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On global location-domination in graphs*

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Abstract

A dominating set S of a graph G is called *locating-dominating*, *LD-set* for short, if every vertex v not in S is uniquely determined by the set of neighbors of v belonging to S. Locating-dominating sets of minimum cardinality are called LD-codes and the cardinality of an LD-code is the *location-domination number* $\lambda(G)$. An LD-set S of a graph G is *global* if it is an LD-set of both G and its complement $\overline{G}$. The *global location-domination number* $\lambda_g(G)$ is introduced as the minimum cardinality of a global LD-set of G.

In this paper, some general relations between LD-codes and the location-domination number in a graph and its complement are presented first. Next, a number of basic properties involving the global location-domination number are showed. Finally, both parameters are studied in-depth for the family of block-cactus graphs.

Keywords: Domination, global domination, locating domination, complement graph, block-cactus.

Math. Subj. Class.: 05C35, 05C69

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1 Introduction

Let $G = (V, E)$ be a simple, finite graph. The *open neighborhood* of a vertex $v \in V$ is $N_G(v) = \{u \in V : uv \in E\}$ and the *close neighborhood* is $N_G[v] = \{u \in V : uv \in E\} \cup \{v\}$. The *complement* of a graph G, denoted by $\overline{G}$, is the graph on the same vertices such that two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G. The distance between vertices $v, w \in V$ is denoted by $d_G(v, w)$. We write $N(u)$ or $d(v, w)$ if the graph G is clear from the context. Assume that G and H is a pair of graphs whose vertex sets are disjoint. The *union* $G + H$ is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. The *join* $G \vee H$ has $V(G) \cup V(H)$ as vertex set and $E(G) \cup E(H) \cup \{uv : u \in v(G)$ and $v \in V(H)\}$ as edge set. For further notation, see [6].

A set $D \subseteq V$ is a *dominating set* if for every vertex $v \in V \setminus D$, $N(v) \cap D \neq \emptyset$. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. A dominating set is *global* if it is a dominating set of both G and its complement graph, $\overline{G}$. The minimum cardinality of a global dominating set of G is the *global domination number* of G, denoted with $\gamma_g(G)$ [3, 4, 18]. If D is a subset of V and $v \in V \setminus D$, we say that v *dominates* D if $D \subseteq N(v)$.

A set $S \subseteq V$ is a *locating set* if every vertex is uniquely determined by its vector of distances to the vertices in S. The *location number* of G $\beta(G)$ is the minimum cardinality of a locating set of G [10, 12, 20].

A set $S \subseteq V$ is a *locating-dominating set*, *LD-set* for short, if S is a dominating set such that for every two different vertices $u, v \in V \setminus S$, $N(u) \cap S \neq N(v) \cap S$. The *location-domination number* of G, denoted by $\lambda(G)$, is the minimum cardinality of a locating-dominating set. A locating-dominating set of cardinality $\lambda(G)$ is called an *LD-code* [21]. Certainly, every LD-set of a non-connected graph G is the union of LD-sets of its connected components and the location-domination number is the sum of the location-domination number of its connected components. Notice also that a locating-dominating set is both a locating set and a dominating set, and thus $\beta(G) \leq \lambda(G)$ and $\gamma(G) \leq \lambda(G)$. LD-codes and the location-domination parameter have been intensively studied during the last decade; see [1, 2, 5, 8, 13, 15] A complete and regularly updated list of papers on locating dominating codes is to be found in [16].

A *block* of a graph is a maximal connected subgraph with no cut vertices. A graph is a *block graph* if it is connected and each of its blocks is complete. A connected graph G is a *cactus* if all its blocks are cycles or complete graphs of order at most 2. Cactus are characterized as those graphs such that two different cycles share at most one vertex. A *block-cactus* is a connected graph such that each of its blocks is either a cycle or a complete graph. The family of block-cactus graphs is interesting because, among other reasons, it contains all cycles, trees, complete graphs, block graphs, unicyclic graphs and cactus (see Figure 1). Cactus, block graphs, and block-cactus have been studied extensively in different contexts, including the domination one; see [7, 11, 17, 22, 23].

The remaining part of this paper is organized as follows. In Section 2, we deal with the problem of relating the locating-dominating sets and the location-domination number of a graph and its complement. Also, *global LD-sets* and *global LD-codes* are defined. In Section 3, we introduce the so-called *global location-domination number*, and show some basic properties for this new parameter. In Section 4, we are concerned with the study of the sets and parameters considered in the preceding sections for the family of *block-cactus* graphs. Finally, the last section is devoted to address some open problems.

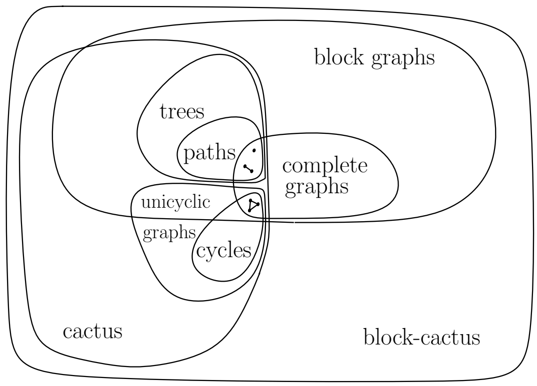

Figure 1: Families of block-cactus.

2 Relating $\lambda(G)$ to $\lambda(\overline{G})$

This section is devoted to approach the relationship between $\lambda(G)$ and $\lambda(\overline{G})$, for any arbitrary graph G. Some of the results we present were previously shown in [13] and we include them for the sake of completeness.

Notice that $N_{\overline{G}}(x) \cap S = S \setminus N_G(x)$ for any set $S \subseteq V$ and any vertex $x \in V \setminus S$. A straightforward consequence of this fact is the following lemma.

Lemma 2.1. *Let $G = (V, E)$ be a graph and $S \subseteq V$. If $x, y \in V \setminus S$, then $N_G(x) \cap S \neq N_G(y) \cap S$ if and only if $N_{\overline{G}}(x) \cap S \neq N_{\overline{G}}(y) \cap S$.*

As an immediate consequence of this lemma, the following result is derived.

