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19 Clifford odd and even objects in even and odd dimensional spaces

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Abstract. In a long series of works I demonstrated, together with collaborators, that the model named the *spin-charge-family* theory offers the explanation for all in the *standard model* assumed properties of the second quantized fermion and boson fields, offering several predictions as well as explanations for several of the observed phenomena. The theory assumes a simple starting action in even dimensional spaces with $d \geq (13+1)$ with massless fermions interacting with gravity only. The internal spaces of fermion and boson fields are described by the Clifford odd and even objects, respectively. This note discusses properties of the internal spaces in odd dimensional spaces, d, $d = (2n+1)$, which differ essentially from the properties in even dimensional spaces.

Povzetek:
V dolgem nizu člankov sem skupaj s sodelavci pokazala, da ponuja teorija , imenovana *spin-charge-family*, razlago za vse v *standardnem modelu* privzete lastnosti fermionskih in bozonskih polj (v drugi kvantizaciji), ponuja pa poleg napovedi tudi razlago za marsikatero od opaženih kozmoloških pojavov. Teorija predlaga preprosto akcijo v sodo-razsežnih prostorih, $d \geq (13+1)$, za brezmasne fermione v interakciji samo z gravitacijskim poljem. Notranje prostore fermionskih in bozonskih polj opišejo Cliffordovi lihi oziroma sodi objekti. Ta prispevek obravnava lastnosti notranjega prostora fermionov in bozonov v prostorih z lihimi razsežnostimi d, $d = (2n+1)$.

Keywords: Second quantization of fermion and boson fields with Clifford algebra; beyond the standard model; Kaluza-Klein-like theories in higher dimensional spaces, Clifford algebra in odd dimensional spaces.

19.1 introduction

My working hypothesis is that "Nature knows all the mathematics", which we have and possibly will ever invent, and "she uses it where needed". Recognizing that there are two kinds of the Clifford algebra objects, γ^a's and $\tilde{\gamma}^a$'s [2], each of them of odd and even

character, I use them to describe the internal spaces of fermion and boson fields [5–9] in even dimensional spaces.
The Clifford odd objects, if they are superposition of odd products of γ^a's, explain in even $d = 2n$ properties of fermion fields. The second kind of the Clifford odd objects, $\tilde{\gamma}^a$'s, can be used, after defining their application on the polynomials of γ^a's (Eq. (7) of my talk in this Proceedings [4]), to equip the irreducible representations of odd polynomials of γ^a's with the family quantum numbers.
The Clifford even objects, if they are superposition of even products of γ^a's, explain in even d properties of boson fields, the gauge fields of the corresponding fermion fields. They do not appear in families [4–9].
In $d = (13 + 1)$ the Clifford odd objects manifest all the properties of the internal space of fermions — of the observed quarks and leptons and antiquarks and antileptons with their families included — and the Clifford even objects explain the gauge fields of the corresponding fermion fields, as well as the Higgs' scalars and Yukawa couplings. The internal space of fermion and boson fields, described by "basis vectors" (they are chosen to be eigenvectors of all the members of the Cartan subalgebra members of the Lorentz group in the internal space of fields), demonstrate properties of the postulates of the second quantization of fermion and boson fields, explaining these postulates [4, 6].
I demonstrate in this note that also in odd dimensional spaces the Clifford odd and the Clifford even objects exist. However, the eigenstates of the operator of handedness are in odd dimensional spaces the superposition of the Clifford odd and the Clifford even objects. This seems to explain the ghost fields appearing in several theories for taking care of the singular contributions in evaluating Feynman graphs.
Next section presents the internal spaces, described by the Clifford odd and the Clifford even "basis vectors" for fermion and boson fields in even dimensional spaces, for $d = (1+1)$ and $d = (3 + 1)$, as well as in odd dimensional spaces, for $d = (0 + 1)$ and $d = (2 + 1)$. This simple cases are chosen to easier demonstrate the difference in properties in even and odd dimensional spaces.
In Refs. [10–12] from 20years ago the authors discuss the question of q time and $d - q$ dimensions in odd and even dimensional spaces, for any q. Using the requirements that the inner product of two fermions is unitary and invariant under Lorentz transformations the authors conclude that odd dimensional spaces are not appropriate due to the existence of fermions of both handedness and correspondingly not mass protected.
In this note the comparison of properties of fermion and boson fields in odd and in even dimensional spaces are made, using the Clifford algebra objects to describe the internal spaces of fermion and boson fields. The recognition of this note might further clarify the "effective" choice of Nature for one time and three space dimensions.
The reader can find more explanation about the properties of internal spaces of fermion and boson fields in even dimensional spaces in my contribution in this proceedings [4].

19.2 "Basis vectors" in $d = 2n$ and $d = 2n + 1$ for $n = 0, 1, 2$

In Ref. [4–9] the reader can find the definition of the "basis vectors" as the eigenstates of the Cartan subalgebra of the Lorentz algebra in internal spaces of fermion and boson fields. "Basis vectors" are written as superposition of the Clifford odd (for fermions) and the Clifford even (for bosons) products of γ^a's. "Basis vectors" for fermions have either left or right handedness, $\Gamma^(d)$ (the handedness is defined in Eq. (19.2), and appear in families (the family quantum numbers are determined by $\tilde{\gamma}^a$'s, with $\tilde{S}^{ab} = \frac{i}{4}\{\tilde{\gamma}^a, \tilde{\gamma}^b\}_-$). The Clifford odd "basis vectors" have their Hermitian conjugated partners in a separate group. "Basis

vectors" for bosons have no families and have their Hermitian conjugated partners within the same group.
Properties of the "basis vectors" in odd dimensional spaces have completely different properties: Only the superposition of the Clifford odd and the Clifford even"basis vectors" have a definite handedness. Correspondingly the eigenvectors of the Cartan subalgebra members have both handedness, $\Gamma^{(2n+1)} = \pm 1$.

