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Upper embeddability of graphs and products of transpositions associated with edges*

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Abstract

Given a graph, we associate each edge with the transposition which exchanges the endvertices. Fixing a linear order on the edge set, we obtain a permutation of the vertices. Dénes proved that the permutation is a full cyclic permutation for any linear order if and only if the graph is a tree.

In this article, we characterize graphs having a linear order such that the associated permutation is a full cyclic permutation in terms of graph embeddings. Moreover, we give a counterexample for Eden's question about an edge ordering whose associated permutation is the identity.

Keywords: Full cyclic permutation ordering, upper-embeddable graph, 2*-cell embedding, rotation system.*

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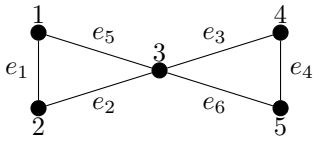


Figure 1: A butterfly graph.



Figure 2: A dumbbell graph.

1 Introduction

In this article, a graph G stands for a *connected multigraph* $G = (V_G, E_G, r_G)$, where

- V_G is a finite set of vertices,
- E_G is a finite set of edges,
- r_G is a map from E_G to $\binom{V_G}{2}$, the collection of subsets of V_G consisting of 2 elements.

Note that $r_G(e)$ represents the endvertices of the edge e. Also note that loops are not allowed.

Let n be a positive integer and suppose that $V_G = [n] \coloneqq \{1, 2, \ldots, n\}$. When an edge $e \in E_G$ satisfies $r_G(e) = \{u, v\}$, we associate the transposition $\tau_e \coloneqq (u\ v) \in S_n$ with the edge e, where S_n denotes the symmetric group of degree n.

An *edge ordering* of a graph G is a linear order $\leq_\omega$ on E_G, denoted as a sequence $\omega = (e_1, \ldots, e_m)$ in which $e_i <_\omega e_j$ if and only if $i < j$. Given an edge ordering $\omega = (e_1, \ldots, e_m)$, we associate the product $\pi_\omega \coloneqq \tau_{e_m} \cdots \tau_{e_1} \in S_n$.

Definition 1.1. A permutation $\sigma \in S_n$ is called a *full cyclic permutation* if σ is a cyclic permutation of length n. An edge ordering $\omega = (e_1, \ldots, e_m)$ of a graph G is a *full cyclic permutation ordering* if the corresponding permutation $\pi_\omega = \tau_{e_m} \cdots \tau_{e_1}$ is a full cyclic permutation.

Dénes proved the following theorem to state a connection between labeled trees and factorization of a full cyclic permutation into transpositions.

Theorem 1.2 (Dénes [1]. See also [7, Section 2] and [11, Lemma 2.1])**.** *Given a graph G, the following are equivalent.*

(i) *Any edge ordering of G is a full cyclic permutation ordering.*

(ii) *G is a tree.*

Note that Theorem 1.2 plays an important role in the studies of the chromatic symmetric functions and the chromatic operator for trees [3, 11]. Also, note that recently the second author [16] studied an analogue of Theorem 1.2 for signed graphs and the hyperoctahedral group.

It is a natural question to ask what graphs admit a full cyclic permutation ordering. For example, let G be the butterfly graph pictured in Figure 1 and define the edge ordering ω by $\omega \coloneqq (e_1, e_2, e_3, e_4, e_5, e_6)$. Then

$$\pi_\omega = \tau_{e_6}\tau_{e_5}\tau_{e_4}\tau_{e_3}\tau_{e_2}\tau_{e_1} = (3\ 5)(1\ 3)(4\ 5)(3\ 4)(2\ 3)(1\ 2) = (1\ 3\ 2\ 5\ 4).$$

Therefore ω is a full cyclic permutation ordering of the butterfly graph G.

Let $\beta(G) := |E_G| - |V_G| + 1$ denote the *Betti number* (also called the *circuit rank*) of the connected graph G. When a graph G has a full cyclic permutation ordering, considering the signature of a full cyclic permutation ordering, one can show that the Betti number $\beta(G)$ is even. However, the converse is false. For example, the dumbbell graph (Figure 2) has no full cyclic permutation orderings although its Betti number is 2.

