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Modifications of the Floyd-Warshall algorithm with nearly quadratic expected-time*

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Abstract

The paper describes two relatively simple modifications of the well-known Floyd-Warshall algorithm for computing all-pairs shortest paths. A fundamental difference of both modifications in comparison to the Floyd-Warshall algorithm is that the relaxation is done in a smart way. We show that the expected-case time complexity of both algorithms is $O(n^2 \log^2 n)$ for the class of complete directed graphs on n vertices with arc weights selected independently at random from the uniform distribution on $[0, 1]$. Theoretically best known algorithms for this class of graphs are all based on Dijkstra's algorithm and obtain a better expected-case bound. However, by conducting an empirical evaluation we prove that our algorithms are at least competitive in practice with best know algorithms and, moreover, outperform most of them. The reason for the practical efficiency of the presented algorithms is the absence of use of priority queue.

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1 Introduction

Finding shortest paths in graphs is a classic problem in algorithmic graph theory. Given a (directed) graph in which arcs are assigned weights, a shortest path between pair of vertices is such a path that infimizes the sum of the weights of its constituent arcs. The problem pops up very frequently also in practice in areas like bioinformatics, logistics, VLSI design (for a more comprehensive list of applications see e.g. [2]). Two of the most common problem's variants are the single-source shortest path problem and the all-pairs shortest path problem (APSP). In the first variant of the problem, we are searching for paths from a fixed vertex to every other vertex, while the APSP asks for a shortest path between every pair of vertices. In this paper we focus exclusively on the APSP variant of the problem.

In general, the APSP can be solved by using the technique of **relaxation**. The relaxation consists of testing whether we can improve the weight of the shortest path from u to v found so far by going via w, and updating it if necessary. In fact, the number of attempts to perform relaxation corresponds to the time complexity under the RAM model. A trivial text-book relaxation-based solution to the APSP is a dynamic programming Floyd-Warshall algorithm [11] running in $O(n^3)$ time on graphs with n vertices.

Moreover, also Dijkstra's algorithm [10] solving single-source shortest path problem is relaxation-based. However, since the order in which the relaxations are performed is greedy, it uses an additional priority queue data structure. Obviously we can solve the APSP running Dijkstra's algorithm from each vertex of the graph obtaining $O(mn \log n)$ solution where m is the number of arcs in the graph, provided we use the binary heap implementation of the priority queue. This is an improvement over the Floyd-Warshal solution for sparse graphs. Asymptotically we get an even better solution by using Fibonacci heaps over binary heaps yielding $O(n^2 \log n + mn)$ time complexity. We refer to such approaches as a **Dijkstra-like**, which inherently use some kind of a priority queue implementation.

However, the described solutions to the APSP using the Dijkstra's algorithm have at least two limitations. The first one is that all arc weights have to be non-negative. This can be alleviated by using Johnson's approach [15], which reweighs all arc weights making them non-negative. On such a graph we can now run Dijkstra's algorithm. The second limitation is related to the efficiency of the solution implementation. Namely, due to computer architecture efficient implementations exploit the issue of data locality; i.e. in consecutive memory accesses they try to access memory locations that are "close together". A similar observation is made in [5] for the solutions to the single-source shortest path problem, where authors show that a Fibonacci heap, as asymptotically better implementation of a priority queue, in practice underperform simple binary heap.

For dense graphs, a slightly better worst-case running time of $O(n^3 \log \log n / \log^2 n)$ over the $O(n^3)$-time Floyd-Warshall algorithm can be achieved by using an efficient matrix multiplication technique [13]. For sparse graphs on n vertices and with m non-negative weighted arcs fastest known solution [20] runs in time $O(mn + n^2 \log \log n)$.

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Considering expected-case running-time of APSP algorithms we can find in the literature a number of good solutions assuming that input instances are generated according to a probability model on the set of complete directed graphs with arc weights. In the uniform model, arc weights are drawn at random, independently of each other, according to a common probability distribution. A more general model is the endpoint-independent model [3, 24], where, for each vertex v, a sequence of $n-1$ non-negative arc weights is generated by a deterministic or stochastic process and then randomly permuted and assigned to the outgoing arcs of v. In the vertex potential model [5, 6], arc weights can be both positive and negative. This is a probability model with arbitrary real arc weights but without negative cycles.

In the uniform model with arc weights drawn from the uniform distribution on $[0, 1]$, the $O(n^2 \log n)$ expected time complexity algorithms for solving the APSP were presented by Karger et al. [16] and Demetrescu and Italiano [8, 9], where the latter was inspired by the former. Another algorithm with the same expected time complexity was presented by Brodnik and Grgurovič [4]. Peres et al. [19] improved the Demetrescu and Italiano algorithm to theoretically optimal $O(n^2)$ by replacing the priority queue implementation with a more involved data structure yielding theoretically desired time complexity. In the endpoint-independent model, Spira [24] proved an expected-case time bound of $O(n^2 \log^2 n)$, which was improved by several authors. Takaoka and Moffat [25] improved the bound to $O(n^2 \log n \log \log n)$. Bloniarz [3] described an algorithm with expected-case running time $O(n^2 \log n \log^* n)$. Finally, Moffat and Takaoka [18] and Mehlhorn and Priebe [17] improved the running time to $O(n^2 \log n)$. In the vertex potential model, Cooper et al. [6] gave an algorithm with an expected-case running time $O(n^2 \log n)$. All the above algorithms use Dijkstra-like approach.

In this paper, we present two modifications of the Floyd-Warshall algorithm, which we name the Tree algorithm and the Hourglass algorithm. A fundamental difference of both modifications in relation to the Floyd-Warshall algorithm is a smarter way to perform the relaxations. This is done by introducing a tree structure that allows us to skip relaxations that do not contribute to the result. The worst-case time complexity of both algorithms remains $O(n^3)$, however, in the analysis we show that their expected running time is substantially better. To simplify the analysis, we consider the uniform model which gives us the following main result.

Theorem 1.1. *For complete directed graphs on n vertices with arc weights selected independently at random from the uniform distribution on $[0, 1]$, the Tree algorithm and the Hourglass algorithm both have an expected-case running time of $O(n^2 \log^2 n)$.*

The proof of our main result relies on the following well-known properties of the complete directed graph on n vertices with uniformly distributed arc weights on $[0, 1]$. First, *a maximum weight* of a shortest path in such a graph is $O(\log n/n)$ with high probability; second, *a longest shortest path* has $O(\log n)$ arcs with high probability; and third, *the maximum outdegree of the subgraph* consisting of all arcs that are shortest paths is $O(\log n)$ with high probability. These properties, together with the observation that if the relaxation on some vertex of the introduced tree structure fails, we can skip relaxations on the entire subtree defined by this vertex (see Lemma 3.1), then give the desired result. Since theoretically best expected-case APSP algorithms are based on Dijkstra's algorithm, it is interesting that a competitive approach can also be obtained by a modification of the Floyd-Warshall algorithm.

To prove the practical competitiveness of our algorithms, we supplement the theoretical analysis with an empirical evaluation. It should be pointed out, that all algorithms mentioned above with $o(n^2 \log^2 n)$ expected-case running time obtain a better theoretical bound. Moreover, Brodnik and Grgurovič in [4] show, for the same family of graphs as studied in this paper, practical supremacy of their algorithm over the algorithms due to Karger et al. [16] and Demetrescu and Italiano [8, 9] and consequently over the algorithm of Peres et al. [19], since its improvement of Demetrescu and Italiano solution does not improve the practical efficiency of the original algorithm. Therefore in the practical evaluation of Tree and Hourglass algorithms we compare them to the algorithm of Brodnik and Grgurovič [4] only. The reason for the practical efficiency of the presented algorithms is the absence of use of priority queue. Indeed, the Tree and Hourglass algorithms are simple to implement and use only simple structures such as vectors and arrays, which also exhibit a high data locality.

The structure of the paper is the following. Section 2 contains the necessary notation and basic definitions to make the paper self-contained. In Section 3 we describe the Tree and Hourglass algorithms. Properties of certain shortest paths in complete graphs with independently and uniformly distributed arc weights are analyzed in Section 4. The proof of the main result is presented in Section 5, while Section 6 contains empirical evaluation of the algorithms. In Section 7 we give some concluding remarks and open problems.

2 Preliminaries

All logarithms are base e unless explicitly stated otherwise. The model of computation used in algorithm design and analysis is the comparison-addition model, where the only allowed operations on arc weights are comparisons and additions.

A **digraph** (or **directed graph**) G is a pair (V, A), where V is a non-empty finite set of **vertices** and $A \subseteq V \times V$ a set of **arcs**. We assume $V = \{v_1, v_2, \ldots, v_n\}$ for some n. The two vertices joined by an arc are called its **endvertices**. For an arc $(u, v) \in A$, we say that u is its **tail**. The **outdegree** of $v \in V$, is the number of arcs in A that have v as their tail. The maximum outdegree in G is denoted by $\Delta(G)$.

A digraph $G' = (V', A')$ is a subdigraph of the digraph $G = (V, A)$ if $V' \subseteq V$ and $A' \subseteq A$. The **(vertex-)induced subdigraph** with the vertex set $S \subseteq V$, denoted by $G[S]$, is the subgraph (S, C) of G, where C contains all arcs $a \in A$ that have both endvertices in S, that is, $C = A \cap (S \times S)$. The **(arc-)induced subdigraph** with the arc set $B \subseteq A$, denoted by $G[B]$, is the subgraph (T, B) of G, where T is the set of all those vertices in V that are endvertices of at least one arc in B.

A path P in a digraph G from $v_{P,0}$ to $v_{P,m}$ is a finite sequence $P = v_{P,0}, v_{P,1}, \ldots, v_{P,m}$ of pairwise distinct vertices such that $(v_{P,i}, v_{P,i+1})$ is an arc of G, for $i = 0, 1, \ldots, m-1$. The **length** of a path P, denoted by $|P|$, is the number of vertices occurring on P. Any vertex of P other than $v_{P,0}$ or $v_{P,m}$ is an **intermediate** vertex. A k**-path** is a path in which all intermediate vertices belong to the subset $\{v_1, v_2, \ldots, v_k\}$ of vertices for some k. Obviously, a 0**-path** has no intermediate vertices.

A **weighted digraph** is a digraph $G = (V, A)$ with a **weight function** $w\colon A \to \mathbb{R}$ that assigns each arc $a \in A$ a **weight** $w(a)$. A weight function w can be extended to a path P by $w(P) = \sum_{i=0}^{m-1} w(v_{P,i}, v_{P,i+1})$. A **shortest path** from u to v, denoted by $u \rightsquigarrow v$, is a path in G whose weight is infimum among all paths from u to v. The **distance** between two vertices u and v, denoted by $D_G(u, v)$, is the weight of a shortest path $u \rightsquigarrow v$ in G.

The **diameter** of G is $\max_{u,v \in V} D_G(u,v)$, that is, the maximum distance between any two vertices in G. Given a subset $S \subseteq V$, the distance between S and a vertex v in G, denoted by $D_G(S,v)$, is $D_G(S,v) = \min_{u \in S} D_G(u,v)$. A shortest k-path from u to v is denoted by $u \overset{k}{\rightsquigarrow} v$. Further, we denote the set of arcs that are shortest k-paths in G by $A^{(k)}$ and the subdigraph $G[A^{(k)}]$ by $G^{(k)}$.

Finally, we will need some tools from combinatorics. In the balls-into-bins process M balls are thrown uniformly and independently into N bins. The maximum number of balls in any bin is called the **maximum load**. Let L_i denote the load of bin i, $i \in \{1, 2, \ldots, N\}$. The next lemma, used in Subsection 4.3, provides an upper bound on the maximum load. It is a simplified version of a standard result, c.f. [23], tailored to our present needs. For completeness we provide the proof.

Lemma 2.1. *If M balls are thrown into N bins where each ball is thrown into a bin chosen uniformly at random, then $\mathbb{P}(\max_{1 \le i \le N} L_i \ge e^2(M/N + \log N)) = O(1/N)$.*

Proof. First, we have $\mu = \mathbb{E}(L_i) = M/N$, $i = 1, 2, \ldots, N$, and we can write each L_i as a sum $L_i = X_{i1} + X_{i2} + \cdots + X_{iM}$, where X_{ij} is a random variable taking value 1, if ball j is in bin i, and 0 otherwise. Next, since L_i is a sum of independent random variables taking values in $\{0, 1\}$, we can apply, for any particular bin i and for every $c > 1$, the multiplicative Chernoff bound [12], which states that

$$\mathbb{P}(L_i \ge c\mu) \le \left(\frac{e^{c-1}}{c^c}\right)^{\mu} \le \left(\frac{e}{c}\right)^{c\mu}.$$

We consider two cases, depending on whether $\mu \ge \log N$ or not. Let $\mu \ge \log N$. Take $c = e^2$. Then,

$$\mathbb{P}(L_i \ge e^2\mu) \le \left(\frac{1}{e}\right)^{e^2\mu} \le \left(\frac{1}{e}\right)^{e^2 \log N} = \frac{1}{N^{e^2}} \le \frac{1}{N^2}.$$

Consider now $\mu < \log N$. Take $c = e^2 \frac{N}{M} \log N$. Since $x^{-x} \le \left(\frac{1}{e}\right)^x$ for all $x \ge e$, we have

$$\begin{aligned}\mathbb{P}\left(L_i \ge \mu e^2 \frac{N}{M} \log N\right) = \mathbb{P}(L_i \ge e^2 \log N) \le \left(\frac{e}{c}\right)^{c\mu} &= \left(\left(\frac{c}{e}\right)^{-\frac{c}{e}}\right)^{e\mu} \\ &\le \left(\left(\frac{1}{e}\right)^{\frac{c}{e}}\right)^{e\mu} = \left(\frac{1}{e}\right)^{e^2 \log N} \le \frac{1}{N^2}.\end{aligned}$$

Putting everything together, we get that

$$\begin{aligned}\mathbb{P}(L_i \ge e^2(\mu + \log N)) &\le \mathbb{P}(L_i \ge e^2\mu \mid \mu \ge \log N) + \mathbb{P}(L_i \ge e^2 \log N \mid \mu < \log N) \\ &\le \frac{1}{N^2} + \frac{1}{N^2} = \frac{2}{N^2}.\end{aligned}$$

This, by the union bound, implies that

$$\mathbb{P}\left(\max_{1 \le i \le N} L_i \ge e^2(\mu + \log N)\right) \le \sum_{i=1}^{N} \mathbb{P}(L_i \ge e^2(\mu + \log N)) \le N \frac{2}{N^2} = O(1/N).$$

□

3 Speeding up the Floyd-Warshall algorithm

The Floyd-Warshall algorithm [11, 26] as presented in Algorithm 1 is a simple dynamic programming approach to solve APSP on a graph $G(V, A)$ represented by a weight matrix W, where $W_{ij} = w(v_i, v_j)$ if $(v_i, v_j) \in A$ and ∞ otherwise. Its running time is $O(n^3)$ due to three nested **for** loops.

Algorithm 1 FLOYD-WARSHALL(W)

```
1 for k := 1 to n do
2     for i := 1 to n do
3         for j := 1 to n do
4             if W_ik + W_kj < W_ij then            ▷ Relaxation
5                 W_ij := W_ik + W_kj
```

3.1 The Tree algorithm

Let us consider iteration k, and let OUT_k denote a shortest path tree rooted at vertex v_k in $G^{(k-1)}$. Intuitively, one might expect that the relaxation in lines 4-5 would not always succeed in lowering the value of W_{ij} which currently contains the weight $w(v_i \overset{k-1}{\rightsquigarrow} v_j)$. This is precisely the observation that we exploit to arrive at a more efficient algorithm: instead of simply looping through every vertex of V in line 3, we perform the depth-first traversal of OUT_k. This permits us to skip iterations which provably cannot lower the current value of W_{ij}. As the following lemma shows, if $w(v_i \overset{k}{\rightsquigarrow} v_j) = w(v_i \overset{k-1}{\rightsquigarrow} v_j)$, then $w(v_i \overset{k}{\rightsquigarrow} v_y) = w(v_i \overset{k-1}{\rightsquigarrow} v_y)$ for all vertices v_y in the subtree of v_j in OUT_k.

Lemma 3.1. *Let $v_j \in V \setminus \{v_k\}$ be some non-leaf vertex in OUT_k, $v_y \neq v_j$ an arbitrary vertex in the subtree of v_j in OUT_k, and $v_i \in V \setminus \{v_k\}$. Consider the path $v_i \overset{k-1}{\rightsquigarrow} v_k \overset{k-1}{\rightsquigarrow} v_j$. If $w(v_i \overset{k-1}{\rightsquigarrow} v_k \overset{k-1}{\rightsquigarrow} v_j) \geq w(v_i \overset{k-1}{\rightsquigarrow} v_j)$, then $w(v_i \overset{k-1}{\rightsquigarrow} v_k \overset{k-1}{\rightsquigarrow} v_y) \geq w(v_i \overset{k-1}{\rightsquigarrow} v_y)$.*

Proof. Since v_j is neither a leaf nor the root of OUT_k, we have $j < k$, and so $v_i \overset{k-1}{\rightsquigarrow} v_j \overset{k-1}{\rightsquigarrow} v_y$ is a $(k-1)$-path between v_i and v_y. Because $v_i \overset{k-1}{\rightsquigarrow} v_y$ is a shortest $(k-1)$-path between v_i and v_y, we have

$$\begin{aligned} w(v_i \overset{k-1}{\rightsquigarrow} v_y) &\leq w(v_i \overset{k-1}{\rightsquigarrow} v_j \overset{k-1}{\rightsquigarrow} v_y) = w(v_i \overset{k-1}{\rightsquigarrow} v_j) + w(v_j \overset{k-1}{\rightsquigarrow} v_y) \\ &\leq w(v_i \overset{k-1}{\rightsquigarrow} v_k \overset{k-1}{\rightsquigarrow} v_j) + w(v_j \overset{k-1}{\rightsquigarrow} v_y) = w(v_i \overset{k-1}{\rightsquigarrow} v_k \overset{k-1}{\rightsquigarrow} v_j \overset{k-1}{\rightsquigarrow} v_y), \end{aligned}$$

where the last inequality follows by the assumption. Finally, since v_y is in the subtree rooted at v_j, we have $v_i \overset{k-1}{\rightsquigarrow} v_k \overset{k-1}{\rightsquigarrow} v_j \overset{k-1}{\rightsquigarrow} v_y = v_i \overset{k-1}{\rightsquigarrow} v_k \overset{k-1}{\rightsquigarrow} v_y$, and so the last term is equal to $w(v_i \overset{k-1}{\rightsquigarrow} v_k \overset{k-1}{\rightsquigarrow} v_y)$. This completes the proof. □

The pseudocode of the modified Floyd-Warshall algorithm augmented with the tree OUT_k, named the Tree algorithm, is given in Algorithm 2. To perform depth first search we first construct the tree OUT_k in line 3 using CONSTRUCTOUT given in Algorithm 3. For the construction of tree OUT_k an additional matrix π, where π_{ij} specifies the penultimate vertex on a k-shortest path from v_i to v_j (i.e. the vertex "before" v_j)[1] is used. More

[1]C.f. $\pi_{ij}^{(k)}$ in [7, Sec. 25.2].

Algorithm 2 TREE(W)

```
1  Initialize π, an n × n matrix, as π_ij := i.
2  for k := 1 to n do
3      OUT_k := CONSTRUCTOUT_k(π)
4      for i := 1 to n do
5          Stack := empty
6          Stack.push(v_k)
7          while Stack ≠ empty do
8              v_x := Stack.pop()
9              for all children v_j of v_x in OUT_k do
10                 if W_ik + W_kj < W_ij then                    ▷ Relaxation
11                     W_ij := W_ik + W_kj
12                     π_ij := π_kj
13                     Stack.push(v_j)
```

Algorithm 3 CONSTRUCTOUT$_k(\pi)$

```
1  Initialize n empty trees: T_1, ..., T_n.
2  for i := 1 to n do
3      T_i.Root := v_i
4  for i := 1 to n do
5      if i ≠ k then
6          Make T_i a subtree of the root of T_{π_ki}.
   return T_k
```

precisely, the tree OUT_k is obtained from π by making v_i a child of $v_{\pi_{ki}}$ for all $i \neq k$. This takes $O(n)$ time. Finally, we replace the iterations in lines 3-5 in Algorithm 1 with depth-first tree traversal of OUT_k in lines 5-13 in Algorithm 2. Note that if, for a given i and a child v_j, the condition in line 10 evaluates to false we do not traverse the subtree of v_j in OUT_k.

Corollary 3.2. *The Tree algorithm correctly computes all-pairs shortest paths.*

Proof. The correctness of the algorithm follows directly from Lemma 3.1. □

Time complexity

Let T_k denote the running time of the algorithm TREE(W) in lines 3-13 at iteration k. As already said, line 3 requires $O(n)$ time. To estimate the time complexity of lines 4-13, we charge the vertex v_x in line 8 by the number of its children. This pays for lines 9-13. Furthermore, this means that on the one hand leaves are charged nothing, while on the other hand nobody is charged for the root v_k. To this end, let $SP_k^{(k)}$ be the set of all shortest k-paths that contain v_k and end at some vertex in the set $\{v_1, v_2, \dots, v_k\}$. Now v_x in line 8 is charged at most $|SP_k^{(k)}|$ times over all iterations of i. Since the number of children of v_x is bounded from above by $\Delta(OUT_k)$, we can bound T_k from above by

$$T_k \leq |SP_k^{(k)}| \cdot \Delta(OUT_k) + O(n). \tag{3.1}$$

Practical improvement

Observe that in Algorithm 2 vertices of OUT_k are visited in a depth-first search (DFS) order, which is facilitated by using the stack. However, this requires pushing and popping of each vertex, as well as reading of all its children in OUT_k. We can avoid this by precomputing two read-only arrays *dfs* and *skip* to support the traversal of OUT_k. The array *dfs* consists of OUT_k vertices as visited in the DFS order. On the other hand, the array *skip* is used to skip OUT_k subtree when relaxation in line 10 of Algorithm 2 does not succeed.

In detail, for a vertex v_z, $skip_z$ contains the index in *dfs* of the first vertex after v_z in the DFS order that is not a descendant of v_z in OUT_k. Utilizing the arrays outlined above, we traverse OUT_k by scanning *dfs* in left-to-right order and using *skip* whenever a relaxation is not made. Consequently, we perform only two read operations per visited vertex. It should be pointed out that the asymptotic time remains the same, as this is solely a technical optimization.

3.2 The Hourglass algorithm

We can further improve the Tree algorithm by using another tree. The second tree, denoted by IN_k, is similar to OUT_k, except that it is a shortest path "tree" for paths $v_i \overset{k-1}{\rightsquigarrow} v_k$ for each $v_i \in V \setminus \{v_k\}$. Strictly speaking, this is not a tree, but if we reverse the directions of the arcs, it turns it into a tree with v_k as the root. Traversal of IN_k is used as a replacement of the **for** loop on variable i in line 4 of Algorithm 2 (in line 2 of Algorithm 1). As the following lemma shows, if $w(v_i \overset{k}{\rightsquigarrow} v_j) = w(v_i \overset{k-1}{\rightsquigarrow} v_j)$, then $w(v_y \overset{k}{\rightsquigarrow} v_j) = w(v_y \overset{k-1}{\rightsquigarrow} v_j)$ for all vertices v_y in the subtree of v_i in IN_k.

Lemma 3.3. *Let $v_i \in V \setminus \{v_k\}$ be some non-leaf vertex in IN_k and let $v_y \neq v_i$ be an arbitrary vertex in the subtree of v_i in IN_k, and $v_j \in V \setminus \{v_k\}$. Consider the path $v_i \overset{k-1}{\rightsquigarrow} v_k \overset{k-1}{\rightsquigarrow} v_j$. If $w(v_i \overset{k-1}{\rightsquigarrow} v_k \overset{k-1}{\rightsquigarrow} v_j) \geq w(v_i \overset{k-1}{\rightsquigarrow} v_j)$, then $w(v_y \overset{k-1}{\rightsquigarrow} v_k \overset{k-1}{\rightsquigarrow} v_j) \geq w(v_y \overset{k-1}{\rightsquigarrow} v_j)$.*

Proof. Due to the choice of v_i and v_y we have: $v_y \overset{k-1}{\rightsquigarrow} v_k = v_y \overset{k-1}{\rightsquigarrow} v_i \overset{k-1}{\rightsquigarrow} v_k$. We want to show, that:

$$w(v_y \overset{k-1}{\rightsquigarrow} v_j) \leq w(v_y \overset{k-1}{\rightsquigarrow} v_i) + w(v_i \overset{k-1}{\rightsquigarrow} v_k \overset{k-1}{\rightsquigarrow} v_j).$$

Observe that $i < k$, since v_i is neither a leaf nor the root of IN_k. Thus we have:

$$w(v_y \overset{k-1}{\rightsquigarrow} v_j) \leq w(v_y \overset{k-1}{\rightsquigarrow} v_i) + w(v_i \overset{k-1}{\rightsquigarrow} v_j).$$

Putting these together we get the desired inequality:

$$w(v_y \overset{k-1}{\rightsquigarrow} v_j) \leq w(v_y \overset{k-1}{\rightsquigarrow} v_i) + w(v_i \overset{k-1}{\rightsquigarrow} v_j) \leq w(v_y \overset{k-1}{\rightsquigarrow} v_i) + w(v_i \overset{k-1}{\rightsquigarrow} v_k \overset{k-1}{\rightsquigarrow} v_j).$$

□

The pseudocode of the modified Floyd-Warshall algorithm augmented with the trees OUT_k and IN_k, named the Hourglass algorithm[2], is given in Algorithms 4 and 5. To construct IN_k efficiently, we need to maintain an additional matrix ϕ_{ij} which stores the second

[2]The hourglass name comes from placing IN_k tree atop the OUT_k tree, which gives it an hourglass-like shape, with v_k being at the neck.

vertex on the path from v_i to v_j (cf. π and π_{ij}). Algorithm 6 constructs IN_k similarly to the construction of OUT_k, except that we use the matrix ϕ_{ik} instead. The only extra space requirement of the Hourglass algorithm that bears any significance is the matrix ϕ, which does not deteriorate the space complexity of $O(n^2)$. The depth-first traversal on IN_k is performed by a recursion on each child of v_k in line 7 of Algorithm 4. In the recursive step, given in Algorithm 5, we can prune OUT_k as follows: if v_i is the parent of v_y in IN_k and $v_i \overset{k-1}{\rightsquigarrow} v_j \leq v_i \overset{k-1}{\rightsquigarrow} v_k \overset{k-1}{\rightsquigarrow} v_j$, then the subtree of v_j can be removed from OUT_k, while inspecting the subtree of v_i in IN_k. Before the return from the recursion the tree OUT_k is reconstructed to the form it was passed as a parameter to the function.

Algorithm 4 HOURGLASS(W)

---

1 Initialize π, an $n \times n$ matrix, as $\pi_{ij} := i$.
2 Initialize ϕ, an $n \times n$ matrix, as $\phi_{ij} := j$.
3 **for** $k := 1$ to n **do**
4     $OUT_k :=$ CONSTRUCTOUT$_k(\pi)$
5     $IN_k :=$ CONSTRUCTIN$_k(\phi)$
6   **for all** children v_i of v_k in IN_k **do**
7   RECURSEIN($W, \pi, \phi, IN_k, OUT_k, v_i$)

Algorithm 5 RECURSEIN($W, \pi, \phi, IN_k, OUT_k, v_i$)

---

1 Stack := empty
2 Stack.push(v_k)
3 **while** Stack $\neq$ empty **do**
4     $v_x :=$ Stack.pop()
5   **for all** children v_j of v_x in OUT_k **do**
6   **if** $W_{ik} + W_{kj} < W_{ij}$ **then** ▷ Relaxation
7             $W_{ij} := W_{ik} + W_{kj}$
8             $\pi_{ij} := \pi_{kj}$
9             $\phi_{ij} := \phi_{ik}$
10   Stack.push(v_j)
11   **else**
12   Remove the subtree of v_j from OUT_k.
13 **for all** children $v_{i'}$ of v_i in IN_k **do**
14   RECURSEIN($W, \pi, \phi, IN_k, OUT_k, v_{i'}$)
15 Restore OUT_k by reverting changes done by all iterations of line 12.

In practice, the recursion can be avoided by using an additional stack, which further speeds up an implementation of the algorithm.

Corollary 3.4. *The Hourglass algorithm correctly computes all-pairs shortest paths.*

Proof. Observe, that lines 5-10 of Algorithm 5 are effectively the same as in Algorithm 2. Line 12 of Algorithm 5 does not affect the correctness of the algorithm due to Lemma 3.3, which states that, for any $v_{i'}$ that is a child of v_i in IN_k, these comparisons can be skipped, as they cannot lead to shorter paths. However, Lemma 3.3 does not apply to a sibling v_{i^*}

Algorithm 6 CONSTRUCTIN$_k(\phi)$

Initialize n empty trees: $T_1, \ldots, T_n$.
for $i := 1$ to n **do**
    T_i.Root $:= v_i$
for $i := 1$ to n **do**
    if $i \neq k$ **then**
        Make T_i a subtree of the root of $T_{\pi_{ki}}$.
return T_k

of v_i, arising from line 6 of Algorithm 4. Therefore line 15 restores the tree OUT_k, which maintains the correctness of the algorithm. □

Finally, note that the worst-case time complexity of the Hourglass (and Tree) algorithm remains $O(n^3)$. The simplest example of this is when all shortest paths are the arcs themselves, at which point all leaves are children of the root and the tree structure never changes.

4 Properties of shortest *k*-paths in complete graphs

Let K_n denote a complete digraph on the vertex set $V = \{v_1, v_2, \ldots, v_n\}$.

4.1 Distances

We assume that arc weights of K_n are exponential random variables with mean 1 and that all $n(n-1)$ random arc weights are independent. Due to the memoryless property, it is easier to deal with exponentially distributed arc weights than directly with uniformly distributed arc weights. The aim of this subsection is to show that the diameter of $K_n^{(k)}$, the subdigraph of K_n consisting of all (weighted) arcs that are shortest k-paths in K_n, is $O(\log n/k)$ with very high probability. We note, however, that by the same argument as given in the beginning of Subsection 4.3, all results derived in this subsection for exponential arc weights also hold, asymptotically for $[0,1]$-uniformly distributed arc weights as soon as $k \geq \log^2 n$.

We start by considering for a fixed $u \in V$, the maximum distance in $K_n^{(k)}$ between u and other vertices in V. To this end, let $S = \{u, v_1, \ldots, v_k\} \subseteq V$, and let $\overline{S} = V \setminus S$. We clearly have

$$\max_{v \in V} D_{K_n^{(k)}}(u, v) \leq \max_{v \in S} D_{K_n[S]}(u, v) + \max_{v \in \overline{S}} D_{K_n[S \times \overline{S}]}(S, v), \tag{4.1}$$

that is, the maximum distance in $K_n^{(k)}$ between u and other vertices in V is bounded above by the sum of the maximum distance in $K_n[S]$ between u and other vertices in S, and by the maximum distance in $K_n[S \times \overline{S}]$ between S and vertices in $\overline{S}$. We note that $K_n[S]$ is a complete digraph on $|S|$ vertices and $K_n[S \times \overline{S}]$ is a complete bipartite digraph with bipartition $(S, \overline{S})$.

To provide an upper bound on $\max_{v \in S} D_{K_n[S]}(u, v)$, we use the following result, which follows from the equation (2.8) in the proof of Theorem 1.1 of Janson [14].

Theorem 4.1 ([14, Theorem 1.1]). *Let $u \in V$ be a fixed vertex of K_n. Then for every $a > 0$, we have*

$$\mathbb{P}\left(\max_{v \in V} D_{K_n}(u, v) \geq \frac{a \log n}{n}\right) = O(e^a n^{2-a} \log^2 n).$$

Lemma 4.2. *Let $8 \leq k \leq n$, and let $S \subseteq V$ with $|S| = k$. Then, for a fixed $u \in S$ and for any constant $c > 0$, we have*

$$\mathbb{P}\left(\max_{v \in S} D_{K_n[S]}(u, v) \geq \frac{c \log n}{k}\right) = O(n^{2-c/2} \log^2 n).$$

Proof. By Theorem 4.1, for any $a > 0$ we have

$$\mathbb{P}\left(\max_{v \in S} D_{K_n[S]}(u, v) \geq \frac{a \log k}{k}\right) = O(e^a k^{2-a} \log^2 k).$$

Setting $a = c \log n / \log k$ we get

$$e^a k^{2-a} \log^2 k = e^{c \log n / \log k} k^2 k^{-c \log n / \log k} \log^2 k \leq (e^{\log n})^{c/2} k^2 (k^{\log_k n})^{-c} \log^2 k.$$

In the last step we used the fact that $1/\log k \leq 1/2$ for $k \geq 8$ and that $\log n / \log k = \log_k n$. Furthermore,

$$(e^{\log n})^{c/2} k^2 (k^{\log_k n})^{-c} \log^2 k = n^{c/2} k^2 n^{-c} \log^2 k = O(n^{2-c/2} \log^2 n),$$

and the result follows. □

Next, we provide an upper bound on $\max_{v \in \overline{S}} D_{K_n[S \times \overline{S}]}(S, v)$.

Lemma 4.3. *Let $1 \leq k \leq n$, let $S \subseteq V$ with $|S| = k$, and let $\overline{S} = V \setminus S$. Then for any constant $c > 0$, we have*

$$\mathbb{P}\left(\max_{v \in \overline{S}} D_{K_n[S \times \overline{S}]}(S, v) \geq \frac{c \log n}{k}\right) = O(n^{1-c} \log n).$$

Proof. Let $Z = \max_{v \in \overline{S}} D_{K_n[S \times \overline{S}]}(S, v)$. Arguing similarly as in the proof of Theorem 1.1 of Janson [14], Z is distributed as

$$\sum_{j=k}^{n-1} X_j,$$

where X_j are independent exponentially distributed random variables with mean $\frac{1}{k(n-j)}$. First, for any constant $c > 0$, the Chernoff bound [12] states that

$$\mathbb{P}(Z \geq c \log n / k) \leq e^{-tc \log n} \mathbb{E}(e^{ktZ}).$$

Further, for $-\infty < t \leq 1$, we have

$$\mathbb{E}(e^{ktZ}) = \prod_{j=k}^{n-1} \mathbb{E}(e^{ktX_j}) = \prod_{j=k}^{n-1} \left(1 - \frac{t}{n-j}\right)^{-1}.$$

Using the inequality $-\log(1-x) \le x + x^2$ for all $0 \le x \le 1/2$, we can bound, for all $0 < t < 1$ and $k \le j \le n-2$, each term $(1 - t/(n-j))^{-1}$ as follows

$$\left(1 - \frac{t}{n-j}\right)^{-1} = \exp\left(-\log\left(1 - \frac{t}{n-j}\right)\right) \le \exp\left(\frac{t}{n-j} + \left(\frac{t}{n-j}\right)^2\right).$$

This gives us

$$\begin{aligned}\mathbb{P}(Z \ge c\log n/k) &\le (1-t)^{-1}\exp\left(-tc\log n + \sum_{j=k}^{n-2}\left(\frac{t}{n-j} + \left(\frac{t}{n-j}\right)^2\right)\right)\\ &= (1-t)^{-1}\exp(-tc\log n + t\log(n-k) + O(1)).\end{aligned}$$

Taking $t = 1 - 1/\log n$, we finally get

$$\mathbb{P}(Z \ge c\log n/k) \le (1/\log n)^{-1}\exp(-c\log n + \log n + O(1)) = O(n^{1-c}\log n).$$

□

We are now ready to show that the diameter of $K_n^{(k)}$ is $O(\log n/k)$ with very high probability.

Theorem 4.4. *Let* $8 \le k \le n$*. Then, for any constant* $c > 0$*, we have*

$$\mathbb{P}\left(\max_{u,v\in V} D_{K_n^{(k)}}(u,v) \ge \frac{c\log n}{k}\right) = O(n^{3-c/4}\log^2 n).$$

Proof. Let $S = \{u, v_1, \dots, v_k\} \subseteq V$, let $\overline{S} = V \setminus S$, and write $\alpha = c\log n/k$. Then, by inequality (4.1), we have

$$\begin{aligned}\mathbb{P}\left(\max_{v\in V} D_{K_n^{(k)}}(u,v) \ge \alpha\right) &\le \mathbb{P}\left(\max_{v\in S} D_{K_n[S]}(u,v) + \max_{v\in \overline{S}} D_{K_n[S\times\overline{S}]}(S,v) \ge \alpha\right)\\ &\le \mathbb{P}\left(\max_{v\in S} D_{K_n[S]}(u,v) \ge \frac{\alpha}{2}\right) + \mathbb{P}\left(\max_{v\in \overline{S}} D_{K_n[S\times\overline{S}]}(S,v) \ge \frac{\alpha}{2}\right).\end{aligned}$$

By Lemma 4.2, we have

$$\mathbb{P}\left(\max_{v\in S} D_{K_n[S]}(u,v) \ge \frac{\alpha}{2}\right) = O(n^{2-c/4}\log^2 n),$$

and, by Lemma 4.3,

$$\mathbb{P}\left(\max_{u\in \overline{S}} D_{K_n[S\times\overline{S}]}(S,v) \ge \frac{\alpha}{2}\right) = O(n^{1-c/2}\log n).$$

Putting everything together, we get

$$\mathbb{P}\left(\max_{v\in V} D_{K_n^{(k)}}(u,v) \ge \alpha\right) = O(n^{2-c/4}\log^2 n),$$

which, by the union bound, implies

$$\mathbb{P}\left(\max_{u,v\in V} D_{K_n^{(k)}}(u,v) \ge \alpha\right) \le n\mathbb{P}\left(\max_{u\in V} D_{K_n^{(k)}}(v,u) \ge \alpha\right) = O(n^{3-c/4}\log^2 n).$$

□

4.2 Lengths

Let all arc weights of K_n be either independent $[0,1]$-uniform random variables or independent exponential random variables with mean 1. In this subsection, we bound the length of the longest shortest k-path in K_n.

The proof of our next lemma follows directly from Theorem 1.1 of Addario-Berry et. al [1] on the longest shortest path in K_n.

Theorem 4.5 ([1, Theorem 1.1])**.** *The following two properties hold:*

(i) For every $t > 0$, we have

$$\mathbb{P}\left(\max_{u,v\in V} |u\rightsquigarrow v| \geq \alpha^* \log n + t\right) \leq e^{\alpha^* + t/\log n} e^{-t},$$

where $\alpha^* \approx 3.5911$ is the unique solution of $\alpha \log \alpha - \alpha = 1$.

(ii) $\mathbb{E}(\max_{u,v\in V} |u\rightsquigarrow v|) = O(\log n)$.

Lemma 4.6. *The following two properties hold:*

(i) For every $c > 5$ and $8 \leq k \leq n$, we have $\mathbb{P}(\max_{u,v\in V} |u\overset{k}{\rightsquigarrow} v| \geq c\log n) = O(n^{2-c/2})$.

(ii) $\mathbb{E}(\max_{u,v\in V} |u\overset{k}{\rightsquigarrow} v|) = O(\log k)$.

Proof. Let $S = \{v_1, v_2, \ldots, v_k\}$, and let $u \to w\overset{k}{\rightsquigarrow} z \to v$ be a shortest k-path in K_n. Since $w\overset{k}{\rightsquigarrow} z$ is a shortest path from w to z in $K_n[S]$, we have

$$\max_{u,v\in V} |u\overset{k}{\rightsquigarrow} v| \leq \max_{w,z\in S} |w\overset{k}{\rightsquigarrow} z| + 2. \tag{4.2}$$

By (i) of Theorem 4.5, for any $t > 0$,

$$\mathbb{P}\left(\max_{w,z\in S} |w\overset{k}{\rightsquigarrow} z| \geq \alpha^* \log k + t\right) \leq e^{\alpha^* + t/\log k} e^{-t},$$

where $\alpha^* \approx 3.5911$ is the unique solution of $\alpha \log \alpha - \alpha = 1$. Using $t = (c-\alpha^*)\log n - 2$ gives us

$$\begin{aligned}\mathbb{P}\left(\max_{w,z\in S} |w\overset{k}{\rightsquigarrow} z| + 2 \geq \alpha^* \log(k/n) + c\log n\right) &\leq e^{\alpha^* - 2/\log k + 2} e^{(c-\alpha^*)(\frac{\log n}{\log k} - \log n)} \\ &\leq e^{\alpha^* - 2/\log k + 2} (e^{\log n})^{1/2(\alpha^* - c)} \\ &= O(n^{2-c/2}).\end{aligned}$$

By inequality (4.2), we have

$$\begin{aligned}\mathbb{P}\left(\max_{u,v\in V} |u\overset{k}{\rightsquigarrow} v| \geq c\log n\right) &\leq \mathbb{P}\left(\max_{w,z\in S} |u\overset{k}{\rightsquigarrow} v| + 2 \geq c\log n\right) \\ &\leq \mathbb{P}\left(\max_{w,z\in S} |w\overset{k}{\rightsquigarrow} z| + 2 \geq \alpha^* \log(k/n) + c\log n\right),\end{aligned}$$

and (i) follows.

To prove (ii), we note that, by (ii) of Theorem 4.5, $\mathbb{E}(\max_{u,v\in S} |u\overset{k}{\rightsquigarrow} v|) = O(\log k)$, and, by inequality (4.2), the result follows.

□

4.3 Maximum outdegree

Let arc weights of K_n be independent $[0,1]$-uniform random variables. Our goal in this subsection is to show that the maximum outdegree of a shortest path tree OUT_k in $K_n^{(k)}$ is $O(\log k + (n-k)/k)$ with high probability for all $k \geq \log^2 n$.

Let now $S = \{v_1, v_2, \ldots, v_k\}$ and $\overline{S} = V \setminus S$. We can consider OUT_k as consisting of the subtree $OUT_k[S]$ to which each vertex from $\overline{S}$ is attached as a leaf. To see how these vertices are attached to $OUT_k[S]$, let us assume for the moment that arc weights are exponentially distributed with mean 1. Then, it is easy to see that a vertex $v \in \overline{S}$ is attached to that one in S with which it forms a shortest arc, say a^v, between S and v. Let $(K_n[S \times \overline{S}])^*$ be the subdigraph of $K_n[S \times \overline{S}]$ with the set V of vertices and the set $\{a^v \mid v \in \overline{S}\}$ of arcs. By observing that $OUT_k[S]$ is a subdigraph of the graph $(K_n[S])^{(k)}$ consisting of all arcs that are shortest paths in $K_n[S]$, we have

$$\Delta(OUT_k) \leq \Delta((K_n[S])^{(k)}) + \Delta((K_n[S \times \overline{S}])^*). \tag{4.3}$$

To extend the latter bound to uniform distribution, we use a standard coupling argument as in [1]. Let U be a random variable uniform on $[0,1]$. Then $-\log(1-U)$ is an exponential random variable with mean 1, and so we can couple the exponential arc weights $W'(u,v)$ to uniform arc weights $W(u,v)$ by setting $W'(u,v) = -\log(1-W(u,v))$. As $x \leq -\log(1-x) \leq x + 2x^2$ for all $0 \leq x \leq 1/2$, we have that, for all arcs (u,v) of K_n, $|W'(u,v) - W(u,v)| = O((W'(u,v))^2)$, uniformly for all $W'(u,v) \leq 1/2$. In particular, if $W'(u,v) \leq 12\log n/k$, say, and $k \geq \log^2 n$, then $|W'(u,v) - W(u,v)| = O(1/\log^2 n)$ for n large enough, and so for a path P with $O(\log n)$ vertices and with $W'(P) \leq 12\log n/k$, we have

$$|W'(P) - W(P)| = O(1/\log n)$$

for n large enough. By Theorem 4.4, with very high probability a shortest $(k-1)$-path in K_n with the exponential arc weights has weight less than $12\log n/k$, while by (i) of Lemma 4.6, with very high probability it has $O(\log n)$ vertices. It then follows easily that, for all n sufficiently large and $k \geq \log^2 n$, the bound as in (4.3) holds for uniform distribution, as well.

The following result on the maximum outdegree in the subgraph $(K_n[S])^{(k)}$ of the complete graph $K_n[S]$ on k vertices with $[0,1]$-uniform arc weights can be found in Peres et al. [19].

Lemma 4.7 ([19, Lemma 5.1])**.** *Let $1 \leq k \leq n$ and let $S \subseteq V$ with $|S| = k$. Then, for every $c > 6$, we have $\mathbb{P}(\Delta((K_n[S])^{(k)}) > c\log k) = O(k^{1-c/6})$.*

The maximum outdegree in $(K_n[S \times \overline{S}])^*$ is directly related to the maximum load in the balls-into-bins process, which is used in the proof of the following lemma.

Lemma 4.8. *Let $1 \leq k \leq n$, let $S \subseteq V$ with $|S| = k$, and let $\overline{S} = V \setminus S$. Then,*

$$\mathbb{P}(\Delta((K_n[S \times \overline{S}])^*) \geq e^2((n-k)/k + \log k)) = O(k^{-1}).$$

Proof. Consider vertices from S as bins and vertices from $\overline{S}$ as balls. For $v \in \overline{S}$, each arc in $S \times v$ is equally likely to be the shortest, so v is thrown into a bin chosen uniformly at random, and the result follows by Lemma 2.1 for $N = k$ and $M = n - k$. □

We are now ready to prove the main result of this subsection.

Theorem 4.9. *For every $k \geq \log^2 n$, we have*

$$\mathbb{P}\left(\Delta(OUT_k) \geq (e^2 + 12)\log k + e^2 \frac{n-k}{k}\right) = O(k^{-1}).$$

Proof. Let $S = \{v_1, v_2, \ldots, v_k\}$ and $\overline{S} = V \setminus S$. Further, let us write $\alpha = 12 \log k$ and $\beta = e^2((n-k)/k \log k)$. By the inequality (4.3), for every $k \geq \log^2 n$, we have

$$\begin{aligned}\mathbb{P}(\Delta(OUT_k) \geq \alpha + \beta) &\leq \mathbb{P}(\Delta((K_n[S])^{(k)}) + \Delta((K_n[S \times \overline{S}])^*) \geq \alpha + \beta) \\ &\leq \mathbb{P}(\Delta((K_n[S])^{(k)}) \geq \alpha) + \mathbb{P}(\Delta((K_n[S \times \overline{S}])^*) \geq \beta).\end{aligned}$$

By Lemma 4.7, we have $\mathbb{P}(\Delta((K_n[S])^{(k)}) \geq \alpha) \leq 1/k$. Similarly, by Lemma 4.8, we have $\mathbb{P}(\Delta((K_n[S \times \overline{S}])^*) \geq \beta) \leq 1/k$. Hence, $\mathbb{P}(\Delta(OUT_k) \geq \alpha + \beta) \leq 1/k + 1/k = O(1/k)$. □

5 Expected-case analysis

We perform an expected-case analysis of the Tree algorithm for the complete directed graphs on n vertices with arc weights selected independently at random from the uniform distribution on $[0, 1]$. Recall that $SP_k^{(k)}$ is the set of all shortest k-paths that contain v_k and end at some vertex in the set $\{v_1, v_2, \ldots, v_k\}$. We first show that the expected number of paths in $SP_k^{(k)}$ is $O(n \log k)$.

Lemma 5.1. *For each $k = 1, 2, \ldots, n$, we have $\mathbb{E}(|SP_k^{(k)}|) = O(n \log k)$.*

Proof. For $v_i \in V$, let $SP_i^{(k)}$ denote the set of all shortest k-paths that contain v_i and end at some vertex in the set $\{v_1, v_2, \ldots, v_k\}$. Note that

$$\sum_{i=1}^{k} |SP_i^{(k)}| \leq \sum_{i=1}^{n} \sum_{j=1}^{k} |v_i \overset{k}{\rightsquigarrow} v_j|.$$

By symmetry, we have $\mathbb{E}(|SP_i^{(k)}|) = \mathbb{E}(|SP_j^{(k)}|)$ for arbitrary $i, j \in \{1, 2, \ldots, k\}$, and hence

$$k\mathbb{E}(|SP_k^{(k)}|) = \sum_{i=1}^{k} \mathbb{E}(|SP_i^{(k)}|) \leq \sum_{i=1}^{n} \sum_{j=1}^{k} \mathbb{E}(|v_i \overset{k}{\rightsquigarrow} v_j|) \leq kn\mathbb{E}(\max_{u,v \in V} |u \overset{k}{\rightsquigarrow} v|).$$

By (ii) of Lemma 4.6, we get that $\mathbb{E}(|SP_k^{(k)}|) = O(n \log k)$. □

We are now ready to analyse the expected time of the Tree algorithm.

Theorem 5.2. *The Tree algorithm has an expected-case running time of $O(n^2 \log^2 n)$ for the complete directed graphs on n vertices with arc weights selected independently at random from the uniform distribution on $[0, 1]$.*

Proof. To estimate the number of comparisons T_k at iteration k, we consider two cases. First, for $k < \log^2 n$ we bound T_k from above by n^2. Second, we estimate $\mathbb{E}(T_k)$ for $k \geq \log^2 n$. For every $c > 0$, we have

$$\begin{aligned}\mathbb{E}(T_k) = \mathbb{E}(T_k \mid \Delta(\mathit{OUT}_k) < c) \cdot \mathbb{P}(\Delta(\mathit{OUT}_k) < c) \\ + \mathbb{E}(T_k \mid \Delta(\mathit{OUT}_k) \geq c) \cdot \mathbb{P}(\Delta(\mathit{OUT}_k) \geq c).\end{aligned}$$

Using inequality (3.1) we get

$$\begin{aligned}\mathbb{E}(T_k \mid \Delta(\mathit{OUT}_k) < c) &\leq \mathbb{E}(|\mathit{SP}_k^{(k)}| \cdot \Delta(\mathit{OUT}_k) + O(n) \mid \Delta(\mathit{OUT}_k) < c) \\ &\leq c \cdot \mathbb{E}(|\mathit{SP}_k^{(k)}|) + O(n).\end{aligned}$$

As T_k is always at most n^2, we have $\mathbb{E}(T_k \mid \Delta(\mathit{OUT}_k) \geq c) \leq n^2$. Further, taking into account that $\mathbb{P}(\Delta(\mathit{OUT}_k) < c) \leq 1$, we get

$$\mathbb{E}(T_k) \leq c \cdot \mathbb{E}(|\mathit{SP}_k^{(k)}|) + O(n) + n^2 \cdot \mathbb{P}(\Delta(\mathit{OUT}_k) \geq c).$$

Take $c = (e^2 + 12)\log k + e^2 \frac{n-k}{k}$. Then, by Lemma 4.9, we have $\mathbb{P}(\Delta(\mathit{OUT}_k) \geq c) = O(k^{-1})$. Moreover, by Lemma 5.1, we have $\mathbb{E}(|\mathit{SP}_k^{(k)}|) = O(n \log k)$, which gives us

$$\begin{aligned}\mathbb{E}(T_k) &= O((e^2 + 12)n \log^2 k + e^2(n-k)n \log k/k) + O(n) + O(n^2/k) \\ &= O(n \log^2 n + n^2 \log n/k).\end{aligned}$$

Putting everything together, we bound the expected time of the algorithm from above as

$$\begin{aligned}\mathbb{E}\left(\sum_{k=1}^{n} T_k\right) &= \sum_{k=1}^{\log^2 n - 1} \mathbb{E}(T_k) + \sum_{k=\log^2 n}^{n} \mathbb{E}(T_k) \\ &\leq \sum_{k=1}^{\log^2 n - 1} n^2 + \sum_{k=\log^2 n}^{n} O(n \log^2 n + n^2 \log n/k) = O(n^2 \log^2 n),\end{aligned}$$

as claimed. □

We conclude the section with a proof of the main theorem.

Proof of Theorem 1.1. The Hourglass algorithm does not have a worse bound than the Tree variant, so the result follows by Theorem 5.2. □

6 Empirical evaluation

All algorithms were implemented in `C++` and compiled using `g++ -march=native -O3`. The tests were ran on an Intel(R) Core(TM) i7-2600@3.40GHz with 8GB RAM running Windows 7 64-bit.

To make the comparison between Floyd-Warshall and its modified versions fairer, we improved the Floyd-Warshall algorithm with a simple modification skipping combinations of i and k where $W_{ik} = \infty$, and consequently reducing the number of relaxations of the algorithm to $R_{FW} \leq n^3$.

The experiments were conducted on the following random digraphs: (i) uniform random digraphs with arc weights uniformly distributed on the interval $[0, 1]$, and (ii) unweighted random digraphs. In both cases, the digraphs were constructed by first setting the desired vertex count and density. Then, a random Hamiltonian cycle was constructed, ensuring the strong connectivity of the digraph. After the cycle was constructed, the remaining $n(n-2)$ arcs were put into a list and randomly permuted, and then added into the digraph until the desired density was reached. Finally, algorithms were executed on the instance, and their running times were recorded. Tests were conducted ten times and averaged, with each test running on a different randomly generated graph.

6.1 Empirical comparison of the number of relaxations

Our motivation when designing the Hourglass and Tree algorithms was to skip relaxations that are not contributing to the result. To verify the theoretical results on the expected number of relaxations in practice we conducted two experiments in which we counted the number of relaxations by different algorithms. For the first experiment we generated a subfamily of digraphs from (i), mentioned above, consisting of complete digraphs of varying size vertex set. On contrary, for the second experiment we generated another subfamily of digraphs from (i), now consisting of sparse digraphs with fixed vertex set and variable arc density. The results of experiments are presented in the plots relative to the number of relaxations performed by the Floyd-Warshall algorithm; i.e. all numbers of relaxations are divided by R_{FW}.

The results of the first experiment, in which $R_{FW} = n^3$ since digraphs are complete, are presented in Figure 1. To relate the theoretical upper bound of $O(n^2 \lg^2 n)$ of the Tree algorithm and the experimental results, we added also the plot of the function $60\frac{n^2 \lg^2 n}{n^3}$. We chose the constant 60 so that the plots of the Tree algorithm and the added function start at the same initial point, namely at 2^8 vertices. The results of the second experiment for

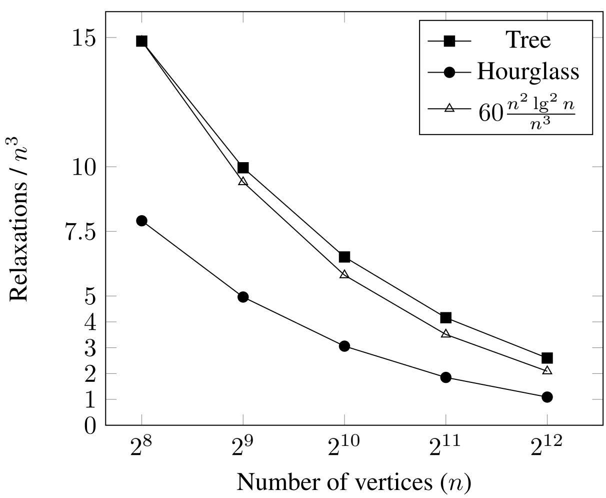

Figure 1: Complete digraphs of various sizes with the number of relaxations of algorithms divided by n^3.

$n = 1024$ vertices and sizes of the arc set varying between $n^2/10$ and $8n^2/10$ are shown in Figure 2.

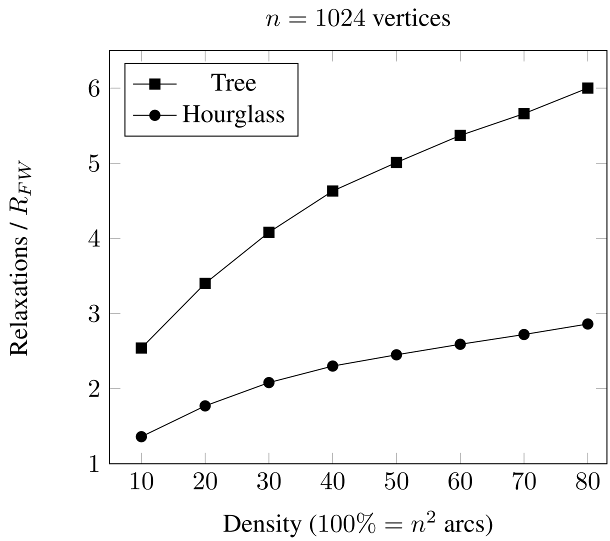

Figure 2: Digraphs with $n = 1024$ vertices and various arc densities with the number of relaxations of algorithms divided by R_{FW}.

In Figure 1 we see a significant reduction of relaxations which also implies the decrease of running time of the Tree and Hourglass algorithms. From the plot we can also see that the experimental results indicate that the theoretical upper bound of the Tree algorithm is asymptotic tight. The experiments on sparse digraphs (see Figure 2) also show a reduction in relaxations as digraphs become sparser.

6.2 Empirical comparison of running times

As discussed in the introduction, we compared the Tree[3] and Hourglass algorithms with the Floyd-Warshall [11, 26] and Dijkstra [10] algorithms, as well as the algorithm of Brodnik and Grgurovič [4], which we refer to as Propagation. These algorithms were chosen since they proved to perform best out of a larger group of algorithms compared in [4].

It should be pointed out, that breadth-first search is extremely efficient in solving APSP on unweighted digraphs. However, we did not include breadth-first search in comparisons, because we consider unweighted graph instances only as the worst-case instances of the general shortest path problem (each arc is part of at least one shortest path in such instances).

The algorithms were tested on the graph families (i) and (ii) described at the beginning of this section, with sizes of the vertex set varying between 512 and 4096, and sizes of the arc set varying between $n^{1.1}$ and n^2. As the priority queue in the Dijkstra and Propagation algorithms we used pairing heaps since they are known to perform especially well in solving APSP in practice [22], even though the amortized complexity of their decrease key

[3] In the tests, we used the implementation of the algorithm with improvements from Subsection 3.1.

operation takes $O(2^{2\sqrt{\lg \lg n}})$ in comparison to O(1) of Fibonacci heaps [21]. We used the implementation of pairing heaps from the Boost Library, version 1.55.

The results for uniform random digraphs presented in Figure 3 show that both, Propagation and Tree, outperform the other algorithms on all vertex and arc densities. As the size n of graphs increases, the running time of Hourglass approaches the running time of Tree, but the constant factors still prove to be too large for Hourglass to prevail because of a more clever exploration strategy. Moreover, it is interesting to see that Floyd-Warshall based Tree and Hourglass outperform Dijkstra on sparse graphs.

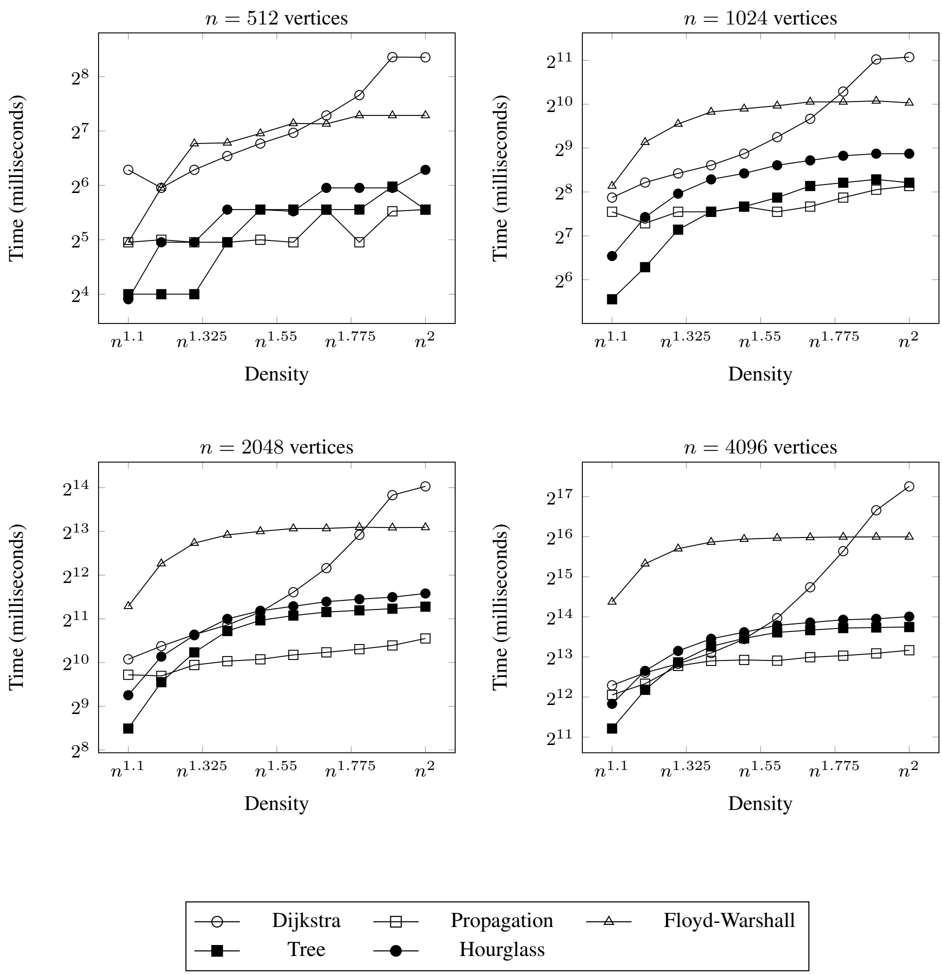

Figure 3: Experimental results on family (i) – uniform digraphs.

The results for unweighted random digraphs are shown in Figure 4. What is interesting is that Tree and Hourglass remain competitive with Dijkstra, and even outperforming it on smaller graphs in some instances. In contrast, the performance of Propagation falls short of Dijkstra because each arc is part of at least one shortest path in these graphs.

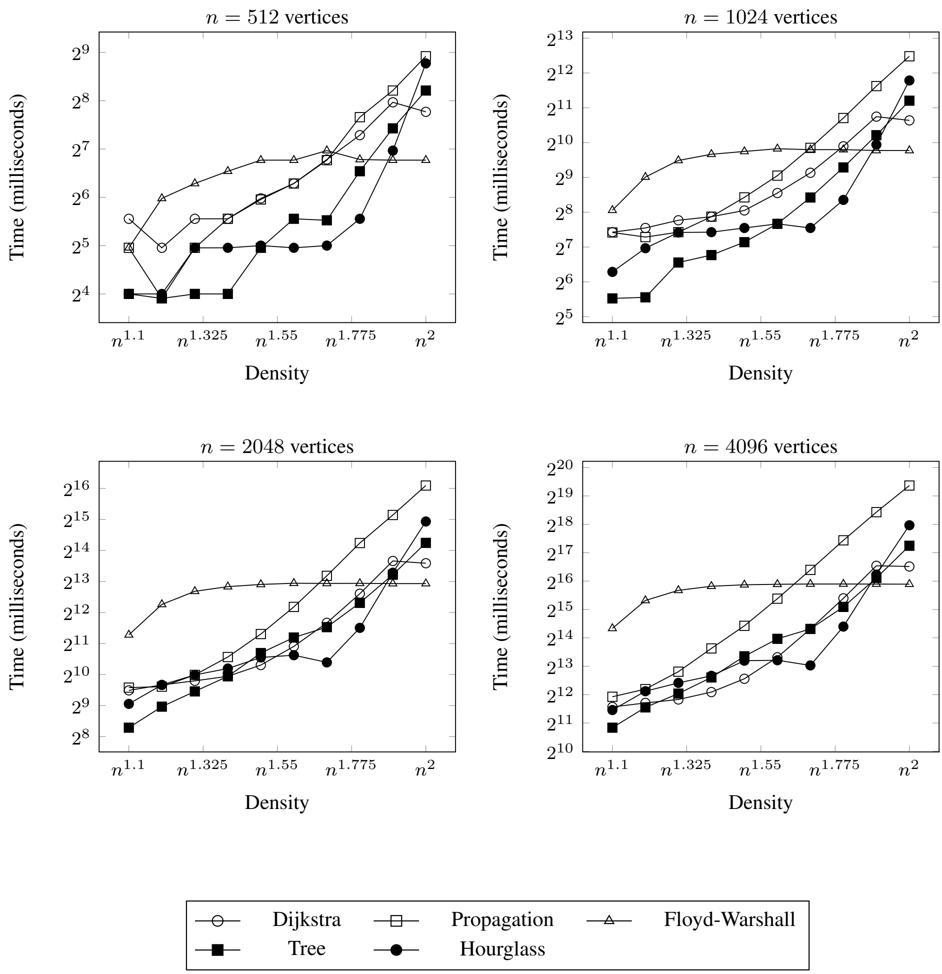

Figure 4: Experimental results on family (ii) – unweighted digraphs.

7 Conclusion

We theoretically analyzed the Tree algorithm which is a relatively simple modification of the Floyd-Warshall algorithm. The analysis gives its expected-case time complexity in the uniform model of $O(n^2 \log^2 n)$, which also explains the algorithm's good practical performance presented in Section 6. We also presented the Hourglass algorithm as a further improvement of the Tree algorithm, but it remains an open question whether its expected-case time complexity in the uniform model is $o(n^2 \log^2 n)$.

Next, since both the Tree and Hourglass algorithms allow negative arc weights, it would be interesting to analyze their expected-case running time complexity for a model that permits negative arcs such as the vertex potential model [5, 6].

Overall, the Tree algorithm is simple to implement and offers very good performance. The Hourglass algorithm has the potential to be even better but probably requires a more complex implementation. It is also worthwhile to note that the space requirement of the Tree algorithm is not worse than the space requirement of any algorithm that reports all shortest paths. The Hourglass algorithm requires an additional matrix of size n^2.

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References

[1] L. Addario-Berry, N. Broutin and G. Lugosi, The longest minimum-weight path in a complete graph, *Comb. Probab. Comput.* **19** (2010), 1–19, doi:10.1017/s0963548309990204.

[2] R. K. Ahuja, T. L. Magnanti and J. B. Orlin, *Network Flows: Theory, Algorithms, and Applications*, Prentice-Hall, Inc., USA, 1993.

[3] P. A. Bloniarz, A shortest-path algorithm with expected time $O(n^2 \log n \log^* n)$, *SIAM J. Comput.* **12** (1983), 588–600, doi:10.1137/0212039.

[4] A. Brodnik and M. Grgurovič, Solving all-pairs shortest path by single-source computations, *Discrete Appl. Math.* **231** (2017), 119–130, doi:10.1016/j.dam.2017.03.008.

[5] B. V. Cherkassky, A. V. Goldberg and T. Radzik, Shortest paths algorithms: theory and experimental evaluation, *Math. Programming* **73** (1996), 129–174, doi:10.1016/0025-5610(95) 00021-6.

[6] C. Cooper, A. Frieze, K. Mehlhorn and V. Priebe, Average-case complexity of shortest-paths problems in the vertex-potential model, *Random Struct. Algorithms* **16** (2000), 33–46, doi: 10.1002/(sici)1098-2418(200001)16:1⟨33::aid-rsa3⟩3.0.co;2-0.

[7] T. H. Cormen, C. Stein, R. L. Rivest and C. E. Leiserson, *Introduction to Algorithms*, McGraw-Hill Higher Education, 2nd edition, 2001.

[8] C. Demetrescu and G. F. Italiano, A new approach to dynamic all pairs shortest paths, *J. ACM* **51** (2004), 968–992, doi:10.1145/1039488.1039492.

[9] C. Demetrescu and G. F. Italiano, Experimental analysis of dynamic all pairs shortest path algorithms, *ACM Trans. Algorithms* **2** (2006), 578–601, doi:10.1145/1198513.1198519.

[10] E. W. Dijkstra, A note on two problems in connexion with graphs, *Numerische Mathematik* **1** (1959), 269–271, doi:10.1007/bf01386390.

[11] R. W. Floyd, Algorithm 97: Shortest path, *Commun. ACM* **5** (1962), 345, doi:10.1145/367766. 368168.

