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A diagram associated with the subconstituent algebra of a distance-regular graph

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Abstract

In this paper we consider a distance-regular graph Γ. Fix a vertex x of Γ and consider the corresponding subconstituent algebra $T = T(x)$. The algebra T is the $\mathbb{C}$-algebra generated by the Bose-Mesner algebra M of Γ and the dual Bose-Mesner algebra M^* of Γ with respect to x. We consider the subspaces $M, M^*, MM^*, M^*M, MM^*M, M^*MM^*, \dots$ along with their intersections and sums. In our notation, MM^* means $\mathrm{Span}\{RS \mid R \in M, S \in M^*\}$, and so on. We introduce a diagram that describes how these subspaces are related. We describe in detail that part of the diagram up to $MM^* + M^*M$. For each subspace U shown in this part of the diagram, we display an orthogonal basis for U along with the dimension of U. For an edge $U \subseteq W$ from this part of the diagram, we display an orthogonal basis for the orthogonal complement of U in W along with the dimension of this orthogonal complement.

Keywords: Subconstituent algebra, Terwilliger algebra, distance-regular graph.

Math. Subj. Class.: 05E30

1 Introduction

In this paper we consider a distance-regular graph Γ. Fix a vertex x of Γ and consider the corresponding subconstituent algebra (or Terwilliger algebra) $T = T(x)$ [32]. The algebra T is the $\mathbb{C}$-algebra generated by the Bose-Mesner algebra M of Γ and the dual Bose-Mesner algebra M^* of Γ with respect to x. The algebra T is finite-dimensional and semisimple [32]. So it is natural to study the irreducible T-modules. These modules are used in the study of hypercubes [14, 26], dual polar graphs [20, 38], spin models [6, 10],

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codes [13, 28], the bipartite property [4, 5, 9, 16, 21, 22, 23, 25, 27], the almost-bipartite property [3, 8, 17], the Q-polynomial property [5, 7, 11, 12, 18, 19, 27, 35], and the thin property [15, 24, 30, 31, 33, 34, 36, 37].

In this paper we discuss the algebra T using a different approach. We consider the subspaces $M, M^*, MM^*, M^*M, MM^*M, M^*MM^*, \dots$ along with their intersections and sums; see Figure 1. We describe the diagram of Figure 1 up to MM^*+M^*M. For each subspace U shown in this part of the diagram, we display an orthogonal basis for U along with the dimension of U. For an edge $U \subseteq W$ from this part of the diagram, we display an orthogonal basis for the orthogonal complement of U in W along with the dimension of this orthogonal complement. Our main results are summarized in Theorems 6.1 and 6.2. In the last part of the paper we summarize what is known about the part of diagram above $MM^* + M^*M$, and we give some open problems.

2 Preliminaries

In this section we recall some facts about distance-regular graphs. We will use the following notation. Let X denote a nonempty finite set. Let $\mathrm{Mat}_X(\mathbb{C})$ denote the $\mathbb{C}$-algebra consisting of the matrices whose rows and columns are indexed by X and whose entries are in $\mathbb{C}$. For $B \in \mathrm{Mat}_X(\mathbb{C})$ let $\overline{B}$, B^t, and $\mathrm{tr}(B)$ denote the complex conjugate, the transpose, and the trace of B, respectively. We endow $\mathrm{Mat}_X(\mathbb{C})$ with the Hermitean inner product $\langle\ ,\ \rangle$ such that $\langle R, S\rangle = \mathrm{tr}(R^t\overline{S})$ for all $R, S \in \mathrm{Mat}_X(\mathbb{C})$. The inner product $\langle\ ,\ \rangle$ is positive definite. Let U, V denote subspaces of $\mathrm{Mat}_X(\mathbb{C})$ such that $U \subseteq V$. The *orthogonal complement* of U in V is defined by $U^\perp = \{v \in V \mid \langle v, u\rangle = 0 \text{ for all } u \in U\}$.

Let $\Gamma = (X, \mathcal{E})$ denote a finite, undirected, connected graph, without loops or multiple edges, with vertex set X and edge set $\mathcal{E}$. Let ∂ denote the shortest path-length distance function for Γ. Define the diameter $D := \max\{\partial(x, y) \mid x, y \in X\}$. For a vertex $x \in X$ and an integer $i \geq 0$ define $\Gamma_i(x) = \{y \in X \mid \partial(x, y) = i\}$. For notational convenience abbreviate $\Gamma(x) = \Gamma_1(x)$. For an integer $k \geq 0$, we say that Γ is *regular with valency* k whenever $|\Gamma(x)| = k$ for all $x \in X$. We say that Γ is *distance-regular* whenever for all integers h, i, j $(0 \leq h, i, j \leq D)$ and $x, y \in X$ with $\partial(x, y) = h$, the number

$$p^h_{ij} := |\Gamma_i(x) \cap \Gamma_j(y)|$$

is independent of x and y. The integers p^h_{ij} are called the *intersection numbers* of Γ. From now on assume that Γ is distance-regular with diameter $D \geq 3$. We abbreviate $k_i := p^0_{ii}$ $(0 \leq i \leq D)$. For $0 \leq i \leq D$ let A_i denote the matrix in $\mathrm{Mat}_X(\mathbb{C})$ with (x, y)-entry

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } \partial(x, y) = i, \\ 0 & \text{if } \partial(x, y) \neq i, \end{cases} \qquad x, y \in X.$$