We regard a graph as a topological space by identifying each edge with the unit interval $[0, 1]$ and gluing them at vertices. Then $\beta(G)$ coincides with the Betti number of G as a topological space. In this article, we will show that having a full cyclic permutation ordering is a topological property as follows.

Let Σ be an orientable closed surface and $\iota\colon G \to \Sigma$ an embedding. We call a connected component of the complement of the image of ι a *face*. An embedding ι is a 2*-cell embedding* if every face is homeomorphic to an open disk. For any 2-cell embedding $\iota\colon G \to \Sigma$,

$$|V_G| - |E_G| + f_\iota = 2 - 2g_\Sigma,$$

where f_ι denotes the number of faces of the 2-cell embedding ι and g_Σ the genus of Σ. Then

$$2g_\Sigma + f_\iota = \beta(G) + 1.$$

Therefore maximizing the genus g_Σ is equivalent to minimizing the number of faces f_ι. Hence the *maximum genus* $\gamma_{\max}(G)$, the maximum of genus g_Σ such that there exists a 2-cell embedding $G \to \Sigma$, satisfies

$$\gamma_{\max}(G) \leq \left\lfloor \frac{\beta(G)}{2} \right\rfloor,$$

where $\lfloor\ \rfloor$ denotes the floor function. If the equality holds, then we say that G is *upper embeddable*.

Upper embeddable graphs are well-studied objects by many researchers [4, 5, 8, 9, 10, 12, 13, 14, 15, 17, 18]. Jungerman and Xuong gave a combinatorial characterization of upper embeddability independently.

Theorem 1.3 (Jungerman [5, Theorem 2], Xuong [18, Theorem A])**.** *A connected graph G with even (odd) Betti number is upper embeddable if and only if there exists a spanning tree T of G such that all (all but one) connected components of $G \setminus T$ consists of an even number of edges.*

Here is the main theorem of this article.

Theorem 1.4. *Given a graph G, the following are equivalent.*

(1) *G has a full cyclic permutation ordering.*

(2) *There exists a 2-cell embedding $\iota\colon G \to \Sigma$ such that $f_\iota = 1$.*

(3) *The Betti number $\beta(G)$ is even and G is upper embeddable, that is, $\beta(G) = 2\gamma_{\max}(G)$.*

(4) *There exists a spanning tree T of G such that every connected components of $G \setminus T$ consists of an even number of edges.*

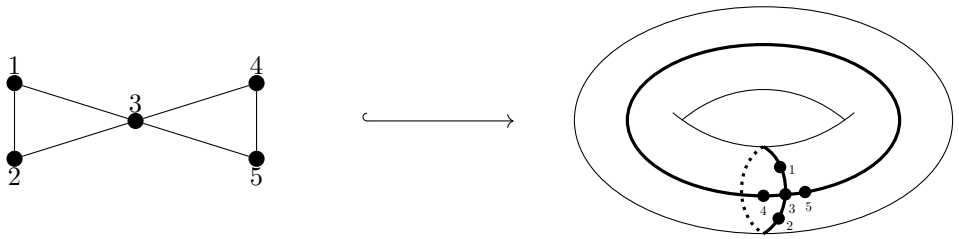


Figure 3: A 2-cell embedding of the butterfly graph into a torus with exactly one face.

Note that the conditions (2), (3), and (4) are equivalent by the definition of upper embeddability and Theorem 1.3. Figure 3 shows how the butterfly graph can be embedded into a torus with exactly one face.

The organization of this article as follows. In Section 2, we will prove that (3) implies (1) and give an example of constructing a full cyclic permutation ordering. In Section 3, we will review the relation of 2-cell embeddings and rotation systems and we will prove that (1) implies (2). Combining the proofs in Section 2 and Section 3, we will complete the proof of Theorem 1.4.

