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The domination and independent domination numbers of some families of snarks*

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Abstract

A dominating set of a graph G is a set $S \subseteq V(G)$ such that every vertex in $V(G)$ either belongs to S or is adjacent to some vertex in S. The domination number is the minimum cardinality of a dominating set of G. An independent dominating set of G is a dominating set that is also independent. The minimum cardinality of an independent dominating set of G is the independent domination number of G. Given the computational complexity of these problems, extensive research has been done on finding bounds or determining these parameters for classes of graphs, especially cubic graphs. Furthermore, determining how far apart these parameters are is also a challenging problem. In this work, we establish some bounds for the domination number and the independent domination number for families of cubic graphs, in particular for Generalized Blanuša Snarks and for two families of Loupekine Snarks known as LP_0-snarks and LP_1-snarks. We also show that the parameters are equal for these graphs and conjecture that this equality holds for every snark.

Keywords: Domination, independent domination, Generalized Blanuša Snarks, Loupekine Snarks.

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1 Introduction

Let G be a finite, undirected and simple graph with vertex set $V(G)$ and edge set $E(G)$. Throughout this text consider $n = |V(G)|$. Denote by $\delta(G)$ the *minimum degree* of G, and by $\Delta(G)$ the *maximum degree* of G. If every vertex has degree three, then G is said to be *cubic*. A non-cubic graph with maximum degree three is called *subcubic*. For $u, v \in V(G)$, the *distance* $dist(u, v)$ between u and v is the number of edges in the shortest path connecting them. The *(open) neighbourhood* of $v \in V(G)$ is $N(v) = \{u \in V(G) : uv \in E(G)$ and $u \neq v\}$; $u \in N(v)$ is a *neighbour* of v. Set $N[v] = N(v) \cup \{v\}$ is the *closed neighbourhood* of v.

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A set $S \subseteq V(G)$ is a *dominating set* of G if, for every $v \in V(G)$, either $v \in S$ or v is adjacent to some vertex in S. The *domination number* $\gamma(G)$ of G is the minimum cardinality of a dominating set of G. If S is a dominating set of G, then S *dominates* G. Moreover, for $v \in V(G)$, we say that v *dominates* all vertices in $N[v]$. For $S' \subseteq S$, let $D(S') = \cup_{v \in S'} N[v]$. Then, we say that S' *dominates* a set H if $H \subseteq D(S')$. A vertex $v \in V(G)$ is *doubly dominated* by S if $|N[v] \cap S| = 2$, i.e., either $v \in S$ and has exactly one neighbour in S or $v \notin S$ and has exactly two neighbours in S. If v is doubly dominated, we say that G has a *double domination*. Similarly, if $|N[v] \cap S| = 3$, then v is said to be *triply dominated* and G has a *triple domination*.

The concept of domination arose in the mid-19th century from a challenge posed by chess enthusiasts; however, it was not until 1958 that the concept of domination in graphs was formally defined. Since then, the area has become a fruitful field of research, with a wide variety of results in the literature [5, 16, 26]. In addition to theoretical progress, the concept of domination has several real-world applications, such as in computer networks [29] and in the study of RNA molecule structures [9].

It is natural that, from the original problem, ramifications and generalisations emerge as the research in the field advances. In fact, there exist currently a large number of variants of the original problem [6, 10, 23]. A set $S \subseteq V(G)$ is *independent* if its elements are pairwise non-adjacent. An *independent dominating set* of G is both dominating and independent. The *independent domination number* $i(G)$ of G is the minimum cardinality of an independent dominating set of G. Also, for independent dominating sets, a wide range of results can be found in the literature, along with practical applications of the problem, such as in wireless networks [24, 25].

Garey and Johnson [12] have shown that, for an arbitrary graph G, determining either $\gamma(G)$ or $i(G)$ is an NP-hard problem. As it is well known, cubic graphs play an important role in Graph Theory as many problems can be reduced to their cubic versions [7, 11] – their structural properties include, generally, the main aspects needed to be analysed. Therefore, it is not surprising that these problems remain NP-hard even when restricted to them [21, 24]. Considering this scenario, much of the research in this area has focused on two main aspects: establishing bounds for and determining the parameters for particular classes of graphs, with cubic or subcubic receiving special attention. In fact, many results that establish bounds for $\gamma(G)$ and $i(G)$ for graphs with special properties have been published [1, 4, 14, 34], including surveys [13, 18] and books [15, 17].

In 1996, Reed [28] proved that connected graphs with $\delta(G) \geq 3$ have $\gamma(G) \leq \frac{3n}{8}$ – this bound is tight – and, in the same article, conjectured that this limit could be improved to $\lceil \frac{n}{3} \rceil$ for connected cubic graphs. Although this conjecture was proved to be false [22] in 2005, it has motivated several works, and finding cubic graphs that verify Reed's Conjecture or improve his suggested bound remains a challenging problem.

In this work, we consider some cubic graphs – families of snarks – establishing their domination and independent domination numbers, and also establishing how far $\gamma(G)$ is from $i(G)$, which is another challenging problem addressed by researchers. Note that, by definition, $\gamma(G) \leq i(G)$. However, for arbitrary graphs, deciding whether $\gamma(G) = i(G)$ is an NP-complete problem [3]. Even for connected cubic graphs, the difference $i(G) - \gamma(G)$ may be unbounded [32]. Although domination problems have been studied extensively for cubic graphs, when it comes to snarks, the (classical) domination problem seems unexplored. To our knowledge, the only papers that approach dominating sets in snarks are those that focus on some other dominating set variants [8, 33]. In this paper, we address the domination number and the independent domination number of Generalized Blanuša Snarks, LP_0-snarks and LP_1-snarks, verifying the equality $\gamma(G) = i(G)$ for these graphs. We also prove this equality for Twisted Goldberg and Goldberg Snarks [27]; as a natural question, we pose the following conjecture.

Conjecture 1.1. *If G is a snark, then $\gamma(G) = i(G)$.*

2 Main results

A *bridge* in a graph G is an edge whose removal increases the number of connected components of G. A *bridgeless* graph is a graph without bridges. *Snarks* are connected bridgeless cubic graphs that are not 3-edge-colourable, i.e., whose edges cannot be assigned three colours such that adjacent edges have distinct colours. Snarks were discovered in the context of the Four-Colour Theorem and play an important role in Graph Theory [30].

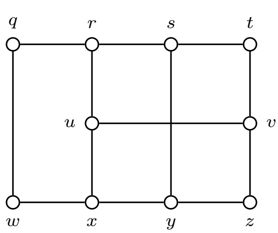
The first known snark, the Petersen Graph (Figure 5(a)), was discovered in 1898 and remained the only known snark until 1946. Since then, new snarks and methods for creating infinite families of snarks, such as the Generalised Blanuša Snark and the Loupekine Snark, have been discovered [19, 31].

Before heading to the main results, we present in the following a known lower bound on the domination number which is used in our proofs.

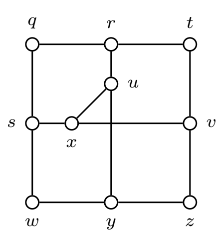
Theorem 2.1 (Acharya et al. [2])**.** *For any graph G, $\gamma(G) \geq \lceil \frac{n}{1+\Delta(G)} \rceil$.*

2.1 Generalized Blanuša Snarks

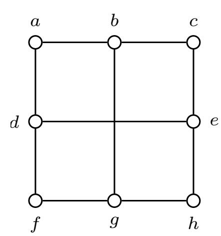
Let $\mathcal{B}^1 = \{B^1_1, B^1_2, B^1_3, \ldots\}$ and $\mathcal{B}^2 = \{B^2_1, B^2_2, B^2_3, \ldots\}$ be the first and the second families of *Generalized Blanuša Snarks*, respectively. The graphs from these families are built using three *blocks*, B^1_0, B^2_0, B, isomorphic to the graphs exhibited in Figures 1(a), 1(b), and 1(c), respectively. Each graph $B^k_i \in \mathcal{B}^k$, $k \in \{1, 2\}$, is the union of B^k_0 and i copies of B, with specific edges added between consecutive blocks, cyclically ordered, as defined in the next paragraph. Then, $|V(B^k_i)| = 8i+10$. The j-th copy of B is denoted B_j and its vertices get attached index j. For simplicity, we denote $V(B^k_i)$ and $E(B^k_i)$ by V^k_i and E^k_i. Analogous notation is used for the vertex and edge sets of blocks B^1_0, B^2_0 and B_j. A vertex of a block is called *internal* if it has no neighbours in other blocks. For B^1_0 we denote the set of its internal vertices as $I(B^1_0)$, i.e., $I(B^1_0) = \{r, s, u, v, x, y\}$. Similarly, $I(B^2_0) = \{r, s, u, v, x, y\}$ and $I(B_j) = \{b_j, d_j, e_j, g_j\}$.



(a) Block B^1_0.



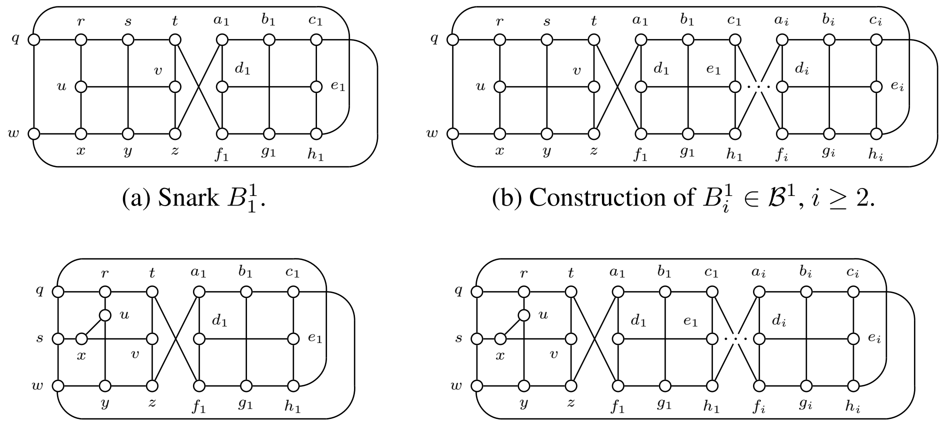
(b) Block B^2_0.



(c) Block B.

Figure 1: Blocks B^1_0, B^2_0 and B used in the construction of $\mathcal{B}^1$ and $\mathcal{B}^2$.

The first and smallest graph in $\mathcal{B}^1$, namely B_1^1, illustrated in Figure 2(a), is the *first Blanuša Snark*. The analogous for $\mathcal{B}^2$ is B_1^2 in Figure 2(c). The remaining graphs from these families can be recursively constructed as follows. For an integer $i \geq 2$, B_i^k is built from B_{i-1}^k and B_i so that: $V_i^k = V_{i-1}^k \cup V_i$; and $E_i^k = (E_{i-1}^k \setminus E_{i-1}^{out}) \cup E_i^{in} \cup E_i$, with $E_{i-1}^{out} = \{qh_{i-1}, wc_{i-1}\}$ and $E_i^{in} = \{c_{i-1}f_i, h_{i-1}a_i, qh_i, wc_i\}$. See Figures 2(b) and 2(d) for examples.



(a) Snark B_1^1.

(b) Construction of $B_i^1 \in \mathcal{B}^1$, $i \geq 2$.

(c) Snark B_1^2.

(d) Construction of $B_i^2 \in \mathcal{B}^2$, $i \geq 2$.

Figure 2: In (a) and (c), the first members of families $\mathcal{B}^1$ and $\mathcal{B}^2$, respectively. In (b) and (d), sketches of the recursive construction of B_i^1 and B_i^2 for $i \geq 2$.

In this section, we first determine the domination number of the Generalized Blanuša Snarks (Theorems 2.3 and 2.4). Afterwards, we show that the independent domination numbers of these graphs are equal to their domination numbers (Theorem 2.5). We start by presenting Lemma 2.2, which establishes an important lower bound for $\gamma(B_i^1)$ when i is odd.

Lemma 2.2. *Let $B_i^1 \in \mathcal{B}^1$, $i \geq 3$ and i odd. Then, $\gamma(B_i^1) \geq 2i + 4$.*

Proof. Consider $B_i^1 \in \mathcal{B}^1$, with $i \geq 3$ and i odd. By construction, B_i^1 comprises blocks B_0^1 and B_j, for $1 \leq j \leq i$. Suppose, by contradiction, that $\gamma(B_i^1) \leq 2i + 3$. Since B_i^1 is cubic, we conclude, by Theorem 2.1, that $\gamma(B_i^1) \geq 2i + 3$. Therefore, we can assume $\gamma(B_i^1) = 2i + 3$.

Let S be a minimum dominating set of B_i^1, with $S = \cup_{l=0}^{i} S_l$, $S_0 = S \cap V_0^1$, and $S_j = S \cap V_j$, for $1 \leq j \leq i$. Each vertex of S dominates four vertices and $|S| = 2i + 3$. Therefore, $|D(S)| \leq 8i + 12$. However, $|V_i^1| = 8i + 10$. Hence, either there exist exactly two double dominations or one triple domination in B_i^1.

In order to prove that $|S| = 2i + 3$ leads to a contradiction, we first prove that $|S_0| \geq 3$. Assume, by contradiction, $|S_0| \leq 2$. By definition, $I(B_0^1)$ can be dominated only by S_0. In particular, u can be dominated only by $I(B_0^1)$. Thus, $|S_0 \cap I(B_0^1)| \geq 1$. Suppose $|S_0 \cap I(B_0^1)| = 1$. Each vertex of $I(B_0^1) \setminus \{u, v\}$ dominates three vertices of $I(B_0^1)$ and v dominates only two vertices of $I(B_0^1)$. Also, note that each vertex of $V_0^1 \setminus I(B_0^1)$ dominates at most two vertices of $I(B_0^1)$. If $S_0 \cap I(B_0^1) \neq \{u\}$, then there exists at least one vertex of $I(B_0^1)$ not dominated. On the other hand, if $u \in S_0$, then it dominates $I(B_0^1) \setminus \{s, y\}$.

Thus, $S_0 \setminus \{u\}$ dominates s and y. This is a contradiction since $|S_0 \setminus \{u\}| \leq 1$ and $N(s) \cap N(y) = \emptyset$. Therefore, $|S_0 \cap I(B_0^1)| = 2$, i.e., $|S_0| = |S_0 \cap I(B_0^1)| = 2$.

The only pairs of vertices of $I(B_0^1)$ that may dominate $I(B_0^1)$ are either $\{s, u\}$ or $\{y, u\}$. Suppose $S_0 = \{s, u\}$. Note that vertex r is doubly dominated and vertices q and w must be dominated by $S \setminus \{s, u\}$. Considering that $|S_0| = 2$, in order to dominate q and w, $\{c_i, h_i\} \subseteq S_i$, which implies that e_i is doubly dominated. We conclude that r and e_i are the only doubly dominated vertices in the graph. Therefore, to dominate d_i we must have $N[d_i] \cap S_i \neq \emptyset$. If d_i or e_i belong to S_i, then e_i is triply dominated, a contradiction since r is doubly dominated. If a_i or f_i belong to S_i, then there is a third doubly dominated vertex, also a contradiction. Therefore, $S_0 \neq \{s, u\}$ and, by symmetry, $S_0 \neq \{y, u\}$. We conclude that $|S_0| \geq 3$.

In fact, $|S_0| = 3$ and $|S_j| = 2$, for every $1 \leq j \leq i$. Observe that $I(B_j)$ cannot be dominated by a single vertex of S_j. Therefore, $|S_j| \geq 2$. Since $|S_0| \geq 3$, $|S_j| \geq 2$ and $|S| = \sum_{l=0}^{i} |S_l| = 2i + 3$, it follows that $|S_0| = 3$ and $|S_j| = 2$, for every $1 \leq j \leq i$.

Now, we show that either $\{q, x\} \subseteq S_0$ or $\{r, w\} \subseteq S_0$. Since $|S_0| = 3$, it is impossible to have both $\{q, x\} \subseteq S_0$ and $\{r, w\} \subseteq S_0$. Also, $|S_0 \cap \{q, r, w, x\}| \neq 3$ since such S_0 would not dominate vertex v. Therefore, $|S_0 \cap \{q, r, w, x\}| \leq 2$.

Suppose $\{q, x\} \not\subseteq S_0$. If $|S_0 \cap \{q, r, w, x\}| = 2$, then $S_0 \cap \{r, w\} \neq \emptyset$. If $\{r, w\} \subseteq S_0$, the result follows. Assume $\{r, w\} \not\subseteq S_0$. Thus, $|S_0 \cap \{q, x\}| = 1$ and $|S_0 \cap \{r, w\}| = 1$. If $\{q, r\} \subseteq S_0$, then at least one vertex of $\{v, x, y\}$ in $I(B_0^1)$ would not be dominated by S_0 since $\{v, x, y\} \not\subseteq D(\{q, r\})$ and $N[v] \cap N[x] \cap N[y] = \emptyset$. Thus, $\{q, r\} \not\subseteq S_0$ and, by symmetry, $\{w, x\} \not\subseteq S_0$. Similarly, if $\{q, w\} \subseteq S_0$, then some vertex of $I(B_0^1) \setminus \{r, x\}$ would not be dominated by S_0. Therefore, $\{q, w\} \not\subseteq S_0$. If $\{r, x\} \subseteq S_0$, then u is doubly dominated. Recall that there exist exactly two double dominations or one triple domination in the graph. Thus, to dominate v, $|S_0 \cap \{t, v, z\}| = 1$, so as not to exceed the maximum number of double and triple dominations. Note that if $|S_0 \cap \{t, v, z\}| = 1$, then we reach the maximum number of double or triple dominations allowed. This implies $c_i \notin S$ and $h_i \notin S$. On the other hand, since $q \notin S_0$, $S_i \cap \{b_i, e_i\} \neq \emptyset$ so as to have c_i dominated. Considering that $|S_i| = 2$, there is no other vertex in B_i that could belong to S_i without creating another double domination. Thus, $\{r, x\} \not\subseteq S_0$. By symmetry, we conclude that if $\{r, w\} \not\subseteq S_0$ then $\{q, x\} \subseteq S_0$.

Now, consider $|S_0 \cap \{q, r, w, x\}| = 1$. Initially, suppose $r \in S_0$. Then, $u \in S_0$ or $y \in S_0$ to dominate vertex x. If $\{r, u\} \subseteq S_0$, then both r and u are doubly dominated. Therefore, it is not possible to dominate y without creating another double or a triple domination. Thus, $u \notin S_0$ and $y \in S_0$. We conclude that either $S_0 = \{r, t, y\}$ or $S_0 = \{r, v, y\}$, so that $I(B_0^1) \subseteq D(S_0)$ and the number of double and triple dominations do not exceed the maximum allowed. In both cases, a_1 must be dominated by some vertex of $S_1 \setminus \{a_1\}$. Since $|S_1| = 2$, we conclude that $S_1 = \{b_1, d_1\}$ or $S_1 = \{b_1, e_1\}$ or $S_1 = \{d_1, g_1\}$. However, all these cases exceed the maximum number of double or triple dominations. Thus, $r \notin S_0$ and, by symmetry, $x \notin S_0$. Next, suppose $q \in S_0$. Using a similar argument, we have $S_0 = \{q, u, s\}$ or $S_0 = \{q, u, y\}$ or $S_0 = \{q, v, y\}$. If $S_0 = \{q, u, s\}$ or $S_0 = \{q, u, y\}$, then the maximum number of double or triple dominations is exceeded. If $S_0 = \{q, v, y\}$, the number of double or triple dominations is reached. Since $|S_1| = 2$, we cannot dominate $I(B_1) \cup \{a_1, f_1\}$ unless we have another double domination in B_1. It follows that $q \notin S_0$. Moreover, by symmetry, $w \notin S_0$. Therefore, $|S_0 \cap \{q, r, w, x\}| \neq 1$.

Finally, consider $|S_0 \cap \{q, r, w, x\}| = 0$. Since $|S_i| = 2$, we conclude that $S_i = \{c_i, h_i\}$ in order to dominate q and w. However, $d_i \notin D(S_i)$ and $d_i \in I(B_i)$, a contradiction. Thus, $|S_0 \cap \{q, r, w, x\}| \neq 0$.

We are now ready to prove that $|S| = 2i + 3$ results in a contradiction. As shown before, $|S_0| = 3$ and $|S_j| = 2$, for every $1 \leq j \leq i$. Furthermore, either $\{q, x\} \subseteq S_0$ or $\{r, w\} \subseteq S_0$. By symmetry, we can assume $\{q, x\} \subseteq S_0$. The only vertices of $I(B_0^1)$ not in $D(\{q, x\})$ are s and v. Since $|S_0| = 3$, the only vertex of $S_0 \setminus \{q, x\}$ that dominates both s and v is t. Therefore, $S_0 = \{q, x, t\}$ and $a_1 \in S_1$ so as to have z dominated. Since $|S_1| = 2$, we conclude that $h_1 \in S_1$ in order to dominate e_1 and g_1. Since c_1 is not dominated by a_1 or h_1, we conclude that $f_2 \in S_2$. As $|S_2| = 2$, $c_2 \in S_2$ so as to dominate b_2 and e_2. By induction, we conclude that

$$S_j = \begin{cases} \{c_j, f_j\} & \text{if } j \text{ is even,} \\ \{a_j, h_j\} & \text{otherwise.} \end{cases}$$

Since $i \geq 3$ and odd, $S_i = \{a_i, h_i\}$. Nonetheless, $c_i \notin D(S_i)$. As c_i is adjacent to w and $w \notin S_0$, it follows that S is not a dominating set of B_i^1, which contradicts the definition of S. □

Theorem 2.3. *Let $B_i^1 \in \mathcal{B}^1$. Then, $\gamma(B_i^1) = 2i + 3$ when $i = 1$ or i is even, and $\gamma(B_i^1) = 2i + 4$ otherwise.*

Proof. Let $G \cong B_i^1$. The lower bound follows from Theorem 2.1 and from Lemma 2.2. For $i = 1$, we note that $\{q, v, y, b_1, d_1\}$ is a dominating set for G (see Figure 2(a)) and the result follows. In order to conclude the proof for $i \geq 2$, we construct a dominating set S for G with $|S| = 2i + 3$ if i is even, and $|S| = 2i + 4$ otherwise.

Let $S = \cup_{l=0}^{i} S_l$ with S_0 and S_j, for $1 \leq j \leq i$, defined as:

$$S_0 = \begin{cases} \{q, t, x\} & \text{if } i \text{ even,} \\ \{t, u, w, y\} & \text{otherwise, and} \end{cases} \qquad S_j = \begin{cases} \{c_j, f_j\} & \text{if } j \text{ odd,} \\ \{a_j, h_j\} & \text{otherwise.} \end{cases}$$

Note that $S_0 \subseteq V_0^1$ and $S_j \subseteq V_j$ (see Figure 3). The proof is by induction on i and based on the recursive construction of the family.

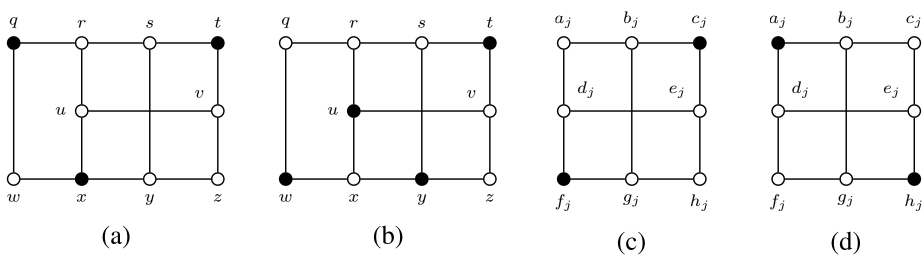


Figure 3: In (a), vertices of S_0, when i is even; in (b), vertices of S_0, when i is odd; in (c), vertices of S_j, when j is even; and in (d), vertices of S_j, when j is odd.

First, note that $D(S_0 \cup S_1) = (V_0^1 \cup V_1) \setminus \{c_1\}$ and, for $1 \leq j < i$, $D(S_j \cup S_{j+1})$ is: $(V_j \cup V_{j+1}) \setminus \{f_j, h_{j+1}\}$ when j is odd; and $(V_j \cup V_{j+1}) \setminus \{a_j, c_{j+1}\}$ otherwise. Thus, $D(S_0 \cup S_1 \cup S_2) = V(G)$ since $N(h_2) \cap S_0 \neq \emptyset$. For $i = 3$, by a similar reasoning, we conclude that $D(S_0 \cup S_1 \cup S_2 \cup S_3) = V(G)$ since c_3 is dominated by S_0.

Consider $i \geq 4$. Then, $i - 2 \geq 2$. By the induction hypothesis, $S' = \cup_{l=0}^{i-2} S_l$ is a dominating set for B_{i-2}^1. Let G' be a graph such that $V(G') = V_{i-2}^1$ and $E(G') = E_{i-2}^1 \setminus E_{i-2}^{out}$, where $E_{i-2}^{out} = \{qh_{i-2}, wc_{i-2}\}$. In order to show that $S = S' \cup S_{i-1} \cup S_i$ is a dominating set for G, we analyse the parity of i.

Consider i even. In this case, $D(S') = V(G') \setminus \{h_{i-2}\}$. Moreover, we have $D(S_{i-1} \cup S_i) = (V_{i-1} \cup V_i) \setminus \{f_{i-1}, h_i\}$ and $\{h_{i-2}, f_{i-1}\} \subseteq D(S_{i-2} \cup S_{i-1})$. Therefore, it remains to consider vertex h_i, for which $N(h_i) \cap S_0 = \{q\}$; we then conclude that S is a dominating set for G.

Similarly, for i odd, $D(S') = V(G') \setminus \{c_{i-2}\}$, $D(S_{i-1} \cup S_i) = (V_{i-1} \cup V_i) \setminus \{a_{i-1}, c_i\}$ and $\{c_{i-2}, a_{i-1}\} \subseteq D(S_{i-2} \cup S_{i-1})$; and for vertex c_i we have $N(c_i) \cap S_0 = \{w\}$. Thus, S is a dominating set for G. □

Theorem 2.4. *Let $B_i^2 \in \mathcal{B}^2$. Then, $\gamma(B_i^2) = 2i + 3$.*

Proof. As in Theorem 2.3, we prove that $\gamma(B_i^2) \leq 2i + 3$ by constructing a dominating set S for B_i^2 with $|S| = 2i + 3$. Then, we conclude $\gamma(B_i^2) = 2i + 3$ by Theorem 2.1. Let $S = \cup_{l=0}^{i} S_l$ with $S_0 \subseteq V_0^2$ and $S_j \subseteq V_j$, $1 \leq j \leq i$, for which

$$S_0 = \begin{cases} \{q, t, u\} & \text{if } i \text{ is even,} \\ \{t, u, w\} & \text{otherwise, and} \end{cases} \qquad S_j = \begin{cases} \{c_j, f_j\} & \text{if } j \text{ is even,} \\ \{a_j, h_j\} & \text{otherwise.} \end{cases}$$

Figures 4(a) and 4(b) illustrate the construction of S_0. For $i = 1$, one can see that $S = \{t, u, w, a_1, h_1\}$ is the dominating set of B_1^2 (see Figure 2(c)). For $i \geq 2$, S_j is the same set used in the proof of Theorem 2.3 (see Figures 3(c) and 3(d)). Thus, in both theorems, $D(\cup_{l=1}^{i} S_l)$ is the same.

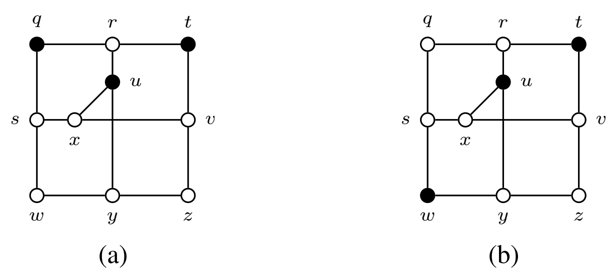


Figure 4: In (a), vertices of S_0 for i even; and, in (b), vertices of S_0 for i odd.

It remains to consider set S_0. For i even, $D(S_0 \cup S_1) = (V_0^2 \cup V_1) \setminus \{w, c_1\}$ and, for i odd, $D(S_0 \cup S_1) = (V_0^2 \cup V_1) \setminus \{q, c_1\}$. Moreover, $N(w) \cap S_i \neq \emptyset$ when i is even, and $N(q) \cap S_i \neq \emptyset$ when i is odd. We conclude that $\{w, q, c_1\} \subseteq D(S)$. Finally, since $|S_0| = 3$ and $|S_j| = 2$ for $1 \leq j \leq i$, we conclude that $|S| = 2i + 3$ and the result follows. □

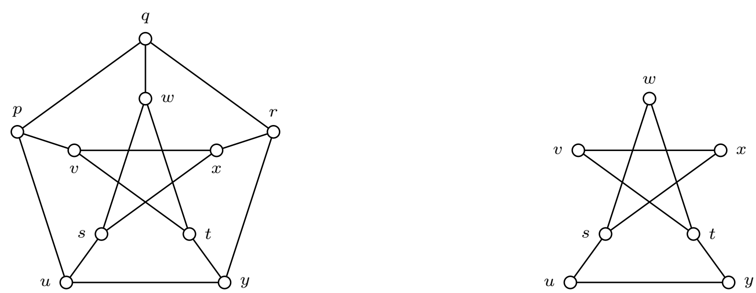
Theorem 2.5. *Let $B_i^k \in \mathcal{B}^k$, $k \in \{1, 2\}$. Then, $\gamma(B_i^k) = i(B_i^k)$.*

Proof. Since $i(B_i^k) \geq \gamma(B_i^k)$, it suffices to prove that the dominating sets presented in Theorems 2.3 and 2.4 are also independent. For $i = 1$, the result follows by inspection of sets $\{q, v, y, b_1, d_1\}$ and $\{t, u, w, a_1, h_1\}$. For $i \geq 2$, let $S = \cup_{l=0}^{i} S_l$, $S_0 \subseteq V_0^k$ and $S_j \subseteq V_j$, $1 \leq j \leq i$, be the sets defined in Theorem 2.3, for $k = 1$, and in Theorem 2.4, for $k = 2$.