Proposition 2.2. *If $S \subseteq V$ is an LD-set of a graph $G = (V, E)$, then S is an LD-set of $\overline{G}$ if and only if S is a dominating set of $\overline{G}$.*

Proposition 2.3 ([13]). *If $S \subseteq V$ is an LD-set of a graph $G = (V, E)$, then S is an LD-set of $\overline{G}$ if and only if there is no vertex in $V \setminus S$ dominating S in G.*

Proof. By Proposition 2.2, S is an LD-set of $\overline{G}$ if and only if S is a dominating set of $\overline{G}$. But S is a dominating set of $\overline{G}$ if and only if $N_{\overline{G}}(u) \cap S \neq \emptyset$, for any vertex $u \in V \setminus S$. This condition is equivalent to $N_G(u) \cap S \neq S$ for any vertex $u \in V \setminus S$. Therefore, S is an LD-set of $\overline{G}$ if and only if there is no vertex $u \in V \setminus S$ such that $S \subseteq N_G(u)$, that is, there is no vertex in $V \setminus S$ dominating S. □

Proposition 2.4 ([13]). *If $S \subseteq V$ is an LD-set of a graph $G = (V, E)$ then there is at most one vertex $u \in V \setminus S$ dominating S, and in the case it exists, $S \cup \{u\}$ is an LD-set of $\overline{G}$.*

Proof. By definition of LD-set of G, there is at most one vertex adjacent to all vertices of S. Moreover, u is the only vertex not adjacent to any vertex of S in $\overline{G}$. Therefore $S \cup \{u\}$ is an LD-set of G and a dominating set of $\overline{G}$. By Proposition 2.2, it is also an LD-set of $\overline{G}$. □

Theorem 2.5 ([13]). *For every graph G, $|\lambda(G) - \lambda(\overline{G})| \leq 1$.*

Proof. If S has an LD-code of G not containing a vertex dominating S, then S is an LD-set of $\overline{G}$ by 2.3. Consequently, $\lambda(\overline{G}) \leq \lambda(G)$. If S is an LD-code of G with a vertex $u \in V \setminus S$ dominating S, then $S \cup \{u\}$ is an LD-set of $\overline{G}$ by 2.4. Hence, $\lambda(\overline{G}) \leq \lambda(G) + 1$. In any case, $\lambda(\overline{G}) - \lambda(G) \leq 1$. By symmetry, $\lambda(G) - \lambda(\overline{G}) \leq 1$, and thus $|\lambda(G) - \lambda(\overline{G})| \leq 1$. □

According to the preceding result, for every graph G, $\lambda(\overline{G}) \in \{\lambda(G)-1, \lambda(G), \lambda(G)+1\}$, all cases being feasible for some connected graph G. For example, it is easy to check that the star $K_{1,n-1}$ of order $n \geq 2$ satisfies $\lambda(\overline{K_{1,n-1}}) = \lambda(K_{1,n-1})$, and the bi-star $K_2(r,s)$, $r, s \geq 2$, obtained by joining the central vertices of two stars $K_{1,r}$ and $K_{1,s}$, satisfies $\lambda(\overline{K_2(r,s)}) = \lambda(K_2(r,s)) - 1$.

We intend to obtain either necessary or sufficient conditions for a graph G to satisfy $\lambda(\overline{G}) > \lambda(G)$, i.e., $\lambda(\overline{G}) = \lambda(G) + 1$. After noticing that this fact is closely related to the existence or not of sets that are simultaneously locating-dominating sets in both G and its complement $\overline{G}$, the following definition is introduced.

Definition 2.6. A set S of vertices of a graph G is a *global LD-set* if S is an LD-set of both G and its complement $\overline{G}$.

Certainly, an LD-set is non-global if and only if there exists a (unique) vertex $u \in V(G) \setminus S$ which dominates S, i.e., such that $S \subseteq N(u)$.

Accordingly, an LD-code S of a graph G is said to be *global* if it is a global LD-set, i.e. if S is both an LD-code of G and an LD-set of $\overline{G}$. In terms of this new definition, a result proved in [13] can be presented as follows.

Proposition 2.7 ([13])**.** *If G is a graph with a global LD-code, then $\lambda(\overline{G}) \leq \lambda(G)$.*

Proposition 2.8. *If G is a graph with a non-global LD-set S and u is the only vertex dominating S, then the following conditions are satisfied:*

1. *The eccentricity of u is $ecc(u) \leq 2$;*
2. *the radius of G is $rad(G) \leq 2$;*
3. *the diameter of G is $diam(G) \leq 4$;*
4. *the maximum degree of G is $\Delta(G) \geq \lambda(G)$.*

Proof. If $x \in N(u)$, then $d(u,x) = 1$. If $x \notin N(u)$, since S is a dominating set of G, then there exists a vertex $y \in S \cap N(x) \subseteq N(u)$. Hence, $ecc(u) \leq 2$. Consequently, $rad(G) \leq 2$ and $diam(G) \leq 4$. By the other hand, $deg_G(u) = |N_G(u)| \geq |S| = \lambda(G)$, implying that $\Delta(G) \geq \lambda(G)$. □

Corollary 2.9. *If G is a graph satisfying $\lambda(\overline{G}) = \lambda(G) + 1$, then G is a connected graph such that $rad(G) \leq 2$, $diam(G) \leq 4$ and $\Delta(G) \geq \lambda(G)$.*

The above result is tight in the sense that there are graphs of diameter 4 and radius 2 (resp. $\Delta(G) = \lambda(G)$), verifying $\lambda(\overline{G}) = \lambda(G) + 1$. The graph displayed in Figure 2 is an example of graph satisfying $rad(G) = 2$, $diam(G) = 4$ and $\lambda(\overline{G}) = \lambda(G) + 1$, and the complete graph K_n is an example of a graph such that $\Delta(G) = \lambda(G)$ and $\lambda(\overline{G}) = \lambda(G) + 1$, since $\lambda(\overline{K_n}) = n$, $\lambda(K_n) = \Delta(K_n) = n - 1$.