We call A_i the *i-th distance matrix* of Γ. We call $A = A_1$ the *adjacency matrix* of Γ. Observe that A_i is real and symmetric for $0 \leq i \leq D$. Note that $A_0 = I$ is the identity matrix in $\mathrm{Mat}_X(\mathbb{C})$. Observe that $\sum_{i=0}^D A_i = J$, where J is the all-ones matrix in $\mathrm{Mat}_X(\mathbb{C})$. Observe that for $0 \leq i, j \leq D$,

$$A_i A_j = \sum_{h=0}^{D} p^h_{ij} A_h. \tag{2.1}$$

For integers h, i, j $(0 \leq h, i, j \leq D)$ we have

$$p^h_{0j} = \delta_{hj}, \tag{2.2}$$
$$p^0_{ij} = \delta_{ij} k_i. \tag{2.3}$$

Let M denote the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by A. By [2, p. 44] the matrices $A_0, A_1, \ldots, A_D$ form a basis for M. We call M the *Bose-Mesner algebra of* Γ. By [1, p. 59, 64], M has a basis $E_0, E_1, \ldots, E_D$ such that

(i) $E_0 = |X|^{-1} J$;

(ii) $\sum_{i=0}^{D} E_i = I$;

(iii) $E_i^t = E_i$ $(0 \leq i \leq D)$;

(iv) $\overline{E_i} = E_i$ $(0 \leq i \leq D)$;

(v) $E_i E_j = \delta_{ij} E_i$ $(0 \leq i, j \leq D)$.

The matrices $E_0, E_1, \ldots, E_D$ are called the *primitive idempotents* of Γ, and E_0 is called the *trivial* idempotent. For $0 \leq i \leq D$ let m_i denote the rank of E_i. For $0 \leq i \leq D$ let θ_i denote an eigenvalue of A associated with E_i. Let λ denote an indeterminate. Define polynomials $\{u_i\}_{i=0}^{D}$ in $\mathbb{C}[\lambda]$ by $u_0 = 1$, $u_1 = \lambda / k$, and

$$\lambda u_i = c_i u_{i-1} + a_i u_i + b_i u_{i+1} \qquad (1 \leq i \leq D-1).$$

By [2, p. 131, 132],

$$A_j = k_j \sum_{i=0}^{D} u_j(\theta_i) E_i \qquad (0 \leq j \leq D), \tag{2.4}$$
$$E_j = |X|^{-1} m_j \sum_{i=0}^{D} u_i(\theta_j) A_i \qquad (0 \leq j \leq D). \tag{2.5}$$

Since $E_i E_j = \delta_{ij} E_i$ and by (2.4) we have $A_j E_i = k_j u_j(\theta_i) E_i = E_i A_j$ $(0 \leq i, j \leq D)$. By [1, Theorem 3.5] we have the orthogonality relations

$$\sum_{i=0}^{D} u_i(\theta_r) u_i(\theta_s) k_i = \delta_{rs} m_r^{-1} |X| \qquad (0 \leq r, s \leq D), \tag{2.6}$$
$$\sum_{r=0}^{D} u_i(\theta_r) u_j(\theta_r) m_r = \delta_{ij} k_i^{-1} |X| \qquad (0 \leq i, j \leq D). \tag{2.7}$$

We recall the Krein parameters of Γ. Let $\circ$ denote the entry-wise multiplication in $\mathrm{Mat}_X(\mathbb{C})$. Note that $A_i \circ A_j = \delta_{ij} A_i$ for $0 \leq i, j \leq D$. So M is closed under $\circ$. By [2, p. 48], there exist scalars $q^h_{ij} \in \mathbb{C}$ such that

$$E_i \circ E_j = |X|^{-1} \sum_{h=0}^{D} q^h_{ij} E_h \qquad (0 \leq i, j \leq D). \tag{2.8}$$

We call the q^h_{ij} the *Krein parameters* of Γ. By [2, Proposition 4.1.5], these parameters are real and nonnegative for $0 \le h, i, j \le D$.

We recall the dual Bose-Mesner algebra of Γ. Fix a vertex $x \in X$. For $0 \le i \le D$ let $E^*_i = E^*_i(x)$ denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{C})$ with (y, y)-entry

$$(E^*_i)_{yy} = \begin{cases} 1 & \text{if } \partial(x, y) = i, \\ 0 & \text{if } \partial(x, y) \ne i, \end{cases} \qquad y \in X.$$

We call E^*_i the *i-th dual idempotent* of Γ with respect to x. Observe that

(i) $\sum_{i=0}^{D} E^*_i = I$;

(ii) $E^{*t}_i = E^*_i$ $(0 \le i \le D)$;

(iii) $\overline{E^*_i} = E^*_i$ $(0 \le i \le D)$;

(iv) $E^*_i E^*_j = \delta_{ij} E^*_i$ $(0 \le i, j \le D)$.

By construction $E^*_0, E^*_1, \ldots, E^*_D$ are linearly independent. Let $M^* = M^*(x)$ denote the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ with basis $E^*_0, E^*_1, \ldots, E^*_D$. We call M^* the *dual Bose-Mesner algebra* of Γ with respect to x.

We now recall the dual distance matrices of Γ. For $0 \le i \le D$ let $A^*_i = A^*_i(x)$ denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{C})$ with (y, y)-entry

$$(A^*_i)_{yy} = |X|(E_i)_{xy} \qquad y \in X. \tag{2.9}$$

We call A^*_i the *dual distance matrix* of Γ with respect to x and E_i. By [32, p. 379], the matrices $A^*_0, A^*_1, \ldots, A^*_D$ form a basis for M^*. Observe that

(i) $A^*_0 = I$;

(ii) $\sum_{i=0}^{D} A^*_i = |X| E^*_0$;

(iii) $A^{*t}_i = A^*_i$ $(0 \le i \le D)$;

(iv) $\overline{A^*_i} = A^*_i$ $(0 \le i \le D)$;

(v) $A^*_i A^*_j = \sum_{h=0}^{D} q^h_{ij} A^*_h$ $(0 \le i, j \le D)$.