In Section 4, we will study another extreme condition, that is, edge orderings ω such that π_ω is the identity permutation, which Eden [2] studied. Eden gave necessary conditions for such orderings and asked whether the condition is also sufficient. We will give a counterexample for this question.

2 Proof that (3) implies (1)

First, we introduce the following lemma.

Lemma 2.1. *Let π be a full cyclic permutation in S_n. If the distinct numbers u, v, w appear in π in this cyclic order, then the product $(u\ v)(v\ w)\pi$ is a full cyclic permutation.*

Proof. By the assumption, we can write

$$\pi = (u\ a_1\ \cdots\ a_r\ v\ b_1\ \cdots\ b_s\ w\ c_1\ \cdots\ c_t),$$

where a_i, b_i, c_i denote distinct numbers in $[n] \setminus \{u, v, w\}$. Then we obtain

$$\begin{aligned}(u\ v)(v\ w)\pi &= (u\ v)(v\ w)(u\ a_1\ \cdots\ a_r\ v\ b_1\ \cdots\ b_s\ w\ c_1\ \cdots\ c_t)\\ &= (u\ a_1\ \cdots\ a_r\ w\ c_1\ \cdots\ c_t\ v\ b_1\ \cdots\ b_s),\end{aligned}$$

which is a full cyclic permutation. □

We say that two edges e and e' are *adjacent* if $r_G(e) \cap r_G(e') \neq \varnothing$, that is, they have a common endvertex. Note that the case $r_G(e) = r_G(e')$ is allowed.

Lemma 2.2. *Let e and e' be two adjacent edges in a graph G. If $G \setminus \{e, e'\}$ has a full cyclic permutation ordering, then G has a full cyclic permutation ordering.*

Proof. Let $\omega' = (e_1, \ldots, e_m)$ be a full cyclic permutation ordering of $G \setminus \{e, e'\}$. If $r_G(e) = r_G(e')$, then $\omega := (e_1, \ldots, e_m, e', e)$ is a full cyclic permutation ordering of G since $\pi_\omega = \pi_{\omega'}$.

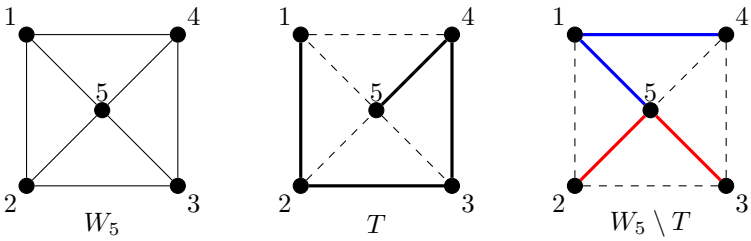


Figure 4: Wheel graph W_5 and its spanning tree T.

Now, suppose that $r_G(e) = \{u, v\}$ and $r_G(e') = \{v, w\}$ with $u \neq w$. If the cycle order of u, v, w in $\pi_{\omega'}$ is u, v, w, then $\omega := (e_1, \ldots, e_m, e', e)$ is a full cyclic permutation ordering of G by Lemma 2.1. In a symmetrical manner, if the cycle order is w, v, u, then let $\omega := (e_1, \ldots, e_m, e, e')$. □

The following lemma is required.

Lemma 2.3 (Xoung [17, Lemma 3])**.** *Suppose that G is upper embeddable and $\beta(G)$ is even. If G is not a tree, then there exist two adjacent edges e and e' such that $G \setminus \{e, e'\}$ is upper embeddable.*

Proof that (3) implies (1) in Theorem 1.4. We will show that G has a full cyclic permutation ordering by induction on the Betti number $\beta(G)$. When $\beta(G) = 0$, G is a tree and has a full cyclic permutation ordering by Dénes' theorem (Theorem 1.2).