19.2.1 Even dimensional spaces $d = (1+1), (3+1)$

To simplify the comparison between even and odd dimensional spaces, simple cases for either even or odd dimensional spaces are discussed. The definition of nilpotents and projectors and the relations among them can be found in App. 19.4.

$d = (1+1)$

There are 4 ($2^{d=2}$) "eigenvectors" of the Cartan subalgebra members, Eq. (19.4), S^{01} and $\mathcal{S}^{01}$ of the Lorentz algebra S^{ab} and $\mathcal{S}^{ab} = S^{01} + \tilde{S}^{01}$ ($S^{ab} = \frac{i}{4}\{\gamma^a, \gamma^b\}_-$ $\tilde{S}^{ab} = \frac{i}{4}\{\tilde{\gamma}^a, \tilde{\gamma}^b\}_-$) representing one Clifford odd "basis vector" $\hat{b}^{1\dagger}_1 = \stackrel{01}{(+i)}$ (m=1), appearing in one family (f=1) and correspondingly one Hermitian conjugated partner $\hat{b}^1_1 = \stackrel{01}{(-i)}$ [1] and two Clifford even "basis vector" ${}^{I}\mathcal{A}^{1\dagger}_1 = \stackrel{01}{[+i]}$ and ${}^{II}\mathcal{A}^{1\dagger}_1 = \stackrel{01}{[-i]}$, each of them is self adjoint.
Correspondingly we have two Clifford odd, Eqs. (19.3, 19.7)

$$\hat{b}^{1\dagger}_1 = \stackrel{01}{(+i)}\,, \quad \hat{b}^1_1 = \stackrel{01}{(-i)}$$

and two Clifford even

$${}^{I}\mathcal{A}^{1\dagger}_1 = \stackrel{01}{[+i]}\,, \quad {}^{II}\mathcal{A}^{1\dagger}_1 = \stackrel{01}{[-i]}$$

"basis vectors".
The two Clifford odd "basis vectors" are Hermitian conjugated to each other. I make a choice that $\hat{b}^{1\dagger}_1$ is the "basis vector", the second Clifford odd object is its Hermitian conjugated partner. Defining the handedness as $\Gamma^{(1+1)} = \gamma^0\gamma^1$, Eq. (19.2), it follows, using Eq. (19.5), that $\Gamma^{(1+1)}\,\hat{b}^{1\dagger}_1 = \hat{b}^{1\dagger}_1$, which means that $\hat{b}^{1\dagger}_1$ is the right handed "basis vector".
We could make a choice of left handed "basis vector" if choosing $\hat{b}^{1\dagger}_1 = \stackrel{01}{(-i)}$, but the choice of handedness would remain only one.

Each of the two Clifford even "basis vectors" is self adjoint (($^{I,II}\mathcal{A}^{1\dagger}_1)^\dagger = {}^{I,II}\mathcal{A}^{1\dagger}_1$).

Let us notice, taking into account Eqs. (19.5, 19.9), that

$$\{\hat{b}^1_1(\equiv\stackrel{01}{(-i)}) *_A \hat{b}^{1\dagger}_1(\equiv\stackrel{01}{(+i)})\}|\psi_{oc}>= {}^{II}\mathcal{A}^{1\dagger}_1(\equiv\stackrel{01}{[-i]})|\psi_{oc}>= |\psi_{oc}>\,,$$

$$\{\hat{b}^{1\dagger}_1(\equiv\stackrel{01}{(+i)}) *_A \hat{b}^1_1(\equiv\stackrel{01}{(-i)})\}|\psi_{oc}>= 0\,,$$

---

[1] It is our choice which one, $\stackrel{01}{(+i)}$ or $\stackrel{01}{(-i)}$, we chose as the "basis vector" $\hat{b}^{1\dagger}_1$ and which one is its Hermitian conjugated partner. The choice of the "basis vector" determines the vacuum state $|\psi_{oc}>$. For $\hat{b}^{1\dagger}_1 = \stackrel{01}{(+i)}$, the vacuum state is $|\psi_{oc}> = \stackrel{01}{[-i]}$ (due to the requirement that $\hat{b}^{1\dagger}_1|\psi_{oc}>$ is nonzero), which is the Clifford even object.

$$^{I}\mathcal{A}^{1\dagger}_{1}(\equiv \stackrel{01}{[+i]}) *_A \hat{b}^{1}_{1}(\equiv \stackrel{01}{(+i)})|\psi_{oc}>= \hat{b}^{1}_{1}(\equiv \stackrel{01}{(+i)})|\psi_{oc}>,$$
$$^{I}\mathcal{A}^{1\dagger}_{1}(\equiv \stackrel{01}{[+i]})\, \hat{b}^{1}_{1}(\equiv \stackrel{01}{(-i)})|\psi_{oc}>= 0\,.$$

We find that

$$^{I}\mathcal{A}^{1\dagger}_{1} *_A {}^{II}\mathcal{A}^{1\dagger}_{1} = 0 = {}^{II}\mathcal{A}^{1\dagger}_{1} *_A {}^{I}\mathcal{A}^{1\dagger}_{1}\,.$$