By construction, S_0 is an independent set of B_0^1 and each S_j, $1 \leq j \leq i$, is an independent set of B_j. It remains to show that there is no vertex of S_j adjacent to vertices of S_{j+1} (indexes taken modulo i). Suppose $j = 0$. Notice that $t \in S_j$ and $z \notin S_j$. Moreover, $a_1 \in S_1$ and $f \notin S_1$. Suppose $1 \leq j \leq i$. If j is even, then $c_j \in S_j$, $h_j \notin S_j$ and: (i) $a_{j+1} \in S_{j+1}$ and $f_{j+1} \notin S_{j+1}$ if $j < i$; (ii) $q \in S_{j+1}$ and $w \notin S_{j+1}$ if $j = i$. If j is odd, then $h_j \in S_j$, $c_j \notin S_j$ and: (i) $f_{j+1} \in S_{j+1}$ and $a_{j+1} \notin S_{j+1}$ if $j < i$; (ii) $w \in S_{j+1}$ and $q \notin S_{j+1}$ if $j = i$. Therefore, S is an independent set of B_i^k. □

2.2 Loupekine Snarks

Loupekine's method for building snarks uses subgraphs of known snarks. In this section, we present two infinite families of Loupekine Snarks built from the Petersen Graph G. A *block* B of G is obtained by removing a path with three vertices from G. Block B is depicted in Figure 5, in which path pqr is removed from G. Note that the removal of any path with three vertices from the Petersen Graph results in a graph isomorphic to B.

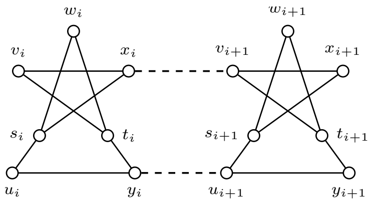


(a) Petersen Graph. (b) Block B.

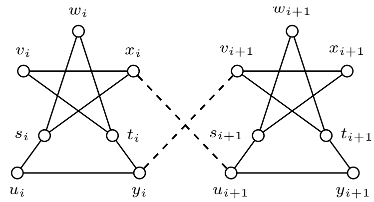
Figure 5: Construction of a block from the Petersen Graph.

Now, for $k \geq 3$, we construct a subcubic graph G_B using k copies of B. Denote each copy of B by B_i, for $0 \leq i \leq k-1$, and attach index i to its vertices. For each i, B_i and B_{i+1} are connected by *parallel link edges* $\{x_i v_{i+1}, y_i u_{i+1}\}$ or *crossing link edges* $\{x_i u_{i+1}, y_i v_{i+1}\}$ (indexes modulo k), as illustrated in Figure 6. The resulting graph G_B, called *block graph*, has exactly k vertices with degree two, namely $w_0, w_1, \ldots, w_{k-1}$. An example of G_B with $k = 3$ is shown in Figure 6(c).

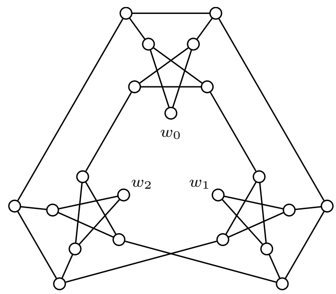
Let G_C be a graph whose connected components are isomorphic to K_2 or $K_{1,3}$, such that G_C has exactly k vertices of degree one, namely $z_0, z_1, \ldots, z_{k-1}$. Vertices of degree three of G_C are called *link vertices*. Let V_L be the set of link vertices of G_C with $\sigma = |V_L|$. Note that $0 \leq \sigma \leq \left\lfloor \frac{k}{3} \right\rfloor$. Graph G_C is called *central graph*. Figure 7(b) presents an example of central graph.



(a) Parallel link edges.



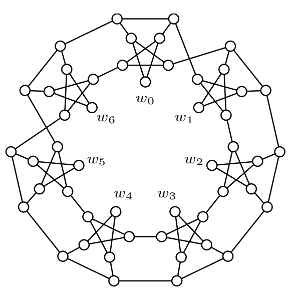
(b) Crossing link edges.



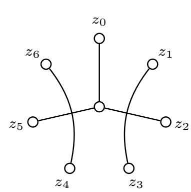
(c) A block graph G_B.

Figure 6: Link edges in (a) and (b). In (c), a block graph G_B built with blocks B_0, B_1 and B_2.

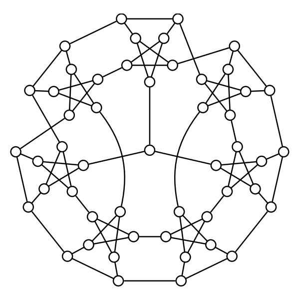
Next, we construct a cubic graph G_L from G_B and G_C as follows. Consider a block graph G_B built with $k \geq 3$ blocks B and a central graph G_C with k vertices of degree one. For each i, $0 \leq i < k$, identify $w_i \in V(G_B)$ with $z_i \in V(G_C)$. Figure 7 exemplifies this construction. Isaacs [20] determined in which cases G_L is a snark and one consequence of this characterization is that, whenever G_L is a snark, we have $\sigma > 0$. If G_L is a snark, then it is called an LP_1-snark. Additionally, if each connected component of G_C is connected to consecutive indexed blocks, then G_L is also called an LP_0-snark.



(a) Block graph G_B.



(b) Central graph G_C.



(c) Graph G_L.

Figure 7: Graph G_L constructed from a block graph G_B and a central graph G_C.

In Theorem 2.6, we establish a simple bound for the domination number and for the independent domination number of LP_1-snarks.

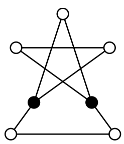
Theorem 2.6. *Let G be an LP_1-snark with $7k + \sigma$ vertices and $0 < \sigma \leq \lfloor \frac{k}{3} \rfloor$. Then,*

$$\gamma(G) \leq i(G) \leq 2k + \sigma.$$

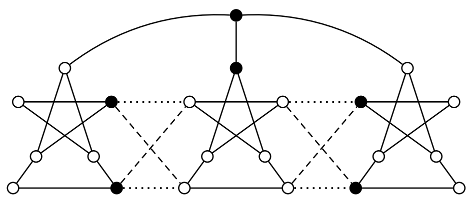
Proof. Let G be an LP_1-snark. By construction, G is built from blocks B_i, for $0 \leq i < k$. Let $S = \cup_{i=0}^{k-1} S_i \cup V_L$ with $S_i = \{s_i, t_i\}$. Set S_i is illustrated in Figure 8(a). Note that, for each S_i, $D(S_i) = V(B_i)$. Then, S is a dominating set of G_B. Moreover, every vertex of $N(s_i) \cup N(t_i)$ is in $V(B_i)$ and $s_i t_i \notin E(G)$. Therefore, S_i is an independent set of B_i.

It remains to analyse the link vertices. By definition, V_L is an independent set for the central graph of G. Since $V_L \subseteq S$, we conclude that S is an independent dominating set of G. Thus, $|S_i| = 2$, $|S| = 2k + \sigma$ and the result follows. □

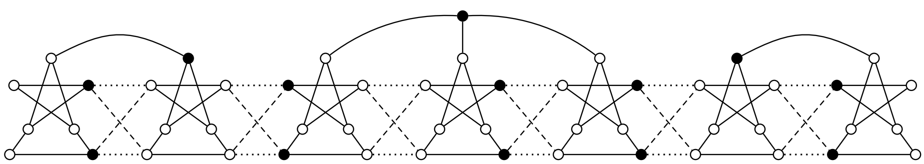
Better bounds for the domination number of LP_1-snarks can be obtained when considering specific properties, as in Theorem 2.7, in which a tighter bound for LP_0-snarks is presented. In order to prove this bound, we use three specific subgraphs, G_T^1, G_T^2, G_T^3, illustrated in Figure 8. In this figure, for each graph, we present a *canonical dominating set*. Let μ be the maximum number of G_T^3 disjoint subgraphs of an LP_0-snark G. The value of μ is used to establish a bound for the domination number of G as follows.



(a) G_T^1.



(b) G_T^2.



(c) G_T^3.

Figure 8: In (a), the G_T^1 subgraph; in (b) the G_T^2 subgraph; and in (c), the G_T^3 subgraph. The link edges between the blocks in G_T^2 and G_T^3 can be either parallel or crossing. For each subgraph, the canonical dominating set is shown in black vertices.

Theorem 2.7. *Let G be an LP_0-snark with $7k + \sigma$ vertices, $0 < \sigma \leq \lfloor \frac{k}{3} \rfloor$, and $\mu \geq 0$. Then, $\gamma(G) \leq 2k - \mu$.*

Proof. Let G be an LP_0-snark with k blocks B_i, $0 \leq i \leq k - 1$, and $\mu \geq 0$. We prove the upper bound by constructing a dominating set S of G with $|S| = 2k - \mu$. Let $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$ be a family of disjoint subgraphs of G such that:

(i) for each $H \in \mathcal{F}_i$, $H \cong G_T^i$;

(ii) $|\mathcal{F}_3| = \mu$;

(iii) $|\mathcal{F}_2| = \sigma - \mu$;

(iv) $|\mathcal{F}_1| = k - 7\mu - 3(\sigma - \mu)$.

Note that families $\mathcal{F}_3$, $\mathcal{F}_2$ and $\mathcal{F}_1$ are built by first maximizing $|\mathcal{F}_3|$, then, $|\mathcal{F}_2|$, leaving to $\mathcal{F}_1$ the remaining blocks. Since $\mathcal{F}$ comprises every block of G, using the canonical dominating sets of G_T^1, G_T^1 and G_T^1, we have a dominating set S for G. By construction, $|S| = 13|\mathcal{F}_3| + 6|\mathcal{F}_2| + 2|\mathcal{F}_1|$. Therefore, $|S| = 2k - \mu$ and the result follows. □

Theorem 2.8 gives a lower bound for the domination number of LP_0-snark.

Theorem 2.8. *Let G be an LP_0-snark. Then, $\gamma(G) \geq \lfloor \frac{n}{4} \rfloor + 1$.*

Proof. Let G be an LP_0-snark of order n. Since $k \geq 3$, $n \geq 22$. Suppose, by contradiction, that $\gamma(G) \leq \lfloor \frac{n}{4} \rfloor$. Let $n = 4t + r$, $r \in \{0, 1, 2, 3\}$. Note that $t \geq 6$ when $r \in \{0, 1\}$, and $t \geq 5$ otherwise. Let S be a minimum dominating set of G. Then, $|S| = \gamma(G) \leq \lfloor \frac{n}{4} \rfloor \leq t$. Since each vertex of S dominates four vertices of G, $V(G)$ is not dominated by S when $r \in \{1, 2, 3\}$. It remains to consider $r = 0$.

Suppose $n = 4t$. By Theorem 2.1, $\gamma(G) \geq t$. Thus, we can assume $|S| = t$, and, since $n = 4t$, we conclude that every vertex in $V(G)$ is dominated by exactly one vertex of S. This implies that, for each $a, b \in S$, $\text{dist}(a, b) > 2$. For every B_i, the vertices of $R = \{v_i, s_i, t_i, w_i, x_i\}$ induce a subgraph isomorphic to cycle C_5. Suppose $|S \cap R| = 0$. This implies $\{u_i, y_i\} \subseteq S$ in order to dominate $\{s_i, t_i\}$, which is impossible, as $\text{dist}(u_i, y_i) = 1$. Then, $|S \cap R| \geq 1$. Also, note that for every $a, b \in R$, $\text{dist}(a, b) \leq 2$. Hence, $|S \cap R| \leq 1$. We conclude that $|S \cap R| = 1$.

Now, we prove that $|S \cap R| = 1$ leads to a contradiction. Since $\sigma \geq 1$, we can assume that w_{i-1} and w_i are adjacent to a link vertex h. If $s_i \in S$, then t_i cannot be dominated since every vertex of $N[t_i]$ is at distance at most two from s_i. Thus $s_i \notin S$ and, symmetrically, $t_i \notin S$. Now, suppose $v_i \in S$. Every vertex in $\{s_i, t_i, w_i, x_i, y_i\}$ is at distance at most two from v_i. Then, $\{h, u_i\} \subseteq S$ in order to have $\{s_i, w_i\}$ dominated. As a result, t_{i-1} cannot be dominated without contradicting the required distance between vertices of S. Therefore, $v_i \notin S$ and, by symmetry, $x_i \notin S$. Finally, suppose $w_i \in S$. In order to dominate $\{u_i, v_i\}$ we have $\{x_{i-1}, y_{i-1}\} \subseteq S$ and, again, w_{i-1} cannot be dominated without a contradiction. Then, $w_i \notin S$, which concludes the proof. □

3 Concluding remarks

In this work, we determined $\gamma(B_i^k)$ and $i(B_i^k)$ for the Generalized Blanuša Snarks B_i^k, $k \in \{1, 2\}$, also showing that $\gamma(B_i^k) = i(B_i^k)$. Moreover, we established bounds for the domination number and independent domination number of the LP_0-snarks and LP_1-snarks. The bounds presented in Theorem 2.6 are not tight since we know examples of LP_1-snarks with smaller dominating sets. For instance, the LP_1-snark G shown in Figure 9 has $\gamma(G) = 13 < 2k + \sigma$. Note that, there are no vertices in block B_0 dominated by vertices in block B_1, and vice versa. This also holds between blocks B_4 and B_5. Therefore, we can construct an infinite family of LP_1-snarks with $\gamma(G) < 2k + \sigma$ by inserting an infinite number of blocks G_T^1, G_T^2 or G_T^3 between B_0 and B_1 (or B_4 and B_5). Defining a partition of the blocks of G as done for LP_0-snarks could be a promising approach to improve this bound.

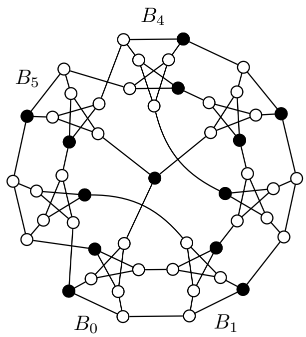
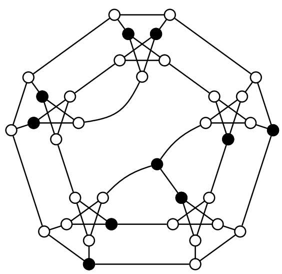
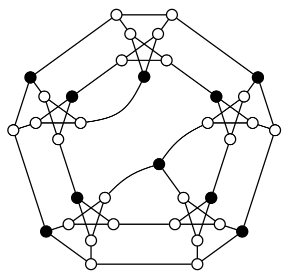


Figure 9: LP_1-snark G with $\gamma(G) = 13 < 2k + \sigma$.

Considering Theorem 2.7, however, we do not know any LP_0-snarks with smaller dominating sets. In Figure 10(a), we exhibit an LP_0-snark G for which the bounds given by Theorems 2.7 and 2.8 are tight. This graph has $n = 36$, $k = 5$, $\mu = 0$ and $\gamma(G) = 10$. Observe that the dominating sets built in Theorems 2.7 and 2.8 are not independent. However, for this graph, $\gamma(G) = i(G)$ as shown in Figure 10(b).



(a) $\gamma(G) = 10$.



(b) $i(G) = 10$.

Figure 10: In (a), a minimum dominating set of graph G; and in (b), a minimum independent dominating set of G.

Based on these observations, we pose the following conjecture.

Conjecture 3.1. *Let G be an LP_0-snark with $7k + \sigma$ vertices, $0 < \sigma \leq \lfloor \frac{k}{3} \rfloor$, and $\mu \geq 0$. Then, $\gamma(G) = 2k - \mu$.*

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Minimal directed strongly regular Cayley graphs over generalized dicyclic groups*

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Abstract

Let G be a group with identity element 1, and let S be a subset of $G \setminus \{1\}$. The subset S is called minimal if $\langle S \rangle = G$ and there exists an element $s \in S$ such that $\langle S \setminus \{s, s^{-1}\} \rangle \neq G$. In this paper, we completely determine all directed strongly regular Cayley graphs $\mathrm{Cay}(G, S)$ for any generalized dicyclic group G, provided that S is a minimal subset of G.

Keywords: Directed strongly regular graph, Cayley graph, generalized dicyclic group.

Math. Subj. Class. (2020): 05C20, 05C25

1 Introduction

An undirected graph Γ is a pair $(V(\Gamma), E(\Gamma))$, where $V(\Gamma)$ is the vertex set, and $E(\Gamma) \subseteq V(\Gamma) \times V(\Gamma)$ is the edge set of Γ such that $(u, v) \in E(\Gamma)$ if and only if $(v, u) \in E(\Gamma)$. In this paper, for the sake of convenience, we call 'undirected graph' as 'graph'. A directed graph Γ is a pair $(V(\Gamma), E(\Gamma))$, where $V(\Gamma)$ is the vertex set, and $E(\Gamma) \subseteq V(\Gamma) \times V(\Gamma)$ is the edge set of Γ such that $(u, v) \in E(\Gamma)$ does not always imply that $(v, u) \in E(\Gamma)$. Thus, an undirected graph can be regarded as a directed graph where each edge connects two vertices bidirectionally.

Let G be a group with identity 1, and let S be a subset of $G \setminus \{1\}$. The *directed Cayley graph* of G with respect to S, denoted by $\mathrm{Cay}(G, S)$, is the directed graph with vertex set G in which for any two vertices $g, h \in G$, there is an arc from g to h if and only if $hg^{-1} \in S$.

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Here S is called the *connection set* of $\mathrm{Cay}(G,S)$. If S is inverse-closed, the directed Cayley graph $\mathrm{Cay}(G,S)$ can be viewed as an undirected graph. Clearly, $\mathrm{Cay}(G,S)$ is a regular directed graph which is connected if and only if $\langle S\rangle = G$. If $\langle S\rangle = G$ and there exists some $s \in S$ such that $\langle S \setminus \{s, s^{-1}\}\rangle \neq G$, then we say that S is *minimal* (with respect to s). In such a case, the directed Cayley graph $\mathrm{Cay}(G,S)$ is called a *minimal directed Cayley graph over* G. In this paper, we only consider finite directed graphs.

Let Γ be a directed graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$. If there is an arc (or directed edge) from vertex u to vertex v in Γ, then we write $u \to v$. If $u \to v$ and $v \to u$, then we say that there is an undirected edge between u and v, and write $u \leftrightarrow v$. For any vertex $v \in V(\Gamma)$, the sets of *in-neighbors*, *out-neighbors* and *io-neighbors* of v are denoted by $N^-_\Gamma(v) = \{u \in V(\Gamma) \mid u \to v\}$, $N^+_\Gamma(v) = \{u \in V(\Gamma) \mid v \to u\}$ and $N^\#_\Gamma(v) = \{u \in V(\Gamma) \mid v \leftrightarrow u\}$, respectively. It is obvious that $N^\#_\Gamma(v) = N^+_\Gamma(v) \cap N^-_\Gamma(v)$. The *neighborhood* of v is $N_\Gamma(v) = N^-_\Gamma(v) \cup N^+_\Gamma(v)$. If $u \in N_\Gamma(v)$, then we write $u \sim v$. The numbers $d^-_\Gamma(v) = |N^-_\Gamma(v)|$, $d^+_\Gamma(v) = |N^+_\Gamma(v)|$ and $d^\#_\Gamma(v) = |N^\#_\Gamma(v)|$ are the *in-degree*, *out-degree* and *io-degree* of v, respectively. The *degree* of v is $d_\Gamma(v) = |N_\Gamma(v)|$. Clearly, $d_\Gamma(v) = d^-_\Gamma(v) + d^+_\Gamma(v) - d^\#_\Gamma(v)$. We usually delete by the subscript Γ in the notations like $N_\Gamma(v)$ if the graph Γ is clear from the context.

The concept of directed strongly regular graph was introduced by Duval [5]. A directed graph Γ is called a *directed strongly regular graph* with parameters (n,k,t,λ,μ) if the following conditions hold:

- $|V(\Gamma)| = n$;
- $d^-(v) = d^+(v) = k$ and $d^\#(v) = t$, for every $v \in V(\Gamma)$;
- $|\{z \in V(\Gamma) \mid x \to z \to y\}| = \begin{cases} \lambda, & \text{if } x \to y \\ \mu, & \text{if } x \not\to y \end{cases}$., for all $x, y \in V(\Gamma)$ with $x \neq y$.

In particular, a directed strongly regular graph with $k = t$ is just a strongly regular graph (see [2, Chapter 9] for the definition). As pointed out in [5], directed strongly regular graph has similar algebraic properties to strongly regular graph. For example, every directed strongly regular graph with $0 < t < k$ has exactly three distinct adjacency eigenvalues, and furthermore, all these eigenvalues are integers.

In [5], Duval built some basic properties for directed strongly regular graph, and constructed several families of such graphs by using special matrices. Since then, a lot of attention has been paid to directed strongly regular graph. Fiedler, Klin, and Muzychuk [8] proved the existence of directed strongly regular graph for the three feasible parameter sets listed in [5]. Klin, Munemasa, Muzychuk, and Zieschang [16] constructed new infinite series of directed strongly regular graph by using coherent algebras. Martínez and Araluze [17] applied partial sum families to the construction of directed strongly regular graph. Brouwer, Olmez, and Song [3] obtained some families of directed strongly regular graph with $b = \mu$ by using antiflags of $1\frac{1}{2}$-designs. Also, it was found that Cayley graph play a key role in the construction of directed strongly regular graph. Hobart and Justin Shaw [12] and Duval and Iourinski [6] obtained new infinite families of directed strongly regular graph via Cayley graphs of dihedral groups and certain semidirect product groups, respectively. By generalizing the method of Duval and Iourinski [6], He and Zhang [11] obtained a larger family of directed strongly regular graphs. For more results on directed strongly regular graphs, we refer the reader to [1, 7, 9, 10, 20].

Jørgensen [14] proved that there are no directed strongly regular Cayley graphs with $0 < t < k$ over any abelian group. This observation motivates the problem of characterizing directed strongly regular graphs over non-abelian groups. However, despite significant efforts, a characterization of directed strongly regular graphs over dihedral groups is still outstanding. Indeed, characterizing directed strongly regular Cayley graphs over a relatively simple non-abelian group like a dihedral group is a challenging task. Instead, we focus our attention on minimal directed Cayley graphs. Miklavič and Šparl [18] provided a characterization of minimal undirected distance regular Cayley graphs over abelian groups. Subsequently, in their recent work, they [19] characterized completely minimal undirected distance regular Cayley graphs over generalized dihedral groups. In this paper, the main idea of the proof, involving distinguishing the two cases depending on whether s is in A or not, analyzing the structure of the directed graph implied by the fact that S is minimal with respect to s, and considering different possibilities regarding the order of s in the case that s is in A, originates from the paper [19]. Furthermore, Huang and Das [13] characterized minimal undirected distance regular Cayley graphs over generalized dicyclic groups. Inspired by these works, we present a characterization of minimal directed strongly regular Cayley graphs over generalized dicyclic groups. Let A be an abelian group of order $2n(n > 1)$ with exactly one involution α, and let G be the *generalized dicyclic group* generated by A and b where $b^2 = \alpha$ and $b^{-1}xb = x^{-1}$ for all $x \in A$. Note that G is a non-abelian group of order $4n$, and α is the unique element of order 2 in G. The main result is as follows.

Theorem 1.1. *Let G be a generalized dicyclic group with identity element 1, and let S be a subset of $G \setminus \{1\}$ that generates G. Suppose there exists an element $s \in S$ such that $\langle S \setminus \{s, s^{-1}\}\rangle \neq G$. Then, the Cayley graph $\mathrm{Cay}(G, S)$ is directed strongly regular if and only if it is isomorphic to one of the following graphs:*

(i) *The complete bipartite graph $K_{4,4}$, which is a directed strongly regular graph with parameters $(8, 4, 4, 0, 4)$;*

(ii) *The directed graph* (a) *shown in* Figure 1, *which is the only directed strongly regular graph with parameters $(12, 4, 2, 0, 2)$;*

(iii) *The directed graph* (b) *shown in* Figure 1, *which is a directed strongly regular graph with parameters $(12, 5, 3, 2, 2)$.*

Remark 1.2. In fact, the graph (a) is the only directed strongly regular graph with parameters $(12, 4, 2, 0, 2)$ according to Jørgensen [15]. Also, according to Jørgensen [15], there are exactly 20 directed strongly regular graphs with parameters $(12, 5, 3, 2, 2)$, and the graph (b) is just one of them.

2 Proof of Theorem 1.1

Let G be a group. For any $g \in G$, we define $o(g)$ to be the order of g. For any subgroup $H \leq G$, we define $[G : H]$ to be the index of H in G, which is the number of distinct right cosets of H in G. Let Γ_1 and Γ_2 be two graphs with vertex sets $V(\Gamma_1)$ and $V(\Gamma_2)$, respectively. The Cartesian product of Γ_1 and Γ_2 is denoted by $\Gamma_1 \Box \Gamma_2$ and defined as the graph with vertex set $V(\Gamma_1) \times V(\Gamma_2)$, where each vertex is represented by an ordered pair (u, v) with $u \in V(\Gamma_1)$ and $v \in V(\Gamma_2)$. Two vertices (u_1, v_1) and (u_2, v_2) in $\Gamma_1 \Box \Gamma_2$ are

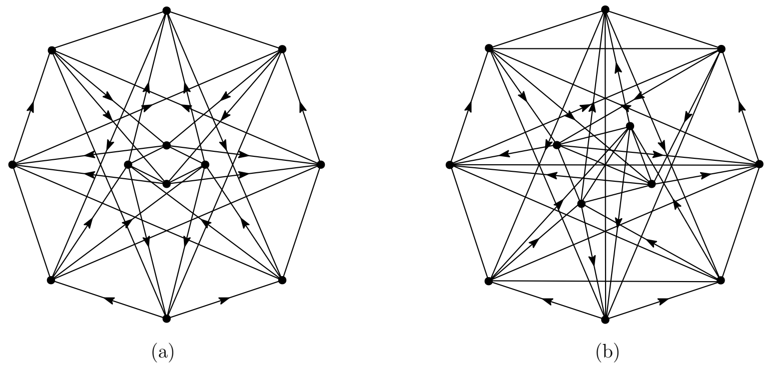

Figure 1: The graph in Theorem 1.1.

adjacent if and only if either $u_1 = u_2$ and v_1 is adjacent to v_2 in Γ_2, or $v_1 = v_2$ and u_1 is adjacent to u_2 in Γ_1.

The *Hamming graph* $H(d, q)$ is the Cartesian product of d copies of the complete graph K_q, where d and q are both positive integers. The *Doobs graph* $D(m, n)$ is the Cartesian product of $H(n, 4)$ and m copies of the Shrikhande graph (cf. [4, 22]), where n is nonnegative and m is positive. If $n = 0$, then it is just the Cartesian product of m copies of the Shrikhande graph. Furthermore, it is known that if a nontrivial connected graph is a Cartesian product, it can be factorized uniquely as a Cartesian product of prime factors, that is, graphs that cannot themselves be decomposed as Cartesian products of graphs [21, 24]. In [19], Miklavič and Šparl gave the following characterization for distance-regular Cartesian products based on a result of Stevanović [23].

Lemma 2.1 ([19])**.** *Let $\Gamma = \Gamma_1 \square \Gamma_2$, where Γ_1 and Γ_2 are nontrivial graphs. If Γ is distance-regular, then Γ is isomorphic to a Hamming graph $H(d, q)$ or to a Doobs graph $D(n, m)$.*

To facilitate clarity of notation, we have adopted the following notation in this section.

- Let $G = \langle A, b \mid b^2 = \alpha, b^{-1}ab = a^{-1}$ for all $a \in A\rangle$ be the *generalized dicyclic group* generated by abelian group A and b, where $|A| = 2n(n > 1)$ and α is the unique element of order 2 in A. Clearly, G is a non-abelian group of order $4n$, and α is the unique element of order 2 in G.
- Let S be a subset of $G \setminus \{1\}$ with $\langle S\rangle = G$.
- Let $s \in S$ be such that $H = \langle S \setminus \{s, s^{-1}\}\rangle \neq \emptyset$ is a proper subgroup of G.
- $\Gamma = \mathrm{Cay}(G, S)$ is a directed strongly regular graph with parameters (n, k, t, λ, μ).
- $\Gamma' = \mathrm{Cay}(H, S \setminus \{s, s^{-1}\})$.

It is clear that $|S| \geq 2$. Since $G = A \cup Ab$, we consider the cases $s \in A$ and $s \in Ab$ in the subsequent subsections, respectively.

2.1 The case $s \in Ab$

In this case, $G = \langle A, b\rangle = \langle A, s\rangle$, $s^2 = b^2 = \alpha$, $s^{-1}as = a^{-1}$ for all $a \in A$ and thus we may assume that $s = b$. Therefore, $H = \langle S \setminus \{b, b^{-1}\}\rangle$, and $H \cap Hb = \emptyset$.