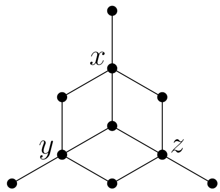

Figure 2: This graph satisfies: $rad(G) = 2$, $diam(G) = 4$, $\lambda(G) = 3$, $\lambda(\overline{G}) = 4$ and $\{x, y, z\}$ is a non-global LD-code.

3 The global location-domination number

Definition 3.1. The *global location-domination number* of a graph G, denoted by $\lambda_g(G)$, is defined as the minimum cardinality of a global LD-set of G.

Notice that, for every graph G, $\lambda_g(\overline{G}) = \lambda_g(G)$, since for every set of vertices $S \subset V(G) = V(\overline{G})$, S is a global LD-set of G if and only if it is a global LD-set of $\overline{G}$.

Proposition 3.2. *For any graph $G = (V, E)$, $\lambda(G) \le \lambda_g(G) \le \lambda(G) + 1$.*

Proof. The first inequality is a consequence of the fact that a global LD-set of G is also an LD-set of G. For the second inequality, suppose that S is an LD-code of G, i.e. $|S| = \lambda(G)$. If S is a global LD-set of G, then $\lambda_g(G) = \lambda(G)$. Otherwise, there exists a vertex $u \in V \setminus S$ dominating S and $S \cup \{u\}$ is an LD-set of $\overline{G}$. Therefore, $\lambda_g(G) \le \lambda(G) + 1$. □

Corollary 3.3. *For any graph $G = (V, E)$, $\max\{\lambda(G), \lambda(\overline{G})\} \le \lambda_g(G) \le \min\{\lambda(G) + 1, \lambda(\overline{G}) + 1\}$.*

Corollary 3.4. *Let $G = (V, E)$ be a graph.*

- *If $\lambda(G) \ne \lambda(\overline{G})$, then $\lambda_g(G) = \max\{\lambda(G), \lambda(\overline{G})\}$.*
- *If $\lambda(G) = \lambda(\overline{G})$, then $\lambda_g(G) \in \{\lambda(G), \lambda(G)+1\}$, and both possibilities are feasible.*

Proof. Both statements are consequence of Proposition 3.2. Next, we give some examples to illustrate all possibilities given. It is easy to check that the complete graph K_2 satisfies $1 = \lambda(K_2) \ne \lambda(\overline{K_2}) = 2$ and $\lambda_g(K_2) = \lambda(\overline{K_2})$; the path P_3 satisfies $\lambda(P_3) = \lambda(\overline{P_3}) = \lambda_g(P_3) = 2$ and the cycle C_5, satisfies $\lambda(C_5) = \lambda(\overline{C_5}) = 2$ and $\lambda_g(C_5) = 3$. □

Proposition 3.5. *For any graph $G = (V, E)$, $\lambda_g(G) = \lambda(G) + 1$ if and only if every LD-code of G is non-global.*

Proof. A global LD-code of G is an LD-set of both G and $\overline{G}$. Hence, if G contains at least a global LD-code, then $\lambda_g(G) = \lambda(G)$. Conversely, if every LD-code of G is non-global, then there is no global LD-set of G of size $\lambda(G)$. Then, $\lambda_g(G) = \lambda(G) + 1$. □

As a consequence of Propositions 2.8 and 3.5, the following corollary holds.

Corollary 3.6. *If G is a graph with $diam(G) \ge 5$, then $\lambda_g(G) = \lambda(G)$.*

We finalize this section by determining the exact values of $\lambda(G)$, $\lambda(\overline{G})$ and $\lambda_g(G)$ for some basic graph families.

Lemma 3.7. *If $n \geq 7$, then $\lambda(\overline{C_n}) = \lambda(\overline{P_n}) = \lambda(P_{n-1})$.*

Proof. Firsty, we prove that $\lambda(\overline{C_n}) \leq \lambda(P_{n-1})$ and $\lambda(\overline{P_n}) \leq \lambda(P_{n-1})$. Suppose that $V(P_{n-1}) = \{1, 2, \ldots, n-1\}$ and $E(P_{n-1}) = \{(i, i+1) : i = 1, 2, ..., n-2\}$ are the vertex set and the edge set of P_{n-1}, respectively. Assume that S is an LD-code of P_{n-1} such that S does not contain vertex 1 neither $n-1$ (it is easy to construct such an LD-code from those given in [1]). Since $n-1 \geq 6$, S has at least 3 vertices and there is no vertex in $V(P_{n-1}) \setminus S$ dominating S in P_{n-1}. Hence, S is an LD-set of $\overline{P_{n-1}}$.

Next, consider the graph G^* obtained by adding to the graph $\overline{P_{n-1}}$ a new vertex u adjacent to the vertices $2, 3, \ldots, n-2$, and may be to 1 or $n-1$. Clearly, by construction, u is adjacent to all vertices of S in G^* and there is no vertex in $\overline{P_{n-1}}$ adjacent to all vertices in S. Therefore, S is an LD-set of G^* and $\lambda(G^*) \leq \lambda(P_{n-1})$. Finally, observe that if u is not adjacent to 1, neither to $n-1$, then G^* is the graph $\overline{C_n}$ and if u is adjacent to exactly one of the vertices 1 or $n-1$, then G^* is the graph $\overline{P_n}$, which proves the inequalities before stated.

Lastly, we prove that $\lambda(P_{n-1}) \leq \lambda(\overline{G})$, when $G \in \{P_n, C_n\}$. Consider an LD-code S of $\overline{G}$. Let x be the only vertex dominating S in $\overline{G}$, if it exists, or any vertex not in S, otherwise. By construction, S is an LD-set of $G - x$, hence $\lambda(G - x) \leq \lambda(\overline{G})$. To end the proof, we distinguish two cases.