From the case $d = (3+1)$ we can learn a little more:

$d = (3+1)$

There are 16 $(2^{d=4})$ "eigenvectors" of the Cartan subalgebra members (S^{03}, S^{12}) and $(\mathcal{S}^{03}, \mathcal{S}^{12})$ of the Lorentz algebras S^{ab} and $\mathcal{S}^{ab}$, Eq. (19.4), in $d = (3+1)$.
There are two families $(2^{\frac{4}{2}-1}$, f=(1,2)) with two $(2^{\frac{4}{2}-1}$, m=(1,2)) members each of the Clifford odd "basis vectors" $\hat{b}^{m\dagger}_{f}$, with $2^{\frac{4}{2}-1} \times 2^{\frac{4}{2}-1}$ Hermitian conjugated partners $\hat{b}^{m}_{f}$ in a separate group (not reachable by S^{ab}).
There are $2^{\frac{4}{2}-1} \times 2^{\frac{4}{2}-1}$ members of the group of $^{I}\mathcal{A}^{m\dagger}_{f}$, which are Hermitian conjugated to each other or are self adjoint, all reachable by $\mathcal{S}^{ab}$ from any starting "basis vector $^{I}\mathcal{A}^{1\dagger}_{1}$.
And there is another group of $2^{\frac{4}{2}-1} \times 2^{\frac{4}{2}-1}$ members of $^{II}\mathcal{A}^{m\dagger}_{f}$, again either Hermitian conjugated to each other or are self adjoint. All are reachable from the starting vector $^{II}\mathcal{A}^{1\dagger}_{1}$ by the application of $\mathcal{S}^{ab}$.
Again we can make a choice of either right or left handed Clifford odd "basis vectors", but not of both handedness. Making a choice of the right handed "basis vectors"

$$\begin{array}{cccc}
f=1 & f=2 & & \\
\tilde{S}^{03}=\frac{i}{2}, \tilde{S}^{12}=-\frac{1}{2}, & \tilde{S}^{03}=-\frac{i}{2}, \tilde{S}^{12}=\frac{1}{2}, & S^{03}, & S^{12} \\
\hat{b}^{1\dagger}_{1} = \stackrel{03}{(+i)}\stackrel{12}{[+]} & \hat{b}^{1\dagger}_{2} = \stackrel{03}{[+i]}\stackrel{12}{(+)} & \frac{i}{2} & \frac{1}{2} \\
\hat{b}^{2\dagger}_{1} = \stackrel{03}{[-i]}\stackrel{12}{(-)} & \hat{b}^{2\dagger}_{2} = \stackrel{03}{(-i)}\stackrel{12}{[-]} & -\frac{i}{2} & -\frac{1}{2}\,,
\end{array}$$

we find for the Hermitian conjugated partners of the above "basis vectors"

$$\begin{array}{cccc}
S^{03}=-\frac{i}{2}, S^{12}=\frac{1}{2}, & S^{03}=\frac{i}{2}, S^{12}=-\frac{1}{2}, & \tilde{S}^{03}, & \tilde{S}^{12} \\
\hat{b}^{1}_{1} = \stackrel{03}{(-i)}\stackrel{12}{[+]} & \hat{b}^{1}_{2} = \stackrel{03}{[+i]}\stackrel{12}{(-)} & -\frac{i}{2} & -\frac{1}{2} \\
\hat{b}^{2}_{1} = \stackrel{03}{[-i]}\stackrel{12}{(+)} & \hat{b}^{2}_{2} = \stackrel{03}{(+i)}\stackrel{12}{[-]} & \frac{i}{2} & \frac{1}{2}\,.
\end{array}$$

Let us notice that if we look at the subspace $SO(1,1)$, with the Clifford odd "basis vector" with the Cartan subalgebra member S^{03} of the space $SO(3,1)$, and neglect the values of S^{12}, we do have $\hat{b}^{1\dagger}_{1} = \stackrel{03}{(+i)}$ and $\hat{b}^{2\dagger}_{2} = \stackrel{03}{(-i)}$, which have opposite handedness $\Gamma^{(1,1)}$ in $d = (1+1)$, but they have different "charges" S^{12} in $d = (3+1)$. In the whole internal space all the Clifford odd "basis vectors" have only one handedness.
We further find that $|\psi_{oc}>= \frac{1}{\sqrt{2}}(\stackrel{03}{[-i]}\stackrel{12}{[+]} + \stackrel{03}{[+i]}\stackrel{12}{[+]})$. All the Clifford odd "basis vectors" are orthogonal: $\hat{b}^{m\dagger}_{f} *_A \hat{b}^{m'\dagger}_{f'} = 0$.
For the Clifford even "basis vectors" we find two groups of either self adjoint members or with the Hermitian conjugated partners within the same group. The members of one group are not reachable by the application of $\mathcal{S}^{03}$ on members of another group. We have for$^{I}\mathcal{A}^{m\dagger}_{f}$, $m = (1,2), f = (1,2)$