Assume $\beta(G) > 0$. By Lemma 2.3, there exist two adjacent edges e and e' such that $G' := G \setminus \{e, e'\}$ is upper embeddable. By the induction hypothesis, G' has a full cyclic permutation ordering. By Lemma 2.2, G has a full cyclic permutation ordering. □

Example 2.4. Consider the wheel graph W_5 pictured in Figure 4. The Betti number of W_5 is 4. Let T be the spanning tree of W_5 consisting of the edges $12, 23, 34, 45$. Then $W_5 \setminus T$ is connected and consisting of 4 edges. Therefore W_5 satisfies the condition (4) in Theorem 1.4 and hence has a full cyclic permutation ordering. We will construct a full cyclic permutation ordering following by the proof of Lemma 2.2.

We partition the edges of $W_5 \setminus T$ into two adjacent pairs $\{25, 35\}$ and $\{14, 15\}$. Define the edge ordering ω_1 of T by $\omega_1 := (12, 23, 34, 45)$. By Dénes' theorem (Theorem 1.2), π_{ω_1} is a full cyclic permutation. Indeed we have

$$\pi_{\omega_1} = (4\ 5)(3\ 4)(2\ 3)(1\ 2) = (2\ 1\ 5\ 4\ 3).$$

Next we define the edge ordering ω_2 of $T \cup \{25, 35\}$. Following the proof above, define ω_2 by $\omega_2 := \omega_1 * (35, 25)$, where $*$ denotes the concatenation. Then

$$\pi_{\omega_2} = (2\ 5)(3\ 5)(2\ 1\ 5\ 4\ 3) = (4\ 2\ 1\ 3\ 5).$$

Similarly, define ω_3 by $\omega_3 := \omega_2 * (15, 14)$. Then

$$\pi_{\omega_3} = (1\ 4)(1\ 5)(4\ 2\ 1\ 3\ 5) = (4\ 2\ 5\ 1\ 3).$$

Thus we obtain a full cyclic permutation ordering $\omega_3 = (12, 23, 34, 45, 35, 25, 15, 14)$ of W_5.

3 Proof that (1) implies (2)

Let $I_G(v)$ denote the set of edges of G incident to a vertex $v \in V_G$. A *rotation system* of G is a collection $\rho = (\rho_v)_{v \in V_G}$ consisting of cyclic orders ρ_v on $I_G(v)$, where a cyclic order on $I_G(v)$ is an equivalence class of linear orders on $I_G(v)$ obtained by identifying $(e_1, e_2, \ldots, e_s)$ with its circular shift $(e_2, \ldots, e_s, e_1)$, denoted by $[e_1, \ldots, e_s]$.

Every embedding of G on an orientable closed surface defines a rotation system with the clockwise ordering for each vertex. Conversely, from a rotation system, we can obtain a 2-cell embedding of G on an orientable closed surface as follows.

Define D_G by

$$D_G \coloneqq \{\, (e, u) \in E_G \times V_G \mid u \in r_G(e) \,\}.$$

We call an element of D_G a *dart*. When $r_G(e) = \{u, v\}$, the dart (e, u) shows an orientation of the edge e from the *source* u to the *target* v. Define the involution α on D_G by $\alpha(e, u) \coloneqq (e, v)$.

Given a rotation system $\rho = (\rho_v)_{v \in V_G}$, we will define bijections σ and ϕ from D_G to itself. Suppose that $\rho_v = [e_1, \ldots, e_s]$. Define σ by $\sigma(e_i, v) \coloneqq (e_{i+1}, v)$, where we consider $e_{s+1} = e_1$. Let $\phi \coloneqq \sigma \circ \alpha$.

For every dart d, the target of d coincides with the source of $\phi(d)$. Therefore each orbit in $D_G/\langle\phi\rangle$ determines a closed walk on G and we can make a polygon whose sides are formed by the darts in the orbit. Gluing the sides of the polygons obtained from the orbits in $D_G/\langle\phi\rangle$ by the involution α, we obtain an embedding of G on a closed surface. One can show that this surface is actually orientable (See [6, Subsection 3.2]) and hence this embedding is the desired 2-cell embedding.

Theorem 3.1 ([6, Theorem 3.2.4])**.** *Given a graph G, there exists a one-to-one correspondence between rotation systems of G and 2-cell embeddings of G on oriented closed surfaces up to orientation-preserving homeomorphism.*

Note that, from the construction, the number of the faces of the embedding corresponding to a rotation system is equal to the number of the orbits in $D_G/\langle\phi\rangle$.