- If G is the cycle C_n, then $G - x$ is the path P_{n-1}, implying that $\lambda(P_{n-1}) \leq \lambda(\overline{C_n})$.
- If G if the path P_n, then $G - x$ is either the path P_{n-1} or the graph $P_r + P_s$, with $r, s \geq 1$ and $r + s = n - 1 \geq 6$. Since, $\lambda(P_r + P_s) = \lambda(P_r) + \lambda(P_s) = \lceil 2r/5 \rceil + \lceil 2s/5 \rceil \geq \lceil 2(r+s)/5 \rceil = \lambda(P_{n-1})$, we conclude that, in any case, $\lambda(P_{n-1}) \leq \lambda(\overline{P_n})$. □

Proposition 3.8. *Let G be a graph of order $n \geq 1$. If G belongs to the set $\{P_n, C_n, W_n, K_n, K_{1,n-1}, K_{r,n-r}, K_2(r, n-r-2)\}$, then the values of $\lambda(G)$ and $\lambda(\overline{G})$ are known and they are displayed in Tables 1 and 2.*

Proof. The values of the location-domination number of all these families, except the wheels, are already known (see [1, 13, 21]). Next, let us calculate the values of the location-domination number for the wheels and for the complements of all these families and also, from the results previously proved, the global location-domination number of them.

- For paths, cycles and wheels of small order, the values of $\lambda(G)$ and $\lambda_g(G)$ can easily be checked by hand (see Table 1).
- If $n \geq 7$, then $\lambda(W_n) = \lambda(C_{n-1}) = \lceil \frac{2n-2}{5} \rceil$, since (i) $W_n = K_1 \vee C_{n-1}$, (ii) every LD-code S of C_{n-1} is an LD-set of W_n, and (iii) every LD-code of C_{n-1} is global.
- $\lambda(\overline{K_n}) = \lambda(K_1 + \cdots + K_1) = \lambda(K_1) + \cdots + \lambda(K_1) = n$.
- $\lambda(\overline{K_{1,n-1}}) = \lambda(K_1 + K_{n-1}) = \lambda(K_1) + \lambda(K_{n-1}) = 1 + (n-2) = n - 1$.
- $\lambda(\overline{K_{r,n-r}}) = \lambda(K_r + K_{n-r}) = \lambda(K_r) + \lambda(K_{n-r}) = (r-1) + (n-r-1) = n - 2$, if $2 \leq r \leq n - r$.
- The complement of the bi-star $K_2(r, s)$, with $s = n - r - 2$, is the graph obtained by joining a vertex v to exactly r vertices of a complete graph of order $r + s$ and joining a vertex w to the remaining s vertices of the complete graph of order $r + s$. It is immediate to verify that the set containing all vertices except w, a vertex adjacent to v and a vertex adjacent to w is an LD-code of $\overline{K_2(r, s)}$ with $n - 3$ vertices. Thus, $\lambda(\overline{K_2(r, s)}) = n - 3$.

G	P_1	P_2	P_3	P_4	P_5	P_6	C_4	C_5	C_6	W_5	W_6	W_7
$\lambda(G)$	1	1	2	2	2	3	2	2	3	2	3	3
$\lambda(\overline{G})$	1	2	2	2	2	3	2	2	3	3	3	4
$\lambda_g(G) = \lambda_g(\overline{G})$	1	2	2	2	3	3	2	3	3	3	3	4

Table 1: The values of $\lambda(G)$, $\lambda(\overline{G})$ and $\lambda_g(G)$ of small paths, cycles and wheels.

- For every $n \geq 7$, $\lambda(\overline{P_n}) = \lambda(\overline{C_n}) = \lceil \frac{2n-2}{5} \rceil$. This result is a direct consequence of Lemma 3.7 and the fact that $\lambda(P_n) = \lambda(C_n) = \lceil \frac{2n}{5} \rceil$.
- According to Lemma 3.7, $\lambda(\overline{W_n}) = \lambda(K_1 + \overline{C_{n-1}}) = \lambda(K_1) + \lambda(\overline{C_{n-1}}) = 1 + \lambda(P_{n-2}) = 1 + \lceil 2(n-2)/5 \rceil = \lceil (2n+1)/5 \rceil$. □

Theorem 3.9. *Let G be a graph of order $n \geq 1$. If G belongs to the set $\{P_n, C_n, W_n, K_n, K_{1,n-1}, K_{r,n-r}, K_2(r, n-r-2)\}$, then $\lambda_g(G)$ is known and it is displayed in Tables 1 and 2.*

Proof. All the cases follow from Corollary 3.4, except $K_{1,n-1}$ and $K_{r,n-r}$, which are trivial. □

G	P_n	C_n	W_n	K_n	$K_{1,n-1}$	$K_{r,n-r}$	$K_2(r, n-r-2)$
order n	$n \geq 7$	$n \geq 7$	$n \geq 8$	$n \geq 2$	$n \geq 4$	$2 \leq r \leq n-r$	$2 \leq r \leq n-r-2$
$\lambda(G)$	$\lceil \frac{2n}{5} \rceil$	$\lceil \frac{2n}{5} \rceil$	$\lceil \frac{2n-2}{5} \rceil$	$n-1$	$n-1$	$n-2$	$n-2$
$\lambda(\overline{G})$	$\lceil \frac{2n-2}{5} \rceil$	$\lceil \frac{2n-2}{5} \rceil$	$\lceil \frac{2n+1}{5} \rceil$	n	$n-1$	$n-2$	$n-3$
$\lambda_g(G) =$ $= \lambda_g(\overline{G})$	$\lceil \frac{2n}{5} \rceil$	$\lceil \frac{2n}{5} \rceil$	$\lceil \frac{2n+1}{5} \rceil$	n	$n-1$	$n-2$	$n-2$

Table 2: The values of $\lambda(G)$, $\lambda(\overline{G})$ and $\lambda_g(G)$ for some families of graphs.

4 Global location-domination in block-cactus

This section is devoted to characterizing those block-cactus G satisfying $\lambda(\overline{G}) = \lambda(G)+1$. By Proposition 2.7, this equality is feasible only for graphs without global LD-codes.