Lemma 2.2. *For any $g \in G$ and $x \in Hg$, if $y \notin Hg$ satisfying $x \sim y$, then $y \in \{bx, b^{-1}x\}$. Furthermore, if $b^{-1} \in S$, then $b^{-1}x \leftrightarrow x \leftrightarrow bx$; if $b^{-1} \notin S$, then $b^{-1}x \to x \to bx$.*

Proof. We only prove the case of $b^{-1} \in S$. The case of $b^{-1} \notin S$ is very similar and we omit the details. If $x \to y$, then there exists $s' \in S$ such that $y = s'x \in s'Hg$. It leads to $s' \in \{b, b^{-1}\}$ since otherwise $s' \in H$ and thus $y \in Hg$, a contradiction. Therefore, $y = bx$ or $b^{-1}x$. If $y \to x$, then there exists $s'' \in S$ such that $s''y = x$ and thus $y = (s'')^{-1}x \in (s'')^{-1}Hg$. It leads to $s'' \in \{b, b^{-1}\}$ since otherwise $s'' \in H$ and thus $(s'')^{-1} \in H$ and $y \in Hg$, a contradiction. Hence, we also have $y = bx$ or $b^{-1}x$. Additionally, by the definition, if $b^{-1} \in S$, then $b^{-1}x \leftrightarrow x \leftrightarrow bx$; if $b^{-1} \notin S$, then $b^{-1}x \to x \to bx$. □

Lemma 2.3. *The parameters (n, k, t, λ, μ) satisfy $\lambda \in \{0, 2\}$. Furthermore, $\lambda = 2$ if and only if $b^{-1} \in S$ and $b^2 \in S$.*

Proof. Suppose that $b^{-1} \notin S$. Consider the vertices $1 \in H$ and $b \in Hb$. Clearly, $1 \to b$ because $b \in S$. Lemma 2.2 indicates that $N^+(1) \cap (V(\Gamma) \setminus H) = \{b\}$ and $N^-(b) \cap (V(\Gamma) \setminus Hb) = \{1\}$. Therefore, we have $\{z \in V(\Gamma) \mid 1 \to z \to b\} = \emptyset$. It leads to $\lambda = 0$ because $1 \to b$. Suppose that $b^{-1} \in S$. Again, consider the vertices $1 \in H$ and $b \in Hb$. By Lemma 2.2, we have $N(1) \cap (V(\Gamma) \setminus H) = \{b, b^{-1}\}$ and $N(b) \cap (V(\Gamma) \setminus Hb) = \{1, b^2\}$. This implies that $\{z \in V(\Gamma) \mid 1 \to z \to b\} \subseteq \{b^{-1}, b^2\}$. Observe that

$$b^{-1} \in \{z \in V(\Gamma) \mid 1 \to z \to b\} \Leftrightarrow b^2 \in S \Leftrightarrow b^2 \in \{z \in V(\Gamma) \mid 1 \to z \to b\}.$$

Therefore, if $b^2 \in S$, then $\{z \in V(\Gamma) \mid 1 \to z \to b\} = \{b^{-1}, b^2\}$ and thus $\lambda = 2$; if $b^2 \notin S$, then $\{z \in V(\Gamma) \mid 1 \to z \to b\} = \emptyset$ and thus $\lambda = 0$. □

Lemma 2.4. *It holds that $b^{-1} \in S$.*

Proof. Suppose to the contrary that $b^{-1} \notin S$. According to Lemma 2.3, we have $\lambda = 0$. Since $b \in \{z \in V(\Gamma) \mid 1 \to z \to b^2\}$ and $\lambda = 0$, we have $1 \nrightarrow b^2$, namely, $b^2 \notin S$. Thus $\mu \geq 1$. Consider the vertices $1 \in H$ and $b^{-1} \in Hb^{-1}$. Clearly, $1 \nrightarrow b^{-1}$ because $b^{-1} \notin S$. Lemma 2.2 indicates that $N^+(1) \cap (V(\Gamma) \setminus H) = \{b\}$ and $N^-(b^{-1}) \cap (V(\Gamma) \setminus Hb^{-1}) = \{b^2\}$. Therefore, $\{z \in V(\Gamma) \mid 1 \to z \to b^{-1}\} \subseteq \{b, b^2\}$. Since $b^2 \notin S$, we have $1 \nrightarrow b^2$ and $b \nrightarrow b^{-1}$. Therefore, $\{z \in V(\Gamma) \mid 1 \to z \to b^{-1}\} = \emptyset$, which contradicts $\mu \geq 1$. □

Lemma 2.5. *If $S \setminus \{b, b^{-1}\} \subseteq A$ and $b^2 \notin S$, then Γ is isomorphic to the complete bipartite graph $K_{4,4}$ or the directed graph* (a) *shown in* Figure 1.

Proof. Lemma 2.4 indicates $b^{-1} \in S$, and Lemma 2.3 indicates $\lambda = 0$ because $b^2 \notin S$. Since $1 \nrightarrow b^2$ because $b^2 \notin S$ and $\{b, b^{-1}\} \subseteq \{z \in V(\Gamma) \mid 1 \to z \to b^2\}$, we have $\mu \geq 2$. Now we take an arbitrary element $x \in S \setminus \{b, b^{-1}\} \subseteq A$. Since $b^2 \notin S$, we have $x \neq b^2$, and thus $xb \neq b^{-1}$. Since $x \in H$, $xb \in Hb \neq H$ and $xb \notin \{b, b^{-1}\}$, we have $xb \notin S$, and thus $1 \nrightarrow xb$. According to Lemma 2.2, we have $N(1) \cap (V(\Gamma) \setminus H) = \{b, b^{-1}\}$ and $N(xb) \cap (V(\Gamma) \setminus Hb) = \{x^{-1}, x^{-1}b^2\}$. Therefore, we have

$$\{z \in V(\Gamma) \mid 1 \to z \to xb\} \subseteq \{b, b^{-1}, x^{-1}, x^{-1}b^2\}. \tag{2.1}$$

Moreover, we have $b \in \{z \in V(\Gamma) \mid 1 \to z \to xb\}$ and

$$\begin{aligned} b^{-1} &\in \{z \in V(\Gamma) \mid 1 \to z \to xb\} \Leftrightarrow xb^2 \in S, \\ x^{-1} &\in \{z \in V(\Gamma) \mid 1 \to z \to xb\} \Leftrightarrow x^{-1} \in S, \\ x^{-1}b^2 &\in \{z \in V(\Gamma) \mid 1 \to z \to xb\} \Leftrightarrow x^{-1}b^2 \in S. \end{aligned} \tag{2.2}$$

Thus, $2 \le \mu \le 4$.

Claim 1. $o(x) \neq 2$.

Proof of Claim 1. If $o(x) = 2$, since b^2 is the unique element of order 2 in A and $x \in A$, we have $b^2 = x \in S$. It contradicts $b^2 \notin S$. Hence, $o(x) \neq 2$. □

Now we divide two cases to discuss.

Case 1. $\mu = 4$.

In this case, from (2.1) and (2.2), we have $\{z \in V(\Gamma) \mid 1 \to z \to xb\} = \{b, b^{-1}, x^{-1}, x^{-1}b^2\}$, and thus $\{xb^2, x^{-1}, x^{-1}b^2\} \subseteq S$. By the arbitrariness of x, we conclude that

$$\{b, b^{-1}, x, x^{-1}, xb^2, x^{-1}b^2 \mid x \in S \setminus \{b, b^{-1}\}\} \subseteq \{z \in V(\Gamma) \mid 1 \to z \to b^2\}.$$

Since $1 \nrightarrow b^2$, $\mu = 4$ and $o(x) \neq 2$, we have $S = \{b, b^{-1}, x, x^{-1}\}$, $x = x^{-1}b^2$ and $x^{-1} = xb^2$. Therefore, $o(x) = 4$ and $G = \langle S \rangle = \{1, b, b^2, b^{-1}, x, x^{-1}, xb, x^{-1}b\}$. Since $\langle S \setminus \{b, b^{-1}\}\rangle \neq G$, we have that S is minimal with respect to b. Hence, as a minimal directed Cayley graph, $\Gamma = \mathrm{Cay}(G, S) \cong K_{4,4}$, is indeed a directed strongly regular graph, and the corresponding parameters are $(8, 4, 4, 0, 4)$.

Case 2. $2 \le \mu \le 3$.

Claim 2. $o(x) \neq 4$ and $x^{-1}b^2 \notin S$.

Proof of Claim 2. Suppose to the contrary that $o(x) = 4$. Then $x^{-1}b^2 = x$ and $x^{-1} = xb^2$. Since $\mu \neq 4$ and $x^{-1}b^2 = x \in S$, from (2.1) and (2.2), we have $\{z \in V(\Gamma) \mid 1 \to z \to xb\} = \{b, x^{-1}b^2\}$, and thus $\mu = 2$. However, there exist three distinct elements $\{b, b^{-1}, x^{-1}b^2\} \subseteq \{z \in V(\Gamma) \mid 1 \to z \to b^2\}$, a contradiction. Next, we prove that $x^{-1}b^2 \notin S$. Otherwise, we have $\{b, b^{-1}, x, x^{-1}b^2\} \subseteq \{z \in V(\Gamma) \mid 1 \to z \to b^2\}$ since $o(x) \neq 4$, a contradiction. □

Claim 3. $\mu = 2$, and either $xb^2 \in S$ or $x^{-1} \in S$.

Proof of Claim 3. Suppose to the contrary that $\mu = 3$. Since $x^{-1}b^2 \notin S$ by Claim 2, (2.1) and (2.2) indicate $\{z \in V(\Gamma) \mid 1 \to z \to xb\} = \{b, b^{-1}, x^{-1}\}$, and thus $xb^2, x^{-1} \in S$. It is impossible since otherwise there exist four distinct elements $\{b, b^{-1}, x^{-1}, xb^2\} \subseteq \{z \in V(\Gamma) \mid 1 \to z \to b^2\}$, a contradiction. Since $x^{-1}b^2 \notin S$ and $\mu = 2$, we have either $\{z \in V(\Gamma) \mid 1 \to z \to xb\} = \{b, b^{-1}\}$ or $\{z \in V(\Gamma) \mid 1 \to z \to xb\} = \{b, x^{-1}\}$. Thus, either $xb^2 \in S$ or $x^{-1} \in S$. □

Claim 4. S is not inverse-closed.

Proof of Claim 4. Suppose to the contrary that S is inverse-closed. According to Claims 1 and 2, $o(x) \neq 2, 4$. Consider the vertices $1 \in H$ and $x^2 \in H$. Note that $\lambda = 0$ and $x \in \{z \in V(\Gamma) \mid 1 \to z \to x^2\}$. It implies that $1 \not\to x^2$. Since $x^2b, x^2b^{-1} \notin H$ and $o(x) \neq 2, 4$, we have $x^2b, x^2b^{-1} \notin S$. Therefore, $b, b^{-1} \notin \{z \in V(\Gamma) \mid 1 \to z \to x^2\}$. Since $\mu = 2$, there exists $y \in S \setminus \{b, b^{-1}\}$ such that $x \neq y$ and $y \in \{z \in V(\Gamma) \mid 1 \to z \to x^2\}$. Also, $y^{-1} \in S$ because S is inverse-closed. Note that $x, y \in A$ and A is an abelian group, which implies that $\{1, x^2, xy\} \subseteq \{z \in V(\Gamma) \mid x \to z \to y\}$. Since $\lambda = 0$, we have $x \not\to y$. Since $\mu = 2$, $o(x) \neq 2$ and $x \neq y$, we have $xy = 1$, and thereby $y = x^{-1}$. Since $y \to x^2$, we have $x^3 = x^2y^{-1} \in S$. It is impossible since otherwise there exist three distinct elements $\{1, x^2, x^{-2}\} \subseteq \{z \in V(\Gamma) \mid x^{-1} \to z \to x\}$, which contradicts either $\lambda = 0$ or $\mu = 2$. □

Claim 5. $S = \{b, b^{-1}, x, xb^2\}$.

Proof of Claim 5. According to Claims 3 and 4, there exists $x \in S \setminus \{b, b^{-1}\} \subseteq A$ such that $x^{-1} \notin S$ but $xb^2 \in S$, and thereby $\{b, b^{-1}, x, xb^2\} \subseteq S$.

Suppose to the contrary that $S \setminus \{b, b^{-1}, x, xb^2\} \neq \emptyset$. We firstly assert that $S \setminus \{b, b^{-1}, x, xb^2\}$ is inverse-closed. Otherwise, there exists $x' \in S \setminus \{b, b^{-1}, x, xb^2\} \subseteq A$ such that $(x')^{-1} \notin S$ and $x'b^2 \in S$. This implies that there exist four distinct elements $\{x, x', xb^2, x'b^2\} \subseteq \{z \in V(\Gamma) \mid 1 \to z \to xx'\}$, which contradicts either $\lambda = 0$ or $\mu = 2$. Take $y \in S \setminus \{b, b^{-1}, x, xb^2\}$, and thus $y^{-1} \in S$. Clearly, $y^2b, y^2b^{-1} \notin S$ because $y^2b, y^2b^{-1} \notin H$ and $o(y) \neq 2, 4$. Consider the vertices 1 and y^2. Note that $y \in \{z \in V(\Gamma) \mid 1 \to z \to y^2\}$ and $\lambda = 0$, which implies $1 \not\to y^2$ and thus $y^2 \notin S$. Since $\mu = 2$ and $y^2b, y^2b^{-1} \notin S$, there exists $a \in S \setminus \{b, b^{-1}, y\}$ such that $a \in \{z \in V(\Gamma) \mid 1 \to z \to y^2\}$. It leads to $y^2a^{-1} \in S$. We next assert that $a \notin \{x, xb^2\}$ or $y^2a^{-1} \notin \{x, xb^2\}$. Otherwise, $a, y^2a^{-1} \in \{x, xb^2\}$, and thus $a^2 = x^2$. Therefore, we have either $a = y^2a^{-1}$ or $a = (y^2a^{-1})b^2$. If $a = y^2a^{-1}$, then $a^2 = y^2$. This implies $x^2 = y^2$. It is impossible since otherwise there exist three distinct elements $\{y, x, xb^2\} \subseteq \{z \in V(\Gamma) \mid 1 \to z \to y^2\}$, which contradicts $\mu = 2$ because $1 \not\to y^2$. If $a = (y^2a^{-1})b^2$, then $a^2 = y^2b^2$. Thus $x^2 = y^2b^2$, namely, $y^2 = x^2b^2$. It is impossible since otherwise there exist three distinct elements $\{y, x, xb^2\} \subseteq \{z \in V(\Gamma) \mid 1 \to z \to y^2\}$, which contradicts $\mu = 2$.

If $a \notin \{x, xb^2\}$, then $a \in S \setminus \{b, b^{-1}, x, xb^2\}$, and thus $a^{-1} \in S$ because $S \setminus \{b, b^{-1}, x, xb^2\}$ is inverse-closed as asserted before. Notice that $y, a \in A$ and A is an abelian group, which implies that $\{1, ya, y^2\} \subseteq \{z \in V(\Gamma) \mid a \to z \to y\}$. Thus, $a \not\to y$ according to $\lambda = 0$. Since $\mu = 2$, $o(y) \neq 2$ and $a \neq y$, we have $ya = 1$, namely, $a = y^{-1}$. If $y^2a^{-1} \notin \{x, xb^2\}$, then $y^2a^{-1} \in S \setminus \{b, b^{-1}, x, xb^2\}$, and thus $ay^{-2} = (y^2a^{-1})^{-1} \in S$. Notice that $y, a \in A$ and A is an abelian group, which implies that $\{1, ya, y^2\} \subseteq \{z \in V(\Gamma) \mid y \to z \to a\}$. Thus, $y \not\to a$ according to $\lambda = 0$. Since $\mu = 2$, $o(y) \neq 2$ and $a \neq y$, we have $ya = 1$, namely, $a = y^{-1}$.

Since $a = y^{-1}$ and $a \to y^2$, we have $y^3 \in S$. Consider the vertices y^{-1} and y. Clearly, $y^{-1} \not\to y$ because $y^2 \notin S$. Since $y^{-1}, y^3 \in S$ and $o(y) \neq 2, 4$, there exist three distinct elements $\{1, y^2, y^{-2}\} \subseteq \{z \in V(\Gamma) \mid y^{-1} \to z \to y\}$, which contradicts $\mu = 2$. □

Combining Claims 1, 2 and 5, we get $o(x) \neq 2, 4$ and $S = \{b, b^{-1}, x, xb^2\}$, which implies that $x^2, x^2b, x^2b^2 \notin S$. It leads to $1 \not\to x^2b$. Consider $1 \in H$ and $x^2b \in Hb$. Lemma 2.2 indicates that $N(1) \cap (V(\Gamma) \setminus H) = \{b, b^{-1}\}$ and $N(x^2b) \cap (V(\Gamma) \setminus Hb) = \{x^{-2}, x^{-2}b^2\}$. Therefore, $\{z \in V(\Gamma) \mid 1 \to z \to x^2b\} \subseteq \{b, b^{-1}, x^{-2}, x^{-2}b^2\}$. By noticing $b \not\to x^2b$ since $x^2 \notin S$, and $b^{-1} \not\to x^2b$ since $x^2b^2 \notin S$, we conclude that

$\{z \in V(\Gamma) \mid 1 \to z \to x^2b\} = \{x^{-2}, x^{-2}b^2\}$ because $\mu = 2$. This implies that $x^{-2}, x^{-2}b^2 \in S$. Therefore, either $x^{-2} = x$ and $x^{-2}b^2 = xb^2$, or $x^{-2} = xb^2$ and $x^{-2}b^2 = x$. Thus, either $o(x) = 3$ or $o(x) = 6$. Both cases lead to

$$G = \langle S \rangle = \{1, b, b^2, b^{-1}, x, x^{-1}, bx, b^2x, bx^{-1}, b^2x^{-1}, b^{-1}x, b^{-1}x^{-1}\}.$$

Since $\langle S \setminus \{b, b^{-1}\}\rangle \neq G$, we have that S is minimal with respect to b. Hence, as a minimal directed Cayley graph, Γ is isomorphic to the directed graph (a) shown in Figure 1, which is a directed strongly regular graph with parameters $(12, 4, 2, 0, 2)$. □

Lemma 2.6. *If $S \setminus \{b, b^{-1}\} \subseteq A$ and $b^2 \in S$, then Γ is isomorphic to the directed graph* (b) *shown in* Figure 1.

Proof. Lemma 2.4 indicates $b^{-1} \in S$, and Lemma 2.3 indicates $\lambda = 2$ because $b^2 \in S$. Now we take an arbitrary element $x \in S \setminus \{b, b^{-1}, b^2\} \subseteq A$. Since $x \neq b^2$, we have $xb \neq b^{-1}$. Since $x \in H$, $xb \in Hb \neq H$ and $xb \notin \{b, b^{-1}\}$, we have $xb \notin S$, and thus $1 \nrightarrow xb$. Similar to Lemma 2.5, the equations (2.1) and (2.2) still hold. Thus, $\mu \leq 4$.

Claim 1. $S \setminus \{b, b^{-1}, b^2\} \neq \emptyset$ and $x^{-1}b^2 \notin S$.

Proof of Claim 1. Since $G = \langle S \rangle$ is non-abelian, we get $S \setminus \{b, b^{-1}, b^2\} \neq \emptyset$. If $x^{-1}b^2 \in S$, then we have $\{b, b^{-1}, x^{-1}b^2\} \subseteq \{z \in V(\Gamma) \mid 1 \to z \to b^2\}$, which contradicts $\lambda = 2$ because $1 \to b^2$. Hence, $x^{-1}b^2 \notin S$. □

Claim 2. $o(x) \neq 2, 4$.

Proof of Claim 2. Note that b^2 is the unique element of order 2 in A and $x \in A$. If $o(x) = 2$, then $x = b^2$, which contradicts $x \neq b^2$. If $o(x) = 4$, then $x^2 = b^2$, and thus $x^{-1}b^2 = x \in S$, which contradicts $x^{-1}b^2 \notin S$ given in Claim 1. □

Claim 3. Either $x^{-1} \in S$ or $xb^2 \in S$.

Proof of Claim 3. If $x^{-1}, xb^2 \in S$, then there exists three distinct elements $\{b, b^{-1}, x^{-1}\} \subseteq \{z \in V(\Gamma) \mid 1 \to z \to b^2\}$, which contradicts $\lambda = 2$. If $x^{-1}, xb^2 \notin S$, then $1 \nrightarrow xb^2$. Since $x^{-1}, xb^2 \notin S$ and $x^{-1}b^2 \notin S$ by Claim 1, (2.1) and (2.2) indicate $\{z \in V(\Gamma) \mid 1 \to z \to xb\} = \{b\}$. Thus, we have $\mu = 1$ because $1 \nrightarrow xb$. However, $1 \nrightarrow xb^2$ but $\{x, b^2\} \subseteq \{z \in V(\Gamma) \mid 1 \to z \to xb^2\}$, a contradiction. Therefore, either $x^{-1} \in S$ or $xb^2 \in S$. □

Claim 4. $\mu = 2$ and S is not inverse-closed.

Proof of Claim 4. Since either $s^{-1} \in S$ or $xb^2 \in S$, (2.1) and (2.2) indicate either $\{z \in V(\Gamma) \mid 1 \to z \to xb\} = \{b, b^{-1}\}$ or $\{z \in V(\Gamma) \mid 1 \to z \to xb\} = \{b, x^{-1}\}$. Thus, $\mu = 2$ because $1 \nrightarrow xb$.

Suppose to the contrary that S is inverse-closed. Therefore, $x^{-1} \in S$. Since $x^2b, x^2b^{-1} \notin H$ and $o(x) \neq 2, 4$, we have $x^2b, x^2b^{-1} \notin S$. Therefore, $b, b^{-1} \notin \{z \in V(\Gamma) \mid 1 \to z \to x^2\}$. Since $\lambda = \mu = 2$, there exists $y \in S \setminus \{b, b^{-1}\}$ satisfying $x \neq y$ and $y \in \{z \in V(\Gamma) \mid 1 \to z \to x^2\}$. Also, $y^{-1} \in S$ because S is inverse-closed. Note that $x, y \in A$ and A is an abelian group, which implies that $\{1, x^2, xy\} \subseteq \{z \in V(\Gamma) \mid x \to z \to y\}$. Therefore, $x \nrightarrow y$. According to $\mu = 2$, $o(x) \neq 2$ and $x \neq y$, we conclude that $xy = 1$, and thereby $y = x^{-1}$. Since $y \to x^2$, we have $x^3 = x^2y^{-1} \in S$. Therefore, there exist three distinct elements $\{1, x^2, x^{-2}\} \subseteq \{z \in V(\Gamma) \mid x^{-1} \to z \to x\}$, which contradicts $\lambda = \mu = 2$. □

Claim 5. $S = \{b, b^{-1}, b^2, x, xb^2\}$.

Proof of Claim 5. According to Claims 3 and 4, there exists $x \in S \setminus \{b, b^{-1}, b^2\}$ such that $x^{-1} \notin S$ but $xb^2 \in S$, and thereby

$$\{b, b^{-1}, b^2, x, xb^2\} \subseteq S.$$

Suppose to the contrary that $S \setminus \{b, b^{-1}, b^2, x, xb^2\} \neq \emptyset$. We firstly assert that $S \setminus \{b, b^{-1}, b^2, x, xb^2\}$ is inverse-closed. Otherwise, there exists $x' \in S \setminus \{b, b^{-1}, b^2, x, xb^2\} \subseteq A$ such that $(x')^{-1} \notin S$, and thus $x'b^2 \in S$ by Claim 3. Note that $x, x', b^2 \in A$ and A is an abelian group, which implies that there exist four distinct elements $\{x, x', xb^2, x'b^2\} \subseteq \{z \in V(\Gamma) \mid 1 \to z \to xx'\}$. It contradicts $\lambda = \mu = 2$.

Take $y \in S \setminus \{b, b^{-1}, b^2, x, xb^2\}$, and thus $y^{-1} \in S$. Clearly, $y^2b, y^2b^{-1} \notin S$ because $y^2b, y^2b^{-1} \notin H$ and $o(y) \neq 2, 4$. It leads to $b, b^{-1} \notin \{z \in V(\Gamma) \mid 1 \to z \to y^2\}$. Since $y \in \{z \in V(\Gamma) \mid 1 \to z \to y^2\}$ and $\lambda = \mu = 2$, there exists $a \in S \setminus \{b, b^{-1}\}$ such that $a \neq y$ and $a \in \{z \in V(\Gamma) \mid 1 \to z \to y^2\}$. We next assert that $a \notin \{x, xb^2\}$ or $y^2a^{-1} \notin \{x, xb^2\}$. Otherwise, $a, y^2a^{-1} \in \{x, xb^2\}$, and thus $a^2 = x^2$. Therefore, we have either $a = y^2a^{-1}$ or $a = y^2a^{-1}b^2$. If $a = y^2a^{-1}$, then $a^2 = y^2$. This implies $x^2 = y^2$. It leads to that there exist three distinct elements $\{y, x, xb^2\} \subseteq \{z \in V(\Gamma) \mid 1 \to z \to y^2\}$, which contradicts $\lambda = \mu = 2$. If $a = y^2a^{-1}b^2$, then $a^2 = y^2b^2$. Thus $x^2 = y^2b^2$, namely, $y^2 = x^2b^2$. It yields that there exists three distinct elements $\{y, x, xb^2\} \subseteq \{z \in V(\Gamma) \mid 1 \to z \to y^2\}$, which also contradicts $\lambda = \mu = 2$.

If $a \notin \{x, xb^2\}$, then $a \in S \setminus \{b, b^{-1}, x, xb^2\}$, and thus $a^{-1} \in S$ because $S \setminus \{b, b^{-1}, x, xb^2\}$ is inverse-closed. Notice that $y, a \in A$ and A is an abelian group, which implies that $\{1, ya, y^2\} \subseteq \{z \in V(\Gamma) \mid a \to z \to y\}$. Since $\lambda = \mu = 2$, $o(y) \neq 2$ and $a \neq y$, we have $ya = 1$, namely, $a = y^{-1}$. If $y^2a^{-1} \notin \{x, xb^2\}$, then $y^2a^{-1} \in S \setminus \{b, b^{-1}, x, xb^2\}$, and thus $ay^{-2} = (y^2a^{-1})^{-1} \in S$. Notice that $y, a \in A$ and A is an abelian group, which implies that $\{1, ya, y^2\} \subseteq \{z \in V(\Gamma) \mid y \to z \to a\}$. Since $\lambda = \mu = 2$, $o(y) \neq 2$ and $a \neq y$, we have $ya = 1$, namely, $a = y^{-1}$.

Since $a = y^{-1}$ and $a \to y^2$, we have $y^3 \in S$. Consider the vertices y^{-1} and y. Since $y^{-1}, y^3 \in S$ and $o(y) \neq 2, 4$, there exist three distinct elements $\{1, y^2, y^{-2}\} \subseteq \{z \in V(\Gamma) \mid y^{-1} \to z \to y\}$, which contradicts either $\lambda = 2$ or $\mu = 2$. ☐

Combining Claims 2 and 5, we get

$$o(x) \neq 2, 4 \textbf{ and } S = \{b, b^{-1}, b^2, x, xb^2\},$$

which implies that $x^2, x^2b, x^2b^2 \notin S$. Consider $1 \in H$ and $x^2b \in Hb$. Clearly, $1 \nrightarrow x^2b$ because $x^2b \notin S$. Lemma 2.2 indicates that $N(1) \cap (V(\Gamma) \setminus H) = \{b, b^{-1}\}$ and $N(x^2b) \cap (V(\Gamma) \setminus Hb) = \{x^{-2}, x^{-2}b^2\}$. Therefore, $\{z \in V(\Gamma) \mid 1 \to z \to x^2b\} \subseteq \{b, b^{-1}, x^{-2}, x^{-2}b^2\}$. By noticing $b \nrightarrow x^2b$ since $x^2 \notin S$, and $b^{-1} \nrightarrow x^2b$ since $x^2b^2 \notin S$, we deduce that $\{z \in V(\Gamma) \mid 1 \to z \to x^2b\} = \{x^{-2}, x^{-2}b^2\}$ because $\mu = 2$. This implies that $x^{-2}, x^{-2}b^2 \in S$. Therefore, either $x^{-2} = x$ and $x^{-2}b^2 = xb^2$, or $x^{-2} = xb^2$ and $x^{-2}b^2 = x$. Thus, either $o(x) = 3$ or $o(x) = 6$. Both cases lead to

$$G = \langle S \rangle = \{1, b, b^2, b^{-1}, x, x^{-1}, bx, b^2x, bx^{-1}, b^2x^{-1}, b^{-1}x, b^{-1}x^{-1}\}.$$

Since $\langle S \setminus \{b, b^{-1}\} \rangle \neq G$, we have that S is minimal with respect to b. Hence, as a minimal directed Cayley graph, Γ is isomorphic to the directed graph (b) shown in Figure 1, which is a directed strongly regular graph with parameters $(12, 5, 3, 2, 2)$. ☐

Lemma 2.7. *If* $\big(S \setminus \{b, b^{-1}\}\big) \cap Ab \neq \emptyset$, *then* Γ *is isomorphic to the complete bipartite graph* $K_{4,4}$.

Proof. Take an arbitrary element $xb \in \big(S \setminus \{b, b^{-1}\}\big) \cap Ab$ with $x \in A$. Since $xb \in H$, we have $b^2 = (xb)(xb) \in H$, and thus $x = (xb)b^3 \in Hb$. Clearly, $o(x) \neq 2$, since otherwise $x = b^2$, and thus $xb = b^{-1}$, a contradiction. According to Lemma 2.4, we have $b^{-1} \in S$. Note that $\lambda \in \{0, 2\}$ and $\{b, b^{-1}, xb\} \subseteq \{z \in V(\Gamma) \mid 1 \to z \to b^2\}$, which implies that $1 \nrightarrow b^2$ and $\mu \geq 3$. Thus, $b^2 \notin S$, and $\lambda = 0$ by Lemma 2.3. Since $\lambda = 0$ and $b^{-1} \in \{z \in V(\Gamma) \mid 1 \to z \to x\}$, we have $1 \nrightarrow x$. Consider the vertices $1 \in H$ and $x \in Hb$. Lemma 2.2 indicates that $N(1) \cap (V(\Gamma) \setminus H) = \{b, b^{-1}\}$ and $N(x) \cap (V(\Gamma) \setminus Hb) = \{x^{-1}b, x^{-1}b^{-1}\}$. Therefore, we have

$$\{z \in V(\Gamma) \mid 1 \to z \to x\} \subseteq \{b, b^{-1}, x^{-1}b, x^{-1}b^{-1}\}. \tag{2.3}$$

Moreover, we have $b^{-1} \in \{z \in V(\Gamma) \mid 1 \to z \to x\}$ and

$$\begin{aligned} b \in \{z \in V(\Gamma) \mid 1 \to z \to x\} &\Leftrightarrow xb^{-1} \in S, \\ x^{-1}b \in \{z \in V(\Gamma) \mid 1 \to z \to x\} &\Leftrightarrow x^{-1}b \in S, \\ x^{-1}b^{-1} \in \{z \in V(\Gamma) \mid 1 \to z \to x\} &\Leftrightarrow x^{-1}b^{-1} \in S. \end{aligned} \tag{2.4}$$

This implies that $\mu \leq 4$, and thus $\mu \in \{3, 4\}$ because $\mu \geq 3$.

In what follows, we divide two cases to discuss.

Case 1. $\mu = 3$.

We claim that $xb^{-1} \notin S$. Otherwise, $xb^{-1} \in S$. Then there exist four distinct elements $\{b, b^{-1}, xb, xb^{-1}\} \subseteq \{z \in V(\Gamma) \mid 1 \to z \to b^2\}$, which contradicts $\mu = 3$. By noticing that $xb^{-1} \notin S$ and $\mu = 3$, (2.3) and (2.4) indicate $\{z \in V(\Gamma) \mid 1 \to z \to x\} = \{b^{-1}, x^{-1}b, x^{-1}b^{-1}\}$, and thus $x^{-1}b, x^{-1}b^{-1} \in S$. It yields that there exist four distinct elements $\{b, b^{-1}, x^{-1}b, x^{-1}b^{-1}\} \subseteq \{z \in V(\Gamma) \mid 1 \to z \to b^2\}$, which contradicts $\mu = 3$. Thus, this case cannot happen.

Case 2. $\mu = 4$.