$$
\begin{array}{llcc}
 & & \mathcal{S}^{03} & \mathcal{S}^{12} \\
{}^{I}\mathcal{A}^{1\dagger}_{1} = & \stackrel{03}{[+i]}\stackrel{12}{[+]} & 0 & 0 \\
{}^{I}\mathcal{A}^{2\dagger}_{1} = & \stackrel{03}{(-i)}\stackrel{12}{(-)} & -i & -1
\end{array}
,
\begin{array}{llcc}
 & & \mathcal{S}^{03} & \mathcal{S}^{12} \\
{}^{I}\mathcal{A}^{1\dagger}_{2} = & \stackrel{03}{(+i)}\stackrel{12}{(+)} & i & 1 \\
{}^{I}\mathcal{A}^{2\dagger}_{2} = & \stackrel{03}{[-i]}\stackrel{12}{[-]} & 0 & 0 ,
\end{array}
$$

and for ${}^{II}\mathcal{A}^{m\dagger}_{f}$, $m = (1, 2)$, $f = (1, 2)$

$$
\begin{array}{llcc}
 & & \mathcal{S}^{03} & \mathcal{S}^{12} \\
{}^{II}\mathcal{A}^{1\dagger}_{1} = & \stackrel{03}{[+i]}\stackrel{12}{[-]} & 0 & 0 \\
{}^{II}\mathcal{A}^{2\dagger}_{1} = & \stackrel{03}{(-i)}\stackrel{12}{(+)} & -i & 1
\end{array}
,
\begin{array}{llcc}
 & & \mathcal{S}^{03} & \mathcal{S}^{12} \\
{}^{II}\mathcal{A}^{1\dagger}_{2} = & \stackrel{03}{(+i)}\stackrel{12}{(-)} & i & 1 \\
{}^{II}\mathcal{A}^{2\dagger}_{2} = & \stackrel{03}{[-i]}\stackrel{12}{[+]} & 0 & 0 .
\end{array}
$$

The Clifford even "basis vectors" have no families.

${}^{I}\mathcal{A}^{m\dagger}_{f} *_{A} {}^{I}\mathcal{A}^{m'\dagger}_{f'} = 0$, for any (m, m′, f, f ′).

Even dimensional spaces have the properties of the fermion and boson second quantized fields, as explained in Ref. [4].

19.2.2 Odd dimensional spaces $d = (0 + 1), (2 + 1)$

In odd dimensional spaces fermions have handedness defined with the odd products of γ^a's, Eq. (19.2). Correspondingly the operator of handedness transforms the Clifford odd "basis vectors" into Clifford even "basis vectors" and the description of either fermions or bosons with the Clifford even and odd "basis vectors" have in odd dimensional spaces different meaning than in even dimensional spaces:

i. While in even dimensional spaces the Clifford odd "basis vectors", $\hat{b}^{m\dagger}_{f}$, have $2^{\frac{d}{2}-1}$ members, m, in $2^{\frac{d}{2}-1}$ families, f, and their Hermitian conjugated partners appear in a separate group of $2^{\frac{d}{2}-1}$ members in $2^{\frac{d}{2}-1}$ families, there are in odd dimensional spaces some of the $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1} = 2^{d-2}$ Clifford odd "basis vectors" self adjoint and yet they have some of the Hermitian conjugated partners in another group with 2^{d-2} members.

ii. In even dimensional spaces the Clifford even "basis vectors" ${}^{i}\hat{\mathcal{A}}^{m\dagger}_{f}, i = (I, II)$, appear in two mutually orthogonal groups, each with $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ members and each with the Hermitian conjugated partners within the same group, $2^{\frac{d}{2}-1}$ of them are self adjoint.
In odd dimensional spaces the Clifford even "basis vectors" appear in two groups, each with $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1} = 2^{d-2}$ members, which are either self adjoint or have their Hermitian conjugated partners in another group. Not all the members of one group are orthogonal to the members of another group, only the self adjoint ones are.

iii. While $\hat{b}^{m\dagger}_{f}$ have in even dimensional spaces one handedness only (either right or left, depending on the definition of handedness), in odd dimensional spaces the operator of handedness is a Clifford odd object — the product of an odd number of γ^a's, Eq. (19.2), (still commuting with S^{ab}) — transforming the Clifford odd "basis vectors" into Clifford even "basis vectors" and opposite. Correspondingly are the eigenvectors of the operator of handedness the superposition of the Clifford odd and the Clifford even "basis vectors", offering in odd dimensional spaces the right and left handed eigenvectors of the operator of handedness.

Let us illustrate the above mentioned properties of the "basis vectors" in odd dimensional spaces, starting with the simplest case:

d=(0+1)

There is one Clifford odd "basis vector", which is self adjoint

$$\hat{b}^{1\dagger}_{1} = \gamma^{0} = (\hat{b}^{1\dagger}_{1})^{\dagger} = \hat{b}^{1}_{1}$$

and one Clifford even "basis vectors"

$${}^{i}\hat{\mathcal{A}}^{1\dagger}_{1} = 1\,.$$

The operator of handedness $\Gamma^{(0+1)} = \gamma^{0}$ transforms $\hat{b}^{1\dagger}_{1}$ into identity ${}^{i}\hat{\mathcal{A}}^{1\dagger}_{1}$ and ${}^{i}\hat{\mathcal{A}}^{1\dagger}_{1}$ into $\hat{b}^{1\dagger}_{1}$.
The two eigenvectors of the operator of handedness are

$$\frac{1}{\sqrt{2}}(\gamma^{0}+1)\,,\quad \frac{1}{\sqrt{2}}(\gamma^{0}-1)\,,$$

with the handedness $(+1,-1)$, that is of right and left handedness. respectively.

d=(2+1)