We will refer in this section to some specific graphs, such as the *paw*, the *bull*; the *banner* P, the *complement of the banner*, $\overline{P}$; the *butterfly* and the *corner* L (see Figure 3).

The block-cactus of order at most 2 are K_1 and K_2. For these graphs we have $\lambda(K_1) = \lambda(\overline{K_1}) = 1$ and $\lambda(K_2) = 1 < 2 = \lambda(\overline{K_2})$.

In [5], all 16 non-isomorphic graphs with $\lambda(G) = 2$ are given. After carefully examining all cases, the following result is obtained (see Figure 4).

Proposition 4.1. *Let $G = (V, E)$ be a block-cactus such that $\lambda(G) = 2$. Then, $\lambda(\overline{G}) \geq \lambda(G)$. Moreover, $\lambda(\overline{G}) = \lambda(G) + 1 = 3$ if and only if G is isomorphic to the cycle of order 3, the paw, the butterfly or the complement of a banner.*

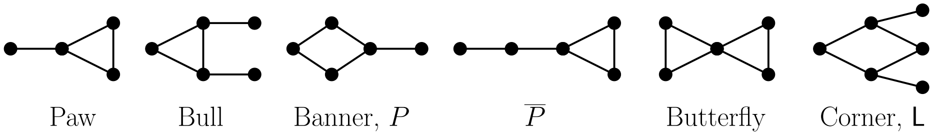

Figure 3: Some special graphs.

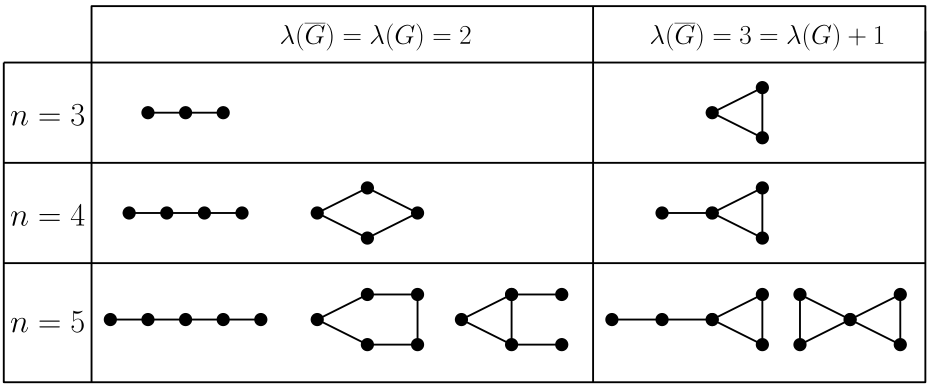

Figure 4: All block-cactus with $\lambda(G) = 2$.

Next, we approach the case $\lambda(G) \geq 3$. First of all, let us present some lemmas, providing a number of necessary conditions for a given block-cactus to have at least a non-global LD-set.

Lemma 4.2. *Let $G = (V, E)$ be a block-cactus and $S \subseteq V$ a non-global LD-set of G. If $u \in V \setminus S$ dominates S, then $G[N(u)]$ is a disjoint union of cliques.*

Proof. Let x, y be a pair of vertices belonging to the same component H of $G[N(u)]$. Suppose that $xy \notin E$ and take an $x - y$ path P in H. Let z be an inner vertex of P. Notice that the set $\{u, x, y, z\}$ is contained in the same block B of G. As B is not a clique, it must be a cycle, a contradiction, since $deg_B(u) \geq 3$. □

Lemma 4.3. *Let $G = (V, E)$ be a block-cactus and $S \subseteq V$ a non-global LD-set of G. If $u \in V \setminus S$ dominates S and $W = V \setminus N[u]$, then, for every vertex $w \in W$, the following properties hold.*

i) $1 \leq | N(u) \cap N(w) | \leq 2$.

ii) *If $N(u) \cap N(w) = \{x\}$, then $x \in S$.*

iii) *If $N(u) \cap N(w) = \{x, y\}$, then $xy \notin E$.*

iv) *If $w' \in W$ and $N(u) \cap N(w) = N(u) \cap N(w') = \{x\}$, then $w' = w$.*

v) *If $w' \in W$, $w' \neq w$ and $|N(u) \cap N(w)| = |N(u) \cap N(w')| = 2$, then $N[w] \cap N[w'] = \emptyset$.*

Proof. i),ii),iii): $|\ N(u) \cap N(w)\ | \geq 1$ as $S \subset N(u)$ and S dominates vertex w. If $N(u) \cap N(w) = \{x\}$, then necessarily $x \in S$. Assume that $|\ N(u) \cap N(w)\ | > 1$. Observe that the set $N[u] \cap N[w]$ is contained in the same block B of G. Certainly, B must be a cycle since $uw \notin E$. Hence, $|\ N(u) \cap N(w)\ | = 2$. Moreover, in this case B is isomorphic to the cycle C_4, which means that, if $V(B) = \{u, x, y, w\}$, then $xy \notin E$.
iv): If $w' \neq w$, then $S \cap N(w) \neq S \cap N(w')$, as S is an LD-set.
v): Suppose that $w \neq w'$, $N(u) \cap N(w) = \{x, y\}$ and $N(u) \cap N(w') = \{z, t\}$. Notice that $\{x, y\} \neq \{z, t\}$, since S is an LD-set. If $y = z$, then the set $\{u, w, w', x, y, t\}$ is contained in the same block B of G, a contradiction, because B is neither a clique, since $uw \notin E$, nor a cycle, as $deg_G(u) \geq 3$. Assume thus that $\{x, y\} \cap \{z, t\} = \emptyset$. If either $ww' \in E$ or $N(w) \cap N(w') \neq \emptyset$, then the set $\{u, w, w', x, y, z, t\}$ is contained in the same block B of G, again a contradiction, because B is neither a clique, since $uw \notin E$, nor a cycle, as $deg_G(u) \geq 4$. □

Lemma 4.4. *Let $G = (V, E)$ be a block-cactus and $S \subseteq V$ a non-global LD-set of G. If $u \in V \setminus S$ dominates S and $W = V \setminus N[u]$, then*

- *Every component of $G[W]$ is isomorphic either to K_1 or to K_2.*
- *If $w, w' \in W$ and $ww' \in E$, then the set $\{w, w'\}$ is contained in the same block, which is isomorphic to C_5.*

Proof. Let w, w' such that $ww' \in E$. According to Lemma 4.3, the set $\{u\} \cup N[w] \cup N[w']$ forms a block B of G, which is isomorphic to the cycle C_5. In particular, no vertex of $W \setminus \{w, w'\}$ is adjacent to w or to w'. □

As a corollary of the previous three lemmas the following proposition is obtained.