Let ω be an edge ordering of G. For each $v \in V_G$, let ω_v denote the linear order on $I_G(v)$ induced by ω. Moreover, let $\rho_{\omega,v}$ be the cyclic order on $I_G(v)$ determined by ω_v. Thus we obtain the rotation system $\rho_\omega \coloneqq (\rho_{\omega,v})_{v \in V_G}$ from an edge ordering ω and hence the corresponding bijections σ_ω and $\phi_\omega = \sigma_\omega \circ \alpha$.

Lemma 3.2. *Let $\omega = (e_1, \ldots, e_m)$ be an edge ordering of a graph G. For each $v \in V_G$, let f_v denote the minimal edge in $I_G(v)$ with respect to ω. Define a map $\Psi\colon V_G/\langle\pi_\omega\rangle \to D_G/\langle\phi_\omega\rangle$ by*

$$\Psi([v]) \coloneqq [(f_v, v)],$$

where the brackets denote equivalence classes. Then Ψ is a bijection.

Proof. First, we will show that the map Ψ is well-defined. It is sufficient to show that $[(f_v, v)] = [(f_{\pi_\omega(v)}, \pi_\omega(v))]$ for each $v \in V_G$.

Fix $v \in V_G$. The edge ordering $\omega = (e_1, \ldots, e_m)$ defines the set T_v as follows.

$$T_v \coloneqq \{\, e_i \in E_G \mid (\tau_i \tau_{i-1} \cdots \tau_1)(v) \neq (\tau_{i-1} \cdots \tau_1)(v) \,\},$$

where we agree with $(\tau_{i-1}\cdots\tau_1)(v) = v$ if $i = 1$. Suppose that $T_v = \{ e_{j_1}, e_{j_2}, \ldots, e_{j_s} \}$ with $j_1 < j_2 < \cdots < j_s$. Then $(e_{j_1}, \ldots, e_{j_s})$ is a trail from v to $\pi_\omega(v)$. Let $v_0 := v$ and for $k \in \{1, \ldots, s\}$ define v_k recursively as the endvertex of e_{j_k} other than v_{k-1}. Note that $v_s = \pi_\omega(v)$.

From the definition of T_v, for each $k \in \{1, \ldots, s-1\}$, $e_{j_{k+1}}$ covers e_{j_k} in $I_G(v_k)$ with respect to ω. Therefore $\phi_\omega(e_{j_k}, v_{k-1}) = (e_{j_{k+1}}, v_k)$. Since e_{j_s} is the maximal element in $I_G(v_s)$ with respect to the order ω, we have $\phi_\omega(e_{j_s}, v_{s-1}) = (f_{v_s}, v_s) = (f_{\pi_\omega(v)}, \pi_\omega(v))$. Thus Ψ is well-defined.

Next, to prove the surjectivity, take an orbit $W \in D_G/\langle\phi_\omega\rangle$. Let f be the minimal element in

$$\{ e \in E_G \mid (e, v) \in W \text{ for some } v \in V_G \}$$

with respect to ω. Suppose $(f, v) \in W$. Assume f is not minimal in $I_G(v)$ with respect to ω_v. Then $f' := \sigma_\omega^{-1}(f)$ is less than f in $I_G(v)$ with respect to ω. Let $r_G(f') = \{v, v'\}$. Then $\phi_\omega(f', v') = (f, v)$ and hence $(f', v') \in W$. This contradicts to the minimality of f. Therefore f is minimal in $I_G(v)$ and hence $f = f_v$. Hence $\Psi([v]) = [(f, v)] = W$.

Finally, we prove the injectivity. Let $u, v \in V_G$ and suppose that $\Psi([u]) = \Psi([v])$. Then we have $\phi_\omega^s(f_u, u) = (f_v, v)$ for some $s \in \mathbb{Z}$. Without loss of generality, we can assume that $s > 0$.