[12] T. Hagerup and C. Rüb, A guided tour of Chernoff bounds, *Inf. Process. Lett.* **33** (1990), 305–308, doi:10.1016/0020-0190(90)90214-I.

[13] Y. Han and T. Takaoka, An $O(n^3 \log\log n/ \log^2 n)$ time algorithm for all pairs shortest paths, in: *Proceedings of the 13th Scandinavian Conference on Algorithm Theory*, Springer-Verlag, Berlin, Heidelberg, SWAT'12, 2012 pp. 131–141, doi:10.1007/978-3-642-31155-0_12.

[14] S. Janson, One, two and three times $\log n/n$ for paths in a complete graph with random weights, *Combin. Probab. Comput.* **8** (1999), 347–361, doi:10.1017/s0963548399003892.

[15] D. B. Johnson, Efficient algorithms for shortest paths in sparse networks, *J. ACM* **24** (1977), 1–13, doi:10.1145/321992.321993.

[16] D. Karger, D. Koller and S. J. Phillips, Finding the hidden path: time bounds for all-pairs shortest paths, *SIAM J. Comput.* **22** (1993), 1199–1217, doi:10.1137/0222071.

[17] K. Mehlhorn and V. Priebe, On the all-pairs shortest-path algorithm of Moffat and Takaoka, *Random Struct. Algorithms* **10** (1997), 205–220, doi:10.1002/(sici)1098-2418(199701/03)10:1/2⟨205::aid-rsa11⟩3.3.co;2-J.

[18] A. Moffat and T. Takaoka, An all pairs shortest path algorithm with expected time $O(n^2 \log n)$, *SIAM J. Comput.* **16** (1987), 1023–1031, doi:10.1137/0216065.

[19] Y. Peres, D. Sotnikov, B. Sudakov and U. Zwick, All-pairs shortest paths in $O(n^2)$ time with high probability, *J. ACM* **60** (2013), Article No. 26 (25 pages), doi:10.1145/2508028.2505988.

[20] S. Pettie, A new approach to all-pairs shortest paths on real-weighted graphs, *Theor. Comput. Sci.* **312** (2004), 47–74, doi:10.1016/s0304-3975(03)00402-x.

[21] S. Pettie, Towards a final analysis of pairing heaps, in: *46th Annual IEEE Symposium on Foundations of Computer Science (FOCS'05)*, IEEE Computer Society, Washington, DC, USA, 2005 pp. 174–183, doi:10.1109/sfcs.2005.75.

[22] S. Pettie, V. Ramachandran and S. Sridhar, Experimental evaluation of a new shortest path algorithm, in: *Proceedings of the 4th International Workshop on Algorithm Engineering and Experiments 2002 (ALENEX '02)*, Springer-Verlag, Berlin, Heidelberg, 2002 pp. 126–142, doi:10.1007/3-540-45643-0_10.

[23] M. Raab and A. Steger, "Balls into bins"—a simple and tight analysis, in: *Randomization and Approximation Techniques in Computer Science (Barcelona, 1998)*, Springer, Berlin, volume 1518 of *Lecture Notes in Computer Science*, pp. 159–170, 1998, doi:10.1007/3-540-49543-6_13.

[24] P. M. Spira, A new algorithm for finding all shortest paths in a graph of positive arcs in average time $O(n^2\log^2 n)$, *SIAM J. Comput.* **2** (1973), 28–32, doi:10.1137/0202004.

[25] T. Takaoka and A. Moffat, An $O(n^2 \log n \log \log n)$ expected time algorithm for the all shortest distance problem, in: *Mathematical Foundations of Computer Science 1980*, Springer, Berlin, Heidelberg, volume 88 of *Lecture Notes in Computer Science*, 1980 pp. 643–655, doi:10.1007/bfb0022539, proceedings of the 9th Symposium Held in Rydzyna, Poland, September 1 – 5, 1980.

[26] S. Warshall, A theorem on boolean matrices, *J. ACM* **9** (1962), 11–12, doi:10.1145/321105.321107.

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Z-oriented triangulations of surfaces

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Abstract

The main objects of the paper are z-oriented triangulations of connected closed 2-dimensional surfaces. A z-orientation of a map is a minimal collection of zigzags which double covers the set of edges. We have two possibilities for an edge – zigzags from the z-orientation pass through this edge in different directions (type I) or in the same direction (type II). Then there are two types of faces in a triangulation: the first type is when two edges of the face are of type I and one edge is of type II and the second type is when all edges of the face are of type II. We investigate z-oriented triangulations with all faces of the first type (in the general case, any z-oriented triangulation can be shredded to a z-oriented triangulation of such type). A zigzag is homogeneous if it contains precisely two edges of type I after any edge of type II. We give a topological characterization of the homogeneity of zigzags; in particular, we describe a one-to-one correspondence between z-oriented triangulations with homogeneous zigzags and closed 2-cell embeddings of directed Eulerian graphs in surfaces. At the end, we give an application to one type of the z-monodromy.

Keywords: Directed Eulerian embedding, triangulation of a surface, zigzag, z-monodromy, z-orientation.

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1 Introduction

Petrie polygons are well-known objects described by Coxeter [5] (see also [13]). These are skew polygons in regular polyhedra such that any two consecutive edges, but not three, are on the same face. Analogs of Petrie polygons for graphs embedded in surfaces are called *zigzags* [7, 10] or *closed left-right paths* [9, 18]. These are sequences of oriented edges defined by the rule described above. Zigzags have many applications, for example, they are successfully exploited to enumerate all combinatorial possibilities for fullerenes [3].

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The case when a map, i.e. an embedding of a graph in a surface, has a unique zigzag is very important [7, 9]. Following [7] we call such maps z-*knotted*. They have nice homological properties and are closely connected to the Gauss code problem [6, 9, 11].

The study of zigzags in 3-regular plane graphs, in particular fullerenes, is one of the main directions of [7]. A large class of z-knotted 3-regular plane graphs is obtained using a computer. The dual objects, i.e. spherical triangulations, have the same zigzag structure. Zigzags in triangulations of surfaces (not necessarily orientable) are investigated in [15, 16, 17]. By [17], every such triangulation admits a z-knotted shredding, i.e. it can be modified to a z-knotted triangulation of the same surface by triangulating of some of its faces.

A z-*orientation* of a map is a minimal collection of zigzags which double covers the set of edges [7]. In the z-knotted case, this collection contains only one zigzag and is unique up to reversing. For every z-orientation we have the following two types of edges: an edge is of type I if the distinguished zigzags pass through this edge in different directions and an edge is of type II if they pass through the edge in the same direction. It is not difficult to prove that for every face in a triangulation with fixed z-orientation one of the following possibilities is realized: the face contains precisely two edges of type I and the third edge is of type II (the first type of face) or all edges are of type II (the second type of face). We observe that every z-oriented triangulation can be shredded to a triangulation where all faces are of the first type (Section 2). In this paper, we restrict ourself to z-oriented triangulations with all faces of the first type.

Let Γ be such a triangulation of a surface M. Then the number of edges of type I is twice the number of edges of type II and we say that a zigzag is *homogeneous* if it contains precisely two edges of type I after each edge of type II. Denote by Γ_{II} the subgraph of Γ formed by all edges of type II. Our first result (Theorem 3.3) states that the following three conditions are equivalent:

(1) all zigzags of Γ are homogeneous,

(2) Γ_{II} is a closed 2-cell embedding of a simple Eulerian digraph such that every face is a directed cycle,

(3) each connected component of $M \setminus \Gamma_{\mathrm{II}}$ is homeomorphic to an open 2-dimensional disk.

Note that directed Eulerian spherical embeddings are known also as *plane alternating dimaps*; they are investigated, for example, in [2, 8, 12]. Directed Eulerian embeddings in arbitrary surfaces are considered in [1, 4].

We will use the following structural property of Γ (without assumption that the zigzags are homogeneous): the connected components of $M \setminus \Gamma_{\mathrm{II}}$ are open disks, cylinders or Möbius strips (the third type of components can be realized only for the non-orientable case) and all these possibilities are realized. We show that the existence of cylinders or Möbius strips contradicts the homogeneity of zigzags.

A z-monodromy of face is a permutation which acts on the oriented edges of this face, the z-monodromy of an edge e is the first oriented edge of the face which occurs in certain zigzag after e. By [17], there are precisely 7 types of z-monodromies (M1) – (M7). For each of the types (M3) – (M5) and (M7) there is a triangulation such that each face has the z-monodromy of this type. The types (M1) and (M2) are exceptional: all faces with z-monodromies of each of these types form a forest [16]. The case (M6) cannot be investigated by the methods of [16] and the authors left it as an open problem. It is easy to see that each face with the z-monodromy (M6) is of the first type for every z-orientation. Using this fact, we construct a series of toric triangulations where all faces have z-monodromies of type (M6).

2 Zigzags and z-orientations of triangulations of surfaces

Let M be a connected closed 2-dimensional surface (not necessarily orientable). A *triangulation* of M is a 2-cell embedding of a connected simple finite graph in M such that all faces are triangles [14, Section 3.1]. Then the following assertions are fulfilled:

(1) every edge is contained in precisely two distinct faces,

(2) the intersection of two distinct faces is an edge or a vertex or empty.

Let Γ be a triangulation of M. A *zigzag* in Γ is a *sequence* of edges $\{e_i\}_{i\in\mathbb{N}}$ satisfying the following conditions for every natural i:

- e_i and e_{i+1} are distinct edges of a certain face (then they have a common vertex, since every face is a triangle),
- the faces containing e_i, e_{i+1} and e_{i+1}, e_{i+2} are distinct and the edges e_i and e_{i+2} are non-intersecting.

Since Γ is finite, there is a natural number $n > 0$ such that $e_{i+n} = e_i$ for every natural i. In what follows, every zigzag will be presented as a cyclic sequence $e_1, \dots, e_n$, where n is the smallest number satisfying the above condition.

Every zigzag is completely determined by any pair of consecutive edges belonging to this zigzag and for any distinct edges e and e' on a face there is a unique zigzag containing the sequence e, e'. If $Z = \{e_1, \dots, e_n\}$ is a zigzag, then the reversed sequence $Z^{-1} = \{e_n, \dots, e_1\}$ also is a zigzag. A zigzag cannot contain a sequence $e, e', \dots, e', e$ which implies that $Z \neq Z^{-1}$ for any zigzag Z, i.e. a zigzag cannot be self-reversed (see, for example, [17]). We say that Γ is *z-knotted* if it contains precisely two zigzags Z and Z^{-1}, in other words, there is a single zigzag up to reversing.

Example 2.1. Zigzags in the Platonic solids (three of them are triangulations of the sphere) are skew polygons without self-intersections and are called *Petrie polygons*.

Example 2.2. Let BP_n be the n-gonal bipyramid, where $1, \dots, n$ denote the consecutive vertices of the base and the remaining two vertices are denoted by a, b (see Figure 1 for $n = 3$).

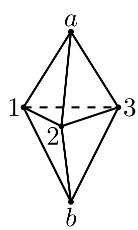

Figure 1: 3-gonal bipyramid.

(a) In the case when $n = 2k+1$, the bipyramid BP_n is z-knotted. If k is odd, then the unique (up to reversing) zigzag is

$$\begin{aligned}&a1, 12, 2b, b3, \ldots, a(n-2), (n-2)(n-1), (n-1)b, bn, n1,\\ &1a, a2, 23, 3b, \ldots, a(n-1), (n-1)n, nb,\\ &b1, 12, 2a, a3, \ldots, b(n-2), (n-2)(n-1), (n-1)a, an, n1,\\ &1b, b2, 23, 3a, \ldots, b(n-1), (n-1)n, na.\end{aligned}$$

If k is even, then this zigzag is

$$\begin{aligned}&a1, 12, 2b, b3, \ldots, b(n-2), (n-2)(n-1), (n-1)a, an, n1,\\ &1b, b2, 23, 3a, \ldots, a(n-1), (n-1)n, nb,\\ &b1, 12, 2a, a3, \ldots, a(n-2), (n-2)(n-1), (n-1)b, bn, n1,\\ &1a, a2, 23, 3b, \ldots, b(n-1), (n-1)n, na.\end{aligned}$$

(b) If $n = 2k$ and k is odd, then the bipyramid contains precisely two zigzags (up to reversing):

$$\begin{aligned}&a1, 12, 2b, b3, 34, \ldots, a(n-1), (n-1)n, nb,\\ &b1, 12, 2a, a3, 34, \ldots, b(n-1), (n-1)n, na\end{aligned}$$

and

$$\begin{aligned}&a2, 23, 3b, b4, 45, \ldots, an, n1, 1b,\\ &b2, 23, 3a, a4, 45, \ldots, bn, n1, 1a.\end{aligned}$$

(c) In the case when $n = 2k$ and k is even, there are precisely four zigzags (up to reversing):

$$\begin{aligned}&a1, 12, 2b, \ldots, b(n-1), (n-1)n, na;\\ &b1, 12, 2a, \ldots, a(n-1), (n-1)n, nb;\\ &a2, 23, 3b, \ldots, bn, n1, 1a;\\ &b2, 23, 3a, \ldots, an, n1, 1b.\end{aligned}$$

See [15, 17] for more examples of z-knotted triangulations. Examples of z-knotted fullerenes can be found in [7].

Suppose that Γ contains precisely k distinct zigzags up to reversing. A *z-orientation* of Γ is a collection τ consisting of k distinct zigzags such that for each zigzag Z we have $Z \in \tau$ or $Z^{-1} \in \tau$. There are precisely 2^k distinct z-orientations of Γ. For every z-orientation $\tau = \{Z_1, \ldots, Z_k\}$ the z-orientation $\tau^{-1} = \{Z_1^{-1}, \ldots, Z_k^{-1}\}$ will be called *reversed* to τ. The triangulation Γ with a z-orientation τ will be denoted by (Γ, τ) and called a *z-oriented triangulation*.

Let τ be a z-orientation of Γ. For every edge e of Γ one of the following possibilities is realized:

- there is a zigzag $Z \in \tau$ such that e occurs in this zigzag twice and other zigzags from τ do not contain e,

- there are two distinct zigzags $Z, Z' \in \tau$ such that e occurs in each of these zigzags only once and other zigzags from τ do not contain e.

In the first case, we say that e is an *edge of type* I if Z passes through e twice in different directions; otherwise, e is said to be an *edge of type* II. Similarly, in the second case: e is an *edge of type* I if Z and Z' pass through e in different directions or e is an *edge of type* II if Z and Z' pass through e in the same direction. In what follows, edges of type II will be considered together with the direction defined by τ. A vertex of Γ is called *of type* I if it belongs only to edges of type I; otherwise, we say that this is a *vertex of type* II.

The following statements hold for any z-orientation τ of Γ.

Lemma 2.3. *For each vertex of type* II *the number of edges of type* II *which enter this vertex is equal to the number of edges of type* II *which leave it.*

Proof. The number of times that the zigzags from τ enter a vertex is equal to the number of times that these zigzags leave this vertex. □

Proposition 2.4. *For every face of* (Γ, τ) *one of the following possibilities is realized:*

(I) *the face contains two edges of type* I *and the third edge is of type* II*, see* Figure 2(a)*;*

(II) *all edges of the face are of type* II *and form a directed cycle, see* Figure 2(b).

A face in a triangulation is said to be *of type* I or *of type* II if the corresponding possibility is realized.

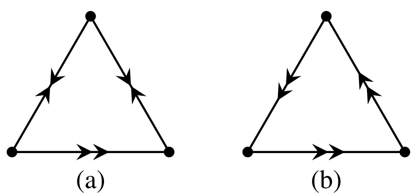

Figure 2: Two types of faces.

Proof of Proposition 2.4. Consider a face whose edges are denoted by e_1, e_2, e_3. Without loss of generality we can assume that the zigzag containing the sequence e_1, e_2 belongs to τ. Let Z and Z' be the zigzags containing the sequences e_2, e_3 and e_3, e_1, respectively. Then $Z \in \tau$ or $Z^{-1} \in \tau$ and $Z' \in \tau$ or $Z'^{-1} \in \tau$. An easy verification shows that for each of these four cases we obtain (I) or (II). □

Example 2.5. If n is odd, then the bipyramid BP_n has a unique z-orientation (up to reversing), see Example 2.2(a). The edges ai and bi, $i \in \{1, \dots, n\}$ are of type I and the edges on the base of the bipyramid are of type II. The vertices a, b are of type I and the vertices on the base are of type II. All faces are of type I. The same happens for the case when $n = 2k$ and k is odd if the z-orientation is defined by the two zigzags presented in Example 2.2(b); however, all faces are of type II if we replace one of these zigzags by the reversed.

Example 2.6. Suppose that $n = 2k$ and k is even. Let Z_1, Z_2, Z_3, Z_4 be the zigzags from Example 2.2(c). For the z-orientation defined by these zigzags all faces are of type I. If the z-orientation is defined by Z_1, Z_2 and Z_3^{-1}, Z_4^{-1}, then all faces are of type II. In the case when the z-orientation is defined by Z_1, Z_2, Z_3 and Z_4^{-1}, there exist faces of both types.

Remark 2.7. If we replace a z-orientation by the reversed z-orientation, then the type of every edge does not change (but all edges of type II reverse the directions), consequently, the types of vertices and faces also do not change. For z-knotted triangulations there is a unique z-orientation (up to reversing) and we can determine the types of edges, vertices and faces without attaching to a z-orientation [15].

A triangulation Γ' of M is a *shredding* of the triangulation Γ if it is obtained from Γ by triangulating some faces of Γ such that all new vertices are contained in the interiors of these faces.

Proposition 2.8. *Any z-oriented triangulation admits a z-oriented shredding with all faces of type* I.

Proof. Let F be a face of type II in a z-oriented triangulation (Γ, τ) and let e_1, e_2, e_3 be edges of F. Suppose that the edges of F are oriented as in Figure 3 and denote by σ the permutation $(1, 2, 3)$.

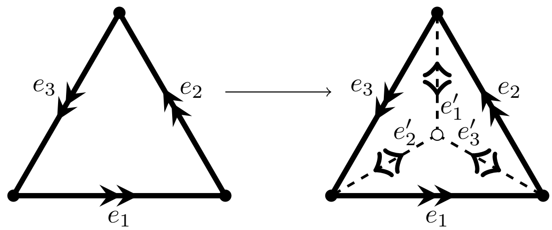

Figure 3: Triangulation of faces of type II.

Zigzags from τ passes through F precisely three times, so the face F separates them into 3 segments of type

$$e_{\sigma^{-1}(i)}, e_i, X_{ij}, e_j, e_{\sigma(j)},$$

where $i, j \in \{1, 2, 3\}$ and the sequence X_{ij} is a maximal part of a zigzag formed by edges occuring between e_i and e_j. Let $\mathcal{X}$ be the set of all such sequences X_{ij} for F and the z-orientation τ. Note that every $X_{ij} \in \mathcal{X}$ is completely determined by the beginning edge e_i and the final edge e_j. Now, we triangulate the face F by adding a vertex in the interior of F and three edges connecting this vertex with the vertices of F. We denote this new triangulation by Γ' and write e'_i for the new edge if it does not has a common vertex with e_i (see Figure 3). Observe that for any $i \in \{1, 2, 3\}$ there exists a zigzag in Γ' containing a subsequence of the form

$$e_i, e'_{\sigma^{-1}(i)}, e'_i, e_{\sigma^{-1}(i)}, X_{\sigma^{-1}(i)j}$$

for certain $j \in \{1, 2, 3\}$ and $X_{\sigma^{-1}(i)j} \in \mathcal{X}$. The edge e_j which occurs in the zigzag directly after this subsequence is the same as the edge after $X_{\sigma^{-1}(i)j}$ in (Γ, τ), since $X_{\sigma^{-1}(i)j}$ does not contain edges of F. Therefore, zigzags of Γ' related to the three faces not contained in

Γ pass through the edges coming from Γ in the same way as zigzags from τ. This implies the existence of a z-orientation of Γ' such that all edges from Γ do not change their types and the three new faces of Γ' contained in F are of type I. Recursively, we eliminate all faces of type II from (Γ, τ) and come to a z-oriented shredding of Γ with all faces of type I and such that the type of any edge from (Γ, τ) is preserved. □

3 Homogeneous zigzags in triangulations with faces of type I

In this section, we will always suppose that Γ is a triangulation with fixed z-orientation τ such that all faces in Γ are of type I, i.e. each face contains precisely two edges of type I and the third edge is of type II. If m is the number of faces, then there are precisely m edges of type I and $m/2$ edges of type II. In other words, the number of edges of type I is twice the number of edges of type II. We say that a zigzag of Γ is *homogeneous* if it is a cyclic sequence $\{e_i, e'_i, e''_i\}_{i=1}^{n}$, where each e_i is an edge of type II and all e'_i, e''_i are edges of type I. If a zigzag is homogeneous, then the reversed zigzag also is homogeneous. Denote by Γ_{II} the subgraph of Γ formed by all vertices and by all edges of type II.

Example 3.1. The zigzags of $\Gamma = BP_n$ are homogeneous if n is odd (the z-knotted case) or n is even and the z-orientation is defined by the two zigzags from Example 2.2(b) or by the four zigzags from Example 2.2(c). Only a and b are vertices of type I and Γ_{II} is the directed cycle formed by the edges of the base of the bipyramid. Conversely, if all zigzags of Γ are homogeneous and there are precisely two vertices of type I, then Γ is a bipyramid (this statement is an easy consequence of Theorem 3.3 which will be presented later).

Example 3.2. Let Γ' be a triangulation of M with a z-orientation such that all faces are of type II (see [16, Example 4] for a z-knotted triangulation of $\mathbb{S}^2$ whose faces are of type II). As in the proof of Proposition 2.8, we consider the shredding Γ'' of Γ' which is obtained by adding a vertex in the interior of each face and three edges connecting this vertex with the vertices of the face. This triangulation Γ'' admits a z-orientation such that all faces are of type I. Every zigzag $e_1, e_2, e_3, \ldots$ in Γ' is extended to a zigzag

$$e_1, e'_1, e''_1, e_2, e'_2, e''_2, e_3, \ldots$$

in Γ'' which passes through edges of Γ' in the opposite directions. All e_i are of type II and all e'_i and e''_i are of type I. So, all zigzags in Γ'' are homogeneous.

An *Eulerian digraph* is a connected digraph such that indegree equals outdegree for every vertex.

Theorem 3.3. *The following three conditions are equivalent:*

(1) *All zigzags of* Γ *are homogeneous.*

(2) Γ_{II} *is a closed* 2*-cell embedding of a simple Eulerian digraph such that every face is a directed cycle.*

(3) *Each connected component of* $M \setminus \Gamma_{\mathrm{II}}$ *is homeomorphic to an open* 2*-dimensional disk.*

The implication (2) ⇒ (3) is obvious. The implications (1) ⇒ (2) and (3) ⇒ (1) will be proved in Section 4 and Section 5, respectively.

4 Proof of the implication (1) $\Rightarrow$ (2) in Theorem 3.3

Now, we generalize the construction described in Proposition 2.8 and Example 3.2. Let Γ' be a closed 2-cell embedding of a connected finite simple graph in the surface M. Then all faces of Γ' are homeomorphic to a closed 2-dimensional disk. For each face F we take a point v_F belonging to the interior of F. We add all v_F to the vertex set of Γ' and connect each v_F with every vertex of F by an edge. We obtain a triangulation of M which will be denoted by $\mathrm{T}(\Gamma')$.

The assumption that our 2-cell embedding is closed cannot be omitted. Indeed, if a certain face of Γ' is not homeomorphic to a closed 2-dimensional disk, then there is a pair of vertices connected by a double edge and $\mathrm{T}(\Gamma')$ is not a triangulation in our sense.

Proposition 4.1. *If all zigzags of Γ are homogeneous, then Γ_{II} is a closed 2-cell embedding of a simple Eulerian digraph such that every face is a directed cycle and $\Gamma = \mathrm{T}(\Gamma_{\mathrm{II}})$. Conversely, if Γ' is a closed 2-cell embedding of a simple Eulerian digraph and every face is a directed cycle, then the triangulation $\mathrm{T}(\Gamma')$ admits a unique z-orientation such that all zigzags of $\mathrm{T}(\Gamma')$ are homogeneous and Γ' is the subgraph of $\mathrm{T}(\Gamma')$ formed by all vertices and edges of type* II.

Proof. (I): Let v be a vertex of Γ. Consider all faces containing v and take the edge on each of these faces which does not contain v. All such edges form a cycle which will be denoted by $C(v)$.

Suppose that all zigzags of Γ are homogeneous and consider any edge e_1 of type II. Let v_1 and v_2 be the vertices of this edge such that e_1 is directed from v_1 to v_2. We choose one of the two faces containing e_1 and take in this face the vertex v which does not belong to e_1. Let e_1' and e_1'' be the edges which contain v and occur in a certain zigzag $Z \in \tau$ immediately after e_1, see Figure 4. Denote by e_2 the third edge of the face containing e_1' and e_1''. This edge contains v_2 and another one vertex, say v_3. Since Z is homogeneous, the edges e_1' and e_1'' are of type I, and consequently, e_2 is of type II. The zigzag which goes through e_1' from v to v_2 belongs to τ (this follows easily from the fact that Z goes through e_1' in the opposite direction and e_1' is an edge of type I). The latter guarantees that the edge e_2 is directed from v_2 to v_3. By our assumption, the edge e_3 which occurs in Z immediately after e_1' and e_1'' is of type II. This edge is directed from v_3 to a certain vertex v_4. So, e_1, e_2, e_3 are consecutive edges of the cycle $C(v)$ and each e_i is directed from v_i to v_{i+1}. Consider the zigzag from τ which contains the sequence e_2, e_1''. The next edge in this zigzag connects v and v_4 (the zigzag goes from v to v_4). Let e_4 be the edge which occurs in the zigzag after it. Then e_4 is an edge of type II (by our assumption), it belongs to $C(v)$ and leaves v_4. Recursively, we establish that $C(v)$ is a directed cycle formed by edges of type II and every edge containing v is of type I, i.e. v is a vertex of type I. Now,

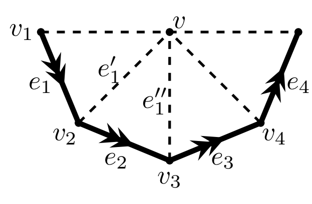

Figure 4: Cycle of edges of type II.

we consider the other face containing e_1 and take the vertex v' of this face which does not belong to e_1. Using the same arguments, we establish that v' is a vertex of type I and $C(v')$ is a directed cycle formed by edges of type II.

For every vertex v of type I we can take a face containing v and the edge of this face which does not contain v. This edge is of type II (since the remaining two edges of the face are of type I). The above arguments show that the following assertions are fulfilled:

(1) vertices of type I exist and for every such vertex v the cycle $C(v)$ is a directed cycle formed by edges of type II;

(2) for every edge of type II there are precisely two vertices v and v' of type I such that this edge is contained in the cycles $C(v)$ and $C(v')$.

Similarly, for every edge e of type I we take a face containing e; this face contains an edge of type II which implies that e connects vertices of different types.

Consider Γ_{II}. Any two vertices of type II in Γ can be connected by a path formed by edges of type II which means that Γ_{II} is connected. Indeed, if a path between two vertices of type II goes through a vertex v of type I, then the edge going into v and the edge leaving v are incident to vertices in the same cycle $C(v)$ and so we can rewrite that part of the path to use edges from $C(v)$ instead of the edges through v. It is easy to see that Γ_{II} is a 2-cell embedding of a simple digraph such that every face is the directed cycle $C(v)$ for a certain vertex v of type I; in particular, this 2-cell embedding is closed. Lemma 2.3 implies that Γ_{II} is an Eulerian digraph. The equality $\Gamma = \mathrm{T}(\Gamma_{\mathrm{II}})$ is obvious.

The following remark will be used to prove the second part of the theorem. The conditions (1) and (2) guarantee that every zigzag of Γ containing an edge of type II is homogeneous. Recall that the number of edges of type I is twice the number of edges of type II. This implies that there is no zigzag containing edges of type I only (since every edge occurs twice in a unique zigzag from τ or it occurs ones in precisely two distinct zigzags from τ). Therefore, every zigzag of Γ is homogeneous if (1) and (2) hold.

(II): Suppose that Γ' is a closed 2-cell embedding of a simple Eulerian digraph such that every face is a directed cycle.

Let $e_1, \dots, e_n$ be the directed cycle formed by all edges of a certain face of Γ'. For every $i \in \{1, \dots, n\}$ we define $j(i) = i + 2(\operatorname{mod} n)$ and denote by e'_i and e''_i the edges containing the vertex v_F in $\mathrm{T}(\Gamma')$ and intersecting e_i and $e_{j(i)}$, respectively. Consider the zigzag of $\mathrm{T}(\Gamma')$ which contains the sequence $e_i, e'_i, e''_i, e_{j(i)}$. It passes through e_i and $e_{j(i)}$ according to the directions of these edges; and the same holds for every edge of Γ' which occurs in this zigzag. Such a zigzag exists for any pair formed by a face of Γ' and an edge on this face. The collection of all such zigzags is a z-orientation of $\mathrm{T}(\Gamma')$ with the following properties: all edges of Γ' are of type II and every v_F is a vertex of type I. This implies that $\mathrm{T}(\Gamma')$ satisfies the conditions (1) and (2) which gives the claim. □

Note that the second part of Proposition 4.1 will be used to prove the implication (3) ⇒ (1).

5 Structure of triangulations with faces of type I

In this section, we describe some structural properties of z-oriented triangulations with faces of type I. As an immediate consequence we obtain the implication (3) ⇒ (1).

As above, we suppose that (Γ, τ) is a z-oriented triangulation of M, where all faces are of type I. As above, we denote by Γ_{II} the subgraph of Γ consisting of all vertices and all edges of type II. From the previous section it follows that if the zigzags of (Γ, τ) are homogeneous, then connected components of $M \setminus \Gamma_{\text{II}}$ are homeomorphic to open 2-dimensional disks. Now, we describe the general case.

Theorem 5.1. *The following assertions are fullfiled:*

(1) *Connected components of $M \setminus \Gamma_{\text{II}}$ are homeomorphic to an open 2-dimensional disk, an open Möbius strip or an open cylinder.*

(2) *A connected component of $M \setminus \Gamma_{\text{II}}$ contains a vertex of type* I *if and only if it is an open 2-dimensional disk; such a vertex of type* I *is unique.*

Proof. Consider two distinct edges e_0 and e_1 of type I contained in a certain face F_1. There is precisely one face containing e_1 and distinct from F_1. Denote this face by F_2 and write e_2 for the other edge of type I on F_2. Recursively, we construct sequences of edges $\{e_i\}_{i\in\mathbb{N}\cup\{0\}}$ and faces $\{F_i\}_{i\in\mathbb{N}}$ such that e_{i-1} is the common edge of F_{i-1}, F_i for every $i \in \mathbb{N}$. For any pair of the faces F_{i-1}, F_i we distinguish the following two cases presented in Figure 5. In the first case, the edges of type II of F_{i-1} and F_i have a common vertex (Figure 5(a)). In the second case (Figure 5(b)), the edges of type II are disjoint.

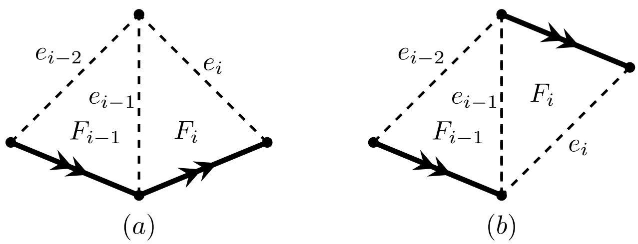

Figure 5: Two possibilities of adjacency for faces of type I.

Let n be the smallest natural number such that $e_n = e_0$ (such a number exists by finiteness). Therefore, the above sequences can be considered as cyclic sequences $\{e_i\}_{i=1}^n$ and $\{F_i\}_{i=1}^n$. The union $\mathcal{F} = \bigcup_{i=1}^n F_i$ will be called a *component* of (Γ, τ). The boundary of $\mathcal{F}$ consists of (not necessarily all) edges of type II belonging to faces F_i.

Denote by e_i^{II} the edge of type II belonging to F_i. We take n disjoint closed triangles $T_1, T_2, \dots, T_n$. For any $i = 1, 2, \dots, n$ there is a homeomorphism $h_i \colon F_i \to T_i$ transferring any vertex and any edge of F_i to a vertex and an edge of T_i, respectively. We identify $h_i(e_i)$ and $h_{i+1}(e_i)$ for any i in such a way that for every vertex v of e_i the vertices $h_i(v)$ and $h_{i+1}(v)$ are identified. We get a 2-dimensional surface $\mathcal{T}$ with boundary. The boundary of $\mathcal{T}$ is the union of the images of all edges of type II, i.e. $\partial\mathcal{T} = \bigcup_{i=1}^n h_i(e_i^{\text{II}})$. Note that $\mathcal{F}$ is not necessarily a surface (since it is possible that for distinct i, j the edges $e_i^{\text{II}}, e_j^{\text{II}}$ have a common vertex). The interior of surface $\mathcal{T}$ is homeomorphic to one of the connected components of $M \setminus \Gamma_{\text{II}}$ and $\mathcal{F}$ can be obtained from $\mathcal{T}$ by gluing of some parts of the boundary.

Suppose that $h_i(e_i)$ and $h_{i+1}(e_i)$ are identified only for $i = 1, 2, \dots, n-1$ (but not $h_1(e_0)$ and $h_n(e_n)$ from T_1 and T_n, respectively). Then we get a space homeomorphic to a closed 2-dimensional disk whose boundary contains $h_1(e_0), h_n(e_n)$. Now, to complete the construction of $\mathcal{T}$, we have to glue $h_1(e_0)$ and $h_n(e_n)$. Precisely one of the following possibilities is realized:

- A union of these sides is connected and by gluing of them we obtain that $\mathcal{T}$ is homeomorphic to a closed 2-dimensional disk (Figure 6(1)).
- The sides are disjoint and by identification of them we get a surface homeomorphic to a closed Möbius strip (Figure 6(2)) or a closed cylinder (Figure 6(3)).

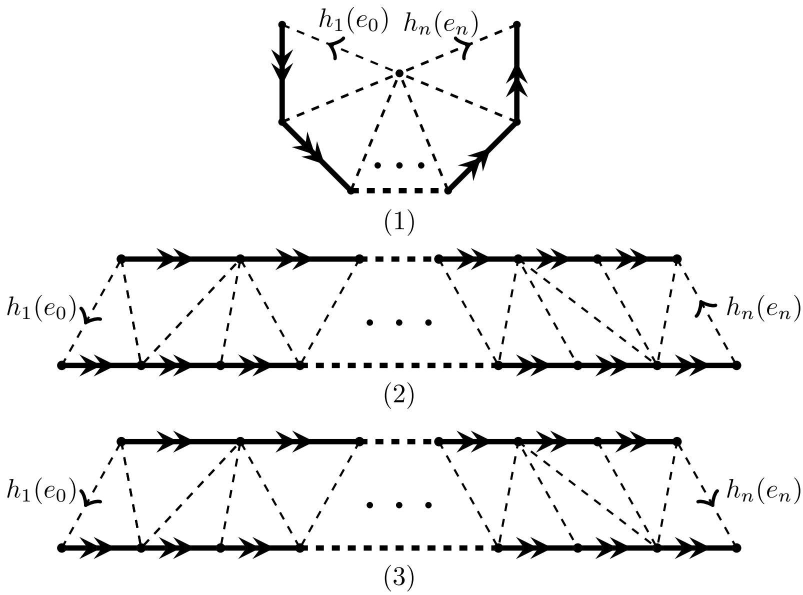

Figure 6: Closed disk, closed Möbius strip and closed cylinder.

Let v_i be the vertex of T_i corresponding to the vertex of F_i not belonging to the edge e_i^{II}. In the first case, the images of edges of type I have the common vertex which is the image of all $h_i(v_i)$; it is clear that this vertex corresponds to the vertex of type I from $\mathcal{F}$, see Figure 6(1). In the remaining cases, any vertex $h_i(v_i)$ is contained in the boundary of $\mathcal{T}$ and correspond to a certain vertex of Γ_{II} (see Figure 6(2) and 6(3)). So, we obtained the statements (1) and (2). □

If a connected component of $M \setminus \Gamma_{\mathrm{II}}$ is homeomorphic to an open 2-dimensional disk, then the corresponding component of (Γ, τ) is homeomorphic to a closed 2-dimensional disk (if this component has some identifications at the boundary, then the vertex of type I in this component is joined by a double edge to a certain vertex at the boundary which is impossible, since we work with embeddings of simple graphs).

Proof of (3) $\Rightarrow$ (1) *in Theorem 3.3.* Assume that each connected component of $M \setminus \Gamma_{\mathrm{II}}$ is a disk. By the above remark, Γ_{II} is a closed 2-cell embedding. Lemma 2.3 shows that this is an embedding of simple Eulerian digraph. The second part of Theorem 5.1 states that each disk contains a unique vertex of type I; as in the proof of Theorem 5.1 we establish that its boundary is an oriented cycle. We have $\Gamma = \mathrm{T}(\Gamma_{\mathrm{II}})$ and the second part of Proposition 4.1 gives the claim. □

The following three examples show that all possibilities for connected components of $M \setminus \Gamma_{\mathrm{II}}$ are realized.

Example 5.2. Consider the following triangulation Γ of a torus $\mathbb{T} = \mathbb{S}^1 \times \mathbb{S}^1$ (see Figure 7). The triangulation Γ admits the z-orientation such that all faces are of type I. The subgraph Γ_{II} has two connected components which are 6-cycles and $\mathbb{T} \setminus \Gamma_{\mathrm{II}}$ consists of two connected components homeomorphic to open cylinders $(-1, 1) \times \mathbb{S}^1$.

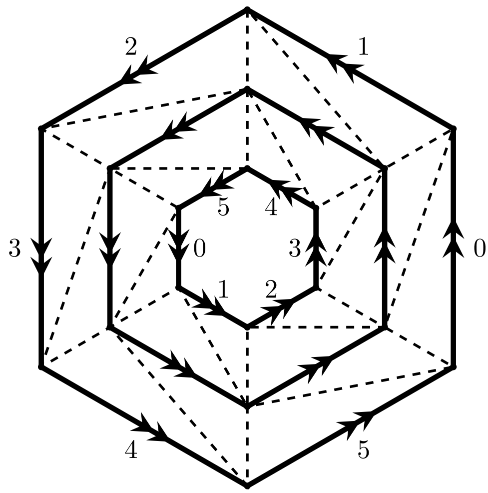

Figure 7: Toric triangulation.

In a similar way, we can construct a z-oriented toric triangulation with connected components of $\mathbb{T} \setminus \Gamma_{\mathrm{II}}$ which are open cylinders of arbitrary length.

Example 5.3. Let $n \in \mathbb{N}$ and let Γ be the triangulation of a real projective plane obtained by gluing of boundaries of a Möbius strip and a closed 2-dimensional disk (see Figure 8). According to the corresponding z-orientation all faces are of type I and the graph Γ_{II} consists of all edges marked by the double arrows and their vertices. Then $\mathbb{R}\mathrm{P}^2 \setminus \Gamma_{\mathrm{II}}$ has two connected components. One of them is homeomorphic to an open 2-dimensional disk and the remaining to an open Möbius strip.

Example 5.4. Suppose that Γ is the triangulation of a sphere obtained by the gluing of the two disks whose boundaries are cycles $e_1, e_2, \ldots, e_6$ (see Figure 9). There is a z-orientation τ such that all faces are of type I. Then $\mathbb{S}^2 \setminus \Gamma_{\mathrm{II}}$ has precisely four connected components: three components are homeomorphic to an open 2-dimensional disk and the remaining to an open cylinder. The components of (Γ, τ) corresponding to the first three connected components are closed 2-dimensional disks. The fourth component of (Γ, τ) is homeomorphic to a closed cylinder $\mathbb{S}^1 \times D^1$, where two points at one of the connected components of the boundary are glued.

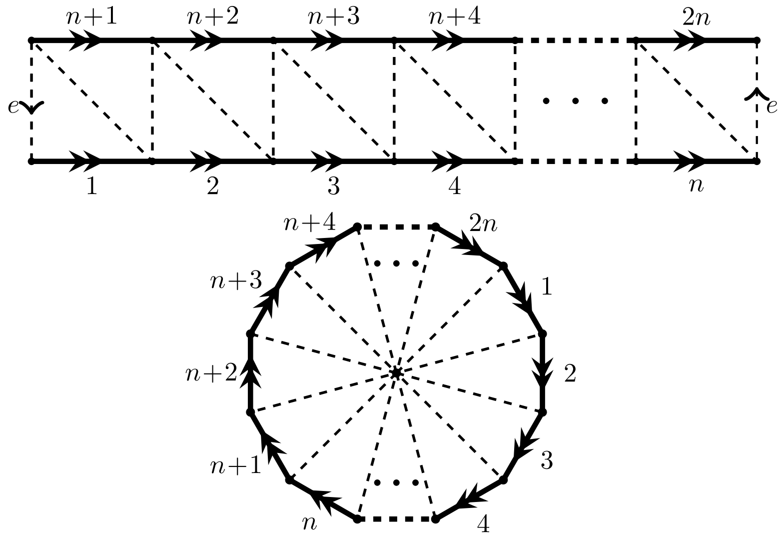

Figure 8: Projective plane triangulation.

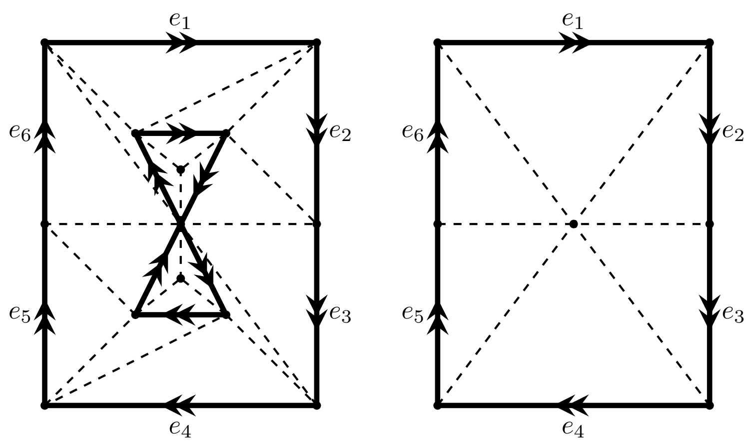

Figure 9: Spherical triangulation.

6 Relations to z-monodromies

Let F be a face in Γ whose vertices are a, b, c. Consider the set $\Omega(F)$ of all oriented edges of F

$$\Omega(F) = \{ab, bc, ca, ac, cb, ba\},$$

where xy is the edge from $x \in \{a, b, c\}$ to $y \in \{a, b, c\}$. If $e = xy$ then we write yx by $-e$. Denote by D_F the following permutation of the set $\Omega(F)$

$$(ab, bc, ca)(ac, cb, ba).$$

The *z-monodromy* (see [16, 17]) of the face F is the permutation M_F defined as follows. For any $e \in \Omega(F)$ we take $e_0 \in \Omega(F)$ such that $D_F(e_0) = e$ and consider the zigzag containing the sequence e_0, e. We define $M_F(e)$ as the first element of $\Omega(F)$ contained in this zigzag after e.

By [17, Theorem 4.4], there are the following possibilities for the z-monodromy M_F and each of them is realized:

(M1) M_F is the identity,

(M2) $M_F = D_F$,

(M3) $M_F = (-e_1, e_2, e_3)(-e_3, -e_2, e_1)$,

(M4) $M_F = (e_1, -e_2)(e_2, -e_1)$, where e_3 and $-e_3$ are fixed points,

(M5) $M_F = (D_F)^{-1}$,

(M6) $M_F = (-e_1, e_3, e_2)(-e_2, -e_3, e_1)$,

(M7) $M_F = (e_1, e_2)(-e_1, -e_2)$, where e_3 and $-e_3$ are fixed points,

where (e_1, e_2, e_3) is one of the cycles in D_F.

Let G_i be the subgraph of the dual Γ^* formed by vertices corresponding to faces in Γ whose z-monodromies are of type (Mi), two vertices of G_i are adjacent if they are adjacent in Γ^*. By [16, Theorem 1], the subgraphs G_1 and G_2 are forests. For (M3), (M4), (M5) and (M7) the above statement fails: z-monodromies of all faces of the bipyramid BP_n are of type

- (M3) for $n = 2k + 1$ where k is odd,
- (M4) for $n = 2k + 1$ where k is even,
- (M7) for $n = 2k$ where k is odd,
- (M5) for $n = 2k$ where k is even.

Proposition 6.1. *If M_F is* (M6)*, then F is of type* I *for any z-orientation of* Γ.

Proof. Let e_1, e_2, e_3 be consecutive oriented edges of the face F. We suppose that the z-monodromy of F is (M6), i.e.

$$M_F = (-e_1, e_3, e_2)(-e_2, -e_3, e_1).$$

There are precisely two zigzags (up to reversing) which contain F

$$e_1, e_2, \ldots, -e_1, -e_3, \ldots \qquad \text{and} \qquad e_2, e_3, \ldots ;$$

since the edge corresponding to the pair $\{e_1, -e_1\}$ is passed in two different directions by the same zigzag, then it is of type I for any orientation of the zigzag. Therefore, F is of type I for any z-orientation. □

Lemma 6.2. *Let F be a face in (Γ, τ) such that there are precisely two zigzags from τ which contain edges from F. Then the following assertions are fulfilled:*

(1) *There is a unique edge* $e \in F$ *which occurs in one of these zigzags twice.*

(2) *The type of* e *does not depend on the choice of* z*-orientation.*

(3) *If* e *is of type* I*, then* M_F *is* (M6)*. If* e *is of type* II*, then* M_F *is* (M7)*.*

Proof. (1): Any face occurs precisely thrice, as a pair of its adjacent edges, in zigzags from the z-orientation τ. By the assumption, there are precisely two zigzags from τ which pass through our face. This is possible only when one of these zigzags passes through it once and the second twice.

(2): The edge e can occur in the same zigzag twice in two ways: the zigzag passes through e the first time in one of directions and the second time in the opposite (type I) or the zigzag passes through e twice in the same direction (type II). It is easy to see that the type of e is the same for any z-orientation of Γ.

(3): By [17, Remark 4.9] the z-monodromy of the face F is (M6) or (M7). In the case (M6) the statement follows from Proposition 6.1. Let e_1, e_2, e_3 be consecutive edges of F and M_F be of type (M7), i.e.

$$M_F = (e_1, e_2)(-e_1, -e_2).$$

In this case, F occurs twice in the zigzag

$$e_2, e_3, \dots, e_3, e_1, \dots$$

and e_3 is of type II for any z-orientation of Γ. □

Now, we can construct a class of toric triangulations, where z-monodromies are of type (M6) for all faces. Our arguments are based on Lemma 6.2.

Example 6.3. Let n, m be odd numbers not less than 3 and let Γ_0 be a $n \times m$ grid where the opposite sides are identified. Then Γ_0 can be embedded into a torus in the natural way. Suppose that $\Gamma = \mathrm{T}(\Gamma_0)$ (see Figure 10 for the case $n = m = 3$).

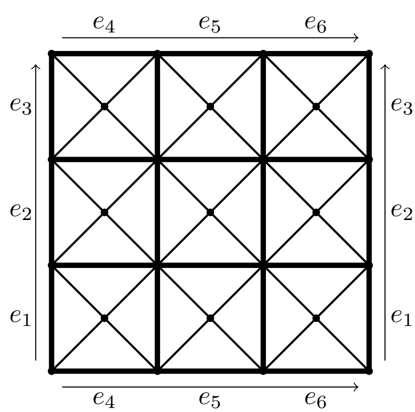

Figure 10: Toric triangulation related to 3×3 grid.

Each zigzag of Γ determines a band formed by n or m squares from the grid (see Figure 11 for a band consisting of 5 squares) and passes through each face of this band twice.

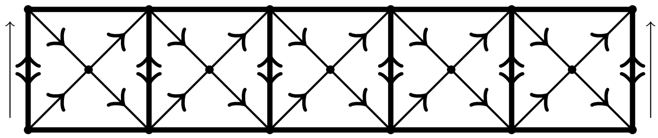

Figure 11: Band consisting of 5 squares.

Observe that the edges common for two consecutive squares from the grid are passed twice (they marked on Figure 11 by the bold line) and are of type I for any z-orientation. Remaining edges are passed by the zigzag once. Therefore, all edges of subgraph Γ_0 are of type I and all faces of Γ are of type I for any z-orientation. It is clear that any edge incident to a vertex in the interior of a square occurs once in two different zigzags. Thus, for any face of Γ there are precisely two zigzags which pass it. Lemma 6.2 guarantees that z-monodromies of all faces of Γ are (M6).

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References

[1] C. P. Bonnington, M. Conder, M. Morton and P. McKenna, Embedding digraphs on orientable surfaces, *J. Comb. Theory Ser. B* **85** (2002), 1–20, doi:10.1006/jctb.2001.2085.

[2] C. P. Bonnington, N. Hartsfield and J. Širáň, Obstructions to directed embeddings of Eulerian digraphs in the plane, *Eur. J. Comb.* **25** (2004), 877–891, doi:10.1016/j.ejc.2003.06.006.

[3] G. Brinkmann and A. W. M. Dress, PentHex puzzles: a reliable and efficient top-down approach to fullerene-structure enumeration, *Adv. in Appl. Math.* **21** (1998), 473–480, doi:10.1006/aama.1998.0608.

[4] Y. Chen, J. L. Gross and X. Hu, Enumeration of digraph embeddings, *Eur. J. Comb.* **36** (2014), 660–678, doi:10.1016/j.ejc.2013.10.003.

[5] H. S. M. Coxeter, *Regular Polytopes*, Dover Publications, New York, 3rd edition, 1973.

[6] H. Crapo and P. Rosenstiehl, On lacets and their manifolds, *Discrete Math.* **233** (2001), 299–320, doi:10.1016/s0012-365x(00)00248-x.

[7] M.-M. Deza, M. Dutour Sikirić and M. I. Shtogrin, *Geometric Structure of Chemistry-Relevant Graphs: Zigzags and Central Circuits*, volume 1 of *Forum for Interdisciplinary Mathematics*, Springer, New Delhi, 2015, doi:10.1007/978-81-322-2449-5.

[8] G. E. Farr, Minors for alternating dimaps, *Q. J. Math.* **69** (2018), 285–320, doi:10.1093/qmath/hax039.

[9] C. Godsil and G. Royle, *Algebraic Graph Theory*, volume 207 of *Graduate Texts in Mathematics*, Springer-Verlag, New York, 2001, doi:10.1007/978-1-4613-0163-9.

[10] S. Lins, Graph-encoded maps, *J. Comb. Theory Ser. B* **32** (1982), 171–181, doi:10.1016/0095-8956(82)90033-8.

[11] S. Lins, E. Oliveira-Lima and V. Silva, A homological solution for the Gauss code problem in arbitrary surfaces, *J. Comb. Theory Ser. B* **98** (2008), 506–515, doi:10.1016/j.jctb.2007.08.007.

[12] T. A. McCourt, Growth rates of groups associated with face 2-coloured triangulations and directed Eulerian digraphs on the sphere, *Electron. J. Comb.* **23** (2016), #P1.51 (23 pages), doi:10.37236/5410.

[13] P. McMullen and E. Schulte, *Abstract Regular Polytopes*, volume 92 of *Encyclopedia of Mathematics and its Applications*, Cambridge University Press, Cambridge, 2002, doi:10.1017/cbo9780511546686.

[14] B. Mohar and C. Thomassen, *Graphs on Surfaces*, Johns Hopkins Studies in the Mathematical Sciences, Johns Hopkins University Press, Baltimore, MD, 2001.

[15] M. Pankov and A. Tyc, Connected sums of z-knotted triangulations, *Eur. J. Comb.* **80** (2019), 326–338, doi:10.1016/j.ejc.2018.02.010.

[16] M. Pankov and A. Tyc, On two types of z-monodromy in triangulations of surfaces, *Discrete Math.* **342** (2019), 2549–2558, doi:10.1016/j.disc.2019.05.030.

[17] M. Pankov and A. Tyc, z-knotted triangulations of surfaces, *Discrete Comput. Geom.* **66** (2021), 636–658, doi:10.1007/s00454-020-00182-3.

[18] H. Shank, The theory of left-right paths, in: A. P. Street and W. D. Wallis (eds.), *Combinatorial Mathematics III*, Springer-Verlag, Berlin-New York, volume 452 of *Lecture Notes in Mathematics*, 1975 pp. 42–54, doi:10.1007/bfb0069542, proceedings of the Third Australian Conference, held at the University of Queensland, St. Lucia, May 16 – 18, 1974.

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The cubical matching complex revisited*

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Abstract

Ehrenborg noted that all tilings of a bipartite planar graph are encoded by its cubical matching complex and claimed that this complex is collapsible. We point out to an oversight in his proof and explain why these complexes can be the disjoint union of two or more collapsible complexes. We also prove that all links in these complexes are suspensions up to homotopy. Furthermore, we extend the definition of a cubical matching complex to planar graphs that are not necessarily bipartite, and show that these complexes are either contractible or a disjoint union of contractible complexes. For a simple connected region that can be tiled with dominoes (2×1 and 1×2) and 2×2 squares, let f_i denote the number of tilings with exactly i squares. We prove that $f_0 - f_1 + f_2 - f_3 + \cdots = 1$ (established by Ehrenborg) is the only linear relation for the numbers f_i.

Keywords: Domino tilings, independence complexes, matching, cubical complexes.

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1 Introduction

Let $G = (V, E)$ be a bipartite planar graph that admits a perfect matching. Assume that G is embedded in the plane. This embedding splits the plane into the *regions*, the connected components of $\mathbb{R}^2 \setminus |G|$ (here $|G|$ denotes the embedding of G into $\mathbb{R}^2$). An *elementary cycle* of G is a cycle that encircles a single region R different from the outer region R^*. Throughout this paper, we identify an elementary cycle with the region it encircles as well as with its set of vertices or edges.

A *tiling* of G is a partition of the vertex set V into disjoint blocks of the following two types:

(1) an edge $\{x, y\}$ of G; or

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(2) an elementary cycle R (the set of vertices of R).

The set of all tilings of G form a cubical complex $\mathcal{C}(G)$ (called the *cubical matching complex*) defined by Ehrenborg in [6]. Note that $\mathcal{C}(G)$ depends not only on G, but also on the choice of the embedding of that graph in the plane.

A face F of $\mathcal{C}(G)$ has the form $F = M_F \cup C_F = (M_F, C_F)$, where C_F is a collection $C_F = \{R_1, R_2, \dots, R_t\}$ of vertex-disjoint elementary cycles of G, and M_F is a perfect matching on $G \setminus \big(R_1 \cup R_2 \cup \cdots \cup R_t\big)$. The dimension of F is $|C_F|$, and the vertices of $\mathcal{C}(G)$ are the perfect matchings of G.

All tilings of G covered by $F = (M_F, C_F)$ can be obtained by deleting an elementary cycle R from C_F, and adding every other edge of R into M_F (there are two possibilities to do this). Therefore, for two faces $F_1 = (M_{F_1}, C_{F_1})$ and $F_2 = (M_{F_2}, C_{F_2})$, we have that

$$\big(F_1 \subset F_2\big) \iff \Big(C_{F_1} \subset C_{F_2} \text{ and } M_{F_1} \supset M_{F_2}\Big). \tag{1.1}$$

Let G° denote the weak dual graph of a planar graph G. The vertices of G° are all bounded regions of G, and two regions that share a common edge are adjacent in G°.

The *independence complex* of a graph H is a simplicial complex $I(H)$ whose faces are the independent subsets of vertices of H. Note that for any face $F = (M_F, C_F)$ of $\mathcal{C}(G)$, the set C_F contains independent vertices of G°, i.e., C_F is a face of $I(G^\circ)$.

Since two elementary cycles of G sharing a common edge cannot be in a common face of $\mathcal{C}(G)$, it may seem at the first glance that $\mathcal{C}(G)$ can be computed from the independence complex $I(G^\circ)$.

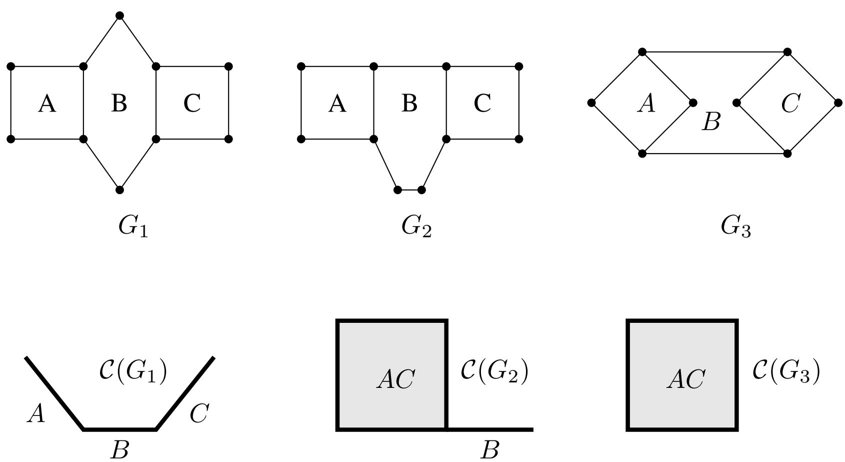


Figure 1: The three graphs with the same weak dual, but different cubical matching complexes.

However, Figure 1 shows three graphs with the same weak dual but different cubical matching complexes. The facets of the complexes on Figure 1 are labeled by corresponding subsets of pairwise disjoint elementary regions. This example points out that the requirement that $G \setminus (R_1 \cup \cdots \cup R_t)$ admits a perfect matching is a substantial one.

Example 1.1. Let $\mathcal{L}_n$ and $\mathcal{C}_n$ denote the independence complexes of P_n and C_n (the path and cycle with n vertices) respectively. The homotopy types of these complexes are determined by Kozlov in [10]:

$$\mathcal{L}_n \simeq \begin{cases} \text{a point}, & \text{if } n = 3k+1; \\ S^{\lfloor \frac{n-1}{3} \rfloor}, & \text{otherwise.} \end{cases} \qquad \mathcal{C}_n \simeq \begin{cases} S^{k-1}, & \text{if } n = 3k \pm 1; \\ S^{k-1} \vee S^{k-1}, & \text{if } n = 3k. \end{cases}$$

We will use these complexes later, see Corollary 2.2 and Remark 2.5. An interested reader can find more details about combinatorial and topological properties of $\mathcal{L}_n$ and $\mathcal{C}_n$ (and about independence complexes in general) in [7, 8] and [9].

There are some cubical complexes that cannot be realized as subcomplexes of the d-cube $C^d = [0,1]^d$, see Chapter 4 of [5].

Proposition 1.2. *Let G be a bipartite planar graph that has a perfect matching. If G has d elementary regions, then its cubical matching complex $\mathcal{C}(G)$ can be embedded into C^d.*

Proof. We use an idea from [12] to describe the coordinates of vertices of $\mathcal{C}(G)$ explicitly. Let $R_1, R_2, \ldots, R_d$ be a fixed linear order of elementary regions of G. We choose an arbitrary perfect matching M_0 of G (a vertex of $\mathcal{C}(G)$) to be the origin $\mathbf{0} = (0, 0, \ldots, 0)$ in $\mathbb{R}^d$. For another vertex M of $\mathcal{C}(G)$ the symmetric difference $M \vartriangle M_0$ is a disjoint union of cycles. Now, to a given perfect matching M of G, we assign the vertex $V_M = (x_1, \ldots, x_d)$ of C^d, where

$$x_i = \begin{cases} 1, & \text{if } R_i \text{ is contained in an odd number of cycles of } M \vartriangle M_0; \\ 0, & \text{otherwise.} \end{cases}$$

If M' and M'' are two perfect matchings of G such that $M' \vartriangle M'' = R_j$ (meaning that these two matchings differ just on an elementary region R_j), then their corresponding vertices $V_{M'}$ and $V_{M''}$ of C^d differ only at the j-th coordinate.

Therefore, all 1-dimensional faces of $\mathcal{C}(G)$ that correspond to the same region R_i are mutually parallel edges of C^d. The face $F = (M_F, C_F)$ of $\mathcal{C}(G)$ is embedded in C^d as the convex hull of its $2^{|C(F)|}$ vertices. □

2 The local structure of $\mathcal{C}(G)$

The *star* of a face F in a cubical complex $\mathcal{C}$ is the set of all faces of $\mathcal{C}$ that contain F

$$\operatorname{star}(F) = \{F' \in \mathcal{C} : F \subset F'\}.$$

The *link* of a vertex v in a cubical complex $\mathcal{C}$ is the simplicial complex $\operatorname{link}_{\mathcal{C}}(v)$ that can be realized in $\mathcal{C}$ as a "small sphere" around the vertex v. More formally, the vertices of $\operatorname{link}_{\mathcal{C}}(v)$ are the edges of $\mathcal{C}$ containing v. A subset of vertices of $\operatorname{link}_{\mathcal{C}}(v)$ is a face of $\operatorname{link}_{\mathcal{C}}(v)$ if and only if the corresponding edges belong to a common face of $\mathcal{C}$.

The *link* of a face F in a cubical complex $\mathcal{C}$ is defined in a similar way. The set of vertices of $\operatorname{link}_{\mathcal{C}}(F)$ is

$$\{F' \in \mathcal{C} : F \subset F' \text{ and } \dim F' = 1 + \dim F\},$$

and a subset A of the set of vertices is a face of $\operatorname{link}_{\mathcal{C}}(F)$ if and only if all elements of A are contained in a same face of $\mathcal{C}$.

Ehrenborg investigated the links of the cubical complexes associated to tilings of a region by dominos or lozenges.

Here we describe the links in the cubical matching complex $\mathcal{C}(G)$ for any bipartite planar graph G. For a face $F = (M_F, C_F)$ of $\mathcal{C}(G)$, let $\mathcal{R}_F$ denote the set of all elementary regions of G for which every second edge is contained in M_F. Further, let G_F denote the subgraph of the weak dual graph G° induced with the regions from $\mathcal{R}_F$.

From the definition of the link in a cubical complex and (1.1), we obtain the next statement.

Proposition 2.1. *For any face $F = (M_F, C_F)$ of $\mathcal{C}(G)$ we have that*

$$\operatorname{link}_{\mathcal{C}}(F) \cong I(G_F).$$

The above proposition explains the appearance of complexes $\mathcal{L}_n$ and $\mathcal{C}_n$ as links in cubical the matching complexes, see Theorem 3.3 and Section 4 in [6].

Assume that all elementary regions of G are quadrilaterals. In that case, for any face F of $\mathcal{C}(G)$, a region in $\mathcal{R}_F$ has exactly two opposite edges in M_F, and the degree of a vertex in G_F is at most two. Therefore, G_F is a disjoint union of paths and cycles. If the regions (quadrilaterals) $R_1, R_2, \dots, R_t$ are vertices of a cycle in G_F, then the edges of these regions that are not in M_F form two cycles of length t in G. As G is bipartite, the length of any cycle in G_F is even.

Corollary 2.2. *If all elementary regions of G are quadrilaterals, then $\operatorname{link}_{\mathcal{C}}(F)$ is a join of complexes $\mathcal{L}_p$ and $\mathcal{C}_{2q}$.*

Theorem 2.3. *Let G be a bipartite planar graph that has a perfect matching. For any face $F = (M_F, C_F)$ of $\mathcal{C}(G)$ the graph G_F is bipartite.*

Proof. Assume that G_F contains an odd cycle $R_1, R_2, \dots, R_{2m+1}$. Recall that R_i is an elementary region of G and the that every second edge of R_i is contained in M_F. Two neighborly regions R_{i-1} and R_i have to share the odd number of edges, the first and the last of their common edges belong to M_F. Therefore, for each region R_i, there is an odd number of common edges of R_i and R_{i-1} that belong to M_F. Obviously, the same holds for R_i and R_{i+1}.

So, we can conclude that there is an odd number of edges of R_i that are between $R_i \cap R_{i-1}$ and $R_i \cap R_{i+1}$ (the first and the last one of these edges are not in M_F). Therefore, the union of these edges (for all regions R_i) forms an odd cycle in G. This is a contradiction with the assumption that G is a bipartite graph. □

Barmak proved in [1] (see also in [11]) that the independence complexes of bipartite graphs are suspensions up to homotopy type. This implies the next result.

Corollary 2.4. *All links in $\mathcal{C}(G)$ are homotopy equivalent to suspensions. Therefore, the link of any face in $\mathcal{C}(G)$ has at most two connected components.*

For any simplicial complex K there exists a bipartite graph G such that the independence complex of G is homotopy equivalent to the suspension over K, see [1]. Skwarski proved in [13] (see also [1]) that there exists a planar graph G whose independence complex is homotopy equivalent to an iterated suspension of K.

We prove that the links of faces in cubical matching complexes are independence complexes of bipartite planar graphs. What can be said about homotopy types of these complexes?

Remark 2.5. There is a natural question, posed by Ehrenborg in [6]: *For what graphs G would the cubical matching complex $\mathcal{C}(G)$ be pure and/or shellable?* The complexes $\mathcal{L}_n$ are non-pure for $n > 4$, and the complexes $\mathcal{C}_n$ are non-shellable for $n > 5$. Therefore, these complexes can be used to show that the cubical matching complex of a concrete graph is non-pure or non-shellable.

3 Collapsibility and contractibility of cubical matching complexes

The next statement that we discuss was the main result of [6]. We identify a problem with the proof, describe counterexamples (infinitely many), recover a weaker result, and give a generalization.

Theorem 3.1 (Theorem 1.2 in [6])**.** *For a planar bipartite graph G that has a perfect matching, the cubical matching complex $\mathcal{C}(G)$ is collapsible.*

The proof of the above statement is based on the following two results:

(i) (Propp, Theorem 2 in [12]) *The set of all perfect matchings of a bipartite planar graph is a distributive lattice (under a certain ordering, the details of which may be found in [12]).*

(ii) (Kalai, see in [14], Solution to Exercise 3.47 c) *The cubical complex associated (see [14]) to a meet-distributive lattice is collapsible.*

Note however that Propp in his proof of (i) assumed the following two additional conditions for bipartite planar graph G:

($*$) Graph G is connected, and

($**$) Any edge of G is contained in some matching of G but not in others.

The next statement is the correct version of Theorem 3.1.

Theorem 3.2. *For a connected planar bipartite graph G that has a perfect matching and whose any edge is contained in some matching of G but not in others, the cubical matching complex $\mathcal{C}(G)$ is collapsible.*

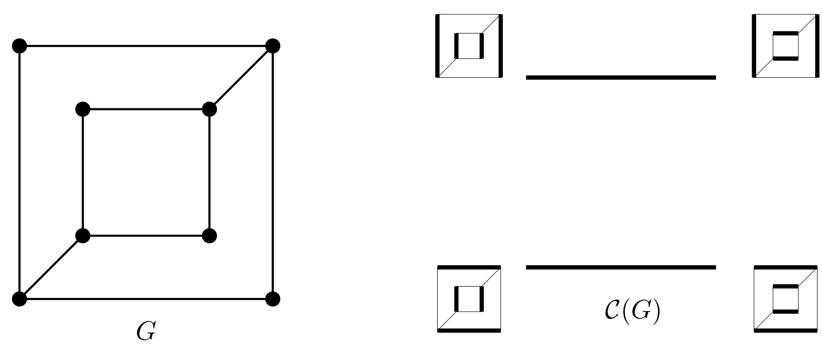


Figure 2: A bipartite planar graph G for which $\mathcal{C}(G)$ is not collapsible.