In this case, from (2.3) and (2.4), we have $\{z \in V(\Gamma) \mid 1 \to z \to x\} = \{b, b^{-1}, x^{-1}b, x^{-1}b^{-1}\}$, and thus $xb^{-1}, x^{-1}b, x^{-1}b^{-1} \in S$. By the arbitrariness of xb, we conclude that

$$\begin{aligned} &\{b, b^{-1}, xb, xb^{-1}, x^{-1}b, x^{-1}b^{-1} \mid xb \in \big(S \setminus \{b, b^{-1}\}\big) \cap Ab\} \\ &\subseteq \{z \in V(\Gamma) \mid 1 \to z \to b^2\}. \end{aligned}$$

Since $1 \nrightarrow b^2$, $\mu = 4$ and $o(x) \neq 2$, we have $\big(S \setminus \{b, b^{-1}\}\big) \cap Ab = \{xb, xb^{-1}\}$, $xb = x^{-1}b^{-1}$ and $xb^{-1} = x^{-1}b$. Therefore, $S \cap Ab = \{b, b^{-1}, xb, xb^{-1}\}$, $o(x) = 4$ and $\{z \in V(\Gamma) \mid 1 \to z \to b^2\} = \{b, b^{-1}, xb, xb^{-1}\}$.

Next we show that $S \cap A = \emptyset$. Suppose to the contrary that there exists $y \in S \cap A$. We have $y \in H$, $yb \in Hb$. Since $\lambda = 0$ and $b \in \{z \in V(\Gamma) \mid 1 \to z \to yb\}$, we have $1 \nrightarrow yb$. Consider the vertices $1 \in H$ and $yb \in Hb$. Lemma 2.2 indicates that $N(1) \cap (V(\Gamma) \setminus H) = \{b, b^{-1}\}$ and $N(yb) \cap (V(\Gamma) \setminus Hb) = \{y^{-1}, y^{-1}b^2\}$. Therefore, $\{z \in V(\Gamma) \mid 1 \to z \to yb\} = \{b, b^{-1}, y^{-1}, y^{-1}b^2\}$ according to $\mu = 4$. By noticing that $y^{-1}b^2 \in S$ since $1 \to y^{-1}b^2$, we have $y^{-1}b^2 \in \{z \in V(\Gamma) \mid 1 \to z \to b^2\}$, which contradicts $\{z \in V(\Gamma) \mid 1 \to z \to b^2\} = \{b, b^{-1}, xb, xb^{-1}\}$.

By the arguments above, $\big(S \setminus \{b, b^{-1}\}\big) \cap Ab = \{xb, xb^{-1}\}$ and $S \cap A = \emptyset$. Therefore, $S = \{b, b^{-1}, xb, xb^{-1}\}$. It leads to $G = \langle S \rangle = \{1, b, b^{-1}, b^2, x, x^{-1}, xb, x^{-1}b\}$ by $o(x) = 4$. Since $\langle S \setminus \{b, b^{-1}\}\rangle \neq G$, we have that S is minimal with respect to b. Hence, as a minimal directed Cayley graph, $\Gamma = \mathrm{Cay}(G, S) \cong K_{4,4}$, is indeed a directed strongly regular graph, and the corresponding parameters are $(8, 4, 4, 0, 4)$. □

2.2 The case $s \in A$

Recall that $\langle S \rangle = G$ and $H = \langle S \setminus \{s, s^{-1}\}\rangle \subset G$. It is straightforward that G can be expressed as the disjoint union of the right cosets Hs^i, where $0 \leq i \leq [G : H] - 1$. Clearly, $[G : H] \mid o(s)$. A result similar to Lemma 2.2 follows from the same method, so we omit the tedious proof.

Lemma 2.8. *For any $g \in G$ and $x \in Hg$, if $y \notin Hg$ and $x \sim y$, then $y \in \{sx, s^{-1}x\}$. Furthermore, if $s^{-1} \in S$, then $s^{-1}x \leftrightarrow x \leftrightarrow sx$; if $s^{-1} \notin S$, then $s^{-1}x \to x \to sx$.*

Lemma 2.9. *It holds that $S \cap Ab \neq \emptyset$ and $o(s) > 2$.*

Proof. Since $G = \langle S \rangle$, we have $S \cap Ab \neq \emptyset$. If $o(s) = 2$, by noticing that b^2 is the unique element of order 2 in A and $s \in A$, then $s = b^2$. For $xb \in S \cap Ab$, we have $s = b^2 = (xb)(xb)$. Therefore, $H = \langle S \setminus \{s, s^{-1}\}\rangle = \langle S \rangle = G$, a contradiction. Hence, $o(s) > 2$. □

Lemma 2.10. *It holds that $\mu \geq 1$. If $(S \setminus \langle s \rangle) \cap A \neq \emptyset$ or $s^{-1} \in S$, then $\mu \geq 2$.*

Proof. Since $S \cap Ab \neq \emptyset$ and $s \in A$, we have $(S \setminus \langle s \rangle) \cap Ab \neq \emptyset$. Let $h \in (S \setminus \langle s \rangle) \cap Ab$. Since $hs \in Hs \neq H$, we have $hs \notin S$. Therefore, $1 \nrightarrow hs$. Notice that $s \in \{z \in V(\Gamma) \mid 1 \to z \to hs\}$, which implies that $\mu \geq 1$.

If $s^{-1} \in S$, then $\{s, h\} \subseteq \{z \in V(\Gamma) \mid 1 \to z \to hs\}$, and therefore $\mu \geq 2$. If $(S \setminus \langle s \rangle) \cap A \neq \emptyset$, let $h' \in (S \setminus \langle s \rangle) \cap A$. Again, since $1 \nrightarrow h's$ and $\{s, h'\} \subseteq \{z \in V(\Gamma) \mid 1 \to z \to h's\}$, we have $\mu \geq 2$. □

Lemma 2.11. *It holds that $s^{-1} \in S$.*

Proof. Suppose to the contrary that $s^{-1} \notin S$. Consider the vertices $1 \in H$ and $s \in Hs$. Clearly, $1 \to s$. Since $s^{-1} \notin S$, Lemma 2.8 indicates that $N^+(1) \cap (V(\Gamma) \setminus H) = \{s\}$ and $N^-(s) \cap (V(\Gamma) \setminus Hs) = \{1\}$. It leads to $\{z \in V(\Gamma) \mid 1 \to z \to s\} = \emptyset$ and $\lambda = 0$. Since $s \in \{z \in V(\Gamma) \mid 1 \to z \to s^2\}$ and $\lambda = 0$, we have $1 \nrightarrow s^2$, and thus $s^2 \notin S$.

If $[G\colon H] \geq 4$, then $o(s) \geq [G\colon H] \geq 4$. Therefore, $s^3 \neq s$ and $s^3 \in Hs^3 \neq H$. It leads to $s^3 \notin S$ because $s^{-1} \notin S$. Consider the vertices $1 \in H$ and $s^3 \in Hs^3$. Clearly, $1 \nrightarrow s^3$. Lemma 2.8 indicates that $N^+(1) \cap (V(\Gamma) \setminus H) = \{s\}$ and $N^-(s^3) \cap (V(\Gamma) \setminus Hs^3) = \{s^2\}$. Therefore $\{z \in V(\Gamma) \mid 1 \to z \to s^3\} \subseteq \{s, s^2\}$. By noticing that $s \nrightarrow s^3$ and $1 \nrightarrow s^2$ since $s^2 \notin S$, we conclude that $\{z \in V(\Gamma) \mid 1 \to z \to s^3\} = \emptyset$. This implies $\mu = 0$ because $1 \nrightarrow s^3$, which contradicts $\mu \geq 1$ by Lemma 2.10.

If $[G\colon H] = 2$, then $o(s) \geq 4$ because $[G\colon H] \mid o(s)$ and $o(s) > 2$. Since $s^3 \in Hs \neq H$ and $s^3 \neq s$, we have $s^3 \notin S$, i.e., $1 \nrightarrow s^3$. Similarly, by considering the vertices $1 \in H$ and $s^3 \in Hs^3 = Hs$, we can obtain $\mu = 0$ in a similar manner, which is impossible because $\mu \geq 1$.

If $[G\colon H] = 3$, then either $o(s) \geq 6$ or $o(s) = 3$ because $[G\colon H] \mid o(s)$. Consider the vertices $1 \in H$ and $s^2 \in Hs^2$. Clearly, $1 \nrightarrow s^2$ because $s^2 \notin S$. Lemma 2.8 indicates that

$N^+(1) \cap (V(\Gamma) \setminus H) = \{s\}$ and $N^-(s^2) \cap (V(\Gamma) \setminus Hs^2) = \{s\}$. It leads to $\{z \in V(\Gamma) \mid 1 \to z \to s^2\} = \{s\}$ and $\mu = 1$ because $1 \nrightarrow s^2$. According to Lemma 2.9, $S \cap Ab \neq \emptyset$. Let $xb \in S \cap Ab$. Since $\lambda = 0$ and $xb \in \{z \in V(\Gamma) \mid 1 \to z \to b^2\}$, we have $1 \nrightarrow b^2$, and thus $S \cap Ab = \{xb\}$ due to $\mu = 1$. We claim that $(xb)^{-1}, (xb)^{-1}s^2 \notin S$. Otherwise, if $(xb)^{-1} \in S$, then $(xb)^{-1} = xb$, namely, $b^2 = 1$, which contradicts $o(b) = 4$; if $(xb)^{-1}s^2 \in S$, then $(xb)^{-1}s^2 = xb$, namely, $s^2 = b^2$, $o(s) = 4$, which contradicts $[G\colon H] \mid o(s)$. Consider the vertices $1 \in H$ and $(xb)^{-1}s \in Hs$. Clearly, $(xb)^{-1}s \notin S$, and thereby $1 \nrightarrow (xb)^{-1}s$. Lemma 2.8 indicates that $N^+(1) \cap (V(\Gamma) \setminus H) = \{s\}$ and $N^-((xb)^{-1}s) \cap (V(\Gamma) \setminus Hs) = \{(xb)^{-1}s^2\}$. Therefore, $\{z \in V(\Gamma) \mid 1 \to z \to (xb)^{-1}s\} \subseteq \{s, (xb)^{-1}s^2\}$. By noticing that $s \nrightarrow (xb)^{-1}s$ since $(xb)^{-1} \notin S$, and $1 \nrightarrow (xb)^{-1}s^2$ since $(xb)^{-1}s^2 \notin S$, we conclude that $\{z \in V(\Gamma) \mid 1 \to z \to (xb)^{-1}s\} = \emptyset$, which contradicts $\mu = 1$.

The proof is completed. □

Lemma 2.12. *It holds that* $[G : H] = 2$.

Proof. According to Lemma 2.11, we have $s^{-1} \in S$, and thus $\mu \geq 2$ by Lemma 2.10.

Claim 1. $[G\colon H] < 4$.

Proof of Claim 1. Suppose to the contrary that $[G\colon H] \geq 4$. Therefore, $o(s) \geq [G\colon H] \geq 4$. It is evident that $S \cap Ab \neq \emptyset$, as otherwise $S \subseteq A$, which is impossible since $A \subset G = \langle S \rangle$. Let $xb \in S \cap Ab$. Thus, $xb \in H$ and $xbs, xbs^2, xbs^3 \notin H$ due to $[G\colon H] = 4$. It leads to $xbs, xbs^2, xbs^3 \notin S$. Consider the vertices $1 \in H$ and $xbs^2 \in Hs^2$. Clearly, $1 \nrightarrow xbs^2$. Lemma 2.8 implies that $N(1) \cap (V(\Gamma) \setminus H) = \{s, s^{-1}\}$ and $N(xbs^2) \cap (V(\Gamma) \setminus Hs^2) = \{xbs^3, xbs\}$. Therefore, $\{z \in V(\Gamma) \mid 1 \to z \to xbs^2\} \subseteq \{s, s^{-1}, xbs^3, xbs\}$. By noticing $s \nrightarrow xbs^2$, $1 \nrightarrow xbs$ since $xbs \notin S$, and $s^{-1} \nrightarrow xbs^2$, $1 \nrightarrow xbs^3$ since $xbs^3 \notin S$, we have $\{z \in V(\Gamma) \mid 1 \to z \to xbs^2\} = \emptyset$ and thus $\mu = 0$, which contradicts $\mu \geq 2$. □

Claim 2. $[G\colon H] \neq 3$.

Proof of Claim 2. Suppose to the contrary that $[G\colon H] = 3$. Since $[G\colon H] \mid o(s)$, we have either $o(s) \geq 6$ or $o(s) = 3$.

If $o(s) \geq 6$, then $s^2 \notin S$ because $s^2 \notin H$ and $s^2 \notin \{s, s^{-1}\}$. Consider the vertices $1 \in H$ and $s^2 \in Hs^2$. Clearly, $1 \nrightarrow s^2$. Lemma 2.8 indicates that $N(1) \cap (V(\Gamma) \setminus H) = \{s, s^{-1}\}$ and $N(s^2) \cap (V(\Gamma) \setminus Hs^2) = \{s, s^3\}$. This implies that $\{z \in V(\Gamma) \mid 1 \to z \to s^2\} \subseteq \{s, s^{-1}, s^3\}$. Therefore, we have $s^3 \in S$, since otherwise $1 \nrightarrow s^3$ and $s^{-1} \nrightarrow s^2$, and thus $\{z \in V(\Gamma) \mid 1 \to z \to s^2\} = \{s\}$, which contradicts $\mu \geq 2$. It leads to $\{z \in V(\Gamma) \mid 1 \to z \to s^2\} = \{s, s^{-1}, s^3\}$ and $\mu = 3$. Taking $xb \in S \cap Ab$, we have $xb \in H$, and thus $xbs, xbs^2 \notin H$. It leads to $xbs, xbs^2 \notin S$. Consider the vertices $1 \in H$ and $xbs \in Hs$. Clearly, $1 \nrightarrow xbs$. Lemma 2.8 indicates that $N(1) \cap (V(\Gamma) \setminus H) = \{s, s^{-1}\}$ and $N(xbs) \cap (V(\Gamma) \setminus Hs) = \{xbs^2, xb\}$. This implies that $\{z \in V(\Gamma) \mid 1 \to z \to xbs\} \subseteq \{s, s^{-1}, xbs^2, xb\}$. By noticing $s^{-1} \nrightarrow xbs$ and $1 \nrightarrow xbs^2$ since $xbs^2 \notin S$, we have $\{z \in V(\Gamma) \mid 1 \to z \to xbs\} = \{s, xb\}$, and thus $\mu = 2$, which contradicts $\mu = 3$.

If $o(s) = 3$, then $s^2 = s^{-1} \in S$. We claim that S is inverse-closed. Since $S \cap \langle s \rangle$ is inverse-closed, we only need to show that $S \setminus \langle s \rangle$ is inverse-closed. For any $x \in S \setminus \langle s \rangle$, we have $x \in H$, and thus $x^{-1}s, x^{-1}s^2 \notin H$. Therefore, $x^{-1}s, x^{-1}s^2 \notin S$. Consider the vertices $1 \in H$ and $x^{-1}s \in Hs$. Clearly, $1 \nrightarrow x^{-1}s$ because $x^{-1}s \notin S$. Lemma 2.8 indicates that $N(1) \cap (V(\Gamma) \setminus H) = \{s, s^{-1}\}$ and $N(x^{-1}s) \cap (V(\Gamma) \setminus Hs) = \{x^{-1}, x^{-1}s^2\}$. This implies that $\{z \in V(\Gamma) \mid 1 \to z \to x^{-1}s\} \subseteq \{s, s^{-1}, x^{-1}, x^{-1}s^2\}$. By noticing

$1 \nrightarrow x^{-1}s^2$ and $s^{-1} \nrightarrow x^{-1}s$ since $x^{-1}s^2 \notin S$, we have $\{z \in V(\Gamma) \mid 1 \to z \to x^{-1}s\} = \{s, x^{-1}\}$ according to $\mu \geq 2$, and thus $x^{-1} \in S$. By the arbitrariness of $x \in S \setminus \langle s \rangle$, S is inverse-closed, and thereby Γ is an undirected graph. Recall that G is the disjoint union of Hs^i, and that the subgraph $\Gamma\left[Hs^i\right]$ of Γ is induced by Hs^i, where $0 \leq i \leq 2$. Now we prove that $\Gamma \cong K_3 \square \Gamma'$. For any $xs^i, ys^i \in Hs^i$, we have $xs^i \sim ys^i$ if and only if $yx^{-1} \in S$. Therefore, $xs^i \sim ys^i$ if and only if $x \sim y$, and thus $\Gamma\left[Hs^i\right] \cong \Gamma[H] = \Gamma'$. Moreover, Lemma 2.8 indicates $N(xs^i) \cap (V(\Gamma) \setminus Hs^i) = \{xs^{i-1}, xs^{i+1}\}$. Therefore, $\Gamma \cong K_3 \square \Gamma'$. Since Γ is undirected, Γ is a strongly regular graph. As a Cartesian product of graphs can be factorized uniquely as a product of prime factors, Lemma 2.1 implies that Γ is isomorphic to the Hamming graph $H(d, 3)$ for some positive integer d. It contradicts that Γ is of even order $4n$. □

According to Claims 1 and 2, we have $[G \colon H] < 4$ and $[G \colon H] \neq 3$. This implies that $[G \colon H] = 2$. The proof is completed. □

Lemma 2.13. *We have*

(i) $\lambda \in \{0, 2\}$, *and* $\lambda = 2$ *if and only if* $s^2 \in S$;

(ii) $\mu \in \{2, 4\}$, *and* $\mu = 4$ *if and only if* $xbs^2 \in S$ *for any* $xb \in S \cap Ab$;

(iii) $o(s) = 4$.

Proof. According to Lemma 2.11, we get $s^{-1} \in S$, and thus $\mu \geq 2$ by Lemma 2.10. Lemma 2.12 indicates $[G \colon H] = 2$. Thus, $s^2 \in H$ and $2 \mid o(s)$ since $[G \colon H] \mid o(s)$. Note that $o(s) > 2$ by Lemma 2.9, which implies that either $o(s) = 4$ or $o(s) \geq 6$. In what follows, we show (i), (ii) and (iii), respectively.

Consider the vertices $1 \in H$ and $s \in Hs$. Clearly, $1 \to s$. Lemma 2.8 indicates that $N(1) \cap (V(\Gamma) \setminus H) = \{s, s^{-1}\}$ and $N(s) \cap (V(\Gamma) \setminus Hs) = \{1, s^2\}$. This implies that $\{z \in V(\Gamma) \mid 1 \to z \to s\} \subseteq \{s^{-1}, s^2\}$. Observe that

$$s^{-1} \in \{z \in V(\Gamma) \mid 1 \to z \to s\} \Leftrightarrow s^2 \in S \Leftrightarrow s^2 \in \{z \in V(\Gamma) \mid 1 \to z \to s\}.$$

Clearly, $s^2 \neq s^{-1}$ because $o(s) \neq 3$. Therefore, if $s^2 \in S$, then $\{z \in V(\Gamma) \mid 1 \to z \to s\} = \{s^{-1}, s^2\}$ and thus $\lambda = 2$; if $s^2 \notin S$, then $\{z \in V(\Gamma) \mid 1 \to z \to s\} = \emptyset$ and thus $\lambda = 0$. Hence, (i) holds.

Take an arbitrary element $xb \in S \cap Ab$ with $x \in A$. Since $xb \in H$, we have $xbs, xbs^3 \in Hs \neq H$. It leads to $xbs, xbs^3 \notin S$. Consider the vertices $1 \in H$ and $xbs \in Hs$. Clearly, $1 \nrightarrow xbs$ because $xbs \notin S$. Lemma 2.8 indicates that $N(1) \cap (V(\Gamma) \setminus H) = \{s, s^{-1}\}$ and $N(xbs) \cap (V(\Gamma) \setminus Hs) = \{xb, xbs^2\}$. This implies that $\{z \in V(\Gamma) \mid 1 \to z \to xbs\} \subseteq \{s, s^{-1}, xb, xbs^2\}$. Observe that $\{s, xb\} \subseteq \{z \in V(\Gamma) \mid 1 \to z \to xbs\}$ and

$$\begin{aligned} s^{-1} \in \{z \in V(\Gamma) \mid 1 \to z \to xbs\} &\Leftrightarrow xbs^2 \in S \\ &\Leftrightarrow xbs^2 \in \{z \in V(\Gamma) \mid 1 \to z \to xbs\}. \end{aligned}$$

Therefore, if $xbs^2 \in S$, then

$$\{z \in V(\Gamma) \mid 1 \to z \to s\} = \{s, s^{-1}, xb, xbs^2\}$$

and thus $\mu = 4$; if $xbs^2 \notin S$, then $\{z \in V(\Gamma) \mid 1 \to z \to xbs\} = \{xb, s\}$ and thus $\mu = 2$. Hence, (ii) holds.

Now we prove that $o(s) = 4$. Suppose to the contrary that $o(s) \geq 6$. By (ii), we have $\mu \in \{2, 4\}$, and $\mu = 4$ if and only if $xbs^2 \in S$ for any $xb \in S \cap Ab$. We claim that $xbs^4 \in S$. Consider the vertices $1 \in H$ and $xbs^3 \in Hs$. Clearly, $1 \nrightarrow xbs^3$ because $xbs^3 \notin S$. Lemma 2.8 indicates that $N(1) \cap (V(\Gamma) \setminus H) = \{s, s^{-1}\}$ and $N(xbs^3) \cap (V(\Gamma) \setminus Hs) = \{xbs^2, xbs^4\}$. This implies that $\{z \in V(\Gamma) \mid 1 \to z \to xbs^3\} \subseteq \{s, s^{-1}, xbs^2, xbs^4\}$. Observe that

$$\begin{aligned} s \in \{z \in V(\Gamma) \mid 1 \to z \to xbs^3\} &\Leftrightarrow xbs^2 \in S \\ &\Leftrightarrow xbs^2 \in \{z \in V(\Gamma) \mid 1 \to z \to xbs^3\}, \\ s^{-1} \in \{z \in V(\Gamma) \mid 1 \to z \to xbs^3\} &\Leftrightarrow xbs^4 \in S \\ &\Leftrightarrow xbs^4 \in \{z \in V(\Gamma) \mid 1 \to z \to xbs^3\}. \end{aligned}$$

Therefore, if $\mu = 4$, then

$$\{z \in V(\Gamma) \mid 1 \to z \to xbs^3\} = \{s, s^{-1}, xbs^2, xbs^4\}$$

and thus $xbs^4 \in S$; if $\mu = 2$, then $\{z \in V(\Gamma) \mid 1 \to z \to xbs^3\} = \{s^{-1}, xbs^4\}$ because $xbs^2 \notin S$ by (ii), and thus $xbs^4 \in S$. Next we divide two cases to reach a contradiction.

Case 1. $\lambda = 2$.

Since $\lambda = 2$, we have $s^2 \in S$. We claim that $s^{-2} \in S$. Consider the vertices $1 \in H$ and $s^{-1} \in Hs^{-1}$. Clearly, $1 \to s^{-1}$ because $s^{-1} \in S$. Lemma 2.8 indicates that $N(1) \cap (V(\Gamma) \setminus H) = \{s, s^{-1}\}$ and $N(s^{-1}) \cap (V(\Gamma) \setminus Hs^{-1}) = \{1, s^{-2}\}$. This implies that $\{z \in V(\Gamma) \mid 1 \to z \to s^{-1}\} \subseteq \{s, s^{-2}\}$. Observe that

$$\begin{aligned} s \in \{z \in V(\Gamma) \mid 1 \to z \to s^{-1}\} &\Leftrightarrow s^{-2} \in S \\ &\Leftrightarrow s^{-2} \in \{z \in V(\Gamma) \mid 1 \to z \to s^{-1}\}. \end{aligned}$$

Therefore, we have $s^{-2} \in S$ according to $\lambda = 2$. If $\mu = 2$, then $xbs^2 \notin S$ and $xbs^4 \in S$ as claimed before. Thus, there exist three distinct elements $\{xb, s^2, xbs^4\} \subseteq \{z \in V(\Gamma) \mid 1 \to z \to xbs^2\}$, which contradicts $\mu = 2$. If $\mu = 4$, then $xbs^2, xbs^4 \in S$ by above arguments. Thus, there exist three distinct elements $\{s, xbs^2, xbs^4\} \subseteq \{z \in V(\Gamma) \mid 1 \to z \to s^2\}$, which contradicts $\lambda = 2$.

Case 2. $\lambda = 0$.

Since $\lambda = 0$, we have $s^2 \notin S$. By noticing $s^3 \notin H$ since $s^2 \in H$, and $s^3 \neq s, s^{-1}$ since $o(s) \geq 6$, we have $s^3 \notin S$. Thus, $1 \nrightarrow s^3$. Consider the vertices $1 \in H$ and $s^3 \in Hs$. Lemma 2.8 indicates that $N(1) \cap (V(\Gamma) \setminus H) = \{s, s^{-1}\}$ and $N(s^3) \cap (V(\Gamma) \setminus Hs) = \{s^2, s^4\}$. This implies that $\{z \in V(\Gamma) \mid 1 \to z \to s^3\} \subseteq \{s, s^{-1}, s^2, s^4\}$. Observe that

$$\begin{aligned} s \in \{z \in V(\Gamma) \mid 1 \to z \to s^3\} &\Leftrightarrow s^2 \in S \\ &\Leftrightarrow s^2 \in \{z \in V(\Gamma) \mid 1 \to z \to s\}, \\ s^{-1} \in \{z \in V(\Gamma) \mid 1 \to z \to s^3\} &\Leftrightarrow s^4 \in S \\ &\Leftrightarrow s^4 \in \{z \in V(\Gamma) \mid 1 \to z \to s\}. \end{aligned}$$

Therefore, according to $s^2 \notin S$ and $\mu \in \{2, 4\}$, we conclude that $\{z \in V(\Gamma) \mid 1 \to z \to s^3\} = \{s^{-1}, s^4\}$ and thus $s^4 \in S$. This implies $s^4 \in \{z \in V(\Gamma) \mid 1 \to z \to xbs^4\}$. Since

$\lambda = 0$ and $s^4 \in \{z \in V(\Gamma) \mid 1 \to z \to xbs^4\}$, we have $1 \nrightarrow xbs^4$ and thus $xbs^4 \notin S$, which contradicts $xbs^4 \in S$ as claimed before.

Hence, (iii) holds. □

Lemma 2.14. *The graph Γ is isomorphic to the complete bipartite graph $K_{4,4}$.*

Proof. According to Lemma 2.12, we get $[G\colon H] = 2$. And by Lemma 2.13(iii), we get $o(s) = 4$, and thus $o(s^2) = 2$. Since $s^2 \in A$ and b^2 is the unique element of order 2 in A, we have $s^2 = b^2$. Clearly, $S \cap Ab \neq \emptyset$. Let $xb \in S \cap Ab$, then $s^2 = (xb)(xb)$. Note that $\lambda \in \{0, 2\}$ and $\{s, s^{-1}, xb\} \subseteq \{z \in V(\Gamma) \mid 1 \to z \to s^2\}$, which implies that $1 \nrightarrow s^2$ and $\mu \geq 3$. Thus, $s^2 \notin S$, and $\mu = 4$ by Lemma 2.13(ii).

Firstly, we prove $S \cap Ab$ is inverse-closed. If there exists $xb \in S \cap Ab$ such that $(xb)^{-1} \notin S$, then $xb \in H$ and $(xb)^{-1}s \in Hs \neq H$. It leads to $(xb)^{-1}s \notin S$. Consider the vertices $1 \in H$ and $(xb)^{-1}s \in Hs$. Clearly, $1 \nrightarrow (xb)^{-1}s$ because $(xb)^{-1}s \notin S$. Lemma 2.8 indicates that $N(1) \cap (V(\Gamma) \setminus H) = \{s, s^{-1}\}$ and $N((xb)^{-1}s) \cap (V(\Gamma) \setminus Hs) = \{(xb)^{-1}, (xb)^{-1}s^2\}$. Therefore, $\{z \in V(\Gamma) \mid 1 \to z \to (xb)^{-1}s\} \subseteq \{s, s^{-1}, (xb)^{-1}, (xb)^{-1}s^2\}$. By noticing $1 \nrightarrow (xb)^{-1}$ and $s \nrightarrow (xb)^{-1}s$ since $(xb)^{-1} \notin S$, we have $\{z \in V(\Gamma) \mid 1 \to z \to (xb)^{-1}s\} \subseteq \{s^{-1}, (xb)^{-1}s^2\}$, which contradicts $\mu = 4$.

Next, we prove $(S \setminus \langle s \rangle) \cap A = \emptyset$. Suppose to the contrary that $(S \setminus \langle s \rangle) \cap A \neq \emptyset$. Let $y \in (S \setminus \langle s \rangle) \cap A$. Thus, $y \in H$ and $y^{-1}s \in Hs \neq H$. It leads to $y^{-1}s \notin S$. Consider the vertices $1 \in H$ and $y^{-1}s \in Hs$. Clearly, $1 \nrightarrow y^{-1}s$ because $y^{-1}s \notin S$. Lemma 2.8 indicates that $N(1) \cap (V(\Gamma) \setminus H) = \{s, s^{-1}\}$ and $N(y^{-1}s) \cap (V(\Gamma) \setminus Hs) = \{y^{-1}, y^{-1}s^2\}$. Therefore, $\{z \in V(\Gamma) \mid 1 \to z \to y^{-1}s\} \subseteq \{s, s^{-1}, y^{-1}, y^{-1}s^2\}$. Since $1 \nrightarrow y^{-1}s$ and $\mu = 4$, we have $\{z \in V(\Gamma) \mid 1 \to z \to y^{-1}s\} = \{s, s^{-1}, y^{-1}, y^{-1}s^2\}$. This implies that $y^{-1}, y^{-1}s^2 \in S$. Thus, there exist five distinct elements $\{s, s^{-1}, y, xb, (xb)^{-1}\} \subseteq \{z \in V(\Gamma) \mid 1 \to z \to s^2\}$, which contradicts $\mu = 4$.

As proved before, $S \cap Ab \neq \emptyset$ is inverse-closed. For any $xb \in S \cap Ab$, we have $(xb)^{-1} \in S$. Notice that $s^2 = (xb)(xb) = (xb)^{-1}(xb)^{-1}$, which indicates that

$$\{s, s^{-1}, xb, (xb)^{-1} \mid xb \in S \cap Ab\} \subseteq \{z \in V(\Gamma) \mid 1 \to z \to s^2\}.$$

Since $\mu = 4$, we conclude that $S \cap Ab = \{xb, (xb)^{-1}\}$. Therefore, $S = \{s, s^{-1}, xb, (xb)^{-1}\}$ according to $(S \setminus \langle s \rangle) \cap A = \emptyset$. It leads to $G = \langle S \rangle = \{1, s, s^{-1}, s^2, xb, xb^{-1}, sxb, s^{-1}xb\}$. Since $\langle S \setminus \{s, s^{-1}\} \rangle \neq G$, we have that S is minimal with respect to s. Hence, as a minimal directed Cayley graph, $\Gamma = \mathrm{Cay}(G, S) \cong K_{4,4}$, is indeed a directed strongly regular graph, and the corresponding parameters are $(8, 4, 4, 0, 4)$. □

Finally, we give the proof of Theorem 1.1.