Recall the edges of G are ordered by $\omega = (e_1, \ldots, e_m)$. We can write the edge f_u as $f_u = e_{j_0}$ with some $j_0 \in \{1, \ldots, m\}$. Moreover we can obtain the walk $(e_{j_0}, e_{j_1}, \ldots, e_{j_s})$ by $(e_{j_k}, v_k) := \phi_\omega(e_{j_{k-1}}, v_{k-1})$ for $k \in \{1, \ldots, s\}$, where $v_0 := u$. Note that $(e_{j_0}, v_0) = (f_u, u)$ and $(e_{j_s}, v_s) = (f_v, v)$. Suppose that

$$\{ k \in \{1, \ldots, s\} \mid j_{k-1} \geq j_k \} = \{p_1, \ldots, p_t\}$$

with $p_1 < \cdots < p_t = s$. Then $\pi_\omega^i(v_0) = v_{j_{p_i}}$ for $i \in \{1, \ldots, t\}$. In particular, $\pi_\omega^t(u) = \pi_\omega^t(v_0) = v_{j_{p_t}} = v_s = v$. Thus $[u] = [v]$ and hence f is injective. □

Now we are ready to prove that (1) implies (2).

Proof that (1) *implies* (2) *in* Theorem 1.4. Let ω be a full cyclic permutation ordering of G. Then the number of faces of the 2-cell embedding corresponding to the rotation system ρ_ω is equal to $|D_G/\langle\phi_\omega\rangle| = |V_G/\langle\pi_\omega\rangle| = 1$. □

4 Identity permutation ordering

In this section, a graph is not necessarily connected. We say that an edge ordering ω of a graph G is an *identity permutation ordering* if $\pi_\omega = \varepsilon$, where ε denotes the identity permutation. Every edgeless graph vacuously has an identity permutation ordering. The minimal example of a non-trivial graph having an identity permutation ordering is the 2-cycle C_2. Eden [2] studied simple graphs that have an identity permutation ordering and mentioned the complete graph K_4 is the minimal example. Figure 5 shows identity permutation orderings of C_2 and K_4.

Eden gave necessary conditions (without proof) for simple connected graphs having an identity permutation ordering as follows.

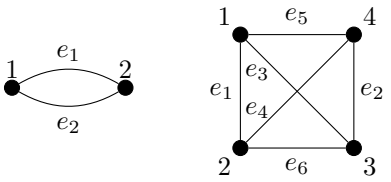


Figure 5: The 2-cycle C_2 and the complete graph K_4 have an identity permutation ordering.

Proposition 4.1 (Eden [2, page 130])**.** *Let G be a simple connected graph on n vertices with m edges. If G has an identity permutation ordering, then the following conditions hold.*

(1) *m is even.*

(2) *There exist a set $\mathcal{C}$ consisting of closed trails and a map $\psi\colon V_G \to \mathcal{C}$ such that the following conditions hold.*

(i) *ψ is bijective.*

(ii) *Every $v \in V_G$ belongs to the closed trail $\psi(v)$.*

(iii) *The sum of the number of edges of closed trails in $\mathcal{C}$ is $2m$.*

(iv) *Each edge of G belongs to exactly two closed trails in $\mathcal{C}$.*

Proof. Suppose that $\omega = (e_1, \dots, e_m)$ be an identity permutation ordering. Since the identity permutation is an even permutation, we have m is even. Let $\Psi\colon V_G \to D_G/\langle\phi_\omega\rangle$ be the bijection considered in Lemma 3.2. Note that $V_G = V_G/\langle\pi_\omega\rangle$ since $\pi_\omega = \varepsilon$.

For each $v \in V_G$, there exists no dart $d \in D_G$ such that both d and $\alpha(d)$ belong to $\Psi(v)$ since π_ω is the identity. Thus, forgetting the direction of each dart in $\Psi(v)$, we obtain the closed trail $\psi(v) \subseteq E_G$. Letting $\mathcal{W} \coloneqq \{\, \psi(v) \mid v \in V_G \,\}$, we have a surjection $\psi\colon V_G \to \mathcal{W}$.