There are twice $2^{d=(3-2)}=2$ Clifford odd "basis vectors". We chose as the Cartan subalgebra member S^{01} of S^{ab}, Eq (19.4): $\hat{b}^{1\dagger}_{1} \stackrel{01}{=[-i]}\,\gamma^{2}$, $\hat{b}^{2\dagger}_{1} \stackrel{01}{=(+i)}$, $\hat{b}^{1\dagger}_{2} \stackrel{01}{=(-i)}$, $\hat{b}^{2\dagger}_{2} \stackrel{01}{=[+i]}\,\gamma^{2}$, with the properties

$$\begin{array}{ccc}
f=1 & f=2 & \\
\tilde{S}^{01}=\frac{i}{2} & \tilde{S}^{01}=-\frac{i}{2}, & S^{01}\\
\hat{b}^{1\dagger}_{1} \stackrel{01}{=[-i]}\,\gamma^{2} & \hat{b}^{1\dagger}_{2} \stackrel{01}{=(-i)} & -\frac{i}{2}\\
\hat{b}^{2\dagger}_{1} \stackrel{01}{=(+i)} & \hat{b}^{2\dagger}_{2} \stackrel{03}{=[+i]}\,\gamma^{2} & \frac{i}{2},
\end{array}$$

$\hat{b}^{1\dagger}_{1}$ and $\hat{b}^{2\dagger}_{2}$ are self adjoint (up to a sign), $\hat{b}^{2\dagger}_{1} \stackrel{01}{=(+i)}$ and $\hat{b}^{1\dagger}_{2} \stackrel{01}{=(-i)}$ are Hermitian conjugated to each other.

In odd dimensional spaces the Clifford odd "basis vectors" describing fermions are not separated from their Hermitian conjugated partners, as it is the case in even dimensional spaces, and do not appear in families. $\hat{b}^{1\dagger}_{1}$ are either self adjoint or have their Hermitian conjugated partners in another family.

The operator of handedness is (chosen up to a sign to be) $\Gamma^{(2+1)} = i\gamma^{1}\gamma^{2}\gamma^{2}$, Eq. (19.2).

There are twice $2^{(d=3)-2} = 2$ Clifford even "basis vectors". We choose as the Cartan subalgebra member $\mathcal{S}^{01}$: ${}^{I}\hat{\mathcal{A}}^{1\dagger}_{1} \stackrel{01}{=[+i]}$, ${}^{I}\hat{\mathcal{A}}^{2\dagger}_{1} \stackrel{01}{=(-i)}\,\gamma^{2}$, ${}^{II}\hat{\mathcal{A}}^{1\dagger}_{2} \stackrel{01}{=[-i]}$, ${}^{II}\hat{\mathcal{A}}^{2\dagger}_{2} \stackrel{01}{=(+i)}\,\gamma^{2}$, with the properties

$$\begin{array}{cccc}
 & \mathcal{S}^{01} & & \mathcal{S}^{01}\\
{}^{I}\hat{\mathcal{A}}^{1\dagger}_{1} \stackrel{01}{=[+i]} & 0 & {}^{II}\hat{\mathcal{A}}^{1\dagger}_{2} \stackrel{01}{=[-i]} & 0\\
{}^{I}\hat{\mathcal{A}}^{2\dagger}_{1} \stackrel{01}{=(-i)}\,\gamma^{2} & -i & {}^{II}\hat{\mathcal{A}}^{2\dagger}_{2} \stackrel{03}{=(+i)}\,\gamma^{2} & i,
\end{array}$$

${}^{I}\hat{\mathcal{A}}^{1\dagger}_{1} \stackrel{01}{=[+i]}$ and ${}^{II}\hat{\mathcal{A}}^{1\dagger}_{2} \stackrel{01}{=[-i]}$ are self adjoint, ${}^{I}\hat{\mathcal{A}}^{2\dagger}_{1} \stackrel{01}{=(-i)}\,\gamma^{2}$ and ${}^{II}\hat{\mathcal{A}}^{2\dagger}_{2} \stackrel{03}{=(+i)}\,\gamma^{2}$ are Hermitian conjugated to each other.

In odd dimensional spaces the two groups of the Clifford even "basis vectors" are not orthogonal.

Let us find the eigenvectors of the operator of handedness $\Gamma^{(2+1)} = i\gamma^0\gamma^1\gamma^2$. Since it is the Clifford odd object its eigenvectors are superposition of Clifford odd and Clifford even "basis vectors".
It follows

$$\Gamma^{(2+1)}\{\stackrel{01}{[-i]} \pm i \stackrel{01}{[-i]}\,\gamma^2\} = \mp\{\stackrel{01}{[-i]} \pm i \stackrel{01}{[-i]}\,\gamma^2\},$$
$$\Gamma^{(2+1)}\{\stackrel{01}{(+i)} \pm i \stackrel{01}{(+i)}\,\gamma^2\} = \mp\{\stackrel{01}{(+i)} \pm i \stackrel{01}{(+i)}\,\gamma^2\},$$
$$\Gamma^{(2+1)}\{\stackrel{01}{[+i]} \pm i \stackrel{01}{[+i]}\,\gamma^2\} = \pm\{\stackrel{01}{[+i]} \pm i \stackrel{01}{[+i]}\,\gamma^2\},$$
$$\Gamma^{(2+1)}\{\stackrel{01}{(-i)}\,\gamma^2 \pm i \stackrel{01}{(-i)}\} = \pm\{\stackrel{01}{(-i)}\,\gamma^2 \pm i \stackrel{01}{(-i)}\},$$