From (2.4) and (2.5) we have

$$A^*_j = m_j \sum_{i=0}^{D} u_i(\theta_j) E^*_i \qquad (0 \le j \le D), \tag{2.10}$$

$$E^*_j = |X|^{-1} k_j \sum_{i=0}^{D} u_j(\theta_i) A^*_i \qquad (0 \le j \le D). \tag{2.11}$$

3 The subconstituent algebra T

In this section we study the subconstituent algebra of a distance-regular graph. For the rest of the paper, fix a distance-regular graph Γ and a vertex x of Γ. Let $T = T(x)$ denote the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by M, M^*. The algebra T is called the *subconstituent algebra* (or *Terwilliger algebra*) [32]. In order to describe T, we consider how M, M^* are related. We will use the following notation. For any two subspaces $\mathcal{R}, \mathcal{S}$ of $\mathrm{Mat}_X(\mathbb{C})$ we define $\mathcal{RS} = \mathrm{Span}\{RS \mid R \in \mathcal{R}, S \in \mathcal{S}\}$. Consider the subspaces $M, M^*, MM^*, M^*M, MM^*M, M^*MM^*, \dots$ along with their intersections and sums. To describe the inclusions among the resulting subspaces we draw a diagram; see Figure 1. In this diagram, a line segment that goes upward from U to W means that W contains U.

Consider the diagram in Figure 1. For each subspace U shown in the diagram, we seek an orthogonal basis for U and the dimension of U. Also, for each edge $U \subseteq W$ shown in the diagram, we seek an orthogonal basis for the orthogonal complement of U in W along with the dimension of this orthogonal complement. We accomplish these goals for that part of the diagram up to $MM^* + M^*M$. Our main results are summarized in Theorems 6.1 and 6.2. Before we get started, we recall a few inner product formulas.

Lemma 3.1 ([11, Lemma 3.1, Lemma 4.1])**.** *For* $0 \le h, i, j, r, s, t \le D$*,*

(i) $\langle E_i^* A_j E_h^*, E_r^* A_s E_t^* \rangle = \delta_{ir}\delta_{js}\delta_{ht} k_h p_{ij}^h$,

(ii) $\langle E_i A_j^* E_h, E_r A_s^* E_t \rangle = \delta_{ir}\delta_{js}\delta_{ht} m_h q_{ij}^h$.

The following result is well-known.

Lemma 3.2 ([32, Lemma 3.2])**.** *For* $0 \le h, i, j \le D$*,*

(i) $E_i^* A_h E_j^* = 0$ *if and only if* $p_{ij}^h = 0$*,*

(ii) $E_i A_h^* E_j = 0$ *if and only if* $q_{ij}^h = 0$*.*

Lemma 3.3 ([29, Lemma 10])**.** *For* $0 \le h, i, j, r, s, t \le D$*,*

$$\langle A_i E_j^* A_h, A_r E_s^* A_t \rangle = \sum_{\ell=0}^{D} k_\ell p_{ir}^\ell p_{js}^\ell p_{ht}^\ell .$$

4 The subspace $M + M^*$

Our goal in this section is to analyze the inclusion diagram up to $M + M^*$. We begin with the trace of elements in M and M^*.

Lemma 4.1. *For* $0 \le i \le D$*,*

(i) $\mathrm{tr}(A_i) = \delta_{0i}|X|$*,*

(ii) $\mathrm{tr}(E_i) = m_i$*,*

(iii) $\mathrm{tr}(E_i^*) = k_i$*,*

(iv) $\mathrm{tr}(A_i^*) = \delta_{0i}|X|$*.*

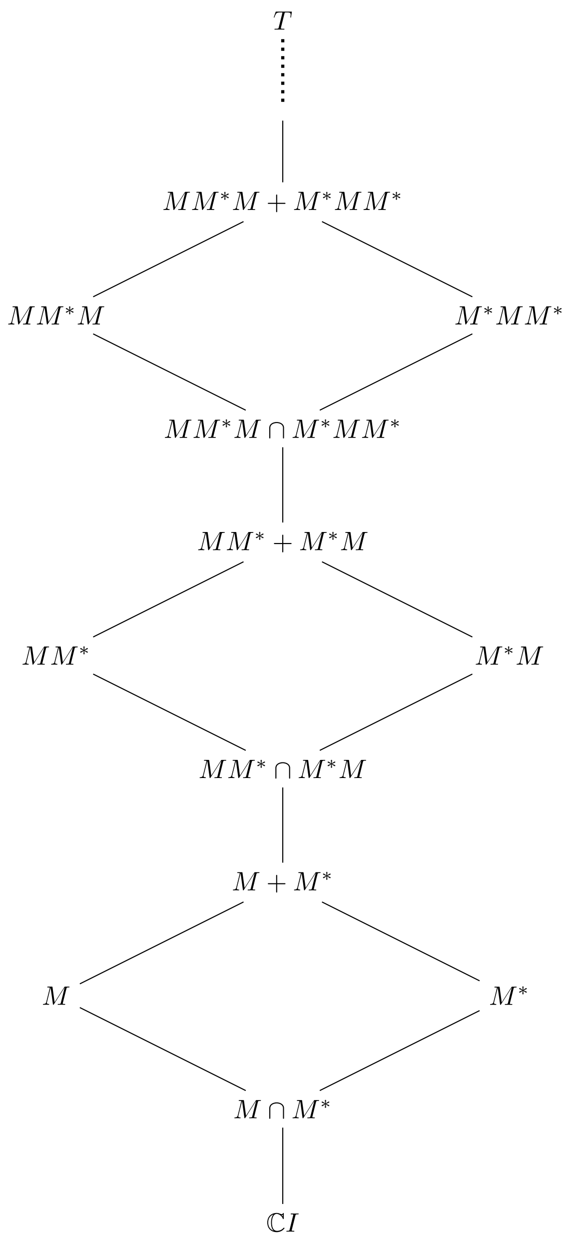

Figure 1: Inclusion diagram.