Proposition 4.5. *Let $G = (V, E)$ be a block-cactus and $S \subseteq V$ a non-global LD-set of G. If $u \in V \setminus S$ dominates S, then every maximal connected subgraph of G such that u is not a cut-vertex is isomorphic to one of the following graphs (see Figure 5):*

a) u is adjacent to every vertex of a complete graph K_r, $r \geq 1$, and each one of the vertices of K_r is adjacent to at most one new vertex of degree 1;

b) u is a vertex of a cycle of order 4, and each neighbor of u is adjacent to at most one new vertex of degree 1;

c) u is a vertex of a cycle of order 5.

In the next theorem, we characterize those block-cactus not containing any global LD-code of order at least 3.

Theorem 4.6. *Let $G = (V, E)$ be a block-cactus such that $\lambda(G) \geq 3$. Then, every LD-code of G is non-global if and only if G is isomorphic to one of the following graphs (see Figure 6):*

a) $K_1 \vee (K_1 + K_r)$, $r \geq 3$;

b) the graph obtained by joining one vertex of K_2 with a vertex of a complete graph of order $r + 1$, $r \geq 3$;

c) K_{r+1}, $r \geq 3$;

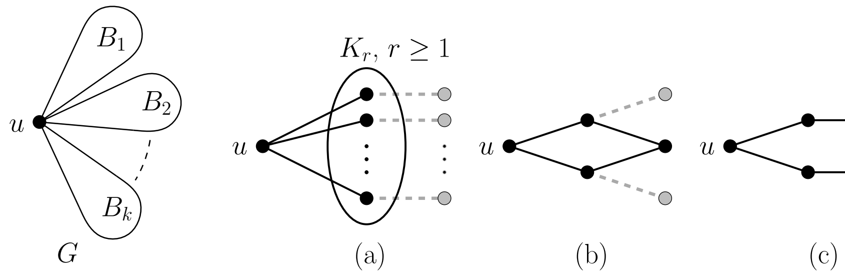

Figure 5: If $B_1, \ldots, B_k$ are the maximal connected subgraphs of G with vertex u not being a cut-vertex, each subgraph B_i is isomorphic to one of the graphs displayed in (a), (b), (c). Gray vertices are optional.

d) the graph obtained by joining a vertex of K_2 with one of the vertices of degree 2 of a corner;

e) if we consider the graph $K_1 \vee (K_{r_1} + \cdots + K_{r_t})$ and t' copies of a corner, with $t + t' \geq 2$ and $r_1, \ldots, r_t \geq 2$, the graph obtained by identifying the vertex u of K_1 with one of the vertices of degree 2 of each copy of the corner.

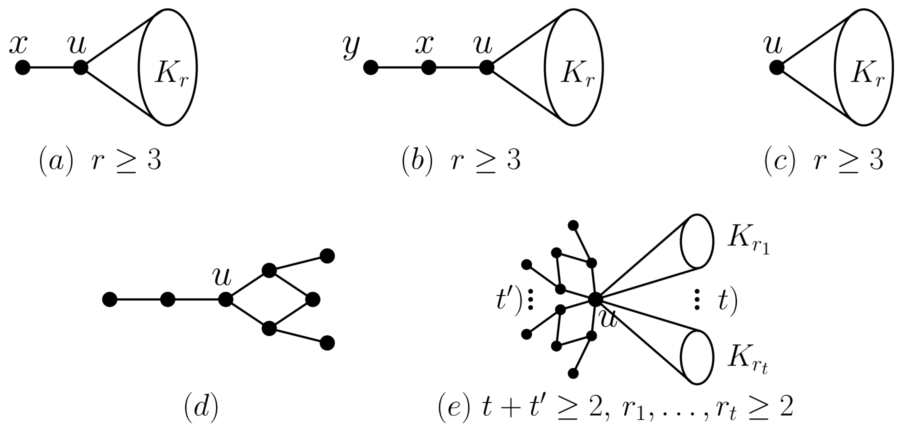

Figure 6: Block-cactus with $\lambda(G) \geq 3$ not containing any global LD-code.

Proof. Firstly, let us show that none of these graphs contains a global LD-code.

a) Let G be the graph showed in Figure 6(a). Observe that $\lambda(G) = r$ and, for every LD-code S, $|S \cap \{x, u\}| = 1$ and $|S \cap K_r| = r - 1$. Let w be the vertex of K_r not in S. If $x \in S$, then $S \subset N(u)$. Otherwise, if $u \in S$, then $S \subset N(w)$.

b) Let G be the graph showed in Figure 6(b). Notice that $\lambda(G) = r$ and, for every LD-code S, $x \in S$ and $|S \cap K_r| = r - 1$. Hence , if S is an LD-code of G, then $S \subset N(u)$.

c) If $G = K_n$ (Figure 6(c)), then G contains no global LD-code.

d) Let G be the graph showed in Figure 6(d). Clearly, the unique LD-code of G is $S = N(u)$.

e) Let G be the graph showed in Figure 6(e). In this graph, every LD-code contains both vertices adjacent to vertex u in each copy of the corner and, for every $i \in \{1, \ldots, t\}$, $r_i - 1$ vertices of K_{r_i}. Thus, for every LD-code S of G, $S \subset N(u)$.