Example 3.3. The figure below shows a bipartite planar graph whose cubical matching complex is not collapsible. Note that the subdividing of the edges between the inner and outer quadrilaterals in Figure 2 gives us an infinite family of counterexamples for Theorem 3.1. Also, we can use this example to obtain a graph whose cubical complex has arbitrarily many connected components. Simply, we continue by inserting a new square into the smallest quadrilateral of the already constructed graph, and connect two non-adjacent vertices of the new square with the corresponding vertices of the old graph. This counterexample is motivated by the Jockusch example (page 27 in [12]). In his example we find a bipartite planar graph with 20 edges, but just 12 of them can be used in a perfect matching, and its cubical matching complex is a disjoint union of four segments.

Now we prove a weaker version of Theorem 3.1.

Theorem 3.4. *For a planar bipartite graph G that has a perfect matching, the cubical matching complex $\mathcal{C}(G)$ is either collapsible or a disjoint union of collapsible complexes.*

The proof will be established in a series of lemmas. Through these lemmas we assume that G is a planar bipartite graph that has a perfect matching.

The edges that do not appear in any perfect matching of G (the forbidden edges) can be deleted. Also, if the edge xy is a forced edge (xy appears in all perfect matching of G), then we may consider the graph $G - \{x, y\}$.

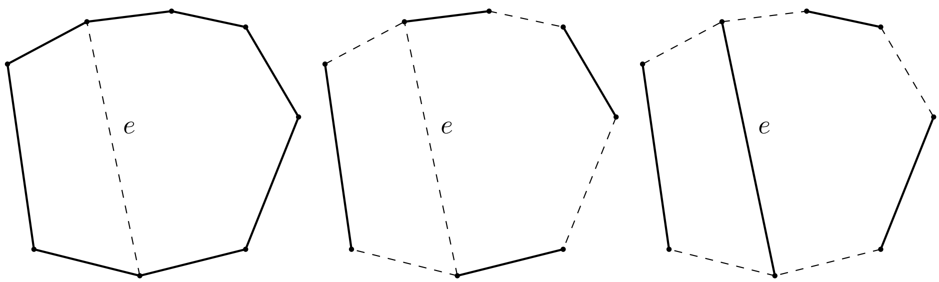


Figure 3: If a new region can be included in a tiling of $G - e$, then e is not forbidden.

Lemma 3.5. *Let e denote a forbidden edge in G and let $G' = G - e$. The possible new elementary region of G', that appears after we delete e, can not be included in a tiling of G'.*

Proof. Assume that a new region R that contains e can be included in a tiling of G'. Then e divides R into two regions of the old graph G, and we can find a perfect matching of G that contains e, see Figure 3. □

In a similar way we may prove that a new region appearing after deleting the vertices of a forced edge can not be included in a tiling of new graph.

Corollary 3.6. *Let $\overline{G}$ denote the graph obtained by deleting all forced and forbidden edges from G. Then the cubical matching complexes of $\overline{G}$ and G are isomorphic.*

Lemma 3.7. *Assume that G is a not connected, and let $G_1, G_2, \ldots, G_k$ be the connected components of G. If these components are separated (there is no component of G that is* *contained in an elementary region of another component) then* $\mathcal{C}(G) \cong \mathcal{C}(G_1) \times \mathcal{C}(G_2) \times \cdots \times \mathcal{C}(G_k)$.

Proof. For a tiling $F = (M_F, C_F)$ of G let $F_i = (M_i, C_i)$ denote the corresponding face of $C(G_i)$ (i.e., $M_i = M_F \cap E(G_i)$ and C_i is the set of regions of G_i that are included into C_F). Then we have that $F \cong F_1 \times F_2 \times \cdots \times F_k$. □

Lemma 3.8. *Assume that* G *has two different connected components* G_1 *and* G_2 *such that* G_1 *is contained in an elementary region* R *of* G_2. *Then we have that*

$$\mathcal{C}(G) = \mathcal{C}(G_1) \times (\mathcal{C}(G_2) \setminus \{R\}) . \tag{3.1}$$

If there exists a tiling of G_2 *that uses the region* R, *then* $\mathcal{C}(G)$ *is a disjoint union of collapsible complexes.*

Here $\mathcal{C}(G_2) \setminus \{R\}$ denotes the cubical complex obtained from $\mathcal{C}(G_2)$ by deleting the faces (tilings) that contain R.

Proof. The proof of (3.1) is the same as in the previous lemma. Recall that $\mathcal{C}(G_2)$ can be embedded in a cube, and that the edges corresponding to R are mutually parallel, see Proposition 1.2. Therefore, $\mathcal{C}(G_2) \setminus \{R\}$ is a disjoint union of collapsible complexes. □

Now, we consider the cubical matching complex for all planar graphs that have a perfect matching (but that are not necessarily bipartite).

Definition 3.9. Let G be a planar graph that admits a perfect matching. A tiling of G is a partition of the vertex set V into disjoint blocks of the following two types:

- an edge $\{x, y\}$ of G; or
- the set of vertices $\{v_1, v_2, \ldots, v_{2m}\}$ of an even elementary cycle R.

Let $\mathcal{C}(G)$ denote the set of all tilings of G. Note that $\mathcal{C}(G)$ is also a cubical complex.

Example 3.10. If G is a graph of a triangular prism (embedded in the plane so that the outer region is a triangle), then $\mathcal{C}(G)$ is a union of three 1-dimensional segments that share the same vertex, see the left side of Figure 4. Each segment of $\mathcal{C}(G)$ corresponds to a rectangle of prism. The link of the common vertex of these segments is a 0-dimensional complex with three points. This situation, where a link has 3 connected components, is not possible in a bipartite planar graph, as shown by Corollary 2.4. Further, the planar graph on the right-hand side on Figure 4 satisfies the conditions $(*)$ and $(**)$, but the corresponding cubical complex is not collapsible, it is a union of three disjoint edges. Therefore, the assumption that G is a bipartite graph is substantial in Theorem 3.2.

The next theorem describe the homotopy type of the cubical matching complex associated to a planar graph that admits a perfect matching.

Theorem 3.11. *Let* G *be a planar graph that has a perfect matching. The cubical complex* $\mathcal{C}(G)$ *is contractible or a disjoint union of contractible complexes.*

While contractibility is weaker than collapsibility, we partly relax the bipartite condition and obtain a weaker version of a corrected Theorem 3.1, with a different proof.

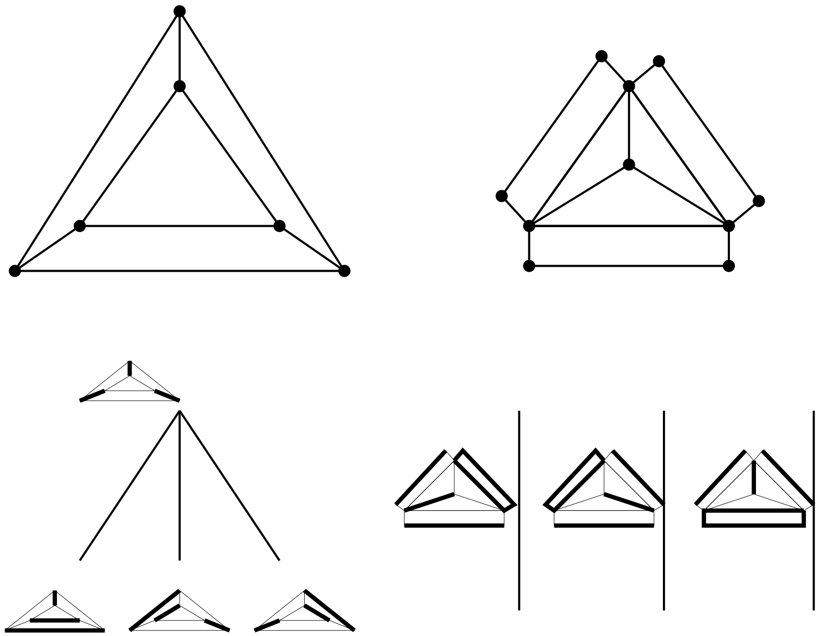


Figure 4: Non-bipartite graphs and their cubical matching complexes.

Proof. We use induction on the number of edges of G. Let $e = xy$ denote an edge that belongs to the outer region R^*. Let $R \neq R^*$ denote the elementary region that contains e. If R is an odd region, then all tilings of G can be divided into two disjoint classes:

(a) The tilings of G that do not use e. These tilings are just the tilings of $G \setminus e$.

(b) The tilings of G that contain e as an edge in a partial matching correspond to the tilings of $G \setminus \{x, y\}$, and the subcomplex of $\mathcal{C}(G)$ generated by these tilings is isomorphic to $\mathcal{C}(G \setminus \{x, y\})$.

In that case we obtain that $\mathcal{C}(G) = \mathcal{C}(G \setminus \{x, y\}) \cup \mathcal{C}(G \setminus e)$ is a disjoint union of contractible complexes by inductive assumption.

If R is an even elementary region, then some tilings of G may contain R, and these tilings are not considered in (a) and (b). Note that there is a bijection between tilings of G that contain R and all tilings of $G \setminus R$ (the graph obtained from G by deleting all vertices from R). The subcomplex of $\mathcal{C}(G)$ generated by tilings that contain R forms a prism over $\mathcal{C}(G \setminus R)$, i.e., this subcomplex is isomorphic to $Prism(\mathcal{C}(G \setminus R)) = \mathcal{C}(G \setminus R) \times [0, 1]$. Therefore, we obtain that

$$\mathcal{C}(G) = \mathcal{C}(G \setminus \{x, y\}) \cup \mathcal{C}(G \setminus e) \cup Prism(\mathcal{C}(G \setminus R)). \tag{3.2}$$

Let $\mathcal{C}_e$ denote the subcomplex of $\mathcal{C}(G \setminus e)$ formed by all tilings that contain every second edge of R (but do not contain e, obviously). Further, let $\mathcal{C}_{x,y}$ denote the subcomplex of $\mathcal{C}(G \setminus \{x, y\})$, defined by tilings that contain every second edge of R (these tilings have to contain e). Note that both of complexes $\mathcal{C}_e$ and $\mathcal{C}_{x,y}$ are isomorphic to $\mathcal{C}(G \setminus R)$, and

$$\mathcal{C}(G \setminus e) \cap Prism(\mathcal{C}(G \setminus R)) = \mathcal{C}_e \quad \text{and} \quad \mathcal{C}(G \setminus \{x,y\}) \cap Prism(\mathcal{C}(G \setminus R)) = \mathcal{C}_{x,y}.$$

The complexes on the right-hand side of (3.2) are disjoint unions of contractible complexes by the inductive hypothesis. Assume that

$$\mathcal{C}(G \setminus \{x,y\}) = A_1 \cup A_2 \cup \cdots \cup A_s \quad \text{and} \quad \mathcal{C}_{x,y} = B_1 \cup B_2 \cup \cdots \cup B_t,$$

where A_i and B_j denote the contractible components of corresponding complexes. Obviously, each complex B_j is contained in some A_i. Now, we need the following lemma.

Lemma 3.12. *Each connected component of $\mathcal{C}(G \setminus \{x,y\})$ contains at most one component of $\mathcal{C}_{x,y}$.*

Proof of Lemma 3.12. Assume that a component of $\mathcal{C}(G \setminus \{x,y\})$ contains two components of $\mathcal{C}_{x,y}$. In that case, there are two vertices of $\mathcal{C}_{x,y}$ (perfect matchings of G that contain xy) that are in different components of $\mathcal{C}_{x,y}$, but in the same component of $\mathcal{C}(G \setminus \{x,y\})$. Assume that M' and M'' are two such vertices, chosen so that the distance between them in $\mathcal{C}(G \setminus \{x,y\})$ is minimal. Let

$$M' = M_0 \overset{R_0}{\text{---}} M_1 \text{---} \ldots \text{---} M_i \overset{R_i}{\text{---}} M_{i+1} \text{---} \ldots \text{---} M_n \overset{R_n}{\text{---}} M_{n+1} = M'' \tag{3.3}$$

denote the shortest path from M' to M'' in $\mathcal{C}(G \setminus \{x,y\})$. The perfect matching M_{i+1} is obtained from M_i by removing the edges of M_i contained in an elementary region R_i, and by inserting the complementary edges. In other words, we have that $M_{i+1} = M_i \bigtriangleup R_i$, for an elementary region R_i contained in $\mathcal{R}_{F_i} \cap \mathcal{R}_{F_{i+1}}$.

Note that R_0 must be adjacent (share a common edge) with R. Otherwise, both of vertices M_0 and M_1 belong to the same component of $\mathcal{C}_{x,y}$, and we obtain a contradiction with the assumption that the distance between M' and M'' is minimal.

In a similar way, we obtain that for any $i = 1, 2, \ldots, n$, the region R_i must be adjacent with at least one of regions $R, R_0, R_1, \ldots, R_{i-1}$. If not, we have that the perfect matching $\overline{M} = M_0 \bigtriangleup R_i$ belongs to $\mathcal{C}_{x,y}$, and $\overline{M}$ and M' are contained in the same component of $\mathcal{C}_{x,y}$. In that case we obtain a contradiction, because the path

$$\overline{M} = \overline{M}_0 \overset{R_0}{\text{---}} \overline{M}_1 \text{---} \ldots \text{---} \overline{M}_{i-1} \overset{R_{i-1}}{\text{------}} \overline{M}_{i+1} \overset{R_{i+1}}{\text{------}} \ldots \text{---} \overline{M}_n \overset{R_n}{\text{---}} \overline{M}_{n+1} = M''$$

is shorter than (3.3). Here we let that $\overline{M}_{j+1} = \overline{M}_j \bigtriangleup R_j$.

Let e' denote a common edge of regions R_0 and R that is contained in M'. Note that e' is not contained in M_1. However, this edge is again contained in M'', and we conclude that the region R_0 has to reappear again in (3.3).

Let $R_{i_0} = R_0$ denote the first appearance of R_0 in (3.3) after the first step. There are the following three possible situations that enable the reappearance of R_0:

(a) The regions $R_1, R_2, \ldots, R_{i_0-1}$ are disjoint with R_0.
   In that case, we can omit the steps in (3.3) labelled by R_0 and R_{i_0}, and obtain a shorter path between M' and M''.

(b) Each of regions that shares at least one edge with R_0 appears an odd number of times between R_0 and R_{i_0}.
   This is impossible, because R (that share an edge with R_0) can not appear in (3.3).

(c) There is $t < i_0$ such that the region $R_t = \bar{R}$ shares an edge with R_0, but the fragment of the sequence (3.3) between R_0 and R_{i_0} does not contain all region that shares an edge with R_0.
Then the same region $\bar{R}$ has to appear again as R_s, for some s such that $t < s < i_0$. Again, if all regions R_j are disjoint with $\bar{R}$ (for $j = t+1, \ldots, s-1$), we can omit R_t and R_s, and obtain a contradiction. If not, there exist indices t' and s' such that $t < t' < s' < s$ and $R'_t = R'_s$. We continue in the same way, and from the finiteness of the path, obtain a shorter path than (3.3). □

Proof of Theorem 3.11, continued: We built $\mathcal{C}(G)$ by starting with $\mathcal{C}(G \setminus e)$, that is a disjoint union of contractible complexes by assumption. Then we glue the components of $Prism(\mathcal{C}(G \setminus R))$ one by one.

After that, we glue all components of $\mathcal{C}(G \setminus \{x, y\})$. At each step we are gluing two contractible complexes along a contractible subcomplex, or we just add a new contractible complex, disjoint with previously added components. From the Gluing Lemma (see Lemma 10.3 in [4]) we obtain that $\mathcal{C}(G)$ is contractible, or a disjoint union of contractible complexes. □

Corollary 3.13. *If G has two odd elementary regions that share the same edge $e = xy$, then its cubical complex $\mathcal{C}(G) = \mathcal{C}(G \setminus \{x, y\}) \cup \mathcal{C}(G \setminus e)$ has at least two connected components. The same holds if there is an odd elementary region of G that shares an edge with the outer region R^*.*

4 The f-vector of domino tilings

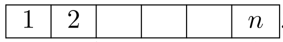
The concept of tilings of a bipartite planar graph generalizes the notion of domino tilings. Let $\mathcal{R}$ be a simple connected region, compound of unit squares in the plane, that can be tiled with domino tiles 1×2 and 2×1. The set of all tilings of $\mathcal{R}$ by domino tiles and 2×2 squares defines a cubical complex, denoted by $\mathcal{C}(\mathcal{R})$. If we consider $\mathcal{R}$ as a planar graph (all of its elementary regions are unit squares), and if G denotes the weak dual graph of $\mathcal{R}$ (the unit squares of $\mathcal{R}$ are vertices of G), then $\mathcal{C}(\mathcal{R})$ is isomorphic to the cubical matching complex $\mathcal{C}(G)$, see Section 3 in [6] for details. Note that the number of i-dimensional faces of $\mathcal{C}(G)$ counts the number of tilings of $\mathcal{R}$ with exactly i squares 2×2.

Ehrenborg used collapsibility of $\mathcal{C}(G)$ to conclude (see Corollary 3.1. in [6]) that the entries of f-vector of $f(\mathcal{C}(G)) = (f_0, f_1, \ldots, f_d)$ satisfy

$$f_0 - f_1 + f_2 - \cdots + (-1)^d f_d = 1. \tag{4.1}$$

Let G denote the weak dual graph of a region $\mathcal{R}$ that admits a domino tiling. Choose a concrete perfect matching M of G, and let $e = xy$ denote the edge in M that contains the vertex (square) in the left corner of the top row of $\mathcal{R}$. The complex $\mathcal{C}(G \setminus \{x, y\})$ is nonempty and contractible by induction. The simple connectivity of $\mathcal{R}$ implies that the other two complexes that appear on the right-hand side of the relation (3.2) are either both empty or contractible (by induction). If both of these complexes are nonempty, when we glue them as in the proof of Theorem 3.11, we obtain that $\mathcal{C}(G)$ is contractible. So, we conclude that the relation (4.1) is true in any case, disregarding possible problems with Theorem 3.1.

In this section we will prove that (4.1) is the only linear relation for f-vectors of cubical complexes of domino tilings. We follow the idea from [2], where Bayer and Billera determine the affine span of the flag f-vectors of polytopes by constructing polytopes whose flag f-vectors are affinely independent. Here we describe $d+1$ simple connected regions whose cubical complexes are d-dimensional and their f-vectors are affinely independent.

For all $n \in \mathbb{N}$, we let G_n denote the following graph 

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This graph (also known as the ladder graph) has $2n+2$ vertices, $3n+1$ edges and n elementary regions (squares). For $i = 1, 2, \dots, n$, let $G_{n,i}$ denote the graph obtained by adding one unit square below the i-th square of G_n.

Now, we describe some recursive relations for f-vectors of $\mathcal{C}(G_n)$ and $\mathcal{C}(G_{n,i})$.

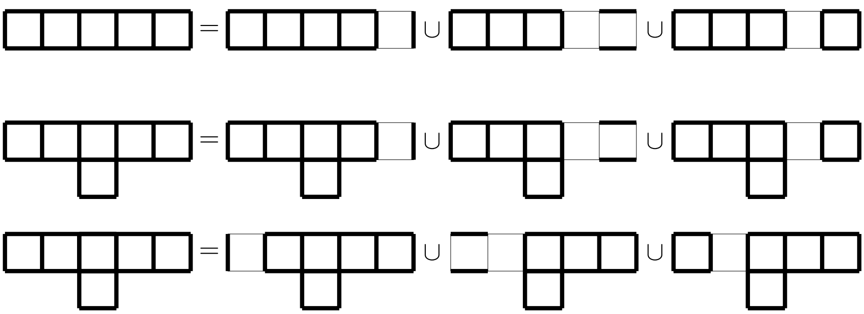
Proposition 4.1. *For all positive integers n the entries of f-vectors of $\mathcal{C}(G_n)$ and $\mathcal{C}(G_{n,i})$ satisfy the following recurrences:*

$$f_i(\mathcal{C}(G_{n+2})) = f_i(\mathcal{C}(G_{n+1})) + f_i(\mathcal{C}(G_n)) + f_{i-1}(\mathcal{C}(G_n)), \tag{4.2}$$

$$f_i(\mathcal{C}(G_{n+2,i})) = f_i(\mathcal{C}(G_{n+1,i})) + f_i(\mathcal{C}(G_{n,i})) + f_{i-1}(\mathcal{C}(G_{n,i})), \tag{4.3}$$

$$f_i(\mathcal{C}(G_{n+2,i})) = f_i(\mathcal{C}(G_{n+1,i-1})) + f_i(\mathcal{C}(G_{n,i-2})) + f_{i-1}(\mathcal{C}(G_{n,i-2})). \tag{4.4}$$

Proof. All formulas follow from relation (3.2), see the proof of Theorem 3.11. To obtain the formula (4.2), we apply (3.2) on G_{n+2}. The rightmost vertical edge and the rightmost unit square in G_{n+2} act as e and R in (3.2), see the first row on the next figure.



(4.2)

(4.3)

(4.4)

Figure 5: The "geometric proof" of recursive relations for $f(\mathcal{C}(G_n))$ and $f(\mathcal{C}(G_{n,i}))$.

In the same way we can prove the remaining two relations. For each relation, we choose an adequate elementary region R, a corresponding edge e of R, and use relation (3.2), see Figure 5. $\square$

The f-vector $(f_0, f_1, f_2, \dots, f_{\lceil \frac{n}{2} \rceil})$ of $\mathcal{C}(G_n)$ can be encoded by the polynomial F_n:

$$F_n = F_{\mathcal{C}(G_n)}(x) = f_0 + f_1 x + f_2 x^2 + \cdots + f_{\lceil \frac{n}{2} \rceil} x^{\lceil \frac{n}{2} \rceil}.$$

Similarly, we define the polynomials $F_{n,i}$ to encode the f-vector of $\mathcal{C}(G_{n,i})$. Directly from (4.2) and (4.3) we obtain that

$$F_{n+2}(x) = F_{n+1}(x) + (x+1)F_n(x), \qquad F_{n+2,i}(x) = F_{n+1,i}(x) + (x+1)F_{n,i}(x).$$

Now, we define new polynomials P_n and $P_{n,i}$ by

$$P_n = P_n(x) = F_n(x-1), \qquad P_{n,i} = P_{n,i}(x) = F_{n,i}(x-1).$$

This is a variant of h-polynomial associated to a cubical complex.

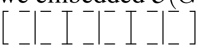
From Proposition 4.1 it follows that the polynomials P_n and $P_{n,i}$ satisfy the following recurrences

$$P_{n+2}(x) = P_{n+1}(x) + xP_n(x), \tag{4.5}$$
$$P_{n+2,i}(x) = P_{n+1,i}(x) + xP_{n,i}(x), \tag{4.6}$$
$$P_{n+2,i}(x) = P_{n+1,i-1}(x) + xP_{n,i-2}(x). \tag{4.7}$$

Remark 4.2. We can use (4.5) to obtain the polynomials P_n explicitly

$$P_{2d-1} = \binom{d}{d}x^d + \cdots + \binom{d+k}{d-k}x^k + \cdots + \binom{2d-1}{1}x + \binom{2d}{0}, \text{ and}$$
$$P_{2d} = \binom{d+1}{d}x^d + \cdots + \binom{d+k+1}{d-k}x^k + \cdots + \binom{2d}{1}x + \binom{2d+1}{0}.$$

Note that the polynomials P_n are related with Fibonacci polynomials, see Section 9.4 in [3] for the definition and a combinatorial interpretation of coefficients. The coefficients of these polynomials are positive integers and the sum of coefficients of P_n is a Fibonacci number. Note that this is just the number of vertices in $\mathcal{C}(G_n)$.

Assume that we embedded $\mathcal{C}(G_n)$ into n-cube as in Proposition 1.2, so that the perfect matching $M_0 =$  of G_n is the vertex in the origin. Now, the coefficient of x^k in P_n counts the number of vertices of $\mathcal{C}(G_n)$ for which the sum of coordinates is k, i.e., it is the number of vertices of $\mathcal{C}(G_n)$ whose distance from M_0 is k.

Also, following [3], we can recognize the coefficient of x^k in P_n as the number of k-element subsets of $[n]$ that do not contain two consecutive integers. Similarly, we can interpret the coefficient of x^k in $P_{n,i}$ as the number of k-element subsets of the multiset $M = \{1, 2, \dots, i-1, i, i, i+1, \dots, n\}$ that do not contain two consecutive integers. Note that the multiplicity of i in M is two, and all other elements have the multiplicity one.

Definition 4.3. Let $\mathcal{P}^d$ denote the vector space of all polynomials of degree at most d. We define the linear map $A_d : \mathcal{P}^d \to \mathcal{P}^{d+1}$ recursively by

$$A_d(x^k) = xA_{d-1}(x^{k-1}) \text{ for all } k > 0, \tag{4.8}$$
$$A_0(1) = 1 + 2x \text{ and } A_d(1) = P_{2d+1} - A_d(P_{2d-1} - 1). \tag{4.9}$$

Lemma 4.4. *For any non-negative integer* d*, we have that*

$$A_d(P_{2d-1}) = P_{2d+1}, A_d(P_{2d}) = P_{2d+2} \quad \textit{and} \quad A_{d+1}(P_{2d}) = P_{2d+2}.$$

Proof. From (4.9) it follows that $A_d(P_{2d-1}) = P_{2d+1}$. For the proof of the second formula we use (4.5), (4.8) and induction

$$A_d(P_{2d}) = A_d(P_{2d-1} + xP_{2d-2}) = P_{2d+1} + xA_{d-1}(P_{2d-2}) = P_{2d+1} + xP_{2d} = P_{2d+2}.$$

The last formula in this lemma follows from (4.5) and earlier proved formulas

$$\begin{aligned} A_{d+1}(P_{2d}) &= A_{d+1}(P_{2d+1} - xP_{2d-1}) = P_{2d+3} - xA_d(P_{2d-1}) \\ &= P_{2d+3} - xP_{2d+1} = P_{2d+2}. \end{aligned}$$

□

Lemma 4.5. *For all integers i and d such that $1 \leqslant i \leqslant \lfloor \frac{d}{2} \rfloor$, the following holds:*

$$A_d(P_{2d-1,i}) = P_{2d+1,i} \quad \textit{and} \quad A_d(P_{2d,i}) = P_{2d+2,i}.$$

Proof. For $i = 1$ and $i = 2$ we apply relation (3.2) in a similar way as in the proof of Proposition 4.1. We just delete the only square in the second row of $G_{n,1}$ and $G_{n,2}$, and obtain that

$$P_{2d-1,1} = P_{2d-1} + xP_{2d-3}, \qquad P_{2d-1,2} = P_{2d-1} + xP_{2d-4}.$$

By using Lemma 4.4, we obtain that

$$\begin{aligned} A_d(P_{2d-1,1}) &= A_d(P_{2d-1} + xP_{2d-3}) = P_{2d+1} + xP_{2d-1} = P_{2d+1,1}, \text{ and} \\ A_d(P_{2d-1,2}) &= A_d(P_{2d-1} + xP_{2d-4}) = P_{2d+1} + xA_{d-1}(P_{2d-4}) \\ &= P_{2d+1} + xP_{2d-2} = P_{2d+1,2}. \end{aligned}$$

In a similar way, we can prove that

$$A_d(P_{2d,1}) = P_{2d+2,1} \quad \text{and} \quad A_d(P_{2d,2}) = P_{2d+2,2}.$$

Assume that the statement of this lemma is true for $P_{2d-1,j}$ and $P_{2d,j}$ when $j < i + 1$. Now, we use (4.7) and induction to calculate

$$\begin{aligned} A_d(P_{2d,i+1}) &= A_d(P_{2d-1,i} + xP_{2d-2,i-1}) = A_d(P_{2d-1,i}) + xA_{d-1}(P_{2d-2,i-1}) \\ &= P_{2d+1,i} + xP_{2d,i-1} = P_{2d+2,i+1}. \end{aligned}$$

From (4.6) we obtain that

$$\begin{aligned} A_d(P_{2d-1,i+1}) &= A_d(P_{2d,i+1} - xP_{2d-2,i+1}) = A_d(P_{2d,i+1}) - xA_{d-1}(P_{2d-2,i+1}) \\ &= P_{2d+2,i+1} - xP_{2d,i+1} = P_{2d+1,i+1}. \end{aligned}$$ □

From Definition 4.3 and Remark 4.2 we can obtain the concrete formula for the linear map A_d.

Proposition 4.6. *For all $d, k \in \mathbb{N}$ such that $d \geqslant k \geqslant 1$, we have that:*

$$A_d(x^k) = x^k \left(1 + 2x - x^2 + 2x^3 - 5x^4 + 14x^5 - 42x^6 + \cdots + (-1)^{d-k} C_{d-k} x^{d-k+1}\right).$$

Here C_m denotes the m-th Catalan number.

Proof. From (4.8) it is enough to prove that

$$A_d(1) = 1 + 2x - x^2 + 2x^3 - 5x^4 + \cdots + (-1)^d C_d x^{d+1}. \tag{4.10}$$

For all integers n and k such that $n \geqslant k \geqslant 1$ (by using the induction and the Pascal's Identity), we can obtain the next relation

$$\binom{n}{k} = \sum_{i=0}^{k} (-1)^i \binom{n+1+i}{k-i} C_i. \tag{4.11}$$

Now, we assume that (4.10) is true for all positive integers less than d, and calculate $A_d(1)$ by definition:

$$\begin{aligned}A_d(1) &= P_{2d+1} - A_d(P_{2d-1} - 1)\\ &= \sum_{i=0}^{d+1} \binom{2d+2-i}{i} x^i - \sum_{i=1}^{d} \binom{2d-i}{i} x^i A_{d-i}(1).\end{aligned}$$

The coefficients of $1, x$ and x^2 in $A_d(1)$ are respectively:

$$\binom{2d+2}{0} = 1, \binom{2d+1}{1} - \binom{2d-1}{1} = 2, \binom{2d}{2} - \binom{2d-2}{2} - 2\binom{2d-1}{1} = -1.$$

For $k > 1$ the coefficient of x^{k+1} in the polynomial $A_d(1)$ is

$$\binom{2d+1-k}{k+1} - \binom{2d-k-1}{k+1} - 2\binom{2d-k}{k} - \sum_{i=1}^{k-1} (-1)^i \binom{2d-k+i}{k-i} C_i.$$

From (4.11) we obtain that the coefficient of x^{k+1} in $A_d(1)$ is $(-1)^k C_k$. □

Corollary 4.7. *For any positive integer d the linear map A_d is injective.*

Now, we consider all simple connected regions for which the degree of the associated polynomial $P_{\mathcal{R}}(x) = F_{\mathcal{R}}(x-1)$ is equal to d. Let $\mathcal{F}^d$ denote the affine subspace of $\mathcal{P}^d$ spanned by these polynomials.

Lemma 4.8. *The polynomial $P_{2d+1,d}$ is not contained in $A_d(\mathcal{F}^d)$.*

Proof. From (4.7) and (4.6) it follows that

$$\begin{aligned}P_{2d+1,d} - P_{2d+1,d-1} &= (P_{2d,d-1} + xP_{2d-1,d-2}) - (P_{2d,d-1} + xP_{2d-1,d-1})\\ &= -x(P_{2d-1,d-1} - P_{2d-1,d-2}) = (-1)^{d+1}(x^{d+1} + x^d).\end{aligned}$$

We know that $P_{2d+1,d-1} = A_d(P_{2d-1,d-1})$. If there exists a polynomial $p \in \mathcal{F}^d$ such that $A_d(p) = P_{2d+1,d}$ then we obtain

$$x^{d+1} + x^d = \pm A_d(p - P_{2d-1,d-1}),$$

which is impossible from Proposition 4.6. □

Theorem 4.9. *The polynomials $P_{2d-1}, P_{2d}, P_{2d-1,1}, \ldots, P_{2d-1,d-1}$ are affinely independent in $\mathcal{F}^d$.*

Proof. We use induction on the degree. Assume that d polynomials $P_{2d-3}, P_{2d-2}, P_{2d-3,1}, \ldots, P_{2d-3,d-2}$ are affinely independent in $\mathcal{F}^{d-1}$. From Lemmas 4.4 and 4.5 and Corollary 4.7, we conclude that P_{2d-1}, P_{2d}, $P_{2d-1,1}$, $\ldots$, $P_{2d-1,d-2}$ are affinely independent. These polynomials span a $(d-1)$-dimensional affine subspace of $\mathcal{F}^d$. From Lemma 4.8 follows that $P_{2d-1,d-1}$ is not contained in $A_{d-1}(\mathcal{F}^{d-1})$. □

Corollary 4.10. *The Euler-Poincare relation (4.1) is the only linear relation for the f-vectors of tilings.*

This answer the question of Ehrenborg question about numerical relations between the numbers of different types of tilings, see [6].

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References

[1] J. A. Barmak, Star clusters in independence complexes of graphs, *Adv. Math.* **241** (2013), 33–57, doi:10.1016/j.aim.2013.03.016.

[2] M. M. Bayer and L. J. Billera, Generalized Dehn-Sommerville relations for polytopes, spheres and Eulerian partially ordered sets, *Invent. Math.* **79** (1985), 143–157, doi:10.1007/bf01388660.

[3] A. T. Benjamin and J. J. Quinn, *Proofs that Really Count: The Art of Combinatorial Proof*, volume 27 of *The Dolciani Mathematical Expositions*, Mathematical Association of America, Washington, DC, 2003, doi:10.5948/9781614442080.

[4] A. Björner, Topological methods, in: R. L. Graham, M. Grötschel and L. Lovász (eds.), *Handbook of Combinatorics, Volume 2*, Elsevier Sci. B. V., Amsterdam, pp. 1819–1872, 1995.

[5] V. M. Buchstaber and T. E. Panov, *Toric Topology*, volume 204 of *Mathematical Surveys and Monographs*, American Mathematical Society, Providence, RI, 2015, doi:10.1090/surv/204.

[6] R. Ehrenborg, The cubical matching complex, *Ann. Comb.* **18** (2014), 75–81, doi:10.1007/s00026-013-0212-7.

[7] R. Ehrenborg and G. Hetyei, The topology of the independence complex, *Eur. J. Comb.* **27** (2006), 906–923, doi:10.1016/j.ejc.2005.04.010.

[8] A. Engström, Complexes of directed trees and independence complexes, *Discrete Math.* **309** (2009), 3299–3309, doi:10.1016/j.disc.2008.09.033.

[9] J. Jonsson, *Simplicial Complexes of Graphs*, volume 1928 of *Lecture Notes in Mathematics*, Springer-Verlag, Berlin, 2008, doi:10.1007/978-3-540-75859-4.

[10] D. N. Kozlov, Complexes of directed trees, *J. Comb. Theory Ser. A* **88** (1999), 112–122, doi:10.1006/jcta.1999.2984.

[11] U. Nagel and V. Reiner, Betti numbers of monomial ideals and shifted skew shapes, *Electron. J. Comb.* **16** (2009), #R3 (59 pages), doi:10.37236/69.

[12] J. Propp, Lattice structure for orientations of graphs, 2020, `arXiv:0209005 [math.CO]`.

[13] J. Skwarski, *Operations on a graph G that shift the homology of the independence complex of G*, Master's thesis, Kungliga Tekniska Högskolan, 2010, `https://www.math.kth.se/xComb/skwarski`.

[14] R. P. Stanley, *Enumerative Combinatorics, Volume I*, The Wadsworth & Brooks/Cole Mathematics Series, Wadsworth & Brooks/Cole Advanced Books & Software, Monterey, CA, 1986, doi:10.1007/978-1-4615-9763-6.

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From Italian domination in lexicographic product graphs to w-domination in graphs*

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Abstract

In this paper, we show that the Italian domination number of every lexicographic product graph $G \circ H$ can be expressed in terms of five different domination parameters of G. These parameters can be defined under the following unified approach, which encompasses the definition of several well-known domination parameters and introduces new ones.

Let $N(v)$ denote the open neighbourhood of $v \in V(G)$, and let $w = (w_0, w_1, \ldots, w_l)$ be a vector of nonnegative integers such that $w_0 \geq 1$. We say that a function $f\colon V(G) \to \{0, 1, \ldots, l\}$ is a w-dominating function if $f(N(v)) = \sum_{u \in N(v)} f(u) \geq w_i$ for every vertex v with $f(v) = i$. The weight of f is defined to be $\omega(f) = \sum_{v \in V(G)} f(v)$. The w-domination number of G, denoted by $\gamma_w(G)$, is the minimum weight among all w-dominating functions on G.

Specifically, we show that $\gamma_I(G \circ H) = \gamma_w(G)$, where $w \in \{2\} \times \{0, 1, 2\}^l$ and $l \in \{2, 3\}$. The decision on whether the equality holds for specific values of $w_0, \ldots, w_l$ will depend on the value of the domination number of H. This paper also provides preliminary results on $\gamma_w(G)$ and raises the challenge of conducting a detailed study of the topic.

Keywords: Italian domination, w-domination, k-domination, k-tuple domination, lexicographic product graph.

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1 Introduction

Let G be a graph, l a positive integer, and $f\colon V(G) \to \{0,\dots,l\}$ a function. For every $i \in \{0,\dots,l\}$, we define $V_i = \{v \in V(G) : f(v) = i\}$. We will identify f with the subsets $V_0,\dots,V_l$ associated with it, and so we will use the unified notation $f(V_0,\dots,V_l)$ for the function and these associated subsets. The weight of f is defined to be

$$\omega(f) = f(V(G)) = \sum_{i=1}^{l} i|V_i|.$$

An *Italian dominating function* (IDF) on a graph G is a function $f(V_0,V_1,V_2)$ satisfying that $f(N(v)) = \sum_{u\in N(v)} f(u) \geq 2$ for every $v \in V_0$, where $N(v)$ denotes the open neighbourhood of v. Hence, $f(V_0,V_1,V_2)$ is an IDF if $N(v)\cap V_2 \neq \varnothing$ or $|N(v)\cap V_1| \geq 2$ for every $v \in V_0$. The *Italian domination number*, denoted by $\gamma_I(G)$, is the minimum weight among all IDFs on G. This concept was introduced by Chellali et al. in [6] under the name of Roman $\{2\}$-domination. The term "Italian domination" comes from a subsequent paper by Henning and Klostermeyer [13].

In this paper we show that the Italian domination number of every lexicographic product graph $G \circ H$ can be expressed in terms of five different domination parameters of G. These parameters can be defined under the following unified approach.

Let $w = (w_0,\dots,w_l)$ be a vector of nonnegative integers such that $w_0 \geq 1$. We say that $f(V_0,\dots,V_l)$ is a *w-dominating function* if $f(N(v)) \geq w_i$ for every $v \in V_i$. The *w-domination number* of G, denoted by $\gamma_w(G)$, is the minimum weight among all w-dominating functions on G. For simplicity, a w-dominating function f of weight $\omega(f) = \gamma_w(G)$ will be called a $\gamma_w(G)$-function.

This unified approach allows us to encompass the definition of several well-known domination parameters and introduce new ones. For instance, we would highlight the following particular cases of known domination parameters that we define here in terms of w-domination.

- The *domination number* of G is defined to be $\gamma(G) = \gamma_{(1,0)}(G) = \gamma_{(1,0,0)}(G)$. Obviously, every $\gamma_{(1,0,0)}(G)$-function $f(V_0,V_1,V_2)$ satisfies that $V_2 = \varnothing$ and V_1 is a dominating set of cardinality $|V_1| = \gamma(G)$, i.e., V_1 is a $\gamma(G)$-set.

- The *total domination number* of a graph G with no isolated vertex is defined to be $\gamma_t(G) = \gamma_{(1,1)}(G) = \gamma_{(1,1,w_2,\dots,w_l)}(G)$, for every $w_2,\dots,w_l \in \{0,1\}$. Notice that there exists a $\gamma_{(1,1,w_2,\dots,w_l)}(G)$-function $f(V_0,V_1,\dots,V_l)$ such that $V_i = \varnothing$ for every $i \in \{2,\dots,l\}$ and V_1 is a total dominating set of cardinality $|V_1| = \gamma_t(G)$, i.e., V_1 is a $\gamma_t(G)$-set.

- Given a positive integer k, the *k-domination number* of a graph G is defined to be $\gamma_k(G) = \gamma_{(k,0)}(G)$. In this case, V_1 is a k-dominating set of cardinality $|V_1| = \gamma_k(G)$, i.e., V_1 is a $\gamma_k(G)$-set. The study of k-domination in graphs was initiated by Fink and Jacobson [8] in 1984.

- Given a positive integer k, the *k-tuple domination number* of a graph G of minimum degree $\delta \geq k-1$ is defined to be $\gamma_{\times k}(G) = \gamma_{(k,k-1)}(G)$. In this case, V_1 is a k-tuple dominating set of cardinality $|V_1| = \gamma_{\times k}(G)$, i.e., V_1 is a $\gamma_{\times k}(G)$-set. In particular, $\gamma_{\times 1}(G) = \gamma(G)$ and $\gamma_{\times 2}(G)$ is known as the *double domination number* of G. This parameter was introduced by Harary and Haynes in [9].

- Given a positive integer k, the *k-tuple total domination number* of a graph G of minimum degree $\delta \geq k$ is defined to be $\gamma_{\times k,t}(G) = \gamma_{(k,k)}(G)$. In particular, $\gamma_{\times 1,t}(G) = \gamma_t(G)$ and $\gamma_{\times 2,t}(G)$ is known as the *double total domination number*, and V_1 is a double total dominating set of cardinality $|V_1| = \gamma_{\times 2,t}(G)$, i.e., V_1 is a $\gamma_{\times 2,t}(G)$-set. The k-tuple total domination number was introduced by Henning and Kazemi in [12].

- The *Italian domination number* of G is defined to be $\gamma_{_I}(G) = \gamma_{(2,0,0)}(G)$. As mentioned earlier, this parameter was introduced by Chellali et al. in [6] under the name of Roman $\{2\}$-domination number. The concept was studied further in [13, 16].

- The *total Italian domination number* of a graph G with no isolated vertex is defined to be $\gamma_{tI}(G) = \gamma_{(2,1,1)}(G)$. This parameter was introduced by Cabrera et al. in [4], and independently by Abdollahzadeh Ahangar et al. in [1], under the name of total Roman $\{2\}$-domination number. The total Italian domination number of lexicographic product graphs was studied in [5].

- The *$\{k\}$-domination number* of G is defined to be $\gamma_{\{k\}}(G) = \gamma_{(k,k-1,\ldots,1,0)}(G)$. This parameter was introduced by Domke et al. in [7] and studied further in [3, 15, 17].

Notice that the concept of Y-dominating function introduced by Bange et al. [2] is quite different from the concept of w-dominating function introduced in this paper. Given a set Y of real numbers, a function $f\colon V(G) \to Y$ is a Y-dominating function if $f(N[v]) = f(v) + \sum_{u\in N(v)} f(u) \geq 1$ for every $v \in V(G)$. The Y-domination number, denoted by $\gamma_{_Y}(G)$, is the minimum weight among all Y-dominating functions on G. Hence, if $Y = \{0, 1, \ldots, l\}$, then $\gamma_{_Y}(G) = \gamma_{(1,0,\ldots,0)}(G) = \gamma(G)$.

For the graphs shown in Figure 1 we have the following values.

- $\gamma_I(G_1) = \gamma_{(2,1,0)}(G_1) = \gamma_{(2,2,0)}(G_1) = 4 < 6 = \gamma_{(2,2,1)}(G_1) = \gamma_{(2,2,2)}(G_1)$.
- $\gamma_I(G_2) = \gamma_{(2,1,0)}(G_2) = \gamma_{(2,2,0)}(G_2) = \gamma_{(2,2,1)}(G_2) = \gamma_{(2,2,2)}(G_2) = 3$.
- $\gamma_I(G_3) = \gamma_{(2,1,0)}(G_3) = 6 < 8 = \gamma_{(2,2,0)}(G_3) = \gamma_{(2,2,1)}(G_3) = \gamma_{(2,2,2)}(G_3)$.

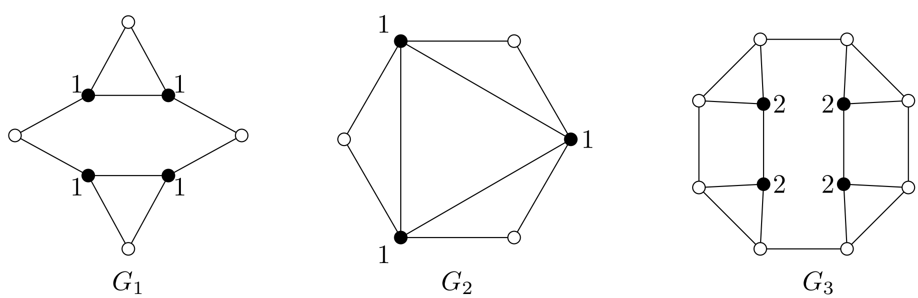

Figure 1: The labels of black-coloured vertices describe a $\gamma_{(2,1,0)}(G_1)$-function, a $\gamma_{(2,2,0)}(G_2)$-function and a $\gamma_{(2,2,2)}(G_3)$-function, respectively.

The remainder of the paper is organized as follows. In Section 2 we show that for any graph G with no isolated vertex and any nontrivial graph H with $\gamma(H) \neq 3$ or $\gamma_I(H) \neq 3$, the Italian domination number of $G \circ H$ equals one of the following parameters: $\gamma_{(2,1,0)}(G)$, $\gamma_{(2,2,0)}(G)$, $\gamma_{(2,2,1)}(G)$ or $\gamma_{(2,2,2)}(G)$. The specific value $\gamma_I(G \circ H)$ takes depends on the value of $\gamma(H)$. For the cases where $\gamma_I(H) = \gamma(H) = 3$, we show that $\gamma_I(G \circ H) = \gamma_{(2,2,2,0)}(G)$. Section 3 is devoted to providing some preliminary results on w-domination. We first describe some general properties of $\gamma_w(G)$ and then dedicate a subsection to each of the specific cases declared of interest in Section 2.

We assume that the reader is familiar with the basic concepts, notation and terminology of domination in graph. If this is not the case, we suggest the textbooks [10, 11, 14]. For the remainder of the paper, definitions will be introduced whenever a concept is needed.

2 Italian domination in lexicographic product graphs

The *lexicographic product* of two graphs G and H is the graph $G \circ H$ whose vertex set is $V(G \circ H) = V(G) \times V(H)$ and $(u, v)(x, y) \in E(G \circ H)$ if and only if $ux \in E(G)$ or $u = x$ and $vy \in E(H)$.

Notice that for any $u \in V(G)$ the subgraph of $G \circ H$ induced by $\{u\} \times V(H)$ is isomorphic to H. For simplicity, we will denote this subgraph by H_u. Moreover, the neighbourhood of $(x, y) \in V(G) \times V(H)$ will be denoted by $N(x, y)$ instead of $N((x, y))$. Analogously, for any function f on $G \circ H$, the image of (x, y) will be denoted by $f(x, y)$ instead of $f((x, y))$.

Lemma 2.1. *For any graph G with no isolated vertex and any nontrivial graph H with $\gamma_I(H) \neq 3$ or $\gamma(H) \neq 3$, there exists a $\gamma_I(G \circ H)$-function f satisfying that $f(V(H_u)) \leq 2$ for every $u \in V(G)$.*

Proof. Given an IDF f on $G \circ H$, we define the set $R_f = \{x \in V(G) : f(V(H_x)) \geq 3\}$. Let f be a $\gamma_I(G \circ H)$-function such that $|R_f|$ is minimum among all $\gamma_I(G \circ H)$-functions. Suppose that $|R_f| \geq 1$. Let $u \in R_f$ such that $f(V(H_u))$ is maximum among all vertices belonging to R_f. Suppose that $f(V(H_u)) > \gamma_I(H)$. In this case we take a $\gamma_I(H)$-function h and construct an IDF g defined on $G \circ H$ as $g(u, y) = h(y)$ for every $y \in V(H)$ and $g(x, y) = f(x, y)$ for every $x \in V(G) \setminus \{u\}$ and $y \in V(H)$. Obviously, $\omega(g) < \omega(f)$, which is a contradiction. Thus, $3 \leq f(V(H_u)) \leq \gamma_I(H_u) = \gamma_I(H)$. Now, we analyse the following two cases.

Case 1. $f(V(H_u)) \geq 4$. Let $u' \in N(u)$ and $v \in V(H)$. We define a function f' on $G \circ H$ as $f'(u, v) = f'(u', v) = 2$, $f'(u, y) = f(u', y) = 0$ for every $y \in V(H) \setminus \{v\}$, and $f'(x, y) = f(x, y)$ for every $x \in V(G) \setminus \{u, u'\}$ and $y \in V(H)$. Notice that f' is an IDF on $G \circ H$ with $\omega(f') \leq \omega(f)$ and $|R_{f'}| < |R_f|$, which is a contradiction.

Case 2. $f(V(H_u)) = 3$. Suppose that $\gamma_I(H) \neq 3$. Since $\gamma_I(H) \geq 4$, there exist $u' \in N(u)$ and $v \in V(H)$ such that $f(u', v) \geq 1$. Hence, the function f' defined in Case 1 is an IDF on $G \circ H$ with $\omega(f') \leq \omega(f)$ and $|R_{f'}| < |R_f|$, which is again a contradiction.

Thus, $\gamma_I(H) = 3$, and so $\gamma(H) \neq 3$, which implies that $\gamma(H) = 2$. Let $\{v_1, v_2\}$ be a $\gamma(H)$-set. Let $u' \in N(u)$ and $v' \in V(H)$ such that $f(u', v') = \max\{f(u', y) : y \in V(H)\}$. Consider the function f' defined as $f'(u, v_1) = f'(u, v_2) = 1$, $f'(u, y) = 0$ for every $y \in V(H) \setminus \{v_1, v_2\}$, $f'(u', v') = \min\{2, f(u', v') + 1\}$, $f'(u', y) = 0$ for every $y \in V(H) \setminus \{v'\}$, and $f'(x, y) = f(x, y)$ for every $x \in V(G) \setminus \{u, u'\}$ and $y \in V(H)$.

Notice that f' is an IDF on $G \circ H$ with $\omega(f') \leq \omega(f)$ and $|R_{f'}| < |R_f|$, which is a contradiction.

Therefore, $R_f = \varnothing$, and the result follows. □

Theorem 2.2. *The following statements hold for any graph G with no isolated vertex and any nontrivial graph H with $\gamma_I(H) \neq 3$ or $\gamma(H) \neq 3$.*

(i) *If $\gamma(H) = 1$, then $\gamma_I(G \circ H) = \gamma_{(2,1,0)}(G)$.*

(ii) *If $\gamma_2(H) = \gamma(H) = 2$, then $\gamma_I(G \circ H) = \gamma_{(2,2,0)}(G)$.*

(iii) *If $\gamma_2(H) > \gamma(H) = 2$, then $\gamma_I(G \circ H) = \gamma_{(2,2,1)}(G)$.*

(iv) *If $\gamma(H) \geq 3$, then $\gamma_I(G \circ H) = \gamma_{(2,2,2)}(G)$.*

Proof. Let $f(V_0, V_1, V_2)$ be a $\gamma_I(G \circ H)$-function which satisfies Lemma 2.1. Let $f'(X_0, X_1, X_2)$ be the function defined on G by $X_1 = \{x \in V(G) : f(V(H_x)) = 1\}$ and $X_2 = \{x \in V(G) : f(V(H_x)) = 2\}$. Notice that $\gamma_I(G \circ H) = \omega(f) = \omega(f')$. We claim that f' is a $\gamma_{(w_0,w_1,w_2)}(G)$-function. In order to prove this and find the values of w_0, w_1 and w_2, we differentiate the following three cases.

Case 1. $\gamma(H) = 1$. Assume that $x \in X_0$. Since $f(V(H_x)) = 0$, for any $y \in V(H)$ we have that $f(N(x, y) \setminus V(H_x)) \geq 2$. Thus, $f'(N(x)) \geq 2$. Now, assume that $x \in X_1$, and let $(x, y) \in V_1$ be the only vertex in $V(H_x)$ such that $f(x, y) > 0$. Since $\gamma(H) = 1$, for any $z \in V(H) \setminus \{y\}$, we have that $f(N(x, z) \setminus V(H_x)) \geq 1$, which implies that $f'(N(x)) \geq 1$. Therefore, f' is a $(2, 1, 0)$-dominating function on G and, as a consequence, $\gamma_I(G \circ H) = \omega(f) = \omega(f') \geq \gamma_{(2,1,0)}(G)$.

Now, for any $\gamma_{(2,1,0)}(G)$-function $g(W_0, W_1, W_2)$ and any universal vertex v of H, the function $g'(W_0', W_1', W_2')$, defined by $W_2' = W_2 \times \{v\}$ and $W_1' = W_1 \times \{v\}$, is an IDF on $G \circ H$. Therefore, $\gamma_I(G \circ H) \leq \omega(g') = \omega(g) = \gamma_{(2,1,0)}(G)$.

Case 2. $\gamma(H) = 2$. As in Case 1 we conclude that $f'(N(x)) \geq 2$ for every $x \in X_0$. Now, assume that $x \in X_1$, and let $(x, y) \in V_1$ be the only vertex in $V(H_x)$ such that $f(x, y) > 0$. Since $\gamma(H) = 2$, there exists a vertex $z \in V(H)$ such that $(x, z) \in V_0 \setminus N(x, y)$. Hence, $f(N(x, z) \setminus V(H_x)) \geq 2$, which implies that $f'(N(x)) \geq 2$. Therefore, f' is a $(2, 2, 0)$-dominating function on G and, as a consequence, $\gamma_I(G \circ H) = \omega(f) = \omega(f') \geq \gamma_{(2,2,0)}(G)$.

Now, if $\gamma_2(H) > \gamma(H) = 2$, then for every $x \in X_2$, there exists $y \in V(H)$ such that $(x, y) \in V_0$ and $f(N(x, y) \cap V(H_x)) \leq 1$, which implies that $f(N(x, y) \setminus V(H_x)) \geq 1$, and so $f'(N(x)) \geq 1$. Hence, f' is a $(2, 2, 1)$-dominating function on G and, as a consequence, $\gamma_I(G \circ H) = \omega(f) = \omega(f') \geq \gamma_{(2,2,1)}(G)$.

On the other side, if $\gamma_2(H) = 2$, then for any $\gamma_{(2,2,0)}(G)$-function $g(W_0, W_1, W_2)$ and any $\gamma_2(H)$-set $S = \{v_1, v_2\}$, the function $g'(W_0', W_1', W_2')$, defined by $W_1' = (W_1 \times \{v_1\}) \cup (W_2 \times S)$ and $W_2' = \varnothing$, is an IDF on $G \circ H$. Therefore, $\gamma_I(G \circ H) \leq \omega(g') = \omega(g) = \gamma_{(2,2,0)}(G)$.

Finally, if $\gamma_2(H) > \gamma(H) = 2$ then we take a $\gamma_{(2,2,1)}(G)$-function $h(Y_0, Y_1, Y_2)$ and a $\gamma(H)$-set $S' = \{v_1', v_2'\}$, and construct a function $h'(Y_0', Y_1', Y_2')$ on $G \circ H$ by making $Y_1' = (Y_1 \times \{v_1'\}) \cup (Y_2 \times S')$ and $Y_2' = \varnothing$. Obviously, h' is an IDF on $G \circ H$, and so we can conclude that $\gamma_I(G \circ H) \leq \omega(h') = \omega(h) = \gamma_{(2,2,1)}(G)$.

Case 3. $\gamma(H) \geq 3$. In this case, for every $x \in V(G)$, there exists $y \in V(H)$ such that $f(N[(x,y)] \cap V(H_x)) = 0$. Hence, $f(N(x,y) \setminus V(H_x)) \geq 2$, which implies that $f'(N(x)) \geq 2$ for every $x \in V(G)$. Therefore, f' is a $(2,2,2)$-dominating function on G and, as a consequence, $\gamma_I(G \circ H) = \omega(f) = \omega(f') \geq \gamma_{(2,2,2)}(G)$.

On the other side, for any $\gamma_{(2,2,2)}(G)$-function $g(W_0, W_1, W_2)$ and any $v \in V(H)$, the function $g'(W_0', W_1', W_2')$, defined by $W_2' = W_2 \times \{v\}$ and $W_1' = W_1 \times \{v\}$, is an IDF on $G \circ H$. Hence, $\gamma_I(G \circ H) \leq \omega(g') = \omega(g) = \gamma_{(2,2,2)}(G)$.

According to the three cases above, the result follows. □

The following result considers the case $\gamma_I(H) = \gamma(H) = 3$.

Theorem 2.3. *If H is a graph with $\gamma_I(H) = \gamma(H) = 3$, then for any graph G,*

$$\gamma_I(G \circ H) = \gamma_{(2,2,2,0)}(G).$$

Proof. Let $f(V_0, V_1, V_2)$ be a $\gamma_I(G \circ H)$-function, and $f'(X_0, X_1, X_2, X_3)$ the function defined on G by $X_1 = \{x \in V(G) : f(V(H_x)) = 1\}$, $X_2 = \{x \in V(G) : f(V(H_x)) = 2\}$ and $X_3 = \{x \in V(G) : f(V(H_x)) \geq 3\}$. We claim that f' is a $(2,2,2,0)$-dominating function on G.

Let $x \in X_0 \cup X_1 \cup X_2$. Since $f(V(H_x)) \leq 2$ and $\gamma(H) = 3$, there exists $y \in V(H)$ such that $f(N[(x,y)] \cap V(H_x)) = 0$. Thus, $f'(N(x)) \geq 2$ for every $x \in X_0 \cup X_1 \cup X_2$, which implies that f' is a $(2,2,2,0)$-dominating function on G. Therefore, $\gamma_I(G \circ H) = \omega(f) \geq \omega(f') \geq \gamma_{(2,2,2,0)}(G)$.

On the other side, let $h(Y_0, Y_1, Y_2, Y_3)$ be a $\gamma_{(2,2,2,0)}(G)$-function, h_1 a $\gamma_I(H)$-function and $v \in V(H)$. We define a function g on $G \circ H$ by $g(x,v) = h(x)$ for every $x \in V(G) \setminus Y_3$, $g(x,y) = 0$ for every $x \in V(G) \setminus Y_3$ and $y \in V(H) \setminus \{v\}$, and $g(x,y) = h_1(y)$ for every $(x,y) \in Y_3 \times V(H)$. A simple case analysis shows that g is an IDF on $G \circ H$. Therefore, $\gamma_I(G \circ H) \leq \omega(g) = \omega(h) = \gamma_{(2,2,2,0)}(G)$. □

The graph shown in Figure 2 satisfies $6 = \gamma_{(2,2,0)}(G) = \gamma_{(2,2,1)}(G) < 7 = \gamma_{(2,2,2,0)}(G) < \gamma_{(2,2,2)}(G) = 8$.

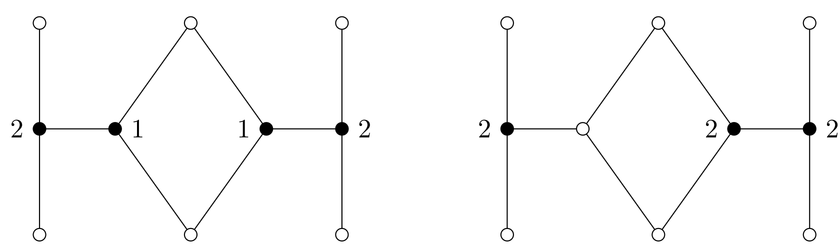

Figure 2: This figure shows two $\gamma_{(2,2,0)}(G)$-functions on the same graph. The function on the left is also a $\gamma_{(2,2,1)}(G)$-function.

3 Preliminary results on w-domination

In this section, we fix the notation $\mathbb{Z}^+ = \{1, 2, 3, \dots\}$ and $\mathbb{N} = \mathbb{Z}^+ \cup \{0\}$ for the sets of positive and nonnegative integers, respectively.

Throughout this section, we will repeatedly apply, without explicit mention, the following necessary and sufficient condition for the existence of a w-dominating function.

Remark 3.1. Let G be a graph of minimum degree δ and let $w = (w_0, \dots, w_l) \in \mathbb{Z}^+ \times \mathbb{N}^l$. If $w_0 \geq \cdots \geq w_l$, then there exists a w-dominating function on G if and only if $w_l \leq l\delta$.

Proof. Let $w = (w_0, \dots, w_l) \in \mathbb{Z}^+ \times \mathbb{N}^l$ such that $w_0 \geq \cdots \geq w_l$. If $w_l \leq l\delta$, then the function f, defined by $f(v) = l$ for every $v \in V(G)$, is a w-dominating function on G, as $V_l = V(G)$ and for any $x \in V_l$, $f(N(x)) \geq l\delta \geq w_l$.

Now, suppose that $w_l > l\delta$. If g is a w-dominating function on G, then for any vertex v of degree δ we have $g(N(v)) \leq \delta l < w_l \leq w_{l-1} \leq \cdots \leq w_0$, which is a contradiction. Therefore, the result follows. □

We will show that in general the w-domination numbers satisfy a certain monotonicity. Given two integer vectors $w = (w_0, \dots, w_l)$ and $w' = (w'_0, \dots, w'_l)$, we say that $w \prec w'$ if $w_i \leq w'_i$ for every $i \in \{0, \dots, l\}$. With this notation in mind, we can state the next remark which is direct consequence of the definition of w-domination number.

Remark 3.2. Let G be a graph of minimum degree δ and let $w = (w_0, \dots, w_l), w' = (w'_0, \dots, w'_l) \in \mathbb{Z}^+ \times \mathbb{N}^l$ such that $w_i \geq w_{i+1}$ and $w'_i \geq w'_{i+1}$ for every $i \in \{0, \dots, l-1\}$. If $w \prec w'$ and $w'_l \leq l\delta$, then every w'-dominating function is a w-dominating function and, as a consequence,

$$\gamma_w(G) \leq \gamma_{w'}(G).$$

We would emphasize the following remark on the specific cases of domination parameters considered in Section 2. Obviously, when we write $\gamma_{(2,2,2)}(G)$ or $\gamma_{(2,2,1)}(G)$, we are assuming that G has minimum degree $\delta \geq 1$.

Remark 3.3. The following statements hold.

(i) $\gamma_I(G) = \gamma_{(2,0,0)}(G) \leq \gamma_{(2,1,0)}(G) \leq \gamma_{(2,2,0)}(G) \leq \gamma_{(2,2,1)}(G) \leq \gamma_{(2,2,2)}(G)$.

(ii) If $w_2 \in \{1, 2\}$, then $\gamma_{(1,0,w_2)}(G) = \gamma_{(1,0,0)}(G) = \gamma(G)$ and $\gamma_{(1,1,w_2)}(G) = \gamma_{(1,1,0)}(G) = \gamma_t(G)$.

(iii) For any integer $k \geq 3$, there exists an infinite family $\mathcal{H}_k$ of graphs such that for every graph $G \in \mathcal{H}_k$, $\gamma_I(G) = \gamma_{(2,0,0)}(G) = \gamma_{(2,1,0)}(G) = \gamma_{(2,2,0)}(G) = \gamma_{(2,2,1)}(G) = \gamma_{(2,2,2)}(G) = k$.

(iv) There exists an infinite family of graphs such that $\gamma_I(G) < \gamma_{(2,1,0)}(G) < \gamma_{(2,2,0)}(G) < \gamma_{(2,2,1)}(G) < \gamma_{(2,2,2)}(G)$.

In order to see that the remark above holds, we just have to construct families of graphs satisfying (iii) and (iv), as (i) is a particular case of Remark 3.2 and (ii) is derived from the definition of (w_0, w_1, w_2)-domination number. In the case of (iii), we construct a family $\mathcal{H}_k = \{G_{k,r} : r \in \mathbb{Z}^+\}$ as follows. Let $k \geq 3$ be an integer, and let N_r be the empty graph of order r. For any positive integer r we construct a graph $G_{k,r} \in \mathcal{H}_k$ from a complete graph K_k and $\binom{k}{2}$ copies of N_r, in such way that for each pair of different vertices $\{x, y\}$ of K_k we choose one copy of N_r and connect every vertex of N_r with x and y, making x and y vertices of degree $(k-1)(r+1)$ in $G_{k,r}$. For instance, the graph $G_{3,1}$ is isomorphic to the graph G_2 shown in Figure 1. It is readily seen that

$\gamma_{_I}(G_{k,r}) = \gamma_{(2,2,2)}(G_{k,r}) = k$. On the other hand, in the case of (iv), we consider the family of cycles of order $n \geq 10$ with $n \equiv 1 \pmod 3$. For these graphs we have that $\gamma_I(C_n) < \gamma_{(2,1,0)}(C_n) < \gamma_{(2,2,0)}(C_n) < \gamma_{(2,2,1)}(C_n) < \gamma_{(2,2,2)}(C_n)$. The specific values of $\gamma_{(w_0,w_1,w_2)}(C_n)$ will be given in Subsections 3.1 – 3.4.

Next we show a class of graphs where $\gamma_{(w_0,\dots,w_l)}(G) = w_0\gamma(G)$ whenever $l \geq w_0 \geq \cdots \geq w_l$. To this end, we need to introduce some additional notation and terminology. Given two graphs G_1 and G_2, the *corona product graph* $G_1 \odot G_2$ is the graph obtained from G_1 and G_2, by taking one copy of G_1 and $|V(G_1)|$ copies of G_2 and joining by an edge every vertex from the i^{th}-copy of G_2 with the i^{th}-vertex of G_1. For every $x \in V(G_1)$, the copy of G_2 in $G_1 \odot G_2$ associated to x will be denoted by $G_{2,x}$. It is well known that $\gamma(G_1 \odot G_2) = |V(G_1)|$ and, if G_1 does not have isolated vertices, then $\gamma_t(G_1 \odot G_2) = \gamma(G_1 \odot G_2) = |V(G_1)|$.

Theorem 3.4. *Let $G \cong G_1 \odot G_2$ be a corona graph where G_1 does not have isolated vertices, and let $w = (w_0, \dots, w_l) \in \mathbb{Z}^+ \times \mathbb{N}^l$. If $l \geq w_0 \geq \cdots \geq w_l$ and $|V(G_2)| \geq w_0$, then*

$$\gamma_w(G) = w_0\gamma(G).$$

Proof. Since G_1 does not have isolated vertices, the upper bound $\gamma_w(G) \leq w_0|V(G_1)| = w_0\gamma(G)$ is straightforward, as the function f, defined by $f(x) = w_0$ for every vertex $x \in V(G_1)$ and $f(x) = 0$ for every $x \in V(G) \setminus V(G_1)$, is a w-dominating function on G.

On the other hand, let f be a $\gamma_w(G)$-function and suppose that there exists $x \in V(G_1)$ such that $f(V(G_{2,x})) + f(x) \leq w_0 - 1$. In such a case, $f(N[y]) \leq w_0 - 1$ for every $y \in V(G_{2,x})$, which is a contradiction, as $|V(G_2)| \geq w_0$. Therefore, $\gamma_w(G) = \omega(f) \geq w_0|V(G_1)| = w_0\gamma(G)$. □

Proposition 3.5. *Let G be a graph of order n. Let $w = (w_0, \dots, w_l) \in \mathbb{Z}^+ \times \mathbb{N}^l$ such that $w_0 \geq \cdots \geq w_l$. If G' is a spanning subgraph of G with minimum degree $\delta' \geq \frac{w_l}{l}$, then*

$$\gamma_w(G) \leq \gamma_w(G').$$

Proof. Let $E^- = \{e_1, \dots, e_k\}$ be the set of all edges of G not belonging to the edge set of G'. Let $G'_0 = G$ and, for every $i \in \{1, \dots, k\}$, let $X_i = \{e_1, \dots, e_i\}$ and $G'_i = G - X_i$, the edge-deletion subgraph of G induced by $E(G) \setminus X_i$. Since any w-dominating function on G'_i is a w-dominating function on G'_{i-1}, we can conclude that $\gamma_w(G'_{i-1}) \leq \gamma_w(G'_i)$. Hence, $\gamma_w(G) = \gamma_w(G'_0) \leq \gamma_w(G'_1) \leq \cdots \leq \gamma_w(G'_k) = \gamma_w(G')$. □

From Proposition 3.5 we obtain the following result.