Proof. (i): Follows from the definition of A_i.
(ii): Since E_i is diagonalizable, we have $\operatorname{tr}(E_i) = \operatorname{rank}(E_i) = m_i$.
(iii): Follows from the definition of E_i^*.
(iv): By (2.5) and since

$$E_0 = |X|^{-1}J = |X|^{-1}\sum_{i=0}^{D} A_i,$$

we have

$$\sum_{i=0}^{D}(1 - u_i(\theta_0))A_i = 0.$$

Since $\{A_i\}_{i=0}^{D}$ are linearly independent, we obtain $u_i(\theta_0) = 1$ for $0 \leq i \leq D$. By (2.6), (2.10) and (iii), we have

$$\operatorname{tr}(A_i^*) = m_i \sum_{j=0}^{D} u_j(\theta_i)\operatorname{tr}(E_j^*) = m_i \sum_{j=0}^{D} u_j(\theta_i)u_j(\theta_0)k_j = \delta_{0i}|X|. \qquad \square$$

Next we obtain some inner products.

Lemma 4.2. *For* $0 \leq i, j \leq D$,

(i) $\langle A_i, A_j\rangle = \delta_{ij}k_i|X|$,

(ii) $\langle E_i, E_j\rangle = \delta_{ij}m_i$,

(iii) $\langle E_i^*, E_j^*\rangle = \delta_{ij}k_i$,

(iv) $\langle A_i^*, A_j^*\rangle = \delta_{ij}m_i|X|$.

Proof. (i): Use (2.1) and Lemma 4.1.
(ii): By Lemma 4.1 and since $E_iE_j = \delta_{ij}E_i$.
(iii): Since $E_i^*E_j^* = \delta_{ij}E_i^*$ ($0 \leq i, j \leq D$) and by Lemma 4.1 (iii).
(iv): By (2.10) and (iii), we obtain

$$\langle A_i^*, A_j^*\rangle = \langle m_i \sum_{h=0}^{D} u_h(\theta_i)E_h^*, m_j \sum_{\ell=0}^{D} u_\ell(\theta_j)E_\ell^*\rangle = m_im_j \sum_{h=0}^{D} u_h(\theta_i)u_h(\theta_j)k_h.$$

By (2.6), we have $\langle A_i^*, A_j^*\rangle = m_im_j\delta_{ij}m_j^{-1}|X| = \delta_{ij}m_i|X|$. $\square$

The algebra M has two bases $\{A_i\}_{i=0}^{D}$ and $\{E_i\}_{i=0}^{D}$. The algebra M^* has two bases $\{A_i^*\}_{i=0}^{D}$ and $\{E_i^*\}_{i=0}^{D}$. Next we show that these bases are orthogonal.

Lemma 4.3. *Each of the following is an orthogonal basis for* M*:*

$$\{A_i\}_{i=0}^{D}, \qquad \{E_i\}_{i=0}^{D}.$$

Moreover, each of the following is an orthogonal basis for M^**:*

$$\{A_i^*\}_{i=0}^{D}, \qquad \{E_i^*\}_{i=0}^{D}.$$

Proof. By Lemma 4.2 and the comment below it. □

Recall that $A_0 = I = A_0^*$. Next we compute some inner products between M and M^*.

Lemma 4.4. *For* $0 \leq i, j \leq D$,

$$\langle A_i, A_j^* \rangle = \delta_{i0}\delta_{0j}|X|k_i.$$

Proof. Observe that $\langle A_i, A_j^* \rangle = \langle A_i A_0^* A_0, A_0 A_j^* A_0 \rangle$. By Lemma 3.3 and (2.2), (2.6) and (2.10), the result follows. □

The next results describe orthogonal bases for $M + M^*$ and $M \cap M^*$.

Lemma 4.5. *The following is an orthogonal basis for* $M + M^*$*:*

$$A_D, \ldots, A_1, I, A_1^* \ldots, A_D^*.$$

Proof. Immediate from Lemmas 4.2 and 4.4. □

Lemma 4.6.

$$\dim(M + M^*) = 2D + 1.$$

Proof. Immediate from Lemma 4.5. □

Lemma 4.7. *We have*

$$M \cap M^* = \mathbb{C}I \quad \textit{and} \quad \dim(M \cap M^*) = 1.$$

Proof. Observe that $I \in M \cap M^*$. By linear algebra, we have

$$\dim(M \cap M^*) = \dim(M) + \dim(M^*) - \dim(M + M^*).$$

By construction $\dim(M) = D + 1$, $\dim(M^*) = D + 1$. By this and Lemma 4.6, $\dim(M \cap M^*) = 1$. The result follows. □

Lemma 4.8. *The following statements hold:*

(i) The matrices $\{A_i\}_{i=1}^D$ *form an orthogonal basis for the orthogonal complement of* $M \cap M^*$ *in* M.

(ii) The matrices $\{A_i^*\}_{i=1}^D$ *form an orthogonal basis for the orthogonal complement of* $M \cap M^*$ *in* M^*.

(iii) The matrices $\{A_i\}_{i=1}^D$ *form an orthogonal basis for the orthogonal complement of* M^* *in* $M + M^*$.

(iv) The matrices $\{A_i^*\}_{i=1}^D$ *form an orthogonal basis for the orthogonal complement of* M *in* $M + M^*$.