In order to prove that these are the only graphs not containing any global LD-code, we previously need to show the following lemmas.

Lemma 4.7. *Let $G = (V, E)$ be a block-cactus and $S \subseteq V$ a non-global LD-set of G. If $u \in V \setminus S$ dominates S, then, for every component H of $G[N(u)]$ of cardinality r, $|V(H) \cap S)| = \max\{1, r - 1\}$.*

Proof. This result is an immediate consequence of Lemma 4.2 ($G[N(u)]$ is a disjoint union of cliques), along with the fact that S is an LD-set. □

Given a cut vertex u of a connected graph G, let Λ_u be the set of all maximal connected subgraphs H of G such that (i) $u \in V(H)$ and (ii) u is not a cut vertex of H. Observe that any subgraph of Λ_u can be obtained from a certain component of the graph $G - u$, by adding the vertex u according to the structure of G.

Lemma 4.8. *Let $G = (V, E)$ be a block-cactus with $\lambda(G) \geq 3$ and let $S \subseteq V$ be a non-global LD-set of G. If $u \in V \setminus S$ dominates S and the set Λ_u contains a graph isomorphic to one of the graphs displayed in Figure 7, then G has a global LD-code.*

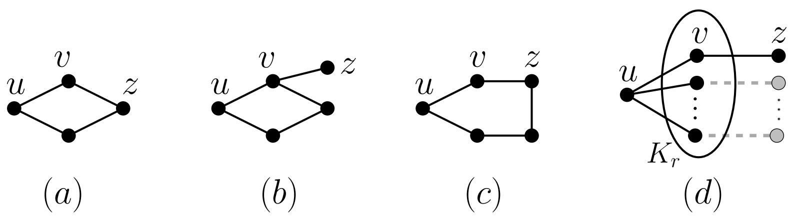

Figure 7: Some possible elements of Λ_u.

Proof. Let v, z the pair of vertices shown in Figure 7. Then, by to Lemma 4.7, $v \in S$ and $S' = (S \setminus \{v\}) \cup \{z\}$ is an LD-set de G having the same cardinality as S. Hence, S' is a global LD-code of G. □

Lemma 4.9. *Let $G = (V, E)$ be a block-cactus with $\lambda(G) \geq 3$ and let $S \subseteq V$ be a non-global LD-set of G. If $u \in V \setminus S$ dominates S and the set Λ_u contains a pair of graphs H_1 and H_2 such that $H_1, H_2 \in \{P_2, P_3\}$, then G has a global LD-code.*

Proof. If H_1 is isomorphic to P_3, with $V(H_1) = \{u, v, z\}$ and $E(H_1) = \{uv, vz\}$, then, according to Lemma 4.7, $v \in S$ and $S' = (S \setminus \{v\}) \cup \{z\}$ is an LD-set de G having the same cardinality as S. Hence, S' is a global LD-code of G.

If both H_1 and H_2 are isomorphic to P_2, and $V(H_1) = \{u,t\}$ and $E(H_1) = \{ut\}$, then, according to Lemma 4.7, $v \in S$ and $S' = (S \setminus \{t\}) \cup \{u\}$ is an LD-set de G having the same cardinality as S. Hence, S' is a global LD-code of G. □

Lemma 4.10. *Let $G = (V,E)$ be a block-cactus and $S \subseteq V$ a non-global LD-set of G whose dominating vertex is u. If Λ_u contains three graphs H_1, H_2 and H_3 such that $H_1 \in \{P_2, P_3\}$ and $H_2, H_3 \in \{K_r, \mathsf{L}\}$, where L denotes the corner graph displayed in Figure 3, then G has a global LD-code.*

Proof. If H_1 is isomorphic to P_2, with $V(H_1) = \{u,t\}$ and $E(H_1) = \{ut\}$, then, according to Lemma 4.7, $v \in S$ and $S' = (S \setminus \{t\}) \cup \{u\}$ is an LD-set de G having the same cardinality as S. Hence, S' is a global LD-code of G.

If H_1 is isomorphic to P_2, $V(H_1) = \{u,v,z\}$ and $E(H_1) = \{uv, vz\}$, then, according to Lemma 4.7, $v \in S$ and $S' = (S \setminus \{v\}) \cup \{z\}$ is an LD-set de G having the same cardinality as S. Hence, S' is a global LD-code of G. □

We are now ready to end the proof of the Theorem 4.6. Suppose that G is a block-cactus such that every LD-code of G is non-global. Let $S \subseteq V$ be an LD-code of G and let $u \in V \setminus S$ be a vertex dominating S. Notice that, according to Proposition 4.5, every graph of Λ_u is isomorphic to one of the graphs displayed in Figure 5. Moreover, having into account the results obtained in Lemma 4.8, Lemma 4.9 and Lemma 4.10, the set Λ_u is one the following sets:

- $\{P_2, K_r\}$. In this case, G is the graph shown in Figure 6(a).
- $\{P_3, K_r\}$. In this case, G is the graph shown in Figure 6(b).
- $\{P_2, \mathsf{L}\}$. Let u, t be the vertices of P_2. Then, according to Lemma 4.7, $t \in S$, and $S' = (S \setminus \{t\}) \cup \{u\}$ is a global LD-code of G.
- $\{P_3, \mathsf{L}\}$. In this case, G is the graph shown in Figure 6(d).
- $\{K_r\}$. In this case, G is the graph shown in Figure 6(c).
- A set of cardinality at least two, being every graph isomorphic either to a clique or to a corner. In this case, G is a graph as shown in Figure 6(e).

This completes the proof of Theorem 4.6. □

As an immediate consequence of Propositions 3.5 and 4.1 and Theorem 4.6, the following corollaries are obtained.

Corollary 4.11. *A block-cactus G satisfies $\lambda_g(G) = \lambda(G)+1$ if and only if G is isomorphic either to one of the graphs described in Figure 6 or it belongs to the set* $\{P_2, P_5, C_3, C_5, \overline{P}, \text{paw}, \text{bull}, \text{butterfly}\}$.

