant.

In both the complete Hamiltonian $\mathbf{p}_1^2/(2m_1) + \mathbf{p}_2^2/(2m_2) + V(r_{12})$ or its reduced version $\mathbf{p}^2/(2\mu) + V(r)$, the individual inverse masses m_i or the inverse reduced mass enter through a positive operator $\mathbf{p}^2$, and linearly. It results that each energy level is an increasing function of this inverse mass m_i^{-1} or μ^{-1}, and that the ground-state energy (or the sum of first levels) is a concave function of this variable. There are many applications. For instance, for the ground-state of the meson with charm and beauty,

$$(b\bar{s}) + (c\bar{c}) - (c\bar{s}) < (b\bar{c}) < (b\bar{b} + c\bar{c})/2. \tag{1}$$

3 Level order of baryon spectra

For many years, the only widespread knowledge of the 3-body problem was the harmonic oscillator. This remains true outside the few-body community. The discussion on baryon excitations is thus often restricted to situations where $V = \sum v(r_{ij})$, with $v(r) = Kr^2 + \delta v$, and δv treated as a correction.

First-order perturbation theory is usually excellent, especially if the oscillator strength K is variationally adjusted to minimise the magnitude of the corrections. However, when first-order perturbation is shown (or claimed) to produce a crossing of levels, one is reasonably worried about higher-order terms, and a more rigorous treatment of the energy spectrum becomes desirable.

A decomposition better than $V = \sum Kr_{ij}^2 + \delta v$ is provided by the generalised partial-wave expansion

$$V = V_0(\rho) + \delta V, \tag{2}$$

where $\rho \propto (r_{12}^2 + r_{23}^2 + r_{31}^2)^{1/2}$ is the hyperradius. The last term δV gives a very small correction to the first levels. With the hyperscalar potential V_0 only, the wave function reads $\Psi = \rho^{-5/2} u(\rho) P_{[L]}(\Omega)$, where the last factor contains the "grand-angular" part. The energy and the hyperradial part are governed by

$$u''(\rho) - \frac{\ell(\ell+1)}{\rho^2} u(\rho) + m[E - V_0(\rho)]u(\rho) = 0, \tag{3}$$

very similar to the usual radial equations of the 2-body problem, except that the effective angular momentum is now $\ell = 3/2$ for the ground-state and its radial excitations and $\ell = 5/2$ for the first orbital excitation with negative parity. The Coulomb theorem holds for non-integer ℓ. If $\Delta V > 0$, then $E(2S) > E(1P)$, i.e., the Roper comes above the orbital excitation. Note that a three-body potential cannot be distinguished from a simple pairwise interaction once it is reduced to its hyperscalar component V_0 by suitable angular integration. It also results from numerical tests that relativistic kinematics does not change significantly the *relative* magnitude of orbital vs. radial excitation energies.

The splitting of levels in the nearly hyperscalar potential (2) is very similar to the famous pattern of the $N = 2$ h.o. multiplet, except that the Roper is disentangled. A similar result is found for higher negative-parity excitation: the split $N = 3$ levels of the nearly harmonic model are separated into a radially excited $L = 1$ and a set of split $L = 3$ levels.

4 Tests of flavour independence for baryons

The analogue for baryons of the inequality between $b\bar{b}$, $c\bar{c}$ and $b\bar{c}$ reads.

$$(QQq) + (Q'Q'q) \leq 2(QQ'q). \tag{4}$$

Unlike the meson case, it requires mild restrictions on the potentials.

For instance, the equal spacing rule $\Omega^- - \Xi^* = \Xi^* - \Sigma^* = \Sigma^* - \Delta$ is understood as follows: the central force gives a concave behaviour, with for instance $\Omega^- - \Xi^* < \Xi^* - \Sigma^*$, but a quasi perfect linearity is restored by the spin–spin interaction which acts more strongly on light quarks. A similar scenario holds for the Gell-Mann–Okubo formula.

Inequalities can also be written for baryons with heavy flavour, some of them being more accessible than others to experimental checks in the near future. Examples are

$$\begin{aligned} &3(bcs) \geq (bbb) + (ccc) + (sss), \\ &2(bcq) \geq (bbq) + (ccq), \qquad 2(cqq) \geq (ccq) + (qqq). \end{aligned} \tag{5}$$

5 Baryons with two heavy flavours

There is a renewed interest in this subject. The recent observation of the $(b\bar{c})$ mesons demonstrates our ability to reconstruct hadrons with two heavy quarks from their decay products.

Baryons with two heavy quarks $(QQ'q)$ are rather fascinating: they combine the adiabatic motion of two heavy quarks as in J/Ψ and Υ mesons with the highly relativistic motion of a light quark as in flavoured mesons D or B.

The wave function of (QQq) exhibits a clear diquark clustering with $r(QQ) \ll r(Qq)$ for the average distances. This does not necessarily mean that for a given potential model, a naive two-step calculation is justified. Here I mean: estimate first the (QQ) mass using the direct potential $v(QQ)$ only, and then solve the $[(QQ)-q]$ 2-body problem using a point-like diquark. If v is harmonic, one would miss a factor 3/2 in the effective spring constant of the (QQ) system, and thus a factor $(3/2)^{1/2}$ in its excitation energy.

On the other hand, it has been checked that the Born–Oppenheimer approximation works extremely well for these (QQq) systems, even when the quark mass ratio Q/q is not very large. This system is the analogue of H_2^+ in atomic physics.

6 The search of multiquarks

A concept of "order" or "disorder" might be introduce to study multiquark stability. This is related to the breaking of permutation symmetry. Consider for instance

$$\begin{aligned} H_4(x) &= \sum_{i=1}^{4} \frac{\mathbf{p}_i^2}{2m} + (1-2x)(V_{12}+V_{34}) + (1+x)(V_{13}+V_{14}+V_{23}+V_{24}) \\ &= H_S + xH_{MS}, \end{aligned} \tag{6}$$

where the parameter x measures the departure from a fully symmetric interaction. From the variational principle, the ground-state energy $E(x)$ is maximal at $x = 0$. In most cases, $E(x)$ will be approximately parabolic, so the amount of binding below $E(0)$ is measured by $|x|$.

In simple colour models of multiquark confinement, the analogue of $|x|$ is larger for the threshold (two mesons) that for a $(\bar{q}\bar{q}qq)$ composite. So a stable multiquark is unlikely.

For the $(\bar{Q}\bar{Q}qq)$ systems presented by our slovenian hosts, and discussed earlier by Ader et al., Stancu and Brink, and others, there is another asymmetry, in the kinetic energy, which now favours multiquark binding. So there is a competition with the colour-dependent potential.

The methods developed for quark studies has been applied for systematic investigations of the stability of three-charge and four-charge systems in atomic physics.

Bibliography

A more comprehensive account of these considerations, including references to original papers or to recent review articles will be found in the Proceedings of the Few-Body Conference held at Evora, Portugal, in September 2000 (to appear as a special issue of Nuclear Physics A). I would like to thank again the organizers of this Workshop for the very pleasant and stimulating environment.