Proof. Follows from definitions of M, M^* along with Lemmas 4.5 and 4.7. □

Lemma 4.9. *Each of the following subspaces has dimension* D*:*

$$(M \cap M^*)^\perp \cap M, \qquad (M \cap M^*)^\perp \cap M^*,$$
$$(M^*)^\perp \cap (M + M^*), \qquad M^\perp \cap (M + M^*).$$

Proof. Immediate from Lemma 4.8. □

5 The subspace $MM^* + M^*M$

Our goal in this section is to analyze the inclusion diagram from $M + M^*$ up to $MM^* + M^*M$. We begin with a few inner product formulas.

Lemma 5.1. *For* $0 \leq i, j, r, s \leq D$,

(i) $\langle A_i A_j^*, A_r^* A_s \rangle = \delta_{is}\delta_{jr}|X|k_i m_j u_i(\theta_j)$,

(ii) $\langle A_i A_j^*, A_r A_s^* \rangle = \delta_{ir}\delta_{js}|X|k_i m_j$,

(iii) $\langle A_i^* A_j, A_r^* A_s \rangle = \delta_{ir}\delta_{js}|X|k_i m_j$.

Proof. (i): Since

$$\langle A_i A_j^*, A_r^* A_s \rangle = \operatorname{tr}(A_j^* A_i A_r^* A_s) = \sum_{y \in X} \sum_{z \in X} (A_j^*)_{yy}(A_i)_{yz}(A_r^*)_{zz}(A_s)_{zy}$$

and by (2.9), it follows that

$$\begin{aligned}\langle A_i A_j^*, A_r^* A_s \rangle &= |X|^2 \sum_{y \in X} \sum_{z \in X} (E_j)_{xy}(A_i)_{yz}(E_r)_{xz}(A_s)_{zy} \\ &= |X|^2 \sum_{y \in X} \sum_{z \in X} (E_j)_{xy}(A_i \circ A_s)_{yz}(E_r)_{zx}.\end{aligned}$$

Since $A_i \circ A_s = \delta_{is} A_i$ $(0 \leq i, s \leq D)$, we get

$$\langle A_i A_j^*, A_r^* A_s \rangle = |X|^2 \delta_{is} \sum_{y \in X} \sum_{z \in X} (E_j)_{xy}(A_i)_{yz}(E_r)_{zx}.$$

Since

$$\sum_{y \in X} \sum_{z \in X} (E_j)_{xy}(A_i)_{yz}(E_r)_{zx} = |X|^{-1} \operatorname{tr}(E_j A_i E_r),$$

we have

$$\langle A_i A_j^*, A_r^* A_s \rangle = |X|\delta_{is} \operatorname{tr}(E_j A_i E_r) = |X|\delta_{is} \operatorname{tr}(E_r E_j A_i).$$

Since $E_i E_j = \delta_{ij} E_i$ $(0 \leq i, j \leq D)$, we obtain

$$\langle A_i A_j^*, A_r^* A_s \rangle = |X|\delta_{is}\delta_{jr} \operatorname{tr}(E_j A_i) = |X|\delta_{is}\delta_{jr}\langle E_j, A_i \rangle.$$

By (2.5) and Lemma 4.2 (i), we get $\langle E_j, A_i \rangle = m_j u_i(\theta_j) k_i$. Hence

$$\langle A_i A_j^*, A_r^* A_s \rangle = \delta_{is}\delta_{jr}|X|k_i m_j u_i(\theta_j).$$

(ii): Since $A_0 = I$, we get $\langle A_i A_j^*, A_r A_s^* \rangle = \langle A_i A_j^* A_0, A_r A_s^* A_0 \rangle$. By (2.10), we obtain

$$\langle A_i A_j^*, A_r A_s^* \rangle = m_j m_s \sum_{h=0}^{D} u_h(\theta_j) \sum_{\ell=0}^{D} u_\ell(\theta_s) \langle A_i E_h^* A_0, A_r E_\ell^* A_0 \rangle.$$

From Lemma 3.3 we have

$$\langle A_i E_h^* A_0, A_r E_\ell^* A_0 \rangle = \sum_{t=0}^{D} k_t p_{ir}^t p_{h\ell}^t p_{00}^t.$$

By (2.2) and (2.3), we obtain

$$\begin{aligned}\langle A_iA_j^*, A_rA_s^*\rangle &= m_jm_s\sum_{h=0}^{D}u_h(\theta_j)\sum_{\ell=0}^{D}u_\ell(\theta_s)k_0p_{ir}^0p_{h\ell}^0\\ &= \delta_{ir}k_im_jm_s\sum_{h=0}^{D}u_h(\theta_j)u_h(\theta_s)k_h.\end{aligned}$$

By (2.6), we get

$$\langle A_iA_j^*, A_rA_s^*\rangle = \delta_{ir}k_im_jm_s\delta_{js}m_s^{-1}|X| = \delta_{ir}\delta_{js}|X|k_im_j.$$

(iii): Since

$$\langle A_i^*A_j, A_r^*A_s\rangle = \operatorname{tr}((A_i^*A_j)^t\overline{(A_r^*A_s)}) = \operatorname{tr}(A_jA_i^*A_r^*A_s) = \operatorname{tr}(A_i^*A_r^*A_sA_j)$$

and

$$A_i^*A_r^* = \sum_{h=0}^{D}q_{ir}^hA_h^*$$

and by (2.1), we get

$$\begin{aligned}\langle A_i^*A_j, A_r^*A_s\rangle &= \sum_{h=0}^{D}\sum_{\ell=0}^{D}q_{ir}^hp_{js}^\ell\operatorname{tr}(A_h^*A_\ell) = \sum_{h=0}^{D}\sum_{\ell=0}^{D}q_{ir}^hp_{js}^\ell\operatorname{tr}(A_\ell A_h^*)\\ &= \sum_{h=0}^{D}\sum_{\ell=0}^{D}q_{ir}^hp_{js}^\ell\operatorname{tr}(A_\ell^t\overline{A_h^*}) = \sum_{h=0}^{D}\sum_{\ell=0}^{D}q_{ir}^hp_{js}^\ell\langle A_\ell, A_h^*\rangle.\end{aligned}$$