Corollary 3.6. *Let G be a graph of order n and $w = (w_0, \dots, w_l) \in \mathbb{Z}^+ \times \mathbb{N}^l$ such that $w_0 \geq \cdots \geq w_l$.*

- *If G is a Hamiltonian graph and $w_l \leq 2l$, then $\gamma_w(G) \leq \gamma_w(C_n)$.*
- *If G has a Hamiltonian path and $w_l \leq l$, then $\gamma_w(G) \leq \gamma_w(P_n)$.*

In order to derive lower bounds on the w-domination number, we need to state the following useful lemma.

Lemma 3.7. *Let G be a graph with no isolated vertex, maximum degree Δ and order n. For any w-dominating function $f(V_0,\dots,V_l)$ on G such that $w_0 \geq \cdots \geq w_l$,*

$$\Delta\omega(f) \geq w_0 n + \sum_{i=1}^{l}(w_i - w_0)|V_i|.$$

Proof. The result follows from the simple fact that the contribution of any vertex $x \in V(G)$ to the sum $\sum_{x\in V(G)} f(N(x))$ equals $\deg(x)f(x)$, where $\deg(x)$ denotes the degree of x. Hence,

$$\begin{aligned}\Delta\omega(f) &= \Delta \sum_{x\in V(G)} f(x)\\ &\geq \sum_{x\in V(G)} \deg(x)f(x)\\ &= \sum_{x\in V(G)} f(N(x))\\ &\geq w_0|V_0| + \sum_{i=1}^{l} w_i|V_i|\\ &= w_0 n + \sum_{i=1}^{l}(w_i - w_0)|V_i|.\end{aligned}$$

Therefore, the result follows. □

Corollary 3.8. *The following statements hold for $k, l \in \mathbb{Z}^+$ and a graph G with minimum degree $\delta \geq 1$, maximum degree Δ and order n.*

(i) *If $k \leq l\delta + 1$ and $w = (\underbrace{k+l-1, k+l-2, \dots, k-1}_{l+1})$, then $\gamma_w(G) \geq \left\lceil \frac{(k+l-1)n}{\Delta+1} \right\rceil$.*

(ii) *If $k \leq l\delta$ and $w = (\underbrace{k,\dots,k}_{l+1})$, then $\gamma_w(G) \geq \left\lceil \frac{kn}{\Delta} \right\rceil$.*

(iii) *If $k \leq l\delta + 1$ and $w = (\underbrace{k, k-1, \dots, k-1}_{l+1})$, then $\gamma_w(G) \geq \left\lceil \frac{kn}{\Delta+1} \right\rceil$.*

(iv) *Let $w = (w_0,\dots,w_l)$ with $w_0 \geq \cdots \geq w_l$. If $l\delta \geq w_l$, then $\gamma_w(G) \geq \left\lceil \frac{w_0 n}{\Delta+w_0} \right\rceil$.*

In the next subsections we shall show that lower bounds above are tight. Corollary 3.8 implies the following known bounds.

$$\gamma(G) \geq \left\lceil \frac{n}{\Delta+1} \right\rceil, \quad \gamma_t(G) \geq \left\lceil \frac{n}{\Delta} \right\rceil, \quad \gamma_{_I}(G) \geq \left\lceil \frac{2n}{\Delta+2} \right\rceil, \quad \gamma_{tI}(G) \geq \left\lceil \frac{2n}{\Delta+1} \right\rceil,$$
$$\gamma_k(G) \geq \left\lceil \frac{kn}{\Delta+k} \right\rceil, \quad \gamma_{\times k}(G) \geq \left\lceil \frac{kn}{\Delta+1} \right\rceil, \quad \gamma_{\{k\}} \geq \left\lceil \frac{kn}{\Delta+1} \right\rceil, \quad \gamma_{\times k,t}(G) \geq \left\lceil \frac{kn}{\Delta} \right\rceil.$$

It is readily seen that $\gamma_{(w_0,\dots,w_l)}(G) = 1$ if and only if $w_0 = 1$, $w_1 = 0$ and $\gamma(G) = 1$. Next we characterize the graphs with $\gamma_{(w_0,\dots,w_l)}(G) = 2$.

Theorem 3.9. *Let $w = (w_0, \dots, w_l) \in \mathbb{Z}^+ \times \mathbb{N}^l$ such that $w_0 \geq \cdots \geq w_l$. For a graph G of order at least three, $\gamma_{(w_0,\dots,w_l)}(G) = 2$ if and only if one of the following conditions holds.*

(i) $w_2 = 0$, $\gamma(G) = 1$ *and either* $w_0 = 2$ *or* $w_0 = w_1 = 1$.

(ii) $w_0 = 1$, $w_1 = 0$ *and* $\gamma(G) = 2$.

(iii) $w_0 = 1$, $w_1 = 1$ *and* $\gamma_t(G) = 2$.

(iv) $w_0 = 2$, $w_1 = 0$ *and* $\gamma_2(G) = 2$.

(v) $w_0 = 2$, $w_1 = 1$ *and* $\gamma_{\times 2}(G) = 2$.

Proof. Assume first that $\gamma_{(w_0,\dots,w_l)}(G) = 2$ and let $f(V_0, \dots, V_l)$ be a $\gamma_{(w_0,\dots,w_l)}(G)$-function. Notice that $w_0 \in \{1, 2\}$ and $|V_2| \in \{0, 1\}$. If $|V_2| = 1$, then $w_2 = 0$ and $V_i = \varnothing$ for every $i \neq 0, 2$. Hence, $\gamma(G) = 1$ and either $w_0 = 2$ or $w_0 = w_1 = 1$. Therefore, (i) follows.

Now we consider the case $V_2 = \varnothing$. Notice that V_1 is a dominating set of cardinality two, $w_1 \in \{0, 1\}$ and $V_i = \varnothing$ for every $i \neq 0, 1$.

Assume first that $w_0 = 1$ and $w_1 = 0$. If $\gamma(G) = 1$, then $\gamma_{(w_0,\dots,w_l)}(G) = 1$, which is a contradiction. Hence, $\gamma(G) = 2$ and so (ii) follows. For $w_0 + w_1 \geq 2$ we have the following possibilities.

If $w_0 = w_1 = 1$, then V_1 is a total dominating set of cardinality two, and so $\gamma_t(G) = 2$. Therefore, (iii) follows.

If $w_0 = 2$ and $w_1 = 0$, then V_1 is a 2-dominating set of cardinality two, which implies that $\gamma_2(G) = 2$. Therefore, (iv) follows.

If $w_0 = 2$ and $w_1 = 1$, then V_1 is a double dominating set of cardinality two, and this implies that $\gamma_{\times 2}(G) = 2$. Therefore, (v) follows.

Conversely, if one of the five conditions holds, then it is easy to check that $\gamma_{(w_0,\dots,w_l)}(G) = 2$, which completes the proof. □

In order to establish the following result, we need to define the following parameter.

$$\nu_{(w_0,\dots,w_l)}(G) = \max\{|V_0| : f(V_0, \dots, V_l) \text{ is a } \gamma_{(w_0,\dots,w_l)}(G)\text{-function}\}.$$

In particular, for $l = 1$ and a graph G of order n, we have that $\nu_{(w_0,w_1)}(G) = n - \gamma_{(w_0,w_1)}(G)$.

Theorem 3.10. *Let G be a graph of minimum degree δ and order n. The following statements hold for any $(w_0, \dots, w_l) \in \mathbb{Z}^+ \times \mathbb{N}^l$ with $w_0 \geq \cdots \geq w_l$.*

(i) *If there exists $i \in \{1, \dots, l-1\}$ such that $i\delta \geq w_i$, then*

$$\gamma_{(w_0,\dots,w_l)}(G) \leq \gamma_{(w_0,\dots,w_i)}(G).$$

(ii) *If $l \geq i + 1 \geq w_0$, then*

$$\gamma_{(w_0,\dots,w_i,0,\dots,0)}(G) \leq (i + 1)\gamma(G).$$

(iii) *Let $k, i \in \mathbb{Z}^+$ such that $l \geq ki$, and let $(w'_0, w'_1, \ldots, w'_i) \in \mathbb{Z}^+ \times \mathbb{N}^l$. If $i\delta \geq w'_i$ and $w_{kj} = kw'_j$ for every $j \in \{0, 1, \ldots, i\}$, then*

$$\gamma_{(w_0,\ldots,w_l)}(G) \leq k\gamma_{(w'_0,\ldots,w'_i)}(G).$$

(iv) *Let $k \in \mathbb{Z}^+$ and $\beta_1, \ldots, \beta_k \in \mathbb{Z}^+$. If $l\delta \geq k + w_l > k$ and $w_0 + k \geq \beta_1 \geq \cdots \geq \beta_k \geq w_1 + k$, then*

$$\gamma_{(w_0+k,\beta_1,\ldots,\beta_k,w_1+k,\ldots,w_l+k)}(G) \leq \gamma_{(w_0,\ldots,w_l)}(G) + k(n - \nu_{(w_0,\ldots,w_l)}(G)).$$

(v) *If $l\delta \geq w_l \geq l \geq 2$, then*

$$\gamma_{(w_0,\ldots,w_l)}(G) \leq l\gamma_{(w_0-l+1,w_l-l+1)}(G).$$

(vi) *If $\delta \geq 1$, $w_0 \leq l - 1$ and $w_{l-1} \geq 1$, then*

$$\gamma_{(w_0,\ldots,w_{l-2},1)}(G) \leq \gamma_{(w_0,\ldots,w_{l-1},0)}(G).$$

Proof. If there exists $i \in \{1, \ldots, l-1\}$ such that $i\delta \geq w_i$, then for any $\gamma_{(w_0,\ldots,w_i)}(G)$-function $f(V_0, \ldots, V_i)$ we define a $(w_0, \ldots, w_l)$-dominating function $g(W_0, \ldots, W_l)$ by $W_j = V_j$ for every $j \in \{0, \ldots, i\}$ and $W_j = \varnothing$ for every $j \in \{i+1, \ldots, l\}$. Hence, $\gamma_{(w_0,\ldots,w_l)}(G) \leq \omega(g) = \omega(f) = \gamma_{(w_0,\ldots,w_i)}(G)$. Therefore, (i) follows.

Now, assume $l \geq i + 1 \geq w_0$. Let S be a $\gamma(G)$-set. Let f be the function defined by $f(v) = i + 1$ for every $v \in S$ and $f(v) = 0$ for the remaining vertices. Since f is a $(w_0, \ldots, w_i, 0 \ldots, 0)$-dominating function, we conclude that $\gamma_{(w_0,\ldots,w_i,0\ldots,0)}(G) \leq \omega(f) = (i+1)|S| = (i+1)\gamma(G)$, which implies that (ii) follows.

In order to prove (iii), assume that $l \geq ki$, $i\delta \geq w'_i$ and $w_{kj} = kw'_j$ for every $j \in \{0, \ldots, i\}$. Let $f'(V'_0, \ldots, V'_i)$ be a $\gamma_{(w'_0,\ldots,w'_i)}(G)$-function. We construct a function $f(V_0, \ldots, V_l)$ as $f(v) = kf'(v)$ for every $v \in V(G)$. Hence, $V_{kj} = V'_j$ for every $j \in \{0, \ldots, i\}$, while $V_j = \varnothing$ for the remaining cases. Thus, for every $v \in V_{kj}$ with $j \in \{0, \ldots, i\}$ we have that $f(N(v)) = kf'(N(v)) \geq kw'_j = w_{kj}$, which implies that f is a $(w_0, \ldots, w_l)$-dominating function, and so $\gamma_{(w_0,\ldots,w_l)}(G) \leq \omega(f) = k\omega(f') = k\gamma_{(w'_0,\ldots,w'_i)}(G)$. Therefore, (iii) follows.

Now, assume that $l\delta \geq k + w_l > k$ and $w_0 + k \geq \beta_1 \geq \cdots \geq \beta_k \geq w_1 + k$. Let $g(W_0, \ldots, W_l)$ be a $\gamma_{(w_0,\ldots,w_l)}(G)$-function. We construct a function $f(V_0, \ldots, V_{l+k})$ as $f(v) = g(v) + k$ for every $v \in V(G) \setminus W_0$ and $f(v) = 0$ for every $v \in W_0$. Hence, $V_{j+k} = W_j$ for every $j \in \{1, \ldots, l\}$, $V_0 = W_0$ and $V_j = \varnothing$ for the remaining cases. Thus, if $v \in V_{j+k}$ and $j \in \{1, \ldots, l\}$, then $f(N(v)) \geq g(N(v)) + k \geq w_j + k$, and if $v \in V_0$, then $f(N(v)) \geq g(N(v)) + k \geq w_0 + k$. This implies that f is a $(w_0 + k, \beta_1, \ldots, \beta_k, w_1 + k, \ldots, w_l + k)$-dominating function, and so

$$\begin{aligned}\gamma_{(w_0+k,\beta_1,\ldots,\beta_k,w_1+k,\ldots,w_l+k)}(G) &\leq \omega(f) = \omega(g) + k\sum_{j=1}^{l}|W_j| \\ &= \gamma_{(w_0,\ldots,w_l)}(G) + k(n - |W_0|) \\ &\leq \gamma_{(w_0,\ldots,w_l)}(G) + k(n - \nu_{(w_0,\ldots,w_l)}(G)).\end{aligned}$$

Therefore, (iv) follows.

Furthermore, if $l\delta \geq w_l \geq l \geq 2$, then by applying (iv) for $k = l-1$, we deduce that

$$\begin{aligned}\gamma_{(w_0,\ldots,w_l)}(G) &\leq \gamma_{(w_0-l+1,w_l-l+1)}(G) + (l-1)(n - \nu_{(w_0-l+1,w_l-l+1)}(G))\\ &= l\gamma_{(w_0-l+1,w_l-l+1)}(G).\end{aligned}$$

Therefore, (v) follows.

From now on, let $\delta \geq 1$, $w_0 \leq l-1$ and $w_{l-1} \geq 1$. Let $f(V_0,\ldots,V_l)$ be a $\gamma_{(w_0,\ldots,w_{l-1},0)}(G)$-function. Assume first $V_l = \varnothing$. Since $w_{l-1} \geq 1$, we have that f is a $(w_0,\ldots,w_{l-2},1)$-dominating function on G, which implies that (vi) follows. Assume now that there exists $v \in V_l$. If $f(N(v)) \geq l-1$, then the function f', defined by $f'(v) = l-1$ and $f'(x) = f(x)$ for every $x \in V(G)\setminus\{v\}$, is a $(w_0,\ldots,w_{l-1},0)$-dominating function with $\omega(f') < \omega(f)$, which is a contradiction. Hence, $f(N(v)) \leq l-2$ for every $v \in V_l$. Since $\delta \geq 1$, for each vertex $x \in V_l$, we fix one vertex $x' \in N(x)$ and we form a set S from them such that $|S| \leq |V_l|$. Let g be the function defined by $g(x) = f(x)+1$ for any $x \in S$, $g(y) = l-1$ for any $y \in V_l$, and $g(z) = f(z)$ for the remaining vertices of G. Since $g(N(x)) \geq l-1 \geq w_i$ for every $x \in S$ and $i \in \{0,\ldots,l-2\}$, $g(N(y)) \geq 1$ for every $y \in V_{l-1}\cup V_l$, and $g(N(z)) \geq w_i$ for every $z \in V_i \setminus (S \cup V_{l-1} \cup V_l)$ and $i \in \{0,\ldots,l-2\}$, we conclude that g is a $(w_0,\ldots,w_{l-2},1)$-dominating function on G. Therefore, $\gamma_{(w_0,\ldots,w_{l-2},1)}(G) \leq \omega(g) \leq \omega(f) = \gamma_{(w_0,\ldots,w_{l-1},0)}(G)$, which completes the proof of (vi). □

In the next subsections we consider several applications of Theorem 3.10 where we show that the bounds are tight. For instance, the following particular cases will be of interest.

Corollary 3.11. *Let G be a graph of minimum degree δ, and let $k, l, w_2,\ldots,w_l \in \mathbb{Z}^+$ with $k \geq w_2 \geq \cdots \geq w_l$.*

(i) *If $\delta \geq k$ and $w = (k+1,k,w_2,\ldots,w_l)$, then $\gamma_w(G) \leq \gamma_{\times k}(G)$.*

(ii) *If $\delta \geq k$ and $w = (k,k,w_2,\ldots,w_l)$, then $\gamma_w(G) \leq \gamma_{\times k,t}(G)$.*

(iii) *If $l\delta \geq k \geq l \geq 2$ and $w = (\underbrace{k+1,k,\ldots,k}_{l+1})$, then $\gamma_w(G) \leq l\gamma_{\times(k-l+2)}(G)$.*

(iv) *If $l\delta \geq k \geq l \geq 2$ and $w = (\underbrace{k,k,\ldots,k}_{l+1})$, then $\gamma_w(G) \leq l\gamma_{\times(k-l+1),t}(G)$.*

(v) *If $l \geq k$, $\delta \geq 1$ and $w = (\underbrace{k,\ldots,k}_{l+1})$, then $\gamma_w(G) \leq k\gamma_t(G)$.*

Proof. If $\delta \geq k$, then by Theorem 3.10(i) we conclude that (i) and (ii) follows.

If $l\delta \geq k \geq l \geq 2$, then by Theorem 3.10(v) we deduce that

$$\gamma_{(\underbrace{k+1,k,\ldots,k}_{l+1})}(G) \leq l\gamma_{(k-l+2,k-l+1)}(G) = l\gamma_{\times(k-l+2)}(G).$$

Hence, (iii) follows. By analogy we derive (iv), as $\gamma_{(k-l+1,k-l+1)}(G) = l\gamma_{\times(k-l+1),t}(G)$.

Finally, if $l \geq k$ and $\delta \geq 1$, then by Theorem 3.10(iii) we deduce that

$$\gamma_{(\underbrace{k,\ldots,k}_{l+1})}(G) \leq k\gamma_{(1,1)}(G) = k\gamma_t(G).$$

Therefore, (v) follows. □

3.1 Preliminary results on $(2,2,2)$-domination

Theorem 3.12. *For any graph G with no isolated vertex, order n and maximum degree Δ,*

$$\left\lceil \frac{2n}{\Delta} \right\rceil \leq \gamma_{(2,2,2)}(G) \leq 2\gamma_t(G).$$

Furthermore, if G has minimum degree $\delta \geq 2$, then

$$\gamma_{(2,2,2)}(G) \leq \gamma_{\times 2,t}(G).$$

Proof. From Corollary 3.8 we deduce the lower bound. The upper bound $\gamma_{(2,2,2)}(G) \leq 2\gamma_t(G)$ follows by Corollary 3.11(v), while, if $\delta \geq 2$, then we apply Corollary 3.11(ii) to deduce that $\gamma_{(2,2,2)}(G) \leq \gamma_{\times 2,t}(G)$. Therefore, the result follows. □

The bounds above are tight. For instance, for the graphs G_2 and G_3 shown in Figure 1 we have that $\lceil \frac{2n}{\Delta} \rceil = \gamma_{(2,2,2)}(G_2) = \gamma_{\times 2,t}(G_2) = 3$ and $\gamma_{(2,2,2)}(G_3) = 2\gamma_t(G_3) = 8$. Notice that every graph $G_{k,r}$ belonging to the infinite family $\mathcal{H}_k$ constructed after Remark 3.3 satisfies the equality $\gamma_{(2,2,2)}(G_{k,r}) = \gamma_{\times 2,t}(G_{k,r}) = k$. Furthermore, from Theorem 3.4 we have that for any corona graph $G \cong G_1 \odot G_2$, where G_1 does not have isolated vertices, $\gamma_{(2,2,2)}(G) = 2\gamma(G) = 2\gamma_t(G)$.

Notice that by Theorem 3.12 we have that $\gamma_{(2,2,2)}(G) \geq \lceil \frac{2n}{\Delta} \rceil \geq 3$ for every graph G with no isolated vertex. Next we characterize all graphs with $\gamma_{(2,2,2)}(G) = 3$. To this end, we need to establish the following lemma.

Lemma 3.13. *For a graph G, the following statements are equivalent.*

(i) $\gamma_{(2,2,2)}(G) = \gamma_{\times 2,t}(G)$.

(ii) *There exists a $\gamma_{(2,2,2)}(G)$-function $f(V_0, V_1, V_2)$ such that $V_2 = \varnothing$.*

Proof. If $\gamma_{(2,2,2)}(G) = \gamma_{\times 2,t}(G)$, then for any $\gamma_{\times 2,t}(G)$-set D, the function $g(W_0, W_1, W_2)$, defined by $W_1 = D$ and $W_0 = V(G) \setminus D$, is a $\gamma_{(2,2,2)}(G)$-function. Therefore, (ii) follows.

Conversely, if there exists a $\gamma_{(2,2,2)}(G)$-function $f(V_0, V_1, V_2)$ such that $V_2 = \varnothing$, then V_1 is a double total dominating set of G, and so $\gamma_{\times 2,t}(G) \leq |V_1| = \omega(f) = \gamma_{(2,2,2)}(G)$. Therefore, Theorem 3.12 leads to $\gamma_{(2,2,2)}(G) = \gamma_{\times 2,t}(G)$. □

Theorem 3.14. *For a graph G, the following statements are equivalent.*

(i) $\gamma_{(2,2,2)}(G) = 3$.

(ii) $\gamma_{\times 2,t}(G) = 3$.

Proof. Assume first that $\gamma_{(2,2,2)}(G) = 3$, and let $f(V_0, V_1, V_2)$ be a $\gamma_{(2,2,2)}(G)$-function. Suppose that there exists $u \in V_2$. Since $f(N(u)) \geq 2$, we deduce that $\gamma_{(2,2,2)}(G) \geq 4$, which is a contradiction. Hence, $V_2 = \varnothing$ and by Lemma 3.13 we conclude that $\gamma_{\times 2,t}(G) = 3$.

Conversely, if $\gamma_{\times 2,t}(G) = 3$, then G has minimum degree $\delta \geq 2$ and so Theorem 3.12 leads to $3 \leq \lceil \frac{2n}{\Delta} \rceil \leq \gamma_{(2,2,2)}(G) \leq \gamma_{\times 2,t}(G) = 3$. Therefore, $\gamma_{(2,2,2)}(G) = 3$. □

Next we consider the case of graphs with $\gamma_{(2,2,2)}(G) = 4$.

Theorem 3.15. *For a graph G, $\gamma_{(2,2,2)}(G) = 4$ if and only if at least one of the following conditions holds.*

(i) $\gamma_{\times 2,t}(G) = 4$.

(ii) $\gamma_t(G) = 2$ *and G has minimum degree* $\delta = 1$.

(iii) $\gamma_t(G) = 2$ *and* $\gamma_{\times 2,t}(G) \geq 4$.

Proof. Assume $\gamma_{(2,2,2)}(G) = 4$. Notice that G does not have isolated vertices. Let $f(V_0, V_1, V_2)$ be a $\gamma_{(2,2,2)}(G)$-function. If $V_2 = \varnothing$, then by Lemma 3.13 we obtain that $\gamma_{\times 2,t}(G) = \gamma_{(2,2,2)}(G) = 4$, and so (i) follows.

From now on, assume that $|V_2| \in \{1, 2\}$. If $|V_2| = 2$, then $V_1 = \varnothing$ and, as a result, V_2 is a total dominating set of G, which implies that $\gamma_t(G) = 2$. On the other side, if $|V_2| = 1$, then $|V_1| = 2$ and both vertices belonging to V_1 are adjacent to the vertex of weight two, and every $v \in V_0$ satisfies $N(v) \cap V_2 \neq \varnothing$ or $V_1 \subseteq N(v)$. This implies that the union of V_2 with a singleton subset of V_1 forms a total dominating set of G, and again $\gamma_t(G) = 2$. Now, if $\delta \geq 2$, then Theorem 3.12 leads to $4 = \gamma_{(2,2,2)}(G) \leq \gamma_{\times 2,t}(G)$. Hence, by Theorem 3.14 we conclude that either $\delta = 1$ or $\gamma_{\times 2,t}(G) \geq 4$. Therefore, either (ii) or (iii) holds.

Conversely, if $\gamma_{\times 2,t}(G) = 4$, then G has minimum degree $\delta \geq 2$ and by Theorem 3.12 we have that $3 \leq \gamma_{(2,2,2)}(G) \leq 4$. Hence, by Theorem 3.14 we deduce that $\gamma_{(2,2,2)}(G) = 4$. Finally, if $\gamma_t(G) = 2$, then Theorem 3.12 leads to $3 \leq \gamma_{(2,2,2)}(G) \leq 4$. Therefore, if $\delta = 1$ or $\gamma_{\times 2,t}(G) \geq 4$, then Theorem 3.14 leads to $\gamma_{(2,2,2)}(G) = 4$. □

Theorem 3.12 implies the next result.

Corollary 3.16. *For any integer $n \geq 3$,*

$$\gamma_{(2,2,2)}(C_n) = n.$$

In order to give the value of $\gamma_{(2,2,2)}(P_n)$, we recall the following well-known result.

Proposition 3.17 ([14]). *For any integer $n \geq 3$,*

$$\gamma_t(P_n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod 4, \\ \frac{n+1}{2} & \text{if } n \equiv 1, 3 \pmod 4, \\ \frac{n}{2} + 1 & \text{if } n \equiv 2 \pmod 4. \end{cases}$$

Lemma 3.18. *If $P_n = u_1 u_2 \dots u_n$ is a path of order $n \geq 6$, then there exists a $\gamma_{(2,2,2)}(P_n)$-function f such that $f(u_n) = f(u_{n-3}) = 0$ and $f(u_{n-1}) = f(u_{n-2}) = 2$.*

Proof. Let $f(V_0, V_1, V_2)$ be a $\gamma_{(2,2,2)}(P_n)$-function such that $|V_2|$ is maximum. Since u_n is a leaf, $f(u_{n-1}) = 2$. Notice that $f(u_n) + f(u_{n-2}) \geq 2$. Hence, we can assume that $f(u_{n-2}) = 2$ and $f(u_n) = 0$. Now, if $f(u_{n-3}) > 0$, then we can define a $(2, 2, 2)$-dominating function f' by $f'(u_{n-3}) = 0$, $f'(u_{n-5}) = \min\{2, f(u_{n-5}) + f(u_{n-3})\}$ and $f'(u_i) = f(u_i)$ for the remaining cases. Since $\omega(f') \leq \omega(f) = \gamma_{(2,2,2)}(P_n)$, either f' is a $\gamma_{(2,2,2)}(P_n)$-function with $f'(u_{n-3}) = 0$ or $f(u_{n-3}) = 0$. In both cases the result follows. □

Proposition 3.19. *For any integer $n \geq 3$,*

$$\gamma_{(2,2,2)}(P_n) = 2\gamma_t(P_n) = \begin{cases} n & \text{if } n \equiv 0 \pmod 4, \\ n+1 & \text{if } n \equiv 1,3 \pmod 4, \\ n+2 & \text{if } n \equiv 2 \pmod 4. \end{cases}$$

Proof. Since Theorem 3.12 leads to $\gamma_{(2,2,2)}(P_n) \leq 2\gamma_t(P_n)$, we only need to prove that $\gamma_{(2,2,2)}(P_n) \geq 2\gamma_t(P_n)$. We proceed by induction on n. It is easy to check that $\gamma_{(2,2,2)}(P_n) = 2\gamma_t(P_n)$ for $n = 3, 4, 5, 6$. This establishes the base case. Now, we assume that $n \geq 7$ and $\gamma_{(2,2,2)}(P_k) \geq 2\gamma_t(P_k)$ for $k < n$. Let $f(V_0, V_1, V_2)$ be a $\gamma_{(2,2,2)}(P_n)$-function which satisfies Lemma 3.18, and let f' be the restriction of f to $V(P_{n-4})$, where $P_n = u_1 u_2 \ldots u_n$ and $P_{n-4} = u_1 u_2 \ldots u_{n-4}$. Hence, by applying the induction hypothesis,

$$\gamma_{(2,2,2)}(P_n) = \omega(f) = \omega(f') + 4 \geq \gamma_{(2,2,2)}(P_{n-4}) + 4 \geq 2\gamma_t(P_{n-4}) + 4 \geq 2\gamma_t(P_n).$$

To conclude the proof we apply Proposition 3.17. □

3.2 Preliminary results on $(2, 2, 1)$-domination

Theorem 3.20. *For any graph G with no isolated vertex, order n and maximum degree Δ,*

$$\left\lceil \frac{2n + \gamma_t(G)}{\Delta + 1} \right\rceil \leq \gamma_{(2,2,1)}(G) \leq \min\{3\gamma(G), 2\gamma_t(G)\}.$$

Furthermore, if G has minimum degree $\delta \geq 2$, then

$$\gamma_{(2,2,1)}(G) \leq \gamma_{\times 2,t}(G).$$

Proof. In order to prove the upper bound $\gamma_{(2,2,1)}(G) \leq 2\gamma_t(G)$, we apply Remark 3.2 and Theorem 3.12, i.e., $\gamma_{(2,2,1)}(G) \leq \gamma_{(2,2,2)} \leq 2\gamma_t(G)$.

Now, let S be a $\gamma(G)$-set. Since G does not have isolated vertex, for each vertex $x \in S$ such that $N(x) \cap S = \varnothing$, we fix one vertex $x' \in N(x)$ and we form a set S' from them. Hence, $S \cup S'$ is a total dominating set and $|S \cup S'| = |S| + |S'| \leq 2\gamma(G)$. Notice that the function $g(X_0, X_1, X_2)$ defined by $X_2 = S$ and $X_1 = S'$, is a $(2, 2, 1)$-dominating function on G. Thus, $\gamma_{(2,2,1)}(G) \leq \omega(g) = 2|S| + |S'| \leq 3\gamma(G)$, and so $\gamma_{(2,2,1)}(G) \leq \min\{2\gamma_t(G), 3\gamma(G)\}$.

On the other side, if G has minimum degree $\delta \geq 2$, then by Corollary 3.11(ii) we have that $\gamma_{(2,2,1)}(G) \leq \gamma_{\times 2,t}(G)$.

In order to prove the lower bound, let $f(V_0, V_1, V_2)$ be a $\gamma_{(2,2,1)}(G)$-function. Since $V_1 \cup V_2$ is a total dominating set, $\gamma_t(G) \leq |V_1| + |V_2|$. Furthermore, from Lemma 3.7 we have, $2n - |V_2| \leq \Delta\gamma_{(2,2,1)}(G)$, which implies that $2n + \gamma_t(G) \leq 2n + |V_1| + |V_2| \leq \Delta\gamma_{(2,2,1)}(G) + |V_1| + 2|V_2| = (\Delta + 1)\gamma_{(2,2,1)}(G)$. Therefore, the lower bound follows. □

The bounds above are tight. For instance, the graph in Figure 3 satisfies $\gamma_{(2,2,1)}(G) = 3\gamma(G) = 9$. Next we show that the remaining two bounds are also achieved.

Corollary 3.21. *Let G be a graph with no isolated vertex, order n and maximum degree Δ. If $\gamma_t(G) < \frac{n+\Delta+1}{\Delta+1/2}$, then*

$$\gamma_{(2,2,1)}(G) = 2\gamma_t(G) \qquad \textit{or} \qquad \gamma_{(2,2,1)}(G) = \left\lceil \frac{2n + \gamma_t(G)}{\Delta + 1} \right\rceil.$$

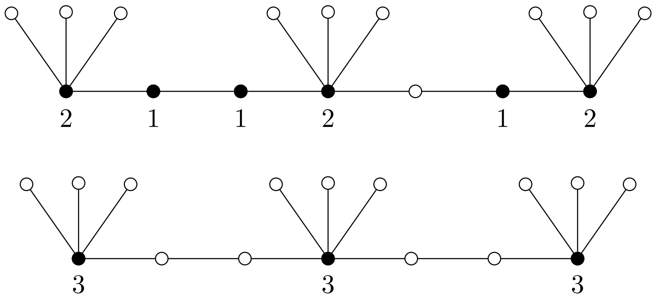

Figure 3: This figure shows a $\gamma_{(2,2,1)}(G)$-function and a $\gamma_{(2,2,2,0)}(G)$-function on the same graph.

Proof. If $\gamma_{(2,2,1)}(G) \neq \left\lceil \frac{2n+\gamma_t(G)}{\Delta+1} \right\rceil$ and $\gamma_{(2,2,1)}(G) \neq 2\gamma_t(G)$, then by Theorem 3.20 we deduce that $\left\lceil \frac{2n+\gamma_t(G)}{\Delta+1} \right\rceil + 1 \leq \gamma_{(2,2,1)}(G) \leq 2\gamma_t(G) - 1$, which implies that $\gamma_t(G) \geq \frac{n+\Delta+1}{\Delta+1/2}$. Therefore, the result follows. □

For the graphs G_2 and G_3 illustrated in Figure 1 we have that $\gamma_t(G_2) = 2 < \frac{22}{9} = \frac{n+\Delta+1}{\Delta+1/2}$ and $\gamma_t(G_3) = 4 < \frac{32}{7} = \frac{n+\Delta+1}{\Delta+1/2}$. Notice that, $\gamma_{(2,2,1)}(G_2) = 3 = \left\lceil \frac{2n+\gamma_t(G_2)}{\Delta+1} \right\rceil$ and $\gamma_{(2,2,1)}(G_3) = 8 = 2\gamma_t(G_3)$.

Below we characterize the graphs with $\gamma_{(2,2,1)}(G) = 3$.

Theorem 3.22. *For a graph G with no isolated vertex, the following statements are equivalent.*

(i) $\gamma_{(2,2,1)}(G) = 3$.

(ii) $\gamma(G) = 1$ *or* $\gamma_{\times 2,t}(G) = 3$.

Proof. Assume first that $\gamma_{(2,2,1)}(G) = 3$, and let $f(V_0, V_1, V_2)$ be a $\gamma_{(2,2,1)}(G)$-function. If $V_2 \neq \varnothing$, then V_2 is a dominating set of cardinality one. Hence, $\gamma(G) = 1$. Now, if $V_2 = \varnothing$, then V_1 is a double total dominating set of cardinality three. Thus, $\gamma_{\times 2,t}(G) = 3$.

On the other side, by Theorem 3.20 we have that $3 \leq \left\lceil \frac{2n+\gamma_t(G)}{\Delta+1} \right\rceil \leq \gamma_{(2,2,1)}(G) \leq 3\gamma(G)$. Hence, if $\gamma(G) = 1$, then $\gamma_{(2,2,1)}(G) = 3$. Now, if $\gamma_{\times 2,t}(G) = 3$, then G has minimum degree $\delta \geq 2$ and by Theorem 3.20 we have that $\gamma_{(2,2,1)}(G) \leq \gamma_{\times 2,t}(G) = 3$. Therefore, $\gamma_{(2,2,1)}(G) = 3$. □

Next we consider the case of graphs with $\gamma_{(2,2,1)}(G) = 4$.

Theorem 3.23. *For a graph G, the following statements are equivalent.*

(i) $\gamma_{(2,2,1)}(G) = 4$.

(ii) $\gamma_t(G) = \gamma(G) = 2$ *or* $\gamma_{\times 2,t}(G) = 4$.

Proof. Assume $\gamma_{(2,2,1)}(G) = 4$. Notice that G does not have isolated vertices and, by Theorem 3.20, we have that $\gamma(G) \geq 2$. Let $f(V_0, V_1, V_2)$ be a $\gamma_{(2,2,1)}(G)$-function. If $V_2 =$

$\varnothing$, then V_1 is a double total dominating set of cardinality four. Hence, $3 \leq \gamma_{\times 2,t}(G) \leq |V_1| = 4$, and Theorem 3.22 implies that $\gamma_{\times 2,t}(G) = 4$.

From now on, assume that $|V_2| \in \{1, 2\}$. If $|V_2| = 2$, then $V_1 = \varnothing$ and, as a result, V_2 is a total dominating set of G, which implies that $\gamma_t(G) = \gamma(G) = 2$. Now, if $|V_2| = 1$, then $|V_1| = 2$ and both vertices belonging to V_1 are adjacent to the vertex of weight two, and every $v \in V_0$ satisfies $N(v) \cap V_2 \neq \varnothing$ or $V_1 \subseteq N(v)$. This implies that the union of V_2 with a singleton subset of V_1 forms a total dominating set of G, and again $\gamma_t(G) = \gamma(G) = 2$.

Conversely, if $\gamma_{\times 2,t}(G) = 4$, then G has minimum degree $\delta \geq 2$ and by Theorem 3.20 we have that $3 \leq \gamma_{(2,2,1)}(G) \leq \gamma_{\times 2,t}(G) = 4$. Hence, by Theorem 3.22 we deduce that $\gamma_{(2,2,1)}(G) = 4$. Finally, if $\gamma_t(G) = 2$, then Theorem 3.20 leads to $3 \leq \gamma_{(2,2,1)}(G) \leq 4$. Therefore, if $\gamma(G) = 2$ then by Theorem 3.22 we conclude that $\gamma_{(2,2,1)}(G) = 4$. □

Lemma 3.24. *For any integer $n \geq 3$,*

$$\gamma_{(2,2,1)}(P_n) \leq \begin{cases} n - \lfloor \frac{n}{7} \rfloor + 1 & \text{if } n \equiv 1, 2 \pmod 7, \\ n - \lfloor \frac{n}{7} \rfloor & \text{otherwise.} \end{cases}$$

Proof. First we show how to construct a $(2, 2, 1)$-dominating function f on P_n for $n \in \{2, \ldots, 8\}$.

- $n = 2$: $f(u_1) = 2$ and $f(u_2) = 1$.
- $n = 3$: $f(u_1) = 0$, $f(u_2) = 2$ and $f(u_3) = 1$.
- $n = 4$: $f(u_1) = f(u_4) = 0$ and $f(u_2) = f(u_3) = 2$.
- $n = 5$: $f(u_1) = f(u_5) = 0$, $f(u_2) = f(u_4) = 2$ and $f(u_3) = 1$.
- $n = 6$: $f(u_1) = f(u_6) = 0$, $f(u_2) = f(u_5) = 2$ and $f(u_3) = f(u_4) = 1$.
- $n = 7$: $f(u_1) = f(u_4) = f(u_7) = 0$, $f(u_2) = f(u_6) = 2$ and $f(u_3) = f(u_5) = 1$.
- $n = 8$: $f(u_1) = f(u_4) = f(u_8) = 0$, $f(u_2) = f(u_6) = f(u_7) = 2$ and $f(u_3) = f(u_5) = 1$.

We now proceed to describe the construction of f for any $n = 7q + r$, where $q \geq 1$ and $0 \leq r \leq 6$. We partition $V(P_n) = \{u_1, \ldots, u_n\}$ into q sets of cardinality 7 and for $r \geq 1$ one additional set of cardinality r, in such a way that the subgraph induced by all these sets are paths.

For any $r \neq 1$, the restriction of f to each of these q paths of length 7 corresponds to the weights associated above with P_7, while for the path of length r (if any) we take the weights associated above with P_r. The case $r = 1$ and $q \geq 2$ is slightly different, as for the first $q - 1$ paths of length 7 we take the weights associated above with P_7 and for the last 8 vertices of P_n we take the weights associated above with P_8.

Notice that, for $n \equiv 1, 2 \pmod 7$, we have that $\gamma_{(2,2,1)}(P_n) \leq \omega(f) = 6q + r + 1 = n - \lfloor \frac{n}{7} \rfloor + 1$, while for $n \not\equiv 1, 2 \pmod 7$ we have $\gamma_{(2,2,1)}(P_n) \leq \omega(f) = 6q + r = n - \lfloor \frac{n}{7} \rfloor$. Therefore, the result follows. □

Lemma 3.25. *Let $P_7 = x_1 \ldots x_7$ be a subgraph of C_n and $X = \{x_1, \ldots, x_7\}$. If f is a $(2, 2, 1)$-dominating function on C_n, then*

$$f(X) \geq 6.$$

Proof. Notice that $f(\{x_1, x_2, x_3\}) \geq 2$ and $f(\{x_4, x_5, x_6, x_7\}) \geq 3$ as f is a $(2, 2, 1)$-dominating function. If $f(\{x_1, x_2, x_3\}) \geq 3$, then we are done. Hence, we assume that $f(\{x_1, x_2, x_3\}) = 2$. In this case, it is not difficult to deduce that $f(\{x_4, x_5, x_6, x_7\}) \geq 4$, which implies that $f(X) \geq 6$, as desired. Therefore, the proof is complete. □

Lemma 3.26. *For any integer $n \geq 3$,*

$$\gamma_{(2,2,1)}(C_n) \geq \begin{cases} n - \lfloor \frac{n}{7} \rfloor + 1 & \text{if } n \equiv 1, 2 \pmod 7, \\ n - \lfloor \frac{n}{7} \rfloor & \text{otherwise.} \end{cases}$$

Proof. It is easy to check that $\gamma_{(2,2,1)}(C_n) = n$ for every $n \in \{3, 4, 5, 6\}$. Now, let $n = 7q + r$, with $0 \leq r \leq 6$ and $q \geq 1$. Let $f(V_0, V_1, V_2)$ be a $\gamma_{(2,2,1)}(C_n)$-function.

If $r = 0$, then by Lemma 3.25 we have that $\omega(f) \geq 6q = n - \lfloor \frac{n}{7} \rfloor$. From now on we assume that $r \geq 1$. By Proposition 3.5 and Lemma 3.24 we deduce that $\gamma_{(2,2,1)}(C_n) \leq \gamma_{(2,2,1)}(P_n) < n$, which implies that $V_2 \neq \varnothing$, otherwise there exists $u \in V(C_n) = V_0 \cup V_1$ such that $N(u) \cap V_0 \neq \varnothing$ and so $|N(u) \cap V_1| \leq 1$, which is a contradiction. Let $x \in V_2$ and, without loss of generality, we can label the vertices of C_n in such a way that $x = u_1$, and $u_2 \in V_1 \cup V_2$ whenever $r \geq 2$. We partition $V(C_n)$ into $X = \{u_1, \ldots, u_r\}$ and $Y = \{u_{r+1}, \ldots, u_n\}$. Notice that Lemma 3.25 leads to $f(Y) \geq 6q$.

Now, if $r \in \{1, 2\}$, then $f(X) \geq r + 1$, which implies that $\omega(f) \geq r + 1 + 6q = n - \lfloor \frac{n}{7} \rfloor + 1$. Analogously, if $r = 3$, then $f(X) \geq r$ and so $\omega(f) \geq r + 6q = n - \lfloor \frac{n}{7} \rfloor$.

Finally, if $r \in \{4, 5, 6\}$, then as f is a $(2, 2, 1)$-dominating function we deduce that $f(X) \geq r$, which implies that $\omega(f) \geq r + 6q = n - \lfloor \frac{n}{7} \rfloor$. □

The following result is a direct consequence of Proposition 3.5 and Lemmas 3.24 and 3.26.

Proposition 3.27. *For any integer $n \geq 3$,*

$$\gamma_{(2,2,1)}(C_n) = \gamma_{(2,2,1)}(P_n) = \begin{cases} n - \lfloor \frac{n}{7} \rfloor + 1 & \text{if } n \equiv 1, 2 \pmod 7, \\ n - \lfloor \frac{n}{7} \rfloor & \text{otherwise.} \end{cases}$$

3.3 Preliminary results on $(2, 2, 0)$-domination

Theorem 3.28. *For any graph G with no isolated vertex, order n and maximum degree Δ,*

$$\left\lceil \frac{2n}{\Delta + 1} \right\rceil \leq \gamma_{(2,2,0)}(G) \leq 2\gamma(G).$$

Furthermore, if G has minimum degree $\delta \geq 2$, then

$$\gamma_{(2,2,0)}(G) \leq \gamma_{\times 2,t}(G).$$

Proof. The upper bound $\gamma_{(2,2,0)}(G) \leq \omega(g) = 2\gamma(G)$ is derived by we applying Theorem 3.10(ii) for $i = 1$ and $l = 2$. Furthermore, if G has minimum degree $\delta \geq 2$, then by Corollary 3.11(ii) we have that $\gamma_{(2,2,0)}(G) \leq \gamma_{\times 2,t}(G)$.

Now, let $f(V_0, V_1, V_2)$ be a $\gamma_{(2,2,0)}(G)$-function. From Lemma 3.7 we deduce that $2(n - |V_2|)) \leq \Delta\gamma_{(2,2,0)}(G)$, which implies that $2n \leq 2n + |V_1| \leq (\Delta + 1)\gamma_{(2,2,0)}(G)$. Therefore, the result follows. □

Theorem 3.28 implies that, if $\gamma(G) = \frac{n}{\Delta+1}$, then $\gamma_{(2,2,0)}(G) = \frac{2n}{\Delta+1}$. It is easy to see that a graph satisfies $\gamma(G) = \frac{n}{\Delta+1}$ if and only if there exists a $\gamma(G)$-set S which is a 2-packing[1] and every vertex in S has degree Δ. The upper bound $\gamma_{(2,2,0)}(G) \leq 2\gamma(G)$ is achieved for the graph G shown in Figure 2, which satisfies $\gamma_{(2,2,0)}(G) = 2\gamma(G) = 6$. Furthermore, by Theorem 3.4 we have that for any corona graph $G \cong G_1 \odot G_2$, where G_1 does not have isolated vertices, $\gamma_{(2,2,0)}(G) = 2\gamma(G)$.

As shown in Theorem 3.9, for a graph G, $\gamma_{(2,2,0)}(G) = 2$ if and only if $\gamma(G) = 1$. Now we consider the case $\gamma_{(2,2,0)}(G) = 3$.

Theorem 3.29. *For a graph G, $\gamma_{(2,2,0)}(G) = 3$ if and only if $\gamma_{\times 2,t}(G) = \gamma(G) + 1 = 3$.*

Proof. Assume $\gamma_{(2,2,0)}(G) = 3$. By Theorem 3.9 we have that $\gamma(G) \geq 2$. Let $f(V_0, V_1, V_2)$ be a $\gamma_{(2,2,0)}(G)$-function. If $|V_2| = 1$ then $|V_1| = 1$, and as f is a $(2,2,0)$-dominating function we deduce that $N[V_2] = V(G)$, i.e., $\gamma(G) = 1$, which is a contradiction. Thus, $V_2 = \varnothing$ and $|V_1| = 3$. Notice that V_1 is a double total dominating set and since $\gamma(G) \geq 2$, it follows that $3 \leq \gamma(G) + 1 \leq \gamma_{\times 2,t}(G) \leq |V_1| = 3$. Hence, $\gamma_{\times 2,t}(G) = \gamma(G) + 1 = 3$, as required.

Conversely, assume $\gamma_{\times 2,t}(G) = \gamma(G) + 1 = 3$. Since G has minimum degree at least two, Theorem 3.28 leads to $2 \leq \gamma_{(2,2,0)}(G) \leq \gamma_{\times 2,t}(G) = 3$, and so Theorem 3.9 implies that $\gamma_{(2,2,0)}(G) = 3$, which completes the proof. □

Theorem 3.30. *For a graph G, $\gamma_{(2,2,0)}(G) = 4$ if and only if one of the following conditions holds.*

(i) *$G \cong K_1 \cup G_1$, where G_1 is a graph with $\gamma(G_1) = 1$.*

(ii) *$\gamma_{\times 2,t}(G) = 4$.*

(iii) *$\gamma(G) = 2$ and G has minimum degree one.*

(iv) *$\gamma(G) = 2$ and $\gamma_{\times 2,t}(G) \geq 4$.*

Proof. If K_1 is a component of G, then by Theorem 3.9 we conclude that $\gamma_{(2,2,0)}(G) = 4$ if and only if $G \cong K_1 \cup G_1$, where G_1 is a graph with $\gamma(G_1) = 1$.

From now on, we consider the case where G is a graph with no isolated vertex. Assume $\gamma_{(2,2,0)}(G) = 4$ and let $f(V_0, V_1, V_2)$ be a $\gamma_{(2,2,0)}(G)$-function. If $V_2 = \varnothing$, then V_1 is a double total dominating set of G. In this case, G has minimum degree $\delta \geq 2$ and by Theorem 3.28 we have that $\gamma_{\times 2,t}(G) \leq |V_1| = 4 = \gamma_{(2,2,0)}(G) \leq \gamma_{\times 2,t}(G)$. Hence (ii) follows.

Now, assume that $|V_2| \in \{1, 2\}$. If $|V_2| = 2$, then $V_1 = \varnothing$, and so $\gamma(G) \leq 2$. Now, if $|V_2| = 1$, then $|V_1| = 2$ and both vertices belonging V_1 are adjacent to the vertex of weight two, and every $v \in V_0$ satisfies $N(v) \cap V_2 \neq \varnothing$ or $V_1 \subseteq N(v)$. This implies that the union of V_2 with a singleton subset of V_1 forms a dominating set of G, and again $\gamma(G) \leq 2$. Thus, from Theorem 3.9 we deduce that $\gamma(G) = 2$. Furthermore, if $\delta \geq 2$, then by Theorem 3.28 we have that $\gamma_{\times 2,t}(G) \geq \gamma_{(2,2,0)} = 4$. Therefore, either (iii) or (iv) holds.

Conversely, if $\gamma_{\times 2,t}(G) = 4$, then Theorem 3.28 leads to $2 \leq \gamma_{(2,2,0)} \leq \gamma_{\times 2,t}(G) = 4$. Hence, by Theorems 3.9 and 3.29 we deduce that $\gamma_{(2,2,0)}(G) = 4$. Analogously, if $\gamma(G) = 2$ and $\delta \geq 1$, then Theorem 3.28 leads to $2 \leq \gamma_{(2,2,0)} \leq 2\gamma(G) = 4$. Thus, by Theorem 3.9 we have that $3 \leq \gamma_{(2,2,0)} \leq 4$. In particular, if $\delta = 1$ or $\gamma_{\times 2,t}(G) \geq 4$, then Theorem 3.29 leads to $\gamma_{(2,2,0)}(G) = 4$, which completes the proof. □

[1]A set $S \subseteq V(G)$ is a 2-packing if $N[u] \cap N[v] = \varnothing$ for every pair of different vertices $u, v \in S$.

Lemma 3.31. *For a graph G, the following statements are equivalent.*

(i) $\gamma_{(2,2,0)}(G) = 2\gamma(G)$.

(ii) *There exists a $\gamma_{(2,2,0)}(G)$-function $f(V_0, V_1, V_2)$ such that $V_1 = \varnothing$.*

Proof. First, we assume that $\gamma_{(2,2,0)}(G) = 2\gamma(G)$ and let D be a $\gamma(G)$-set. Hence, the function $f(V_0, V_1, V_2)$, defined by $V_2 = D$ and $V_0 = V(G) \setminus D$, is a $\gamma_{(2,2,0)}(G)$-function which satisfies (ii), as desired.

Finally, we assume that there exists a $\gamma_{(2,2,0)}(G)$-function $f(V_0, V_1, V_2)$ such that $V_1 = \varnothing$. This implies that V_2 is a dominating set of G. Hence, $\gamma_{(2,2,0)}(G) \leq 2\gamma(G) \leq 2|V_2| = \gamma_{(2,2,0)}(G)$, and the desired equality holds, which completes the proof. □

The following result provides the $(2, 2, 0)$-domination number of paths and cycles.

Proposition 3.32. *For any integer $n \geq 3$,*

$$\gamma_{(2,2,0)}(P_n) = \gamma_{(2,2,0)}(C_n) = 2\left\lceil \frac{n}{3} \right\rceil.$$

Proof. We first prove that $\gamma_{(2,2,0)}(C_n) \geq 2\left\lceil \frac{n}{3} \right\rceil$. Let $f(V_0, V_1, V_2)$ be a $\gamma_{(2,2,0)}(C_n)$-function. If $V_1 = \varnothing$, then by Lemma 3.31 it follows that $\gamma_{(2,2,0)}(C_n) = 2\gamma(C_n) = 2\left\lceil \frac{n}{3} \right\rceil$. If $V_1 \neq \varnothing$, then $1 + 2|V_2| \leq |V_1| + 2|V_2| = \gamma_{(2,2,0)}(C_n) \leq 2\gamma(C_n) = 2\left\lceil \frac{n}{3} \right\rceil$, which leads to $|V_2| \leq \left\lceil \frac{n}{3} \right\rceil - 1$. By Lemma 3.7 we have that $\gamma_{(2,2,0)}(C_n) \geq n - |V_2| \geq n - \left\lceil \frac{n}{3} \right\rceil + 1 \geq 2\left\lceil \frac{n}{3} \right\rceil$, as desired.

Therefore, by the inequality above, Proposition 3.5 and Theorem 3.28 we deduce that $2\lceil \frac{n}{3} \rceil \leq \gamma_{(2,2,0)}(C_n) \leq \gamma_{(2,2,0)}(P_n) \leq 2\gamma(P_n) = 2\lceil \frac{n}{3} \rceil$. Thus, we have equalities in the inequality chain above, which implies that the result follows. □

3.4 Preliminary results on $(2, 1, 0)$-domination

Given a graph G, we use the notation $L(G)$ and $S(G)$ for the sets of leaves and support vertices, respectively.

Theorem 3.33. *For any graph G with no isolated vertex, order n and maximum degree Δ,*

$$\left\lceil \frac{2n}{\Delta + 1} \right\rceil \leq \gamma_{(2,1,0)}(G) \leq \min\{\gamma_{\times 2}(G) - |L(G)| + |S(G)|, 2\gamma(G)\}.$$

Proof. If $f(V_0, V_1, V_2)$ is a $\gamma_{(2,1,0)}(G)$-function, then from Lemma 3.7 we conclude that $2n - |V_1| - 2|V_2| \leq \Delta\gamma_{(2,1,0)}(G)$. Hence, $2n \leq \Delta\gamma_{(2,1,0)}(G) + \omega(f) = (\Delta+1)\gamma_{(2,1,0)}(G)$. Therefore, the lower bound follows.

Let D be a $\gamma_{\times 2}(G)$-set. Notice that $S(G) \cup L(G) \subseteq D$. Since $|N[v] \cap D| \geq 2$ for every $v \in V(G)$, the function $g(V_0, V_1, V_2)$ defined by $V_1 = D \setminus (L(G) \cup S(G))$ and $V_2 = S(G)$, is a $(2, 1, 0)$-dominating function. Hence, $\gamma_{(2,1,0)}(G) \leq \omega(g) = \gamma_{\times 2}(G) - |L(G)| + |S(G)|$.

By Remark 3.2, $\gamma_{(2,1,0)}(G) \leq \gamma_{(2,2,0)}(G)$, hence the upper bound $\gamma_{(2,1,0)}(G) \leq 2\gamma(G)$ is derived from Theorem 3.28. Therefore, $\gamma_{(2,1,0)}(G) \leq \min\{\gamma_{\times 2}(G) - |L(G)| + |S(G)|, 2\gamma(G)\}$. □

The bounds above are tight. For instance, for the graph G_1 shown in Figure 1 we have that $\gamma_{(2,1,0)}(G_1) = \left\lceil \frac{2n}{\Delta+1} \right\rceil = \gamma_{\times 2}(G_1) = 2\gamma(G_1) = 4$. As an example of graph of minimum degree one where $\gamma_{(2,1,0)}(G) = \gamma_{\times 2}(G) - |L(G)| + |S(G)|$ we take the graph G obtained from a star graph $K_{1,r}$, $r \geq 3$, by subdividing one edge just once. In such a case, $\gamma_{(2,1,0)}(G) = 4 = \gamma_{\times 2}(G) - |L(G)| + |S(G)|$. Another example is the graph shown in Figure 2 which satisfies $\gamma_{(2,1,0)}(G) = \gamma_{\times 2}(G) - |L(G)| + |S(G)| = 6$.

Notice that $\gamma_{(2,1,0)}(G) \geq \left\lceil \frac{2n}{\Delta+1} \right\rceil \geq 2$. As shown in Theorem 3.9, $\gamma_{(2,1,0)}(G) = 2$ if and only if $\gamma(G) = 1$. Next we characterize the graph satisfying $\gamma_{(2,1,0)}(G) = 3$.

Theorem 3.34. *For a graph G, $\gamma_{(2,1,0)}(G) = 3$ if and only if $\gamma_{\times 2}(G) = \gamma(G) + 1 = 3$.*

Proof. Assume $\gamma_{(2,1,0)}(G) = 3$. By Theorem 3.9 we have that $\gamma(G) \geq 2$. Let $f(V_0, V_1, V_2)$ be a $\gamma_{(2,1,0)}(G)$-function. If $|V_2| = 1$ then $N[V_2] = V(G)$, i.e., $\gamma(G) = 1$, which is a contradiction. Thus, $V_2 = \varnothing$ and $|V_1| = 3$, which implies that V_1 is a double dominating set. Hence, $3 \leq \gamma(G) + 1 \leq \gamma_{\times 2}(G) \leq |V_1| = 3$. Therefore, $\gamma_{\times 2}(G) = \gamma(G) + 1 = 3$.

Conversely, assume $\gamma_{\times 2}(G) = \gamma(G) + 1 = 3$. Notice that G has minimum degree $\delta \geq 1$ and so by Theorems 3.9 and 3.33 we have that $3 \leq \gamma_{(2,1,0)}(G) \leq \gamma_{\times 2}(G) = 3$, which implies that $\gamma_{(2,1,0)}(G) = 3$. □

Next we consider the case of graphs with $\gamma_{(2,1,0)}(G) = 4$.

Theorem 3.35. *For a graph G, $\gamma_{(2,1,0)}(G) = 4$ if and only if one of the following conditions is satisfied.*

(i) *$G \cong K_1 \cup G_1$, where G_1 is a graph with $\gamma(G_1) = 1$.*

(ii) *$\gamma_{\times 2}(G) = 4$.*

(iii) *$\gamma(G) = 2$ and $\gamma_{\times 2}(G) \geq 4$.*

Proof. If K_1 is a component of G, then by Theorem 3.9 we conclude that $\gamma_{(2,1,0)}(G) = 4$ if and only if $G \cong K_1 \cup G_1$, where G_1 is a graph with $\gamma(G_1) = 1$.

From now on, we consider the case where G is a graph with no isolated vertex. Assume $\gamma_{(2,1,0)}(G) = 4$. By Theorem 3.33 we deduce that $\gamma_{\times 2}(G) \geq 4$ and $\gamma(G) \geq 2$. Let $f(V_0, V_1, V_2)$ be a $\gamma_{(2,1,0)}(G)$-function. If $V_2 = \varnothing$, then V_1 is a double dominating set of G, which implies that $\gamma_{\times 2}(G) \leq |V_1| = 4$. Hence, (ii) follows. From now on, assume $|V_2| \in \{1, 2\}$. If $|V_2| = 2$, then $V_1 = \varnothing$ and so, V_2 is a dominating set of G, which implies that $\gamma(G) = 2$. If $|V_2| = 1$, then for every $v \in V_1$ we have that $V_2 \cup \{v\}$ is a dominating set of G. Hence, $\gamma(G) = 2$. Therefore, (iii) follows.

Conversely, if (ii) or (iii) holds, then by Theorems 3.33 we have that $2 \leq \gamma_{(2,1,0)}(G) \leq 4$. Therefore, by Theorems 3.9 and 3.34 we deduce that $\gamma_{(2,1,0)}(G) = 4$, which completes the proof. □

The formulas on the $\{k\}$-dominating number of cycles and paths were obtained in [17]. We present here the particular case of $k = 2$, as $\gamma_{\{2\}}(G) = \gamma_{(2,1,0)}(G)$.

Proposition 3.36 ([17]). *For any integer $n \geq 3$,*

$$\gamma_{\{2\}}(C_n) = \left\lceil \frac{2n}{3} \right\rceil \qquad \text{and} \qquad \gamma_{\{2\}}(P_n) = 2\left\lceil \frac{n}{3} \right\rceil.$$

3.5 Preliminary results on $(2, 2, 2, 0)$-domination

The following result is a direct consequence of Theorem 3.10(i), (ii) and (vi).

Corollary 3.37. *For any graph G with no isolated vertex,*

$$\gamma_{(2,2,1)}(G) \leq \gamma_{(2,2,2,0)}(G) \leq \min\{3\gamma(G), \gamma_{(2,2,2)}(G)\}.$$

The bounds above are tight. For instance, every graph $G_{k,r}$ belonging to the infinite family $\mathcal{H}_k$ constructed after Remark 3.3 satisfies the equalities $\gamma_{(2,2,1)}(G_{k,r}) = \gamma_{(2,2,2)}(G_{k,r}) = \gamma_{(2,2,2,0)}(G_{k,r}) = k$. In contrast, the graph shown in Figure 2 satisfies $\gamma_{(2,2,1)}(G) = 6 < 7 = \gamma_{(2,2,2,0)}(G) < 8 = \gamma_{(2,2,2)}(G)$. Moreover, Figure 3 illustrates a graph G with $\gamma_{(2,2,1)}(G) = \gamma_{(2,2,2,0)}(G) = 3\gamma(G) = 9$.

In order to characterize the graphs with $\gamma_{(2,2,2,0)}(G) \in \{3, 4\}$, we need to establish the following lemma.

Lemma 3.38. *For a graph G, the following statements are equivalent.*

(i) $\gamma_{(2,2,2,0)}(G) = \gamma_{(2,2,2)}(G)$.

(ii) *There exists a $\gamma_{(2,2,2,0)}(G)$-function $f(V_0, V_1, V_2, V_3)$ such that $V_3 = \varnothing$.*

Proof. If $\gamma_{(2,2,2,0)}(G) = \gamma_{(2,2,2)}(G)$, then for any $\gamma_{(2,2,2)}(G)$-function $f(V_0, V_1, V_2)$, there exists a $\gamma_{(2,2,2,0)}(G)$-function $g(W_0, W_1, W_2, W_3)$ defined by $W_0 = V_0$, $W_1 = V_1$, $W_2 = V_2$ and $W_3 = \varnothing$. Therefore, (i) implies (ii).

Conversely, if there exists a $\gamma_{(2,2,2,0)}(G)$-function $f(V_0, V_1, V_2, V_3)$ such that $V_3 = \varnothing$, then the function $g(W_0, W_1, W_2)$, defined by $W_0 = V_0$, $W_1 = V_1$ and $W_2 = V_2$, is a $(2, 2, 2)$-dominating function on G, and so $\gamma_{(2,2,2)}(G) \leq \omega(g) = \omega(f) = \gamma_{(2,2,2,0)}(G)$. Therefore, Corollary 3.37 leads to $\gamma_{(2,2,2,0)}(G) = \gamma_{(2,2,2)}(G)$, which completes the proof. □

Theorem 3.39. *For a graph G, the following statements are equivalent.*

(i) $\gamma_{(2,2,2,0)}(G) = 3$.

(ii) $\gamma(G) = 1$ *or* $\gamma_{\times 2,t}(G) = 3$.

Proof. Assume first that $\gamma_{(2,2,2,0)}(G) = 3$, and let $f(V_0, V_1, V_2, V_3)$ be a $\gamma_{(2,2,2,0)}(G)$-function. Notice that $|V_3| \in \{0, 1\}$. If $|V_3| = 1$, then $V_1 \cup V_2 = \varnothing$, which implies that V_3 is a dominating set of cardinality one. Hence, $\gamma(G) = 1$.

If $V_3 = \varnothing$, then by Lemma 3.38 we have that $\gamma_{(2,2,2)}(G) = \gamma_{(2,2,2,0)}(G) = 3$, and by Theorem 3.14 we deduce that $\gamma_{\times 2,t}(G) = 3$.

Conversely, if $\gamma(G) = 1$, then Corollary 3.37 leads to $3 \leq \gamma_{(2,2,2,0)}(G) \leq 3\gamma(G) = 3$. Moreover, if $\gamma_{\times 2,t}(G) = 3$, then G has minimum degree $\delta \geq 2$ and so Theorem 3.10(i) leads to $3 \leq \gamma_{(2,2,2,0)}(G) \leq \gamma_{(2,2,2)}(G) \leq \gamma_{\times 2,t}(G) = 3$. Therefore, $\gamma_{(2,2,2,0)}(G) = 3$. □

Theorem 3.40. *For a graph G, $\gamma_{(2,2,2,0)}(G) = 4$ if and only if at least one of the following conditions holds.*

(i) $\gamma_{\times 2,t}(G) = 4$.

(ii) $\gamma(G) = \gamma_t(G) = 2$ *and G has minimum degree $\delta = 1$.*

(iii) $\gamma(G) = \gamma_t(G) = 2$ *and* $\gamma_{\times 2,t}(G) \geq 4$.

Proof. Assume $\gamma_{(2,2,2,0)}(G) = 4$. Let $f(V_0, V_1, V_2, V_3)$ be a $\gamma_{(2,2,2,0)}(G)$-function. Hence, $|V_3| \in \{0, 1\}$. If $|V_3| = 1$, then V_3 is a dominating set of cardinality one. Hence, $\gamma(G) = 1$, which is a contradiction with Theorem 3.39. Hence, $V_3 = \varnothing$, and so, Lemma 3.38 leads to $\gamma_{(2,2,2)}(G) = \gamma_{(2,2,2,0)}(G) = 4$. Thus, by Theorems 3.15 and 3.39 we deduce (i) – (iii).

Conversely, if conditions (i) – (iii) hold, then by Theorem 3.14 we have that $\gamma_{(2,2,2)}(G) = 4$. Corollary 3.37 leads to $3 \leq \gamma_{(2,2,2,0)}(G) \leq \gamma_{(2,2,2)}(G) = 4$. Notice that if $\delta \geq 2$, then $\gamma(G) \geq 2$ and $\gamma_{\times 2,t}(G) \geq 4$. Hence, Theorem 3.39 leads to $\gamma_{(2,2,2,0)}(G) = 4$. □

Proposition 3.41. *For any integer* $n \geq 3$,

$$\gamma_{(2,2,2,0)}(C_n) = n.$$

Proof. By Corollaries 3.16 and 3.37 we have that $\gamma_{(2,2,2,0)}(C_n) \leq \gamma_{(2,2,2)}(C_n) = n$. We only need to prove that $\gamma_{(2,2,2,0)}(C_n) \geq n$. Let $f(V_0, V_1, V_2, V_3)$ be a $\gamma_{(2,2,2,0)}(G)$-function such that $|V_3|$ is minimum. If $V_3 = \varnothing$, then by Lemma 3.38 and Corollary 3.16 we conclude that $\gamma_{(2,2,2,0)}(C_n) = n$. Assume $V_3 \neq \varnothing$. If $v \in V_3$, then $N(v) \subseteq V_0$ as otherwise, by choosing one vertex $u \in N(v) \setminus V_0$, the function f' defined by $f'(v) = 2$, $f'(u) = \min\{2, f(u) + 1\}$ and $f'(x) = f(x)$ for the remaining vertices, is a $(2, 2, 2, 0)$-dominating function with $\omega(f') \leq \omega(f)$ and $|V_3'| < |V_3|$, which is a contradiction. Hence, $\sum_{x \in V_3} f(N[x]) = 3|V_3|$. Now, we observe that

$$2 \sum_{x \in V(C_n) \setminus N[V_3]} f(x) \geq \sum_{x \in V(C_n) \setminus N[V_3]} \left(\sum_{u \in N(x)} f(u) \right) \geq 2(n - 3|V_3|).$$

Therefore,

$$\begin{aligned} \gamma_{(2,2,2,0)}(C_n) = \omega(f) &= \sum_{x \in V_3} f(N[x]) + \sum_{x \in V(C_n) \setminus N[V_3]} f(x) \\ &\geq 3|V_3| + (n - 3|V_3|) = n, \end{aligned}$$

and the result follows. □

Proposition 3.42. *For any integer* $n \geq 3$,

$$\gamma_{(2,2,2,0)}(P_n) = \begin{cases} 6 & \textit{if } n = 5, \\ n & \textit{otherwise}. \end{cases}$$

Proof. It is easy to check that $\gamma_{(2,2,2,0)}(P_n) = n$ for $n = 3, 4, 6, 7, 8$, and also $\gamma_{(2,2,2,0)}(P_5) = 6$. From now on, assume $n \geq 9$. By Propositions 3.5 and 3.41 we have that $n = \gamma_{(2,2,2,0)}(C_n) \leq \gamma_{(2,2,2,0)}(P_n)$. Hence, we only need to prove that $\gamma_{(2,2,2,0)}(P_n) \leq n$. To this end, we proceed to construct a $(2, 2, 2, 0)$-dominating function $f(V_0, V_1, V_2, V_3)$ on $P_n = v_1 v_2 \dots v_n$ such that $\omega(f) = n$.