We can conclude that neither Clifford odd nor Clifford even "basis vectors" have in odd dimensional spaces the properties which they do demonstrate in even dimensional spaces, the properties which empower the Clifford odd "basis vectors" to represent fermions and the Clifford even "basis vectors" to represent the corresponding gauge fields.
i. In odd dimensional spaces the Clifford odd "basis vectors" are not separated from their Hermitian conjugated partners, they instead are either self adjoint or have their Hermitian conjugated in another family. We can not define creation and annihilation operators as a tensor products of "basis vectors" and basis in momentum space so that they would manifest the creation and annihilation operators fulfilling the postulates of the second quantized fermions.
In odd dimensional spaces the two groups of the Clifford even "basis vectors" are not orthogonal, only the self adjoint "basis vectors" are orthogonal, the rest of "basis vectors" have their Hermitian conjugated partners in another group.
ii. The Clifford odd operator of handedness allows left and right handed superposition of Clifford odd and Clifford even "basis vectors".

19.3 Discussion

This note discusses the properties of the internal spaces of fermion and boson fields in even and odd dimensional spaces, if the internal spaces are described by the Clifford odd and even "basis vectors", which are the superposition of odd or even products of operators γ^a's. "Basis vectors" are arranged into algebraic products of nilpotents and projectors, which are eigenvectors of the Cartan subalgebra of the Lorentz algebra S^{ab} in the internal space of fermions and bosons.
The Clifford odd "basis vectors", which are products of an odd number of nilpotents and the rest of projectors, offer in even dimensional spaces the description of the internal space of fermion fields.
Each irreducible representation of the Lorentz algebra is equipped with the family quantum number determined by the second kind of the Clifford operators $\tilde{\gamma}^a$'s. The Clifford odd "basis vectors" anticommute. Their Hermitian conjugated partners appear in a different group. In a tensor product with the basis in ordinary space the "basis vectors" and their Hermitian conjugated partners form the creation and annihilation operators which fulfil the anticommutation relations postulated for second quantized fermion fields.
In $d = 2(2n+1), n \geq 7$, these creation and annihilation operators, applying on the vacuum state, or on the Hilbert space, offer the description of all the properties of the observed

quarks and leptons and antiquarks and antileptons. The massless fermion fields are of one handedness only.

The Clifford even "basis vectors", which are products of an even number of nilpotents and the rest of projectors, offer in even dimensional spaces the description of the internal space of boson fields, the gauge fields of the corresponding fermion fields. The Clifford even "basis vectors" commute. They do not appear in families and have their Hermitian conjugated partners in the same group. In a tensor product with the basis in ordinary space the "basis vectors" form the creation and annihilation operators which fulfil the commutation relations postulated for second quantized boson fields fields. In $d = 2(2n+1), n \geq 7$, these creation and annihilation operators offer the description of all the properties of the observed gauge fields as well as of the scalar Higgs's field, explaining also the Yukawa couplings.
This way of describing internal space of boson fields with the Clifford even "basis vectors", although very promising, needs further studies to understand what new it can bring into second quantization of fermion and boson fields. In particular, it must be understood what does it bring if we replace in a simple starting action in $d = 2(2n+1), n \geq 7$

$$\begin{aligned}
\mathcal{A} &= \int d^d x \; E \; \frac{1}{2} \, (\bar{\psi} \, \gamma^a p_{0a} \psi) + h.c. + \\
& \int d^d x \; E \; (\alpha \, R + \tilde{\alpha} \, \tilde{R}) \, , \\
p_{0a} &= f^{\alpha}{}_a p_{0\alpha} + \frac{1}{2E} \, \{p_\alpha, E f^{\alpha}{}_a\}_- \, , \\
p_{0\alpha} &= p_\alpha - \frac{1}{2} S^{ab} \omega_{ab\alpha} - \frac{1}{2} \tilde{S}^{ab} \tilde{\omega}_{ab\alpha} \, , \\
R &= \frac{1}{2} \{ f^{\alpha[a} f^{\beta b]} \; (\omega_{ab\alpha,\beta} - \omega_{ca\alpha} \, \omega^{c}{}_{b\beta}) \} + h.c. \, , \\
\tilde{R} &= \frac{1}{2} \{ f^{\alpha[a} f^{\beta b]} \; (\tilde{\omega}_{ab\alpha,\beta} - \tilde{\omega}_{ca\alpha} \, \tilde{\omega}^{c}{}_{b\beta}) \} + h.c. \, . \qquad (19.1)
\end{aligned}$$

Here [2] $f^{\alpha[a} f^{\beta b]} = f^{\alpha a} f^{\beta b} - f^{\alpha b} f^{\beta a}$. f^{a}_{α}, and the two kinds of the spin connection fields, $\omega_{ab\alpha}$ (the gauge fields of S^{ab}) and $\tilde{\omega}_{ab\alpha}$ (the gauge fields of $\tilde{S}^{ab}$), manifest in $d = (3+1)$ as the known vector gauge fields and the scalar gauge fields taking care of masses of quarks and leptons and antiquarks and antileptons and the weak boson fields [3], if we replace the covariant derivative $p_{0\alpha}$

$$p_{0\alpha} = p_\alpha - \frac{1}{2} S^{ab} \omega_{ab\alpha} - \frac{1}{2} \tilde{S}^{ab} \tilde{\omega}_{ab\alpha}$$