We will prove ψ is injective. Assume that there exist distinct vertices u, v such that $\psi(u) = \psi(v)$. Let $\omega' = (f_1, \dots, f_r)$ be the induced order of ω on $\psi(u)$. Since $\pi_{\omega'}(u) = u$, the edges f_1 and f_r are incident to u. Also, f_1 and f_r are incident to v by the same reason. Hence f_1 and f_r are parallel edges between u and v. This contradicts that G is simple. Therefore ψ is injective and hence bijective.

By the definition of maps Ψ and ψ, every $v \in V_G$ belongs to $\psi(v)$. Moreover,

$$\sum_{v \in V_G} |\psi(v)| = \sum_{v \in V_G} |\Psi(v)| = |D_G| = 2m.$$

For any $e \in E_G$, the two darts on e belongs distinct orbits $\Psi(u)$ and $\Psi(v)$. Then e belongs to $\psi(u)$ and $\psi(v)$ and the other trails do not contain e. Thus the map $\psi\colon V_G \to \mathcal{C}$ has the desired properties. □

Example 4.2. For the complete graph K_4 in Figure 5, the following map ψ satisfies the conditions in Proposition 4.1.

$$\psi(1) = \{e_1, e_4, e_5\},\ \psi(2) = \{e_1, e_3, e_6\},\ \psi(3) = \{e_2, e_4, e_6\},\ \psi(4) = \{e_2, e_3, e_5\}.$$

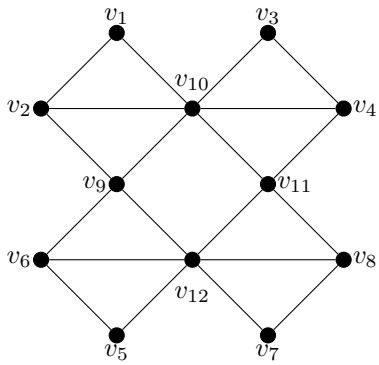


Figure 6: A counterexample for Eden's question.

Eden asked whether the necessary conditions in Proposition 4.1 are also sufficient. We will give a counterexample for this question. Let G be the graph on 12 vertices with 20 edges pictured in Figure 6. Define a map ψ by

$$\begin{aligned}
&\psi(v_1) = v_1v_2v_{10}v_1, \quad \psi(v_2) = v_2v_9v_{10}v_2, \quad \psi(v_3) = v_3v_{10}v_4v_3,\\
&\psi(v_4) = v_4v_{10}v_{11}v_4, \quad \psi(v_5) = v_5v_6v_{12}v_5, \quad \psi(v_6) = v_6v_9v_{12}v_6,\\
&\psi(v_7) = v_7v_8v_{12}v_7, \quad \psi(v_8) = v_8v_{11}v_{12}v_8, \quad \psi(v_9) = v_9v_2v_1v_{10}v_9,\\
&\psi(v_{10}) = v_{10}v_3v_4v_{11}v_{10}, \ \psi(v_{11}) = v_{11}v_8v_7v_{12}v_{11}, \ \psi(v_{12}) = v_{12}v_5v_6v_9v_{12}.
\end{aligned}$$

The conditions in Proposition 4.1 are satisfied. Suppose that G has an identity permutation ordering ω. By Lemma 3.2 there exists a 2-cell embedding ι with $f_\iota = |D_G/\langle\phi_\omega\rangle| = |V_G| = 12$ faces. Then the genus g_ι satisfies

$$2 - 2g_\iota = |V_G| - |E_G| + f_\iota = 12 - 20 + 12 = 4.$$

Therefore $g_\iota = -1$, which is a contradiction.

Problem 4.3. Characterize graphs that have an identity permutation ordering.

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References

[1] J. Dénes, The representation of a permutation as the product of a minimal number of transpositions, and its connection with the theory of graphs, *Publ. Math. Inst. Hungar. Acad. Sci.* **4** (1959), 63–71.