From Lemma 4.4, we have

$$\sum_{h=0}^{D}\sum_{\ell=0}^{D}q_{ir}^hp_{js}^\ell\langle A_\ell, A_h^*\rangle = |X|\sum_{h=0}^{D}\sum_{\ell=0}^{D}q_{ir}^hp_{js}^\ell\delta_{\ell 0}\delta_{h0}k_\ell = |X|q_{ir}^0p_{js}^0k_0 = |X|q_{ir}^0p_{js}^0.$$

By (2.3) and since $q_{ir}^0 = \delta_{ir}m_i$, we obtain

$$\langle A_i^*A_j, A_r^*A_s\rangle = \delta_{ir}\delta_{js}|X|k_jm_i. \qquad \square$$

Next we obtain orthogonal bases for MM^* and M^*M.

Lemma 5.2. *The following statements hold:*

(i) The matrices $\{A_iA_j^* \mid 0 \le i, j \le D\}$ form an orthogonal basis for MM^*.

(ii) The matrices $\{A_j^*A_i \mid 0 \le i, j \le D\}$ form an orthogonal basis for M^*M.

Proof. Immediate from Lemma 5.1. □

Lemma 5.3. *Each of the following subspaces has dimension $(D+1)^2$:*

$$MM^*, \qquad\qquad M^*M.$$

Proof. Immediate from Lemma 5.2. □

Our next goal is to obtain an orthogonal basis for $MM^* + M^*M$.

Lemma 5.4. *We have*

$$MM^* + M^*M = \sum_{i=0}^{D}\sum_{j=0}^{D} \mathrm{Span}\{A_iA_j^*, A_j^*A_i\} \qquad \textit{(orthogonal direct sum).}$$

Proof. Immediate from Lemma 5.1. □

Corollary 5.5. *We have*

$$\dim(MM^* + M^*M) = \sum_{i=0}^{D}\sum_{j=0}^{D} \dim(\mathrm{Span}\{A_iA_j^*, A_j^*A_i\}).$$

Proof. Immediate from Lemma 5.4. □

Definition 5.6. For $0 \le i,j \le D$ let $H_{i,j}$ denote the 2×2 matrix of inner products for $A_iA_j^*, A_j^*A_i$.

Lemma 5.7. *For* $0 \le i,j \le D$,

$$H_{i,j} = |X|k_im_j \begin{pmatrix} 1 & u_i(\theta_j) \\ u_i(\theta_j) & 1 \end{pmatrix}.$$

Proof. Immediate from Lemma 5.1 and Definition 5.6. □

Lemma 5.8. *For* $0 \le i,j \le D$ *we have*

$$\det(H_{i,j}) = |X|^2k_i^2m_j^2(1-(u_i(\theta_j))^2).$$

Proof. Immediate from Lemma 5.7. □

Corollary 5.9. *For* $0 \le i,j \le D$, $\det(H_{i,j}) = 0$ *if and only if* $u_i(\theta_j) = \pm 1$.

Proof. Immediate from Lemma 5.8. □

Lemma 5.10. *The following elements are orthogonal for* $0 \le i,j \le D$*:*

$$A_iA_j^* + A_j^*A_i, \qquad\qquad A_iA_j^* - A_j^*A_i.$$

Moreover

$$||A_iA_j^* + A_j^*A_i||^2 = 2|X|k_im_j(1+u_i(\theta_j)),$$
$$||A_iA_j^* - A_j^*A_i||^2 = 2|X|k_im_j(1-u_i(\theta_j)).$$

Proof. Immediate from Lemma 5.7. □

Lemma 5.11. *The following statements hold for* $0 \le i,j \le D$*:*

(i) Assume $u_i(\theta_j) = 1$. *Then* $A_iA_j^* = A_j^*A_i$ *and this common value is nonzero.*

(ii) Assume $u_i(\theta_j) = -1$. Then $A_i A_j^* = -A_j^* A_i$ and this common value is nonzero.

(iii) Assume $u_i(\theta_j) \neq \pm 1$. Then $A_i A_j^*, A_j^* A_i$ are linearly independent.

Proof. (i), (ii): Immediate from Lemma 5.10.
(iii): Immediate from Lemma 5.8. □

Lemma 5.12. *For $0 \leq i, j \leq D$ we give an orthogonal basis for* $\operatorname{Span}\{A_i A_j^*, A_j^* A_i\}$ *in Table 1.*

Table 1: An orthogonal basis for $\operatorname{Span}\{A_i A_j^*, A_j^* A_i\}$.

Case	Orthogonal basis	Dimension
$u_i(\theta_j) = \pm 1$	$A_i A_j^*$	1
$u_i(\theta_j) \neq \pm 1$	$A_i A_j^* + A_j^* A_i$, $A_i A_j^* - A_j^* A_i$	2

Proof. Follows from Definition 5.6 and Lemmas 5.7 and 5.11. □

Corollary 5.13. *The following is an orthogonal basis for $MM^* + M^*M$:*

$$\{A_i A_j^* + A_j^* A_i,\ A_i A_j^* - A_j^* A_i \mid 0 \leq i, j \leq D,\ u_i(\theta_j) \neq \pm 1\} \cup \{A_i A_j^* \mid 0 \leq i, j \leq D,\ u_i(\theta_j) = \pm 1\}.$$

Proof. Immediate from Lemmas 5.4 and 5.12. □

Our next goal is to find the dimension of $MM^* + M^*M$.