- If $n \equiv 0 \pmod 3$, then we set $V_3 = \bigcup_{i=1}^{n/3} \{v_{3i-1}\}$ and $V_0 = V(G) \setminus V_3$.
- If $n \equiv 1 \pmod 3$, then we set $V_3 = \bigcup_{i=1}^{(n-4)/3} \{v_{3i-1}\}$, $V_2 = \{v_{n-2}, v_{n-1}\}$ and $V_0 = V(G) \setminus (V_2 \cup V_3)$.

- If $n \equiv 2 \pmod 3$, then we set $V_3 = \bigcup_{i=1}^{(n-8)/3}\{v_{3i-1}\}$, $V_2 = \{v_{n-6}, v_{n-5}, v_{n-2}, v_{n-1}\}$ and $V_1 = \varnothing$.

Notice that in the three cases above, f is a $(2,2,2,0)$-dominating function of weight $\omega(f) = n$, as required. Therefore, the proof is complete. □

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References

[1] H. Abdollahzadeh Ahangar, M. Chellali, S. M. Sheikholeslami and J. C. Valenzuela-Tripodoro, Total Roman $\{2\}$-dominating functions in graphs, *Discuss. Math. Graph Theory* (2020), doi:10.7151/dmgt.2316, in press.

[2] D. W. Bange, A. E. Barkauskas, L. H. Host and P. J. Slater, Generalized domination and efficient domination in graphs, *Discrete Math.* **159** (1996), 1–11, doi:10.1016/0012-365x(95)00094-d.

[3] B. Brešar, M. A. Henning and S. Klavžar, On integer domination in graphs and Vizing-like problems, *Taiwanese J. Math.* **10** (2006), 1317–1328, doi:10.11650/twjm/1500557305.

[4] S. Cabrera García, A. Cabrera Martínez, F. A. Hernández Mira and I. G. Yero, Total Roman $\{2\}$-domination in graphs, *Quaest. Math.* **44** (2021), 411–434, doi:10.2989/16073606.2019.1695230.

[5] A. Cabrera Martínez, S. Cabrera García and J. A. Rodríguez-Velázquez, Double domination in lexicographic product graphs, *Discrete Appl. Math.* **284** (2020), 290–300, doi:10.1016/j.dam.2020.03.045.

[6] M. Chellali, T. W. Haynes, S. T. Hedetniemi and A. A. McRae, Roman $\{2\}$-domination, *Discrete Appl. Math.* **204** (2016), 22–28, doi:10.1016/j.dam.2015.11.013.

[7] G. S. Domke, S. T. Hedetniemi, R. C. Laskar and G. Fricke, Relationships between integer and fractional parameters of graphs, in: Y. Alavi, G. Chartrand, O. R. Oellermann and A. J. Schwenk (eds.), *Graph Theory, Combinatorics, and Applications, Volume 1*, Wiley, New York, A Wiley-Interscience Publication, pp. 371–387, 1991, proceedings of the Sixth Quadrennial International Conference on the Theory and Applications of Graphs held at Western Michigan University, Kalamazoo, Michigan, May 30–June 3, 1988.

[8] J. F. Fink and M. S. Jacobson, n-domination in graphs, in: Y. Alavi, G. Chartrand, D. R. Lick and C. E. Wall (eds.), *Graph Theory with Applications to Algorithms and Computer Science*, John Wiley & Sons, New York, A Wiley-Interscience Publication, 1985 pp. 283–300.

[9] F. Harary and T. W. Haynes, Double domination in graphs, *Ars Combin.* **55** (2000), 201–213.

[10] T. W. Haynes, S. T. Hedetniemi and P. J. Slater (eds.), *Domination in Graphs: Advanced Topics*, volume 209 of *Monographs and Textbooks in Pure and Applied Mathematics*, Marcel Dekker, New York, 1998, doi:10.1201/9781315141428.

[11] T. W. Haynes, S. T. Hedetniemi and P. J. Slater, *Fundamentals of Domination in Graphs*, volume 208 of *Monographs and Textbooks in Pure and Applied Mathematics*, Marcel Dekker, New York, 1998, doi:10.1201/9781482246582.

[12] M. A. Henning and A. P. Kazemi, k-tuple total domination in graphs, *Discrete Appl. Math.* **158** (2010), 1006–1011, doi:10.1016/j.dam.2010.01.009.

[13] M. A. Henning and W. F. Klostermeyer, Italian domination in trees, *Discrete Appl. Math.* **217** (2017), 557–564, doi:10.1016/j.dam.2016.09.035.

[14] M. A. Henning and A. Yeo, *Total Domination in Graphs*, Springer Monographs in Mathematics, Springer, New York, 2013, doi:10.1007/978-1-4614-6525-6.

[15] X. Hou and Y. Lu, On the $\{k\}$-domination number of Cartesian products of graphs, *Discrete Math.* **309** (2009), 3413–3419, doi:10.1016/j.disc.2008.07.030.

[16] W. F. Klostermeyer and G. MacGillivray, Roman, Italian, and 2-domination, *J. Combin. Math. Combin. Comput.* **108** (2019), 125–146.

[17] C.-M. Lee and M.-S. Chang, Variations of Y-dominating functions on graphs, *Discrete Math.* **308** (2008), 4185–4204, doi:10.1016/j.disc.2007.08.080.

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Sum-list-colouring of θ-hypergraphs

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Abstract

Given a hypergraph $\mathcal{H}$ and a function $f\colon V(\mathcal{H}) \to \mathbb{N}$, we say that $\mathcal{H}$ is f-choosable if there is a proper vertex coloring ϕ of $\mathcal{H}$ such that $\phi(v) \in L(v)$ for all $v \in V(\mathcal{H})$, where $L\colon V(\mathcal{H}) \to 2^{\mathbb{N}}$ is any assignment of $f(v)$ colors to a vertex v. The sum choice number $\chi_{sc}(\mathcal{H})$ of $\mathcal{H}$ is defined to be the minimum of $\sum_{v\in V(\mathcal{H})} f(v)$ over all functions f such that $\mathcal{H}$ is f-choosable. A trivial upper bound on $\chi_{sc}(\mathcal{H})$ is $|V(\mathcal{H})| + |\mathcal{E}(\mathcal{H})|$. The class Γ_{sc} of hypergraphs that achieve this bound is induced hereditary. We analyze some properties of hypergraphs in Γ_{sc} as well as properties of hypergraphs in the class of forbidden hypergraphs for Γ_{sc}. We characterize all θ-hypergraphs in Γ_{sc}, which leads to the characterization of all θ-hypergraphs that are forbidden for Γ_{sc}.

Keywords: Hypergraphs, sum-list-colouring, θ-hypergraphs.

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1 Introduction

A hypergraph is a very natural generalization of a graph. It always motivates the extension of a problem first posed in the class of graphs to the class of hypergraphs. If it is a vertex colouring problem, then there is additional motivation. Indeed, a lot of scientists consider different concepts of vertex colouring of graphs (for example: list-colouring, sum-colouring, equitable-colouring), starting, in each case, from proper colouring, and next, analyzing some improper variants, in which a graph induced by vertices of a colour class is not necessarily edgeless. If we assume that each colour class has to induce a graph with some property (for example: acyclic, with a bounded degree, and so on) and this property is closed with respect to induced subgraphs, then, in each of these concepts, the problem of improper colouring of a graph is equivalent to the problem of proper colouring of a unique

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hypergraph constructed for this graph. Clearly, the construction of such a hypergraph can be difficult, but this approach gives the possibility to solve the problem. Moreover, in this case, each of the results obtained for hypergraphs can be applied to different variants of the same concept of graph colouring. Consequently, it can produce many special results.

The concept of sum-list-colouring of graphs is motivated by real problems and was first introduced in [6, 8]. Erdős, Rubin and Taylor [6] considered the so called size functions whose values for vertices of a graph represented the sizes of the lists assigned to them. Isaak [8] was the first to analyze the minimum sum of the list sizes that guarantees the existence of any particular proper vertex list colouring if lists are of these sizes. Such an invariant was determined in [11], with the help of Hall's Theorem, for complete graphs, and then, for a few other classes of graphs [9, 12]. In [9] the upper bound on the minimum sum of the list sizes was determined. Graphs that meet this bound, known as sc-*greedy*, led themselves to a very popular line of investigation in the literature [2, 3, 7, 9, 12]. In [5], the authors analyzed sum-list-colouring concept assuming that colour classes need not be edgeless. This investigation shows some differences between proper and improper cases and uses hypergraph theory tools. We continue this consideration herein, focusing on hypergraphs, believing that the following results will be used for various variants of the colouring concept. We extend the sc-greedy notion from graphs to hypergraphs and characterize all θ-hypergraphs that are sc-greedy (Theorem 4.11). This yields the characterization of all θ-hypergraphs that are forbidden for the class of sc-greedy hypergraphs (Corollary 4.12).

2 Preliminaries

In general, we follow the notation and terminology of [1, 4]. A hypergraph $\mathcal{H}$ consists of a non-empty finite set $V(\mathcal{H})$ of *vertices* and a finite set $\mathcal{E}(\mathcal{H})$ of at least 2-element subsets of $V(\mathcal{H})$, called *edges*. A hypergraph is *simple* if none of its edges is a subset of another edge. A hypergraph is *linear* if any two of its edges have no more than one vertex in common.

Let $\mathcal{H}$ be a hypergraph. A hypergraph $\mathcal{H}'$ is a *subhypergraph* of $\mathcal{H}$ if $V(\mathcal{H}') \subseteq V(\mathcal{H})$ and $\mathcal{E}(\mathcal{H}') \subseteq \mathcal{E}(\mathcal{H})$. For $V' \subseteq V(\mathcal{H})$, a *subhypergraph of a hypergraph $\mathcal{H}$ induced by V'*, denoted by $\mathcal{H}[V']$, has the vertex set V' and the edge set $\{E \in \mathcal{E}(\mathcal{H}) : E \subseteq V'\}$. We use $\mathcal{H} - V'$ notation instead of $\mathcal{H}[V(\mathcal{H}) \setminus V']$ and even $\mathcal{H} - v$ instead of $\mathcal{H} - \{v\}$.

Let $\mathcal{H}$ be a hypergraph, $v \in V(\mathcal{H})$ and $\mathcal{E}(v) = \{E \in \mathcal{E}(\mathcal{H}) : v \in E\}$. By $\mathcal{H}(v)$ we denote a hypergraph with the vertex set $\cup_{E\in\mathcal{E}(v)} E$ and with the edge set $\mathcal{E}(v)$. The *degree* of v in $\mathcal{H}$, denoted by $\deg_{\mathcal{H}}(v)$, is defined as the number of edges of $\mathcal{H}(v)$. The *β-degree* of v in $\mathcal{H}$, denoted by $\deg^{\beta}_{\mathcal{H}}(v)$, is the largest number of edges of a linear subhypergraph of $\mathcal{H}(v)$. The $\mathcal{H}_1 \cup \mathcal{H}_2$ symbol denotes the *union* of disjoint hypergraphs $\mathcal{H}_1, \mathcal{H}_2$. By the *identification* of two non-adjacent vertices v_1 and v_2 (in a hypergraph $\mathcal{H}$ into a vertex w) we mean the result of the following operations on $\mathcal{H}$: the removal of vertices v_1, v_2, the addition of a new vertex w, the replacement of each edge containing either v_1 or v_2 by an edge in which w substitutes v_1, v_2, respectively, and the removal of multiple edges if the current hypergraph has such edges. Note that v_1, v_2 can be vertices of different components, say $\mathcal{H}_1, \mathcal{H}_2$, of $\mathcal{H}$. In this case, sometimes, instead of the identification of vertices in $\mathcal{H}_1 \cup \mathcal{H}_2$ we may talk about the identification of vertices of two disjoint hypergraphs $\mathcal{H}_1, \mathcal{H}_2$.

The 1-vertex hypergraph is a *hypertree* without edges. Next, a hypergraph that has one edge consisting of all its vertices is a *hypertree* with one edge. A *hypertree* with m edges

($m \geq 2$) can be constructed from a hypertree $\mathcal{H}_1$ with m_1 edges and a hypertree $\mathcal{H}_2$ with $m - m_1$ edges, $0 < m_1 < m$, by the identification of an arbitrary vertex of $\mathcal{H}_1$ and an arbitrary vertex of $\mathcal{H}_2$. Note that each hypertree is linear.

A hypertree $\mathcal{H}$ is a *hyperpath* if there is an ordering (called *canonical*) of $V(\mathcal{H})$ such that each edge of $\mathcal{H}$ consists of some consecutive vertices (with respect to this ordering). The length of a hyperpath is the number of its edges. By a *hypercycle* we mean a hypergraph obtained from a hyperpath of length of at least three by the identification of the vertex with the first index and the vertex with the last index in an arbitrary canonical ordering of the vertex set of this hyperpath. The length of a hypercycle is the same as the length of a hyperpath that was used in the construction. Moreover, if $v_1, \ldots, v_n$ is a canonical ordering of the vertex set of the hyperpath, then $v_1, \ldots, v_{n-1}$ is a *canonical* ordering of the vertex set of the resulting hypercycle. Let $k \in \mathbb{N}$. By a *k-edge, k^+-edge* (*of a hypergraph $\mathcal{H}$*) we mean an edge of $\mathcal{H}$ consisting of k, at least k vertices, respectively. A hypergraph is *k-uniform* if each of its edges is a k-edge. Thus 2-uniform hypergraphs are *graphs* and especially, 2-uniform hypertrees, hyperpaths, hypercycles are *trees*, *paths*, *cycles*, respectively.

3 Sum-choice-number of hypergraphs

Let $\mathcal{H}$ be a hypergraph. A *proper colouring of $\mathcal{H}$* is a mapping $\phi\colon V(\mathcal{H}) \to \mathbb{N}$ such that for every edge E of $\mathcal{H}$ there are at least two different vertices v_1, v_2 in E such that $\phi(v_1) \neq \phi(v_2)$. Given a mapping $L\colon V(\mathcal{H}) \to 2^{\mathbb{N}}$ we call a mapping $\phi\colon V(\mathcal{H}) \to \mathbb{N}$ *an L-colouring of $\mathcal{H}$* if for every vertex $v \in V(\mathcal{H})$ it holds that $\phi(v) \in L(v)$. Let f be a function from $V(\mathcal{H})$ to the set of positive integers, a mapping $L\colon V(\mathcal{H}) \to 2^{\mathbb{N}}$ such that $|L(v)| = f(v)$ for every vertex v in $V(\mathcal{H})$ is called an *f-assignment* for $\mathcal{H}$. The hypergraph $\mathcal{H}$ is *f-choosable* if for each f-assignment L for $\mathcal{H}$ there is a proper L-coloring of $\mathcal{H}$. Thus, $\mathcal{H}$ is f-choosable if $\mathcal{H}$ is properly L-colourable for each f-assignment L for $\mathcal{H}$. The *sum-choice-number* $\chi_{sc}(\mathcal{H})$ of $\mathcal{H}$ is defined as the minimum of $\sum_{v \in V(\mathcal{H})} f(v)$ taken over all f such that $\mathcal{H}$ is f-choosable. Hence

$$\chi_{sc}(\mathcal{H}) = \min{}_f \left\{ \sum_{v \in V(\mathcal{H})} f(v) : \mathcal{H} \text{ is } f\text{-choosable} \right\}.$$

If $\mathcal{H}$ is f-choosable and $\chi_{sc}(\mathcal{H}) = \sum_{v \in V(\mathcal{H})} f(v)$, then we say that f *realizes* $\chi_{sc}(\mathcal{H})$.

The definition of the sum-choice-number of a hypergraph implies some immediate observations.

Fact 3.1. *If a hypergraph $\mathcal{H}_1$ is f-choosable for some function f with a domain $V(\mathcal{H}_1)$, then each subhypergraph $\mathcal{H}_2$ of $\mathcal{H}_1$ is $f|_{V(\mathcal{H}_2)}$-choosable.*

Fact 3.2. *If $\mathcal{H}_1, \mathcal{H}_2$ are vertex disjoint hypergraphs, then*

$$\chi_{sc}(\mathcal{H}_1 \cup \mathcal{H}_2) = \chi_{sc}(\mathcal{H}_1) + \chi_{sc}(\mathcal{H}_2).$$

Fact 3.3. *If $\mathcal{H}_2$ is a subhypergraph of a hypergraph $\mathcal{H}_1$, then*

$$|V(\mathcal{H}_1)| - |V(\mathcal{H}_2)| + \chi_{sc}(\mathcal{H}_2) \leq \chi_{sc}(\mathcal{H}_1).$$

Applying the reasoning provided in the proof of Theorem 8 from [5] we obtain the following theorem.

Theorem 3.4. *If $\mathcal{H}$ is a hypergraph and $v_1, \dots, v_n$ is an arbitrary ordering of $V(\mathcal{H})$, then*

$$\chi_{sc}(\mathcal{H}) \leq \sum_{i=1}^{n} \deg^{\beta}_{\mathcal{H}_i}(v_i) + n,$$

where $\mathcal{H}_i = \mathcal{H}[\{v_1, \dots, v_i\}]$.

Proof. Given the ordering $v_1, \dots, v_n$, let $f(v_i) = \deg^{\beta}_{\mathcal{H}_i}(v_i) + 1$. To finish the proof we will show that $\mathcal{H}$ is f-choosable. Let L be any f-assignment for $\mathcal{H}$. We colour the vertices of $\mathcal{H}$ greedily, in accordance with the ordering $v_1, \dots, v_n$. Namely, in the i^{th} step we assign to v_i the least colour from $L(v_i)$ such that for each $a \in \mathbb{N}$ the hypergraph induced by the vertices coloured with a in the hypergraph $\mathcal{H}_i$ is edgeless. Note that such a colouring exists for each of i^{th} steps, $i \in \{1, \dots, n\}$, since, there are at most $\deg^{\beta}_{\mathcal{H}_i}(v_i)$ colours in $L(v_i)$ for which $\mathcal{H}_i$ has an edge that would be monochromatic if we assigne this colour to v_i. □

Using the reasoning presented in the final part of the above proof we have the following property.

Lemma 3.5. *If $\mathcal{H}$ is a hypergraph and f is a function that realizes $\chi_{sc}(\mathcal{H})$, then $f(v) \leq \deg^{\beta}_{\mathcal{H}}(v) + 1$ for each $v \in V(\mathcal{H})$.*

Proof. Suppose, for contradiction purposes, that f satisfies the assumptions of the lemma and there is a vertex $u \in \mathcal{H}$ such that $f(u) \geq \deg_{\mathcal{H}}(u) + 2$. We will show that $\mathcal{H}$ is f'-choosable for f' defined by $f'(v) = f(v)$ for $v \in V(\mathcal{H}) \setminus \{u\}$ and $f'(u) = f(u) - 1$, giving a contradiction with the assumptions about f. Let $\mathcal{H}' = \mathcal{H} - u$ and let f'-assignment L' for $\mathcal{H}$ be given. Since $f'|_{V(\mathcal{H}')} = f|_{V(\mathcal{H}')}$ we know that there is a proper $L'|_{V(\mathcal{H}')}$-colouring ϕ' of $\mathcal{H}'$, by Fact 3.1. Clearly ϕ' can be extended to a proper L'-colouring of $\mathcal{H}$ since $f'(u) \geq \deg^{\beta}_{\mathcal{H}}(u) + 1$. □

Observe that the bound given in Theorem 3.4 mostly depends on the ordering of vertices. For example, consider a hypergraph $\mathcal{H}$ such that $V(\mathcal{H}) = \{v_1, \dots, v_5\}$ and $\mathcal{E}(\mathcal{H}) = \{E_1, E_2, E_3\}$, where $E_1 = \{v_1, v_2, v_3, v_4\}$, $E_2 = \{v_1, v_2, v_3, v_5\}$ and $E_3 = \{v_1, v_2, v_4, v_5\}$. Let $\pi_1\colon v_1, v_2, v_3, v_4, v_5$ and $\pi_2\colon v_3, v_4, v_5, v_1, v_2$ be two different orderings of $V(\mathcal{H})$. Thus Theorem 3.4 gives the upper bound of 7 on $\chi_{sc}(\mathcal{H})$ when we use π_1 and of 6 when we use π_2. On the other hand $\deg^{\beta}_{\mathcal{H}}(v) \leq \deg_{\mathcal{H}}(v)$ for every vertex v of a hypergraph $\mathcal{H}$. Moreover, for any ordering $v_1, \dots, v_n$ of vertices of an n-vertex hypergraph $\mathcal{H}$ we have $\sum_{i=1}^{n} \deg_{\mathcal{H}_i}(v_i) = |\mathcal{E}(\mathcal{H})|$, where $\mathcal{H}_i = \mathcal{H}[\{v_1, \dots, v_i\}]$. Hence Theorem 3.4 implies the following fact.

Fact 3.6. *If $\mathcal{H}$ is a hypergraph, then*

$$\chi_{sc}(\mathcal{H}) \leq |V(\mathcal{H})| + |\mathcal{E}(\mathcal{H})|.$$

A hypergraph $\mathcal{H}$ is called *sc-greedy* if $\chi_{sc}(\mathcal{H}) = |V(\mathcal{H})| + |\mathcal{E}(\mathcal{H})|$. In brief, in the following, we denote the number $|V(\mathcal{H})| + |\mathcal{E}(\mathcal{H})|$ by $GB(\mathcal{H})$. The notion of sc-greediness was previously introduced for graphs in [2]. Observe that if $\mathcal{H}$ is an sc-greedy hypergraph, then for every ordering $v_1, \dots, v_n$ of $V(\mathcal{H})$ it holds that $\deg^{\beta}_{\mathcal{H}_i}(v_i) = \deg_{\mathcal{H}_i}(v_i)$ for each permissible i and $\mathcal{H}_i = \mathcal{H}[\{v_1, \dots, v_i\}]$. Now suppose that a hypergraph has at least two edges E_1, E_2 that have at least two vertices, in common, say $\{v_1, v_2\} \subseteq E_1 \cap E_2$. We construct an ordering of vertices of $\mathcal{H}$ putting first the vertices from $E_1 \setminus \{v_1, v_2\}$, next the vertices from $E_2 \setminus \{v_1, v_2\}$, next v_1, v_2, and finally the remaining vertices. Clearly, $\deg^{\beta}_{\mathcal{H}_j}(v_2) < \deg_{\mathcal{H}_j}(v_2)$, where $j = |E_1 \cup E_2|$. Hence we conclude the following fact.

Fact 3.7. *Each sc-greedy hypergraph is linear.*

The literature on sc-greediness of graphs is very rich. We try to comment this property in the class of hypergraphs, especially those hypergraphs that are not graphs. Let Γ_{sc} denote the family of all sc-greedy hypergraphs. First note that Γ_{sc} is not closed while taking subhypergraphs. Indeed, the $K_{2,3}$ graph is not sc-greedy, but a graph resulting from $K_{2,3}$ by the addition of an edge, which joins two vertices of degree 3, is sc-greedy [12]. On the other hand, Γ_{sc} is closed while taking induced subhypergraphs (it is a well-known fact for sc-greedy graphs). To see it, suppose that there is an sc-greedy hypergraph $\mathcal{H}$ having an induced subhypergraph $\mathcal{H}'$, which is not sc-greedy. We construct a function f such that $\mathcal{H}$ is f-choosable and $\sum_{v\in V(\mathcal{H})} f(v) \leq GB(\mathcal{H}) - 1$ based on the function f' that realizes $\chi_{sc}(\mathcal{H}')$. Actually, $f|_{V(\mathcal{H}')} = f'$ and for $i \in \{1, \ldots, p\}$ we put $f(v_i) = \deg^{\beta}_{\mathcal{H}_i}(v_i) + 1$, where $v_1, \ldots, v_p$ is an arbitrary ordering of $V(\mathcal{H}) \setminus V(\mathcal{H}')$ and $\mathcal{H}_i = \mathcal{H}[V(\mathcal{H}') \cup \{v_1, \ldots, v_i\}]$. It implies that $\mathcal{H}$ is not sc-greedy, contradicting our assumption. Thus Γ_{sc} is an induced hereditary class and there is a family $\mathcal{C}(\Gamma_{sc})$ of hypergraphs, each of which is not sc-greedy and whose each proper induced subhypergraph is sc-greedy. The elements of $\mathcal{C}(\Gamma_{sc})$ are called *forbidden hypergraphs for* Γ_{sc} and they uniquely determine Γ_{sc}. Note that Γ_{sc} contains only linear hypergraphs. The class $\mathcal{C}(\Gamma_{sc})$ does not have this property. For example, each non-linear hypergraph $\mathcal{H}$ defined by $V(\mathcal{H}) = E_1 \cup E_2$, $\mathcal{E}(\mathcal{H}) = \{E_1, E_2\}$, where $|E_1 \cap E_2| \geq 2$, $E_1 \setminus E_2 \neq \emptyset$, $E_2 \setminus E_1 \neq \emptyset$, is an element of $\mathcal{C}(\Gamma_{sc})$.

In the next part of the paper we focus our attention on linear hypergraphs in $\mathcal{C}(\Gamma_{sc})$.

Lemma 3.8. *Let $\mathcal{H}$ be a linear hypergraph in $\mathcal{C}(\Gamma_{sc})$ and $v \in V(\mathcal{H})$. If f is a function that realizes $\chi_{sc}(\mathcal{H})$, then*

i) $f(v) \leq \deg_{\mathcal{H}}(v)$, *and*

ii) $\deg_{\mathcal{H}}(v) \geq 2$ *implies* $f(v) \geq 2$ *provided that each edge in* $\mathcal{E}(\mathcal{H}(v))$ *contains in* $\mathcal{H}$ *at most two vertices of degree greater than one.*

Proof. To show i) suppose that there is at least one vertex u in $V(\mathcal{H})$ such that $f(u) \geq \deg_{\mathcal{H}}(u) + 1$. Lemma 3.5 says that $f(u) = \deg_{\mathcal{H}}(u) + 1$. Now we define $\mathcal{H}' = \mathcal{H} - u$. Clearly $\mathcal{H}'$ is a proper induced subhypergraph of $\mathcal{H}$ and consequently is sc-greedy, by the definition of $\mathcal{C}(\Gamma_{sc})$. From the construction we know that $|V(\mathcal{H}')| = |V(\mathcal{H})| - 1$ and $|\mathcal{E}(\mathcal{H}')| = |\mathcal{E}(\mathcal{H})| - \deg_{\mathcal{H}}(u)$. Thus $\chi_{sc}(\mathcal{H}') = GB(\mathcal{H}') = GB(\mathcal{H}) - (\deg_{\mathcal{H}}(u) + 1)$. As a subhypergraph of $\mathcal{H}$, the hypergraph $\mathcal{H}'$ is $f|_{V(\mathcal{H}')}$-choosable, by Fact 3.1. It follows that $\chi_{sc}(\mathcal{H}') \leq \sum_{v\in V(\mathcal{H}')} f(v) = \sum_{v\in V(\mathcal{H})} f(v) - (\deg_{\mathcal{H}}(u) + 1) \leq GB(\mathcal{H}) - 1 - (\deg_{\mathcal{H}}(u)+1)$. Thus $GB(\mathcal{H})-1-(\deg_{\mathcal{H}}(u)+1) \geq \chi_{sc}(\mathcal{H}') = GB(\mathcal{H})-(\deg_{\mathcal{H}}(u)+1)$, i.e. a contradiction.

To show ii) suppose that $\mathcal{H}$ and $u \in V(\mathcal{H})$ satisfy the assumptions and $f(u) = 1$. If there is an edge $E \in \mathcal{E}(\mathcal{H}(u))$ that contains only one vertex of degree greater than one (only u), then for each vertex in E the value of f is equal to one, and consequently $\mathcal{H}$ is not f-choosable, a contradiction. Let $\{E_1, \ldots, E_k\} = \mathcal{E}(\mathcal{H}(u))$. Thus for each $i \in \{1, \ldots, k\}$ there is exactly one vertex u_i different from u such that $u_i \in E_i$ and $\deg_{\mathcal{H}}(u_i) \geq 2$. Let $\mathcal{H}' = \mathcal{H}[(V(\mathcal{H}) \setminus \cup_{i=1}^{k} E_i) \cup \{u_1, \ldots, u_k\}]$. Note that $|\mathcal{E}(\mathcal{H}')| = |\mathcal{E}(\mathcal{H})| - k$ and $|V(\mathcal{H}')| =$

$|V(\mathcal{H})| - t$ for some $t \in \mathbb{N}$. We define $f' \colon V(\mathcal{H}') \to \mathbb{N}$ such that $f'(v) = f(v)$ if $v \notin \{u_1, \ldots, u_k\}$ and $f'(u_i) = f(u_i) - 1$ for $i \in \{1, \ldots, k\}$. Note that $\sum_{v \in V(\mathcal{H}')} f'(v) \leq \sum_{v \in V(\mathcal{H})} f(v) - k - t$. Since $\mathcal{H} \in \mathcal{C}(\Gamma_{sc})$, it holds that $\sum_{v \in V(\mathcal{H})} f(v) \leq GB(\mathcal{H}) - 1$, and consequently $\sum_{v \in V(\mathcal{H}')} f'(v) \leq GB(\mathcal{H}) - 1 - k - t = GB(\mathcal{H}') - 1$. Thus $\mathcal{H}'$ is not f'-choosable. Let L' be an f'-assignment for $\mathcal{H}'$ such that there is no proper L'-colouring of $\mathcal{H}'$ and let $a \notin \bigcup_{v \in V(\mathcal{H}')} L'(v)$. We define an f-assignment L for $\mathcal{H}$ in the following way: $L(v) = L'(v)$ for $v \in V(\mathcal{H}') \setminus \{u_1, \ldots, u_k\}$, next $L(u_i) = L'(u_i) \cup \{a\}$ for $i \in \{1, \ldots, k\}$ and $L(v) = \{a\}$ for $v \in V(\mathcal{H}) \setminus V(\mathcal{H}')$. It is very easy to see that there is no proper L-colouring of $\mathcal{H}$, which means that $\mathcal{H}$ is not f-choosable contradicting the assumption. Hence $f(u) \geq 2$ in this case. □

Let us continue the investigation concerning sc-greedy hypergraphs. We start with the observation that Theorem 1 in [2] (referring to graphs) can be extended to hypergraphs. In fact, the same proof, in which the words graphs are substituted by hypergraphs, works to obtain the following statement.

Theorem 3.9. *Let $\mathcal{H}_1$, $\mathcal{H}_2$ be two disjoint hypergraphs and $v_1 \in V(\mathcal{H}_1)$, $v_2 \in V(\mathcal{H}_2)$. If $\mathcal{H}$ is the hypergraph obtained by the identification of v_1 and v_2 in $\mathcal{H}_1 \cup \mathcal{H}_2$, then*

$$\chi_{sc}(\mathcal{H}) = \chi_{sc}(\mathcal{H}_1) + \chi_{sc}(\mathcal{H}_2) - 1.$$

Note that each hypertree with one edge is sc-greedy. The recursion used in the definition of a hypertree with an arbitrary number of edges and Theorem 3.9 give us the following consequence.

Corollary 3.10. *Each hypertree is sc-greedy.*

Using Lemma 3.8 concerning hypergraphs in $\mathcal{C}(\Gamma_{sc})$, we have the next result.

Theorem 3.11. *Each hypercycle is sc-greedy.*

Proof. Clearly, if there is a hypercycle $\mathcal{H}$ that is not sc-greedy, then $\mathcal{H} \in \mathcal{C}(\Gamma_{sc})$. It follows by Corollary 3.10 and Fact 3.2, since each component of every proper induced subhypergraph of $\mathcal{H}$ is a hypertree (actually it is a hyperpath). Suppose that f realizes $\chi_{sc}(\mathcal{H})$. By Lemma 3.8 i), ii) we have $f(v) \geq \deg_{\mathcal{H}}(v)$ for each vertex of $\mathcal{H}$, and consequently $\sum_{v \in \mathcal{H}} f(v) \geq GB(\mathcal{H})$, a contradiction. □

Let $\mathcal{F}$ be the class of recursively defined hypergraphs such that: all hypercycles and all hypertrees are in $\mathcal{F}$, and, giving any two disjoint hypergraphs $\mathcal{H}_1$, $\mathcal{H}_2$ in $\mathcal{F}$ and vertices $v_1 \in V(\mathcal{H}_1)$, $v_2 \in V(\mathcal{H}_2)$, a hypergraph obtained by the identification of v_1 and v_2 in $\mathcal{H}_1 \cup \mathcal{H}_2$ is also in $\mathcal{F}$.

The following result is a consequence of Corollary 3.10 and Theorems 3.11, 3.9.

Corollary 3.12. *If $\mathcal{H} \in \mathcal{F}$, then $\mathcal{H}$ is sc-greedy.*

4 θ-hypergraphs

Let $k_1, k_2, k_3 \in \mathbb{N}$. By $\theta^h_{k_1,k_2,k_3}$ we denote the hypergraph consisting of two vertices of degree 3 connected by three internally disjoint hyperpaths of lengths k_1, k_2, k_3. In what follows, we sometimes use the notion of a hyperpath of $\theta^h_{k_1,k_2,k_3}$ of length k_i, $i \in \{1, 2, 3\}$, meaning the hyperpath of length k_i, used in the definition of $\theta^h_{k_1,k_2,k_3}$. By a θ-hypergraph we mean an arbitrary hypergraph $\theta^h_{k_1,k_2,k_3}$. Observe that if at least two of the numbers k_1, k_2, k_3 are equal to one, then $\theta^h_{k_1,k_2,k_3}$ is not linear. Additionally, if one of the hyperpaths of length one is created by a 2-edge, $\theta^h_{k_1,k_2,k_3}$ is even not simple. When $\theta^h_{k_1,k_2,k_3}$ is a graph, we can denote it by θ_{k_1,k_2,k_3} since such notation is present in the literature. In [7] Heinold found the values $\chi_{sc}(\theta_{k_1,k_2,k_3})$ for all simple graphs θ_{k_1,k_2,k_3}. We recall here this result.

Theorem 4.1 ([7]). *Let $k_1, k_2, k_3 \in \mathbb{N}$ and at most one of k_1, k_2, k_3 is equal to one. A graph θ_{k_1,k_2,k_3} is not sc-greedy if and only if $k_1 = k_2 = 2$ and k_3 is even. Moreover, if θ_{k_1,k_2,k_3} is not sc-greedy, then*

$$\chi_{sc}(\theta_{k_1,k_2,k_3}) = GB(\theta_{k_1,k_2,k_3}) - 1.$$

Theorem 4.1 shows that the sum-choice-number of each simple graph θ_{k_1,k_2,k_3}, is always less by one or equal to the sum of the numbers of vertices and edges of this graph. Fortunately, θ-hypergraphs have the same property.

Lemma 4.2. *If $k_1, k_2, k_3 \in \mathbb{N}$ and at most two of the numbers k_1, k_2, k_3 are equal to one, then $\chi_{sc}(\theta^h_{k_1,k_2,k_3}) \geq GB(\theta^h_{k_1,k_2,k_3}) - 1$.*

Proof. Let $\mathcal{H} = \theta^h_{k_1,k_2,k_3}$ and let E be an arbitrary edge of $\mathcal{H}$ that is one of the edges of the shortest hyperpath among three hyperpaths that compose $\mathcal{H}$. Next let $\mathcal{H}'$ be a subhypergraph of $\mathcal{H}$ such that $V(\mathcal{H}') = V(\mathcal{H})$ and $\mathcal{E}(\mathcal{H}') = \mathcal{E}(\mathcal{H}) \setminus \{E\}$. Note that $\mathcal{H}'$ is sc-greedy, by Corollary 3.12 and Fact 3.2. It means $\chi_{sc}(\mathcal{H}') \geq GB(\theta^h_{k_1,k_2,k_3}) - 1$. Thus the statement follows from Fact 3.3. □

It is worth mentioning that $\chi_{sc}(\theta^h_{1,1,1}) = GB(\theta^h_{1,1,1}) - 2$. Indeed, let v be one of two vertices of degree 3 in $V(\theta^h_{1,1,1})$ and let $f\colon V(\theta^h_{1,1,1}) \to \mathbb{N}$ be defined by: $f(v) = 2$ and $f(u) = 1$ for every $u \in V(\theta^h_{1,1,1}) \setminus \{v\}$. Clearly, $\theta^h_{1,1,1}$ is f-choosable since, for every f-assignment L, each colouring, in which the colours of v and the other vertex of degree 3 are different, is a proper L-colouring of $\theta^h_{1,1,1}$. Thus, $\chi_{sc}(\theta^h_{1,1,1}) \leq GB(\theta^h_{1,1,1}) - 2 = |V(\theta^h_{1,1,1})| + 1$. Because $\theta^h_{1,1,1}$ has edges, we have $\chi_{sc}(\theta^h_{1,1,1}) \geq |V(\theta^h_{1,1,1})| + 1$.

Let $f\colon V(\mathcal{H}) \to \mathbb{N}$ and L be an f-assignment for $\mathcal{H}$. In what follows, if $L(v) = \{a_1, \ldots, a_{f(v)}\}$, then we always assume that elements $a_1, \ldots, a_{f(v)}$ are pairwise different. Thus, among others, in Lemma 4.3 the integers a, b, c are pairwise different. Furthermore, if $i_1, \ldots, i_p$ are consecutive integers and $i_1 > i_p$, then the set $\{i_1, \ldots, i_p\}$ is empty.

Lemma 4.3. *Let $k \in \mathbb{N}$ and $\mathcal{H}$ be a hypercycle of length $2k$. Next let $v_1, \ldots, v_n$ be an arbitrary canonical ordering of $V(\mathcal{H})$, where $\{v_{i_1} = v_1, \ldots, v_{i_{2k}}\}$ is the set of all vertices of degree two in $\mathcal{H}$ with $i_j < i_l$ for $j < l$.*

If $f\colon V(\mathcal{H}) \to \mathbb{N}$ is defined by $f(v) = \deg_{\mathcal{H}}(v)$ and L is an f-assignment for $\mathcal{H}$ such that

i) $L(v_{i_j}) = \{a, b\}$ *for* $j \in \{1, \ldots, 2k-2\}$, *and*

ii) $L(v_{i_{2k-1}}) = \{b, c\}$, *and*

iii) $L(v_{i_{2k}}) = \{a, c\}$, *and*

iv) $L(v_s) = \{c\}$ *for* $s \in \{i_{2k-1}+1, \ldots, i_{2k}-1\}$, *and*

v) $L(v_s) = \{a\}$ *for* $s \in \{i_{2k}+1, \dots, n\}$ *or for* $s \in \{i_r+1, \dots, i_{r+1}-1\}$ *with odd* r, $1 \leq r \leq 2k-3$, *and*

vi) $L(v_s) = \{b\}$ *for* $s \in \{i_r+1, \dots i_{r+1}-1\}$ *with even* r, $2 \leq r \leq 2k-2$,

then in each proper L*-colouring* ϕ *of* $\mathcal{H}$ *it holds that* $\phi(v_1) = b$.

Proof. Suppose that there is a proper L-colouring ϕ of $\mathcal{H}$ such that $\phi(v_1) = a$. Consequently, it must be $\phi(v_{i_{2p}}) = b$ for all $p \in \{1, \dots, k-1\}$, next $\phi(v_{i_{2k-1}}) = c$ and $\phi(v_{i_{2k}}) = a$. Hence the edge $\{v_{i_{2k}}, \dots, v_n, v_1\}$ is monochromatic in ϕ, a contradiction. □

To avoid repetitions, we skip the simple proof of the next lemma that can be done in the same manner as the proof of Lemma 4.3.

Lemma 4.4. *Let* $k \in \mathbb{N}$ *and* $\mathcal{H}$ *be a hypercycle of length* $2k+1$. *Next let* $v_1, \dots, v_n$ *be an arbitrary canonical ordering of* $V(\mathcal{H})$, *where* $\{v_{i_1} = v_1, \dots, v_{i_{2k+1}}\}$ *is the set of all vertices of degree two in* $\mathcal{H}$ *with* $i_j < i_k$ *for* $j < k$.

If $f\colon V(\mathcal{H}) \to \mathbb{N}$ *is defined by* $f(v) = \deg_{\mathcal{H}}(v)$ *and* L *is an* f*-assignment for* $\mathcal{H}$ *such that*

i) $L(v_{i_j}) = \{a, b\}$ *for* $j \in \{1, \dots, 2k+1\}$, *and*

ii) $L(v_s) = \{a\}$ *for* $s \in \{i_{2k+1}+1, \dots, n\}$ *or for* $s \in \{i_r+1, \dots, i_{r+1}-1\}$ *with odd* r, $1 \leq r \leq 2k-1$, *and*

iii) $L(v_s) = \{b\}$ *for* $s \in \{i_r+1, \dots i_{r+1}-1\}$ *and even* r, $2 \leq r \leq 2k$,

then in each proper L*-colouring* ϕ *of* $\mathcal{H}$ *it holds that* $\phi(v_1) = b$.

Lemma 4.5. *Let* $k_1, k_2, k_3 \in \mathbb{N}$. *If the hyperpath of length* k_2 *of* $\theta^h_{k_1,k_2,k_3}$ *has only* 2*-edges, and either*

i) $k_1 + k_2$ *and* $k_2 + k_3$ *are odd numbers and at least one of the inequalities* $k_1 \geq 2$, $k_3 \geq 2$ *holds, or*

ii) $k_1 + k_2$ *is an odd number and* $k_2 + k_3$ *is an even number and* $k_3 \geq 3$, *or*

iii) $k_1 + k_2$ *is an even number and* $k_2 + k_3$ *is an even number and* $k_1 \geq 3$ *and* $k_3 \geq 3$,

then $\theta^h_{k_1,k_2,k_3}$ *is sc-greedy.*

Proof. Let $\mathcal{H} = \theta^h_{k_1,k_2,k_3}$ and let $\mathcal{H}$ satisfies the assumptions of the lemma. Observe that at most one of integers k_1, k_2, k_3 is equal to one since otherwise, $\mathcal{H}$ does not satisfy the assumptions of the lemma, so $\mathcal{H}$ is linear.

Suppose, for a contradiction, that $\mathcal{H}$ is not sc-greedy. Since each component of each proper induced subhypergraph of $\mathcal{H}$ is in $\mathcal{F}$, we obtain $\mathcal{H} \in \mathcal{C}(\Gamma_{sc})$, by Corollary 3.12 and Fact 3.2. Let f be a function that realizes $\chi_{sc}(\mathcal{H})$. Lemma 3.8 i) implies that $f(v) = 1$ if $\deg_{\mathcal{H}}(v) = 1$ and Lemma 3.8 ii) implies that $f(v) \geq 2$ for each vertex v of degree greater than one. Thus f has values in $\{1, 2\}$ and is fixed, since $\sum_{v \in V(\mathcal{H})} f(v) = GB(\mathcal{H}) - 1$ (see Lemma 4.2).

Now we shall construct an f-assignment L for $\mathcal{H}$ such that $\mathcal{H}$ is not properly L-colourable. Assume that $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ are three hyperpaths of lengths k_1, k_2, k_3, respectively, from which $\theta^h_{k_1,k_2,k_3}$ is composed. Next let C_1 be a hypercycle that is a subhypergraph of $\theta^h_{k_1,k_2,k_3}$ composed from vertices and edges of the hyperpaths $\mathcal{P}_1, \mathcal{P}_2$. Similarly, let C_2 be a hypercycle that is a subhypergraph of $\theta^h_{k_1,k_2,k_3}$ composed from vertices and edges of the hyperpaths $\mathcal{P}_2, \mathcal{P}_3$. Thus lengths of C_1, C_2 are $k_1 + k_2$ and $k_2 + k_3$, respectively. Now we define canonical orderings π_1 of C_1 and π_2 of C_2, both starting with the same fixed vertex of degree three in $\mathcal{H}$, say v_1, and both proceeding consecutively, first along the vertices of $\mathcal{P}_2$, and next, along the vertices of either $\mathcal{P}_1$ or $\mathcal{P}_3$, respectively.

Next we construct an f-assignment L_1 for C_1 and π_1 either in accordance with Lemma 4.3 or in accordance with Lemma 4.4, depending on the parity of the length of C_1 (the parity of $k_1 + k_2$). Similarly, we construct an f-assignment L_2 for C_2 and π_2 either in accordance with Lemma 4.3 or Lemma 4.4, but in this case we exchange the meaning of colours a, b. Namely, we substitute a by b and b by a in each value of L_2 (given by the corresponding lemma).

Observe that the assumptions on numbers k_1, k_2, k_3 and the fact that $\mathcal{P}_2$ has only 2-edges imply that $L_1 = L_2$ on vertices of $\mathcal{P}_2$. Define an f-assignment L for $\mathcal{H}$ such that

$$L(v) = \begin{cases} L_1(v), & \text{if } v \in V(C_1) \setminus V(\mathcal{P}_2), \\ L_2(v), & \text{if } v \in V(C_2) \setminus V(\mathcal{P}_2), \\ L_1(v) = L_2(v), & \text{if } v \in V(\mathcal{P}_2). \end{cases}$$

Suppose, for a contradiction, that ϕ is a proper L-colouring of $\mathcal{H}$. Fact 3.1 implies that $\phi|_{V(C_1)}$ is a proper L_1-colouring of C_1 and $\phi|_{V(C_2)}$ is a proper L_2-colouring of C_2. By Lemma 4.3 or Lemma 4.4 (depending on the parity of $k_1 + k_2$) we have $\phi(v_1) = b$ and by one of Lemmas 4.3, 4.4 (depending on the parity of $k_2 + k_3$) we have $\phi(v_1) = a$, a contradiction. □

For forthcoming Lemmas 4.6, 4.7, 4.8, 4.9, 4.10, we introduce the following notations. Let $\mathcal{P}$ be a hyperpath on at least two vertices and let $v_1, \ldots, v_n$ be a canonical ordering of $V(\mathcal{P})$. Next let $f^*\colon V(\mathcal{P}) \to \mathbb{N}$ be defined by $f^*(v) = \deg_{\mathcal{P}}(v)$ for $v \notin \{v_1, v_n\}$ and $f^*(v_1) = f^*(v_n) = 2$.

Giving an f^*-assignment L for $\mathcal{P}$ and $(\alpha, \gamma) \in L(v_1) \times L(v_n)$, we say that a pair (α, γ) is *extendable* (*for* $\mathcal{P}$) if there is a proper L-colouring ϕ of $\mathcal{P}$ such that $\phi(v_1) = \alpha$, $\phi(v_n) = \gamma$. The pair $(\alpha, \gamma) \in L(v_1) \times L(v_n)$ that is not extendable for $\mathcal{P}$ is called *forbidden* (*for* $\mathcal{P}$).

The next lemma is a generalization of the lemma that was proven in [3].

Lemma 4.6. *Let $\mathcal{P}$ be a hyperpath on at least two vertices, let $v_1, \ldots, v_n$ be a canonical ordering of $V(\mathcal{P})$ and let $\{v_{i_1}, \ldots, v_{i_k}\}$ be the set of all vertices of degree two in $\mathcal{P}$ with $i_p < i_s$ for $p < s$. If L is an f^*-assignment for $\mathcal{P}$ such that $L(v_1), L(v_{i_1}), \ldots, L(v_{i_k}), L(v_n)$ are not identical, then at most one pair in $L(v_1) \times L(v_n)$ is forbidden for $\mathcal{P}$.*

Proof. We will show that at most one pair in $L(v_1) \times L(v_n)$ is forbidden for the path P, where $V(P) = \{v_1 = v_{i_0}, v_{i_1}, \ldots, v_{i_k}, v_n = v_{i_{k+1}}\}$ and $E(P) = \{v_{i_j} v_{i_{j+1}}\colon j \in \{0, \ldots, k\}\}$. Note that P is a graph. Since every proper $L|_{V(P)}$-colouring of P can be extended to a proper L-colouring of $\mathcal{P}$, the lemma will follow. We prove this statement by the induction on the number of edges in P (the number $k+1$). Observe that it is true for a path P with one edge. Suppose that $|E(P)| \geq 2$. Since the lists of $v_1, v_{i_1}, \ldots, v_{i_k}, v_n$ are not identical, the lists of $v_1, v_{i_1}, \ldots, v_{i_k}$ or the lists of $v_{i_1}, \ldots, v_{i_k}, v_n$ are not identical either. Say the lists of $v_{i_1}, \ldots, v_{i_k}, v_n$ are not identical. Let $L(v_1) = \{a, b\}$, $L(v_n) =$

$\{c, d\}$, $L(v_{i_1}) = \{\alpha, \beta\}$ and assume that $a \neq \beta$ and $b \neq \alpha$. By inductive assumptions, at least three pairs in $L(v_{i_1}) \times L(v_n)$ are extendable for $P - v_1$, say $(\alpha, c), (\alpha, d), (\beta, c)$. Thus pairs $(b, c), (b, d), (a, c)$ are extendable for P, and hence, they are extendable for $\mathcal{P}$. □

Lemma 4.7. *If $\mathcal{P}$ is a hyperpath with at least one 3^+-edge, $v_1, \dots, v_n$ is a canonical ordering of $V(\mathcal{P})$ and L is an f^*-assignment for $\mathcal{P}$, then at most one pair in $L(v_1) \times L(v_n)$ is forbidden for $\mathcal{P}$.*

Proof. We prove the assertion by the induction on the number of edges in $\mathcal{E}(\mathcal{P})$. Observe that the lemma trivially holds when $\mathcal{P}$ has one edge. Suppose that $\mathcal{P}$ is a hyperpath with at least one 3^+-edge and $|\mathcal{E}(\mathcal{P})| \geq 2$. Let $\{v_{i_1}, \dots, v_{i_k}\}$ be the set of all vertices of degree two in $\mathcal{P}$ with $i_p < i_s$ for $p < s$. If the lists of $v_1, v_{i_1}, \dots, v_{i_k}, v_n$ are not identical, then the statement follows from Lemma 4.6. Thus we may assume that $L(v_1) = L(v_{i_1}) = \cdots = L(v_{i_k}) = L(v_n) = \{a, b\}$. Let $E_1 = \{v_1, v_2, \dots, v_{i_1}\}$. Renaming vertices, if it is necessary, we may assume that $\mathcal{P}' = \mathcal{P} \setminus \{v_1, \dots, v_{i_1-1}\}$ contains a 3^+-edge. Thus $\mathcal{P}'$ is a hyperpath satisfying inductive assumptions, and so, at least three pairs in $L(v_{i_1}) \times L(v_n)$ are extendable for $\mathcal{P}'$. Note that if we colour v_1 with a and v_{i_1} with b or if we colour v_1 with b and v_{i_1} with a, then we can extend such a colouring to a proper L-colouring of $\mathcal{P}$. Hence, in $L(v_1) \times L(v_n)$ there are three pairs that are extendable for $\mathcal{P}$. □

Lemmas 4.6, 4.7 immediately imply the following fact.

Lemma 4.8. *If $\mathcal{P}$ is a hyperpath on at least two vertices, $v_1, \dots, v_n$ is a canonical ordering of $V(\mathcal{P})$ and L is an f^*-assignment for $\mathcal{P}$, then at most two pairs in $L(v_1) \times L(v_n)$ are forbidden for $\mathcal{P}$. Moreover,*

i) *exactly two pairs are forbidden for $\mathcal{P}$ if and only if $\mathcal{P}$ contains only 2-edges and $L(v_1) = \cdots = L(v_n)$, and*

ii) *if there are exactly two forbidden pairs for $\mathcal{P}$ and $\mathcal{P}$ is of even length and $L(v_1) = \{a, b\}$, then (a, a) and (b, b) are extendable for $\mathcal{P}$, and*

iii) *if there are exactly two forbidden pairs for $\mathcal{P}$ and $\mathcal{P}$ is of odd length and $L(v_1) = \{a, b\}$, then (a, b) and (b, a) are extendable for $\mathcal{P}$.*

Lemma 4.9. *If $\mathcal{P}$ is a hyperpath of length two, $v_1, \dots, v_n$ is a canonical ordering of $V(\mathcal{P})$, L is an f^*-assignment for $\mathcal{P}$ and $L(v_1) = L(v_n) = \{a, b\}$, then (a, a) and (b, b) are extendable for $\mathcal{P}$.*

Based on Lemmas 4.6, 4.7, 4.8, 4.9 we have the following result.

Lemma 4.10. *Let $k_1, k_2, k_3 \in \mathbb{N}$. If $\theta^h_{k_1,k_2,k_3}$ is sc-greedy, then one of the hyperpaths of $\theta^h_{k_1,k_2,k_3}$, say the hyperpath of length k_2, has only 2-edges, and, under this assumption, one of the following conditions is satisfied:*

i) *$k_1 + k_2$ and $k_2 + k_3$ are odd numbers and at least one of the inequalities $k_1 \geq 2$, $k_3 \geq 2$ holds, or*

ii) *$k_1 + k_2$ is an odd number and $k_2 + k_3$ is an even number and $k_3 \geq 3$, or*

iii) *$k_1 + k_2$ is an even number and $k_2 + k_3$ is an even number and $k_1 \geq 3$ and $k_3 \geq 3$.*

Proof. Let $\mathcal{H} = \theta^h_{k_1,k_2,k_3}$. Suppose that for each possible permutation $k_{i_1}, k_{i_2}, k_{i_3}$ of numbers k_1, k_2, k_3 either the hyperpath of $\mathcal{H}$ of length k_{i_2} contains 3^+-edge or $\theta^h_{k_{i_1},k_{i_2},k_{i_3}}$ satisfies no of the conditions i), ii), iii) of the lemma. The aim is to prove that $\mathcal{H}$ is not sc-greedy. We may assume that at most one of the numbers k_1, k_2, k_3 is equal to one. Otherwise, $\mathcal{H}$ is not linear and the statement follows by Fact 3.7. We define f so that $f(v) = 1$ if $\deg_{\mathcal{H}}(v) = 1$ and $f(v) = 2$ if $\deg_{\mathcal{H}}(v) \geq 2$. Next, we will show that for each f-assignment L for $\mathcal{H}$ there is a proper L-colouring of $\mathcal{H}$. Since $\sum_{v \in V(\mathcal{H})} f(v) = GB(\mathcal{H}) - 1$, the theorem will follow. Let $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ be the hyperpaths of $\mathcal{H}$ of lengths k_1, k_2, k_3, respectively, and let v_1, v_n be vertices of degree 3 in $\mathcal{H}$. Let L be an arbitrary f-assignment for $\mathcal{H}$. First observe that if each hyperpath $\mathcal{P}_i$, $i \in \{1,2,3\}$, has at least one 3^+-edge, then for each hyperpath $\mathcal{P}_i$ at most one pair in $L(v_1) \times L(v_n)$ is forbidden, by Lemma 4.7. Since we have four possible pairs in $L(v_1) \times L(v_n)$, at least one pair can be extended to a proper L-colouring of $\mathcal{H}$. Thus at least one hyperpath contains only 2-edges, so it is the path. Moreover, the lists of vertices of this path must be identical, by Lemma 4.8 i). We may assume that each vertex of this path has the list $\{a, b\}$, and so, $L(v_1) = L(v_n) = \{a, b\}$. Let us consider two cases.

Case 1. The numbers k_1, k_2, k_3 are all of the same parity. Definitely, $\mathcal{H}$ does not fulfill neither the property i) nor ii). Since $\mathcal{H}$ does not have the property iii) either, at least one of integers k_1, k_2, k_3 is less than or equal to two. Assume that $k_1 \leq 2$.

Subcase 1.1. All of the numbers k_1, k_2, k_3 are even. Thus $k_1 = 2$. If $k_2 \geq 3$ and $k_3 \geq 3$, then $\mathcal{P}_1$ has to contain at least one 3^+-edge. Otherwise, $\mathcal{H}$ satisfies iii) for a permutation $k_{i_1}, k_{i_2}, k_{i_3}$, where $k_{i_2} = k_1$. By our initial assumption $\mathcal{P}_2$ or $\mathcal{P}_3$ has only 2-edges and pairs $(a, a), (b, b)$ are extendable for this path, by Lemma 4.8 ii). Without loss of generality assume that $\mathcal{P}_2$ contains only 2-edges and pairs $(a, a), (b, b)$ are extendable for $\mathcal{P}_2$. Note that pairs $(a, a), (b, b)$ are also extendable for $\mathcal{P}_1$, by Lemma 4.9. From Lemma 4.8, at most two pairs are forbidden for $\mathcal{P}_3$, and if exactly two pairs are forbidden for $\mathcal{P}_3$, then $(a, a), (b, b)$ are extendable for $\mathcal{P}_3$. Thus both pairs $(a, a), (b, b)$ can be extended to a proper L-colouring of $\mathcal{H}$. If at most one pair is forbidden for $\mathcal{P}_3$, then at least one of pairs $(a, a), (b, b)$ can be extended to a proper L-colouring of $\mathcal{H}$.

Now suppose that at least two hyperpaths have lengths two, say $k_1 = 2$ and $k_2 = 2$. From Lemma 4.9, (a, a) and (b, b) are extendable for both $\mathcal{P}_1$ and $\mathcal{P}_2$. From Lemma 4.8, at most two pairs are forbidden for $\mathcal{P}_3$. If exactly two pairs are forbidden for $\mathcal{P}_3$, then again by Lemma 4.8 ii), both pairs $(a, a), (b, b)$ are extendable for $\mathcal{P}_3$. Otherwise, at most one pair is forbidden for $\mathcal{P}_3$. Thus (a, a) or (b, b) is extendable for all $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$.

Subcase 1.2. All of the numbers k_1, k_2, k_3 are odd. Thus $k_1 = 1$. Since $\mathcal{H}$ is linear, we have $k_2 \geq 3$ and $k_3 \geq 3$. Furthermore, $\mathcal{P}_1$ is a 3^+-edge, otherwise, $\mathcal{H}$ satisfies iii). Again, without loss of generality, assume that $\mathcal{P}_2$ contains only 2-edges and pairs $(a, b), (b, a)$ are extendable for $\mathcal{P}_2$, by Lemma 4.8 iii). Thus $(a, b), (b, a)$ are extendable for both $\mathcal{P}_1$, $\mathcal{P}_2$. If there is at most one pair forbidden for $\mathcal{P}_3$, then one of pairs $(a, b), (b, a)$ is extendable for all $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$. Otherwise, two pairs are forbidden for $\mathcal{P}_3$, however, then both pairs $(a, b), (b, a)$ are extendable for $\mathcal{P}_3$, by Lemma 4.8 iii). So, both pairs $(a, b), (b, a)$ can be extended to a proper L-colouring of $\mathcal{H}$.

Case 2. The numbers k_1, k_2, k_3 are not of the same parity. In this case either one hyperpath in $\{\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3\}$ is of odd length and two of them are of even length or one hyperpath in $\{\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3\}$ is of even length and two are of odd length. So, $\mathcal{H}$ does not have the property iii). The hyperpath of odd length in the first case and the hyperpath of even length in the second case has a 3^+-edge, since otherwise, $\mathcal{H}$ has the property i). Let us consider the following two possible subcases.

Subcase 2.1. The number k_1 is odd and $\mathcal{P}_1$ has a 3^+-edge, k_2, k_3 are even. Since $\mathcal{H}$ does not have the property ii), $k_2 = 2$ or $k_3 = 2$. Say $k_3 = 2$. If $k_2 > 2$, then $\mathcal{P}_3$ has 3^+-edge, as otherwise, $\mathcal{H}$ has the property ii). Thus $\mathcal{P}_2$ must have only 2-edges and pairs $(a, a), (b, b)$ are extendable for $\mathcal{P}_2$, by Lemma 4.8 ii). From Lemma 4.9, pairs $(a, a), (b, b)$ are extendable for $\mathcal{P}_3$, so, $(a, a), (b, b)$ are extendable for both $\mathcal{P}_2$, $\mathcal{P}_3$. By Lemma 4.7, at most one of these pairs is forbidden for $\mathcal{P}_1$, so there is a proper L-colouring of $\mathcal{H}$. Suppose now that $k_2 = 2$, so we have $k_2 = k_3 = 2$. From Lemma 4.9 both pairs $(a, a), (b, b)$ are extendable for both $\mathcal{P}_2$, $\mathcal{P}_3$. Since at most one of these pairs is forbidden for $\mathcal{P}_1$, the statement follows.

Subcase 2.2. The number k_1 is even and $\mathcal{P}_1$ has a 3^+-edge, k_2, k_3 are odd. Since $\mathcal{H}$ does not have the property ii), $k_2 = 1$ or $k_3 = 1$. Say $k_3 = 1$. Furthermore, the previous consideration leads to $k_2 \geq 3$. If $\mathcal{P}_3$ has exactly 2-edge, then $\mathcal{H}$ has the property ii). Thus $\mathcal{P}_3$ must have a 3^+-edge, and so, $\mathcal{P}_2$ must have only 2-edges. Hence, pairs $(a, b), (b, a)$ are extendable for $\mathcal{P}_2$. Thus (a, b) and (b, a) are extendable for both $\mathcal{P}_2$, $\mathcal{P}_3$. Furthermore, at most one of pairs is forbidden for $\mathcal{P}_1$, and so, at least one of the pairs $(a, a), (b, b)$ is extendable for all $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$, which creates a proper L-colouring of $\mathcal{H}$ and proves the lemma. □

Now, we are in a position to present the main result of the paper that immediately follows from Lemmas 4.5, 4.10. Next, the consequence of this result is formulated.

Theorem 4.11. *Let $k_1, k_2, k_3 \in \mathbb{N}$. A hypergraph $\theta^h_{k_1,k_2,k_3}$ is sc-greedy if and only if one of the hyperpaths of $\theta^h_{k_1,k_2,k_3}$, say the hyperpath of length k_2, has only 2-edges and, under this assumption, one of the following conditions holds:*

i) *$k_1 + k_2$ and $k_2 + k_3$ are odd numbers and at least one of the inequalities $k_1 \geq 2$, $k_3 \geq 2$ holds, or*

ii) *$k_1 + k_2$ is an odd number and $k_2 + k_3$ is an even number and $k_3 \geq 3$, or*

iii) *$k_1 + k_2$ is an even number and $k_2 + k_3$ is an even number and $k_1 \geq 3$ and $k_3 \geq 3$.*

Note that for arbitrary parameters k_1, k_2, k_3 such that $(k_1, k_2, k_3) \neq (1, 1, 1)$ and arbitrary vertex of $\theta^h_{k_1,k_2,k_3}$ each component of $\theta^h_{k_1,k_2,k_3} - v$ is in $\mathcal{F}$. Hence the following fact is valid.

Corollary 4.12. *If $k_1, k_2, k_3 \in \mathbb{N}$ and at most two of the numbers k_1, k_2, k_3 are equal to one, then $\theta^h_{k_1,k_2,k_3} \in \mathcal{C}(\Gamma_{sc})$ if and only if $\theta^h_{k_1,k_2,k_3} \notin \Gamma_{sc}$.*

5 Concluding remarks and open problems

A connected hypergraph is 2-*connected* if it cannot be a result of identification of a vertex of $\mathcal{H}_1$ and a vertex of $\mathcal{H}_2$, where $\mathcal{H}_1$, $\mathcal{H}_2$ are some disjoint hypergraphs, each on at least two vertices.

Note that, based on Fact 3.2 and Theorem 3.9, both, the union operation and the identification operation (applied to vertices of two disjoint hypergraphs) keep sc-greediness of hypergraphs. Hence, analyzing sc-greediness of hypergraphs it is enough to focus on 2-connected ones. Additionally, each sc-greedy hypergraph must be linear, by Fact 3.7. Thus the following question seems to be interesting.

Problem 5.1. How to characterize all hypergraphs in Γ_{sc} that are 2-connected and linear?

To support the discussion this question we start with some notes concerning graphs. The famous theorem that characterizes all linear (equivalently, simple) 2-connected graphs can be found in [4]. To cite it we need the following notion.

Let G be a graph on at least two vertices. By *adding* a G-path to G, we mean the result of two operations of identification applied to the graph G and an arbitrary path P with a canonical ordering $v_1, \ldots, v_n$ of $V(P)$, $n \geq 2$ (G and P are disjoint). More precisely, it is a result of identification of v_1 and x and also v_n and y, where x, y are two different vertices of G.

Lemma 5.2 ([4]). *A simple graph is* 2*-connected if and only if it can be constructed from a cycle by successively adding* G*-paths to graphs* G *already constructed.*

Observe that each cycle is 2-connected and sc-greedy. Next, adding a G-path to the cycle G we obtain a θ-graph. Theorem 4.1 characterizes all θ-graphs that are in Γ_{sc}. The question is, whether we should expect that an sc-greedy graph be obtained by adding a G-path to an sc-greedy θ-graph. In [3] the authors proved that $\chi_{sc}(G) + 2s \leq \chi_{sc}(G_1) \leq \chi_{sc}(G) + 2s + 1$ if G_1 is the result of adding a G-path on $s + 2$ vertices to an arbitrary simple graph G, assuming that $s \geq 1$. They did not consider the case when a G-path has 2 vertices, which seems to be very important. However, based on this observation, they gave the characterization of all graphs in Γ_{sc} that are generalized θ-graphs. For $r \geq 3$, a *generalized* θ*-graph* is a simple graph, denoted by $\theta_{k_1,k_2,\ldots,k_r}$, consisting of two vertices connected by r internally vertex disjoint paths of lengths $k_1, k_2, \ldots, k_r$. In [3] the authors showed that $\theta_{k_1,k_2,\ldots,k_r}$ is not sc-greedy if $r \geq 5$. Moreover, θ_{k_1,k_2,k_3,k_4} is not sc-greedy if and only if it contains an induced subgraph $\theta_{2,2,k_i}$ with even k_i and $i \in \{3, 4\}$ or, if all numbers k_1, k_2, k_3, k_4 have the same parity. It follows that starting with a θ-graph G and adding a G-path two times, in this special case (to obtain a generalized θ-graph), we always obtain a graph that is not sc-greedy. On the other hand, the graph presented in Figure 1 that needs to be added a G-path 3 times is still sc-greedy (see the graph $G_{10,12}$ in [10]). It leads to formulating the following problem.

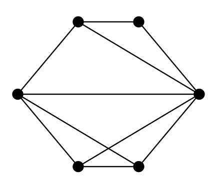

Figure 1: An sc-greedy graph that needs the application of adding a G-path 3 times.

Problem 5.3. Under which conditions can we obtain an sc-greedy graph by adding a G-path to an sc-greedy graph G?

In a way similar for graphs we can define the operation of adding an $\mathcal{H}$-path to a hypergraph $\mathcal{H}$ and pose the problem similar to Problem 5.3 in the class of hypergraphs.

Problem 5.4. Under which conditions can we obtain an sc-greedy hypergraph by adding an $\mathcal{H}$-path to an sc-greedy hypergraph $\mathcal{H}$?