Proof of Theorem 1.1. If $s \in Ab$, then Lemmas 2.5 and 2.6 imply that Γ is isomorphic to the complete bipartite graph $K_{4,4}$, or the directed graph (a) or (b) shown in Figure 1. If $s \in A$, then from Lemma 2.14, we deduce that Γ is isomorphic to the complete bipartite graph $K_{4,4}$. □

3 Concluding remarks

In this paper, we investigate directed strongly regular Cayley graphs over generalized dicyclic groups and prove that, under the condition that the corresponding connection set is minimal, the complete bipartite graph $K_{4,4}$ and the directed graphs (a) and (b) shown in Figure 1 are the only such graphs over generalized dicyclic groups.

We restrict our attention to generalized dicyclic groups $G = \langle A, b\rangle$ with exactly one involution in A. If A has more involutions, the conditions for our investigation become much more challenging. Nevertheless, we think that our approach is feasible, albeit requiring more detailed and laborious analysis. In our future research, we aim to explore more efficient methods to tackle such problems.

In general, a subset $S \subseteq G$ is called a minimal generating set if $\langle S\rangle = G$ but $\left\langle S\backslash\{s, s^{-1}\}\right\rangle < G$ for any $s \in S$. However, in this manuscript, we say S is minimal if $\langle S\rangle = G$ and there exists $s \in S$ such that $\left\langle S\backslash\{s, s^{-1}\}\right\rangle$. Therefore, our condition is weaker but results in a stronger outcome. Nonetheless, characterizing all directed strongly regular Cayley graphs over generalized dicyclic groups remains an open problem. In fact, even for even dihedral groups, this problem is yet to be solved. It will be a major focus of our future research.

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Super-symmetric maps from dihedral groups*

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Abstract

In 1976, S. Wilson proposed to study a family of regular self-dual and self-Petrie-dual maps arising from groups of order $8n^3$ defined by a specific presentation. Later on, in 2014, D. Archdeacon, M. Conder and J. Širáň proved that these maps are super-symmetric, that is, not only exhibiting all self-dualities but also all admissible exponents. Furthermore, in 2016, G. A. Jones suggested that it should be possible to obtain the same family by the means of a parallel product of maps arising from 2-extensions of dihedral groups of order $2n$. In this paper we verify this suggestion for odd values of n; for even n we show that the parallel product construction gives maps that are quotients of Wilson's maps by a normal subgroup of order 2.

Keywords: Regular map, duality, exponent, super-symmetry.

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1 Introduction

A *regular map* is a cellular embedding of a connected graph on some surface, with the property that the automorphism group of the embedding is transitive on the flag set of the embedded graph. Regularity of the map implies that all vertices of its underlying graph have the same valency, say, d, and all its faces are bounded by closed walks of the same length, say, ℓ; we then say that the map is of *type* (d, ℓ). By the general theory of regular maps developed e.g. in [2], any regular map M of type (d, ℓ) can be identified with a presentation of its automorphism group $G = \mathrm{Aut}(M)$ in the form

$$G = \langle x, y, z \mid x^2, y^2, z^2, (xz)^2, (zy)^d, (yx)^\ell \dots \rangle \tag{1.1}$$

and conversely, every abstract group $G = \langle x, y, z \rangle$ generated by 3 involutions and presented as in (1.1) determines a unique (up to isomorphism) regular map $M = \mathrm{Map}(G; x, y, z)$. It follows that the group $\mathrm{Aut}(M)$ is a quotient of the free product $\Gamma = (C_2 \times C_2) * C_2$ of the Klein four-group with the cyclic group of order 2, with presentation

$$\Gamma = \langle X, Y, Z \mid X^2, Y^2, Z^2, (XZ)^2 \rangle = \langle X, Z \rangle * \langle Y \rangle \tag{1.2}$$

by a torsion-free normal subgroup (of finite index if the map M is finite).

A number of further details about regular maps, their automorphism groups and the outlined correspondence will be given in Section 2. For now we will proceed with alluding to the three basic types of operators producing new regular maps from old, which are duality, Petrie duality, and hole operators [5], which are also known as rotational powers. The first two are well known, and the third one produces a new regular map from a given regular map of valency d by replacing, for any j relatively prime to d, all the cyclic orderings of edges emanating from any vertex by their j-th powers. In general, these operators do not preserve the (isomorphism types of) maps, but if any of them does, the map is said to be self-dual, self-Petrie-dual, and to have exponent j, respectively. Such map-preserving operators are also known as *external symmetries.*

Regular maps that admit all possible external symmetries, i.e., which have all feasible exponents and are both self-dual and self-Petrie dual, have been called *kaleidoscopic, with trinity symmetry* in [1]; for simplicity we will call such regular maps *super-symmetric*.

In the course of preparation of his doctoral dissertation, S. Wilson [4] found out that for any positive integer $n \le 100$ the group W_n of order $8n^3$ with presentation

$$W_n = \langle x, y, z \mid x^2, y^2, z^2, (xz)^2, (zy)^{2n}, (yx)^{2n}, (zyx)^{2n}, [(zy)^2, (yx)^2] \rangle \tag{1.3}$$

determines a self-dual and self-Petrie dual regular map $MW_n = \mathrm{Map}(W_n; x, y, z)$ of type $(2n, 2n)$ on an orientable surface. (The original presentation of Wilson was slightly different but equivalent to the above.) Much later, Archdeacon, Conder and Širáň [1] proved that for all n the same regular map is not only invariant to the two dualities but also to all admissible hole operators, establishing thus super-symmetry of the maps MW_n.

It was then noted by Jones [3, Section 9] that the super-symmetric map MW_n can be explicitly described in a different but equivalent way, namely, as a quotient $\Gamma/\Gamma''(\Gamma')^{(n)}$ of the group Γ from (1.2) by its characteristic subgroup formed by the product of second-order commutator subgroup $\Gamma'' < \Gamma$ by the subgroup $(\Gamma')^{(n)} < \Gamma$ generated by all n-th powers of elements of the commutator subgroup of Γ. This can be seen by realizing that Γ' is the normalizer of the subgroup of Γ generated by the commutators $(zy)^2$ and $(yx)^2$, combined with inspecting the presentation of Wilson's group W_n in which the only relators except

x^2, y^2, z^2 and $(xz)^2$ are the n-th powers of $(zy)^2$ and $(yx)^2$ together with the n-th power of the product $(zy)^2(yx)^2 = (zyx)^2$, and the commutator $[(zy)^2, (yx)^2]$.

Jones' paper [3] actually offers a unifying approach to a number of problems that can be stated in terms of combinatorial categories, including questions on super-symmetry of maps. For example, a general construction of super-symmetric maps from [3] proceeds as follows. Let G be a fixed specimen of a finite group generated by a triple of involutions two of which commute, and let $K = K(G)$ be the intersection of kernels of all epimorphisms $\Gamma \to G$ taking the generating triple (X, Y, Z) of Γ to a triple (x, y, z) of generating involutions of G such that x commutes with z. Then, by [3], the quotient Γ/K is the automorphism group of a regular super-symmetric map.

This general construction by Jones brings us back to Wilson's group W_n from (1.3) and the associated super-symmetric map in a rather surprising way. Namely, along with noting that $W_n \cong \Gamma/\Gamma''(\Gamma')^{(n)}$, in [3, Section 9] Jones remarks that for the same group one has $W_n \cong \Gamma/K(G)$, where for G one may take a natural extension of a dihedral group of order $4n$ by an automorphism of order 2 inverting the two obvious generators of the dihedral group. (The wording of the second paragraph of [3, Section 9] mentions dihedral groups only, but a further comment in the next paragraph makes it clear that it is their natural 2-extensions G that need to be taken as epimorphic images of Γ to create $K(G)$.)

The above remark by Jones has been the starting point of the second paragraph of the research presented in this paper, as there appears to be no a priori reason why the groups $\Gamma''(\Gamma')^{(n)}$ and $K(G)$ for G as above should give isomorphic quotients of Γ.

Before proceeding, let us explain how Jones' construction [3] of super-symmetric maps by means of intersection of kernels of epimorphisms is linked to an earlier work of Wilson [6] on parallel products of groups. Again, for a fixed specimen of a finite group G consider the (finite) family $\mathcal{G}$ of all the presentations $G(x, y, z)$ of G as in (1.1); assume that there are a total of m such presentations of the form $G(x_i, y_i, z_i)$, $1 \le i \le m$. The *parallel product* of the family of presentations $\mathcal{G}$ is the subgroup $\mathcal{H}$ of the direct product $\Pi_{1\le i\le m} G(x_i, y_i, z_i)$ generated by the triple $(\tilde{x}, \tilde{y}, \tilde{z})$ such that $\tilde{x} = \Pi_{1\le i\le m}(x_i)$, $\tilde{y} = \Pi_{1\le i\le m} i(y_i)$ and $\tilde{z} = \Pi_{1\le i\le m}(z_i)$. The regular map $\tilde{M} = \mathrm{Map}(\mathcal{H}; \tilde{x}, \tilde{y}, \tilde{z})$ determined by this parallel product is super-symmetric, because (by the nature of how it has been defined) it is preserved by every external symmetry.

The parallel product construction originating in [6] and the construction of [3] involving intersection of epimorphism kernels give, in fact, isomorphic maps. Indeed, there is an isomorphism from $\mathrm{Aut}(\tilde{M})$ onto the quotient Γ/K, where K is the intersection of kernels of epimorphisms $\Gamma \to G(x_i, y_i, z_i)$ taking the generating triple (X, Y, Z) of Γ to (x_i, y_i, z_i), $1 \le i \le m$, such that the isomorphism takes the generating triple $(\tilde{x}, \tilde{y}, \tilde{z})$ of $\mathrm{Aut}(\tilde{M})$ to the corresponding generating triple of Γ/K. Perhaps the easiest way to see this is to realize that a word in the generators $\tilde{x}, \tilde{y}, \tilde{z}$ is a relator in a presentation of the group $\mathrm{Aut}(\tilde{M})$ if and only if, for every $i \in \{1, 2, \ldots, m\}$, the projection of the word onto the i-th coordinate is a relator in a presentation of G in the generators x_i, y_i, z_i.

By another observation made in [3], the kernels of a pair of epimorphisms $\Gamma \to G$ sending the generating triple (X, Y, Z) to (x_i, y_i, z_i) and (x_j, y_j, z_j) coincide if and only if the epimorphisms differ by an automorphism of G taking the triple (x_i, y_i, z_i) to (x_j, y_j, z_j). In terms of maps, the last condition means that the corresponding regular maps $M_i = \mathrm{Map}(G_i; x_i, y_i, z_i)$ and $M_j = \mathrm{Map}(G_j; x_j, y_j, z_j)$ are isomorphic. Thus, Jones' construction is equivalent to Wilson's even when the latter is restricted to a parallel product of automorphism groups of regular maps that are pairwise non-isomorphic.

The theoretical background collected above offers a way to verify the motivating remark made by Jones [3, Section 9] on isomorphism of Wilson's group W_n with $\Gamma/K(G)$, where G (of order $8n$) is the extension of a dihedral group of order $4n$ as described earlier. After giving a more detailed introduction into regular maps on 2-extensions of dihedral groups in Section 2, we will describe in Section 3 the kernel $K(G)$ for extended dihedral groups G by an explicit counting of epimorphism from the 'universal' group Γ onto G. On the 'parallel product' counterpart, in Section 4 we will determine the automorphism groups of the corresponding super-symmetric maps. Since throughout the first four sections we assume that $n > 1$, in Section 5 we also consider the case $n = 1$ for completeness.

As the outcome it will turn out that Jones' remark [3, Section 9] on isomorphism of the Wilson group W_n with the quotient of Γ generated by the intersection of kernels of epimorphisms from Γ onto the automorphism groups of the above three maps (preserving the obvious generating sets) is valid only for odd $n > 1$; for even n the result happens to be isomorphic to a quotient of W_n by a normal subgroup of order 2.

We add a remark on the methodology used. In principle, calculation of the kernel $K(G)$ can be approached in two ways: either by considering generating triples of involutions of G, or by determining images of generating triples of Γ in epimorphisms $\Gamma \to G$, in both cases under equivalence induced by the action of $\mathrm{Aut}(G)$. Both ways require sorting out numerous possibilities for generating triples modulo automorphisms of G, and in effect they result in approximately the same amount of work (Section 3). Our preference to use the 'epimorphisms' approach was motivated by Jones' suggestion about isomorphism of the Wilson group with the quotient of Γ by intersection of kernels of *epimorphisms* $\Gamma \to G$.

2 Regular maps and dihedral groups

Let M be a regular map of type (d, ℓ) with automorphism group $\mathrm{Aut}(M)$. Let f be a flag of M contained in a face F such that f also contains a vertex v a 'half' of an edge e incident with v. By regularity, there are unique map automorphisms r_0, r_1 and r_2, taking f onto the flag incident with f along the edge e, across the edge e, and to the unique third flag distinct from e and incident to f, containing v and being contained in F, respectively. The three automorphisms are involutory, acting as reflections in the three 'sides' of the flag f. The compositions r_2r_1, r_1r_0 and r_0r_2 are, respectively, rotations about v of order d, about the centre of F of order ℓ, and about the centre of e of order 2. By connectivity, the three involutions generate the automorphism group of M, leading to its presentation of the form

$$\mathrm{Aut}(M) = \langle r_0, r_1, r_2 \mid r_0^2, r_1^2, r_2^2, (r_0r_2)^2, (r_2r_1)^d, (r_1r_0)^\ell, \ldots \rangle \,. \tag{2.1}$$

Conversely, as stated in the introduction, any abstract group G generated by three involutions two of which commute and presented in the form equivalent to (2.1), that is,

$$G = \langle x, y, z \mid x^2, y^2, z^2, (xz)^2, (zy)^d, (yx)^\ell, \ldots \rangle \tag{2.2}$$

is the automorphism group of a regular map of type (d, ℓ), denoted $\mathrm{Map}(G; x, y, z)$ earlier. Its flag set may be identified with the set of elements of G and its edges, vertices and faces may be identified with left cosets of the subgroups $\langle x, z\rangle$, $\langle y, z\rangle$ and $\langle x, y\rangle$. The carrier surface of a regular map $M = \mathrm{Map}(G) = \mathrm{Map}(x, y, z)$ for a group G as in (2.2) is orientable if and only if the group $\langle xy, yz\rangle = \langle xz, yz\rangle$ has index 2 in G. Two such regular maps $\mathrm{Map}(G_1; x_1, y_1, z_1)$ and $\mathrm{Map}(G_2; x_2, y_2, z_2)$ are isomorphic if and only if there is a group isomorphism $G_1 \to G_2$ taking the ordered triple (x_1, y_1, z_1) to (x_2, y_2, z_2).

We describe next in more detail the three map operators we have mentioned in the introduction: duality, Petrie duality and rotational powers (known also as hole operators). Letting a group $G = \langle x, y, z\rangle$ be as in (2.2), the *dual* of the regular map $M = \mathrm{Map}(G) = \mathrm{Map}(x, y, z)$ is the map $D(M) = \mathrm{Map}(x_D, y_D, z_D)$ where $x_D = z$, $y_D = y$ and $z_D = x$. The *Petrie dual* of the same map M is the map $P(M) = \mathrm{Map}(x_P, y_P, z_P)$ with $x_P = xz$, $y_P = y$ and $z_P = z$. For the third operator, if d is the valency of M, let j be a positive integer coprime to d and smaller than d. The j-th rotational power of M (or, the result of the j-th hole operator applied to M) is the map $M^{(j)} = \mathrm{Map}(x_j, y_j, z_j)$ in which $x_j = x$, $y_j = z(zy)^j$ and $z_j = z$. All these map share with M the same automorphism group, and hence all of them are regular. We say that M is *self-dual*, *self-Petrie dual*, and has *exponent* j if M is, respectively, isomorphic to $D(M)$, $P(M)$, and $M^{(j)}$.

Beyond Abelian groups the simplest non-trivial groups to consider as automorphism groups of regular maps are perhaps the dihedral groups and their 2-extensions by an 'obvious' involutory automorphism, which we now describe. For reasons to be explained later, we will be working with dihedral groups the cyclic part of which has even order. Thus, for an arbitrary positive integer n, we will consider the dihedral group D_{2n} of order $4n$, with a cyclic subgroup of order $2n$, defined by the presentation

$$D_{2n} = \langle r, s \mid r^{2n}, s^2, (rs)^2\rangle\ . \tag{2.3}$$

This group can be extended by an obvious automorphism t of order 2, inverting both r and s (or, equivalently, inverting r and commuting with s), giving a group D^*_{2n} with presentation

$$D^*_{2n} = \langle r, s, t \mid r^{2n}, s^2, t^2, (rs)^2, (rt)^2, (st)^2\rangle\ . \tag{2.4}$$

Letting $x = st$, $y = rt$ and $z = t$, the presentation (2.4) defines a regular map $M_{2n} = \mathrm{Map}(G; x, y, z) = \mathrm{Map}(st, rt, t)$ of type $(2n, 2)$, with automorphism group $G = D^*_{2n} = \langle x, y, z\rangle = \langle r, s, t\rangle$. The map M_{2n} is simply a dipole of valency $2n$ on a sphere, with its two vertices being (say) the north and the south pole, and with each of the $2n$ faces bounded by a 2-cycle. Elements r and s from the presentation (2.4) may be visualized as a $2n$-fold rotation of the map about its axis through the poles and if s is a 180-degree rotation about an axis joining the centre of an edge e with the centre of the edge opposite to e. (See Figure 1.) The subgroup $\langle r, s\rangle \cong D_{2n}$ given by (2.3) is the *rotation group* of the map, consisting of all its orientation-preserving automorphisms.

In general, a map on an orientable surface is *orientably-regular* if the group of all its orientation-preserving automorphisms is regular on arcs (vertex-edge incident pairs) of the map. In this terminology, the map M_{2n} is both orientably-regular and regular; such maps are also called *reflexible*, and half of their automorphisms are orientation-reversing. As an example of an orientation-reversing automorphism of M_{2n} one may take a reflection in a plane through the north-south pole axis that bisects a face incident with the edge e; this can be taken to be the element t in the presentation (2.4) that inverts both r and s.

The dual $D(M_{2n}) = \mathrm{Map}(t, rt, st)$ of the regular map M_{2n} has type $(2, 2n)$ and can be visualized on a sphere as a cycle of length $2n$ displaced around the equator. The Petrie dual $P(M_{2n}) = \mathrm{Map}(s, rt, t)$ of the regular map M_{2n} has type $(2n, 2n)$ and its visualization is more tricky, because by Euler's formula its carrier surface is orientable, of genus $n - 1$. It may be instructive to note that the rotation group of $P(M_{2n})$ is generated by r and st and hence is Abelian, isomorphic to $C_{2n} \times C_2$, while the rotation group of the orientably-regular map M_{2n} is non-Abelian for $n > 1$. Clearly, the three maps M_{2n}, $P(M_{2n})$ and $D(M_{2n})$ are pairwise non-isomorphic.

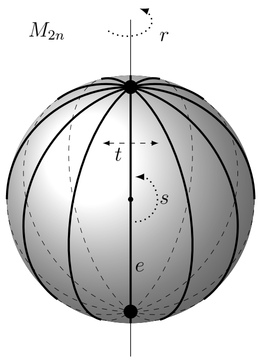

Figure 1: A dipole with $2n$ parallel edges embedded on the sphere.

To describe the action of the entire operator group $\langle D, P\rangle \cong S_3$ on M_{2n}, note that the remaining three images of M_{2n} under this group are $PD(M_{2n}) = \mathrm{Map}(s, rt, st)$, $DP(M_{2n}) = \mathrm{Map}(t, rt, s)$ and $DPD(M_{2n}) = PDP(M_{2n}) = \mathrm{Map}(st, rt, s)$. Because of the obvious automorphism of the group D^*_{2n} of (2.4) fixing rt and interchanging s with t it follows that $M_{2n} \cong DPD(M_{2n}) \cong PDP(M_{2n})$, $P(M_{2n}) \cong DP(M_{2n})$ and $D(M_{2n}) \cong PD(M_{2n})$.

In principle, all the above material could have been developed also for dihedral groups with cyclic part of odd order. But in such a case the Petrie dual of the dual of such a map is non-orientable (it has just one face hence its Euler characteristic is equal to 1), and also the connection to the Wilson group from (1.3) would be lost.

Our last remark in this section concerns exponents of the map M_{2n}. By our earlier description of rotational powers, if j is an arbitrary unit mod $2n$, the j-th rotational power $M_{2n}^{(j)}$ is the regular map $\mathrm{Map}(st, r^j t, t)$. Since the assignment $r \mapsto r^j$, $s \mapsto s$, $t \mapsto t$ extends to an automorphism of the group D^*_{2n}, it follows that the regular maps M_{2n} and $M_{2n}^{(j)}$ are isomorphic, and so every unit mod $2n$ is an exponent of M_{2n}. In the terminology of [1], the map M_{2n} is kaleidoscopic.

3 Epimorphisms onto extended dihedral groups

We will describe the intersection $K(D^*_{2n})$ of all kernels of epimorphisms $f \in \mathrm{Epi}(\Gamma \to D^*_{2n})$ from the group Γ given by the presentation (1.2) onto an extended dihedral group D^*_{2n} from (2.4). As already mentioned (and noted in [3]), for a general group G a pair of epimorphisms $f_1, f_2 \colon \Gamma \to G$ have the same kernels if and only if there is an automorphism ϕ of G such that $\phi \circ f_1 = f_2$. The 'if' direction is obvious, and for the other one it is sufficient to observe that the mapping $\phi\colon G \to G$ given by $\phi\colon f_1(g) \mapsto f_2(g)$ for every $g \in \Gamma$ is a well-defined automorphism of G. It follows that the number of distinct kernels entering the intersection $K(D^*_{2n})$ is equal to

$$|\mathrm{Epi}(\Gamma \to D^*_{2n})|/|\mathrm{Aut}(D^*_{2n})|. \tag{3.1}$$

To facilitate the task we will work with a different but equivalent presentation of Γ which resembles the presentation of D^*_{2n}. Instead of the original involutory generators X, Y, Z

of Γ we introduce three new generators R, S, T by letting $R = YZ$, $S = XZ$ and $T = Z$; the corresponding equivalent presentation of Γ then is

$$\Gamma = \langle R, S, T \mid S^2, T^2, (RT)^2, (ST)^2 \rangle. \tag{3.2}$$

In this notation, determination of all epimorphisms $f \in \mathrm{Epi}(\Gamma \to D_{2n}^*)$ amounts to identification of all possible assignments of the ordered triple (R, S, T) of generators of Γ to ordered triples of elements in D_{2n}^* such that the assignment induces a mapping of relators appearing in the presentation (3.2) onto (consequences of) relators that appear in the presentation (2.4) of D_{2n}^*, that is, in the presentation $\langle r, s, t \mid r^{2n}, s^2, t^2, (rs)^2, (rt)^2, (st)^2 \rangle$.

To avoid trivial considerations, assume throughout that $n > 1$. The order of D_{2n}^* is $8n$ and its centre is $Z(D_{2n}^*) = \{1, r^n, st, r^n st\} \cong C_2 \times C_2$. All the remaining involutions in D_{2n}^* have the form $r^i s$ or $r^j t$ for some $i, j \in \{0, 1, \ldots, 2n-1\}$. On the other end of the spectrum, the largest order of an element in D_{2n}^* is $2n$, and all elements of this order have the form r^i, $1 \le i < 2n$, with $\gcd(i, 2n) = 1$, and $r^i st$, $1 \le i < 2n$, with $\gcd(i, n) = 1$.

We continue with proving four claims, in all of which we assume that $f : \Gamma \to D_{2n}^*$ is an epimorphism, that is, in particular, with $f(\Gamma)$ having order $8n$, with $n > 1$.

Claim 3.1. *Exactly one of the involutions $f(S)$, $f(T)$ and $f(ST)$ is central in G, and if $f(R)$ is an involution, then exactly one of $f(R)$, $f(T)$ and $f(RT)$ is central.*

Proof. Suppose $f(S)$ and $f(T)$ are non-central, that is, $f(S) = r^i g$ and $f(T) = r^j h$ for some $g, h \in \{s, t\}$ and $i, j \in \{0, 1, \ldots, 2n-1\}$. From $(ST)^2 = 1$ it follows that $f(S)$ and $f(T)$ commute, but it is easy to check that this happens if and only if $i \in \{j, j+n\} \bmod 2n$. The product of the two involutions then evaluates to $f(S)f(T) = r^i g r^j h = r^{i-j} gh$ with $i - j \in \{0, n\}$ and $gh \in \{1, st\}$, so that $f(ST) \in Z(G)$. If two (and hence all three) of the f-images of S, T and ST were central, then $f(\Gamma)$ would be generated by $f(R)$ and by the central elements $f(S)$ and $f(T)$ and hence it would be Abelian, a contradiction. The argument is completely analogous for the second part of the claim, with S replaced by R and assuming that $f(R)$ is an involution. □

Claim 3.2. *The element $f(R)$ is not central, and if $f(R)$ is a non-central involution, then $f(S)$ is not central while $f(T)$ is.*

Proof. If $f(R)$ is central, then it commutes with involutions $f(S)$ and $f(T)$ and since $(ST)^2 = 1$ the last two elements commute, so that the f-image of Γ would be Abelian, a contradiction. Let now $f(R)$ be a non-central involution. Clearly, one cannot have both $f(S)$ and $f(T)$ central. Suppose that $f(T)$ is non-central; Claim 3.1 then implies that $f(RT)$ is central. Now, depending on whether $f(S)$ is central or not (and in the second case $f(ST)$ is central, by Claim 3.1), the image $f(\Gamma)$ would be generated (say) by $f(R)$ and either by the pair of central elements $f(RT)$ and $f(S)$, or by the pair of central elements $f(RT)$ and $f(ST)$, giving a contradiction as above. Claim 3.2 follows. □

Claim 3.3. *Let $f(R) = r^{i+j} g$ and $f(S) = r^j h$ be non-central involutions for $g, h \in \{s, t\}$ and $i, j \in \{0, 1, \ldots, 2n-1\}$ mod $2n$. Then, $f(T)$ is a central involution satisfying one of the conditions (i) – (iii):*

(i) $f(T) \in \{st, r^n st\}$ *if n is even (independently on g, h), or if n is odd and $g = h$, in both cases with* $\gcd(i, 2n) = 1$,

(ii) $f(T) \in \{r^n, r^n st\}$ *if* n *is odd,* i *is even, and* $g \neq h$*, with* $\gcd(i, n) = 1$*,*

(iii) $f(T) \in \{r^n, st\}$ *if both* n *and* i *are odd, and still* $g \neq h$*, with* $\gcd(i, n) = 1$*.*

Proof. In Claim 3.2 we have shown that if $f(R)$ is a non-central involution, then $f(S)$ is non-central while $f(T)$ is central; let $f(R)$ and $f(S)$ be as above. Consider the element $f(RS) = f(R)f(S) = r^i gh$ for $gh \in \{1, st\} \in Z(D^*_{2n})$, and let H be the cyclic subgroup of G generated by $f(RS)$. Since conjugation by $f(S)$ inverts $f(RS)$, the group $H_1 = H\langle f(S)\rangle = \langle f(R), f(S)\rangle$ is dihedral, of order $2|H|$. If $f(T) \notin H_1$, adjoining $f(T)$ to H_1 results in a group H_2 of order $4|H|$. But f is assumed to be surjective, so H_2 must be the entire group $f(\Gamma)$, of order $8n$, which means that $|H| = 2n$. It follows that the order of $r^i gh$ must be equal to $2n$, and this is the case if and only if either n is even and $\gcd(i, 2n) = 1$ (with no restriction on $g, h \in \{s, t\}$), or n is odd and $\gcd(i, 2n) = 1$ for $g = h$, or else n is odd and $\gcd(i, n) = 1$ for $g \neq h$.

It remains to clarify circumstances under which the central element $f(T)$ is contained in H_1, as this means that f is not an epimorphism. By the above calculations it is sufficient to do this when the dihedral group H_1 has order $4n$, that is, when the generator $r^i gh$ of its cyclic part H has order $2n$. For the unique central element contained in H_1 out of $Z(D^*_{2n})\backslash\{1\} = \{r^n, st, r^n st\}$ we then have the following three possibilities, which we accompany with the corresponding gcd conditions listed above:

(i) $r^n \in H_1$ if n is even (independently on g, h), or if n is odd and $g = h$, in both cases with $\gcd(i, 2n) = 1$,

(ii) $st \in H_1$ if n is odd, i is even, and $g \neq h$, with $\gcd(i, n) = 1$,

(iii) $r^n st \in H_1$ if both n and i are odd, and still $g \neq h$, with $\gcd(i, n) = 1$.

Our arguments imply that (i) – (iii) are the only cases in which f is not an epimorphism. This completes the proof. □

Claim 3.4. *If* $f(R)$ *is not an involution, then* $f(T)$ *is not central and* $f(R)$ *has order* $2n$*. In more detail,* $f(T) = r^j h$ *for some* $h \in \{s, t\}$ *and* $j \in \{0, 1, \ldots, 2n-1\}$*, and one of the possibilities (i) – (iii) occur, for* $i \in \{1, 2, \ldots, 2n-1\}$ *as stated:*

(i) $f(R) = r^i$ *with* $\gcd(i, 2n) = 1$*, or* n *is even and* $f(R) = r^i st$ *with* $\gcd(i, n) = 1$*, and in both cases either* $f(S)$ *or* $f(ST)$ *(but not both) is arbitrarily chosen from* $\{st, r^n st\}$*,*

(ii) n *is odd,* $f(R) = r^i st$ *with* i *even and* $\gcd(i, n) = 1$*, and either* $f(S)$ *or* $f(ST)$ *(but not both) is an arbitrary central involution from* $\{r^n, r^n st\}$*,*

(iii) n *is odd,* $f(R) = r^i st$ *with* i *odd and* $\gcd(i, n) = 1$*, and either* $f(S)$ *or* $f(ST)$ *(but not both) is an arbitrary involution from* $\{r^n, r^n st\}$*.*

Proof. By Claim 3.2 the element $f(R)$ is not central, and if it is not involutory, then $f(R) \in \{r^i, r^i st\}$ for some $i \in \{0, 1, \ldots, 2n-1\}$. If T is a central involution of D^*_{2n}, from $(RT)^2 = 1$ it follows that $f(R)^2 = 1$ and so $f(R) \in Z(D^*_{2n})$, contrary to the first part of Claim 3.2. We thus may assume that $f(T) = r^j h$ for some $h \in \{s, t\}$ and $0 \le j \le 2n-1$.