Corollary 4.12. *Every tree T other than P_2 and P_5 satisfies $\lambda(T) = \lambda_g(T)$.*

Corollary 4.13. *Every unicyclic graph G different from the one displayed in Figure 6(d) and not belonging to the set* $\{C_3, C_5, \overline{P}, \text{paw}, \text{bull}\}$ *satisfies* $\lambda(G) = \lambda_g(G)$.

If G is a block-cactus of order at least 2, we have obtained the following characterization.

Theorem 4.14. *If $G = (V, E)$ is a block-cactus of order at least 2, then $\lambda(\overline{G}) = \lambda(G) + 1$ if and only if G is isomorphic to one of the following graphs (see Figure 8):*

(a) $K_1 \vee (K_1 + K_r)$, $r \geq 2$;

(b) the graph obtained by joining one vertex of K_2 with a vertex of a complete graph of order $r + 1$, $r \geq 2$;

(c) K_{r+1}, $r \geq 1$;

(d) $K_1 \vee (K_{r_1} + \cdots + K_{r_t})$, $t \geq 2$, $r_1, \ldots, r_t \geq 2$.

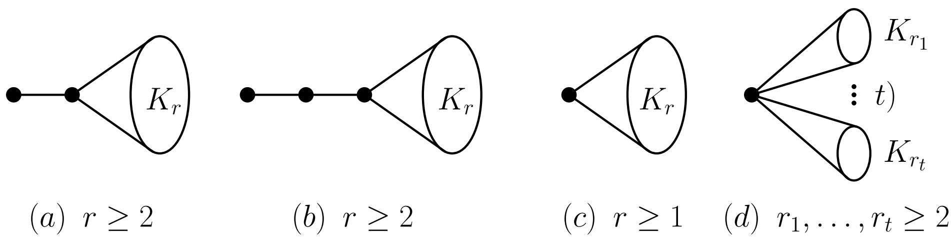

Figure 8: Block-cactus satisfying $\lambda(\overline{G}) = \lambda(G) + 1$.

Proof. Let us see first that all graphs described above satisfy $\lambda(G) < \lambda(\overline{G})$. Recall that if W is a set of twin vertices of a graph G, then every LD-set must contain at least all but one of the vertices of W. Consider one of the graphs described in (a), $G \cong K_1 \vee (K_1 + K_r)$, $r \geq 2$. The complement of G is the graph $K_1 + K_{1,r}$. It is easy to verify that $\lambda(G) = r < r + 1 = \lambda(\overline{G})$. If G is one of the graphs described in b), then $\lambda(G) = r < r + 1 = \lambda(\overline{G})$. Finally, if $G \cong K_1 \vee +(K_{r_1} + \cdots + K_{r_t})$ is a graph of order n, with $t \geq 1$ and $r_1, \ldots, r_t \geq 2$, then we have $\lambda(G) = n - t - 1 < n - t = \lambda(\overline{G})$.

Now, suppose that $G = (V, E)$ is a block-cactus of order at least 3 satisfying $\lambda(\overline{G}) = \lambda(G) + 1$.

If $\lambda(G) = 1$, as the order of G is at least 2, then G is the 2-path P_2, which satisfies $2 = \lambda(\overline{P_2}) = \lambda(P_2) + 1$. This case is described under (c) when r=1 (see Figure 8).

If $\lambda(G) = 2$, then by Proposition 4.1 the graph G is the paw, the complement of the banner, the 3-cycle C_3 or the butterfly, and these graphs are described, respectively, under (a) when $r = 2$; (b) when $r = 2$; (c) when $t = 1$ and $r_1 = 2$ and (d) when $t = r_1 = r_2 = 2$ (see Figure 8).

If $\lambda(G) \geq 3$, by Proposition 2.7, G does not contain a global LD-code, and therefore it must be one of those graphs described in Theorem 4.6. Hence, it suffices to prove that the graphs described under items d) or e) with $t' > 0$, in Theorem 4.6, do not satisfy $\lambda(\overline{G}) = \lambda(G) + 1$. The graph G described in item d) satisfies $\lambda(G) = \lambda(\overline{G}) = 3$, since an LD-code of G is the set containing the three vertices adjacent to the three vertices of degree 1 in G and an LD-code of $\overline{G}$ is the set containing the three vertices adjacent to the three vertices of degree 3 in G. Finally, if G is one of the graphs described in item e) obtained from t copies of complete graphs and t' copies of corners, $t' \geq 1$, then the set of vertices including all but one vertex of each complete graph and the two vertices of degree 3 of each copy of the corner, is an LD-code of G. If we change exactly one of the vertices of degree

3 of a copy of the corner by the vertex of degree 2 in this copy, then we obtain an LD-code of $\overline{G}$. Thus, $\lambda(G) = \lambda(\overline{G}) = 2t' + (r_1 - 1) + \cdots + (r_t - 1)$. □

Corollary 4.15. *Every tree T other than P_2 satisfies $\lambda(\overline{T}) \leq \lambda(T)$.*

Corollary 4.16. *Every unicyclic graph G not belonging to the set $\{C_3, \overline{P}, \mathrm{paw}\}$ satisfies $\lambda(\overline{G}) \leq \lambda(G)$.*

5 Further research

This work can be continued in several directions. Next, we propose a few of them.

- We have completely solved the equality $\lambda(\overline{G}) = \lambda(G) + 1$ for the block-cactus family. In [14], a similar study has been done for the family of bipartite graphs. We suggest to approach this problem for other families of graphs, such as outerplanar graphs, chordal graphs and cographs.
- Characterizing those trees T satisfying $\lambda(\overline{T}) = \lambda(T) = \lambda_g(T)$.
- We have proved that every tree other than P_2 and P_5, every cycle other than C_3 and C_5, and every complete bipartite graph satisfies the equality $\lambda(G) = \lambda_g(G)$. We propose to find other families of graphs with this same behaviour.

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