Unfortunately, there are 2-connected linear hypergraphs that cannot be obtained from a hypercycle by successively adding $\mathcal{H}$-paths.

On the other hand, there is a relatively large subclass of the class of 2-connected linear hypergraphs, for which, the problem of belonging to Γ_{sc} can be solved, with help of consideration concerning graphs.

A hypergraph $\mathcal{G}$ is a *blown of a graph* G if $\mathcal{G}$ is a result of a substitution of each edge of G by a 3^+-edge of $\mathcal{G}$ that contains vertices of substituted edge of G (see for example Figure 2). Let $\mathcal{F}'$ be the class consisting of all hypergraphs that are all possible blowns of 2-connected graphs, except cycles. Clearly, every hypergraph in $\mathcal{F}'$ is 2-connected, by definition.

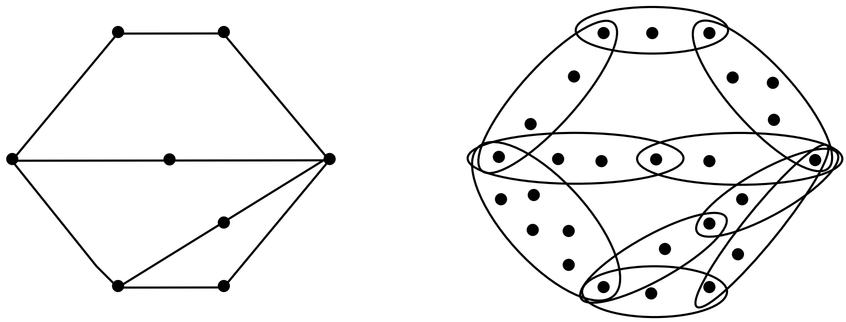

Figure 2: A graph and its blown.

Theorem 5.5. *No hypergraph in $\mathcal{F}'$ is sc-greedy.*

Proof. First observe that if $\mathcal{H} \in \mathcal{F}'$, then $\mathcal{H}$ is a blown of some 2-connected graph H that is not a cycle. From Lemma 5.2, H contains a subgraph that is a θ-graph. By the definition of $\mathcal{F}'$ we know that $\mathcal{H}$ contains an induced subhypergraph $\mathcal{H}'$ that is a θ-hypergraph, and moreover, $\mathcal{H}'$ has no hyperpath having only 2-edges. Thus $\mathcal{H}'$ is not sc-greedy, by Lemma 4.10. Finally, $\mathcal{H}$ is not sc-greedy, since Γ_{sc} is closed when taking induced subhypergraphs. □

It is worth mentioning that hypergraphs in $\mathcal{F}'$ can or cannot be blowns of sc-greedy graphs. It follows that probably, to be an sc-greedy hypergraph is a relatively rare feature in the class of hypergraphs with only 3^+-edges. It motivates the following separate subproblem of Problem 5.1.

Problem 5.6. How to characterize all hypergraphs in Γ_{sc} that are 2-connected, linear and have only 3^+-edges.

Finally, observe that each hypergraph in $\mathcal{C}(\Gamma_{sc})$ is 2-connected, but this class includes linear and non-linear hypergraphs. Thus, referring to the $\mathcal{C}(\Gamma_{sc})$ class, we obtain the following open question and its subquestion.

Problem 5.7. How to characterize all hypergraphs in $\mathcal{C}(\Gamma_{sc})$?

Problem 5.8. How to characterize all non-linear hypergraphs in $\mathcal{C}(\Gamma_{sc})$?

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References

[1] C. Berge, *Hypergraphs: Combinatorics of Finite Sets*, volume 45 of *North-Holland Mathematical Library*, North-Holland, Amsterdam, 1989.

[2] A. Berliner, U. Bostelmann, R. A. Brualdi and L. Deaett, Sum list coloring graphs, *Graphs Combin.* **22** (2006), 173–183, doi:10.1007/s00373-005-0645-9.

[3] C. Brause, A. Kemnitz, M. Marangio, A. Pruchnewski and M. Voigt, Sum choice number of generalized θ-graphs, *Discrete Math.* **340** (2017), 2633–2640, doi:10.1016/j.disc.2016.11.028.

[4] R. Diestel, *Graph Theory*, volume 173 of *Graduate Texts in Mathematics*, Springer-Verlag, New York, 2nd edition, 2000, doi:10.1007/b100033.

[5] E. Drgas-Burchardt and A. Drzystek, General and acyclic sum-list-colouring of graphs, *Appl. Anal. Discrete Math.* **10** (2016), 479–500, doi:10.2298/aadm161011026d.

[6] P. Erdős, A. L. Rubin and H. Taylor, Choosability in graphs, in: P. Z. Chinn and D. McCarthy (eds.), *Proceedings of the West Coast Conference on Combinatorics, Graph Theory and Computing*, Utilitas Mathematica, Winnipeg, Man., volume XXVI of *Congressus Numerantium*, 1980 pp. 125–157, held at Humboldt State University, Arcata, Calif., September 5 – 7, 1979.

[7] B. Heinold, *Sum list coloring and choosability*, Ph.D. thesis, Lehigh University, 2006.

[8] G. Isaak, Sum list coloring $2 \times n$ arrays, *Electron. J. Combin.* **9** (2002), #N8, 7 pp., doi: 10.37236/1669.

[9] G. Isaak, Sum list coloring block graphs, *Graphs Combin.* **20** (2004), 499–506, doi:10.1007/s00373-004-0564-1.

[10] A. Kemnitz, M. Marangio and M. Voigt, Sum list colorings of small graphs, *Congr. Numer.* **227** (2016), 27–50.

[11] J. Kratochvíl, Zs. Tuza and M. Voigt, New trends in the theory of graph colorings: choosability and list coloring, in: R. L. Graham, J. Kratochvíl and F. S. Roberts (eds.), *Contemporary Trends in Discrete Mathematics*, American Mathematical Society, Providence, RI, volume 49 of *DIMACS Series in Discrete Mathematics and Theoretical Computer Science*, pp. 183–197, 1999, doi:10.1090/dimacs/049/13, proceedings of the DIMATIA-DIMACS Conference on The Future of Discrete Mathematics held at Štiřín Castle, May 19 – 25, 1997.

[12] M. Lastrina, *List-coloring and sum-list-coloring problems on graphs*, Ph.D. thesis, Iowa State University, 2012.

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Cospectrality of multipartite graphs*

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Abstract

Let G be a graph on n vertices and consider the adjacency spectrum of G as the ordered n-tuple whose entries are eigenvalues of G written decreasingly. Let G and H be two non-isomorphic graphs on n vertices with spectra S and T, respectively. Define the distance between the spectra of G and H as the distance of S and T to a norm N of the n-dimensional vector space over real numbers. Define the cospectrality of G as the minimum of distances between the spectrum of G and spectra of all other non-isomorphic n vertices graphs to the norm N. In this paper we investigate copsectralities of the cocktail party graph and the complete tripartite graph with parts of the same size to the Euclidean or Manhattan norms.

Keywords: Spectra of graphs, cospectrality of graphs, adjacency matrix of a graph, Euclidean norm, Manhattan norm.

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1 Introduction and results

All graphs considered here are simple, that is finite and undirected without loops and multiple edges. Let G be a graph with vertex set $\{v_1, \dots, v_n\}$. The adjacency matrix of G is an $n \times n$ matrix $A(G) = [a_{ij}]$ such that $a_{ij} = 1$ if v_i and v_j are adjacent, and $a_{ij} = 0$ otherwise. By the eigenvalues of G, we mean those of its adjacency matrix. We denote by $Spec(G)$ the multiset of the eigenvalues of the graph G.

Richard Brualdi proposed in [24] the following problem:

Problem ([24, Problem AWGS.4])**.** Let G_n and G'_n be two non-isomorphic graphs on n vertices with spectra

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \quad \text{and} \quad \lambda'_1 \geq \lambda'_2 \geq \cdots \geq \lambda'_n,$$

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respectively. Define the distance between the spectra of G_n and G'_n as

$$\lambda(G_n, G'_n) = \sum_{i=1}^{n} (\lambda_i - \lambda'_i)^2 \quad \left(\text{or use } \sum_{i=1}^{n} |\lambda_i - \lambda'_i|\right).$$

Define the cospectrality of G_n by

$$\mathrm{cs}(G_n) = \min\{\lambda(G_n, G'_n) \colon G'_n \text{ not isomorphic to } G_n\}.$$

Let

$$\mathrm{cs}_n = \max\{\mathrm{cs}(G_n) \colon G_n \text{ a graph on } n \text{ vertices}\}.$$

This function measures how far apart the spectrum of a graph with n vertices can be from the spectrum of any other graph with n vertices.

Problem A. Investigate $\mathrm{cs}(G_n)$ for special classes of graphs.

Problem B. Find a good upper bound on cs_n.

In [15], Jovanović et al. studied the spectral distance between certain graphs to the ℓ^1-norm i.e. $\sigma(G_n, G'_n) = \sum_{i=1}^{n} |\lambda_i - \lambda'_i|$. In [1], Abdollahi et al. completely answered Problem B to any ℓ^p-norm by proving that $\mathrm{cs}_n = 2$ for all $n \geq 2$, whenever $1 \leq p < \infty$ and $\mathrm{cs}_n = 1$ to the ℓ^∞-norm. In [2, 20], the authors studied Problem A to the Euclidean norm (the ℓ^2-norm) and determined the cospectralities of classes of complete graphs and complete bipartite graphs. In [3] we compute the cospectralities to the ℓ^1-norm of complete graphs and complete bipartite graphs with parts of the same size. In [4, 10, 11, 13, 14, 16, 17, 18], Problems A or B are studied based on different matrix representations. To find some applications of the cospectrality of graphs, we refer to [6, 25, 27].

In this paper we study Problem A and investigate the cospectralities of CP_n and $K_{n,n,n}$, $(n \geq 3)$, to the ℓ^1- and ℓ^2-norms i.e. $\sigma(G_n, G'_n) = \sum_{i=1}^{n} |\lambda_i - \lambda'_i|$ and $\lambda(G_n, G'_n) = \sum_{i=1}^{n} (\lambda_i - \lambda'_i)^2$, respectively. We find some conditions for the eigenvalues of a graph H such that $\mathrm{cs}(G) = \sigma(G, H)$ and G is isomorphic to CP_n or $K_{n,n,n}$. Also we give some computational results and conjectures to find $\mathrm{cs}(CP_n)$ and $\mathrm{cs}(K_{n,n,n})$.

In the last section we consider cospectralities of null graphs, complete graphs and complete bipartite graphs using the ℓ^p-norm for $p > 2$ and we see that similar known conclusions using with ℓ^1 and ℓ^2-norms (see [2, 3, 11, 20]) hold more or less valid.

Let us first introduce some notations. For a graph G, $V(G)$ and $E(G)$ denote the vertex set and edge set of G, respectively; By the order of G we mean the number of vertices; Denote by $\overline{G}$ the complement of G. The degree of a vertex of a graph is the number of edges that are incident with the vertex and Δ is the maximum degree of the vertices. An r-regular graph is a graph where all vertices have degree r.

For two graphs G and H with disjoint vertex sets, $G + H$ denotes the graph with the vertex set $V(G) \cup V(H)$ and the edge set $E(G) \cup E(H)$, i.e. the disjoint union of two graphs G and H. The complete product (join) $G \nabla H$ of graphs G and H is the graph obtained from $G + H$ by joining every vertex of G with every vertex of H. In particular, nG denotes $\underbrace{G + \cdots + G}_{n}$ and $\nabla_n G$ denotes $\underbrace{G \nabla \cdots \nabla G}_{n}$. The coalescence $G \cdot H$ is obtained by the disjoint union of two graphs G and H by identifying a vertex u of G with a vertex v of H.

For positive integers $n_1, \ldots, n_\ell$, $K_{n_1,\ldots,n_\ell}$ denotes the complete multipartite graph with ℓ parts of sizes $n_1, \ldots, n_\ell$. Let K_n denote the complete graph on n vertices, $nK_1 = \overline{K_n}$ denote the null graph on n vertices and P_n denote the path with n vertices. The cocktail party graph CP_n has $2n$ vertices and it is a complement of nK_2. So for $n = 1, CP_1 = K_{1,1}$ and for $n \geq 2$ we have $CP_n = K_{\underbrace{2,\ldots,2}_{n}}$.

Since CP_n and $K_{n,n,n}$ are regular graphs, by Propositions 3 and 6 of [9], CP_n and $K_{n,n,n}$ are determined by their spectrum. So we can compute the values of $\mathrm{cs}(CP_n)$ and $\mathrm{cs}(K_{n,n,n})$.

Our main results are as follows.

Theorem 1.1. *If $n \geq 2$ and $cs(CP_n) = \sigma(CP_n, H)$ for some graph H with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_{2n}$, then*

(1) *If H is a connected graph, then $2n-3 \leq \lambda_1 < 2n-1$. Otherwise $2n-3 \leq \lambda_1 < 2n-2$ and H has two connected components such that one of them is K_1.*

(2) $0 \leq \lambda_2 \leq 1$,

(3) $-1 \leq \lambda_i \leq \frac{1}{2}$, *for any integer i, $3 \leq i \leq n+1$, and if $n \geq 13$, then $0 \leq \lambda_3 \leq \frac{1}{2}$,*

(4) $-3 \leq \lambda_{n+2} \leq -1$,

(5) $-3 \leq \lambda_i \leq \frac{-3}{2}$, *for any integer i, $n+3 \leq i \leq 2n$.*

Theorem 1.2. *Let $n \geq 4$ and $cs(K_{n,n,n}) = \sigma(K_{n,n,n}, H)$ for some graph H with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_{3n}$. For all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have*

(1) $2n - \frac{\sqrt{3}}{3} - \frac{\varepsilon}{2} < \lambda_1 < 2n + \frac{\sqrt{3}}{3} + \frac{\varepsilon}{2}$,

(2) $\sqrt{2}-1 < \lambda_2 < \frac{\sqrt{3}}{3} + \frac{\varepsilon}{2}$ *or $\lambda_2 = 0$ and $H \cong tK_1 + K_{p,q,r}$ for some positive integers p, q and r such that at least one of them is greater than* 1,

(3) $0 \leq \lambda_3 < \frac{\sqrt{3}}{6} + \frac{\varepsilon}{4}$,

(4) $-\frac{\sqrt{3}}{3} - \frac{\varepsilon}{2} < \lambda_i < \frac{\sqrt{3}}{6} + \frac{\varepsilon}{4}$, *for any integer i, $4 \leq i \leq 3n-2$,*

(5) $-n - \frac{\sqrt{3}}{3} - \frac{\varepsilon}{2} < \lambda_{3n-1} < -n + \frac{\sqrt{3}}{3} + \frac{\varepsilon}{2}$,

(6) $-n - \frac{\sqrt{3}}{3} - \frac{\varepsilon}{2} < \lambda_{3n} < -n + \frac{\sqrt{3}}{6} + \frac{\varepsilon}{4}$.

2 Cospectrality of cocktail party graphs

In this section $\mathrm{cs}(CP_n)$ is investigated to the ℓ^1- and ℓ^2-norms. We need the following results in the sequel. The proofs of next two results are similar to those of Lemma 2.2 and Corollary 2.3 of [18]. We give them here for the reader's convenience.

Lemma 2.1. *Let $a_1 \geq a_2 \geq \cdots \geq a_n$ and $b_1 \geq b_2 \geq \cdots \geq b_n$ be two sequences with $\sum_{i=1}^n a_i = \sum_{i=1}^n b_i = 0$. If there exist some $1 \leq j \leq n$ and a real positive number α such that $|a_j - b_j| > \alpha$, then $\sum_{i=1}^n |a_i - b_i| > 2\alpha$.*

Proof. Without loss of generality, we may assume that $a_j - b_j > \alpha$. Suppose that $a_{i_1} \geq b_{i_1}, \ldots, a_{i_s} \geq b_{i_s}$ and $a_{i_{s+1}} \leq b_{i_{s+1}}, \ldots, a_{i_n} \leq b_{i_n}$, then

$$\begin{aligned}\sum_{i=1}^{n} |a_i - b_i| &= \sum_{t=1}^{s}(a_{i_t} - b_{i_t}) + \sum_{t=s+1}^{n}(b_{i_t} - a_{i_t}) \\ &= 2\sum_{t=1}^{s}(a_{i_t} - b_{i_t}) \\ &\geq 2(a_j - b_j) \\ &> 2\alpha. \end{aligned}$$

□

Corollary 2.2. *Let $a_1 \geq a_2 \geq \cdots \geq a_n$ and $b_1 \geq b_2 \geq \cdots \geq b_n$ be two sequences with $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i = 0$. If there exist $1 \leq j_1 \neq j_2 \leq n$ and a real positive number α such that $a_{j_1} - b_{j_1} + a_{j_2} - b_{j_2} > \alpha$, then $\sum_{i=1}^{n} |a_i - b_i| > 2\alpha$.*

Proof. If either $a_{j_1} - b_{j_1} > \alpha$ or $a_{j_2} - b_{j_2} > \alpha$, then by Lemma 2.1, the result holds. So we may assume that $0 < a_{j_1} - b_{j_1} \leq \alpha$ and $0 < a_{j_2} - b_{j_2} \leq \alpha$. Let $a'_{j_1} = a_{j_1} + a_{j_2}$, $b'_{j_1} = b_{j_1} + b_{j_2}$, $a'_i = a_i$ and $b'_i = b_i$ for $i \neq j_1, j_2$. So $\sum_{i=1, i\neq j_2}^{n} a'_i = \sum_{i=1, i\neq j_2}^{n} b'_i = 0$ and $a'_{j_1} - b'_{j_1} > \alpha$. Thus the result follows from Lemma 2.1. □

Theorem 2.3. *Let G be a graph with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$. If $cs(G) = \sigma(G, H)$ for some graph H with eigenvalues $\lambda'_1 \geq \cdots \geq \lambda'_n$, then for all integers i and j, $1 \leq j < i \leq n$,*

(1) $|\lambda_i - \lambda'_i| \leq 1$,

(2) $\lambda_i - \lambda'_j \leq \frac{1}{2}$.

Proof. By Theorem 1.1 of [1], $\mathrm{cs}_n = 2$ for all $n \geq 2$, so $\mathrm{cs}(G) \leq 2$. Now the result follows from Lemma 2.1 and Corollary 2.2. □

Theorem 2.4 ([5, Theorem 1])**.** *Let G be a simple graph of order n without isolated vertices. If $\lambda_2(G)$ is the second largest eigenvalue of G, then*

(1) $\lambda_2(G) = -1$ *if and only if G is a complete graph with at least two vertices,*

(2) $\lambda_2(G) = 0$ *if and only if G is a complete k-partite graph with $2 \leq k \leq n-1$,*

(3) *there exists no graph G such that $-1 < \lambda_2(G) < 0$.*

Theorem 2.5 ([21, Theorem 3.8])**.** *Let G be a graph of order n. If $\lambda_3(G) < 0$, then G has at least $n - 12$ eigenvalues -1.*

Theorem 2.6 ([7, Theorem 3.2.1])**.** *Let λ_1 be the greatest eigenvalue of the graph G, and let $\overline{d}$ and Δ be its average degree and maximum degree, respectively. Then*

$$\overline{d} \leq \lambda_1 \leq \Delta.$$

Moreover, $\overline{d} = \lambda_1$ if and only if G is regular. For a connected graph G, $\lambda_1 = \Delta$ if and only if G is regular.

Proof of Theorem 1.1. Since

$$Spec(CP_n) = \{2n-2, \underbrace{0,\ldots,0}_{n}, \underbrace{-2,\ldots,-2}_{n-1}\},$$

we have

$$\sigma(CP_n, H) = |2n-2-\lambda_1| + \sum_{i=2}^{n+1} |\lambda_i| + \sum_{i=n+2}^{2n} |2+\lambda_i|.$$

If cs$(CP_n) = \sigma(CP_n, H)$, then by Theorem 1.1 of [1], cs$(CP_n) \le 2$. By Theorems 2.3, 2.4, 2.5 and Corollary 2.2, we obtain (2) – (5) and $2n-3 \le \lambda_1 \le 2n-1$.

If H is a connected graph and $\lambda_1 = 2n-1$, then by Theorem 2.6, $H \cong K_{2n}$, a contradiction. So $2n-3 \le \lambda_1 < 2n-1$. Now suppose that H is not connected. Let $H_1,\ldots,H_k$ be the connected components of H. There exists an unique i, $1 \le i \le k$, such that $\lambda_1(H) = \lambda_1(H_i)$. We can assume that $\lambda_1(H) = \lambda_1(H_1)$. Thus $\lambda_1(H_j) \le \lambda_2(H) \le 1$, for every j, $2 \le j \le k$. So $\lambda_1(H_j) = 0$ or $\lambda_1(H_j) = 1$, $2 \le j \le k$. Since $-1 \le \lambda_3(H) \le \frac{1}{2}$, there exists at most one connected component with $\lambda_1(H_j) = 1$, $2 \le j \le k$. Therefore $H \cong H_1 + tK_1$ or $H \cong H_1 + K_2 + sK_1$, for some integers $t > 0$ and $s \ge 0$. By Theorem 2.6, $2n-3 \le \lambda_1(H) = \lambda_1(H_1) \le \Delta \le 2n-1$, where Δ is the maximum degree of the vertices of H. If $\Delta = 2n-1$, then, by Theorem 2.6, $H_1 \cong K_{2n}$, a contradiction. Let $\Delta = 2n-3$. Therefore by Theorem 2.6, $H_1 \cong K_{2n-2}$, a contradiction. Now suppose that $\Delta = 2n-2$. If $\lambda_1(H_1) = 2n-2$, then by Theorem 2.6, $H_1 \cong K_{2n-1}$, a contradiction. Hence we can assume that $H \cong H_1 + K_1$ and $2n-3 \le \lambda_1(H) < 2n-2$. This completes the proof. □

Remark 2.7. Let H be a connected graph with m edges. If cs$(CP_n) = \sigma(CP_n, H)$, then, by Theorem 1.1 and Theorem 1 in [26], it is not hard to see that $2n^2 - 5n + 4 \le m < 2n^2 - n$.

Now we find $\sigma(CP_n, (CP_{n-1} \bigtriangledown K_1)\cdot K_2)$ and $\lambda(CP_n, CP_n \setminus e)$ and propose two conjectures. We need the following results.

Theorem 2.8 ([7, Theorem 2.1.8]). *If G_1 is r_1-regular with n_1 vertices, and G_2 is r_2-regular with n_2 vertices, then the characteristic polynomial of the join $G_1 \bigtriangledown G_2$ is given by*

$$P_{G_1 \bigtriangledown G_2}(x) = \frac{P_{G_1}(x)P_{G_2}(x)}{(x-r_1)(x-r_2)}((x-r_1)(x-r_2) - n_1 n_2).$$

Theorem 2.9 ([7, Theorem 2.2.3]). *Let $G \cdot H$ be the coalescence in which the vertex u of G is identified with the vertex v of H. Then*

$$P_{G\cdot H}(x) = P_G(x)P_{H-v}(x) + P_{G-u}(x)P_H(x) - xP_{G-u}(x)P_{H-v}(x).$$

Lemma 2.10. *If $(CP_{n-1} \bigtriangledown K_1)\cdot K_2$ is the coalescence of K_2 with $CP_{n-1} \bigtriangledown K_1$ with its vertex of maximum degree as distinguished vertex, then for $n \ge 3$,*

$$Spec((CP_{n-1} \bigtriangledown K_1)\cdot K_2) = \{x_1, x_2, \underbrace{0,\ldots,0}_{n-1}, x_3, \underbrace{-2,\cdots,-2}_{n-2}\},$$

such that $x_1 > x_2 > 0 > x_3$ are the roots of the polynomial $x^3 + (4-2n)x^2 + (1-2n)x + 2n - 4$.

Proof. Since $P_{CP_{n-1}}(x) = x^{n-1}(x+2)^{n-2}(x-2n+4)$ and $P_{K_1}(x) = x$, Theorem 2.8 implies that

$$P_{CP_{n-1}\triangledown K_1}(x) = x^{n-1}(x+2)^{n-2}(x^2+(4-2n)x+2-2n).$$

Since $P_{K_2}(x) = x^2-1$, it follows from Theorem 2.9,

$$P_{(CP_{n-1}\triangledown K_1)\cdot K_2}(x) = x^{n-1}(x+2)^{n-2}(x^3+(4-2n)x^2+(1-2n)x+2n-4).$$

Thus $(CP_{n-1} \triangledown K_1)\cdot K_2$ has $n-1$ and $n-2$ eigenvalues 0 and -2, respectively. The remaining eigenvalues are the roots of the polynomial $x^3+(4-2n)x^2+(1-2n)x+2n-4$. If

$$\begin{aligned}
a &= \Big(8n^3-30n^2+24n+8+3(-60n^4+312n^3-648n^2+606n-237)^{\frac{1}{2}}\Big)^{\frac{1}{3}},\\
b &= -\frac{4}{9}n^2+\frac{10}{9}n-\frac{13}{9},\\
r &= \big((8n^3-30n^2+24n+8)^2+540n^4-2808n^3+5832n^2-5454n+2133\big)^{\frac{1}{6}},\\
\theta &= \frac{1}{3}\arctan\left(\frac{3(60n^4-312n^3+648n^2-606n+237)^{\frac{1}{2}}}{8n^3-30n^2+24n+8}\right).
\end{aligned}$$

Then

$$\begin{aligned}
x_1 &= \frac{2n}{3}-\frac{4}{3}+\frac{a}{3}-\frac{3b}{a},\\
x_2 &= \frac{2n}{3}-\frac{4}{3}+(\frac{3b}{2r}-\frac{r}{6})\cos\theta-\sqrt{3}(\frac{3b}{2r}-\frac{r}{6})\sin\theta,\\
x_3 &= \frac{2n}{3}-\frac{4}{3}+(\frac{3b}{2r}-\frac{r}{6})\cos\theta+\sqrt{3}(\frac{3b}{2r}-\frac{r}{6})\sin\theta.
\end{aligned}$$

This completes the proof. □

Lemma 2.11. $\lim_{n\longrightarrow\infty}\sigma\big(CP_n,(CP_{n-1}\triangledown K_1)\cdot K_2\big) = 2$, *whenever* $(CP_{n-1}\triangledown K_1)\cdot K_2$ *is the coalescence of* K_2 *with* $CP_{n-1}\triangledown K_1$ *with its vertex of maximum degree as distinguished vertex.*

Proof. By Lemma 2.10 and using the symbolic computational software Maple [19] (see https://data.amc-journal.eu/cospectrality/maplecode1.mw), the result follows. □

Theorem 2.12 ([7, Theorem 2.1.5])**.** *Let G, H be graphs with n_1, n_2 vertices respectively. The characteristic polynomial of the join $G\triangledown H$ is given by the relation*

$$\begin{aligned}
P_{G\triangledown H}(x) = (-1)^{n_2}P_G(x)P_{\overline{H}}(-x-1) &+ (-1)^{n_1}P_H(x)P_{\overline{G}}(-x-1)\\
&- (-1)^{n_1+n_2}P_{\overline{G}}(-x-1)P_{\overline{H}}(-x-1).
\end{aligned}$$

Lemma 2.13. *For $n\geq 3$ and any edge e,*

$$Spec(CP_n\setminus e) = \left\{x_1, \frac{\sqrt{5}-1}{2}, \underbrace{0,\dots,0}_{n-2}, x_2, -\frac{\sqrt{5}+1}{2}, \underbrace{-2,\dots,-2}_{n-3}, x_3\right\},$$

where $x_1 > 0 > x_2 > x_3$ are the roots of the polynomial $x^3-(2n-5)x^2-(6n-9)x-2n+2$.

Proof. For any edge e, $CP_n \setminus e = P_4 \triangledown CP_{n-2}$. Let $G = P_4$ and $H = CP_{n-2}$. Thus $\overline{G} = G$ and $\overline{H} = (n-2)K_2$. We have

$$P_G(x) = P_{\overline{G}}(x) = x^4 - 3x^2 + 1,$$
$$P_H(x) = (x - 2n + 6)x^{n-2}(x+2)^{n-3},$$
$$P_{\overline{H}}(x) = (x^2-1)^{n-2}.$$

Therefore

$$P_{CP_n \setminus e} = P_{G \triangledown H}(x) = x^{n-2}(x+2)^{n-3}(x^2+x-1)(x^3-(2n-5)x^2-(6n-9)x-2n+2).$$

It follows $CP_n \setminus e$ has $n-2$ and $n-3$ eigenvalues 0 and -2, respectively. The remaining eigenvalues are $\frac{\sqrt{5}-1}{2}$, $-\frac{\sqrt{5}+1}{2}$ and the roots of $x^3 - (2n-5)x^2 - (6n-9)x - 2n + 2$. If

$$a = \big(64n^3 - 48n^2 - 312n + 404 + 12(-240n^4 + 528n^3 + 396n^2 - 1740n + 1137)^{\frac{1}{2}}\big)^{\frac{1}{3}},$$
$$b = -\frac{4}{9}n^2 + \frac{2}{9}(n+1),$$
$$r = \big((64n^3 - 48n^2 - 312n + 404)^2 + 34560n^4 - 76032n^3 - 57024n^2 + 250560n - 163728\big)^{\frac{1}{6}},$$
$$\theta = \frac{1}{3}\arctan\left(\frac{12(240n^4 - 528n^3 - 396n^2 + 1740n - 1137)^{\frac{1}{2}}}{64n^3 - 48n^2 - 312n + 404}\right).$$

Then

$$x_1 = \frac{2n}{3} - \frac{5}{3} + \frac{a}{6} - \frac{6b}{a},$$
$$x_2 = \frac{2n}{3} - \frac{5}{3} + (\frac{3b}{r} - \frac{r}{12})\cos\theta - \sqrt{3}(\frac{3b}{r} - \frac{r}{12})\sin\theta,$$
$$x_3 = \frac{2n}{3} - \frac{5}{3} + (\frac{3b}{r} - \frac{r}{12})\cos\theta + \sqrt{3}(\frac{3b}{r} - \frac{r}{12})\sin\theta,$$

and we are done. □

Lemma 2.14. $\lim_{n\longrightarrow\infty} \lambda(CP_n, CP_n \setminus e) = 10 - 4\sqrt{5}$.

Proof. By Lemma 2.13 and using the symbolic computational software Maple [19] (see https://data.amc-journal.eu/cospectrality/maplecode2.mw), the result follows. □

We have the following conjectures:

Conjecture 2.15. *For every integer $n \geq 2$, cs$(CP_n) = \sigma(CP_n, H)$ for some graph H if and only if $H \cong (CP_{n-1} \triangledown K_1) \cdot K_2$, whenever $(CP_{n-1} \triangledown K_1) \cdot K_2$ is the coalescence of K_2 with $CP_{n-1} \triangledown K_1$ with its vertex of maximum degree as distinguished vertex.*

Conjecture 2.16. *For every integer $n \geq 4$, cs$(CP_n) = \lambda(CP_n, H)$ for some graph H if and only if $H \cong CP_n \setminus e$, for any edge e.*

For $n = 2$ and $n = 3$, cs$(CP_n) = \lambda(CP_n, H)$ if and only if $H \cong (CP_{n-1} \triangledown K_1) \cdot K_2$. Our computational results confirm Conjectures 2.15 and 2.16 for all graphs of order at most 10.

3 Cospectrality of complete tripartite graphs

In this section we investigate $\mathrm{cs}(K_{n,n,n})$, for $n \geq 3$, to the ℓ^1- and ℓ^2-norms. First we need the following results.

Theorem 3.1 ([12, Theorem 9.1.1]). *Let G be a graph of order n and H be an induced subgraph of G with order m. Suppose that $\lambda_1(G) \geq \cdots \geq \lambda_n(G)$ and $\lambda_1(H) \geq \cdots \geq \lambda_m(H)$ are the eigenvalues of G and H, respectively. Then for every i, $1 \leq i \leq m$, $\lambda_i(G) \geq \lambda_i(H) \geq \lambda_{n-m+i}(G)$.*

Theorem 3.2 (See [23] and also [8, Theorem 6.7]). *A graph has exactly one positive eigenvalue if and only if its non-isolated vertices form a complete multipartite graph.*

Lemma 3.3 ([22, Lemma 7]). $\lambda_2((K_1 + K_{r,s})\nabla\overline{K_q}) \leq \sqrt{2} - 1$ $(r \leq s)$ *if and only if one of the conditions* $1 - 10$ *holds:*

(1) $r > 1, s \geq r, q = 1;$

(2) $r = 1, s \geq 1, q \geq 2;$

(3) $r = 2, s \geq 2, q = 2;$

(4) $r = 2, 2 \leq s \leq 3, q \geq 3;$

(5) $r = 2, s = 4, 3 \leq q \leq 7;$

(6) $r = 2, s = 5, 3 \leq q \leq 4;$

(7) $r = 2, 6 \leq s \leq 8, q = 3;$

(8) $r = 3, s = 3, 2 \leq q \leq 4;$

(9) $r = 3, 4 \leq s \leq 7, q = 2;$

(10) $r = 4, s = 4, q = 2.$

Lemma 3.4 ([22, Lemma 8]). $\lambda_2((K_1 + K_{r,s})\nabla K_{p,q}) \leq \sqrt{2} - 1$ $(r \leq s,\ p \leq q)$ *if and only if one of the conditions* $1 - 5$ *holds:*

(1) $r = 1, s = 1, p \geq 1, q \geq p;$

(2) $r = 1, s = 2, 1 \leq p \leq 2, q \leq p;$

(3) $r = 1, s = 2, p = 3, 3 \leq q \leq 7;$

(4) $r = 1, s = 2, p = 4, q = 4;$

(5) $r = 1, s = 3, p = 1, q = 1.$

Theorem 3.5 ([22, Theorem]). *Let G be a graph without isolated vertices and let $\lambda_2(G)$ be the second largest eigenvalue of G. Then $0 < \lambda_2(G) \leq \sqrt{2} - 1$ if and only if one of the following holds:*

(1) $G \cong (\nabla_t(K_1 + K_2))\nabla K_{n_1,\ldots,n_m}$,

(2) $G \cong (K_1 + K_{r,s})\nabla\overline{K_q}$, *and parameters q, r and s satisfy one of the conditions* (1) – (10) *from* Lemma 3.3,

(3) $G \cong (K_1 + K_{r,s})\nabla K_{p,q}$, *and parameters p, q, r and s satisfy one of the conditions* (1) – (5) *from* Lemma 3.4.

Lemma 3.6. *Let $n \geq 3$ and $x_1 > 0 > x_2 > x_3$ be the roots of the polynomial $x^3 - (3n^2 - 1)x - 2n^3 + 2n$. Then*

$$Spec(K_{n-1,n,n+1}) = \{x_1, \underbrace{0, \ldots, 0}_{3n-3}, x_2, x_3\}.$$

Proof. Since $P_{K_{n_1,\ldots,n_k}}(x) = x^{\sum_{i=1}^{k} n_i - k}\left(1 - \sum_{i=1}^{k} \frac{n_i}{x+n_i}\right)\prod_{i=1}^{k}(x+n_i)$,

$$P_{K_{n-1,n,n+1}}(x) = x^{3n-3}(x^3 - (3n^2-1)x - 2n^3 + 2n).$$

Thus $K_{n-1,n,n+1}$ has $3n-3$ eigenvalues 0 and 3 eigenvalues

$$\begin{aligned} x_1 &= \frac{a^2 + 9n^2 - 3}{3a}, \\ x_2 &= \left(\frac{-r}{6} + \frac{1-3n^2}{2r}\right)\cos\theta - \sqrt{3}\left(\frac{-r}{6} + \frac{1-3n^2}{2r}\right)\sin\theta, \\ x_3 &= \left(\frac{-r}{6} + \frac{1-3n^2}{2r}\right)\cos\theta + \sqrt{3}\left(\frac{-r}{6} + \frac{1-3n^2}{2r}\right)\sin\theta, \end{aligned}$$

where

$$\begin{aligned} a &= \left(27n^3 - 27n + 3(-81n^4 + 54n^2 + 3)^{\frac{1}{2}}\right)^{\frac{1}{3}}, \\ r &= \left((27n^3 - 27n)^2 + 729n^4 - 486n^2 - 27\right)^{\frac{1}{6}}, \\ \theta &= \frac{1}{3}\arctan\left(\frac{(81n^4 - 54n^2 - 3)^{\frac{1}{2}}}{9n^3 - 9n}\right). \end{aligned}$$ □

Lemma 3.7. $\lim_{n\longrightarrow\infty} \sigma(K_{n,n,n}, K_{n-1,n,n+1}) = \frac{2\sqrt{3}}{3}$.

Proof. Since $Spec(K_{n,n,n}) = \{2n, \underbrace{0,\ldots,0}_{3n-3}, -n, -n\}$, by Lemma 3.6 and using the computational software Maple [19] (see https://data.amc-journal.eu/cospectrality/maplecode3.mw), the result follows. □

Proof of Theorem 1.2. Note that

$$\sigma(K_{n,n,n}, H) = |2n - \lambda_1| + \sum_{i=2}^{3n-2} |\lambda_i| + |n + \lambda_{3n-1}| + |n + \lambda_{3n}|.$$

By Lemma 3.7, for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $\mathrm{cs}(K_{n,n,n}) < \frac{2\sqrt{3}}{3} + \varepsilon$. By Lemma 2.1, Corollary 2.2, Theorems 2.4 and 2.5, we obtain (1), (3) – (6) and $0 \leq \lambda_2 < \frac{\sqrt{3}}{3} + \frac{\varepsilon}{2}$. Suppose that $0 < \lambda_2 \leq \sqrt{2} - 1$. Hence Theorem 3.5 can be applied.
Case 1: $H \cong (\nabla_t(K_1 + K_2))\nabla K_{n_1,\ldots,n_m}$. If $t \geq 2$, then $(K_1 + K_2)\nabla(K_1 + K_2)$ is an induced subgraph of H. Since

$$Spec((K_1 + K_2)\nabla(K_1 + K_2)) = \{3.73205, .41421, .26795, -1, -1, -2.41421\},$$

by Theorem 3.1, $\lambda_{3n-2} \leq -1$, a contradiction. Now, suppose that $t = 1$. If $m = 1$, then $H \cong (K_1 + K_2)\nabla\overline{K_{3n-3}}$. We have $P_H(x) = x^{3n-4}f(x)$, whenever $f(x) =$

$x^4-(9n-8)x^2-(6n-6)x+3n-3$. So the non-zero eigenvalues of H are the roots of $f(x)=0$. By computing the roots, it implies that $\lambda_{3n-1}=-1$, a contradiction. Therefore $m\geq 2$. If $n_1=\cdots=n_m=1$, then $H\cong(K_1+K_2)\nabla K_{3n-3}$. So $(K_1+K_2)\nabla K_2$ is an induced subgraph of H. Since

$$Spec((K_1+K_2)\nabla K_2)=\{3.32340,.35793,-1,-1,-1.68133\},$$

by Theorem 3.1, $\lambda_{3n-2}\leq -1$, a contradiction. Now, we can assume that $n_i\geq 2$, for some $1\leq i\leq m$. Thus $(K_1+K_2)\nabla K_{1,2}$ is an induced subgraph of H. Since

$$Spec((K_1+K_2)\nabla K_{1,2})=\{4.06779,.36162,0,-1,-1.24464,-2.18477\},$$

by Theorem 3.1, $\lambda_{3n-2}\leq -1$, a contradiction.
Case 2: $H\cong(K_1+K_{r,s})\nabla\overline{K_q}$ and parameters q,r and s satisfy conditions 1–10 from Lemma 3.3. We have $P_H(x)=x^{3n-4}f(x)$ whenever $f(x)=x^4-(q+qr+qs+rs)x^2-2qrsx+qrs$. The non-zero eigenvalues of H are determined by equation $f(x)=0$. By computing the roots, we have $\lambda_1=-\lambda_{3n}$ and $\lambda_2=-\lambda_{3n-1}$, a contradiction.
Case 3: $H\cong(K_1+K_{r,s})\nabla K_{p,q}$, and parameters p,q,r and s satisfy conditions 1–5 from Lemma 3.4. In this case, H can be isomorphic to one of these graphs: $(K_1+K_{1,2})\nabla K_{3,5}$, $(K_1+K_{1,2})\nabla K_{4,4}$ and $(K_1+K_{1,1})\nabla K_{p,q}$ whenever $q\geq p\geq 1$ and $p+q=3n-3$. All of these graphs have $(K_1+K_{1,1})\nabla K_{1,2}$ as an induced subgraph. Since

$$Spec((K_1+K_{1,1})\nabla K_{1,2})=\{4.06779,.36162,0,-1,-1.24464,-2.18477\},$$

by Theorem 3.1, $\lambda_{3n-2}\leq -1$, a contradiction.

So $\sqrt{2}-1<\lambda_2<\frac{\sqrt{3}}{3}+\frac{\varepsilon}{2}$ or $\lambda_2=0$. If $\lambda_2=0$, then, by Theorem 3.2, there are some positive integers $k,n_1,\ldots,n_k$ and an integer $t\geq 0$ such that $H\cong tK_1+K_{n_1,\ldots,n_k}$. If $k=1$, then $H\cong\overline{K_{3n}}$, a contradiction. If $k=2$, then $H\cong tK_1+K_{r,s}$. Since

$$Spec(H)=\{\sqrt{rs},\underbrace{0,\ldots,0}_{3n-2},-\sqrt{rs}\},$$

$\lambda_{3n-1}=0$, a contradiction. Thus $k\geq 3$. Suppose that $k\geq 4$. If $n_1=\cdots=n_k=1$, then $H\cong tK_1+K_{3n-t}$. We have

$$Spec(H)=\{3n-t-1,\underbrace{0,\ldots,0}_{t},\underbrace{-1,\ldots,-1}_{3n-t-1}\}.$$

Hence $\lambda_{3n}=-1$, a contradiction. If there exists an unique i, $1\leq i\leq k$, such that $n_i\geq 2$, then $K_{1,1,1,2}$ is an induced subgraph of H. Since

$$Spec(K_{1,1,1,2})=\{3.64575,0,-1,-1,-1.64575\},$$

by Theorem 3.1, $\lambda_{3n-2}\leq -1$, a contradiction. Thus there exist i and j such that $n_i,n_j\geq 2$. Hence H has $K_{1,1,2,2}$ as an induced subgraph. We have

$$Spec(K_{1,1,2,2})=\{4.37228,0,0,-1,-1.37228,-2\}.$$

So by Theorem 3.1, $\lambda_{3n-2}\leq -1$, a contradiction. Therefore we can assume that $k=3$ and $H\cong tK_1+K_{p,q,r}$, for some positive integers p,q and r. If $p=q=r=1$, then, by similar argument given in $k\geq 4$, we have $\lambda_{3n}=-1$, a contradiction. So $H\cong tK_1+K_{p,q,r}$ such that at least one of p,q and r is greater than 1. This completes the proof. □

Lemma 3.8. $\lim_{n\longrightarrow\infty} \lambda(K_{n,n,n}, K_{n-1,n,n+1}) = \frac{2}{3}$.

Proof. By Lemma 3.6 and using the symbolic computational software Maple [19] (see https://data.amc-journal.eu/cospectrality/maplecode4.mw), the result follows. □

The graph H in Figure 1 is the only unique graph such that $\sigma(K_{3,3,3}, H)$ and $\lambda(K_{3,3,3}, H)$ have the minimum possible values. For $n \geq 4$, we have the following conjectures:

Conjecture 3.9. *For every integer $n \geq 4$, cs$(K_{n,n,n}) = \sigma(K_{n,n,n}, H)$ for some graph H if and only if $H \cong K_{n-1,n,n+1}$.*

Conjecture 3.10. *For every integer $n \geq 4$, cs$(K_{n,n,n}) = \lambda(K_{n,n,n}, H)$ for some graph H if and only if $H \cong K_{n-1,n,n+1}$.*

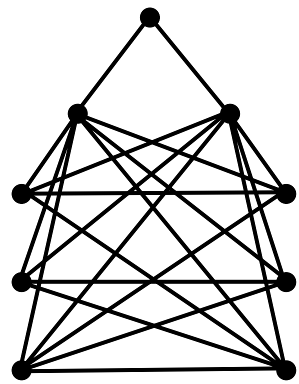

Figure 1: The graph which is closest to $K_{3,3,3}$ both in the ℓ^1- and ℓ^2-norm.

4 Cospectrality of some families of graphs using ℓ^p-norm for $p > 2$

Let $p > 2$ be an arbitrary positive integer. First we determine the cospectrality of the null graphs on n vertices.

Theorem 4.1. *For every integer $n \geq 2$, cs$(nK_1) = 2$. Moreover, cs$(nK_1) = \lambda^{(p)}(nK_1, H)$ for some graph H if and only if $H \cong K_2 + (n-2)K_1$.*

Proof. It is not hard to see that $\lambda^{(p)}(nK_1, K_2 + (n-2)K_1) = 2$. Let H be a simple graph of order n. Thus $\text{cs}(nK_1) = \lambda^{(p)}(nK_1, H) \leq 2$. So $|\lambda_1(H)| \leq \sqrt[p]{2}$, where $\lambda_1(H)$ is the greatest eigenvalue of H. Since the greatest eigenvalue of a graph is always non-negative and $H \ncong nK_1$, we have $0 < \lambda_1(H) \leq \sqrt[p]{2}$. Moreover, there is no graph whose greatest eigenvalue lies in the intervals $(0, 1)$ and $(1, \sqrt{2})$. Hence $\lambda_1(H) = 1$. Thus $H \cong K_2 + (n-2)K_1$. □

In the following we show that the minimum value of $\lambda^{(p)}(K_n, H)$ occurs whenever $H \cong K_n \setminus e$, where $K_n \setminus e$ is the graph obtaining from K_n by deletion one edge e. First we need the following results.

Lemma 4.2. *$\lambda^{(p)}(K_2, K_2 \setminus e) = 2$ and for every integer $n \geq 3$ and every edge e of K_n, $\lambda^{(p)}(K_n, K_n \setminus e) < 2$.*

Proof. It is easy to see that $\lambda^{(p)}(K_2, K_2 \setminus e) = 2$. By Corollary 3.4 and Lemma 3.6 in [2], one can obtain the result. □

Theorem 4.3. *For every integer $n \geq 2$, $cs(K_n) = \lambda^{(p)}(K_n, H)$ for some graph H if and only if $H \cong K_n \setminus e$ for any edge e, where $K_n \setminus e$ is the graph obtaining from K_n by deletion one edge e.*

Proof. For $n = 2$ and $n = 3$, It is easy to see that $\mathrm{cs}(K_n) = \lambda^{(p)}(K_n, K_n \setminus e)$. Let $n \geq 4$. We show that if H is not isomorphic to K_n and $K_n \setminus e$, then $\lambda^{(p)}(K_n, H) \geq 2$.

Let $\lambda_1 \geq \cdots \geq \lambda_n$ be the eigenvalues of H. Therefore

$$\lambda^{(p)}(K_n, H) = |\lambda_1 - n + 1|^p + \sum_{i=2}^{n} |\lambda_i + 1|^p.$$

One can obtain this if one of the following cases holds, then $\lambda^{(p)}(K_n, H) \geq 2$.

Case 1: $\lambda_1 - n + 1 \leq -\sqrt[3]{2}$.

Case 2: $\lambda_2 + 1 \geq \sqrt[3]{2}$.

Case 3: $\lambda_3 \geq 0$.

Now suppose that none of the above cases occurs. Thus we can assume that $\lambda_1 > n - 1 - \sqrt[3]{2}$, $\lambda_2 < \sqrt[3]{2} - 1$ and $\lambda_3 < 0$. If $\lambda_2 \leq 0$, then, by Lemma 3.9 in [2], $H \cong K_{n-1} + K_1$ and $\lambda^{(p)}(K_n, H) = 2$.

Now suppose that $\lambda_2 > 0$. Since $0 < \lambda_2 < \sqrt[3]{2} - 1 < \frac{1}{3}$, by Theorem 2 in [5], there exists an integer t such that $H \cong tK_1 + (K_1 + K_2)\nabla\overline{K_{n-3-t}}$ where $0 \leq t \leq n - 4$.

If $n - 3 - t > 1$, then $(K_1 + K_2)\nabla\overline{K_2}$ is an induced subgraph of H. Since

$$Spec((K_1 + K_2)\nabla\overline{K_2}) = \{2.85577, 0.32164, 0, -1, -2.17741\},$$

by Theorem 3.1, $\lambda_3 \geq 0$, a contradiction. If $n - 3 - t = 1$, then $H \cong (n - 4)K_1 + (K_1 + K_2)\nabla\overline{K_1}$. Since

$$Spec(H) = \{2.17009, 0.31111, \underbrace{0, \ldots, 0}_{n-4}, -1, -1.48119\},$$

$\lambda^{(p)}(K_n, H) > 2$. Therefore by Lemma 4.2, $\mathrm{cs}(K_n) = \lambda^{(p)}(K_n, K_n \setminus e)$. This completes the proof. □

In the following, we investigate the cospectrality of complete bipartite graphs. The proofs of Lemmas 2.5 and 2.7 and Theorem 2.8 in [20] are also working for $p > 2$, an arbitrary positive integer. First we need the following results, the "$\ell^p-version$" of Lemmas 2.5 and 2.7 in [20].

Lemma 4.4. *Let m and n be two positive integers and G be a graph of order $m + n$. If G has $K_{1,1,2}$ or $(K_1 + K_2)\nabla K_1$ as an induced subgraph, then $\lambda^{(p)}(G, K_{m,n}) \geq 1$.*

Lemma 4.5. *Let m and n be two positive integers and G be a graph of order $m+n$. Suppose that there are no positive integers r, s and a non-negative integer t such that $G \cong K_{r,s} + tK_1$. If $\lambda_2(G) \leq \sqrt{2}-1$, then $\lambda^{(p)}(G, K_{m,n}) \geq 1$.*

Theorem 4.6. *Let m and n be two positive integers such that $(m,n) \neq (1,1)$. Then*

$$cs(K_{m,n}) = \lambda^{(p)}(K_{m,n}, K_{r,s} + tK_1),$$

for some integers $r, s \geq 1$ and $t \geq 0$ such that $r+s+t = m+n$ and $r, s \neq m, n$. Moreover, if $cs(K_{m,n}) = \lambda^{(p)}(K_{m,n}, H)$ for some graph H, then $H \cong K_{i,j} + hK_1$, where $i, j \geq 1$ and $h \geq 0$ are some integers so that $i+j+h = m+n$.

Proof. It is easy to see that $\mathrm{cs}(K_{1,2}) = \lambda^{(p)}(K_{1,2}, K_{1,1} + K_1)$. So we can assume that $m+n \geq 4$. Let $i, j \geq 1$ and $h \geq 0$ be some integers such that $i+j+h = m+n$. Thus $\lambda^{(p)}(K_{m,n}, K_{i,j} + hK_1) = 2|\sqrt{mn} - \sqrt{ij}|^p$. By Lemma 2.4 in [20], there are some positive integers r and s such that $r+s \leq m+n$ and $\{r,s\} \neq \{m,n\}$ so that $|\sqrt{mn} - \sqrt{rs}|^p < (\frac{\sqrt{2}-1}{\sqrt{2}})^p$. Let $t = m+n-r-s$. Hence we obtain $\lambda^{(p)}(K_{m,n}, K_{r,s} + tK_1) < (\sqrt{2}-1)^p$. Therefore $\mathrm{cs}(K_{m,n}) < (\sqrt{2}-1)^p < 1$. Now suppose that H is a graph such that $\mathrm{cs}(K_{m,n}) = \lambda^{(p)}(K_{m,n}, H)$. Thus $\lambda^{(p)}(K_{m,n}, H) < (\sqrt{2}-1)^p$. Let $\lambda_2(H)$ be the second largest eigenvalue of H. So we have $|\lambda_2(H)| < \sqrt{2}-1$. Since $\lambda^{(p)}(K_{m,n}, H) < 1$, by Lemma 4.5, there are some integers $r, s \geq 1$ and $t \geq 0$ such that $H \cong K_{r,s} + tK_1$. This completes the proof. □

Theorem 4.7. *Let $n \geq 1$ be an integer. Then, the following hold:*

(1) *$cs(K_{1,1}) = \lambda^{(p)}(K_{1,1}, 2K_1) = 2$,*

(2) *$cs(K_{1,2}) = \lambda^{(p)}(K_{1,2}, K_{1,1} + K_1) = 2|\sqrt{2}-1|^p$,*

(3) *If $n \geq 3$ is a prime number, then*

$$cs(K_{1,n}) = \lambda^{(p)}(K_{1,n}, K_{2,\frac{n+1}{2}} + \frac{n-3}{2}K_1) = 2|\sqrt{n+1} - \sqrt{n}|^p,$$

(4) *If $n \geq 3$ is not a prime number, then*

$$cs(K_{1,n}) = \lambda^{(p)}(K_{1,n}, K_{r,s} + (n+1-r-s)K_1) = 0,$$

where r and s are some positive integers such that $r, s < n$ and $n = rs$.

Proof. The method is similar to that of Theorem 2.10 in [20]. □

By Theorem 4.6, one can easily obtain the following results.

Theorem 4.8. *For every integer $n \geq 2$, $cs(K_{n,n}) = 2|n - \sqrt{n^2-1}|^p$. Moreover, $cs(K_{n,n})$ $= \lambda^{(p)}(K_{n,n}, H)$ for some graph H if and only if $H \cong K_{n-1,n+1}$.*

Theorem 4.9. *For every integer $n \geq 2$, $cs(K_{n,n+1}) = 2|\sqrt{n^2+n} - \sqrt{n^2+n-2}|^p$. Moreover, $cs(K_{n,n+1}) = \lambda^{(p)}(K_{n,n+1}, H)$ for some graph H if and only if $H \cong K_{n-1,n+2}$.*

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References

[1] A. Abdollahi, S. Janbaz and M. R. Oboudi, Distance between spectra of graphs, *Linear Algebra Appl.* **466** (2015), 401–408, doi:10.1016/j.laa.2014.10.020.

[2] A. Abdollahi and M. R. Oboudi, Cospectrality of graphs, *Linear Algebra Appl.* **451** (2014), 169–181, doi:10.1016/j.laa.2014.02.052.

[3] J. Abdollahi, A. and N. Zakeri, ℓ^1-cospectrality of graphs, 2019, `arXiv:1907.11874 [math.CO]`.

[4] M. Afkhami, M. Hassankhani and K. Khashyarmanesh, Distance between the spectra of graphs with respect to normalized Laplacian spectra, *Georgian Math. J.* **26** (2019), 227–234, doi:10.1515/gmj-2017-0051.

[5] D. S. Cao and H. Yuan, Graphs characterized by the second eigenvalue, *J. Graph Theory* **17** (1993), 325–331, doi:10.1002/jgt.3190170307.

[6] H. Choi, H. Lee, Y. Shen and Y. Shi, Comparing large-scale graphs based on quantum probability theory, *Appl. Math. Comput.* **358** (2019), 1–15, doi:10.1016/j.amc.2019.03.061.

[7] D. Cvetković, P. Rowlinson and S. Simić, *An introduction to the theory of graph spectra*, volume 75 of *London Mathematical Society Student Texts*, Cambridge University Press, Cambridge, 2010.

[8] D. M. Cvetković, M. Doob and H. Sachs, *Spectra of Graphs, Theory and Application*, Academic Press, Inc., New York, 1979.

[9] E. R. Van Dam and W. H. Haemers, Which graphs are determined by their spectrum?, *Linear Algebra Appl.* **373** (2003), 241–272, doi:10.1016/s0024-3795(03)00483-x.

[10] K. C. Das and S. Sun, Distance between the normalized Laplacian spectra of two graphs, *Linear Algebra Appl.* **530** (2017), 305–321, doi:10.1016/j.laa.2017.05.025.

[11] M. Ghorbani and M. Hakimi-Nezhaad, Co-spectrality distance of graphs, *J. Inf. Optim. Sci.* **40** (2019), 1221–1235, doi:10.1080/02522667.2018.1480464.

[12] C. Godsil and G. Royle, *Algebraic graph theory*, volume 207 of *Graduate Texts in Mathematics*, Springer-Verlag, New York, 2001, doi:10.1007/978-1-4613-0163-9.

[13] J. Gu, B. Hua and S. Liu, Spectral distances on graphs, *Discrete Appl. Math.* **190/191** (2015), 56–74, doi:10.1016/j.dam.2015.04.011.

[14] M. Hakimi-Nezhaad and A. R. Ashrafi, Laplacian and normalized Laplacian spectral distances of graphs, *Southeast Asian Bull. Math.* **37** (2013), 731–744.

[15] I. Jovanović and Z. Stanić, Spectral distances of graphs, *Linear Algebra Appl.* **436** (2012), 1425–1435, doi:10.1016/j.laa.2011.08.019.

[16] I. Jovanović and Z. Stanić, Spectral distances of graphs based on their different matrix representations, *Filomat* **28** (2014), 723–734, doi:10.2298/fil1404723j.

[17] I. M. Jovanović, Some results on spectral distances of graphs, *Rev. Un. Mat. Argentina* **56** (2015), 95–117.

[18] H. Lin, D. Li and K. C. Das, Distance between distance spectra of graphs, *Linear Multilinear Algebra* **65** (2017), 2538–2550, doi:10.1080/03081087.2017.1278737.

[19] a. d. o. W. M. I. Maplesoft, Maple, computer algebra system, `https://www.maplesoft.com/products/maple/`.

[20] M. R. Oboudi, Cospectrality of complete bipartite graphs, *Linear Multilinear Algebra* **64** (2016), 2491–2497, doi:10.1080/03081087.2016.1162133.

[21] M. R. Oboudi, On the third largest eigenvalue of graphs, *Linear Algebra Appl.* **503** (2016), 164–179, doi:10.1016/j.laa.2016.03.037.

[22] M. Petrović, On graphs whose second largest eigenvalue does not exceed $\sqrt{2} - 1$, *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat.* **4** (1993), 70–75.

[23] J. H. Smith, Symmetry and multiple eigenvalues of graphs, *Glasnik Mat. Ser. III* **12(32)** (1977), 3–8.

[24] D. Stevanović, Research problems from the Aveiro Workshop on Graph Spectra, *Linear Algebra Appl.* **423** (2007), 172–181, doi:10.1016/j.laa.2006.11.027.

[25] R. C. Wilson and P. Zhu, A study of graph spectra for comparing graphs and trees, *Pattern Recognition* **41** (2008), 2833–2841, doi:https://doi.org/10.1016/j.patcog.2008.03.011.

[26] H. Yuan, A bound on the spectral radius of graphs, *Linear Algebra Appl.* **108** (1988), 135–139, doi:10.1016/0024-3795(88)90183-8.

[27] K. Zając and J. Piersa, Eigenvalue spectra of functional networks in fmri data and artificial models, in: *Artificial Intelligence and Soft Computing*, Springer Berlin Heidelberg, 2013 doi: 10.1007/978-3-642-38658-9_19.

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Oriented regular representations of out-valency two for finite simple groups*

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Abstract

In this paper, we show that every finite simple group of order at least 5 admits an oriented regular representation of out-valency 2.

Keywords: Finite simple group, DRR, ORR.

Math. Subj. Class. (2020): 05C25, 05C20, 20B25

1 Introduction

All groups and digraphs in this paper are finite. A *digraph* Γ consists of a set of vertices $\mathrm{V}(\Gamma)$ and a set of arcs $\mathrm{A}(\Gamma)$, each arc being an ordered pair of vertices. A digraph is *proper* if (u, v) being an arc implies that (v, u) is not an arc. The automorphisms of Γ are the permutations of $\mathrm{V}(\Gamma)$ that preserve $\mathrm{A}(\Gamma)$. Under composition, they form the automorphism group $\mathrm{Aut}(\Gamma)$ of Γ.

Let G be a group and $S \subseteq G$. The *Cayley digraph* $\mathrm{Cay}(G, S)$ on G with connection set S is the digraph with vertex set G and (u, v) being an arc whenever $vu^{-1} \in S$. Note that $\mathrm{Cay}(G, S)$ is a proper digraph if and only if $S \cap S^{-1} = \emptyset$. Note also that every vertex u in $\mathrm{Cay}(G, S)$ is contained in exactly $|S|$ arcs of the form (u, v). We thus say that $\mathrm{Cay}(G, S)$ has *out-valency* $|S|$.

It is easy to see that $\mathrm{Aut}(\mathrm{Cay}(G, S))$ contains the right regular representation of G. If this containment is actually equality, then $\mathrm{Cay}(G, S)$ is called a *digraphical regular*

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representation (or DRR) of G. A DRR that is a proper digraph is called an *oriented regular representation* (or ORR).

Babai proved that, apart from five small groups, all groups admit a DRR [1, Theorem 2.1]. He also asked which groups admit ORRs [1, Problem 2.7]. This was answered by Morris and Spiga [9, 10, 12] who showed that apart from generalised dihedral groups and a small list of exceptions, all groups admit ORRs.

In view of the above, a natural problem is to find "nice" DRRs and ORRs, say of "small" out-valency. Clearly, only cyclic groups can have DRRs of out-valency 1, so out-valency 2 is the smallest interesting case. In this paper, we give the most satisfactory answer to this question in the case of simple groups.

Theorem 1.1. *Every finite simple group of order at least* 5 *has a ORR of out-valency* 2*.*

A corollary of Theorem 1.1 is that every nonabelian simple group has a DRR of out-valency 2. However, the latter conclusion is an immediate consequence of the fact that every nonabelian simple group is generated by an involution and a non-involution (even by an involution and an element of odd prime order, see [8, Theorem 1]. Indeed, consider a Cayley digraph on a nonabelian simple group with connection set consisting of such a generating pair. This digraph has out-valency 2, but one out-neighbour of every vertex is also an in-neighbour while the other out-neighbour is not. This implies that fixing a vertex must also fix its out-neighbours and, by connectedness, the whole digraph, and the digraph is a DRR.

Note that Cayley digraphs of out-valency two of simple groups were previously studied in [4]. Another interesting variant of this question would be to consider undirected graphs. In this case, the smallest interesting valency is 3. The question of which simple groups admit graphical regular representations of valency 3 has received some attention but is still open [11, 13, 14, 15].

2 Preliminaries

2.1 Generation of finite simple groups

In this section we present some generation properties of finite simple groups, which will be needed in the proof of Theorem 1.1. The following result is due to Guralnick and Kantor [7, Corollary].

Theorem 2.1 (Guralnick-Kantor)**.** *Every nontrivial element of a finite simple group belongs to a pair of elements generating the group.*

Note that Theorem 2.1 depends on the classification of finite simple groups.

Corollary 2.2. *Let* G *be a finite nonabelian simple group with an element* x *of order* 3*. Then there exists* $y \in G$ *such that* $|y| \geq 4$ *and* $G = \langle x, y\rangle$*.*

Proof. By Theorem 2.1 there exists $z \in G$ such that $G = \langle x, z\rangle$. Note that $\langle x, z\rangle = \langle x, xz\rangle$ hence, if either z or xz has order at least 4, then the conclusion holds (by taking $y = z$ or $y = xz$). We may thus assume that z and xz both have order at most 3. This implies that G is a quotient of the finitely presented group

$$\langle x, z \mid x^3, z^m, (xz)^n\rangle$$

with $m, n \leq 3$. This is the "ordinary" $(3, m, n)$ triangle group which is well known to be solvable when $m, n \leq 3$ (see for example [3]) and therefore so is G, which is a contradiction. $\square$

The only nonabelian simple groups with no elements of order 3 are the Suzuki groups (see [6, Page 8, Table I]), which we now consider. For a positive integer m and prime number p, a prime number r is called a *primitive prime divisor* of $p^m - 1$ if r divides $p^m - 1$ but does not divide $p^k - 1$ for any positive integer $k < m$. By Zsigmondy's theorem [17], $p^m - 1$ has a primitive prime divisor whenever $m \geq 3$ and $(p, m) \neq (2, 6)$.

Proposition 2.3. *Let* $G = \mathrm{Sz}(q)$ *with* $q = 2^{2n+1} \geq 8$ *and let* r *be a primitive prime divisor of* $q^4 - 1$*. Then* $r \geq 5$*,* G *has an element* y *of order* r *and, for each such* y*, there exists* $x \in G$ *such that* $|x| = 4$*,* $|xy| \geq 3$ *and* $G = \langle x, y \rangle$*.*

Proof. First, recall that $|G| = q^2(q^2+1)(q-1)$ (see [6, Page 8, Table I]). Since r is a primitive prime divisor of $q^4 - 1$, it divides $q^4 - 1$ but not $q^2 - 1$ and thus must divide q^2+1. It follows that G has an element y of order r and that $r \geq 5$. We will now prove that there exists an element x of order 4 with the required properties, essentially by a somewhat crude counting argument.

We denote by E_q the elementary abelian group of order q and, for an integer $n \geq 2$, by C_n the cyclic group of order n and D_{2n} the dihedral group of order $2n$.

Up to conjugation, the maximal subgroups of G are the following (see for instance [2, Table 8.16]):

- $(\mathrm{E}_q.\mathrm{E}_q) \rtimes \mathrm{C}_{q-1}$,
- $\mathrm{D}_{2(q-1)}$,
- $\mathrm{C}_{q+\sqrt{2q}+1} \rtimes \mathrm{C}_4$,
- $\mathrm{C}_{q-\sqrt{2q}+1} \rtimes \mathrm{C}_4$,
- $\mathrm{Sz}(q_0)$, where $q_0 = q^{1/d} > 2$ for some prime divisor d of $2n+1$.

Recall that r is odd, does not divide $q - 1$ nor $q_0^4 - 1$ and thus does not divide its factor $(q_0^2+1)(q_0-1)$. This implies that r does not divide $|\mathrm{Sz}(q_0)| = q_0^2(q_0^2+1)(q_0-1)$. It follows that a maximal subgroup M of G containing y must be of the form $\mathrm{C}_{q\pm\sqrt{2q}+1} \rtimes \mathrm{C}_4$. Since every subgroup of a cyclic group is characteristic, $\langle y \rangle$ is normal in M and thus M is the only maximal subgroup of G containing y (for otherwise $\langle y \rangle$ would be normal in another maximal subgroup N of G and thus normal in $\langle M, N \rangle = G$).

Let Q be a Sylow 2-subgroup of G. Then $Q = \mathrm{E}_q.\mathrm{E}_q$ and $|\mathbf{N}_G(Q)| = (\mathrm{E}_q.\mathrm{E}_q) \rtimes \mathrm{C}_{q-1}$. Hence the number n of Sylow 2-subgroups of G is

$$n = \frac{|G|}{|\mathbf{N}_G(Q)|} = \frac{q^2(q^2+1)(q-1)}{q^2(q-1)} = q^2 + 1.$$

Let n_2 and n_4 denote the numbers of elements of order 2 and 4, respectively, in G. According to [5, Lemma 3.2], there are $q - 1$ involutions and $q^2 - q$ elements of order 4 in Q, and different conjugates of Q have trivial intersection. Then

$$n_2 = n(q-1) = (q^2+1)(q-1)$$

and

$$n_4 = n(q^2 - q) = (q^2+1)(q^2-q).$$

Let

$$I = \{g \in G : |gy| \leq 2\}$$

and

$$J = \{g \in G : \langle g, y\rangle \neq G\}.$$

Then $|I| = n_2 + 1$ and, since M is the unique maximal subgroup of G containing y, $|J| \leq |M|$. Since

$$\begin{aligned} |I| + |J| &\leq n_2 + |M| + 1 \\ &= (q^2+1)(q-1) + 4(q \pm \sqrt{2q} + 1) + 1 \\ &\leq (q^2+1)(q-1) + 4(q + \sqrt{2q} + 1) + 1 \\ &< (q^2+1)(q^2-q) = n_4, \end{aligned}$$

it follows that there exists $x \in G$ with $|x| = 4$ and $x \notin I \cup J$, as required. □

2.2 Constructing ORRs of out-valency 2

Lemma 2.4. *Let* $G = \langle x, y\rangle$. *If* $|x| = 3$ *and* $|y| \geq 4$, *then* $\mathrm{Cay}(G, \{x, y\})$ *is an ORR, unless* $|y| = 6$ *and* $x = y^4$, *and* $G \cong \mathrm{C}_6$.