Definition 5.14. Define an integer P as follows:

$$P = |\{(i, j) \mid 1 \leq i, j \leq D, u_i(\theta_j) = \pm 1\}|.$$

Remark 5.15. Recall that $u_0(\theta_j) = 1$ and $u_i(\theta_0) = 1$ for $0 \leq i, j \leq D$. By [2, A.5], the graph Γ is primitive if and only if Γ_i is connected for $1 \leq i \leq D$. From Definition 5.14 and [2, Proposition 4.4.7] we have $P = 0$ if and only if Γ is primitive.

Lemma 5.16.

$$\dim(MM^* + M^*M) = 2D^2 + 2D + 1 - P.$$

Proof. Immediate from Corollary 5.13 and Definition 5.14. □

Our next goal is to obtain an orthogonal basis for $MM^* \cap M^*M$.

Lemma 5.17.

$$\dim(MM^* \cap M^*M) = 2D + 1 + P.$$

Proof. By linear algebra, we have

$$\dim(MM^* \cap M^*M) = \dim(MM^*) + \dim(M^*M) - \dim(MM^* + M^*M).$$

By Lemmas 5.3 and 5.16, the result follows. □

Lemma 5.18. *The following is an orthogonal basis for $MM^* \cap M^*M$:*

$$\{A_i A_j^* \mid 0 \le i, j \le D,\, u_i(\theta_j) = \pm 1\}.$$

Proof. Immediate from Lemmas 5.11 and 5.17. □

We now have orthogonal bases for MM^*, M^*M, $MM^* \cap M^*M$ and $MM^* + M^*M$. The next results establish an orthogonal basis for certain orthogonal complements along with the dimension for these orthogonal complements.

Lemma 5.19. *The matrices $\{A_i A_j^* \mid 1 \le i, j \le D, u_i(\theta_j) = \pm 1\}$ form an orthogonal basis for the orthogonal complement of $M + M^*$ in $MM^* \cap M^*M$.*

Proof. Follows from Lemmas 4.5 and 5.18. □

Lemma 5.20. *The following statements hold:*

- *(i) The matrices $\{A_i A_j^* \mid 1 \le i, j \le D, u_i(\theta_j) \ne \pm 1\}$ form an orthogonal basis for the orthogonal complement of $MM^* \cap M^*M$ in MM^*.*
- *(ii) The matrices $\{A_j^* A_i \mid 1 \le i, j \le D, u_i(\theta_j) \ne \pm 1\}$ form an orthogonal basis for the orthogonal complement of $MM^* \cap M^*M$ in M^*M.*

Proof. Follows from Lemmas 5.2 and 5.18. □

Lemma 5.21. *The following statements hold:*

- *(i) The matrices $\{u_i(\theta_j) A_i A_j^* - A_j^* A_i \mid 1 \le i, j \le D, u_i(\theta_j) \ne \pm 1\}$ form an orthogonal basis for the orthogonal complement of MM^* in $MM^* + M^*M$.*
- *(ii) The matrices $\{A_i A_j^* - u_i(\theta_j) A_j^* A_i \mid 1 \le i, j \le D, u_i(\theta_j) \ne \pm 1\}$ form an orthogonal basis for the orthogonal complement of M^*M in $MM^* + M^*M$.*

Proof. (i): By Lemma 5.1, for $0 \le i, j, r, s \le D$

$$\begin{aligned}
&\langle A_r A_s^* + A_s^* A_r, u_i(\theta_j) A_i A_j^* - A_j^* A_i \rangle \\
&\quad = u_i(\theta_j) \langle A_r A_s^*, A_i A_j^* \rangle - \langle A_r A_s^*, A_j^* A_i \rangle + u_i(\theta_j) \langle A_s^* A_r, A_i A_j^* \rangle - \langle A_s^* A_r, A_j^* A_i \rangle \\
&\quad = \delta_{ir}\delta_{js} |X| k_i m_j u_i(\theta_j) - \delta_{ir}\delta_{js} |X| k_i m_j u_i(\theta_j) \\
&\qquad\qquad + \delta_{ir}\delta_{js} |X| k_i m_j (u_i(\theta_j))^2 - \delta_{ir}\delta_{js} |X| k_i m_j \\
&\quad = \delta_{ir}\delta_{js} |X| k_i m_j ((u_i(\theta_j))^2 - 1).
\end{aligned}$$

By similar arguments,

$$\langle A_r A_s^* - A_s^* A_r, u_i(\theta_j) A_i A_j^* - A_j^* A_i \rangle = \delta_{ir}\delta_{js} |X| k_i m_j (1 - (u_i(\theta_j))^2)$$

for $0 \le i, j, r, s \le D$. By Lemma 5.1, for $0 \le i, j, r, s \le D$

$$\begin{aligned}
\langle A_r A_s^*, u_i(\theta_j) A_i A_j^* - A_j^* A_i \rangle &= u_i(\theta_j) \langle A_r A_s^*, A_i A_j^* \rangle - \langle A_r A_s^*, A_j^* A_i \rangle \\
&= \delta_{ir}\delta_{js} |X| k_i m_j u_i(\theta_j) - \delta_{ir}\delta_{js} |X| k_i m_j u_i(\theta_j) \\
&= 0.
\end{aligned}$$

By Lemma 5.2 and Corollary 5.13, the result follows.