The cyclic subgroup $H = \langle f(R)\rangle$ of $f(\Gamma)$ is normalized by the involution $f(T)$ and so the order of $H_1 = H\langle f(T)\rangle = \langle f(R), f(T)\rangle$ is $2|H|$. By Claim 3.1, exactly one element $U \in \{f(S), f(ST)\}$ is central. If $U \notin H_1$, then the product $H_1\langle U\rangle$ of order $4|H|$ must be the entire group $f(\Gamma)$ of order $8n$, so that $|H| = 2n$. It follows that the order of $f(R)$ is $2n$, which reduces to the conditions $\gcd(i, 2n) = 1$ if $f(R) = r^i$ and $\gcd(i, n) = 1$ if $f(R) = r^i st$.

The condition $U \notin H_1$ needed for f to be an epimorphism can be examined as done in Claim 3.3. The unique central involution in the (dihedral) group H_1 is r^n if $f(R) = r^i$, and again r^n if n is even and $f(R) = r^i st$. If n is odd and $f(R) = r^i st$, the unique central involution in H_1 is st if i is even, and $r^n st$ if i is odd. The rest of the argument is as in the proof of Claim 3.3. □

We are now ready to prove our result on counting epimorphisms, in terms of n and the Euler function $\varphi(n)$.

Proposition 3.5. *For $n > 1$ the number of epimorphisms $\Gamma \to D^*_{2n}$ is $96\, n\varphi(n)$ if n is even, and $72\, n\varphi(n)$ if n is odd.*

Proof. Let $f\colon \{R, S, T\} \to D^*_{2n}$ be an assignment of a triple of elements of D^*_{2n} to generators of Γ, and suppose that f extends to an epimorphism $\Gamma \to D^*_{2n}$. By Claim 3.2, the image $f(R)$ is not a central element of D^*_{2n}. It follows that $f(R)$ is either a non-central involution (and then so is $f(S)$ while $f(T)$ is a central involution, by Claim 3.2) or it has the form described in Claim 3.4.

Suppose f is such that $f(R)$ is a non-central involution. In all cases listed in parts (i) – (iii) of Claim 3.3, there are $2n \times 2$ possibilities to freely choose an exponent $j \in \{0, 1, \ldots, 2n-1\}$ and an element $h \in \{s, t\}$ in the image $f(S) = r^j h$.

If n is even, by the first part of Claim 3.3(i), apart from the $4n$ choices of $f(S)$, any such epimorphism is completely determined also by independent $\varphi(2n) \times 2$ choices of the exponent i mod $2n$ with $\gcd(i - j, 2n) = 1$ and the element $g \in \{s, t\}$ in $f(R) = r^i g$, and this can further be combined with 2 choices of $f(T)$. This gives a subtotal of $16\, n\varphi(2n) = 32\, n\varphi(n)$ epimorphisms in this branch.

If n is odd, the second part of Claim 3.3(i) gives epimorphisms f completely determined (apart from $4n$ choices of j and h in $f(S)$) by $\varphi(2n) = \varphi(n)$ choices of i mod $2n$ such that $\gcd(i, 2n) = 1$ in $f(R) = r^{i+j}h$ (note that now $g = h$), and by 2 choices of $f(T)$, giving $8n\varphi(n)$ possibilities. Following now the Claims 3.3(ii) and (iii), one obtains epimorphisms f completely determined by the $4n$ choices of $f(S)$ as above, combined with $2 \times \varphi(n)$ choices of i mod $2n$ such that $\gcd(i, n) = 1$ in the expression $f(R) = r^i g$ (accounting for both parities of i mod $2n$ for odd n and bearing in mind that $g \neq h$), and still combined with 2 choices of $f(T)$. This results in further $16\, n\varphi(n)$ possibilities and hence in a subtotal of $24\, n\varphi(n)$ epimorphisms in this branch.

Suppose now that the epimorphism f is such that $f(R)$ is not an involution. This situation is addressed by Claim 3.4, throughout which one has again $2n \times 2$ possibilities to freely choose an exponent $j \in \{0, 1, \ldots, 2n-1\}$ and an element $h \in \{s, t\}$ in the image $f(T) = r^j h$. As before, we will distinguish two cases, depending on the parity of n.

If n is even, the first part of Claim 3.4(i) tells us that besides the $4n$ choices of $f(T)$, any of our epimorphisms f is completely determined by independent $\varphi(2n) \times 2$ choices of the exponent i mod $2n$ with $\gcd(i, 2n) = 1$ in both $f(R) \in \{r^i, r^i st\}$, combined with 4 choices of $f(S)$ or $f(ST)$, giving a subtotal of $32\, n\varphi(2n) = 64\, n\varphi(n)$ epimorphisms in this branch of the argument.

If n is odd, the first part of Claim 3.4(i) still applies, contributing by $4n \times \varphi(2n) = 4n\varphi(n)$ choices of i and j together with 4 choices of $f(S)$ or $f(ST)$. From Claim 3.4(ii) and (iii), along with the $4n$ choices of $f(T)$ one has $2 \times \varphi(n)$ choices of i mod $2n$ such that $\gcd(i, n) = 1$ (again taking into the account both parities of i mod $2n$ and recalling that n is odd), combined with 4 choices of $f(S)$ or $f(ST)$. This gives further $32n\varphi(n)$ possibilities and hence in a subtotal of $48\, n\varphi(n)$ epimorphisms in this branch.

It may be checked that all the discussed possibilities of assigning elements of D^*_{2n} to the generators R, S and T of Γ by f indeed extend to epimorphisms $\Gamma \to D^*_{2n}$. This gives the grand total of $96\, n\varphi(n)$ epimorphisms if n is even, and $72\, n\varphi(n)$ epimorphisms if n is odd and grater than 1, as claimed. □

As we shall see, counting automorphisms of the extended dihedral group given by (2.4), that is, $D^*_{2n} = \langle r, s, t \mid r^{2n}, s^2, t^2, (rs)^2, (rt)^2, (st)^2\rangle$, is much easier.

Proposition 3.6. *For $n > 1$ the number of automorphisms of the group D^*_{2n} is $32\, n\varphi(n)$ if n is even, and $24\, n\varphi(n)$ if n is odd.*

Proof. In what follows we *assume* that f is an arbitrary automorphism of D^*_{2n}. Since $Z(D^*_{2n}) = \{1, r^n, st, r^n st\}$, it is of advantage to consider D^*_{2n} to be generated by r, s and st, because f must preserve the centre, and in particular $f(st) \in \{r^n, st, r^n st\}$. Further, the f-image of r must be an element of order $2n$ and so either $f(r) = r^i$ for some i mod $2n$ such that $\gcd(i, 2n) = 1$, or $f(r) = r^i st$ for some i mod $2n$ such that $\gcd(i, n) = 1$. Also, the f-image of s must be a non-central involution, that is, $f(s) = r^j h$ for some $j \in \{0, 1, \ldots, 2n-1\}$ and $h \in \{s, t\}$. As there is an obvious *swap automorphism* of D^*_{2n} fixing r and swapping s with t, up to this automorphism we may (and will) assume that $f(s) = r^j s$. Keeping to this assumption throughout, we will distinguish several cases.

Case 1. Suppose $f(r) = r^i$, where $\gcd(i, 2n) = 1$; in particular, such an i mod $2n$ must be odd. Then, $f(r^n) = r^{in} = r^n$, so that $f(st) \in \{st, r^n st\}$, and any such f is completely determined by one of the $\varphi(2n)$ choices of i, one of the $2n$ choices of j in $f(s) = r^j s$, and one of the two choices of $f(st)$. It follows that, taking into the account the swap automorphism as well, there is a subtotal of $\varphi(2n) \times 2n \times 2 \times 2 = 8\, n\varphi(2n)$ choices of f such that $f(r)$ is a power of r; note that this applies to n of both parities.

Case 2. We continue with the situation when $f(r) = r^i st$, $\gcd(i, n) = 1$ and n is even. The condition $\gcd(i, n) = 1$ is now equivalent to $\gcd(i, 2n) = 1$ and then again i mod $2n$ must be odd and $f(r^n) = r^n$. By the same token as above, for even n we have a further subtotal of $8\, n\varphi(2n)$ automorphisms f such that $f(r) = r^i st$.

Case 3. It remains to consider the possibility $f(r) = r^i st$ for n odd. Then, $f(r^n) = r^{in}(st)^n = r^{in} st$, which is equal to st if i mod $2n$ is even, and to $r^n st$ if i mod $2n$ is odd; in both cases $\gcd(i, n) = 1$. Since for odd n there are $\varphi(n)$ choices if i mod $2n$ with i even, for any such choice and any of the $2n$ choices of j in $f(s) = r^j s$ and any choice of $f(st)$ distinct from the value $f(r^n) = st$, that is, for $f(st) \in \{r^n, r^n st\}$, there are $\varphi(n) \times 2n \times 2 \times 2 = 8\, n\varphi(n)$ choices for f encountered so far in this paragraph (taking into the account the swapping automorphism). The same counting applies to the case when i mod $2n$ is odd, with the only exception that now $f(st)$ has to be distinct from $f(r^n) = r^n st$, that is, here the choices are $f(st) \in \{st, r^n\}$, giving further $8\, n\varphi(n)$ choices for f. For n odd and $f(r)$ of the form $r^i st$ we thus obtained a subtotal of $16\, n\varphi(n)$ choices.

Summing up, for even n the above analysis resulted in a total of $16\,n\varphi(2n) = 32\,n\varphi(n)$ choices for f, while for odd $n > 1$ we obtain a total of $8\,n\varphi(2n) + 16\,n\varphi(n) = 24\,n\varphi(n)$ choices of f. It can be checked that any of these choices indeed define an automorphism of D^*_{2n}, completing the proof. □

Propositions 3.5 and 3.6 have the following immediate consequence for the formula (3.1).

Corollary 3.7. *The epimorphisms $\Gamma \to D^*_{2n}$ have exactly three distinct kernels.*

One of the referees suggested to give an alternative proof to Corollary 3.7 in terms of inequivalent presentations of the extended dihedral group D^*_{2n} of the form (2.2), that is, $\langle x, y, z \mid x^2, y^2, z^2, (xz)^2, (zy)^d, (yx)^\ell, \ldots\rangle$. Here and in what follows, two such presentation of D^*_{2n} in terms of generating triples (x, y, z) and (x', y', z') are equivalent if there is an automorphism f of D^*_{2n} taking the ordered triple (x, y, z) to (x', y', z'). The fact that the number of distinct kernels of epimorphisms $\Gamma \to D^*_{2n}$ is the same as the number of inequivalent presentations of D^*_{2n} as in (2.2) follows from the fact that two epimorphisms $\Gamma \to D^*_{2n}$ have identical kernels if and only if the epimorphisms differ by an automorphism of D^*_{2n}.

For such an alternative proof we begin by determining the number of inequivalent presentations of D^*_{2n} of the form (2.4), that is,

$$\langle r, s, t \mid r^{2n}, s^2, t^2, (rs)^2, (rt)^2, (st)^2\rangle \cong \langle r\rangle \rtimes \langle s, t\rangle \cong C_{2n} \rtimes (C_2 \times C_2) \qquad (3.3)$$

where equivalence of presentations is defined analogously as above with respect to automorphisms of D^*_{2n}. This will be preceded by determining the f-images of the generating triple (r, s, t) of the presentation (3.3) for all the automorphism f of D^*_{2n} determined in the proof of Proposition 3.6.

Proposition 3.8. *Images of the generating triple (r, s, t) for the presentation (3.3) of the group D^*_{2n} for $n > 1$ under automorphisms f of this group, together with their numbers, are listed in the following two tables (where it is assumed that $j \in \{0, 1, \ldots, 2n-1\}$ is arbitrary):*

Even n		
$(f(r), f(s), f(t))$	Conditions	# (up to $s \leftrightarrow t$)
$(r^i, r^j s, r^j t)$	$0 < i < 2n$, $\gcd(i, 2n) = 1$	$\varphi(2n) \times 2n \times 2$
$(r^i, r^j s, r^{j+n} t)$	$0 < i < 2n$, $\gcd(i, 2n) = 1$	$\varphi(2n) \times 2n \times 2$
$(r^i st, r^j s, r^{j+n} t)$	$0 < i < 2n$, $\gcd(i, 2n) = 1$	$\varphi(2n) \times 2n \times 2$
$(r^i st, r^j s, r^j t)$	$0 < i < 2n$, $\gcd(i, 2n) = 1$	$\varphi(2n) \times 2n \times 2$

Odd $n > 1$		
$(f(r), f(s), f(t))$	Conditions	# (up to $s \leftrightarrow t$)
$(r^i, r^j s, r^j t)$	$0 < i < 2n$, $\gcd(i, 2n) = 1$	$\varphi(n) \times 2n \times 2$
$(r^i, r^j s, r^{j+n} t)$	$0 < i < 2n$, $\gcd(i, 2n) = 1$	$\varphi(n) \times 2n \times 2$
$(r^i st, r^j s, r^{j+n} s)$	$0 < i < 2n$, $\gcd(i, n) = 1$	$2 \times \varphi(n) \times 2n \times 2$
$(r^i st, r^j s, r^{j+n} t)$	i even, $0 < i < 2n$, $\gcd(i, n) = 1$	$\varphi(n) \times 2n \times 2$
$(r^i st, r^j s, r^j t)$	i odd, $0 < i < 2n$, $\gcd(i, n) = 1$	$\varphi(n) \times 2n \times 2$

Proof. We will follow the proof of Proposition 3.6, in which we assumed without loss of generality (up to the swapping automorphism $s \leftrightarrow t$) that $f(s) = r^j s$ for an arbitrary, but fixed $j \in \{0, 1, \dots, 2n-1\}$.

In the Case 1 the automorphism f was given by $f(r) = r^i$ for $0 < i < 2n$ with $\gcd(i, 2n) = 1$, $f(s) = r^j$, subject to $f(st) \in \{st, r^n st\}$. If $f(st) = st$, then from $t = sf(st) = sf(s)f(t) = sr^j sf(t) = r^{-j} f(t)$ it follows that $f(t) = r^j t$. The second possibility $f(st) = r^n st$ similarly yields $f(t) = r^{j+n}t$, and the two values of $f(t)$ give the first two lines of both tables, irrespective of parity of n.

The Case 2 of the proof of Proposition 3.6 dealt with f satisfying $f(r) = r^i st$ for even $n \in \{2, \dots, 2n-2\}$ such that $\gcd(i, 2n) = 1$, $f(s) = r^j$ and $f(st) \in \{st, r^n st\}$. The last condition implies as in the Case 1 that $f(t) \in \{r^j t, r^{j+n} t\}$, giving the two presentations in the last two lines of the first table.

Finally, in the Case 3, f was given by $f(r) = r^i st$ and $\gcd(i, n) = 1$, $0 < i < 2n$, but this time for n odd, with $f(st) \in \{r^n, r^n st\}$ if i is even while $f(st) \in \{r^n, st\}$ if i is odd, and with $f(s) = r^j s$ throughout. Evaluating $f(t)$ as in the previous cases one obtains $f(t) \in \{r^{j+n}s, r^{j+n}t\}$ for even i, and $f(t) \in \{r^{j+n}s, r^j t\}$ for odd i, and these imply the entries in the last three rows of the second table.

The counts for the presentations in the tables follow from the proof of Proposition 3.6. □

This enables us to state and prove a useful uniqueness consequence for presentations of generalized dihedral groups of type (3.3).

Corollary 3.9. *Up to equivalence induced by automorphisms, for $n > 1$ the group D^*_{2n} has a unique presentation of the form* (3.3).

Proof. Let $\langle \rho, \sigma, \tau \mid \rho^{2n}, \sigma^2, \tau^2, (\rho\sigma)^2, (\rho\tau)^2, (\sigma\tau)^2 \rangle$ be a presentation of D^*_{2n} of the form (3.3), with $D^*_{2n} \cong \langle \rho \rangle \rtimes \langle \sigma, \tau \rangle \cong C_{2n} \rtimes (C_2 \times C_2)$. The group is a union of the four cosets $\langle r \rangle$, $\langle r \rangle s$, $\langle r \rangle t$ and $\langle r \rangle st$. The only elements of order $2n$ that can be chosen for ρ are r^i for $\gcd(i, 2n) = 1$ and $r^i st$ for $\gcd(i, n) = 1$, and the only involutions for $n > 1$ in D^*_{2n} are the three non-trivial central elements r^n, st and $r^n st$ together with the $4n$ elements in the cosets $\langle r \rangle s$ and $\langle r \rangle t$. Out of the latter $4n$ involutions, up to the automorphism $s \leftrightarrow t$ those that generate a subgroup isomorphic to $C_2 \times C_2$ are $\langle r^j s, r^{j+n} s \rangle$, $\langle r^j s, r^j t \rangle$ and $\langle r^j s, r^{j+n} t \rangle$ for $j \in \{0, 1, \dots, 2n-1\}$; note that products of generators in each of these three cases give the non-trivial central elements. Since none of σ, τ can be a central element, it follows that the choices for the *set* $\{\sigma, \tau\}$ are $\{r^j s, r^{j+n} s\}$, $\{r^j s, r^j t\}$ and $\{r^j s, r^{j+n} t\}$, $0 \le j \le 2n-1$.

A routine checking shows that in order for one of these three types of subgroups isomorphic to $C_2 \times C_2$ to *generate* the entire group D^*_{2n} by adjoining an element ρ of order $2n$ as listed above, one arrives at the same conditions for i that appear in the two tables in Proposition 3.8. But the presentations from Proposition 3.8 are mutually equivalent, and our claim follows. □

We are now ready to prove a statement about the number of inequivalent presentations of type (2.2) of the generalized dihedral group by three involutions, two of which commute.

Proposition 3.10. *For $n > 1$, presentations of D^*_{2n} of the form* (2.2)*, that is, given by $\langle x, y, z \mid x^2, y^2, z^2, (xz)^2, (zy)^d, (yx)^\ell, \dots \rangle$, fall into three equivalence classes with respect to the action of the automorphism group of D^*_{2n}.*

Proof. In a presentation as in the statement the subgroup $\langle x, z\rangle$ is isomorphic to $C_2 \times C_2$. But from the proof of Proposition 3.8 and Corollary 3.9 one sees that all such subgroups are, up to automorphisms of D^*_{2n}, equivalent to the subgroup $\langle s, t\rangle$. Consequently, one may assume that $\langle x, z\rangle = \langle s, t\rangle$ and hence, up to the swapping automorphism $s \leftrightarrow t$, the ordered pair (x, z) may be assumed to coincide with one of the three ordered pairs (st, t), (t, st) and (s, t). The involution y cannot be central and hence must have the form $r^j s$ or $r^j t$ for some $j \in \{1, \dots, 2n-1\}$, but invoking the automorphisms from the proof of Proposition 3.6 one is free to choose $j = 1$, so that $y \in \{rs, rt\}$. Finally, one may apply the automorphism from the last line of both tables of Proposition 3.8 for $i = 1$ and $j = 0$, sending r to rst and fixing s and t (and hence interchanging rs with rt). This shows that one may choose $y = rt$, so that the three representatives (x, y, z) of our equivalence classes may be taken to be the triples (st, rt, t), (t, rt, st) and (s, rt, t). □

In Section 2 we have singled out the map $M_{2n} = \mathrm{Map}(G; x, y, z) = \mathrm{Map}(st, rt, t)$ for $G = D^*_{2n}$ formed by a dipole of $2n$ edges embedded on a sphere (Figure 1). Observe that for the dual and the Petrie dual of M, also introduced in Section 2, one has $D(M_{2n}) = \mathrm{Map}(G; x_D, y_D, z_D)$ for $(x_D, y_D, z_D) = (t, rt, st)$ and $P(M_{2n}) = \mathrm{Map}(G; x_P, y_P, z_P)$ for $(x_P, y_P, z_P) = (s, rt, t)$. Proposition 3.10 therefore allows for a restatement as follows.

Corollary 3.11. *For $n > 1$, the representative triples of equivalence classes of presentations of D^*_{2n} from* Proposition 3.10 *may be taken to be those defining the map M_{2n} of a dipole from* Figure 1*, its dual and its Petrie dual.*

4 Parallel product of maps from dihedral groups

By the theory outlined in the Introduction and by Corollary 3.7, the three distinct kernels induced by the family of all epimorphisms $\Gamma \to D^*_{2n}$ correspond to a choice of three pairwise non-isomorphic regular maps arising from M_{2n} by applying the operators duality, Petrie duality and rotational powers (and combinations of these). The most obvious choice here appears to be to take the triple of map M_{2n}, $P(M_{2n})$ and $D(M_{2n})$, which are pairwise non-isomorphic if $n > 1$; this is now also explicitly backed by Corollary 3.11. Invoking the summary in the Introduction again, the parallel product of these three regular maps is isomorphic to the regular map determined by the group Γ/K, where K is the intersection of the three distinct kernels referred to in Corollary 3.7.

Recall that for the map $M_{2n} = \mathrm{Map}(G; x, y, z)$ for $(x, y, z) = (st, rt, t)$, its Petrie dual and its dual are given by $P(M_{2n}) = \mathrm{Map}(G; x_P, y_P, z_P)$ for $(x_P, y_P, z_P) = (s, rt, t)$, and $D(M_{2n}) = \mathrm{Map}(G; x_D, y_D, z_D)$ for $(x_D, y_D, z_D) = (t, rt, st)$; there is a good reason why we choose this order of the three maps and we comment on this in due place. The parallel product $M_{2n} \times P(M_{2n}) \times D(M_{2n})$ of the regular maps defined by the subgroup of $G \times G \times G$ for $G = D^*_{2n} = \langle x, y, z\rangle$ is uniquely determined by the subgroup H of $G \times G \times G$ generated by the involutions $(x, x_P, x_D) = (st, s, t)$, $(y, y_P, y_D) = (rt, rt, rt)$ and $(z, z_P, z_D) = (t, t, st)$.

Thus, letting a, b and c be elements of $G \times G \times G$ defined by $a = (st, s, t)$, $b = (rt, rt, rt)$ and $c = (t, t, st)$, our aim is to determine the subgroup $H = \langle a, b, c\rangle$ of $G \times G \times G$, and this way the entire regular map $\mathrm{Map}(H) = \mathrm{Map}(a, b, c)$.

The group H turns out to have a large Abelian subgroup. To see this, let $u = (bc)^2$, $v = (ab)^2$ and $w = a(uv)a$; an evaluation of these gives $u = (bc)^2 = (r^2, r^2, 1)$, $v = (ab)^2 = (1, r^{-2}, r^{-2})$ and $w = a(uv)a = (r^2, 1, r^2)$. Since r is of order $2n$, the order of all the (mutually commuting) elements u, v and w is n. It follows that the group $L = \langle u, v, w\rangle$ is a subgroup of the product of cyclic groups $C_n \times C_n \times C_n$. Since $uvw^{-1} = (1, 1, r^{-4})$, it follows that $L = \langle u, v, w\rangle = \langle u, v, uvw^{-1}\rangle$.

The elements u, v and uvw^{-1} have orders n, n and $2n/\gcd(2n, 4) = n/\gcd(n, 2)$, respectively; the latter is equal to n if n is odd, and to $n/2$ if n is even. From the fact that the first coordinate of v and the first two coordinates of uvw^{-1} are equal to 1 it follows that $|L| = n^3$ if n is odd (and greater than 1), while $|L| = n^3/2$ if n is even. (A cautious reader will have realized that the ordering $M_{2n} \times P(M_{2n}) \times D(M_{2n})$ in the parallel product enabled us to determine L in such a quick way.) In fact, L can be written in the form

$$L = \left\{ \left(r^{2(\alpha_1+\alpha_3)}, r^{2(\alpha_1+\alpha_2)}, r^{2(\alpha_2+\alpha_3)} \right) \mid \alpha_1, \alpha_2, \alpha_3 \in \{0, 1, \dots, n-1\} \right\}.$$

By evaluating conjugates of u, v and w by the involutions a, b and c we obtain

$$\begin{array}{lll} aua = vw & bub = v^{-1} & cuc = u^{-1} \\ ava = v^{-1} & bvb = v^{-1} & cvc = uw^{-1} \\ awa = uv & bwb = w^{-1} & cwc = u^{-1}v^{-1} \end{array}$$

thus we can see that L is a normal subgroup of H. In the quotient H/L the involutions

$$\begin{aligned} aL &= \left\{ \left(str^{2(\alpha_1+\alpha_3)}, sr^{2(\alpha_1+\alpha_2)}, tr^{2(\alpha_2+\alpha_3)}\right) \mid \alpha_1, \alpha_2, \alpha_3 \in \{0, 1, \dots, n-1\} \right\}, \\ bL &= \left\{ \left(tr^{2(\alpha_1+\alpha_3)-1}, tr^{2(\alpha_1+\alpha_2)-1}, tr^{2(\alpha_2+\alpha_3)-1}\right) \mid \alpha_1, \alpha_2, \alpha_3 \in \{0, 1, \dots, n-1\} \right\}, \\ cL &= \left\{ \left(tr^{2(\alpha_1+\alpha_3)}, tr^{2(\alpha_1+\alpha_2)}, str^{2(\alpha_2+\alpha_3)}\right) \mid \alpha_1, \alpha_2, \alpha_3 \in \{0, 1, \dots, n-1\} \right\} \end{aligned}$$

and $(ab)L$, $(ac)L$, $(bc)L$, $(abc)L$ turn out to be distinct and mutually commuting. It follows that $H/L \cong C_2 \times C_2 \times C_2$, which means that $|H| = 8|L|$, and hence $|H| = 8n^3$ if n is odd and greater than 1, and $|H| = 4n^3$ if n is even.

It follows that if n is odd and $n > 1$, then the full automorphism group H of the regular map arising from the parallel product $M_{2n} \times P(M_{2n}) \times D(M_{2n})$ has order $8n^3$ and a presentation of the form

$$H = \langle a, b, c \mid a^2, b^2, c^2, (ac)^2, (ab)^{2n}, (bc)^{2n}, [(ab)^2, (bc)^2]\rangle \tag{4.1}$$

which is obtained from the above analysis by using $u = (bc)^2$, $v = (ab)^2$ and the obvious relations $u^n = v^n = [u, v] = 1$. Observe that a consequence of $(uv)^n = 1$ is the relation $(cba)^{2n} = 1$. This all implies that if $n > 1$ is odd then the parallel product $M_{2n} \times P(M_{2n}) \times D(M_{2n})$ is isomorphic to the regular map determined by the Wilson group W_n with presentation (1.3); this also shows that the relator $(zyx)^{2n}$ in Wilson's presentation (1.3) is redundant.

If n is even, however, then our analysis implies that the full automorphism group H of the regular map arising from the parallel product $M_{2n} \times P(M_{2n}) \times D(M_{2n})$ has order only $4n^3$ and admits a presentation of the form

$$H = \langle a, b, c \mid a^2, b^2, c^2, (ac)^2, (ab)^{2n}, (bc)^{2n}, [(ab)^2, (bc)^2], (a(bc)^n)^2\rangle \tag{4.2}$$

the last relator being equivalent to the relation $(uvw^{-1})^{n/2} = 1$ simplified with the help of the fact that both $u = (bc)^2$ and $v = (ab)^2$ commute with both w and $(acb)^2$, which are all easy consequences of the above findings. It follows that if n is even, the group from (4.2) is a quotient of the Wilson group (1.3), or, equivalently, the group defined by (4.1) but for n even, by the normal subgroup of order 2 generated by the involution $a(bc)^n$.

We collect the facts accumulated above in the following form, using the notation introduced throughout.

Theorem 4.1. *Let $n > 1$ and let M be the fully regular orientable map arising from the parallel product $M_{2n} \times P(M_{2n}) \times D(M_{2n})$. If n is odd, the full automorphism group of M has order $8n^3$ and admits a presentation (4.1); this group is isomorphic to the Wilson group given by (1.3). If n is even, then the full automorphism group of M has order $4n^3$ and has a presentation of the form (4.2); the group is in this case a quotient of the Wilson group (1.3) by a normal subgroup of order 2. In both cases, M is super-symmetric.*

Referring back to the Introduction and to the motivation of this paper, Theorem 4.1 implies that the remark made by Jones in [3] on isomorphism of the group $W_n \cong \Gamma/\Gamma''(\Gamma')^{(n)}$ with the group Γ/K is valid only for odd $n > 1$; for even n one has $\Gamma/K \cong W_n/C_2$ for a normal subgroup of W_n of order 2.

Super-symmetry of the parallel product M from Theorem 4.1 also follows from the facts that the original M_{2n} is kaleidoscopic, and so is the Petrie-dual $P(M_{2n})$ because the operator of Petrie-duality commutes with rotational powers, and the dual of M_{2n} is kaleidoscopic for trivial reasons. Nevertheless, our goal was to follow and verify the theory outlined in Jones' paper [3].

5 The special case $n = 1$, and a concluding remark

Throughout the paper we assumed that $n > 1$. For completness, in this section we provide analogous results as above for the case $n = 1$. In this case, the starting group from (2.4) has the form

$$D_2^* = \langle r, s, t \mid r^2, s^2, t^2, (rs)^2, (rt)^2, (st)^2 \rangle \cong C_2 \times C_2 \times C_2. \tag{5.1}$$

The corresponding map M_2 is a dipole, in which the two vertices are joined by two parallel edges. The dual $D(M_2)$ and the Petrie-dual $P(M_2)$ (and also $DP(M_2)$, $PD(M_2)$ and $DPD(M_2)$) are isomorphic to M_2, proving that the map M_2 is super-symmetric. On the other hand, the parallel product $M_2 \times P(M_2) \times D(M_2)$ is not so interesting, because it is giving a regular map whose automorphism group is isomorphic to the starting group D_2^*.

The number of epimorphisms f from the group Γ given in (1.2) onto D_2^* is equal to 168, since $f(X)$ can be chosen as one of the 7 non-identity elements (involutions, in fact) in D_2^*, after that $f(Y)$ can be chosen from the remaining 6 involutions. Finally, $f(Z) \notin \{f(X), f(Y), f(XY)\}$, hence for $f(Z)$ we have 4 choices. It is well-known that $\mathrm{Aut}(C_2 \times C_2 \times C_2) \cong \mathrm{GL}_3(2)$, thus $|\mathrm{Aut}(D_2^*)| = |\mathrm{Aut}(C_2 \times C_2 \times C_2)| = 168$. This means that the number of distinct kernels is $|\mathrm{Epi}(\Gamma \to D_2^*)|/|\mathrm{Aut}(D_2^*)| = 1$, which is consistent with super-symmetry of M_2.