Proof. Let $\Gamma = \mathrm{Cay}(G, \{x, y\})$ and let $A = \mathrm{Aut}(\Gamma)$. Note that Γ is a strongly connected proper digraph. Figure 1 shows all the directed paths of length at most 3 in Γ starting at 1.

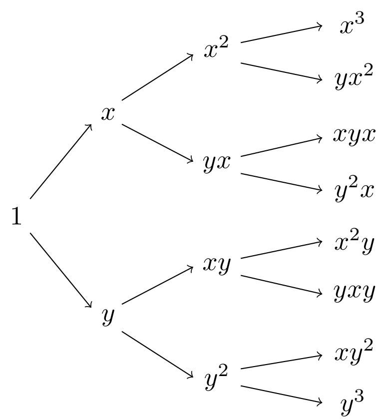

Figure 1: All the directed paths of length at most 3 in $\mathrm{Cay}(G, \{x, y\})$.

Since $|y| \geq 4$, we have $y^3 \neq 1$ and $y \neq x^{-2}$. Moreover, if $y^2 = x^{-1}$, then $|y| = 6$ and thus $x = y^4$ and the result holds. We thus assume this is not the case. Since $x^3 = 1$, this implies that $(1, x, x^2, x^3)$ is the only directed cycle of length 3 starting at 1. This implies that the stabiliser A_1 of the vertex 1 also fixes x. As 1 only has one out-neighbour other than x, it must also be fixed. By vertex-transitivity, we find that fixing a vertex fixes its out neighbours and, using connectedness, we conclude that $A_1 = 1$ and thus Γ is an ORR. □

Lemma 2.5. *Let* $G = \langle x, y\rangle$. *If* $|x| = 4$, $|y| \geq 5$ *and* $|xy| \geq 3$, *then* $\mathrm{Cay}(G, \{x, y\})$ *is an ORR, unless* $|y| = 12$ *and* $x = y^9$, *and* $G \cong \mathrm{C}_{12}$.

Proof. Let $\Gamma = \mathrm{Cay}(G, \{x, y\})$ and let $A = \mathrm{Aut}(\Gamma)$. Note that Γ is a strongly connected proper digraph. Figure 2 shows all the directed paths of length at most 4 in Γ starting at 1.

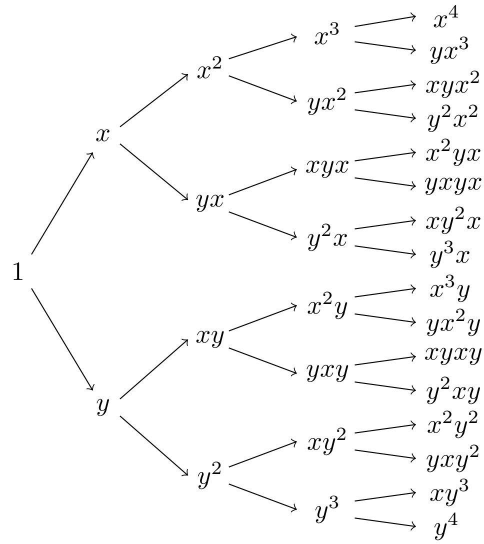

Figure 2: All the directed paths of length at most 4 in $\mathrm{Cay}(G, \{x, y\})$.

Since $|y| \geq 5$, we have $y^4 \neq 1$, $y \neq x^{-3}$ and $y^2 \neq x^{-2}$. Similarly, $|xy| \geq 3$ implies that $(xy)^2 \neq 1 \neq (yx)^2$. Moreover, if $y^3 = x^{-1}$, then $|y| = 12$ and thus $x = y^9$ and the result holds. We thus assume this is not the case. Since $x^4 = 1$, this implies that $(1, x, x^2, x^3, x^4)$ is the only directed cycle of length 4 starting at 1 and, as in the previous lemma, Γ is an ORR. □

3 Proof of Theorem 1.1

Let G be a finite simple group with $|G| \geq 5$. We first suppose that $G = \mathbb{F}_p^+$ for some prime $p \geq 5$. Let $x, y \in \mathbb{F}_p \setminus \{0\}$ such that $x \neq \pm y$ and let $\Gamma = \mathrm{Cay}(G, \{x, y\})$. Note that Γ is a proper digraph of out-valency 2. By [16, Proposition 1.3 and Example 2.2], Γ is an ORR if and only if the only solution to

$$\{\lambda x, \lambda y\} = \{x, y\} \tag{3.1}$$

with $\lambda \in \mathbb{F}_p^\times$ is $\lambda = 1$. Suppose otherwise, that is (3.1) holds with $\lambda \neq 1$. This implies that $\lambda x = y$ and $\lambda y = x$, which yields that

$$\lambda x^2 = (\lambda x)x = y(\lambda y) = \lambda y^2,$$

and hence $x^2 = y^2$, contradicting $x \neq \pm y$. Thus we conclude that Γ is an ORR, as required.

We may now assume that G is nonabelian. If G has an element x of order 3 then, by Corollary 2.2 there exists $y \in G$ such that $|y| \geq 4$ and $G = \langle x, y \rangle$. By Lemma 2.4, $\mathrm{Cay}(G, \{x, y\})$ is an ORR. We may thus assume that G does not have an element of order

3 and thus $G = \mathrm{Sz}(q)$ for some $q = 2^{2n+1} \geq 8$. Let r be a primitive prime divisor of $q^4 - 1$. By Proposition 2.3, G contains elements x and y such that $|x| = 4$, $|y| = r \geq 5$, $|xy| \geq 3$ and $G = \langle x, y \rangle$. By Lemma 2.5, $\mathrm{Cay}(G, \{x, y\})$ is an ORR. □

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References

[1] L. Babai, Finite digraphs with given regular automorphism groups, *Period. Math. Hungar.* **11** (1980), 257–270, doi:10.1007/bf02107568.

[2] J. N. Bray, D. F. Holt and C. M. Roney-Dougal, *The Maximal Subgroups of the Low-Dimensional Finite Classical Groups*, volume 407 of *London Mathematical Society Lecture Note Series*, Cambridge University Press, Cambridge, 2013, doi:10.1017/cbo9781139192576.

[3] M. D. E. Conder, Some results on quotients of triangle groups, *Bull. Austral. Math. Soc.* **30** (1984), 73–90, doi:10.1017/s0004972700001738.

[4] X.-G. Fang, Z.-P. Lu, J. Wang and M.-Y. Xu, Cayley digraphs of finite simple groups with small out-valency, *Comm. Algebra* **32** (2004), 1201–1211, doi:10.1081/agb-120027974.

[5] X. G. Fang and C. E. Praeger, Finite two-arc transitive graphs admitting a Suzuki simple group, *Comm. Algebra* **27** (1999), 3727–3754, doi:10.1080/00927879908826659.

[6] D. Gorenstein, R. Lyons and R. Solomon, *The Classification of the Finite Simple Groups*, volume 40 of *Mathematical Surveys and Monographs*, American Mathematical Society, Providence, RI, 1994, doi:10.1090/surv/040.1.

[7] R. M. Guralnick and W. M. Kantor, Probabilistic generation of finite simple groups, *J. Algebra* **234** (2000), 743–792, doi:10.1006/jabr.2000.8357.

[8] C. S. H. King, Generation of finite simple groups by an involution and an element of prime order, *J. Algebra* **478** (2017), 153–173, doi:10.1016/j.jalgebra.2016.12.031.

[9] J. Morris and P. Spiga, Every finite non-solvable group admits an oriented regular representation, *J. Comb. Theory Ser. B* **126** (2017), 198–234, doi:10.1016/j.jctb.2017.05.003.

[10] J. Morris and P. Spiga, Classification of finite groups that admit an oriented regular representation, *Bull. Lond. Math. Soc.* **50** (2018), 811–831, doi:10.1112/blms.12177.

[11] P. Spiga, Cubic graphical regular representations of finite non-abelian simple groups, *Comm. Algebra* **46** (2018), 2440–2450, doi:10.1080/00927872.2017.1383997.

[12] P. Spiga, Finite groups admitting an oriented regular representation, *J. Comb. Theory Ser. A* **153** (2018), 76–97, doi:10.1016/j.jcta.2017.08.001.

[13] B. Xia, Cubic graphical regular representations of $\mathrm{PSL}_3(q)$, *Discrete Math.* **343** (2020), 111646 (9 pages), doi:10.1016/j.disc.2019.111646.

[14] B. Xia, On cubic graphical regular representations of finite simple groups, *J. Comb. Theory Ser. B* **141** (2020), 1–30, doi:10.1016/j.jctb.2019.06.002.

[15] B. Xia and T. Fang, Cubic graphical regular representations of $\mathrm{PSL}_2(q)$, *Discrete Math.* **339** (2016), 2051–2055, doi:10.1016/j.disc.2016.03.008.

[16] M.-Y. Xu, Automorphism groups and isomorphisms of Cayley digraphs, *Discrete Math.* **182** (1998), 309–319, doi:10.1016/s0012-365x(97)00152-0.

[17] K. Zsigmondy, Zur Theorie der Potenzreste, *Monatsh. Math. Phys.* **3** (1892), 265–284, doi:10.1007/bf01692444.

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New strong divisibility sequences*

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Abstract

We provide new families of divisibility and strong divisibility sequences based on some factorization properties of Chebyshev polynomials.

Keywords: Strong divisibility sequences, Chebyshev polynomials, tridiagonal matrices, determinant.

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1 Introduction

A sequence of any integer numbers $\{a_n\}$ is said to be a *divisibility sequence* if

$$a_m \mid a_n, \quad \text{whenever } m \mid n,$$

and is called a *strong divisibility sequence* if

$$\gcd(a_m, a_n) = a_{\gcd(m,n)}.$$

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The strong divisibility sequences and its weaker version have been studied for more than one century. Actually, the Fibonacci numbers

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, \ldots$$

are perhaps the best known non-trivial strong divisibility sequence. For earlier questions, open problems, and general characterizations, the reader is referred to [4, 10, 11, 12, 21, 22].

As a particular case of the *general conditional recurrence sequences* defined in [16], recently it was proposed in [20] the study of the *conditional recurrence sequences* $\{f_n\}$ satisfying the recurrence relations of integers

$$f_n = \begin{cases} a_1 f_{n-1} + b_2 f_{n-2}, & \text{if } n \text{ is odd,} \\ a_2 f_{n-1} + b_1 f_{n-2}, & \text{if } n \text{ is even.} \end{cases}$$

for $n \geqslant 2$, with $f_0 = 1$ and $f_1 = a_1$, aiming to generate new strong divisibility sequences. Indeed, the authors were able to obtain sufficient conditions for which certain subsequences of $\{f_n\}$ are strong divisible.

Theorem 1.1 ([20]). *Let* $\tilde{f}_n = f_{2n-1}$. *If* $a_1 = 1$ *and* $\gcd(a_1a_2 + b_1 + b_2, b_1b_2) = 1$, *then*

$$\gcd(\tilde{f}_m, \tilde{f}_n) = \tilde{f}_{\gcd(m,n)}.$$

Corollary 1.2 ([20]). *Let* $\tilde{f}_n = f_{2n-1}$. *If* $\gcd(a_1a_2 + b_1 + b_2, b_1b_2) = 1$, *then* $\{\tilde{f}_n\}$ *is a strong divisibility sequence.*

Theorem 1.3 ([20]). *Let* $\tilde{f}_n = f_{2n-1}$. *Thus* $\tilde{f}_m \mid \tilde{f}_n$, *whenever* $m \mid n$.

For example, setting $a_1 = 3, a_2 = 1 = b_1$, and $b_2 = 2$, we get

n	1	2	3	4	5	6	7	8	9
f_n	3	4	18	22	102	124	576	700	3252

This means that the first terms of the subsequence of odd indices of $\{f_n\}$ are

n	1	2	3	4	5	6
$\tilde{f}_n$	3	18	102	576	3252	18360

While $\{\tilde{f}_n\}$ is a divisibility sequence, it is clear that is not strong.

Another interesting result obtained in [20] is the following:

Theorem 1.4. *Let* $\tilde{f}_1 = 1$ *and* $\tilde{f}_n = f_{n-1}$, *for* $n > 1$. *If* $a_1 = 1$, $b_1 = b_2$, *and* $\gcd(a_2, b_1) = 1$, *then* $\{\tilde{f}_n\}$ *is a strong divisibility sequence.*

For the weaker divisibility, the following general result was obtained:

Corollary 1.5. *Under the conditions of Theorem 1.4,* $\{\tilde{f}_n\}$ *is a divisibility sequence.*

Our aim here is to extend the above results to a more general setting, namely for the sequences of integers defined by the recurrence relations

$$f_n = \begin{cases} a_1 f_{n-1} + b_k f_{n-2}, & \text{if } n \equiv 1 \pmod{k}, \\ a_2 f_{n-1} + b_1 f_{n-2}, & \text{if } n \equiv 2 \pmod{k}, \\ a_3 f_{n-1} + b_2 f_{n-2}, & \text{if } n \equiv 3 \pmod{k}, \\ \cdots & \cdots \\ a_{k-1} f_{n-1} + b_{k-2} f_{n-2}, & \text{if } n \equiv k-1 \pmod{k}, \\ a_k f_{n-1} + b_{k-1} f_{n-2}, & \text{if } n \equiv 0 \pmod{k}, \end{cases} \tag{1.1}$$

for $n \geqslant 2$, with $f_0 = 1$ and $f_1 = a_1$. The previous results will be recovered by making $k = 2$. Consequently, we answer to the open problem proposed in [20].

In this paper, we will relate (1.1) with the so-called *periodic continuants* [6, 18] (for recent applications, the reader is referred to [1, 2, 3]). This relation is established by using Chebyshev polynomials of the second kind. Then, from $\{f_n\}$ we can, under certain conditions, generate new strong divisibility sequences. At the same time, we can recover the connection between the sequences defined by recurrence relations with two terms and the determinants of tridiagonal matrices. This is effectively in the spirit of some ideas we can find in [15], proposed by Édouard Lucas back to 1878.

2 The determinant of a tridiagonal k-Toeplitz matrix

The matrices of the form

$$A_n = \begin{pmatrix} a_1 & b_1 & & & & & & & & \\ c_1 & \ddots & \ddots & & & & & & & \\ & \ddots & a_k & b_k & & & & & & \\ & & c_k & a_1 & b_1 & & & & & \\ & & & c_1 & \ddots & \ddots & & & & \\ & & & & \ddots & a_k & b_k & & & \\ & & & & & c_k & a_1 & b_1 & & \\ & & & & & & c_1 & \ddots & \ddots & \\ & & & & & & & \ddots & & \end{pmatrix}_{n \times n},$$

i.e., tridiagonal matrices $A_n = (a_{ij})$ with entries satisfying

$$a_{i+k,j+k} = a_{ij}, \quad \text{for } i, j = 1, 2, \ldots, n-k,$$

are known as *tridiagonal k-Toeplitz*. The determinant of such matrix is known as a *periodic continuant* [18].

For $k = 1$, we get a tridiagonal Toeplitz matrix and its determinant was known in [18] as a *continuant*. The characteristic polynomial of such a matrix was found by V. Lovass-Nagy and P. Rózsa [13, 14], in 1963. Notwithstanding, the particular case when $k = 2$ and the two subdiagonals are constant equal to 1, had been considered in 1947 in D. E. Rutherford's seminal paper [19], followed soon after by J. F. Elliott with his Master's thesis [5, Section IV.4]. In 1966, Rózsa held a seminar at the University of Hamburg on tridiagonal k-Toeplitz matrices motivated mainly by problems of lattice dynamics, of ladder networks, and of structural analysis. In that year, L. Elsner and R. M. Redheffer [6] studied A_n for special cases of k and, two years later, P. Rózsa in [18] originally proved a general formula for the determinant of A_n. Independently, the spectrum of a tridiagonal 2-Toeplitz matrix was also studied by M. J. C. Gover in 1994 [9]. In [7], it is considered the case when $k = 3$ and, later on, the characteristic polynomial of A_n was stated, for any k, when analyzing the invertibility conditions for A_n based on orthogonal polynomials theory (cf. [8]).

We recall now Rózsa's solution. Let $\Delta_{i_1,i_2,\ldots,i_p}$ be the principal minor of A_n indexed by the rows and columns $i_1, i_2, \ldots, i_p$. The determinant of A_n is given in [18] as

$$\det A_n = (\sqrt{b_1c_1\cdots b_kc_k})^q\Bigg(\Delta_{1\ldots,r}U_q(x) + \frac{\sqrt{b_kc_kb_1c_1\cdots b_rc_r}}{\sqrt{b_{r+1}c_{r+1}\cdots b_{k-1}c_{k-1}}}\Delta_{r+2,\ldots,k-1}\,U_{q-1}(x)\Bigg)$$

with $n = qk + r$ and

$$x = \frac{\Delta_{1,\ldots,k} - b_kc_k\Delta_{2,\ldots,k-1}}{2\sqrt{b_1c_1\cdots b_kc_k}},$$

assuming that $\Delta_{1,\ldots,r} = 1$ and $\Delta_{2,\ldots,r} = 0$, for $r = 0$, and with $\{U_n(x)\}_{n\geqslant 0}$ standing for the Chebyshev polynomials of the second kind. These polynomials satisfy the three-term recurrence relation

$$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x), \quad \text{for all } n = 1, 2, \ldots, \tag{2.1}$$

with initial conditions $U_0(x) = 1$ and $U_1(x) = 2x$. We recall that the main explicit formula for the Chebyshev polynomials of the second kind could be

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \quad \text{with } x = \cos\theta \quad (0 \leqslant \theta < \pi), \tag{2.2}$$

for all $n = 0, 1, 2 \ldots$. While (2.2) is more common to find in the orthogonal polynomials theory, there are other explicit representations and relations for $U_n(x)$. Among them, the most frequent are

$$U_n(x) = \frac{\left(x+\sqrt{x^2-1}\right)^{n+1} - \left(x-\sqrt{x^2-1}\right)^{n+1}}{2\sqrt{x^2-1}},$$

an immediate consequence of de Moivre's formula, and

$$U_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor}(-1)^k\binom{n-k}{k}(2x)^{n-2k}.$$

Taking into account the definition of A_n, we can redefine (1.1) in terms of the determinant of A_n, namely,

$$f_n = \begin{vmatrix} a_1 & b_1 & & & & & & & & & \\ -1 & \ddots & \ddots & & & & & & & & \\ & \ddots & a_k & b_k & & & & & & & \\ & & -1 & a_1 & b_1 & & & & & & \\ & & & -1 & \ddots & \ddots & & & & & \\ & & & & \ddots & a_k & b_k & & & & \\ & & & & & -1 & a_1 & b_1 & & & \\ & & & & & & -1 & \ddots & \ddots & & \\ & & & & & & & \ddots & & & \end{vmatrix}_{n\times n}. \tag{2.3}$$

That means that the determinant (2.3) is

$$f_n = (i^k\sqrt{b_1\cdots b_k})^q\left(\Delta_{1,\dots,r}\,U_q(x) + \frac{i^{r+1}\sqrt{b_kb_1\cdots b_r}}{i^{k-r-1}\sqrt{b_{r+1}\cdots b_{k-1}}}\,\Delta_{r+2,\dots,k-1}\,U_{q-1}(x)\right),$$

where

$$x = \frac{\Delta_{1,\dots,k} + b_k\Delta_{2,\dots,k-1}}{i^k 2\sqrt{b_1\cdots b_k}}.$$

In particular, if $r = k-1$, then

$$f_n = (i^k\sqrt{b_1\cdots b_k})^q\Delta_{1,\dots,k-1}\,U_q(x), \tag{2.4}$$

which we will explore to generate new strong divisibility sequences in the next sections.

Before that, we recall a general result relating distinct minors, which can be found for example in [18].

Lemma 2.1. *For any positive integer n and $i < j$,*

$$\Delta_{1,\dots,j-1}\Delta_{i+1,\dots,n} - (-1)^{j-i}b_i\cdots b_{j-1}\Delta_{1,\dots,i-1}\Delta_{j+1,\dots,n} = \Delta_{1,\dots,n}\Delta_{i+1,\dots,j-1}.$$

In fact, Lemma 2.2 in [20] is a particular case of Lemma 2.1.

3 New divisibility sequences

In [20], the authors asked for conditional (strong) divisibility sequences for $r > 2$, i.e., satisfying (1.1). We start with the weaker condition.

Let us recall several factorization properties for Chebyshev polynomials disclosed in [17].

Theorem 3.1 ([17]). *Let $m \geqslant n$ be two positive integers. Then $U_m(x)$ is a multiple of $U_n(x)$ if and only if $m = (\ell+1)n + \ell$, for some nonnegative integer ℓ. More precisely, if ℓ is even, then*

$$U_m(x) = U_n(x)\left(2\sum_{k=0}^{\frac{\ell}{2}} T_{m-(2k+1)n-2k}(x) - 1\right),$$

and if ℓ is odd, then

$$U_m(x) = 2U_n(x) \sum_{k=0}^{\frac{\ell-1}{2}} T_{m-(2k+1)n-2k}(x).$$

In Theorem 3.1, $\{T_n(x)\}_{n\geqslant 0}$ stands for the Chebyshev polynomial of the first kind. These polynomials satisfy the same recurrence (2.1), here with initial conditions $T_0(x) = 1$ and $T_1(x) = x$. An explicit formula for such polynomials is $T_n(x) = \cos n\theta$, with $x = \cos\theta$.

The next two results, naturally connected to those in Section 1, can be found in [17].

Theorem 3.2. *Let m and n be two nonnegative integers and $d = \gcd(m, n)$. Then*

$$\gcd(U_{m-1}(x), U_{n-1}(x)) = U_{d-1}(x).$$

Corollary 3.3. *If m and n are coprime, then* $\gcd(U_{m-1}(x), U_{n-1}(x)) = 1$.

The general sequences that we consider are

$$f_n = (\pm\sqrt{b})^{n-1} U_{n-1}\left(\frac{a}{\pm 2\sqrt{b}}\right),$$

where a, b are nonzero integers (possibly with $b < 0$), for $n \geqslant 1$. In particular, $f_0 = 0$, $f_1 = 1$ and $f_2 = a$.

It is worth mentioning that the symbol $\pm$ can be ignored, that is to say:

$$f_n = (\pm\sqrt{b})^{n-1} U_{n-1}\left(\frac{a}{\pm 2\sqrt{b}}\right) = (\sqrt{b})^{n-1} U_{n-1}\left(\frac{a}{2\sqrt{b}}\right), \tag{3.1}$$

since the Chebyshev polynomials of the second kind $U_n(x)$ have the same parities as n.

We may now state our first main result.

Theorem 3.4. *For any integers a and b, $\{f_n\}$ as defined in* (3.1) *is a divisibility sequence.*

Proof. Assume that $n \mid m$, say $m = sn$, where $s \geqslant 1$. For simplicity, set $x = \frac{1}{2\sqrt{b}}$. So

$$f_n = \frac{U_{n-1}(ax)}{(2x)^{n-1}} \quad \text{and} \quad f_m = \frac{U_{sn-1}(ax)}{(2x)^{sn-1}},$$

which implies that

$$\frac{f_m}{f_n} = \frac{U_{sn-1}(ax)}{(2x)^{(s-1)n} U_{n-1}(ax)}.$$

Set $\ell = s - 1$, we have $sn - 1 = (\ell + 1)(n - 1) + \ell$. From Theorem 3.1, $U_{sn-1}(x)$ is a multiple of $U_{n-1}(x)$. More precisely, when s is even,

$$U_{sn-1}(x) = 2U_{n-1}(x) \sum_{t=0}^{\frac{s-2}{2}} T_{(s-2t-1)n}(x),$$

and when s is odd,

$$U_{sn-1}(x) = U_{n-1}(x)\left(2\sum_{t=0}^{\frac{s-1}{2}} T_{(s-2t-1)n}(x) - 1\right).$$

Therefore

$$\frac{f_m}{f_n} = \frac{U_{sn-1}(ax)}{(2x)^{(s-1)n}U_{n-1}(ax)} = \frac{2}{(2x)^{(s-1)n}}\sum_{t=0}^{\frac{s-2}{2}} T_{(s-2t-1)n}(ax)$$

when s is even, and

$$\frac{f_m}{f_n} = \frac{U_{sn-1}(ax)}{(2x)^{(s-1)n}U_{n-1}(ax)} = \frac{2}{(2x)^{(s-1)n}}\sum_{t=0}^{\frac{s-1}{2}} T_{(s-2t-1)n}(ax) - \frac{1}{(2x)^{(s-1)n}}$$

when s is odd.

We will prove $\frac{U_{sn-1}(ax)}{(2x)^{(s-1)n}U_{n-1}(ax)}$ is an integer whether s is even or odd, by involving with the following two claims.

Claim 1. $2T_{(s-2t-1)n}\left(\frac{a}{2}\right)$ *is an integer, for any* $0 \leqslant t \leqslant \lfloor\frac{s-1}{2}\rfloor$.

This claim follows immediately from the recurrence relation about $T_n(x)$ as shown in (2.1).

Claim 2. $(\sqrt{b})^{(s-1)n}T_{(s-2t-1)n}\left(\frac{1}{\sqrt{b}}\right)$ *is an integer, for any* $0 \leqslant t \leqslant \lfloor\frac{s-1}{2}\rfloor$.

Observe that among all the terms in $T_{(s-2t-1)n}\left(\frac{1}{\sqrt{b}}\right)$, the maximum degree of denominator is $(\sqrt{b})^{(s-1)n}$, which means that all the denominators of $T_{(s-2t-1)n}\left(\frac{1}{\sqrt{b}}\right)$ would be canceled by $(\sqrt{b})^{(s-1)n}$. It leads to this claim.

Combining the above claims, it leads to

$$\frac{2}{(2x)^{(s-1)n}}\sum_{t=0}^{\lfloor\frac{s-1}{2}\rfloor} T_{(s-2t-1)n}(ax) = 2(\sqrt{b})^{(s-1)n}\sum_{t=0}^{\lfloor\frac{s-1}{2}\rfloor} T_{(s-2t-1)n}\left(\frac{a}{2\sqrt{b}}\right)$$

is an integer. When s is even, $f_n \mid f_m$ follows now. When s is odd, together with the fact that $\frac{1}{(2x)^{(s-1)n}} = (\sqrt{b})^{(s-1)n}$ is an integer, $f_n \mid f_m$ also holds. □

4 Strong divisibility sequences

The sequence $\{f_n\}$ defined in (3.1) can have negative terms. Therefore, in our strongly divisibility definition, we are assuming that $\gcd(a_m, a_n) = |a_{\gcd(m,n)}|$. Since we are interested in positive conditional recurrence sequences (1.1), all the terms of $\{f_n\}$ will be considered as positive or, equivalently, $a > 0$ and $a^2 - 4b \geqslant 0$. Notice that the zeros of the Chebyshev polynomials of the second kind are in the interval $(-1, 1)$ and, from its definition, $\lim_{x\to+\infty} U_n(x) = +\infty$.

In order to provide our characterization to the strong divisibility property of $\{f_n\}$, let us state several straightforward relations involving f_n, as defined in (3.1). From (2.1), we have

$$U_n\left(\frac{a}{2\sqrt{b}}\right) = \frac{a}{\sqrt{b}}U_{n-1}\left(\frac{a}{2\sqrt{b}}\right) - U_{n-2}\left(\frac{a}{2\sqrt{b}}\right)$$

and

$$f_n = af_{n-1} - bf_{n-2}. \tag{4.1}$$

A more general identity can be obtained from (2.1), namely

$$U_{s+t}(x) = U_s(x)\,U_t(x) - U_{s-1}(x)\,U_{t-1}(x),$$

and then,

$$f_{s+t} = f_{s+1}f_t - bf_sf_{t-1}. \tag{4.2}$$

The next result is an extension of some other results we can find in the literature, as for example related to the Fibonacci numbers.

Lemma 4.1. *If* $\gcd(a, b) = 1$*, then* $\gcd(f_n, f_{n+1}) = 1$ *for any* $n \geqslant 1$*.*

Proof. We claim that $\gcd(f_n, b) = 1$, for any $n \geqslant 1$, which can be proved by induction. From $f_1 = 1$ and $f_2 = a$, this claim holds when $n = 1, 2$. Assume that $\gcd(f_{n-1}, b) = 1$ and $\gcd(f_{n-2}, b) = 1$. Suppose to the contrary that $\gcd(f_n, b) = s$, where $s > 1$. From (4.1), $s \mid af_{n-1}$. Notice that $\gcd(s, a) = 1$, otherwise it is a contradiction to the hypothesis that $\gcd(a, b) = 1$. So $s \mid f_{n-1}$. However, this is another contradiction to the inductive hypothesis stating $\gcd(f_{n-1}, b) = 1$.

Now we are ready to show that $\gcd(f_n, f_{n+1}) = 1$. Again, from $f_1 = 1$ and $f_2 = a$, we know that $\gcd(f_n, f_{n+1}) = 1$ is true when $n = 1$. Suppose to the contrary that $\gcd(f_{n-2}, f_{n-1}) = 1$, for some $n \geqslant 3$, but $\gcd(f_{n-1}, f_n) = t$ with $t > 1$. From (4.1), $t \mid bf_{n-2}$. Note that $\gcd(t, f_{n-2}) = 1$, otherwise we get a contradiction with $\gcd(f_{n-2}, f_{n-1}) = 1$. Thus, $t \mid b$ means that t is a common divisor of b and f_n, a contradiction to the above claim that $\gcd(f_n, b) = 1$.

The proof is now completed. □

We are now able to prove the main result of this section.

Theorem 4.2. *The sequence* $\{f_n\}$ *defined in* (3.1) *is strongly divisible if and only if* $\gcd(a, b) = 1$.

Proof. The necessity part is easy. Assume that $\{f_n\}$ is a strong divisibility sequence. Suppose to the contrary that $\gcd(a, b) \neq 1$. From (4.1), we may obtain the first few values: $f_1 = 1$, $f_2 = a$, $f_3 = a^2 - b$, $f_4 = a^3 - 2ab$. Clearly, $\gcd(f_3, f_4) \neq 1 = f_1$ follows from $\gcd(a, b) \neq 1$, which is a contradiction to the strong divisibility property of $\{f_n\}$.

Now we prove the part of sufficiency. Suppose that $\gcd(a, b) = 1$. Set $g = \gcd(n, m)$ and $d = \gcd(f_n, f_m)$. We would like to show that $\gcd(f_n, f_m) = |f_{\gcd(n,m)}|$, i.e., $d = |f_g|$, which comes from $f_g \mid d$ and $d \mid f_g$.

On one hand, from $g \mid n$ and $g \mid m$, we get $f_g \mid f_n$ and $f_g \mid f_m$, since $\{f_n\}$ is a divisibility sequence from Theorem 3.4. Thus, $f_g \mid d$.

On the other hand, we still need to show that $d \mid f_g$. Since, $g = \gcd(n, m)$, we may assume that there exist positive integers s, k such that $sn = g + km$. From (4.2), we have

$$f_{sn} = f_{g+km} = f_gf_{km+1} - bf_{g-1}f_{km}.$$

From $d \mid f_n$, and $f_n \mid f_{sn}$ (since $\{f_n\}$ is a divisibility sequence), we get $d \mid f_{sn}$. Similarly, we have $d \mid f_{km}$. Therefore, $d \mid f_g f_{km+1}$. Notice that $\gcd(d, f_{km+1}) = 1$, otherwise, together with $d \mid f_{km}$, it leads to $\gcd(f_{km}, f_{km+1}) \neq 1$, which is a contraction to Lemma 4.1. Now, it follows that $d \mid f_g$.

Combining $f_g \mid d$ and $d \mid f_g$, we obtain $d = |f_g|$, which reveals the strong divisibility property of $\{f_n\}$. □

5 Examples

In this final section, from the above results, we provide several examples of new (conditional) strong divisibility sequences.

Setting $k = 3$, $r = 2$, we have

$$x = \frac{\Delta_{1,\dots,k} + b_k \Delta_{2,\dots,k-1}}{i^k 2\sqrt{b_1 \cdots b_k}}.$$

In particular, if $r = k - 1$, then

$$f_n = (-i\sqrt{b_1 b_2 b_3})^q (a_1 a_2 + b_1)\, U_q \left(\frac{a_1 a_2 a_3 + a_3 b_1 + a_1 b_2 + a_2 b_3}{-i2\sqrt{b_1 b_2 b_3}} \right).$$

So, if we consider the sequence defined by

$$f_n = \begin{cases} f_{n-1} + 3f_{n-2}, & \text{if} n \equiv 1 \pmod 3, \\ 2f_{n-1} + f_{n-2}, & \text{if} n \equiv 2 \pmod 3, \\ 4f_{n-1} + 2f_{n-2}, & \text{if } n \equiv 0 \pmod 3, \end{cases}$$

we have

$$f_n = (-i\sqrt{6})^q\, 3\, U_q \left(\frac{20}{-i2\sqrt{6}} \right).$$

Now set

$$g_{q+1} = (-i\sqrt{6})^q\, U_q \left(\frac{20}{-i2\sqrt{6}} \right),$$

for $q \geqslant 0$. The first terms are:

n	g_n
1	1
2	20
3	406
4	8240
5	167236
6	3394160
7	68886616
8	1398097280
9	28375265296
10	575893889600

Now, we can check, for example, that $g_3 \mid g_6$ or $g_5 \mid g_{10}$. However,

$$\gcd(g_8, g_{10}) = 320.$$

Instead, we take the recurrence relation

$$f_n = \begin{cases} 2f_{n-1} + 3f_{n-2}, & \text{if } n \equiv 1 \pmod 3, \\ f_{n-1} + f_{n-2}, & \text{if } n \equiv 2 \pmod 3, \\ 4f_{n-1} + 2f_{n-2}, & \text{if } n \equiv 0 \pmod 3. \end{cases}$$

Setting

$$g_{q+1} = (-i\sqrt{6})^q \, U_q\left(\frac{19}{-i2\sqrt{6}}\right),$$

for $q \geqslant 0$, the first terms are:

n	g_n
1	1
2	19
3	367
4	7087
5	136855
6	2642767
7	51033703
8	985496959
9	19030644439
10	367495226095

Now we can check, for example, that $g_4 \mid g_8$ or $g_5 \mid g_{10}$. Moreover,

$$\gcd(g_8, g_{10}) = g_2 \quad \text{or} \quad \gcd(g_6, g_9) = g_3,$$

and, of course,

$$\gcd(g_4, g_9) = g_1.$$

Let us consider now two more elaborated examples, for $k = 4$. We start with the following one

$$f_n = \begin{cases} 2f_{n-1} + 4f_{n-2}, & \text{if } n \equiv 1 \pmod 4, \\ f_{n-1} + 3f_{n-2}, & \text{if } n \equiv 2 \pmod 4, \\ 2f_{n-1} + f_{n-2}, & \text{if } n \equiv 3 \pmod 4, \\ 3f_{n-1} + f_{n-2}, & \text{if } n \equiv 0 \pmod 4. \end{cases}$$

Setting

$$g_{q+1} = (\sqrt{12})^q \, U_q\left(\frac{53}{2\sqrt{12}}\right),$$

for $q \geqslant 0$, the first terms are:

n	g_n
1	1
2	53
3	2797
4	147605
5	7789501
6	411072293
7	21693357517
8	1144815080885
9	60414878996701
10	3188250805854533

Straightforward verification shows, for example, that $g_4 \mid g_8$ or $g_5 \mid g_{10}$. Furthermore,

$$\gcd(g_8, g_{10}) = g_2 \quad \text{or} \quad \gcd(g_6, g_9) = g_3,$$

and, of course,

$$\gcd(g_4, g_9) = g_1.$$

Finally, we study

$$f_n = \begin{cases} 2f_{n-1} + 4f_{n-2}, & \text{if } n \equiv 1 \pmod 4, \\ f_{n-1} + 2f_{n-2}, & \text{if } n \equiv 2 \pmod 4, \\ 2f_{n-1} + f_{n-2}, & \text{if } n \equiv 3 \pmod 4, \\ 3f_{n-1} + f_{n-2}, & \text{if } n \equiv 0 \pmod 4. \end{cases}$$

Setting

$$g_{q+1} = (\sqrt{8})^q \, U_q \left(\frac{46}{2\sqrt{8}} \right),$$

for $q \geqslant 0$, the first terms are:

n	g_n
1	1
2	46
3	2108
4	96600
5	4426736
6	202857056
7	9296010688
8	425993635200
9	19521339133696
10	894573651068416

Now we can check, for instance, that $g_4 \mid g_8$ or $g_5 \mid g_{10}$. However,

$$\gcd(g_8, g_{10}) = 2944.$$

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References

[1] M. Anđelić and C. M. da Fonseca, Some comments on tridiagonal (p, r)-toeplitz matrices, *Linear Algebra Appl.* **572** (2019), 46–50, doi:10.1016/j.laa.2019.03.001.

[2] M. Anđelić, C. M. da Fonseca and R. Mamede, On the number of p-vertices of some graphs, *Linear Algebra Appl.* **434** (2011), 514–525, doi:10.1016/j.laa.2010.09.017.

[3] M. Anđelić, Z. Du, C. M. da Fonseca and E. Kılıç, A matrix approach to some second-order difference equations with sign-alternating coefficients, *J. Differ. Equ. Appl.* **26** (2020), 149–162, doi:10.1080/10236198.2019.1709180.

[4] J. Bézivin, A. Pethö and A. J. van der Poorten, A full characterisation of divisibility sequences, *Am. J. Math.* **112** (1990), 985–1001, doi:10.2307/2374733.

[5] J. Elliott, *The characteristic roots of certain real symmetric matrices*, Master's thesis, University of Tennessee, 1953, https://trace.tennessee.edu/utk_gradthes/2384/.

[6] L. Elsner and R. M. Redheffer, Remarks on band matrices, *Numer. Math.* **10** (1967), 153–161, doi:10.1007/bf02174148.

[7] C. M. da Fonseca and J. Petronilho, Explicit inverses of some tridiagonal matrices, *Linear Algebra Appl.* **325** (2001), 7–21, doi:10.1016/s0024-3795(00)00289-5.

[8] C. M. da Fonseca and J. Petronilho, Explicit inverse of a tridiagonal k-toeplitz matrix, *Numer. Math.* **100** (2005), 457–482, doi:10.1007/s00211-005-0596-3.

[9] M. J. C. Gover, The eigenproblem of a tridiagonal 2-toeplitz matrix, *Linear Algebra Appl.* **197-198** (1994), 63–78, doi:10.1016/0024-3795(94)90481-2.

[10] M. Hall, Divisibility sequences of third order, *Am. J. Math.* **58** (1936), 577–584, doi:10.2307/2370976.

[11] V. E. Hoggatt, Jr. and C. T. Long, Divisibility properties of generalized fibonacci polynomials, *Fibonacci Q.* **12** (1974), 113–120, https://www.fq.math.ca/Scanned/12-2/hoggatt1.pdf.

[12] C. Kimberling, Strong divisibility sequences and some conjectures, *Fibonacci Q.* **17** (1979), 13–17, https://www.fq.math.ca/Scanned/17-1/kimberling1.pdf.

[13] V. Lovass-Nagy and P. Rózsa, Matrix analysis of transient voltage distributions in alternating ladder networks, *Proc. Inst. Electr. Eng.* **110** (1963), 1663–1670, doi:10.1049/piee.1963.0236.

[14] V. Lovass-Nagy and P. Rózsa, Die Berechnung von Ausgleichvorgängen auf längskompensierten Fernleitungen, *Arch. Elektrotechn.* **49** (1964), 260–270, doi:10.1007/bf01407243.

[15] E. Lucas, Theorie des fonctions numeriques simplement periodiques, *Am. J. Math.* **1** (1878), 184–196, doi:10.2307/2369308.

[16] D. Panario, M. Sahin, Q. Wang and W. Webb, General conditional recurrences, *Appl. Math. Comput.* **243** (2014), 220–231, doi:10.1016/j.amc.2014.05.108.

[17] M. O. Rayes, V. Trevisan and P. S. Wang, Factorization properties of Chebyshev polynomials, *Comput. Math. Appl.* **50** (2005), 1231–1240, doi:10.1016/j.camwa.2005.07.003.

[18] P. Rózsa, On periodic continuants, *Linear Algebra Appl.* **2** (1969), 267–274, doi:10.1016/0024-3795(69)90030-5.

[19] D. E. Rutherford, Some continuant determinants arising in physics and chemistry, *Proc. R. Soc. Edinb. Sect. A* **62** (1947), 229–236, doi:10.1017/s0080454100006634.

[20] M. Sahin and E. Tan, Conditional (strong) divisibility sequences, *Fibonacci Q.* **56** (2018), 18–31, `https://www.fq.math.ca/Papers/56-1/SahinTan05252017rev.pdf`.

[21] M. Ward, Linear divisibility sequences, *Trans. Amer. Math. Soc.* **41** (1937), 276–286, doi:10.2307/1989623.

[22] W. A. Webb and E. A. Parberry, Divisibility properties of fibonacci polynomials, *Fibonacci Q.* **7** (1969), 457–463, `https://www.fq.math.ca/Scanned/7-5/webb.pdf`.

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Maximal order group actions on Riemann surfaces

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Abstract

A natural problem is to determine, for each value of the integer $g \geq 2$, the largest order of a group that acts on a Riemann surface of genus g. Let $N(g)$ (respectively $M(g)$) be the largest order of a group of automorphisms of a Riemann surface of genus $g \geq 2$ preserving the orientation (respectively possibly reversing the orientation) of the surface.

The basic inequalities comparing $N(g)$ and $M(g)$ are $N(g) \leq M(g) \leq 2N(g)$. There are well-known families of extended Hurwitz groups that provide an infinite number of integers g satisfying $M(g) = 2N(g)$. It is also easy to see that there are solvable groups which provide an infinite number of such examples.

We prove that, perhaps surprisingly, there are an infinite number of integers g such that $N(g) = M(g)$. Specifically, if p is a prime satisfying $p \equiv 1 \pmod 6$ and $g = 3p + 1$ or $g = 2p + 1$, there is a group of order $24(g - 1)$ that acts on a surface of genus g preserving the orientation of the surface. For all such values of g larger than a fixed constant, there are no groups with order larger than $24(g - 1)$ that act on a surface of genus g.

Keywords: Riemann surface, genus, group action, NEC group, strong symmetric genus.

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1 Introduction

A finite group G can be represented as a group of automorphisms of a compact Riemann surface. In most of the classical work, the group actions were required to preserve the orientation of the Riemann surface. It is also possible to allow the group actions to reverse the orientation of the surfaces.

Among the most interesting group actions for a particular value of the genus g are those such that the orders of the groups are "large" relative to the genus g. A natural problem,

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then, is to determine, for each value of the integer $g \geq 2$, the largest order of a group that acts on a Riemann surface of genus g.

Let $N(g)$ (respectively $M(g)$) be the largest order of a group of automorphisms of a Riemann surface of genus $g \geq 2$ preserving the orientation (respectively possibly reversing the orientation) of the surface. Now suppose the group G acts on the Riemann surface X of genus $g \geq 2$ (possibly reversing the orientation of X). Let G^+ be the subgroup of G consisting of the orientation preserving automorphisms. Then $|G^+| \leq N(g)$ and

$$|G| \leq 2|G^+| \leq 2N(g). \tag{1.1}$$

Consequently, we obtain the basic inequalities comparing $N(g)$ and $M(g)$.

$$N(g) \leq M(g) \leq 2N(g). \tag{1.2}$$

The classical upper bound of Hurwitz shows that, for all $g \geq 2$,

$$N(g) \leq 84(g-1) \text{ and } \mathrm{M(g)} \leq 168(\mathrm{g}-1). \tag{1.3}$$

A group G of order $84(g-1)$ is called a Hurwitz group if it acts on a surface of genus g preserving orientation. If the Hurwitz group has an extension G^* of order $2|G|$ that acts on the same surface, then G^* is an extended Hurwitz group. If g is a genus for which there is an extended Hurwitz group, then $N(g) = 84(g-1)$ and $M(g) = 2N(g)$. These groups have generated considerable interest; see especially [5] but also [3, 4] and [22]. There are known infinite families of extended Hurwitz groups. For example, Conder showed that all symmetric groups Σ_n for $n > 167$ are extended Hurwitz groups [8, p. 75]. Consequently, the bounds in (1.3) and the upper bound for $M(g)$ in (1.2) are attained for infinitely many g.

On the other hand, the general lower bound for $N(g)$ is

$$N(g) \geq 8(g+1) \tag{1.4}$$

for all $g \geq 2$. Further, this lower bound is the best possible, that is, there are infinitely many g such that $N(g) = 8(g+1)$. These results were established independently by Accola [1] and Maclachlan [13].

The corresponding lower bound for $M(g)$ is easy to establish.

Theorem 1.1. *For all integers $g \geq 2$, $M(g) \geq 16(g+1)$. Further, there are infinitely many g such that $M(g) = 16(g+1)$.*

The family of groups used by Accola [1] and Maclachlan [13] to establish the results about the lower bound (1.4) can be extended following the approach of Singerman [22, p. 22]. This yields, for each $g \geq 2$, the construction of a group of order $16(g+1)$ that acts on a Riemann surface of genus g so that $M(g) \geq 16(g+1)$.

These groups are another family of groups such that $M(g) = 2N(g)$ for infinitely many genera g. Indeed, intuitively, one expects $M(g)$ to "often" be equal to $2N(g)$. But it is certainly possible that $M(g) < 2N(g)$. For example, the two smallest values of g for which $M(g) < 2N(g)$ are 17 and 20 with $N(17) = 1344$, $M(17) = 1536$ and $N(20) = 228$, $M(20) = 336$. Also, the classification of orientably regular maps of genus $p+1$ [9] and the Belolipetsky-Jones group of order $12p$ for prime p [2, p. 382] shows that $M(g) < 2N(g)$ for infinitely g.

However, for some values of g, $N(g) = M(g)$. The two smallest values of g satisfying $N(g) = M(g)$ are 27 and 28 with $N(27) = M(27) = 624$ and $N(28) = M(28) = 1296$. Surprisingly, this equality holds for infinitely many g. Our main result is the following.

Theorem 1.2. *There are infinitely many g such that $M(g) = N(g)$.*

Specifically, if p is a prime satisfying $p \equiv 1 \pmod 6$ and $g = 3p + 1$, there is a group of order $24(g-1)$ that acts on a surface of genus g preserving the orientation of the surface. For all such values of $g = 3p + 1$ larger than a fixed constant, there are no groups with order larger than $24(g-1)$ that act on a surface of genus g (including those that reverse orientation). Similar results hold if $p \equiv 1 \pmod 6$ and $g = 2p + 1$.

Here we acknowledge our debt to the data on large group actions on surfaces of low genus calculated by Conder [6]. This data was quite helpful in conjecturing Theorem 5.6 and its corollary Theorem 1.2.

We would also like to express our sincere gratitude to the referee for numerous helpful comments. These led to significant improvements in the first three sections.

2 Background results

Much of the following background information is taken from [18]; also see [10, Section 2]. Let the finite group G act on the (compact) Riemann surface X of genus $g \geq 2$. Then represent $X = U/K$, where K is a Fuchsian surface group and obtain an NEC group Γ and a homomorphism $\phi\colon \Gamma \to G$ onto G such that $K = kernel\ \phi$. Associated with the NEC group Γ are its signature and canonical presentation.

Further, the non-euclidean area $\mu(\Gamma)$ of a fundamental region for Γ can be calculated directly from its signature. Here see [21, p.235], where $\mu(\Gamma)$ is given in terms of the topological genus of the quotient surface U/Γ and the periods and link periods of Γ. Then the genus of the surface X on which G acts is given by

$$g = 1 + |G| \cdot \mu(\Gamma)/4\pi. \tag{2.1}$$

The simpler, classical case is that G acts on X preserving orientation. This is the case if and only if Γ is a Fuchsian group and G is generated by elements a_i, b_i for $1 \leq i \leq h$ and x_j of order m_j for $1 \leq j \leq k$ with relation $x_1 \cdots x_k[a_1, b_1] \cdots [a_h, b_h] = 1$. Then the application of (2.1) yields the classical Riemann-Hurwitz equation

$$2g - 2 = |G| \left(2h - 2 + \sum_{j=1}^{k} \left(1 - \frac{1}{m_j} \right) \right). \tag{2.2}$$

The group G acts reversing the orientation of X in case Γ is a proper NEC group. Then it is necessary to check that the surface group K does not contain orientation-reversing elements, or equivalently, the image $\phi(\Gamma^+)$ has index two in G [20, Theorem 1, p. 52]. If this condition holds, then we will say that G has a particular partial presentation *with the Singerman subgroup condition*. The Riemann-Hurwitz equation in this case is more complicated and is in [10, p. 274], for instance. In this case, though, $|G| = 2|G^+|$ and (2.2) can be employed to calculate the relationship between the genus g and $|G|$.

In connection with group actions on surfaces, there are two natural parameters associated with each finite group. The *symmetric genus* $\sigma(G)$ of the group G is the minimum genus of any Riemann surface on which G acts faithfully (possibly reversing orientation). The *strong symmetric genus* $\sigma^0(G)$ of G is the minimum genus of any Riemann surface on which G acts faithfully preserving orientation.

Next we quickly survey the NEC groups with relatively small non-euclidean area. We use the notation of [18]. First, an (ℓ, m, n) triangle group is a Fuchsian group Λ with signature

$$(0; +; [\ell, m, n]; \{\}), \text{ where } 1/\ell + 1/m + 1/n < 1.$$

If the group G is a quotient of Λ by a surface group, then G has a presentation of the form

$$X^\ell = Y^m = (XY)^n = 1. \tag{2.3}$$

We will say that G has partial presentation $T(\ell, m, n)$.

There are two types of NEC groups with a triangle group as canonical Fuchsian subgroup. A full (or extended) (ℓ, m, n) triangle group is an NEC group Γ with signature

$$(0; +; [\,]; \{(\ell, m, n)\}), \text{ where } 1/\ell + 1/m + 1/n < 1.$$

If G is a quotient of Γ (by a surface group), then G has a presentation of the form

$$A^2 = B^2 = C^2 = (AB)^\ell = (BC)^m = (CA)^n = 1, \tag{2.4}$$

and, further, the subgroup generated by AB and BC (the image of Γ^+) has index 2. The partial presentation (2.4) will be denoted $FT(\ell, m, n)$.

A hybrid $(m; n)$ triangle group is an NEC group Γ with signature

$$(0; +; [m]; \{(n)\}), \text{ where } 2/m + 1/n < 1.$$

The canonical Fuchsian subgroup Γ^+ is a (m, m, n) triangle group. If G is a quotient of Γ, then G has a presentation of the form

$$C^2 = X^m = [C, X]^n = 1, \tag{2.5}$$

and the subgroup generated by X and CXC has index 2. This partial presentation will be denoted $HT(m; n)$.

An (ℓ, m, n, t) quadrilateral group is a Fuchsian group Λ with signature

$$(0; +; [\ell, m, n, t]; \{\ \}), \text{ where } 1/\ell + 1/m + 1/n + 1/t < 2.$$

A quotient group G of Λ has a presentation of the form

$$X^\ell = Y^m = Z^n = (XYZ)^t = 1 \tag{2.6}$$

We will denote this partial presentation $Q(\ell, m, n, t)$. If a group has presentation (2.6) with ℓ, m, n, t all equal to 2, then the group acts on a torus.

Suppose G is a group that acts on a Riemann surface X of genus $g \geq 2$, where X is represented $X = U/K$ and $G = \Gamma/K$. Particularly important here is the case in which $|G| > 24(g-1)$, and we will say G is a *large* group of automorphisms of X. There is, of course, a corresponding restriction on the non-euclidean area of the NEC group Γ and the types of partial presentations that Γ can have. The area restriction is $\mu(\Gamma)/2\pi < 1/12$, which is fairly limiting. A careful check of the signatures gives the following. This result appears in [18, Theorem 2] and also [10, p. 275]. Here we have added the specific Riemann-Hurwitz equation for each case. For example, if G has the partial presentation $FT(2, 4, s)$, then $\mu(\Gamma)/2\pi = (s-4)/8s$. Then using equation (2.2) gives $16(g-1) = |G|(s-4)/s$.

Theorem A. *Let G be a group that acts on a Riemann surface of genus $g \geq 2$. Then $|G| > 24(g-1)$ if and only if G has a partial presentation (with the relations fulfilled) of type $T(2,4,5)$ or $T(2,3,s)$, where $7 \leq s \leq 11$, or one of the following types with the Singerman subgroup condition satisfied. The application of the Riemann-Hurwitz equation is included for each case.*

1. $FT(2,3,s)$, $24(g-1) = |G|(s-6)/s$ *where* $s \geq 7$,
2. $FT(2,4,s)$, $16(g-1) = |G|(s-4)/s$ *where* $5 \leq s \leq 11$,
3. $FT(2,5,s)$, $20(g-1) = |G|(3s-10)/s$ *where* $5 \leq s \leq 7$,
4. $FT(3,3,s)$, $12(g-1) = |G|(s-3)/s$ *where* $4 \leq s \leq 5$,
5. $HT(3;4)$, $48(g-1) = |G|$,
6. $HT(3;5)$, $30(g-1) = |G|$,
7. $HT(5;2)$, $40(g-1) = |G|$.

3 Basic lower bound for $M(g)$

We begin by constructing the family of groups that provides the lower bound in Theorem 1.1.

Fix the integer $m \geq 3$, and let L_m be the group defined by the presentation

$$x^2 = y^4 = z^{2m} = xyz = 1, (z^2)^x = z^{-2}. \tag{3.1}$$

It is easy to see that L_m is an extension of the cyclic group Z_m by the dihedral group D_4 and consequently $|L_m| = 8m$. Then the group L_m has partial presentation $T(2,4,2m)$. Then a calculation using (2.1) shows that L_m acts on a Riemann surface X of genus $g = m-1$ preserving the orientation of the surface. The group L_m has order $8(g+1)$. This family of groups is certainly not new. The family L_m was used, independently, to establish the lower bound $8(g+1)$ by both Accola and Maclachlan; here see [1, p. 400] and [13, Theorem 4, p. 266]. The construction of this family also appears in [2, p. 384].

Next we construct an extension of the group L_m by Z_2, following the approach in [15, p. 128]. To L_m adjoin an element t of order 2 that transforms the elements of L_m according to the automorphism

$$\alpha(x) = x^{-1}, \alpha(y) = y^{-1}. \tag{3.2}$$

Then the extension L_m^* has presentation

$$t^2 = x^2 = y^4 = z^{2m} = xyz = (tx)^2 = (ty)^2 = 1, (z^2)^x = z^{-2}. \tag{3.3}$$

The extension L_m^* of L_m has order $2|L_m|$ and has partial presentation $FT(2,4,2m)$. Thus the group L_m^* is a group of order $16(g+1)$ that acts on the surface X of genus g. This extended family was described by Singerman in [22, p. 24]. Now it is easy to prove Theorem 1.1.

Proof. Fix $g \geq 2$, and set $m = g+1$. Then $M(g) \geq |L_m^*| = 16(g+1)$. Also, there are infinitely many values of g such that $N(g) = 8(g+1)$; here see [1, Theorem 4, p. 407] or [13, Theorem 5, p. 272]. Then for such a value of g, $M(g) \leq 2N(g) = 16(g+1)$ using the basic inequality (1.2). Hence there are infinitely many values of g such that $M(g) = 16(g+1)$. □

Before proving Theorem 1.2, we establish an interesting result about the family of groups L_m. We have seen that L_m acts preserving orientation on a Riemann surface of genus $g = m - 1$. In fact, this value is the strong symmetric genus of the group L_m.

First, we get rid of a redundant generator in the definition of L_m and obtain the presentation

$$x^2 = z^{2m} = (zx)^4 = 1, (z^2)^x = z^{-2}. \tag{3.4}$$

Theorem 3.1. $\sigma^0(L_m) = m - 1$.

Proof. Let L_m have generators x and z and relations (3.4) and be generated by u and v. Define $N = \langle z^2, (zx)^2\rangle$. The element $(zx)^2$ is in the center of L_m and since conjugation by x inverts z^2, N is a normal subgroup of L_m. Since $L_m/N \cong Z_2 \times Z_2$, uN, vN and uvN are the same as the cosets xN, zN and zxN in some order. All elements of L_m in the set xN have order 2 and all elements in the set zxN have order 4. The elements of the set zN are of the form z^k or $z^k(zx)^2$ and have order $2m/d$, where $d = gcd(k, 2m)$.

Let u be an element from xN and v an element from zxN. The product is contained in zN. So $uv = (xz)^2z^t = ((xz)^2z)^t$ or $uv = z^t$, where t is odd. Next, suppose the product uv has order smaller than $2m$. So $gcd(t, m) = d > 2$. Let $M = \langle (xz)^2, z^t\rangle$. Since $xzx = (xz)^2z^{-1}$, M is a normal subgroup of L_m of order $4m/d$. It follows that $\langle u, v\rangle = \langle u, uv\rangle \subseteq \langle u, M\rangle$ and since $|\langle u, M\rangle| = 8m/d \neq 8m$, the elements u and v do not generate L_m. Therefore, the product of two generators, u of order 2 and v of order 4, must have order $2m$.

Hence L_m has presentation $T(2, 4, 2m)$, and the corresponding triangle group is the only one that maps faithfully onto L_m.

Suppose that L_m has partial presentation $Q(2, 2, 2, 2)$ and acts on a torus. Let L_m be generated by involutions s, t, u and v satisfying $stuv = 1$. Let $N = \langle (zx)^2, z^2\rangle$. All elements of order 2 are either contained in the coset xN or are in the normal subgroup $V = \langle (zx)^2, z^m\rangle$ of L_m. Since there are no elements of order 2 in the coset zxN, an even number of s, t, u or v must be from xN. If none of the generators are from xN, then $\langle s, t, u, v\rangle \subseteq \langle z, N\rangle \neq L_m$. If all four generators are in xN, then $\langle s, t, u, v\rangle \subseteq \langle x, N\rangle \neq L_m$. Suppose that only two of the generators are from xN, say u and v. Then $\langle s, t, u, v\rangle \subseteq \langle x, V\rangle \neq L_m$ and again we get a contradiction. It also follows that $\sigma^0(L_m) \neq 1$ for all $m > 2$.

Suppose that L_m has partial presentation $Q(2, 2, 2, 3)$. Let L_m be generated by involutions s, t, u and the element v of order 3. The element v must be contained in $\langle z^2\rangle \subseteq N$. Since there are no elements of order 2 in the coset zxN and $(sN)(tN)(uN) = (1N)$, we can't have one of the cosets be xN and another be zN, since then the third would be in zxN. If one or more of s, t and u are in xN, then $\langle s, t, u, v\rangle \subseteq \langle x, N\rangle \neq L_m$. If one or more of s, t and u are in zN, then $\langle s, t, u, v\rangle \subseteq \langle z, N\rangle \neq L_m$. So L_m does not have presentation $\Gamma(2, 2, 2, 3)$. No other Fuchsian group has small enough non-euclidean area and the proof is complete. □

Theorem 3.1 shows that there is at least one group with strong symmetric genus g for all g, which is the main result of [16]. One interesting thing here is that the well-known groups of Theorem 3.1 provide an alternate proof of [16, Theorem 1], which was established using groups of the form $Z_k \times D_n$.

Theorem 3.1 also has a consequence for the function that counts the number of groups of each genus. Using direct products and dicyclic groups, it was shown that there are at least four groups of strong symmetric genus g for all $g \geq 0$ [14, Theorem 1]. It is not hard to see that the group L_{g+1} is another group of genus g, and we have the following.

Theorem 3.2. *If g is a non-negative integer, then there are at least 5 groups of strong symmetric genus g.*

We remark here that these families L_m and L_m^* are groups that act on the torus. These groups are in in Proulx classes (g) and (k) respectively; for the associated partial presentations, see [12, pp. 291,292]. The orientation preserving subgroup of the action of L_m^* on the torus is not L_m, even though L_m is the orientation preserving subgroup of the action of L_m^* on the surface of genus $m-1$. Consequently, the two families L_m and L_m^* are of no help in filling the symmetric genus spectrum. The groups $Z_k \times D_n$ used in [16] to fill all the gaps in the strong symmetric genus spectrum are also groups that act on the torus and have symmetric genus one.

4 A family of $24(g-1)$ automorphisms

Our main task here is to show there are infinitely many values of g such that $M(g) = N(g)$. This result was something of a surprise, to us at least, and it is not easy to prove.

We start with the construction of another family of groups. Let p be a prime satisfying $p \equiv 1 \pmod 6$ and m an integer satisfying $m^3 \equiv 1 \pmod p$ and not congruent to 1 $\pmod p$. Define the groups J_p by the presentation

$$x^3 = u^3 = v^2 = z^p = (uv)^4 = [x,u] = [x,v] = [z,u] = 1, \tag{4.1}$$
$$z^x = z^m, z^v = z^{-1}.$$

It is easy to see that J_p is the semidirect product of the cyclic group Z_p by the group $Z_3 \times \Sigma_4$, namely $Z_p \times_\phi (Z_3 \times \Sigma_4)$ where ϕ is a homomorphism mapping x into $z \to z^m$, u into $z \to z$ and v into $z \to z^{-1}$.

Theorem 4.1. *The group J_p has partial presentation $T(2,3,12)$ and hence acts on a surface of genus $1 + 3p$.*

Proof. We will use the presentation (4.1). First, $o(v) = 2$, $o(ux) = 3$ and $o(vux) = 12$. In addition, $\langle v, ux\rangle = \langle x, u, v\rangle \cong Z_3 \times \Sigma_4$.

Define $r = vz$ and $w = z^{-1}ux$. Clearly, $o(r) = 2$. It is easy to verify that $z^{-k}x = xz^{-km}$. Therefore, $w^3 = (z^{-1}x)^3 = z^{-(m^2+m+1)} = 1$, since $m^2 + m + 1 \equiv 0 \pmod p$. It follows that $o(w) = 3$ and $o(rw) = 12$.

Next, we need to show that $J_p = \langle r, w\rangle$. First, $[r,w] = z^{-(m+1)}[v,u]$. Next, we show that $[r,w]^3 = z^{-3(m+1)}$. If $m \equiv -1 \pmod p$, then $m^3 \equiv -1 \pmod p$ and this is false. Therefore, $z \in \langle [r,w]\rangle$ and it follows that $\langle r,w\rangle = \langle z,x,u,v\rangle = J_p$. Thus J_p has partial presentation $T(2,3,12)$.

□

It is not difficult to see that, in fact, $\sigma^0(J_p) = 1 + 3p$. An obvious consequence of Theorem 4.1 is the following.

Theorem 4.2. *Let p be a prime such that $p \equiv 1 \pmod 6$, and let $g = 3p + 1$. Then the group J_p is a group of order $24(g-1)$ that acts on a surface of genus g preserving the orientation of the surface. Consequently, for any such g,*

$$M(g) \geq N(g) \geq 24(g-1). \tag{4.2}$$

There are, of course, infinitely many such g. We will show that, for most of these values of g, $M(g) = N(g) = 24(g-1)$, establishing Theorem 1.2.

5 Large groups of automorphisms

Assume p is a prime such that $p \equiv 1 \pmod 6$, and let $g = 3p + 1$. Then Theorem 4.2 shows that there is a group of order $24(g-1)$ that acts on a surface of genus g preserving the orientation of the surface, and inequality (4.2) holds. The hard part of the proof of Theorem 1.2 is to show that, for most of these values of g, there are no large groups of automorphisms, that is, no groups with order larger than $24(g-1)$. We use Theorem A. In this section we do not assume that $p \equiv 1 \pmod 6$. However, in the proof it is necessary to assume that that the prime p is not small. This will enable us to apply the following useful result of Accola [1, Lemma 5, p. 402].

Accola's Lemma. *Let G be a non-abelian image of the triangle group $T(2,3,\lambda)$ of order $\mu\lambda$. Then $\lambda \leq \mu^2$.*

Let X be a Riemann surface of genus g, and suppose that G were a large group of automorphisms of X. Then $|G| > 24(g-1) = 72p$, and G has one of the partial presentations in Theorem A. We show that, in fact, G cannot have any of these partial presentations. While it is necessary to consider each presentation, we describe the overall outline of the argument but omit some details. In addition, to apply Accola's Lemma, it is necessary to assume that the prime p is not small, and we assume that $p > (36)^2$.

Lemma 5.1. *If the prime $p > (36)^2$, then p divides $|G|$ but p^2 does not.*

Proof. Suppose first that G has any of the partial presentations in Theorem A except $FT(2,3,s)$. In these cases, the Riemann-Hurwitz formulas in Theorem A give $|G|$ in terms of the parameter s, and for the values of s that can occur, $|G|$ is a multiple of p but p^2 does not divide $|G|$ (for large p). For example, suppose G has partial presentation $FT(2,5,s)$, where s is 5, 6 or 7. Then if s is 5 or 6, then G is $120p$ or $90p$, respectively, and p^2 does not divide $|G|$ if $p > 5$. If $s = 7$, then $|G|$ is not an integer.

Suppose now that G has partial presentation $FT(2,3,s)$ where $s \geq 7$. In this case, $|G| = 72ps/(s-6)$ so that $72ps = |G|(s-6)$. First, for small s, $7 \leq s \leq 12, s \neq 11$, $|G|$ is a multiple of p but p^2 does not divide $|G|$ (for large p). For example, if $s = 8$, $|G| = 288p$. If $s = 11$, $|G|$ is not an integer.

Assume then that G has partial presentation $FT(2,3,s)$ where $s > 12$, the hard case. Now by Euclid's Lemma, either p divides $|G|$ or p divides $(s-6)$.

Assume that p divides $(s-6)$ and write $s - 6 = mp$ for some integer $m \geq 1$. Now $s = mp + 6 > p > (36)^2$ (by assumption). But on the other hand, $|G| = 72ps/mp = 72s/m$. Then $|G^+| = 36s/m$. The group of orientation preserving automorphisms G^+ is a $T(2,3,s)$ group of order cs, where $c = 36/m \leq 36$. Now by Accola's Lemma, $p < s \leq c^2 \leq (36)^2$, an obvious contradiction. Thus, if G is a $FT(2,3,s)$ group (and $p > (36)^2$), then p divides $|G|$.

Finally, we have $|G|/p = 72s/(s-6)$. With $s > 12$, $s/(s-6) < 2$ so that $|G|/p = 72s/(s-6) < 144$. Hence, p^2 does not divide $|G|$ for large p. □

Lemma 5.2. *The Sylow p-subgroup $S_p \cong Z_p$ of G is normal in G.*

Proof. A review of the calculations in the previous proof shows that in each case that is arithmetically possible, $|G| = cp$ for some constant $c < p$ for large p. The constant c depends on the presentation, of course. Now, obviously, $|S_p| = p$. Also, the number n_p of Sylow p-subgroups of G is $\equiv 1 \pmod p$ and is a divisor of $|G|$. Then $n_p \geq (p+1)$ is clearly not possible. Hence S_p is normal in G. □

Now let S_p act on X with $Y = X/S_p$ the quotient space, γ the genus of Y and $\pi\colon X \to Y$ the quotient map.