in Eq. (19.1) with

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[2] $f^{\alpha}{}_a$ are inverted vielbeins to $e^{a}{}_{\alpha}$ with the properties $e^{a}{}_{\alpha} f^{\alpha}{}_b = \delta^{a}{}_b, \; e^{a}{}_{\alpha} f^{\beta}{}_a = \delta^{\beta}_{\alpha}$, $E = \det(e^{a}{}_{\alpha})$. Latin indices $a, b, .., m, n, .., s, t, ..$ denote a tangent space (a flat index), while Greek indices $\alpha, \beta, .., \mu, \nu, .. \sigma, \tau, ..$ denote an Einstein index (a curved index). Letters from the beginning of both the alphabets indicate a general index ($a, b, c, ..$ and $\alpha, \beta, \gamma, ..$), from the middle of both the alphabets the observed dimensions $0, 1, 2, 3$ ($m, n, ..$ and $\mu, \nu, ..$), indexes from the bottom of the alphabets indicate the compactified dimensions ($s, t, ..$ and $\sigma, \tau, ..$). We assume the signature $\eta^{ab} = diag\{1, -1, -1, \cdots, -1\}$.

[3] Since the multiplication with either γ^a's or $\tilde{\gamma}^a$'s changes the Clifford odd "basis vectors" into the Clifford even objects, and even "basis vectors" commute, the action for fermions can not include an odd numbers of γ^a's or $\tilde{\gamma}^a$'s, what the simple starting action of Eq. (19.1) does not. In the starting action γ^a's and $\tilde{\gamma}^a$'s appear as $\gamma^0 \gamma^a \hat{p}_a$ or as $\gamma^0 \gamma^c \, S^{ab} \omega_{abc}$ and as $\gamma^0 \gamma^c \, \tilde{S}^{ab} \tilde{\omega}_{abc}$.

$$p_{0\alpha} = p_\alpha - \sum_{mf} {}^{I}\hat{\mathcal{A}}^{m\dagger}_{f} {}^{I}\mathcal{C}^{m}_{f\alpha} - \sum_{mf} {}^{I}\hat{\widetilde{\mathcal{A}}}^{m\dagger}_{f} {}^{I}\widetilde{\mathcal{C}}^{m}_{f\alpha}\,,$$

where the relation among ${}^{I}\hat{\widetilde{\mathcal{A}}}^{m\dagger}_{f} {}^{I}\widetilde{\mathcal{C}}^{m}_{f\alpha}$ and ${}^{II}\hat{\widetilde{\mathcal{A}}}^{m\dagger}_{f} {}^{II}\widetilde{\mathcal{C}}^{m}_{f\alpha}$ with respect to $\omega_{ab\alpha}$ and $\tilde{\omega}_{ab\alpha}$, not discussed directly in this article, needs additional study and explanation.

While in any even dimensional space the superposition of odd products of γ^a's, forming the Clifford odd "basis vectors", offer the description of the internal space of fermions with the half integer spins (manifesting in $d = (3+1)$ properties of quarks and leptons and antiquarks and antileptons, with the families included if $d = (13+1)$), the superposition of even products of γ^a's, forming the Clifford even "basis vectors", offer the description of the internal space of boson fields with integer spins, manifesting as gauge fields of the corresponding Clifford odd "basis vectors".

The Clifford odd and even "basis vectors" exist also in odd dimensional spaces. In this case their properties differ a lot from the "basis vectors" in even dimensional spaces. The eigenvectors of the operator of handedness are the superposition of the odd and even "basis vectors", offering both handedness, left and right. These basis vectors resembles the ghosts, needed in Feynman diagrams to get read of singularities. This study just starts and needs further comments and understanding.

19.4 Some useful formulas

This appendix contains some equations, needed in this note. More detailed explanations can be found in this proceedings in my talk [4].

The operator of handedness Γ^d is for fermions determined as follows

$$\Gamma = \prod_a (\sqrt{\eta^{aa}}\gamma^a) \cdot \begin{cases} (i)^{\frac{d}{2}}\,, & \text{for d even}\,,\\ (i)^{\frac{d-1}{2}}\,, & \text{for d odd}\,, \end{cases} \tag{19.2}$$

The Clifford objects γ^a's and $\tilde{\gamma}^a$'s fulfil the relations

$$\begin{aligned} \{\gamma^a, \gamma^b\}_+ &= 2\eta^{ab} = \{\tilde{\gamma}^a, \tilde{\gamma}^b\}_+\,,\\ \{\gamma^a, \tilde{\gamma}^b\}_+ &= 0\,, \quad (a,b) = (0,1,2,3,5,\cdots,d)\,,\\ (\gamma^a)^\dagger &= \eta^{aa}\gamma^a\,, \quad (\tilde{\gamma}^a)^\dagger = \eta^{aa}\tilde{\gamma}^a\,. \end{aligned} \tag{19.3}$$

The choice of the Cartan subalgebra members is made

$$\begin{aligned} &\mathcal{S}^{03}, \mathcal{S}^{12}, \mathcal{S}^{56}, \cdots, \mathcal{S}^{d-1\;d}\,,\\ &S^{03}, S^{12}, S^{56}, \cdots, S^{d-1\;d}\,,\\ &\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56}, \cdots, \tilde{S}^{d-1\;d}\,,\\ &\mathcal{S}^{ab} = S^{ab} + \tilde{S}^{ab} = i\,(\theta^a \frac{\partial}{\partial \theta_b} - \theta^b \frac{\partial}{\partial \theta_a})\,. \end{aligned} \tag{19.4}$$