[2] M. Eden, On a relation between labeled graphs and permutations, *J. Comb. Theory* **2** (1967), 129–134, doi:10.1016/S0021-9800(67)80093-0, https://doi.org/10.1016/S0021-9800(67)80093-0.

[3] A. M. Foley, J. Kazdan, L. Kröll, S. M. Alberga, O. Melnyk and A. Tenenbaum, Transplanting trees: Chromatic symmetric function results through the group algebra of S_n, 2021, arXiv:2112.09937 [math.CO].

[4] Y. Huang and Y. Liu, Face size and the maximum genus of a graph. I. Simple graphs, *J. Comb. Theory Ser. B* **80** (2000), 356–370, doi:10.1006/jctb.2000.1990, https://doi.org/10.1006/jctb.2000.1990.

[5] M. Jungerman, A characterization of upper-embeddable graphs, *Trans. Amer. Math. Soc.* **241** (1978), 401–406, doi:10.2307/1998852, https://doi.org/10.2307/1998852.

[6] B. Mohar and C. Thomassen, *Graphs on Surfaces*, Johns Hopkins Studies in the Mathematical Sciences, Johns Hopkins University Press, Baltimore, MD, 2001.

[7] P. Moszkowski, A solution to a problem of Dénes: a bijection between trees and factorizations of cyclic permutations, *European J. Comb.* **10** (1989), 13–16, doi:10.1016/S0195-6698(89)80028-9, https://doi.org/10.1016/S0195-6698(89)80028-9.

[8] L. Nebeský, Every connected, locally connected graph is upper embeddable, *J. Graph Theory* **5** (1981), 205–207, doi:10.1002/jgt.3190050211, https://doi.org/10.1002/jgt.3190050211.

[9] L. Nebeský, A note on upper embeddable graphs, *Czechoslovak Math. J.* **33** (1983), 37–40, doi:10.21136/CMJ.1983.101853, https://doi.org/10.21136/CMJ.1983.101853.

[10] L. Nebeský, On locally quasiconnected graphs and their upper embeddability, *Czechoslovak Math. J.* **35** (1985), 162–166, doi:10.21136/CMJ.1985.102005, https://doi.org/10.21136/CMJ.1985.102005.

[11] B. Pawlowski, Chromatic symmetric functions via the group algebra of S_n, *Algebr. Comb.* **5** (2022), 1–20, doi:10.5802/alco.134, https://doi.org/10.5802/alco.134.

[12] C. Payan and N. H. Xuong, Upper embeddability and connectivity of graphs, *Discrete Math.* **27** (1979), 71–80, doi:10.1016/0012-365X(79)90070-0, https://doi.org/10.1016/0012-365X(79)90070-0.

[13] M. Škoviera, The maximum genus of graphs of diameter two, *Discrete Math.* **87** (1991), 175–180, doi:10.1016/0012-365X(91)90046-5, https://doi.org/10.1016/0012-365X(91)90046-5.

[14] M. Škoviera, The decay number and the maximum genus of a graph, *Math. Slovaca* **42** (1992), 391–406.

[15] M. Škoviera and R. Nedela, The maximum genus of vertex-transitive graphs, *Discrete Math.* **78** (1989), 179–186, doi:10.1016/0012-365X(89)90175-1, https://doi.org/10.1016/0012-365X(89)90175-1.

[16] R. Uchiumi, Signed graphs and signed cycles of hyperoctahedral groups, *Electron. J. Graph Theory Appl.* **11** (2023), 419–429, doi:10.5614/ejgta.2023.11.2.7, https://doi.org/10.5614/ejgta.2023.11.2.7.

[17] N. H. Xuong, How to determine the maximum genus of a graph, *J. Comb. Theory Ser. B* **26** (1979), 217–225, doi:10.1016/0095-8956(79)90058-3, https://doi.org/10.1016/0095-8956(79)90058-3.

[18] N. H. Xuong, Upper-embeddable graphs and related topics, *J. Comb. Theory Ser. B* **26** (1979), 226–232, doi:10.1016/0095-8956(79)90059-5, https://doi.org/10.1016/0095-8956(79)90059-5.