Lemma 5.3. *The quotient map π is unramified, and the quotient space $Y = X/S_p$ has genus $\gamma = 4$. Further, the quotient group $Q = G/S_p$ is a large group of automorphisms of Y.*

Proof. Let τ be the number of branch points of π. Then the Riemann-Hurwitz formula gives

$$2(g-1)/p = 2(\gamma - 1) + \tau(p-1)/p. \tag{5.1}$$

Then $2(g-1) = 2p(\gamma-1) + \tau(p-1)$ and we have $g - 1 = 3p$. Now $\tau(p-1) = 6p - 2p(\gamma - 1) = 2p(4-\gamma)$. Since $\tau(p-1) \geq 0$, $4 \geq \gamma$. If $\gamma = 4$, then $\tau = 0$. Assume $\gamma < 4$. Then $p-1$ divides $p(8-2\gamma)$. Since $p-1$ and p are relatively prime, $p-1$ divides $(8-2\gamma)$ so that $p - 1 \leq 8 - 2\gamma \leq 8$. Now $p \leq 9$ contradicting the assumption that p is large. Thus $\gamma = 4$ and the number of branch points $\tau = 0$, that is, the quotient map π is unramified.

Now the quotient group Q acts on the surface Y of genus 4. Since G is a large group of automorphisms of X, $|G| > 24(g-1) = 72p$. Then $|Q| = |G|/p > 72 = 24(4-1)$ and Q is a large group of automorphisms of Y. □

The large group actions on Riemann surfaces of genus 4 have been classified, and these are presented in Table 1. These group actions were considered in determining the groups of symmetric genus 4; here see [18, pp. 4089,4090] and [10, p. 285]. With a single exception, these actions correspond to groups of reflexible regular maps. A description of the connection between groups of regular maps and large groups of automorphisms of Riemann surfaces is in [17, p. 24]. The regular maps of genus 4 were first classified by Garbe [11, p. 53]. These maps also appear in [7, Table 1]. In Table 1, we give the group number in the MAGMA small groups library. Map symbols are from [7].

Table 1: Large Group Actions on Surfaces of Genus 4.

Group	Order	Library Number	Partial Presentation	Map Symbol	G/G′
$\Sigma_3 \times \Sigma_4$	144	183	$FT(2,3,12)$	R4.1	$(Z_2)^2$
$Z_2 \times \Sigma_5$	240	189	$FT(2,4,5)$	R4.2	$(Z_2)^2$
Σ_5	120	34	$T(2,4,5)$	R4.2	Z_2
	144	186	FT(2,4,6)	R4.3	$(Z_2)^3$
$D_4 \times D_5$	80	39	FT(2,4,10)	R4.4	$(Z_2)^3$
$Z_2 \times A_5$	120	35	FT(2,5,5)	R4.6	Z_2
Σ_5	120	34	$HT(5;2)$		Z_2

The group G is an extension of $S_p \cong Z_p$ by Q. Since $|Q|$ is relatively prime to p, the group G is a semidirect product, by the Schur-Zassenhaus Lemma.

Lemma 5.4. $G \cong Z_p \times_\phi Q$.

The following is important here. The proof is an exercise using the definition of semidirect product.

Lemma 5.5. *Let H be the semidirect product $K \times_\theta Q$, and let $L = kernel(\theta)$. Then L is normal in the big group H.*

Theorem 5.6. *Let p be a prime such that $p > (36)^2$. There are no large groups of automorphisms that act on a surface of genus $g = 3p + 1$.*

Proof. For each of the possibilities for Q, we show that G cannot have the relevant partial presentation.

First suppose there is a group G of order $144p$ with partial presentation $FT(2, 3, 12)$. In particular, G is generated by involutions. Then $G \cong Z_p \times_\phi Q$, where $Q \cong \Sigma_3 \times \Sigma_4$. Let $L = kernel(\phi)$. Since $\phi\colon Q \to Aut(Z_p) \cong Z_{p-1}$, Q/L is cyclic. It follows that $Q' \subset L \subset Q$. Now a calculation shows that the commutator quotient group $Q/Q' \cong (Z_2)^2$. Hence L must have index 1 or 2 in Q, and L is normal in G by Lemma 5.5. If $L = Q$, then $G \cong Z_p \times Q$. Then G is obviously not generated by involutions, since Z_p is not. Hence $[Q : L] = 2$ and the quotient group G/L has order $2p$ so that G/L is isomorphic to either Z_{2p} or the dihedral group D_p. Since Z_{2p} is not generated by involutions, we must have $G/L \cong D_p$. But D_p is not a quotient of a $FT(2, 3, 12)$ group (the product of reflections in D_p has order p or 1). Thus there is no group of order $144p$ with partial presentation $FT(2, 3, 12)$.

Essentially the same proof (using the same notation) shows that there are no groups of order $80p$ with presentation $FT(2, 4, 10)$ and also none of order $144p$ with presentation $FT(2, 4, 6)$. The only difference in each of the cases is that the commutator quotient group $Q/Q' \cong (Z_2)^3$. But it still follows that L has index 1 or 2 in Q.

The proof is very similar but even easier in case there were a group G of order $120p$ with presentation $FT(2, 5, 5)$. Then $G \cong Z_p \times_\phi Q$, where $Q \cong Z_2 \times A_5$. Now $Q' \cong A_5$ so that either $L = Q$ or $L = Q'$. Again, as in the previous cases, L has index 1 or 2 in Q, and it follows in the same way that this case is not possible either.

Now suppose there were such a group G with order $120p$ with partial presentation $T(2, 4, 5)$. Then G is generated by two elements of orders 2 and 5. Then $G \cong Z_p \times_\phi Q$, where $Q \cong \Sigma_5$. Then $Q' \cong A_5$, and thus either $L = Q$ or $L = Q'$. If $L = Q$, then $G \cong Z_p \times Q$. Then G is obviously not a quotient of a $T(2, 4, 5)$ group, since Z_p is not (Z_p has no elements of order 2). Hence $L = Q'$ and the quotient group G/L has order $2p$ so that G/L is isomorphic to either Z_{2p} or the dihedral group D_p. Neither group is a quotient of $T(2, 4, 5)$ groups; neither group has an element of order 5. Thus there is no group of order $120p$ with partial presentation $T(2, 4, 5)$.

Further, there is no group G of order $240p$ with partial presentation $FT(2, 4, 5)$. If there were such a group G acting on a surface X of genus $3p + 1$, then the group G^+ of orientation preserving automorphisms would be a $T(2, 4, 5)$ group acting on X. But we have just seen that this is not possible. Hence there is no $FT(2, 4, 5)$ group of order $240p$.

Finally, assume that there were a group G of order $120p$ with partial presentation $HT(5; 2)$. Then G is generated by two elements of orders 2 and 5, and $G \cong Z_p \times_\phi Q$, where $Q \cong \Sigma_5$. Now $Q' \cong A_5$, and either $L = Q$ or $L = Q'$. If $L = Q$, then $G \cong Z_p \times Q$. Then G is obviously not a quotient of a $HT(5; 2)$ group, since Z_p is not (Z_p has no elements of order 2). Hence $L = Q'$ and the quotient group G/L has order $2p$ so that G/L is isomorphic to either Z_{2p} or the dihedral group D_p. Neither of these groups is a quotient of a $HT(5;2)$ group; neither group has an element of order 5. Thus there is no group of order $120p$ with partial presentation $HT(5;2)$.

In summary, none of the partial presentations listed in Table 1 are possible, and there is no large group action on a surface of genus $g = 3p + 1$ for large $p > (36)^2$. □

Combining Theorems 4.2 and 5.6 gives the following.

Theorem 5.7. *Let p be a prime such that $p \equiv 1 \pmod 6$ and $p > (36)^2$, and let $g = 3p + 1$. Then for any such g,*

$$M(g) = N(g) = 24(g - 1). \tag{5.2}$$

Applying Dirichlet's Theorem about the number of primes in an arithmetic sequence establishes Theorem 1.2.

6 Another family of $24(g - 1)$ automorphisms

There is another interesting family of groups that can be used to determine an infinite sequence of odd values of g such that $M(g) = N(g)$. This provides an alternate proof of Theorem 1.2, and it may be established using arguments similar to those in the two previous sections. But there is no improvement to Theorem 1.2, of course, and technically the proof is somewhat harder. We describe this family of groups but only comment very briefly on the arguments in this case.

Let p be a prime satisfying $p \equiv 1 \pmod 6$ and m an integer satisfying $m^3 \equiv 1 \pmod p$ and not congruent to $1 \pmod p$. Define the groups K_p by the presentation

$$u^3 = v^2 = (uv)^3(u^{-1}v)^3 = z^p = 1, z^u = z^m, z^v = z^{-1}. \tag{6.1}$$

It is easy to see that K_p is the semidirect product of the cyclic group Z_p by the group P_{48}. The group P_{48} has order 48 and contains $SL(2,3)$ as a subgroup; a presentation is in [15, p. 116]. It is one of the groups of symmetric genus 2 [15, Theorem 4]. The group K_p has partial presentation $T(2,3,12)$ and acts on a surface of genus $1 + 2p$. This gives the following analog of Theorem 4.2.

Theorem 6.1. *Let p be a prime such that $p \equiv 1 \pmod 6$, and let $g = 2p + 1$. Then the group K_p is a group of order $24(g - 1)$ that acts on a surface of genus g preserving the orientation of the surface. Consequently, for any such g,*

$$M(g) \geq N(g) \geq 24(g - 1). \tag{6.2}$$

Using the approach (and notation) of Section 6, it is possible to show that there are no large groups of automorphisms for most of these values of g. The analog of Lemma 5.1 holds; it is necessary to assume that $p > (24)^2$ to apply Accola's result. Then the analog of Lemma 5.2 is easy to establish. The result corresponding to Lemma 5.3 holds with a similar proof. There is an important difference here, though. The quotient space Y has genus $\gamma = 3$. It is necessary, then, to consider the large group actions on Riemann surfaces of genus 3. These actions have been classified; see [18, p. 4089] and [10, p. 285]. The regular maps of genus 3 were classified by Sherk [19]; also see [7, Table 1]. There are 10 large group actions in all. Eight of these are map groups, but there are also two groups of 96 to consider, a $FT(3,3,4)$ group and a $HT(3;4)$ group. The analogs of Lemmas 5.4 and 5.5 continue to hold, as does the following companion to Theorem 5.6.

Theorem 6.2. *Let p be a prime such that $p > (24)^2$. There are no large groups of automorphisms that act on a surface of genus $g = 2p + 1$.*

It is necessary to consider the ten possibilities for the quotient group Q. In nine of the cases, as in the proof of Theorem 5.6, an argument using the commutator quotient group suffices; in these cases, Q/Q' is isomorphic to 1, Z_2, $(Z_2)^2$, or $(Z_2)^3$. The exceptional case is the one in which G is a group of order $96p$ with partial presentation $HT(3; 4)$. In this case, $Q/Q' \cong Z_6$; this case can be handled by considering the group G^+. Then it is not hard to show that there is no $T(3, 3, 4)$ group of order $48p$ and hence no $HT(3; 4)$ group of order $96p$.

Combining Theorems 6.1 and 6.2 gives the following.

Theorem 6.3. *Let p be a prime such that $p \equiv 1 \pmod 6$ and $p > (24)^2$, and let $g = 2p + 1$. Then for any such g,*

$$M(g) = N(g) = 24(g - 1). \tag{6.3}$$

Finally, it is worth noting that there are genera in which $N(g) = M(g)$ but the genus g does not have either the form $2p + 1$ or the form $3p + 1$ for a prime p. Two examples are genus 28 and genus 37.

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References

[1] R. D. M. Accola, On the number of automorphisms of a closed Riemann surface, *Trans. Am. Math. Soc.* **131** (1968), 398–408, doi:10.2307/1994955.

[2] M. Belolipetsky and G. A. Jones, Automorphism groups of Riemann surfaces on genus $p + 1$, where p is prime, *Glasg. Math. J.* **47** (2005), 379–393, doi:10.1017/s0017089505002612.

[3] M. Conder, Groups of minimal genus including C_2 extensions of $PSL(2, q)$ for certain q, *Q. J. Math., Oxf. II. Ser.* **38** (1987), 449–460, doi:10.1093/qmath/38.4.449.

[4] M. Conder, Hurwitz groups: A brief survey, *Bull. Am. Math. Soc., New Ser.* **23** (1990), 359–370, doi:10.1090/s0273-0979-1990-15933-6.

[5] M. Conder, An update on Hurwitz groups, *Groups Complex. Cryptol.* **2** (2010), 35–49, doi:10.1515/gcc.2010.002.

[6] M. Conder, *Maximum orders of group actions on compact Riemann surfaces of genus 2 to 301*, University of Auckland, 2019, {https://www.math.auckland.ac.nz/~conder/}.

[7] M. Conder and P. Dobcsányi, Determination of all regular maps of small genus, *J. Comb. Theory, Ser. B* **81** (2001), 224–242, doi:10.1006/jctb.2000.2008.

[8] M. D. E. Conder, Generators for alternating and symmetric groups, *J. Lond. Math. Soc., II. Ser.* **22** (1980), 75–86, doi:10.1112/jlms/s2-22.1.75.

[9] M. D. E. Conder, J. Širáň and T. W. Tucker, The genera, reflexibility and simplicity of regular maps, *J. Eur. Math. Soc. (JEMS)* **12** (2010), 343–364, doi:10.4171/jems/200.

[10] M. D. E. Conder and T. W. Tucker, The symmetric genus spectrum of finite groups, *Ars Math. Contemp.* **4** (2011), 271–289, doi:10.26493/1855-3974.127.eb9.

[11] D. Garbe, Über die regulären Zerlegungen geschlossener orientierbarer Flächen, *J. Reine Angew. Math.* **237** (1969), 39–55, doi:10.1515/crll.1969.237.39.

[12] J. L. Gross and T. W. Tucker, *Topological Graph Theory*, Wiley-Interscience Series in Discrete Mathematics and Optimization, John Wiley & Sons Inc., New York, 1987.

[13] C. Maclachlan, A Bound for the Number of Automorphisms of a Compact Riemann Surface, *J. London Math. Soc.* **s1-44** (1969), 265–272, doi:10.1112/jlms/s1-44.1.265.

[14] C. L. May, The strong symmetric genus of direct products, *J. Algebra Its Appl.* **10** (2011), 901–914, doi:10.1142/s0219498811005026.

[15] C. L. May and J. Zimmerman, Groups of small symmetric genus, *Glasg. Math. J.* **37** (1995), 115–129, doi:10.1017/s0017089500030457.

[16] C. L. May and J. Zimmerman, There is a group of every strong symmetric genus, *Bull. Lond. Math. Soc.* **35** (2003), 433–439, doi:10.1112/s0024609303001954.

[17] C. L. May and J. Zimmerman, The groups of strong symmetric genus 4, *Houston J. Math.* **31** (2005), 21–35, `https://www.math.uh.edu/~hjm/Vol31-1.html`.

[18] C. L. May and J. Zimmerman, The groups of symmetric genus $\sigma \leq 8$, *Commun. Algebra* **36** (2008), 4078–4095, doi:10.1080/00927870802174793.

[19] F. A. Sherk, The regular maps on a surface of genus three, *Can. J. Math.* **11** (1959), 452–480, doi:10.4153/CJM-1959-046-9.

[20] D. Singerman, Automorphisms of compact non-orientable Riemann surfaces, *Glasg. Math. J.* **12** (1971), 50–59, doi:10.1017/s0017089500001142.

[21] D. Singerman, On the structure of non-Euclidean crystallographic groups, *Proc. Camb. Philos. Soc.* **76** (1974), 233–240, doi:10.1017/s0305004100048891.

[22] D. Singerman, Symmetries of Riemann surfaces with large automorphism group, *Math. Ann.* **210** (1974), 17–32, doi:10.1007/bf01344543.

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Signed graphs with two eigenvalues and vertex degree five*

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Abstract

It is known that a signed graph with exactly 2 eigenvalues must be regular, and all those whose vertex degree does not exceed 4 are known. In this paper we characterize all signed graphs with 2 eigenvalues and vertex degree 5. We also determine all signed graphs with 2 eigenvalues and 12 or 13 vertices, which is a natural step since those with a fewer number of vertices are known.

Keywords: Regular signed graph, adjacency matrix, weighing matrix, bipartite double.

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1 Introduction

A *signed graph* $\dot{G}$ is a pair (G, σ), where $G = (V, E)$ is a simple graph, called the *underlying graph*, and $\sigma\colon E \longrightarrow \{1, -1\}$ is the *signature*. The number of vertices of $\dot{G}$ is denoted by n. The edge set of $\dot{G}$ is composed of subsets of positive and negative edges. Two vertices are positive (resp. negative) neighbours if they are joined by a positive (resp. negative) edge.

The *adjacency matrix* $A_{\dot{G}}$ of $\dot{G}$ is obtained from the adjacency matrix of its underlying graph by reversing the sign of all 1s which correspond to negative edges. The *eigenvalues* of $\dot{G}$ are identified as the eigenvalues of $A_{\dot{G}}$, and they form the *spectrum* of $\dot{G}$.

We interpret a graph as a signed graph with all the edges being positive and, where no confusion arises, we write '$\dot{G}$ has k eigenvalues' to mean that $\dot{G}$ has exactly k distinct eigenvalues.

Signed graphs with 2 eigenvalues have been investigated in [5, 7, 8, 11] and some related references. They are known to be regular, moreover every connected signed graph

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with 2 eigenvalues is strongly regular in the sense of [10]. All signed graphs with 2 eigenvalues and (vertex) degree at most 4 are explicitly determined in [7, 11], and they can also be deduced from the results reported in [8]. In particular, there is an infinite family of those with degree 4. In [8] this family is determined in terms of cyclotomic matrices, and in [11] the same family is determined in terms of signed line graphs. Signed graphs with 2 eigenvalues and at most 11 vertices are reported in [11]. Those whose least eigenvalue is greater than -2 and those which are signed line graphs are also known and can be found in the same reference. There are also some sporadic results related to other classes of signed graphs with 2 eigenvalues [7, 10, 11]. Lastly, the Seidel matrix of a simple graph G can be seen as the adjacency matrix of the complete signed graph whose negative edges correspond to the edges of G. Accordingly, many results of [2, 3] concerning graphs with exactly 2 eigenvalues of the Seidel matrix can be interpreted in the context of signed graphs. Moreover, in this paper we use a similar approach.

Since all signed graphs with 2 eigenvalues and degree at most 4 are known, the next natural step is to consider those with degree 5. In Section 3 we characterize all of them. Moreover, we explicitly determine all except those that belong to some of the two particular infinite families, which remain undetermined but well characterized by certain structural properties.

In Section 4 we determine all signed graphs with 2 eigenvalues and 12 or 13 vertices, which is an extension of the aforementioned result on those with at most 11 vertices.

We start with a preparatory section in which we give some terminology, notation and known results. The paper is concluded by the Appendix that contains certain inequivalent weighing matrices which are frequently used in this paper.

2 Preliminaries

We say that a signed graph is connected, regular or bipartite if the same holds for its underlying graph. The *negation* $-\dot{G}$ is obtained by reversing the sign of every edge of $\dot{G}$. The *degree* of a vertex is equal to the number of edges incident with it. In particular, the *negative degree* is the number of negative edges incident with it.

We say that signed graphs $\dot{G}_1$ and $\dot{G}_2$ are *switching isomorphic* if there is a monomial $(0, 1, -1)$-matrix P such that $A_{\dot{G}_2} = P^{-1} A_{\dot{G}_1} P$. (We recall that $P^{-1} = P^{\intercal}$.) In this case we write $\dot{G}_1 \cong \dot{G}_2$. Switching isomorphic signed graphs share the same spectrum and in many considerations they are identified.

The *product* $\dot{G}_1 \times \dot{G}_2$ of $\dot{G}_1$ and $\dot{G}_2$ is the signed graph with the vertex set $V(\dot{G}_1) \times V(\dot{G}_2)$ in which two vertices (u_1, u_2) and (v_1, v_2) are adjacent if and only if u_i and v_i are adjacent in G_i, for $1 \leq i \leq 2$. The sign of an edge of the product is the product of the signs of the corresponding edges of $\dot{G}_1$ and $\dot{G}_2$. The adjacency matrix $A_{\dot{G}_1 \times \dot{G}_2}$ is equal to the Kronecker product $A_{\dot{G}_1} \otimes A_{\dot{G}_2}$. We recall that the Kronecker product is not symmetric, but the resulting matrices are permutation equivalent, which means that the corresponding signed graphs are switching isomorphic. In particular, $\dot{G} \times K_2$ is called the *bipartite double* (or the *bipartite double cover*) of $\dot{G}$ and denoted by $\mathrm{bd}(\dot{G})$. The bipartite double is always bipartite, and it is connected if and only if $\dot{G}$ is connected and non-bipartite. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $\dot{G}$ (with possible repetitions), then the eigenvalues of $\mathrm{bd}(\dot{G})$ are $\pm\lambda_1, \pm\lambda_2, \ldots, \pm\lambda_n$.

A *weighing matrix* N of order n and weight r is an $n \times n$ $(0, 1, -1)$-matrix satisfying $N^{\intercal} N = rI$. We say that two rows of a weighing matrix intersect in k places if their non-zero entries match in exactly k positions. Two weighing matrices M and N are said to be *equivalent* if there are $(0, 1, -1)$-monomial matrices P, Q such that $N = PMQ$.

Results reported in the following two sections rely on the classification of weighing matrices of weight 5 and weighing matrices of order 12 obtained by Harada and Munemasa [6]. The first classification is given in the following theorem, while the corresponding matrices can be found in [4, 6]. To make the paper more self-contained we list them in the Appendix. The notation is transferred from the mentioned references.

Theorem 2.1 ([6]). *Any weighing matrix of weight* 5 *is equivalent to a matrix which is a direct sum of an arbitrary number of matrices* $W(6,5)$*,* $W(8,5)$*,* $W_{12,5}$*,* $W_{14,5}$*,* $D(16,5)$*,* $E(4t_i+2,5)$*,* $F(4t_j+4,5)$*, where* $t_i, t_j \geq 2$*.*

Every matrix of the previous theorem determines a bipartite signed graph with the adjacency matrix

$$\begin{pmatrix} O & N^{\intercal} \\ N & O \end{pmatrix},$$

where N is the matrix in question. Throughout the paper we denote the corresponding signed graphs by $\dot{W}(6,5)$, $\dot{W}(8,5)$, $\dot{W}_{12,5}$, $\dot{W}_{14,5}$, $\dot{D}(16,5)$, $\dot{E}(4t+2,5)$, $\dot{F}(4t+4,5)$. In other words, a dot indicates that we are dealing with the signed graph. The number of vertices in each of these signed graphs is twice of the first parameter.

3 Connected signed graphs with 2 eigenvalues and degree 5

Here is the main result of this section.

Theorem 3.1. *A connected signed graph with degree* 5 *has* 2 *eigenvalues if and only if it is switching isomorphic to*

(i) K_6, $-K_6$,

(ii) *one of the bipartite signed graphs* $\dot{W}(6,5)$*,* $\dot{W}(8,5)$*,* $\dot{W}_{12,5}$*,* $\dot{W}_{14,5}$*,* $\dot{D}(16,5)$*,* $\dot{E}(4t+2,5)$*,* $\dot{F}(4t+4,5)$*,*

(iii) *one of the non-bipartite signed graphs illustrated in* Figure 1 *or a non-bipartite signed graph* $\dot{G}$ *such that either* $\mathrm{bd}(\dot{G}) \cong \dot{E}(4t+2,5)$ *or* $\mathrm{bd}(\dot{G}) \cong \dot{F}(4t+4,5)$*,*

where, wherever it occurs, the parameter t *satisfies* $t \geq 2$*.*

Every signed graph of (ii) or (iii) has the eigenvalues $\pm\sqrt{5}$.

Evidently, a disconnected signed graph with 2 eigenvalues and degree 5 is a disjoint union of connected ones. The proof of the previous theorem is based on the subsequent lemmas. In the first one we consider the case in which the eigenvalues are asymmetric (i.e., they are not equal in absolute value).

Lemma 3.2. *Every connected signed graph with degree* 5 *and* 2 *asymmetric eigenvalues is switching isomorphic to* K_6 *or* $-K_6$*.*

Proof. By considering the minimal polynomial we deduce that the eigenvalues must be integral and the negation of their product is equal to the degree, i.e., 5. This gives just 2 possibilities which further produce the desired solutions. □

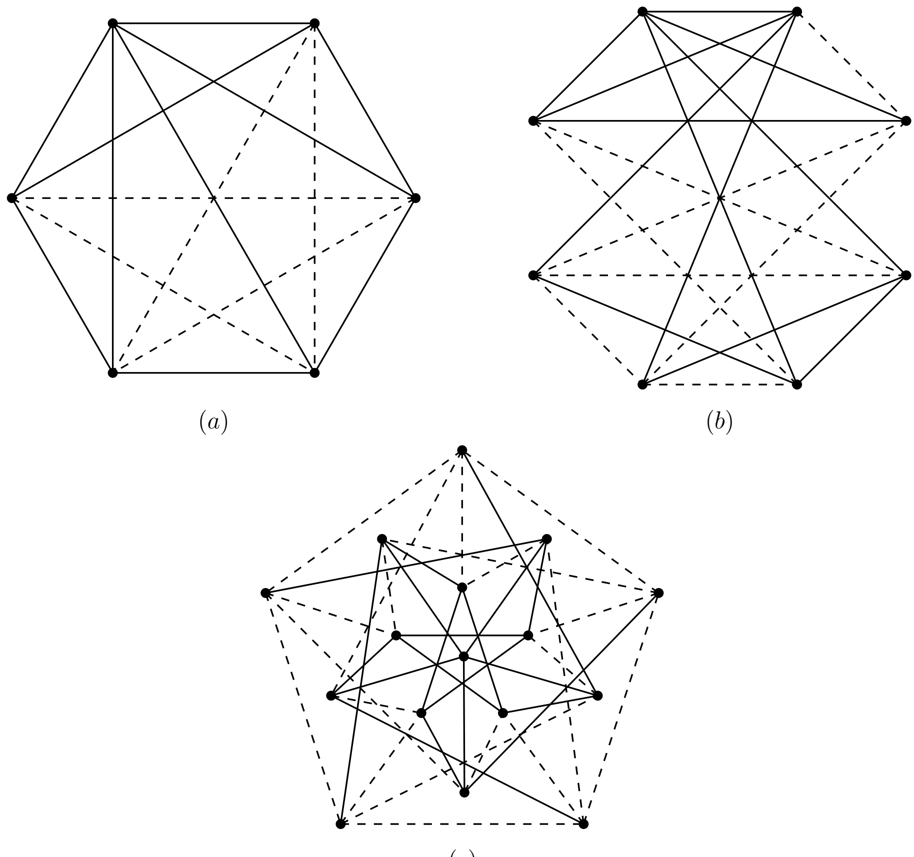

Figure 1: Signed graphs of Theorem 3.1(iii). In this and the forthcoming figures, negative edges are dashed.

In what follows we assume that the eigenvalues are symmetric. In this case they are $\pm\sqrt{5}$, so our task is to consider all signed graphs with these particular eigenvalues. We first consider the bipartite case.

Lemma 3.3. *Every connected bipartite signed graph with eigenvalues $\pm\sqrt{5}$ is switching isomorphic to either $\dot{W}(6,5)$, $\dot{W}(8,5)$, $\dot{W}_{12,5}$, $\dot{W}_{14,5}$, $\dot{D}(16,5)$, $\dot{E}(4t{+}2,5)$ or $\dot{F}(4t{+}4,5)$ (for $t \geq 2$).*

Proof. Let

$$A_{\dot{G}} = \begin{pmatrix} O & M^\intercal \\ M & O \end{pmatrix}$$

be the adjacency matrix of a signed graph under consideration. It follows that M is a square matrix satisfying $M^\intercal M = 5I$, i.e., M is a weighing matrix of weight 5. Since $\dot{G}$ is connected, M is equivalent to one of the seven matrices listed in the formulation of this lemma. If we denote this matrix by N, we have $M = PNQ$ for some $(0,1,-1)$-monomial matrices P and Q, which implies

$$A_{\dot{G}} = \begin{pmatrix} Q^\intercal & O \\ O & P \end{pmatrix} \begin{pmatrix} O & N^\intercal \\ N & O \end{pmatrix} \begin{pmatrix} Q & O \\ O & P^\intercal \end{pmatrix} = \begin{pmatrix} Q & O \\ O & P^\intercal \end{pmatrix}^{-1} \begin{pmatrix} O & N^\intercal \\ N & O \end{pmatrix} \begin{pmatrix} Q & O \\ O & P^\intercal \end{pmatrix},$$

and the result follows. □

The essential part of the proof of Theorem 3.1 refers to the non-bipartite case. Observe that every connected non-bipartite signed graph $\dot{G}$ with eigenvalues $\pm\sqrt{5}$ is extracted from a decomposition of a bipartite one (with the same eigenvalues) having the form $\dot{G} \times K_2$, i.e., which is a bipartite double of $\dot{G}$. We remark that not every bipartite signed graph is a bipartite double, and a decomposition does not need to be unique (in the sense that it may produce switching non-isomorphic signed graphs). We also have $\mathrm{bd}(\dot{G}) \cong \mathrm{bd}(-\dot{G})$. In what follows we consider possible decompositions of signed graphs of Lemma 3.3. For this purpose we need the following simple but very useful result. It helps us to determine the common neighbourhood of the particular vertices of $\dot{G}$.

Lemma 3.4. *If* W_1 *and* W_2 *are the colour classes of* $\mathrm{bd}(\dot{G})$*, then there are bijections* $f_1\colon V(\dot{G}) \longrightarrow W_1$ *and* $f_2\colon V(\dot{G}) \longrightarrow W_2$ *such that* u, v *are adjacent in* $\dot{G}$ *if and only if* $f_1(u), f_2(v)$ *and* $f_2(u), f_1(v)$ *are adjacent in* $\mathrm{bd}(\dot{G})$.

Proof. The result follows by definition of the Kronecker product. □

In the following two lemmas we consider $\dot{W}(6,5)$ and $\dot{W}(8,5)$. Despite the corresponding results can be obtained by computer search, we give theoretical proofs. They illustrate a technique which is used in the forthcoming considerations.

Lemma 3.5. $\mathrm{bd}(\dot{G}) \cong \dot{W}(6,5)$ *holds if and only if* $\dot{G}$ *is switching isomorphic to the signed graph illustrated in* Figure 1(a).

Proof. Since every two rows of $W(6,5)$ intersect in 4 places, we conclude (by Lemma 3.4) that $\dot{G}$ is a complete signed graph with 6 vertices. Since its eigenvalues are $\pm\sqrt{5}$, its adjacency matrix is equivalent to $W(6,5)$, and (up to switching) there is the unique possibility, $W(6,5)$ itself, which leads us to the signed graph of Figure 1(a).

The opposite implication is immediate. □

Considering the minimal polynomial, we get that if $\dot{G}$ has the eigenvalues $\pm\sqrt{5}$, then for every pair of its vertices we have

$$w_2(u,v) = \begin{cases} 5 & \text{if } u = v \\ 0 & \text{if } u \neq v, \end{cases} \tag{3.1}$$

where $w_2(u,v)$ denotes the difference between the number of 2-walks between u and v which traverse edges of the same sign and the number of 2-walks between the same vertices which traverse edges of different sign.

We proceed with $\dot{W}(8,5)$.

Lemma 3.6. $\mathrm{bd}(\dot{G}) \cong \dot{W}(8,5)$ *holds if and only if* $\dot{G}$ *is switching isomorphic to the signed graph illustrated in* Figure 1(b).

Proof. Since the first (resp. last) four rows of $W(8,5)$ intersect each other in the first (resp. last) 4 places, from Lemma 3.4, we conclude that the vertices of $\dot{G}$ are partitioned into two sets of equal size in such a way that each vertex of the first set is adjacent to each vertex of the second. We claim that, up to switching, there is a unique distribution of the edges between the vertices of the mentioned sets as given in Figure 1(b), where the vertices of the first set are drawn in left part. Indeed, without loss of generality, we may assume that $\dot{G}$ contains a vertex whose negative degree is 0 (the top-left vertex in the figure), and then every remaining vertex in its set has two positive and two negative neighbours in the other set, which together with (3.1) gives the desired edge distribution. It remains to insert the remaining 4 edges, 2 in each set. Again, on the basis of (3.1) we arrive at the unique possibility, which gives $\dot{G}$.

The uniqueness (up to switching) follows by the way of construction, and the opposite implication is immediate. □

Observe that, according to (3.1), every pair of vertices of $\dot{G}$ has an even number of common neighbours. Consequently, every pair of rows of $A_{\mathrm{bd}(\dot{G})}$ intersects in an even number of places. Since the degree is 5, they intersect in 0, 2 or 4 places. In what follows, we consider one particular case. We say that a signed graph is triangle-free if it does not contain a triangle as a subgraph.

Lemma 3.7. *If every pair of rows of* $A_{\mathrm{bd}(\dot{G})}$ *intersects in either* 0 *or* 2 *places, then* $\dot{G}$ *is triangle-free.*

Proof. Assume that $\dot{G}$ is not triangle-free. Then, every edge of $\dot{G}$ that belongs to some triangle, in fact, belongs to exactly 2 triangles. Indeed, by (3.1) applied to the vertices incident with such an edge, we conclude that it must belong to an even number of triangles which, by Lemma 3.4 and the assumption of this lemma, makes this number equal to 2. If so, then $\dot{G}$ contains a tetrahedron, or an octahedron, or an icosahedron as an induced subgraph. Moreover, the octahedron is eliminated immediately since it contains a pair of vertices with 4 common neighbours, which would imply the existence of two rows in $A_{\mathrm{bd}(\dot{G})}$ that intersect in 4 places.

First let $\dot{G}$ contain a tetrahedron, and let its vertices be denoted by a, b, c, d. Assume that the negative degree of a is 0. Under the assumption of this lemma, a, b have no common neighbours outside the fixed tetrahedron, and thus $w_2(a,b) = 0$ yields $\sigma(bc) = -\sigma(bd)$. Similarly, $w_2(a,c) = 0$ gives $\sigma(cb) = -\sigma(cd)$, which implies $\sigma(bd) = \sigma(cd)$. But from $w_2(a,d) = 0$, we get $\sigma(bd) = -\sigma(cd)$, which contradicts the previous equality.

If $\dot{G}$ contains an icosahedron, then it also contains (as an induced subgraph) a pentagon and an additional vertex adjacent to all the vertices of the pentagon. By assuming that the negative degree of the additional vertex is 0, we arrive at a contradiction in a very similar way as before. □

We now eliminate $\dot{W}_{12,5}$ and $\dot{W}_{14,5}$ (as candidates for $\mathrm{bd}(\dot{G})$) and consider $\dot{D}(16,5)$.

Lemma 3.8. *There is no signed graph* $\dot{G}$ *such that* $\mathrm{bd}(\dot{G}) \cong \dot{W}_{12,5}$ *or* $\mathrm{bd}(\dot{G}) \cong \dot{W}_{14,5}$.

Proof. Assume by way of contradiction that either $\mathrm{bd}(\dot{G}) \cong \dot{W}_{12,5}$ or $\mathrm{bd}(\dot{G}) \cong \dot{W}_{14,5}$. In both cases $A_{\mathrm{bd}(\dot{G})}$ satisfies the assumption of Lemma 3.7, and thus $\dot{G}$ is triangle-free, and of course the same holds for its underlying graph G.

The spectrum of the underlying graph of $\dot{W}_{12,5}$ is $[\pm 5, (\pm\sqrt{5})^6, (\pm 1)^5]$. Since the non-integral algebraic conjugates are equal in multiplicity, there is just one candidate for the spectrum of G: $[5, (\pm\sqrt{5})^3, (-1)^5]$. According to [12], there is exactly one graph with this spectrum (known as the icosahedron), but it contains a triangle – a contradiction.

Similarly, the spectrum of the underlying graph of $\dot{W}_{14,5}$ is $[\pm 5, \pm 3, (\pm(1+\sqrt{2}))^6, (\pm(1-\sqrt{2}))^6]$. There is just one candidate for the spectrum of G that passes the numerical condition $\operatorname{tr}(A_G) = 0$: $[5, 3, 1\pm\sqrt{2}, (-1\pm\sqrt{2})^5]$. But since in this case we have $\operatorname{tr}(A_G^3) \neq 0$, we deduce that G contains a triangle – a contradiction. □

Lemma 3.9. $\operatorname{bd}(\dot{G}) \cong \dot{D}(16,5)$ *holds if and only if* $\dot{G}$ *is switching isomorphic to the signed graph illustrated in* Figure 1(c).

Proof. Since $A_{\operatorname{bd}(\dot{G})}$ satisfies the assumption of Lemma 3.7, we get that G is triangle-free. The spectrum of the underlying graph of $\dot{D}(16,5)$ is $[\pm 5, (\pm 3)^5, (\pm 1)^{10}]$, which together with $\operatorname{tr}(A_G) = 0, \operatorname{tr}(A_G^3) = 0$, leads to the unique possibility for the spectrum of G: $[5, 1^{10}, (-3)^5]$. There is exactly one graph with this spectrum known as the Clebsch graph (the underlying graph of the signed graph illustrated in the figure).

In what follows we consider the signature σ defined on $E(G)$ which would produce the desired $\dot{G}$. From this point we use a descriptive terminology and refer the reader to follow the corresponding figure. As in the previous proofs, without loss of generality, we assume that (at least) one vertex is not incident with negative edges; let this be the central vertex of the figure, denoted here by a. Observe that (in the figure) a is surrounded by the 5 vertices which form a pentagram. Denote the set of these vertices by P and the set of the remaining vertices at distance 2 from a by Q. Now, we have the following:

- Every neighbour of a has 2 neighbours in P and exactly one of them is a positive neighbour (otherwise, $w_2(a,b) \neq 0$ for at least one $b \in P$);
- Every neighbour of a has 2 neighbours in Q and exactly one of them is a positive neighbour (for a similar reason);
- For b, c being neighbours of a such that b, c have a common neighbour d in P, we have $\sigma(bd) = -\sigma(cd)$. The same holds if $d \in Q$.

Using these conditions we arrive at a unique (up to switching) signature for the edges incident with neighbours of a. Since $w_2(b,c) = 0$ for $b, c \in P$, we get that the edges between the vertices of P have the same sign. If they are positive (as in the figure), then all the remaining (undecided) edges are negative (due to (3.1) applied to non-neighbours of a), and we get the desired $\dot{G}$. If the mentioned edges are taken to be negative, we arrive at a switching isomorphic signed graph.

The opposite implication is immediate. □

We now consider the families of signed graphs illustrated in Figure 2. We first explain their structure. The signed graph of Figure 2(a) has $4t+2$ ($t \geq 2$) vertices, and the vertices distinct from a, b are partitioned into the isomorphic blocks with 4 vertices. For example, the vertices c, d, e, f belong to one block. There are no negative edges between two blocks. The signed graph of Figure 2(b) has $4t+4$ ($t \geq 2$) vertices, and is obtained from the previous one by deleting the 4 edges between two blocks, then inserting two vertices adjacent by a positive edge along with the edges between them and each of the 4 vertices of degree 3 of the corresponding blocks in such a way that the negative edges are

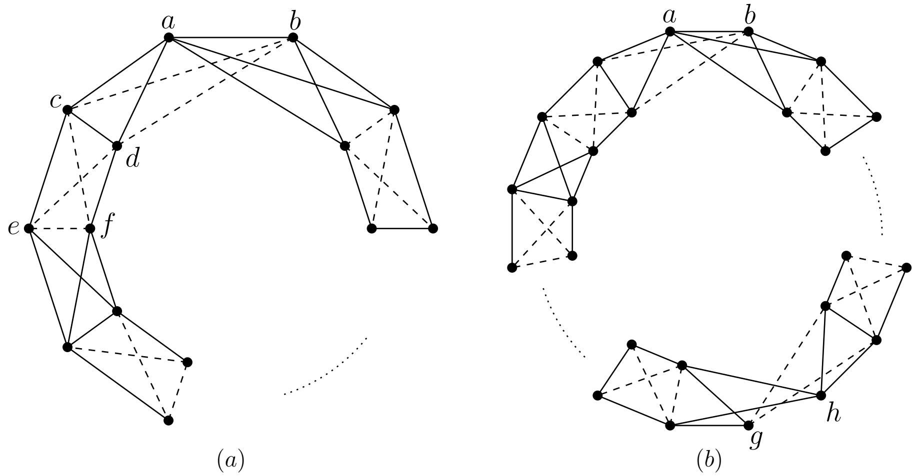

Figure 2: Signed graphs of Lemma 3.10.

just the two edges between one of the new vertices and the block vertices adjacent by a positive edge. If (in the figure) a, b are transferred from the previous signed graph, then the new vertices are g, h.

Lemma 3.10. *If* $\dot{G}$ *is switching isomorphic to the signed graph illustrated in* Figure 2(a) (resp. Figure 2(b))*, then* $\mathrm{bd}(\dot{G}) \cong \dot{E}(4t+2, 5)$ *(resp.* $\mathrm{bd}(\dot{G}) \cong \dot{F}(4t+4, 5)$*).*

Proof. In both cases, $\dot{G}$ is regular of degree 5 and $w_2(u, v) = 0$ holds for every pair of its distinct vertices, which is verified easily. Therefore, $A^2_{\dot{G}} = 5I$, which means that the eigenvalues of $\dot{G}$ are $\pm\sqrt{5}$. Since, obviously, $\dot{G}$ is non-bipartite, its bipartite double is connected and has the same eigenvalues. In other words, $\mathrm{bd}(\dot{G})$ is one of the bipartite signed graphs listed in Lemma 3.3. Now, $\dot{W}(6, 5), \dot{W}(8, 5)$ are eliminated since $\mathrm{bd}(\dot{G})$ has at least 20 vertices, $\dot{W}_{12,5}, \dot{W}_{14,5}$ are eliminated by Lemma 3.8, and $\dot{D}(16, 5)$ is eliminated since, contrary to $\mathrm{bd}(\dot{G})$, it does not contain a pair of vertices with 4 common neighbours. Therefore, either $\mathrm{bd}(\dot{G}) \cong \dot{E}(4t+2, 5)$ or $\mathrm{bd}(\dot{G}) \cong \dot{F}(4t+4, 5)$. Comparing the numbers of vertices, we deduce that the first possibility occurs precisely if $\dot{G}$ is the first signed graph of the figure. □

We prove the main result of this section.

Proof of Theorem 3.1. If the eigenvalues are asymmetric, then Lemma 3.2 leads to (i). If the eigenvalues are symmetric, from Lemma 3.3 we get (ii), while from Lemmas 3.5, 3.6, 3.8, 3.9 we get (iii). □

Lemma 3.10 shows that the presence of $\dot{E}(4t+2, 5)$ and $\dot{F}(4t+4, 5)$ in the formulation of Theorem 3.1(iii) is essential. This lemma leads to the question on how we get the signed graphs of Figure 2, and the answer is simple: We arrive at them by following a simple reasoning based on (3.1) and Lemma 3.4. To determine all the signed graphs that can be extracted from decompositions of $\dot{E}(4t + 2, 5)$ and $\dot{F}(4t + 4, 5)$, one should proceed with the similar reasoning, which in fact becomes complicated especially under the assumption that there exist non-adjacent vertices with 4 common neighbours. An example is given in the next section.

4 Signed graphs with 2 eigenvalues and 12 or 13 vertices

We have mentioned in Section 1 that signed graphs with 2 eigenvalues and at most 11 vertices are known. Those with at most 10 vertices are obtained by computer search reported in [11]. According to the same reference, there are exactly 2 such signed graphs with 11 vertices up to switching: K_{11} and its negation.

It occurs that, for $n = 12$, a computer search which considers all regular signed graphs takes a long time, and so this task requires the application of more sophisticated methods. In what follows we report the results obtained on the basis of a theoretical and computational search on connected signed graphs with 2 eigenvalues and 12 vertices. Let $\dot{G}$ denote such a signed graph and let λ, μ be its eigenvalues. Recall that vertex degree of $\dot{G}$ is $r = -\lambda\mu$.

Assume first that the eigenvalues λ, μ are asymmetric (and then they must be integral). Considering r, we easily eliminate the possibility $|\lambda|, |\mu| \geq 3$. If $|\mu| = 1$, we arrive at K_{12} and $-K_{12}$. If $\mu = -2$, by taking into account $\operatorname{tr}(A_{\dot{G}}) = 0$ and $\operatorname{tr}(A_{\dot{G}}^2) = rn$, we arrive at the unique possible spectrum $[4^4, (-2)^8]$. Now, since the least eigenvalue is not less than -2, the corresponding signed graph is either the signed line graph (in the sense of a definition given in [1, 11]) or the so-called exceptional signed graph. Moreover, it cannot be exceptional since the multiplicity of -2 in every exceptional signed graph is $n-6$, or $n-7$, or $n-8$, as confirmed in [11]. On the other hand, all signed line graphs with 2 eigenvalues are determined in the same reference, and accordingly $\dot{G}$ is switching isomorphic to the line graph of a signed multigraph obtained by inserting a negative edge between every pair of vertices of the complete graph K_4. For all details (including definition of the signed line graph) we refer the reader to the corresponding reference, but we also illustrate $\dot{G}$ in Figure 3(a). Of course, case $\mu = 2$ produces its negation, and we are done.

Assume now that λ, μ are symmetric. Since the case $r \leq 4$ is resolved for every n, we may also assume that $r \geq 5$. For $r = 5$, we immediately get $\dot{W}(6, 5)$ and the signed graph of Figure 2(b) (obtained for $n = 12$). We redraw this signed graph in Figure 3(b). To conclude the list we consider other $\dot{G}$'s satisfying $\operatorname{bd}(\dot{G}) \cong \dot{F}(12, 5)$. This can be performed either by hand or by a brute force, i.e., a computer search. In either way, we arrive at another solution illustrated in Figure 3(c).

We proceed with $r \geq 6$. According to [6], there are 8 inequivalent weighing matrices of order 12 and weight $6, 3$ of weight $7, 7$ of weight $8, 4$ of weight $9, 5$ of weight 10 and 1 of weight 11. They determine all connected bipartite signed graphs with 24 vertices, 2 symmetric eigenvalues and degree r, where $6 \leq r \leq 11$. Using the method exploited in the previous section, we consider possible decompositions of each of them. In each case we start from the given weighing matrix (the data on them can be found in [6]), then compute the spectrum of the underlying graph of the corresponding bipartite signed graph, and then compute the spectrum of a putative graph which can be extracted from it.

In this way we arrive at the data of Table 1. So, the 8 weighing matrices of weight 6 produce the underlying bipartite graphs with 4 distinct spectra (listed in the second column), and each of these spectra gives spectra (listed in the third column) of the underlying graph G of a putative signed graph $\dot{G}$. The remaining weights are considered in the same way, and all the possible spectra of the third column satisfy $\operatorname{tr}(A_G) = 0$, $\operatorname{tr}(A_G^2) = 12r$. Note that in one case for weight 8, there is no spectrum that obeys these numerical conditions.

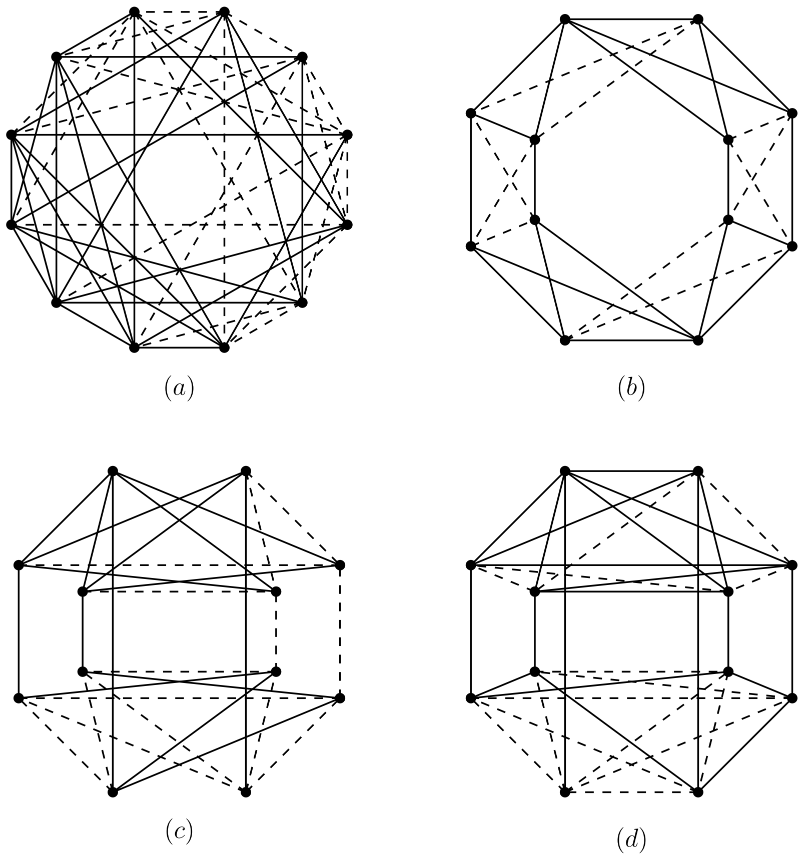

Figure 3: Signed graphs of Theorem 4.1.

The table omits the data for weight 11, since this case is resolved directly. Namely, if the corresponding bipartite signed graph is a bipartite double of a signed graph $\dot{G}$, then $\dot{G}$ is the complete signed graph with 12 vertices and eigenvalues $\pm\sqrt{11}$, but such a signed graph does not exist, which can be confirmed by examining a list of Seidel's strong graphs – an appropriate reference is [9].

It remains to consider the spectra of the third column. (Observe that they are enumerated by 1–15). First, there is no graph with spectrum $4, 5, 9, 11$ or 14, which can easily be confirmed by inspecting regular graphs with 12 vertices. There are 2 graphs with spectrum 7, and each contains pairs of vertices with 5 common neighbours which implies the non-existence of $\dot{G}$ in these cases. There is exactly 1 graph with spectrum $1, 10, 12, 13$ or 15, and the non-existence of $\dot{G}$ is confirmed either by hand (as these graphs have a high level of symmetry along with a large number of vertices with the same neighbourhood) or by computer search. There are exactly 4 graphs with spectrum $3, 1$ graph with spectrum 6 and 1 graph with spectrum 8. The non-existence of $\dot{G}$ is confirmed by computer search. Finally, there is exactly 1 graph with spectrum 2. By computer search we find exactly 1 (up to switching) $\dot{G}$ – it is illustrated in Figure 3(d); with consistent vertex labellings its

weight	spectrum of $G \times K_2$	spectrum of G
6	$\pm 6, (\pm 3.24)^3, (\pm 1.24)^3, 0^{10}$	1. $6, 1.24^3, 0^5, (-3.24)^3$
	$\pm 6, \pm 4, (\pm 2)^5, 0^{10}$	2. $6, 4, 0^5(-2)^5$ 3. $6, 2^2, 0^5, (-2)^3, -4$
	$\pm 6, \pm 3.86, \pm 2.83, (\pm 2)^3, \pm 1.04, 0^{10}$	4. $6, 3.86, 0^5, -1.04, (-2)^3, -2.83$ 5. $6, 2.83, 1.04, 0^5, (-2)^3, -3.86$
	$\pm 6, \pm 3.46, (\pm 2)^3, 0^{12}$	6. $6, 3.46, 0^6, (-2)^3, -3.46$
7	$\pm 7, (\pm 3)^3, (\pm 1)^8$	7. $7, 1^5, (-1)^3, (-3)^3$ 8. $7, 3, 1^2, (-1)^6, (-3)^2$
	$\pm 7, (\pm 2.24)^6, (\pm 1)^5$	9. $7, 2.24^3, -2.24^3, (-1)^5$
8	$\pm 8, (\pm 4)^2, 0^{18}$	10. $8, 0^9, (-4)^2$
	$\pm 8, \pm 3.46, (\pm 2)^2, 0^{14}$	11. $8, 3.46, 0^7, (-2)^2, -3.46$
	$\pm 8, \pm 4, (\pm 2)^4, 0^{12}$	12. $8, 2, 0^6, (-2)^3, -4$
	$\pm 8, \pm 4, (\pm 2.83)^2, 0^{16}$	
9	$\pm 9, (\pm 3)^2, (\pm 1)^9$	13. $9, 1^3, (-1)^6, (-3)^2$ 14. $9, 3, (-1)^9, -3$
10	$\pm 10, (\pm 2)^5, 0^{12}$	15. $10, 0^6, (-2)^5$

Table 1: Data for the search on connected non-bipartite signed graphs with 2 symmetric eigenvalues and 12 vertices.

adjacency matrix is

$$\begin{pmatrix} W(6,5) & I \\ I & -W(6,5) \end{pmatrix}.$$

The results are summarized in the following theorem.

Theorem 4.1. *A connected signed graph with* 12 *vertices and degree at least* 5 *has* 2 *eigenvalues if and only if it is switching isomorphic to either* K_{12}, $-K_{12}$, $\dot{W}(6,5)$, *one of signed graphs illustrated in* Figure 3 *or the negation of the signed graph of* Figure 3(a).

Finally, we quickly resolve the case $n = 13$.

Theorem 4.2. *Every signed graph with* 2 *eigenvalues and* 13 *vertices is switching isomorphic to* K_{13} *or* $-K_{13}$.

Proof. Since the number of vertices is odd, the eigenvalues λ, μ must be asymmetric, and then the proof is similar to the proof of Lemma 3.2 or to above consideration related to the asymmetric case for $n = 12$. Accordingly, for $|\mu| = 1$ we arrive at the desired signed graphs. Up to negation, the remaining possibilities are $\mu = -2, \lambda \in \{3,4,5,6\}$ and $(\mu, \lambda) = (-3, 4)$, and both are eliminated by the conditions $\mathrm{tr}(A_{\dot{G}}) = 0$, $\mathrm{tr}(A_{\dot{G}}^3) = 13r$. □

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References

[1] F. Belardo and S. K. Simić, On the Laplacian coefficients of signed graphs, *Linear Algebra Appl.* **475** (2015), 94–113, doi:10.1016/j.laa.2015.02.007.

[2] A. E. Brouwer, A. M. Cohen and A. Neumaier, *Distance-Regular Graphs*, volume 18 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3)*, Springer-Verlag, Berlin, 1989, doi:10.1007/978-3-642-74341-2.

[3] P. J. Cameron and J. H. van Lint, *Designs, Graphs, Codes and Their Links*, volume 22 of *London Mathematical Society Student Texts*, Cambridge University Press, Cambridge, 1991, doi:10.1017/cbo9780511623714.

[4] H. C. Chan, C. A. Rodger and J. Seberry, On inequivalent weighing matrices, *Ars Comb.* **21** (1986), 299–333.

[5] E. Ghasemian and G. H. Fath-Tabar, On signed graphs with two distinct eigenvalues, *Filomat* **31** (2017), 6393–6400, doi:10.2298/fil1720393g.

[6] M. Harada and A. Munemasa, On the classification of weighing matrices and self-orthogonal codes, *J. Comb. Des.* **20** (2012), 40–57, doi:10.1002/jcd.20295.

[7] Y. Hou, Z. Tang and D. Wang, On signed graphs with just two distinct adjacency eigenvalues, *Discrete Math.* **342** (2019), 111615, 8, doi:10.1016/j.disc.2019.111615.

[8] J. McKee and C. Smyth, Integer symmetric matrices having all their eigenvalues in the interval $[-2, 2]$, *J. Algebra* **317** (2007), 260–290, doi:10.1016/j.jalgebra.2007.05.019.

[9] E. Spence, Regular two-graphs on 36 vertices, *Linear Algebra Appl.* **226/228** (1995), 459–497, doi:10.1016/0024-3795(95)00158-n.

[10] Z. Stanić, On strongly regular signed graphs, *Discrete Appl. Math.* **271** (2019), 184–190, doi:10.1016/j.dam.2019.06.017.

[11] Z. Stanić, Spectra of signed graphs with two eigenvalues, *Appl. Math. Comput.* **364** (2020), 124627, 9, doi:10.1016/j.amc.2019.124627.

[12] E. R. van Dam and E. Spence, Small regular graphs with four eigenvalues, *Discrete Math.* **189** (1998), 233–257, doi:10.1016/s0012-365x(98)00085-5.

A Appendix

Here we list the inequivalent weighing matrices of weight 5 mentioned in Theorem 2.1. The first parameter denotes the order, and the second one denotes the weight. In the schematic representations we write $+$ for 1, $-$ for -1 and dot for 0. For $E(4t+2,5), F(4t+4,5)$ we have $t \geq 2$ and every unspecified entry is 0.

$$\begin{pmatrix}
\cdot & + & + & + & + & + \\
+ & \cdot & + & - & - & + \\
+ & + & \cdot & + & - & - \\
+ & - & + & \cdot & + & - \\
+ & - & - & + & \cdot & + \\
+ & + & - & - & + & \cdot
\end{pmatrix}$$

$W(6,5)$

$$\begin{pmatrix}
+ & + & + & + & + & \cdot & \cdot & \cdot \\
+ & + & - & - & \cdot & + & \cdot & \cdot \\
+ & - & - & + & \cdot & \cdot & + & \cdot \\
+ & - & + & - & \cdot & \cdot & \cdot & + \\
+ & \cdot & \cdot & \cdot & - & - & - & - \\
\cdot & + & \cdot & \cdot & - & - & + & + \\
\cdot & \cdot & + & \cdot & - & + & + & - \\
\cdot & \cdot & \cdot & + & - & + & - & +
\end{pmatrix}$$

$W(8,5)$

$$\begin{pmatrix}
+ & + & + & + & + & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
+ & - & \cdot & \cdot & \cdot & + & + & + & \cdot & \cdot & \cdot & \cdot \\
+ & \cdot & - & \cdot & \cdot & - & \cdot & \cdot & + & + & \cdot & \cdot \\
+ & \cdot & \cdot & - & \cdot & \cdot & - & \cdot & - & \cdot & + & \cdot \\
+ & \cdot & \cdot & \cdot & - & \cdot & \cdot & - & \cdot & - & - & \cdot \\
\cdot & + & - & \cdot & \cdot & \cdot & \cdot & + & \cdot & - & \cdot & + \\
\cdot & + & \cdot & - & \cdot & + & \cdot & \cdot & \cdot & + & - & \cdot \\
\cdot & + & \cdot & \cdot & - & \cdot & + & \cdot & \cdot & \cdot & + & - \\
\cdot & \cdot & + & - & \cdot & - & + & \cdot & \cdot & \cdot & \cdot & + \\
\cdot & \cdot & + & \cdot & - & \cdot & - & + & + & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & + & - & \cdot & \cdot & \cdot & - & + & \cdot & + \\
\cdot & \cdot & \cdot & \cdot & \cdot & + & \cdot & - & + & \cdot & + & +
\end{pmatrix}$$

$W_{12,5}$

$$\begin{pmatrix}
+ & + & + & + & + & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
+ & - & \cdot & \cdot & \cdot & + & + & + & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
+ & \cdot & - & \cdot & \cdot & - & \cdot & \cdot & + & + & \cdot & \cdot & \cdot & \cdot \\
+ & \cdot & \cdot & - & \cdot & \cdot & - & \cdot & - & \cdot & + & \cdot & \cdot & \cdot \\
+ & \cdot & \cdot & \cdot & - & \cdot & \cdot & - & \cdot & - & - & \cdot & \cdot & \cdot \\
\cdot & + & - & \cdot & \cdot & + & \cdot & \cdot & \cdot & \cdot & \cdot & + & + & \cdot \\
\cdot & + & \cdot & - & \cdot & \cdot & \cdot & + & \cdot & \cdot & - & \cdot & - & \cdot \\
\cdot & + & \cdot & \cdot & - & \cdot & + & \cdot & \cdot & \cdot & + & - & \cdot & \cdot \\
\cdot & \cdot & + & - & \cdot & \cdot & \cdot & \cdot & + & \cdot & \cdot & \cdot & + & - \\
\cdot & \cdot & + & \cdot & - & \cdot & \cdot & \cdot & \cdot & + & \cdot & + & \cdot & + \\
\cdot & \cdot & \cdot & + & - & \cdot & - & + & \cdot & \cdot & \cdot & \cdot & \cdot & - \\
\cdot & \cdot & \cdot & \cdot & \cdot & + & - & \cdot & + & \cdot & \cdot & - & \cdot & + \\
\cdot & \cdot & \cdot & \cdot & \cdot & + & \cdot & - & \cdot & + & \cdot & \cdot & - & - \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & + & - & + & + & - & \cdot
\end{pmatrix}$$

$W_{14,5}$

$$\begin{pmatrix}
+ & + & + & + & + & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
+ & - & \cdot & \cdot & \cdot & + & + & + & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
+ & \cdot & - & \cdot & \cdot & - & \cdot & \cdot & + & + & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
+ & \cdot & \cdot & - & \cdot & \cdot & - & \cdot & - & \cdot & + & \cdot & \cdot & \cdot & \cdot & \cdot \\
+ & \cdot & \cdot & \cdot & - & \cdot & \cdot & - & \cdot & - & - & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & + & - & \cdot & \cdot & + & \cdot & \cdot & \cdot & \cdot & \cdot & + & + & \cdot & \cdot & \cdot \\
\cdot & + & \cdot & - & \cdot & \cdot & + & \cdot & \cdot & \cdot & \cdot & - & \cdot & + & \cdot & \cdot \\
\cdot & + & \cdot & \cdot & - & \cdot & \cdot & + & \cdot & \cdot & \cdot & \cdot & - & - & \cdot & \cdot \\
\cdot & \cdot & + & - & \cdot & \cdot & \cdot & \cdot & + & \cdot & \cdot & + & \cdot & \cdot & + & \cdot \\
\cdot & \cdot & + & \cdot & - & \cdot & \cdot & \cdot & \cdot & + & \cdot & \cdot & + & \cdot & - & \cdot \\
\cdot & \cdot & \cdot & + & - & \cdot & \cdot & \cdot & \cdot & \cdot & + & \cdot & \cdot & + & + & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & + & - & \cdot & + & \cdot & \cdot & - & \cdot & \cdot & \cdot & + \\
\cdot & \cdot & \cdot & \cdot & \cdot & + & \cdot & - & \cdot & + & \cdot & \cdot & - & \cdot & \cdot & - \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & + & - & \cdot & \cdot & + & \cdot & \cdot & - & \cdot & + \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & + & - & + & \cdot & \cdot & \cdot & - & - \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & + & - & + & - & +
\end{pmatrix}$$

$D(16,5)$

$$\begin{pmatrix}
+ & + & + & + & + & \cdot \\
+ & + & - & - & \cdot & + \\
+ & - & \cdot & \cdot & \cdot & \cdot & + & + & + & \cdot \\
+ & - & \cdot & \cdot & \cdot & \cdot & - & - & \cdot & + \\
+ & \cdot & \cdot & \cdot & - & - & \cdot & \cdot & - & - \\
\cdot & + & \cdot & \cdot & - & - & \cdot & \cdot & + & + \\
 & & + & - & \cdot & \cdot \\
 & & + & - & \cdot & \cdot \\
 & & + & \cdot & - & + \\
 & & \cdot & + & - & + \\
 & & & & & & & & & & \ddots \\
 & & & & & & & & & & & & & & & + & + & + & \cdot \\
 & & & & & & & & & & & & & & & - & - & \cdot & + \\
 & & & & & & & & & & & & & & & \cdot & \cdot & - & - \\
 & & & & & & & & & & & & & & & \cdot & \cdot & + & + \\
 & & & & & & & & & & & + & - & \cdot & \cdot & + & \cdot & - & + \\
 & & & & & & & & & & & + & - & \cdot & \cdot & \cdot & - & + & - \\
 & & & & & & & & & & & + & \cdot & - & + & - & + & \cdot & \cdot \\
 & & & & & & & & & & & \cdot & + & - & + & + & - & \cdot & \cdot
\end{pmatrix}$$

$E(4t+2,5)$

$$\begin{pmatrix}
+ & + & + & + & + & \cdot \\
+ & + & - & - & \cdot & + \\
+ & - & \cdot & \cdot & \cdot & \cdot & + & + & + & \cdot \\
+ & - & \cdot & \cdot & \cdot & \cdot & - & - & \cdot & + \\
+ & \cdot & \cdot & \cdot & - & - & \cdot & \cdot & - & - \\
\cdot & + & \cdot & \cdot & - & - & \cdot & \cdot & + & + \\
 & & + & - & \cdot & \cdot \\
 & & + & - & \cdot & \cdot \\
 & & + & \cdot & - & + \\
 & & \cdot & + & - & + \\
 & & & & & & & & & & \ddots \\
 & & & & & & & & & & & & & & & + & + & + & \cdot \\
 & & & & & & & & & & & & & & & - & - & \cdot & + \\
 & & & & & & & & & & & & & & & \cdot & \cdot & - & - \\
 & & & & & & & & & & & & & & & \cdot & \cdot & + & + \\
 & & & & & & & & & & & + & - & \cdot & \cdot & + & - & \cdot & \cdot & + & \cdot \\
 & & & & & & & & & & & + & - & \cdot & \cdot & - & + & \cdot & \cdot & \cdot & + \\
 & & & & & & & & & & & + & \cdot & - & + & \cdot & \cdot & \cdot & \cdot & - & - \\
 & & & & & & & & & & & \cdot & + & - & + & \cdot & \cdot & \cdot & \cdot & + & + \\
 & & & & & & & & & & & & & & & + & \cdot & - & + & - & + \\
 & & & & & & & & & & & & & & & \cdot & + & - & + & + & -
\end{pmatrix}$$

$F(4t+4,5)$

Author Guidelines

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Papers should be written in English, prepared in LaTeX, and must be submitted as a PDF file. The title page of the submissions must contain:

- *Title*. The title must be concise and informative.
- *Author names and affiliations*. For each author add his/her affiliation which should include the full postal address and the country name. If avilable, specify the e-mail address of each author. Clearly indicate who is the corresponding author of the paper.
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Maribor Graph Theory Conference (MGTC 2022)

Maribor, Slovenia, 11–15 September 2022

https://conferences.matheo.si/event/37/

The conference aims to bring together researchers from all areas of graph theory and related topics with an emphasis on the topics studied by Maribor graph theorists. These topics are largely influenced by the work of Sandi Klavžar whose 60th birthday will be celebrated during the event. In addition, we will devote one of the sessions to Wilfried Imrich, an honorary doctor of the University of Maribor.

Keynote speakers:

- László Babai (University of Chicago, USA)
- Marthe Bonamy (Université de Bordeaux, France)
- Michael Henning (University of Johannesburg, South Africa)
- Kolja Knauer (Universitat de Barcelona, Spain)
- Douglas Rall (Furman University, USA)
- Mariusz Woźniak (AGH University of Science and Technology, Poland)
- Xuding Zhu (Zhejiang Normal University, China)

Special invited speakers:

- Paul Dorbec (Université de Caen Normandy, France)
- Ismael G. Yero (Universidad de Cadiz, Spain)
- Andreas M. Hinz (LMU München, Germany)
- Martyn Mulder (Erasmus University Rotterdam, Netherlands)
- Balázs Patkós (Alfréd Rényi Institute of Mathematics, Hungary)

Contributed talks are welcome!

Venue: University of Maribor, Faculty of Natural Sciences and Mathematics. The faculty is located on the edge of the city of Maribor not far from the Drava river. The city center is just a half an hour walk from the conference venue.

Organized by: University of Maribor, Faculty of Natural Sciences and Mathematics in collaboration with the Institute of Mathematics, Physics and Mechanics, and NM Klub.

Program and organizing committee:
Boštjan Brešar (chair), Tanja Dravec, Marko Jakovac

University of Maribor
Faculty of Natural Sciences and Mathematics

MGTC 2022

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