(ii): Similar to the proof of (i). □

Lemma 5.22. *The following subspace has dimension P:*

$$(M + M^*)^\perp \cap (MM^* \cap M^*M).$$

Proof. Immediate from Definition 5.14 and Lemma 5.19. □

Lemma 5.23. *Each of the following subspaces has dimension $D^2 - P$:*

$$(MM^* \cap M^*M)^\perp \cap MM^*, \qquad (MM^* \cap M^*M)^\perp \cap M^*M,$$
$$(MM^*)^\perp \cap (MM^* + M^*M), \qquad (M^*M)^\perp \cap (MM^* + M^*M).$$

Proof. Immediate from Definition 5.14 and Lemmas 5.20 and 5.21. □

6 Summary of main results

In Sections 4 and 5 we obtained an orthogonal basis and the dimension for each subspace U in the diagram of Figure 1 up to $MM^* + M^*M$. Also, for each edge $U \subseteq W$ shown in this part of the diagram of Figure 1, we obtained an orthogonal basis for the orthogonal complement of U in W along with the dimension of this orthogonal complement. The results are summarized in this section.

Theorem 6.1. *In each row of Table 2 we describe a subspace U in the diagram of Figure 1. We give an orthogonal basis for U along with the dimension of U.*

Table 2: An orthogonal basis for each subspace U in the diagram of Figure 1 along with its dimension.

Subspace U	Orthogonal basis for U	Dimension of U
$M \cap M^*$	I	1
M	$\{A_i\}_{i=0}^D$	$D+1$
M^*	$\{A_i^*\}_{i=0}^D$	$D+1$
$M + M^*$	$\{A_D, \dots, A_1, I, A_1^*, \dots, A_D^*\}$	$2D+1$
$MM^* \cap M^*M$	$\{A_iA_j^* \mid 0 \le i,j \le D, u_i(\theta_j) = \pm 1\}$	$2D+1+P$
MM^*	$\{A_iA_j^* \mid 0 \le i,j \le D\}$	$(D+1)^2$
M^*M	$\{A_j^*A_i \mid 0 \le i,j \le D\}$	$(D+1)^2$
$MM^* + M^*M$	$\{A_iA_j^* + A_j^*A_i, A_iA_j^* - A_j^*A_i \mid$ $0 \le i,j \le D, u_i(\theta_j) \ne \pm 1\}$ $\cup \{A_iA_j^* \mid 0 \le i,j \le D, u_i(\theta_j) = \pm 1\}$	$2D^2+2D+1-P$

Theorem 6.2. *In each row of Table 3 we describe an edge $U \subseteq W$ from the diagram of Figure 1. We give an orthogonal basis for the orthogonal complement of U in W along with the dimension of this orthogonal complement.*

Table 3: An orthogonal basis for the orthogonal complement of U in W in the diagram of Figure 1 along with the dimension of this orthogonal complement.

U	W	Orthogonal basis for $U^\perp \cap W$	Dimension of $U^\perp \cap W$
$M \cap M^*$	M	$\{A_i\}_{i=1}^D$	D
$M \cap M^*$	M^*	$\{A_i^*\}_{i=1}^D$	D
M	$M + M^*$	$\{A_i^*\}_{i=1}^D$	D
M^*	$M + M^*$	$\{A_i\}_{i=1}^D$	D
$M + M^*$	$MM^* \cap M^*M$	$\{A_iA_j^* \mid 1 \leq i,j \leq D, u_i(\theta_j) = \pm 1\}$	P
$MM^* \cap M^*M$	MM^*	$\{A_iA_j^* \mid 1 \leq i,j \leq D, u_i(\theta_j) \neq \pm 1\}$	$D^2 - P$
$MM^* \cap M^*M$	M^*M	$\{A_j^*A_i \mid 1 \leq i,j \leq D, u_i(\theta_j) \neq \pm 1\}$	$D^2 - P$
MM^*	$MM^* + M^*M$	$\{u_i(\theta_j)A_iA_j^* - A_j^*A_i \mid 1 \leq i,j \leq D,\ u_i(\theta_j) \neq \pm 1\}$	$D^2 - P$
M^*M	$MM^* + M^*M$	$\{A_iA_j^* - u_i(\theta_j)A_j^*A_i \mid 1 \leq i,j \leq D,\ u_i(\theta_j) \neq \pm 1\}$	$D^2 - P$

7 Open problems

In this section, we give some open problems and suggestions for future research. Earlier in the paper we discussed the diagram of Figure 1. In this discussion we analyzed the diagram up to $MM^* + M^*M$. The remaining part of the diagram is not completely understood. We mention what is known. By Lemma 3.1 the subspace M^*MM^* has an orthogonal basis $\{E_i^*A_jE_h^* \mid 0 \leq h,i,j \leq D,\ p_{ij}^h \neq 0\}$. Similarly, the subspace MM^*M has an orthogonal basis $\{E_iA_j^*E_h \mid 0 \leq h,i,j \leq D,\ q_{ij}^h \neq 0\}$.

Problem 7.1. Find an orthogonal basis for the following subspaces:

(i) $MM^*M \cap M^*MM^*$,

(ii) $MM^*M + M^*MM^*$.

Problem 7.2. In each row of Table 4 we give an edge $U \subseteq W$ from the diagram of Figure 1. Find an orthogonal basis for the orthogonal complement of U in W for the following cases.

Table 4: Subspaces U and W from the diagram of Figure 1.

U	W
$MM^* + M^*M$	$MM^*M \cap M^*MM^*$
$MM^*M \cap M^*MM^*$	MM^*M
$MM^*M \cap M^*MM^*$	M^*MM^*
MM^*M	$MM^*M + M^*MM^*$
M^*MM^*	$MM^*M + M^*MM^*$

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