Remark 5.1. We reiterate that if our goal was just to construct super-symmetric maps from generalized dihedral groups D_{2n}^* by parallel products, without putting under scrutiny the remark by Jones made in Section 9 of [3], it would have been sufficient to restrict to a determination of equivalence classes of the corresponding generating triples of D_{2n}^* under the action of the automorphism group $\mathrm{Aut}(D_{2n}^*)$ of D_{2n}^*, which was done for completeness in Proposition 3.10 and Corollary 3.11.

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The 2-rainbow domination number of Cartesian product of cycles

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Abstract

A k*-rainbow dominating function* (kRDF) of G is a function that assigns subsets of $\{1, 2, ..., k\}$ to the vertices of G such that for vertices v with $f(v) = \emptyset$ we have $\bigcup_{u \in N(v)} f(u) = \{1, 2, ..., k\}$. The *weight* $w(f)$ of a kRDF f is defined as $w(f) = \sum_{v \in V(G)} |f(v)|$. The minimum weight of a kRDF of G is called the k*-rainbow domination number* of G, which is denoted by $\gamma_{rk}(G)$. In this paper, we study the 2-rainbow domination number of the Cartesian product of two cycles. Exact values are given for a number of infinite families and we prove lower and upper bounds for all other cases.

Keywords: *2-rainbow domination, domination number, Cartesian product.*

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1 Introduction

The Cartesian product is one of the standard graph products [13]. For example, meshes, tori, hypercubes and some of their generalizations are Cartesian products.

Graph domination is one of the most popular topics in graph theory [15, 16, 17]. There are many variants motivated by interesting applications. The k-rainbow domination problem was first studied in [2] and has attracted a lot of attention. For example, in [3], the authors proved that the concept of 2-rainbow domination is equivalent to ordinary domination

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in the prism $G\Box K_2$ and established the NP-completeness of determining whether a graph has a 2-rainbow dominating function with a certain weight. Furthermore, in [4] the authors characterize the pairs of graphs G and H for which $\gamma(G\Box H) = \min\{V(G), V(H)\}$. There are also many papers that observe 2-rainbow domination on generalized Petersen graphs, for example [5, 9, 32, 33]. In recent years, research on the 2-rainbow domination and its variants has expanded even further. For example, in [22] the k-rainbow domination on regular graphs was investigated. Meybodi et al. [23] investigated k-rainbow domination in graphs with bounded tree-width. In [18] Kim investigated k-rainbow domination in middle graphs in the context of operations research. In [6] an independent variant of k-rainbow domination on the lexicographic products of graphs was investigated. Recently, Kosari and Asgharsharghi [21] studied the l-distance k-rainbow domination numbers of graphs. For further references, see [1].

In this paper we study 2-rainbow domination numbers of the Cartesian product of two cycles. We provide exact values for a number of infinite families and prove lower and upper bounds for all other cases.

Our main results are summarized in the following two theorems.

For $n \equiv 0 \pmod 6$ the first theorem gives exact values of $\gamma_{r2}(C_m\Box C_n)$ for $m \equiv 0, 2 \pmod 3$ and bounds with gap at most $\frac{1}{2}n$ for the case $m \equiv 1 \pmod 3$.

Theorem 1.1. *Let $m \geq 3$ and $n \geq 6$, $n \equiv 0 \pmod 6$. Then we have*

(a) *if $m \equiv 0 \pmod 3$ then $\gamma_{r2}(C_m\Box C_n) = \dfrac{m}{3}n$.*

(b) *if $m \equiv 1 \pmod 3$ then*

$$\left(\frac{m-1}{3} + \frac{1}{2}\right)n \leq \gamma_{r2}(C_m\Box C_n) \leq \frac{m+2}{3}n\,.$$

(c) *if $m \equiv 2 \pmod 3$ then $\gamma_{r2}(C_m\Box C_n) = \dfrac{m+1}{3}n$.*

The second theorem is a summary of the lower and upper bounds of the products of cycles, covering all cases. Note that the gap is at most $\frac{1}{2}n + 2\left\lceil\frac{m}{3}\right\rceil$.

Theorem 1.2. *Let $m \geq 3$ and $n \geq 6$. Then*

$$\left(\left\lfloor\frac{m}{3}\right\rfloor + \alpha\right)n \leq \gamma_{r2}(C_m\Box C_n) \leq \min\left\{\left\lceil\frac{m}{3}\right\rceil(n+\beta), \left\lceil\frac{n}{3}\right\rceil(m+\gamma)\right\},$$

where

$$\alpha = \begin{cases} 0, & m \equiv 0 \pmod 3 \\ \frac{1}{2} & m \equiv 1 \pmod 3 \\ 1, & m \equiv 2 \pmod 3 \end{cases}, \qquad \beta = \begin{cases} 0, & n \equiv 0 \pmod 6 \\ 1, & n \equiv 1,2,3,5 \pmod 6 \\ 2, & n \equiv 4 \pmod 6 \end{cases},$$

$$\textit{and}\quad \gamma = \begin{cases} 0, & n \equiv 0 \pmod 6 \\ 1, & n \equiv 5 \pmod 6 \\ 2, & n \equiv 1,2,3,4 \pmod 6. \end{cases}$$

The upper bounds are given in alternative form as Corollary 4.9.

The rest of the paper is organized as follows. In the next section we recall some basic definitions and some useful previously known results. In Section 3 we prove lower bounds. In Section 4, we study two patterns that allow constructions that yield upper bounds. The final section contains a number of ideas for future research.

2 Preliminaries

A finite, simple and undirected graph $G = (V(G), E(G))$ is given by a set of vertices $V(G)$ and a set of edges $E(G)$. As usual, the edges $\{i, j\} \in E(G)$ are shortly denoted by ij.

A set S is a dominating set if every vertex in the complement $V(G) \setminus S$ is adjacent to a vertex in S. The minimum cardinality of a dominating set of G is called the domination number $\gamma(G)$.

The Cartesian product of two graphs, $G\Box H$, is the graph with vertex set $V(G)\times V(H)$, in which two vertices are adjacent if and only if they are equal in one coordinate and adjacent in the other. The Cartesian product of graphs is one of the standard graph products [13]. The Cartesian product is commutative. In other words: $C_m\Box C_n$ is isomorphic to $C_n\Box C_m$. So if we consider the product of the cycles $C_m\Box C_n$, we can assume $m \leq n$.

For a given vertex $v \in V(G)$, the open neighborhood $N(v)$ consists of the vertices adjacent to v. The degree of vertex v equals $\deg_G(v) = |N(v)|$. The minimum and the maximum degree of a graph G are denoted by $\delta(G)$ and $\Delta(G)$.

Let f be a function that assigns to each vertex a set of colors chosen from the set $\{1, 2, \ldots, k\} = [k]$, with the property that for each $v \in V(G)$ with $f(v) = \emptyset$ we have

$$\bigcup_{u\in N(v)} f(u) = [k].$$

Such a function f is called a k-rainbow dominating function (kRDF) of G. The weight of f, denoted by $w(f)$, is defined as

$$w(f) = \sum_{v\in V(G)} |f(v)|.$$

Recall that $f(v)$ is a set of colors and $|f(v)|$ denotes the number of elements in $f(v)$. The minimum weight of a kRDF on G is called the $k-$rainbow domination number of G, $\gamma_{rk}(G)$ and in this case the function is called $\gamma_{rk}(G)$-*function*. It is clear that for $k = 1$ this definition corresponds to the usual domination.

The following theorems, which connect rainbow domination with (ordinary) domination, will be of interest here.

Theorem 2.1 ([2])**.** *For any graph G we have $\gamma_{rk}(G) = \gamma(G\Box K_k)$.*

Theorem 2.2 ([14])**.** *For any graph G we have $\gamma_{rk}(G) \leq k\gamma(G)$.*

In [19], it was shown that $\gamma(C_3\Box C_n) = n - \lfloor \frac{n}{4} \rfloor$, $\gamma(C_4\Box C_n) = n$ for $n \geq 4$ and

$$\gamma(C_5\Box C_n) = \begin{cases} n, & n \equiv 0 \pmod 5 \\ n+1, & n \equiv 1,2,4 \pmod 5. \end{cases}$$

This result was supplemented in [8], where it was shown that $\gamma(C_5 \Box C_n) = n + 2$ for $n \equiv 3 \pmod 5$ and also exact values for $\gamma(C_6 \Box C_n)$ and $\gamma(C_7 \Box C_n)$ were given. In [7] it was proved that $\lceil \frac{9n}{5} \rceil \leq \gamma(C_8 \Box C_n) \leq \lceil \frac{9n}{5} \rceil + 1$ for $n \geq 8$ and exact value for $\gamma(C_9 \Box C_n)$ was given.

Considering 2-rainbow domination number of the Cartesian product of two cycles, the well-known inequality is (see [29])

$$\frac{mn}{3} \leq \gamma_{r2}(C_m \Box C_n) \leq 2\gamma(C_m \Box C_n).$$

The 2-rainbow domination number of the products $C_3 \Box C_n$ and $C_5 \Box C_n$ were studied in [29, 30]. In [31] a complete characterization of graphs $C_m \Box C_n$ was given, for which the 2-rainbow domination number is equal to $\frac{mn}{3}$. A summary of the then known results on the k-rainbow domination of the Cartesian product of cycles appears in [12]. In Table 1 we recall the previously known formulas for 2-rainbow domination numbers for $C_m \Box C_n$.

Result	Ref.
$\gamma_{r2}(C_3 \Box C_n) = \begin{cases} n, & n \equiv 0 \pmod 6 \\ n+1, & n \equiv 1,2,3,5 \pmod 6 \\ n+2, & n \equiv 4 \pmod 6 \end{cases}$	[29]
$\gamma_{r2}(C_4 \Box C_n) = \begin{cases} \lfloor \frac{3n}{2} \rfloor, & n \equiv 0 \pmod 8 \\ \lfloor \frac{3n}{2} \rfloor + 1, & n \equiv 2,4,5 \pmod 8 \\ \lfloor \frac{3n}{2} \rfloor + 2, & n \equiv 1,3,6,7 \pmod 8 \end{cases}$	[27]
$\gamma_{r2}(C_5 \Box C_n) = 2n$	[30]
$\gamma_{r2}(C_8 \Box C_n) = 3n$	[27]
$\gamma_{r2}(C_m \Box C_n) = \frac{mn}{3}$, if and only if either $m \equiv 0 \pmod 3, n \equiv 0 \pmod 6$ or $m \equiv 0 \pmod 6, n \equiv 0 \pmod 3$	[31]

Table 1: A summary of previously known results.

3 Lower bounds for 2-rainbow domination of $C_m \Box C_n$

For simplicity, we introduce some more notations. The vertices of $V(C_m \Box C_n)$ are denoted by (i, j) for $i \in [m]$ and $j \in [n]$. The coordinates i and j are taken modulo m and n respectively, so that we identify m and 0, for example. For a fixed (small) m, the set of vertices is $\mathcal{C}^i = \{(i, 1), (i, 2), \cdots, (i, n)\}$, $i \in [m]$ is called the i-th column of $C_m \Box C_n$.

Let f be a 2RDF of $C_m \Box C_n$ and $s_i = \sum_{x \in \mathcal{C}^i} |f(x)|$. The sequence $(s_1, s_2, \cdots, s_m)$, is called the 2RDF sequence that corresponds to f. We also use $f(i, j) = f(v)$ to denote the value of f at vertex $v = (i, j)$ for $i \in [m]$ and $j \in [n]$.

First, we recall a general bound for regular graphs. We believe that it is well known, although we have not found a reference with a proof. Therefore, for the sake of completeness, we provide a short proof.

Lemma 3.1. *Let G be an r-regular graph. Then* $\gamma_{rk}(G) \geq \dfrac{k}{r+k}|V(G)|$.

Proof. Assume that f is a kRDF and that n^* vertices are colored. Then double count to obtain $rw(f) \geq (|V(G)| - n^*)k$. Apply $n^* \leq w(f)$ and the conclusion follows. □

Cartesian products of cycles are 4-regular graphs, and we consider 2-rainbow domination, so we need a special case of Lemma 3.1, namely $k = 2$ and $r = 4$.

Corollary 3.2. *Let G be a 4-regular graph. Then* $\gamma_{r2}(G) \geq \dfrac{1}{3}|V(G)|$.

Note that the statement also follows from [22, Lemma 2.2, Case (6)].

The next lemma will be useful to obtain better lower bounds for Cartesian products of cycles. In particular, for bounds of $\gamma_{r2}(C_m \Box C_n)$. Recall that $s_i = \sum_{x \in \mathcal{C}^i} |f(x)|$.

Lemma 3.3. *Let f be a $\gamma_{r2}(C_m \Box C_n)$-function. Write $m = 3k + \ell$, where $\ell \equiv m \pmod 3$. Then*

(a) $s_{i-1} + s_{i+1} \geq 2m - 4s_i = 6k + 2\ell - 4s_i$,

(b) *if $k \geq s_{min} = \min\{s_{i-1}, s_{i+1}\}$, then*

$$s_{max} \geq 2m - 4s_i - s_{min} \geq 5k + 2\ell - 4s_i ,$$

where $s_{max} = \max\{s_{i-1}, s_{i+1}\}$.

Proof. Note that at most s_i vertices of the column $\mathcal{C}^i$ are colored (this holds in the case when all $|f(v)| = 1$). Other (uncolored) vertices in $\mathcal{C}^i$, at least $m - s_i$ of them, have a total demand at least $2(m - s_i)$. Since at most $2s_i$ of this demand can be fulfilled by the colored vertices of $\mathcal{C}^i$, we must have at least $2m - 4s_i$ colors in the neighborhood of $\mathcal{C}^i$. Equivalent to this is $s_{i-1} + s_{i+1} \geq 2m - 4s_i$. So if we use $m = 3k + \ell$, we have

$$s_{i-1} + s_{i+1} \geq 2m - 4s_i = 6k + 2\ell - 4s_i ,$$

as required. Finally, if $k \geq s_{min} = \min\{s_{i-1}, s_{i+1}\}$, then

$$s_{max} \geq 2m - 4s_i - s_{min} = 6k + 2\ell - 4s_i - s_{min} \geq 5k + 2\ell - 4s_i ,$$

and the proof is complete. □

The next observation provides lower bounds. The proof is based on the discharging argument and follows the ideas of [27] and [28].

Proposition 3.4. *Let $m \geq 3$ and $n \geq 3$. Write $m = 3k + \ell$, where $\ell \equiv m \pmod 3$. Then*

$$\gamma_{r2}(C_m \Box C_n) \geq kn + \ell\frac{n}{2} = \frac{mn}{3} + \ell\frac{n}{6} .$$

Proof. Note that when $m = 3k$, the proof follows directly from Lemma 3.1 In the following, we write the proof for the case when $m = 3k + 1$, since the proof for the case when $m = 3k + 2$ is similar and can therefore be omitted.

Let f be a γ_{r2}-function on the vertex set of $C_m \Box C_n$ and let $(s_1, s_2, \ldots, s_m)$, be the 2RDF sequence corresponding to f. We define a discharging rule in which the columns with sufficiently large s_i give half of their overweight to one or both of the neighboring columns. For this purpose, let f' be a function on the vertex set of $C_m \Box C_n$ that assigns a positive real number to each vertex. Denote by $s'_i = \sum_{x \in \mathcal{C}^i} f'(x)$ and let $(s'_1, s'_2, \ldots, s'_m)$ be the sequence corresponding to f'. Moreover, we define f' such that the following holds:

If $s_i > k + \frac{1}{2}$ then set $s'_i = k + \frac{1}{2}$. If $s_i \leq k + \frac{1}{2}$, then

- if $s_{i-1} > k+\frac{1}{2}$ and $s_{i+1} > k+\frac{1}{2}$, then $s'_i = s_i+\frac{1}{2}(s_{i-1}-(k+\frac{1}{2}))+\frac{1}{2}(s_{i+1}-(k+\frac{1}{2}))$,
- if $s_{i-1} > k + \frac{1}{2}$ and $s_{i+1} < k + \frac{1}{2}$, then $s'_i = s_i + \frac{1}{2}(s_{i-1} - (k + \frac{1}{2}))$,
- if $s_{i-1} < k + \frac{1}{2}$ and $s_{i+1} > k + \frac{1}{2}$, then $s'_i = s_i + \frac{1}{2}(s_{i+1} - (k + \frac{1}{2}))$.

We claim that $s'_i \geq k+\frac{1}{2}$ for all i. Assume $s_i \leq k+\frac{1}{2}$. Note that, since s_i is an integer, $s_i \leq k + \frac{1}{2}$ implies $s_i \leq k$. Again, if $s_{i-1} > k$ and $s_{i+1} > k$ then, by Lemma 3.3,

$$\begin{aligned} s'_i &= s_i + \frac{1}{2}(s_{i-1} - (k + \frac{1}{2})) + \frac{1}{2}(s_{i+1} - (k + \frac{1}{2})) \\ &= s_i + \frac{1}{2}(s_{i-1} + s_{i+1}) - (k + \frac{1}{2}) \\ &\geq s_i + 3k + 1 - 2s_i - k - \frac{1}{2} = 2k + \frac{1}{2} - s_i \\ &= k + \frac{1}{2} + (k - s_i) \geq k + \frac{1}{2}\,. \end{aligned}$$

or, when $s_{min} = \min\{s_{i-1}, s_{i+1}\} \leq k$,

$$\begin{aligned} s'_i &= s_i + \frac{1}{2}(s_{max} - (k + \frac{1}{2})) \\ &\geq s_i + 2k + 1 - 2s_i - \frac{1}{4} = 2k + \frac{3}{4} - s_i\,. \end{aligned}$$

Recall that s_i is an integer, so $s_i \leq k + \frac{1}{2}$ is equivalent to $s_i \leq k$, and hence

$$s'_i = 2k + \frac{3}{4} - s_i = k + \frac{3}{4} + (k - s_i) > k + \frac{1}{2},$$

which implies $\gamma_{r2}(C_m \Box C_n) = \sum_i s_i \geq \sum_i s'_i \geq n(k + \frac{1}{2})$.

Summarizing, we get

(a) $\gamma_{r2}(C_m \Box C_n) \geq kn$ when $\ell = 0$,

(b) $\gamma_{r2}(C_m \Box C_n) \geq kn + \frac{n}{2}$ when $\ell = 1$, and

(c) $\gamma_{r2}(C_m \Box C_n) \geq kn + n$ when $\ell = 2$.

which in turn implies

$$\gamma_{r2}(C_m \Box C_n) \geq kn + \ell\frac{n}{2} = \frac{mn}{3} + \ell\frac{n}{6}$$

as claimed. □

4 Upper bounds

Recall the characterization of the products where the general lower bound is attained [31]. More precisely, the result is given in the next theorem.

Theorem 4.1 ([31]). *If either* $m \equiv 0 \pmod 3$ *and* $n \equiv 0 \pmod 6$*, or* $m \equiv 0 \pmod 6$ *and* $n \equiv 0 \pmod 3$*, then*

$$\gamma_{r2}(C_m \Box C_n) = \frac{1}{3}mn\,.$$

For later reference, observe such 2RDF may be based on the pattern

$$\begin{bmatrix} \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & \dots \\ \dots & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & \dots \\ \dots & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & \dots \\ \dots & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & \dots \\ \dots & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & \dots \\ \dots & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}. \tag{4.1}$$

Moreover, it is easy to write explicit formula for the values, namely

$$f_1(i,j) = \begin{cases} 0, & i \not\equiv j \pmod 3 \\ 2{-}i \bmod 2, & i \equiv j \pmod 3). \end{cases}$$

The alternative is to define a 2RDF as

$$f_2(i,j) = \begin{cases} 0, & i \not\equiv j \pmod 3 \\ 2{-}j \bmod 2, & i \equiv j \pmod 3) \end{cases}$$

which results in the pattern

$$\begin{bmatrix} \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & \dots \\ \dots & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & \dots \\ \dots & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & \dots \\ \dots & 2 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & \dots \\ \dots & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & \dots \\ \dots & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 2 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}. \tag{4.2}$$

It is easy to see that the first pattern results in 2RDF's with $\gamma_{r2}(C_m \Box C_n) = \frac{mn}{3}$, if $m \equiv 0 \pmod 3$, $n \equiv 0 \pmod 6$. The second pattern provides 2RDF's with $\gamma_{r2}(C_m \Box C_n) = \frac{mn}{3}$ if $m \equiv 0 \pmod 6$, $n \equiv 0 \pmod 3$. Note that $m \geq 6$ is required for the second pattern, while the first pattern can be applied if $m \geq 3$.

Remark. It is worth noting that in both cases we have $s_i = \frac{m}{3}$.

Now we outline constructions that directly imply some upper bounds.

Proposition 4.2. *Let* $m \equiv 2 \pmod 3$ *and* $n \equiv 0 \pmod 6$. *Write* $m = 3k + 2$. *Then*

$$\gamma_{r2}(C_m \square C_n) \le kn + n.$$

Proof. First, we provide a 2RDF proving that $\gamma_{r2}(C_5 \square C_n) \le 2n$. Start with the pattern (4.2), use the first six rows and replace the 2nd and 3rd row with the union of them.

$$\begin{bmatrix} \dots & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & \dots \\ \dots & \mathbf{0} & \mathbf{2} & \mathbf{1} & \mathbf{0} & \mathbf{2} & \mathbf{1} & \mathbf{0} & \mathbf{2} & \mathbf{1} & \dots \\ \dots & 2 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & \dots \\ \dots & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & \dots \\ \dots & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 2 & \dots \end{bmatrix}$$

Is it obvious that the same construction gives 2RDF's proving that

$$\gamma_{r2}(C_{3k+2} \square C_n) \le kn + n,$$

as claimed. □

Proposition 4.3. *Let* $m \equiv 1 \pmod 3$ *and* $n \equiv 0 \pmod 6$. *Write* $m = 3k + 1$. *Then*

$$\gamma_{r2}(C_m \square C_n) \le kn + n.$$

Proof. First, we provide a 2RDF proving that $\gamma_{r2}(C_4 \square C_n) \le 2n$. Start with the pattern (4.2), use the first six rows, replace the 2nd and 3rd row with the union of them, and replace the 4th and 5th row with the union of them.

$$\begin{bmatrix} \dots & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & \dots \\ \dots & \mathbf{0} & \mathbf{2} & \mathbf{1} & \mathbf{0} & \mathbf{2} & \mathbf{1} & \mathbf{0} & \mathbf{2} & \mathbf{1} & \dots \\ \dots & \mathbf{2} & \mathbf{1} & \mathbf{0} & \mathbf{2} & \mathbf{1} & \mathbf{0} & \mathbf{2} & \mathbf{1} & \mathbf{0} & \dots \\ \dots & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 2 & \dots \end{bmatrix}$$

Is it obvious that the same construction gives 2RDF's proving that

$$\gamma_{r2}(C_{3k+1} \square C_n) \le kn + n,$$

as claimed. □

To summarize, we can combine the Propositions 4.2 and 4.3 with Theorem 4.1 to obtain

Proposition 4.4. *Let* $n \equiv 0 \pmod 6$ *and* $m \ge 3$. *Then* $\gamma_{r2}(C_m \square C_n) \le \lceil \frac{m}{3} \rceil n$.

The next propositions provide general upper bounds for the cases when $n \not\equiv 0 \pmod 6$. Below we provide constructions based on the previously studied 2RDF for each possible reminder $b = 0, 1, 2, 3, 4, 5$ where $n \equiv b \pmod 6$. We start with the case $m \equiv 0 \pmod 3$.

Proposition 4.5. *Let* $m \ge 3$, $m \equiv 0 \pmod 3$, *and* $n \ge 6$, $n \equiv b \pmod 6$. *Hence* $n = 6a + b$ *for some integer* $a \ge 0$. *Then*

(a) *if* $b = 5$ *then* $\gamma_{r2}(C_m \square C_n) \le \gamma_{r2}(C_m \square C_{6a}) + 2m = \frac{m}{3}(n + 1)$,

(b) *if* $b = 4$ *then* $\gamma_{r2}(C_m \square C_n) \le \gamma_{r2}(C_m \square C_{6a}) + 2m = \frac{m}{3}(n + 2)$,

(c) *if* $b = 3$ *then* $\gamma_{r2}(C_m \square C_n) \leq \gamma_{r2}(C_m \square C_{6a}) + 4\frac{m}{3} = \frac{m}{3}(n+1)$,

(d) *if* $b = 2$ *then* $\gamma_{r2}(C_m \square C_n) \leq \gamma_{r2}(C_m \square C_{6a}) + m = \frac{m}{3}(n+1)$,

(e) *if* $b = 1$ *then* $\gamma_{r2}(C_m \square C_n) \leq \gamma_{r2}(C_m \square C_{6a}) + 2\frac{m}{3} = \frac{m}{3}(n+1)$.

Proof. In the following we give explicit constructions for the case $m = 6 = 2 \times 3$ and various n. It is obvious that in general we can simply repeat the pattern of three consecutive rows. The weight of a column is $\frac{m}{3}$, hence the bounds given in proposition.

(a) if $b = 5$, then replace two columns of the 2RDF $(C_m \square C_{6a+6})$ by their union and observe that the table gives a 2RDF of $(C_m \square C_{6a+5})$.

$$\left[\begin{array}{ccccccc|ccc|ccc} \dots & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 \\ \dots & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 \\ \dots & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 2 \\ \dots & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 \\ \dots & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 \\ \dots & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 2 \end{array}\right] \rightarrow$$

$$\left[\begin{array}{ccccccc|ccc|cc} \dots & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & \mathbf{2} & 0 \\ \dots & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & \mathbf{1} & 0 \\ \dots & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 2 \\ \dots & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & \mathbf{2} & 0 \\ \dots & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & \mathbf{1} & 0 \\ \dots & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 2 \end{array}\right].$$

So if we look at the last 6 columns, which have shrunk to 5 columns, we see that the number of colors used does not change. If instead of $m = 6$ we consider $m = 3k$, three rows, e.g. rows $4 - 6$, are repeated $(k-2)$ times and the same construction is applied. The last 6 columns therefore contain $6 \times k = 2m$ colors.

In the remaining cases, we only give the tables containing the constructions that alter the rightmost columns (in the tables $m = 6$ is chosen).

(b) if $b = 4$, then take (for example) the last four columns and replace them with two columns, each of which is the union of two columns.

$$\left[\begin{array}{ccccccc|cc|cccc} \dots & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 \\ \dots & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 \\ \dots & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 2 \\ \dots & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 \\ \dots & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 \\ \dots & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 2 \end{array}\right] \rightarrow$$

$$\left[\begin{array}{ccccccc|cc|cc} \dots & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 2 \\ \dots & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 2 & \mathbf{1} & 0 \\ \dots & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & \mathbf{2} \\ \dots & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 2 \\ \dots & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 2 & \mathbf{1} & 0 \\ \dots & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & \mathbf{2} \end{array}\right].$$

(c) if $b = 3$, then take (for example) the last six columns and replace them with three columns, as follows

$$\begin{bmatrix} \dots & 1 & 0 & 0 & 2 & 0 & 0 & | & 1 & 0 & 0 & 2 & 0 & 0 \\ \dots & 0 & 2 & 0 & 0 & 1 & 0 & | & 0 & 2 & 0 & 0 & 1 & 0 \\ \dots & 0 & 0 & 1 & 0 & 0 & 2 & | & 0 & 0 & 1 & 0 & 0 & 2 \\ \dots & 1 & 0 & 0 & 2 & 0 & 0 & | & 1 & 0 & 0 & 2 & 0 & 0 \\ \dots & 0 & 2 & 0 & 0 & 1 & 0 & | & 0 & 2 & 0 & 0 & 1 & 0 \\ \dots & 0 & 0 & 1 & 0 & 0 & 2 & | & 0 & 0 & 1 & 0 & 0 & 2 \end{bmatrix} \to$$

$$\begin{bmatrix} \dots & 1 & 0 & 0 & 2 & 0 & 0 & | & 1 & 0 & 0 \\ \dots & 0 & 2 & 0 & 0 & 1 & 0 & | & 0 & \{\mathbf{1},\mathbf{2}\} & 0 \\ \dots & 0 & 0 & 1 & 0 & 0 & 2 & | & 0 & 0 & 2 \\ \dots & 1 & 0 & 0 & 2 & 0 & 0 & | & 1 & 0 & 0 \\ \dots & 0 & 2 & 0 & 0 & 1 & 0 & | & 0 & \{\mathbf{1},\mathbf{2}\} & 0 \\ \dots & 0 & 0 & 1 & 0 & 0 & 2 & | & 0 & 0 & 2 \end{bmatrix}.$$

Note that the 2RDF in this case is not a singleton 2RDF. A singleton 2RDF either assigns a singleton to the empty set [10]. We do not know whether there is a singleton 2RDF with the same weight.

(d) if $b = 2$, then take (for example) the last six columns and replace them with two columns, as follows

$$\begin{bmatrix} \dots & 1 & 0 & 0 & 2 & 0 & 0 & | & 1 & 0 & 0 & 2 & 0 & 0 \\ \dots & 0 & 2 & 0 & 0 & 1 & 0 & | & 0 & 2 & 0 & 0 & 1 & 0 \\ \dots & 0 & 0 & 1 & 0 & 0 & 2 & | & 0 & 0 & 1 & 0 & 0 & 2 \\ \dots & 1 & 0 & 0 & 2 & 0 & 0 & | & 1 & 0 & 0 & 2 & 0 & 0 \\ \dots & 0 & 2 & 0 & 0 & 1 & 0 & | & 0 & 2 & 0 & 0 & 1 & 0 \\ \dots & 0 & 0 & 1 & 0 & 0 & 2 & | & 0 & 0 & 1 & 0 & 0 & 2 \end{bmatrix} \to$$

$$\begin{bmatrix} \dots & 1 & 0 & 0 & 2 & 0 & 0 & | & 1 & 0 \\ \dots & 0 & 2 & 0 & 0 & 1 & 0 & | & 0 & \mathbf{2} \\ \dots & 0 & 0 & 1 & 0 & 0 & 2 & | & 0 & 2 \\ \dots & 1 & 0 & 0 & 2 & 0 & 0 & | & 1 & 0 \\ \dots & 0 & 2 & 0 & 0 & 1 & 0 & | & 0 & \mathbf{2} \\ \dots & 0 & 0 & 1 & 0 & 0 & 2 & | & 0 & 2 \end{bmatrix}.$$

(e) if $b = 1$, then replace the last six columns with an altered column.

$$\begin{bmatrix} \dots & 1 & 0 & 0 & 2 & 0 & 0 & | & 1 & 0 & 0 & 2 & 0 & 0 \\ \dots & 0 & 2 & 0 & 0 & 1 & 0 & | & 0 & 2 & 0 & 0 & 1 & 0 \\ \dots & 0 & 0 & 1 & 0 & 0 & 2 & | & 0 & 0 & 1 & 0 & 0 & 2 \\ \dots & 1 & 0 & 0 & 2 & 0 & 0 & | & 1 & 0 & 0 & 2 & 0 & 0 \\ \dots & 0 & 2 & 0 & 0 & 1 & 0 & | & 0 & 2 & 0 & 0 & 1 & 0 \\ \dots & 0 & 0 & 1 & 0 & 0 & 2 & | & 0 & 0 & 1 & 0 & 0 & 2 \end{bmatrix} \to$$

$$\left[\begin{array}{ccccccc|c} \dots & 1 & 0 & 0 & 2 & 0 & 0 & \mathbf{1} \\ \dots & 0 & 2 & 0 & 0 & 1 & 0 & 0 \\ \dots & 0 & 0 & 1 & 0 & 0 & 2 & \mathbf{2} \\ \dots & 1 & 0 & 0 & 2 & 0 & 0 & \mathbf{1} \\ \dots & 0 & 2 & 0 & 0 & 1 & 0 & 0 \\ \dots & 0 & 0 & 1 & 0 & 0 & 2 & \mathbf{2} \end{array}\right].$$

□

Now we generalize Proposition 4.5 to arbitrary m.