Nilpotents and projectors are defined as follows [2,13,14]

$$\overset{ab}{(k)}:= \frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b)\,, \quad \overset{ab}{[k]}:= \frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b)\,, \tag{19.5}$$

with $k^2 = \eta^{aa}\eta^{bb}$.
One finds, taking Eq. (19.3) into account and the assumption

$$\{\tilde{\gamma}^a B = (-)^B \, i \, B \gamma^a\} |\psi_{oc}>\,, \tag{19.6}$$

with $(-)^B = -1$, if B is (a function of) an odd products of γ^a's, otherwise $(-)^B = 1$ [14], $|\psi_{oc}>$ is defined in Eq. (19.8), the eigenvalues of the Cartan subalgebra operators

$$\begin{aligned} S^{ab} \stackrel{ab}{(k)} &= \frac{k}{2} \stackrel{ab}{(k)}\,, & \tilde{S}^{ab} \stackrel{ab}{(k)} &= \frac{k}{2} \stackrel{ab}{(k)}\,, \\ S^{ab} \stackrel{ab}{[k]} &= \frac{k}{2} \stackrel{ab}{[k]}\,, & \tilde{S}^{ab} \stackrel{ab}{[k]} &= -\frac{k}{2} \stackrel{ab}{[k]}\,. \end{aligned} \tag{19.7}$$

The vacuum state for the Clifford odd "basis vectors", $|\psi_{oc}>$, is defined as

$$|\psi_{oc}> = \sum_{f=1}^{2^{\frac{d}{2}-1}} \hat{b}^{m}_{f}{}_{*_A} \hat{b}^{m\dagger}_{f} \,|\,1\,>\,. \tag{19.8}$$

Taking into account Eq. (19.3) it follows

$$\begin{aligned}
&\gamma^a \stackrel{ab}{(k)} = \eta^{aa} \stackrel{ab}{[-k]}, \quad \gamma^b \stackrel{ab}{(k)} = -ik \stackrel{ab}{[-k]}, \quad \gamma^a \stackrel{ab}{[k]} = \stackrel{ab}{(-k)}, \quad \gamma^b \stackrel{ab}{[k]} = -ik\eta^{aa} \stackrel{ab}{(-k)}\,, \\
&\tilde{\gamma}^a \stackrel{ab}{(k)} = -i\eta^{aa} \stackrel{ab}{[k]}, \quad \tilde{\gamma}^b \stackrel{ab}{(k)} = -k \stackrel{ab}{[k]}, \quad \tilde{\gamma}^a \stackrel{ab}{[k]} = \; i \stackrel{ab}{(k)}, \quad \tilde{\gamma}^b \stackrel{ab}{[k]} = -k\eta^{aa} \stackrel{ab}{(k)}\,, \\
&\stackrel{ab}{(k)}^{\dagger} = \eta^{aa} \stackrel{ab}{(-k)}\,, \quad (\stackrel{ab}{(k)})^2 = 0\,, \quad \stackrel{ab}{(k)}\stackrel{ab}{(-k)} = \eta^{aa} \stackrel{ab}{[k]}\,, \\
&\stackrel{ab}{[k]}^{\dagger} = \stackrel{ab}{[k]}\,, \quad (\stackrel{ab}{[k]})^2 = \stackrel{ab}{[k]}\,, \quad \stackrel{ab}{[k]}\stackrel{ab}{[-k]} = 0\,, \\
&\stackrel{ab}{(k)}\stackrel{ab}{[k]} = 0\,, \quad \stackrel{ab}{[k]}\stackrel{ab}{(k)} = \stackrel{ab}{(k)}\,, \quad \stackrel{ab}{(k)}\stackrel{ab}{[-k]} = \stackrel{ab}{(k)}\,, \quad \stackrel{ab}{[k]}\stackrel{ab}{(-k)} = 0\,, \\
&\stackrel{ab}{(\tilde{k})}^{\dagger} = \eta^{aa} \stackrel{ab}{(\widetilde{-k})}\,, \quad (\stackrel{ab}{(\tilde{k})})^2 = 0\,, \quad \stackrel{ab}{(\tilde{k})}\stackrel{ab}{(\widetilde{-k})} = \eta^{aa} \stackrel{ab}{[\tilde{k}]}\,, \\
&\stackrel{ab}{[\tilde{k}]}^{\dagger} = \stackrel{ab}{[\tilde{k}]}\,, \quad (\stackrel{ab}{[\tilde{k}]})^2 = \stackrel{ab}{[\tilde{k}]}\,, \quad \stackrel{ab}{[\tilde{k}]}\stackrel{ab}{[\widetilde{-k}]} = 0\,, \\
&\stackrel{ab}{(\tilde{k})}\stackrel{ab}{[\tilde{k}]} = 0\,, \quad \stackrel{ab}{[\tilde{k}]}\stackrel{ab}{(\tilde{k})} = \stackrel{ab}{(\tilde{k})}\,, \quad \stackrel{ab}{(\tilde{k})}\stackrel{ab}{[\widetilde{-k}]} = \stackrel{ab}{(\tilde{k})}\,, \quad \stackrel{ab}{[\tilde{k}]}\stackrel{ab}{(\widetilde{-k})} = 0\,.
\end{aligned} \tag{19.9}$$

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