Proposition 4.6. *Let $m \geq 3$ and $n \geq 6$. If $n \equiv 4 \pmod 6$, then $\gamma_{r2}(C_m \Box C_n) \leq \lceil \frac{m}{3} \rceil (n+2)$. Otherwise, $\gamma_{r2}(C_m \Box C_n) \leq \lceil \frac{m}{3} \rceil (n+1)$.*

Proof. (sketch) The bounds are obtained by constructions that combine the ideas from Propositions 4.2, 4.3 and 4.5. The main idea is the following. Start with $C_{\tilde{m}} \Box C_n$ where $\tilde{m} = 3\lceil \frac{m}{3} \rceil$. Note that there is at most $\lceil \frac{m}{3} \rceil$ colors in each column. Apply the constructions as in the proofs of Propositions 4.2, 4.3 and 4.5. Recall that in each of these constructions some columns are deleted and we replace one or two rows by unions of two rows. The total weight is preserved in this way, so the proposition holds. □

The upper bounds provided in Propositions 4.5 and 4.6 have a similar form, and can be written in a condensed way as follows.

Corollary 4.7. *Let $m \geq 3$ and $n \geq 6$. Then*

$$\gamma_{r2}(C_m \Box C_n) \leq \left\lceil \frac{m}{3} \right\rceil (n+\beta), \quad \textit{where} \quad \beta = \begin{cases} 0, & n \equiv 0 \pmod 6 \\ 1, & n \equiv 1,2,3,5 \pmod 6 \\ 2, & n \equiv 4 \pmod 6). \end{cases}$$

The construction used in Propositions 4.5, 4.6, and Corollary 4.7 are based on the basic assignment (4.1). Constructions based on (4.2) can be used in a similar way and result in slightly different upper bounds.

Proposition 4.8. *Let $m \geq 6$ and $n \geq 3$. Then*

$$\gamma_{r2}(C_m \Box C_n) \leq \left\lceil \frac{n}{3} \right\rceil (m+\gamma), \quad \textit{where} \quad \gamma = \begin{cases} 0, & m \equiv 0 \pmod 6 \\ 1, & m \equiv 5 \pmod 6 \\ 2, & m \equiv 1,2,3,4 \pmod 6). \end{cases}$$

Proof. We give only a brief outline of the proof and omit the detailed arguments, because the ideas are analogous to those previously elaborated in the proofs of Propositions 4.5, 4.6 and Corollary 4.7,

Recall first that for $m \equiv 0 \bmod 6$ and $n \equiv 0 \bmod 3$ Pattern (4.2) returns a 2RDF with weight $\frac{mn}{3}$.

Let us now assume that $n \equiv 0 \bmod 3$ and let $m \equiv d \bmod 6$. We claim that if $d = 5$ then $\gamma_{r2}(C_m \Box C_n) \leq \lceil \frac{n}{3} \rceil (m+1)$, otherwise, $\gamma_{r2}(C_m \Box C_n) \leq \lceil \frac{n}{3} \rceil (m+2)$. If $d = 5$ then one row is deleted, and the colors of the deleted row are given to one neighboring rows. Formally, row m is defined as a union of the rows m and $m+1$ of the pattern. In any other case, a 2RDF is obtained by deleting some rows and replacing rows 1 and m with unions.

We have thus seen that the cases $n \not\equiv 0 \bmod 3$ can be handled by deleting one or two columns in the pattern. The colors of the deleted column(s) are then used to complete the assignment of columns 1 and n. And we have the upper bound as claimed. □

It seems obvious that the two upper bounds are not equivalent. Now we compare them more closely. To this end we write

$$\begin{aligned} \mathrm{B1}(m,n) &= \left\lceil \frac{m}{3} \right\rceil (n+\beta) \\ &= \frac{1}{3}(m+a)(n+\beta) = \frac{1}{3}mn + \frac{1}{3}an + \frac{1}{3}\beta m + \frac{1}{3}a\beta \qquad (4.3) \\ \mathrm{B2}(m,n) &= \left\lceil \frac{n}{3} \right\rceil (m+\gamma) \\ &= \frac{1}{3}(n+c)(m+\gamma) = \frac{1}{3}mn + \frac{1}{3}\gamma n + \frac{1}{3}cm + \frac{1}{3}\gamma c \qquad (4.4) \end{aligned}$$

where

$$a = \begin{cases} 0, & m \equiv 0 \pmod 3 \\ 1, & m \equiv 2 \pmod 3 \\ 2, & m \equiv 1 \pmod 3 \end{cases}, \quad \beta = \begin{cases} 0, & n \equiv 0 \pmod 6 \\ 1, & n \equiv 1,2,3,5 \pmod 6 \\ 2, & n \equiv 4 \pmod 6 \end{cases},$$

$$\gamma = \begin{cases} 0, & n \equiv 0 \pmod 6 \\ 1, & n \equiv 5 \pmod 6 \\ 2, & n \equiv 1,2,3,4 \pmod 6 \end{cases}, \quad \text{and} \quad c = \begin{cases} 0, & n \equiv 0 \pmod 3 \\ 1, & n \equiv 2 \pmod 3 \\ 2, & n \equiv 1 \pmod 3. \end{cases}$$

Note that both B1 and B2 are of the form $\frac{1}{3}mn + \frac{1}{3}(x,y,z)(n,m,1)$, and let us write the values of (x,y,z) in two tables for easier comparison. (See Table 2 and Table 3.)

B1(m,n)	n mod 6	0	1	2	3	4	5
m mod 6	$a\backslash\beta$	0	1	1	1	2	1
0	0	(0,0,0)	(0,1,0)	(0,1,0)	(0,1,0)	(0,2,0)	(0,1,0)
1	2	(2,0,0)	(2,1,2)	(2,1,2)	(2,1,2)	(2,2,4)	(2,1,2)
2	1	(1,0,0)	(1,1,1)	(1,1,1)	(1,1,1)	(1,2,2)	(1,1,1)
3	0	(0,0,0)	(0,1,0)	(0,1,0)	(0,1,0)	(0,2,0)	(0,1,0)
4	2	(2,0,0)	(2,1,2)	(2,1,2)	(2,1,2)	(2,2,4)	(2,1,2)
5	1	(1,0,0)	(1,1,1)	(1,1,1)	(1,1,1)	(1,2,2)	(1,1,1)

Table 2: B1 as a function of m and n.

B2(m,n)	n mod 6	0	1	2	3	4	5
m mod 6	$\gamma\backslash c$	0	2	1	0	2	1
0	0	(0,0,0)	(0,2,0)	(0,1,0)	(0,0,0)	(0,2,0)	(0,1,0)
1	2	(2,0,0)	(2,2,4)	(2,1,2)	(2,0,0)	(2,2,4)	(2,1,2)
2	2	(2,0,0)	(2,2,4)	(2,1,2)	(2,0,0)	(2,2,4)	(2,1,2)
3	2	(2,0,0)	(2,2,4)	(2,1,2)	(2,0,0)	(2,2,4)	(2,1,2)
4	2	(2,0,0)	(2,2,4)	(2,1,2)	(2,0,0)	(2,2,4)	(2,1,2)
5	1	(1,0,0)	(1,2,2)	(1,1,1)	(1.0,0)	(1,2,2)	(1,1,1)

Table 3: B2 as a function of m and n.

In fourteen cases B1 $<$ B2, in other words the first pattern gives rise a better 2RDF. In four cases, B2 $<$ B1. Note that in two cases, the triples are no comparable. In particular, when $m \equiv 2 \pmod 6$ and $n \equiv 3 \pmod 6$ we have

$$\mathrm{B1} = \frac{1}{3}mn + \frac{1}{3}(n+m+1) <> \mathrm{B2} = \frac{1}{3}mn + \frac{1}{3}2m$$

and hence

$$\mathrm{B1} >=< \mathrm{B2} \quad \Longleftrightarrow \quad n+1 >=< m\,.$$

Similarly, when $m \equiv 3 \pmod 6$ and $n \equiv 3 \pmod 6$,

$$\mathrm{B1} = \frac{1}{3}mn + \frac{1}{3}n <> \mathrm{B2} = \frac{1}{3}mn + \frac{1}{3}2m$$

and hence

$$\mathrm{B1} >=< \mathrm{B2} \quad \Longleftrightarrow \quad n >=< 2m\,.$$

We summarize the observations in Table 4.

$m\backslash n$ mod 6	0	1	2	3	4	5
0	=	B1(m,n)	=	B2(m,n)	=	=
1	=	B1(m,n)	=	B2(m,n)	=	=
2	B1(m,n)	B1(m,n)	B1(m,n)	$>=<$	B1(m,n)	B1(m,n)
3	B1(m,n)	B1(m,n)	B1(m,n)	$>=<$	B1(m,n)	B1(m,n)
4	=	B1(m,n)	=	B2(m,n)	=	=
5	=	B1(m,n)	=	B2(m,n)	=	=

Table 4: Comparison of B1 and B2.

Finally, we recall that the Cartesian product is commutative, $C_m \Box C_n \simeq C_n \Box C_m$. Therefore, the best upper bound for $\gamma_{r2}(C_m \Box C_n)$ is based on the constructions considered here and is the minimum of the bounds $\mathrm{B1}(m,n), \mathrm{B2}(m,n), \mathrm{B1}(n,m)$, and $\mathrm{B2}(n,m)$. The results are written in Table 5.

$m\backslash n$ mod 6	0	1	2	3	4	5
0	=	B1(m,n)	B1(m,n)	B2(m,n)	=	=
1		$*$	B1(n,m)	B1(n,m)	B1(n,m)	B1(n,m)
2			B1(m,n)	B1(n,m)	B1(m,n)	B1(m,n)
3				$*$	B1(m,n)	B1(m,n)
4					=	=
5						=

where $* = \min\{\mathrm{B1}(m,n), \mathrm{B1}(n,m)\}$.

Table 5: Upper bounds using the commutativity of the Cartesian product.

Explicitly, the best upper bounds for $\gamma_{r2}(C_m \Box C_n)$ are of the form

$$\frac{1}{3}mn + \frac{1}{3}(n,m,1)(x,y,z)$$

with values of (x,y,z) from Table 6.

The bounds can be summarized as follows.

$m\backslash n$ mod 6	0	1	2	3	4	5
0	(0,0,0)	(0,1,0)	(0,1,0)	(0,0,0)	(0,2,0)	(0,1,0)
1		(2,1,2) or (1,2,2)	(1,1,1)	(1,0,0)	(1,2,2)	(0,1,0)
2			(1,1,1)	(1,0,0)	(1,2,2)	(1,1,1)
3				(0,1,0) or (1,0,0)	(0,2,0)	(0,1,0)
4					(2,2,4)	(2,1,2)
5						(1,1,1)

Table 6: Upper bounds in terms of (x, y, z).

Corollary 4.9. *Let $m \geq 6$ and $n \geq 6$. As the Cartesian product is commutative, we can assume $m \geq n$. Then*

$$\gamma_{r2}(C_m \Box C_n) \leq \frac{1}{3}mn + \frac{1}{3}\delta\,,$$

where δ can be read from Table 7.

$m\backslash n$ mod6	0	1	2	3	4	5
0	0	m	m	0	$2m$	m
1		min $\{n+2m+2, 2n+m+2\}$	$n+m+1$	n	$n+2m+2$	m
2			$n+m+1$	n	$n+2m+2$	$n+m+1$
3				$\min\{m,n\}$	$2m$	m
4					$2n+2m+4$	$2n+m+2$
5						$n+m+1$

Table 7: The values of δ as a function of m and n in Corollary 4.9.

5 Conclusions and future work

We have provided lower and upper bounds for the 2-rainbow domination number of $C_m \Box C_n$ with a gap of at most $\frac{1}{3}(2m + 2n + 4)$. The proof of the lower bound is based on a discharging argument and seems to be close to the best possible in most cases. The upper bound, on the other hand, is based on two constructions that are quite rough in some cases, and we believe that it can be improved by carefully analyzing special cases. We conjecture that the lower bounds differ from the exact values by at most one constant, which depends on m and is independent of n

At least for examples with small m, we claim that it is possible to avoid the tedious analysis by applying an algebraic method that can be used for various graph invariants including the domination type problems [11, 20, 24, 26]. Such a research task remains a challenge for future work.

Another interesting line of research, which is a natural extension of this study, is a generalization of the results presented here to graph bundles, a natural generalization of graph products [25].

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Families of association schemes on triples from two-transitive groups*

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Abstract

Association schemes on triples (ASTs) are ternary analogues of classical association schemes. Similar to how Schurian association schemes arise from transitive groups, ASTs arise from two-transitive groups. In this paper, we obtain the third valencies and the number of relations of the ASTs obtained from two-transitive permutation groups. Further, we obtain the intersection numbers of the ASTs produced by $P\Gamma L(k,n)$, $PSL(2,n)$, $A\Gamma L(k,n)$, and the sporadic two-transitive groups. In particular, the ASTs from the actions of $P\Gamma L(k,n)$, $PSL(2,n)$, and the sporadic groups are commutative.

Keywords: Association scheme on triples, permutation group, ternary algebra, algebraic combinatorics.

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1 Introduction

An association scheme is a set $X = \{R_i\}_{i=0}^m$ of binary relations on a finite nonempty set Ω that partitions $\Omega \times \Omega$ and which satisfies certain symmetry requirements. These requirements are flexible enough to accommodate a variety of mathematical objects yet rigid enough to afford desirable algebraic and combinatorial properties. For instance, the adjacency algebras of association schemes and their duals under the Hadamard product are semisimple, and the parameters of these algebras may be used in the classification of certain types of graphs [3].

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The datasets generated and/or analysed during the current study, along with the source codes for obtaining these, are available from the corresponding author on reasonable request.

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In the 1990 paper [10], Mesner and Bhattacharya extended the notion of association schemes to association schemes on triples (ASTs). Here, the underlying relations are ternary instead of binary. By generalizing the usual $m \times m$ binary square matrix product $(A, B) \mapsto AB$ to an $m \times m \times m$ ternary cubic hypermatrix product $(A, B, C) \mapsto ABC$, the resulting adjacency hypermatrices form a non-associative ternary algebra. In the same paper, the authors showed that a two-transitive action of a group G on a set Ω induces an AST, denoted $X = \{R_i\}_{i=0}^{m}$, by letting the relations R_i in X be the orbits of the induced action of G on $\Omega \times \Omega \times \Omega$. In particular, the authors gave the sizes (number of relations) of the ASTs obtained from the affine general linear groups $AGL(1, n)$, the projective special linear groups $PSL(2, n)$, the Suzuki groups $Sz(2^{2k+1})$, and the Higman-Sims group HS. Moreover, they also determined some intersection numbers of the ASTs obtained from $AGL(1, n)$ and $PSL(2, n)$ by computing for some products of the adjacency hypermatrices of the corresponding ASTs.

Other works on ASTs include generalizations of identity elements and inverse elements for the ternary algebras of ASTs [11], ASTs with relations invariant under a transitive cyclic subgroup of the symmetric group [4], an algorithm for classifying ASTs whose relations are invariant under a given group action [1], and ASTs obtained by decomposing or taking the unions of relations of an existing AST [2]. However, much remains unknown for ASTs. For instance, it is not known whether the ternary algebras of ASTs satisfy analogues for the semisimplicity and duality properties of the adjacency algebras of classical association schemes.

To further the study of ASTs, we extend the work in [10] by obtaining the sizes and third valencies of the ASTs obtained from the two-transitive permutation groups. These are given in Theorem 1.1.

Theorem 1.1. *The third valencies and number of relations of the ASTs obtained from the symmetric groups S_k, the projective semilinear groups $P\Gamma L(k, n)$, the projective special linear groups $PSL(2, n)$, the Suzuki groups $Sz(2^{2k+1})$, the Ree groups $Ree(3^{2k+1})$, the affine semilinear groups $A\Gamma L(k, n)$, the projective unitary groups $PGU(3, n)$, the symplectic groups $Sp(2k, 2)$, and the sporadic two-transitive groups are given in* Table 1.

Theorem 1.1 is obtained by determining the orbits of the groups' two-point stabilizers. Indeed, the number of nontrivial relations of an AST obtained from a two-transitive group is equal to the number of orbits of a two-point stabilizer while the third valencies are the sizes of these orbits.

Furthermore, by determining which elements of the underlying set can be sent to one another through certain elements of the acting group, we extend some of the results in [10] regarding the intersection numbers of ASTs obtained from $PSL(2, n)$ and $AGL(1, n)$ to the intersection numbers of ASTs obtained from $P\Gamma L(k, n)$ and $A\Gamma L(k, n)$. Additionally, we complete the intersection numbers of ASTs from $PSL(2, n)$ which were partially obtained in [10]. Through GAP 4.11.1 [6], we also determine the intersection numbers of the ASTs obtained from the sporadic two-transitive groups. These intersection numbers correct some errors in [10] and are given in Theorem 1.2.

Theorem 1.2. *The intersection numbers of the ASTs obtained from the affine semilinear groups $A\Gamma L(k, n)$, the projective groups $P\Gamma L(k, n)$ and $PSL(2, n)$, and the sporadic two-transitive groups are given in* Tables 2 *to* 8.

2 Preliminaries

This section is based mostly on [10] and [5]. We define association schemes on triples (ASTs), state relevant properties of ASTs, and describe relationships between ASTs and two-transitive groups.

2.1 Association schemes on triples

Similar to a classical association scheme, an AST on a set Ω is a set of ternary relations $X = \{R_i\}_{i=0}^{m}$ that partitions $\Omega \times \Omega \times \Omega$ and which satisfies certain symmetry requirements. More precisely, we have the following definition.

Definition 2.1. Let Ω be a finite set with at least 3 elements. An association scheme on triples (AST) on Ω is a partition $X = \{R_i\}_{i=0}^{m}$ of $\Omega \times \Omega \times \Omega$ with $m \geq 4$ such that the following conditions hold.

(1) For each $i \in \{0, \ldots, m\}$, there exists an integer $n_i^{(3)}$ such that for each pair of distinct $x, y \in \Omega$, the number of $z \in \Omega$ with $(x, y, z) \in R_i$ is $n_i^{(3)}$.

(2) (Principal Regularity Condition.) For any $i, j, k, l \in \{0, \ldots, m\}$, there exists a constant p_{ijk}^{l} such that for any $(x, y, z) \in R_l$, the number of w such that $(w, y, z) \in R_i$, $(x, w, z) \in R_j$, and $(x, y, w) \in R_k$ is p_{ijk}^{l}.

(3) For any $i \in \{0, \ldots, m\}$ and any $\sigma \in S_3$, there exists a $j \in \{0, \ldots, m\}$ such that

$$R_j = \{(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}) : (x_1, x_2, x_3) \in R_i\}.$$

(4) The first four relations are

$$\begin{aligned} R_0 &= \{(x, x, x) : x \in \Omega\}, \\ R_1 &= \{(x, y, y) : x, y \in \Omega, x \neq y\}, \\ R_2 &= \{(y, x, y) : x, y \in \Omega, x \neq y\}, \text{ and} \\ R_3 &= \{(y, y, x) : x, y \in \Omega, x \neq y\}. \end{aligned}$$

Analogous to the valency of a classical association scheme, we name $n_i^{(3)}$ the third valency of R_i. By conditions (1) and (3), there exist for each i the constants $n_i^{(1)} = |\{z \in \Omega : (z, x, y) \in R_i\}|$ and $n_i^{(2)} = |\{z \in \Omega : (x, z, y) \in R_i\}|$ independent of any pair of distinct elements $x, y \in \Omega$. We term $n_i^{(1)}$ and $n_i^{(2)}$ the first and second valency of R_i, respectively. We call p_{ijk}^{l} the intersection number corresponding to R_i, R_j, R_k, and R_l. The relations R_0, R_1, R_2, and R_3 are called the trivial relations. The other relations are the nontrivial relations.

Similar to classical association schemes, we can represent an AST in terms of hypermatrices. Let $\nu = |\Omega|$. To each $R_i \in X$ we associate the $\nu \times \nu \times \nu$ cubic hypermatrix A_i whose entries are indexed by Ω. For $i \in \{0, \ldots, m\}$, define A_i by

$$(A_i)_{xyz} = \begin{cases} 1, & \text{if } (x, y, z) \in R_i, \\ 0, & \text{otherwise.} \end{cases}$$

Here, $(A_i)_{xyz}$ is the (x, y, z)-entry of A_i. To emphasize their particular nature, the hypermatrices A_0, A_1, A_2, and A_3 corresponding to the trivial relations R_0, R_1, R_2, and R_3 may also be denoted by I_0, I_1, I_2, and I_3, respectively.

The set of $\nu \times \nu \times \nu$ hypermatrices with complex entries indexed by Ω forms a complex vector space under componentwise addition and scalar multiplication. To provide it with a ternary algebra structure, we have the following ternary operation.

Definition 2.2. Let A, B, and C be $\nu \times \nu \times \nu$ hypermatrices with complex entries indexed by Ω. The ternary product $D = ABC$ is the $\nu \times \nu \times \nu$ hypermatrix given by

$$(D)_{xyz} = \sum_{w \in \Omega} (A)_{wyz}(B)_{xwz}(C)_{xyw}.$$

This ternary operation is multilinear and generally not associative [10] (in the sense of Lister in [9]). With this ternary operation, the vector space generated by the hypermatrices $\{A_i\}_{i=0}^m$ of an AST $X = \{R_i\}_{i=0}^m$ becomes a ternary algebra. The structure constants of this ternary algebra are the intersection numbers p_{ijk}^l, as stated in the following theorem.

Theorem 2.3 ([10, Theorem 1.4])**.** *Let $X = \{R_i\}_{i=0}^m$ be an AST on a set Ω and $\{A_i\}_{i=0}^m$ be the corresponding adjacency hypermatrices. Then for any $i, j, k \in \{0, \dots, m\}$, one has*

$$A_i A_j A_k = \sum_{l=0}^{m} p_{ijk}^l A_l.$$

This theorem allows us to state results regarding the intersection numbers of ASTs via expressing them as equations in the adjacency hypermatrices. In relation to this, we state the following theorem from [10], which tells us that the adjacency matrices of the nontrivial relations generate a ternary subalgebra of the adjacency algebra. In particular, $p_{ijk}^l = 0$ when $i, j, k > 3$, and $l \leq 3$.

Theorem 2.4 ([10, Corollary 2.8])**.** *In an AST with relations $\{R_i\}_{i=0}^m$, the adjacency hypermatrices $\{A_i\}_{i=4}^m$ of the nontrivial relations generate a ternary subalgebra of the ternary algebra generated by the adjacency hypermatrices $\{A_i\}_{i=0}^m$.*

According to [10], the most interesting intersection numbers are those that arise from the subalgebra in Theorem 2.4. Indeed, other theorems and remarks in [10] provide values and restrictions for p_{ijk}^l when $\{i, j, k, l\} \cap \{0, 1, 2, 3\} \neq \varnothing$. For instance, we state without proof the following consequences of Proposition 2.7 of [10].

Lemma 2.5. *Let $X = \{R_i\}_{i=0}^m$ be an AST on a set Ω. The following statements hold.*

(1) *Let R_i be a trivial relation, R_j and R_k be nontrivial relations, and R_l be any relation. Then the only intersection numbers of the form p_{ijk}^l, p_{jik}^l, or p_{jki}^l that may be nonzero are p_{1jk}^1, p_{j2k}^2, and p_{jk3}^3.*

(2) *If $m = 4$, then $A_4A_4A_4 = (|\Omega| - 3)A_4$, $I_1A_4A_4 = (|\Omega| - 2)I_1$, $A_4I_2A_4 = (|\Omega| - 2)I_2$, and $A_4A_4I_3 = (|\Omega| - 2)I_3$. Moreover, $n_4^{(3)} = |\Omega| - 2$.*

2.2 Two-transitive permutation groups and ASTs

Transitive group actions induce classical association schemes called Schurian association schemes. The theorem below states that any two-transitive group action analogously produces ASTs.

Theorem 2.6 ([10, Theorem 4.1]). *Let G be a two-transitive group acting on a set Ω. Then the orbits of the natural action of G on $\Omega \times \Omega \times \Omega$ form the relations of an AST X on Ω.*

Notice that X has only one nontrivial relation if and only if all triples of pairwise distinct elements belong to the same orbit. This yields the following remark.

Remark 2.7. The AST X obtained from a two-transitive action of a group G on Ω has only one nontrivial relation if and only if G acts three-transitively on Ω.

Further, if G is a group acting on a set Ω and H is a subgroup of G, then the orbits of G on Ω are unions of orbits of H. This bears the following consequence.

Remark 2.8. If G is a group acting two-transitively on a set Ω, and the restriction of the action to a subgroup H of G remains two-transitive on Ω, then the ASTs obtained from G and from H are equal if and only if the ASTs have the same number of relations.

We include here a lemma from [10] that is useful in determining the sizes of the ASTs obtained from two-transitive groups.

Lemma 2.9 ([10, Lemma 4.2]). *Let G be a two-transitive group acting on a set Ω. Let $\Delta = [(x, y, z)]$ be an orbit of G on $\Omega \times \Omega \times \Omega$ with x, y, and z pairwise distinct. For $a, b \in \Omega$, $a \neq b$, let*

$$\Delta(a, b) = \{c : (a, b, c) \in \Delta\}.$$

Then $\Delta(a, b)$ is a $G_{a,b}$-orbit on $\Omega \setminus \{a, b\}$. Furthermore, the map $\Delta \mapsto \Delta(a, b)$ is a bijection between the nontrivial G-orbits on $\Omega \times \Omega \times \Omega$ and the $G_{a,b}$-orbits on $\Omega \setminus \{a, b\}$.

The following remark gives additional information obtained from Lemma 2.9 and the bijection $\Delta \mapsto \Delta(a, b)$.

Remark 2.10. Let G be a group acting two-transitively on a set Ω and $X = \{R_i\}_{i=0}^{m}$ be the induced AST.

(1) For $a, b \in \Omega$ with $a \neq b$, the representatives of $R_i = [(a, b, c)]$ of the form (a, b, d) for some $d \in \Omega$ are those triples with d in the orbit of c under the action of $G_{a,b}$ on $\Omega \setminus \{a, b\}$. In particular, $n_i^{(3)}$ is the size of the orbit of c under the action of $G_{a,b}$.

(2) By varying the first coordinate or the second coordinate instead of the third coordinate, we obtain the values of the first and second valencies $n_i^{(1)}$ and $n_i^{(2)}$. Indeed, given nontrivial relations $R_i = [(a, b, c)]$, $R_j = [(a, c, b)]$, and $R_k = [(c, a, b)]$, the valencies $n_i^{(3)} = n_j^{(2)} = n_k^{(1)}$ are all equal to the size of the orbit of c under the action of $G_{a,b}$.

As a particular application of Lemma 2.9, the authors of [10] determined that the size of the AST obtained from $PSL(2, n)$ is 5 if n is even and 6 if n is odd. Further, they showed that the size of the AST obtained from $AGL(1, n)$ is precisely $n + 2$ for n a prime power larger than 2. They also determined intersection numbers of these ASTs in terms of their adjacency hypermatrices. Moreover, they determined that the size of the AST obtained from the Suzuki group $Sz(2^{2k+1})$ is $2^{2k+1}+5$ while the size of the AST obtained from the Higman-Sims group HS is 7.

3 ASTs from two-transitive groups

We prove Theorems 1.1 and 1.2 in this section. A list of the groups considered and their explicit descriptions can be found in [5]. For each group, we obtain a two-point stabilizer and the orbits of this stabilizer. Lemma 2.9 and Remark 2.10 then yield the nontrivial relations and the third valencies of the corresponding ASTs. The third valencies of the nontrivial relations and the total number of relations of the ASTs are given in Table 1. Meanwhile, by working with the orbits of the groups on the underlying Cartesian triple product, we obtain the intersection numbers of the ASTs listed in Theorem 1.2. These are summarized in Tables 2 to 8. The tables utilize definitions and notations from [5, 10], and the relevant subsections below.

Group	Third valencies	AST size
S_k, $k \geq 3$	$k-2$	5
$P\Gamma L(k,n)$, $k \geq 3$	$n-1$, $\frac{n^2(n^{k-2}-1)}{n-1}$	6
$PSL(2,n)$, n odd	$\frac{n-1}{2}$	6
$A\Gamma L(k,n)$, $k \geq 2$	$\deg_{GF(p)}(a)$ where $a \in GF(n)$, $n^k - n$	$3 + \frac{1}{\alpha}\sum_{\beta=1}^{\alpha} p^{\gcd(\alpha,\beta)}$
$Sz(2^{2k+1})$, $k \geq 0$	$2^{2k+1}-1$	$2^{2k+1}+5$
$Ree(3^{2k+1})$, $k \geq 0$	$3^{2k+1}-1$, $\frac{3^{2k+1}-1}{2}$	$(3^{2k+1})^2 + 3^{2k+1} + 6$
$PGU(3,n)$	n^2-1, $n-1$	$n+5$
$Sp^{\varepsilon}(2k,2)$, $k \geq 3$	$2^{2k-2}+\varepsilon 2^{k-1}-2$, 2^{2k-2}	6
$PSL(2,11)$ (degree 11)	$3, 6$	6
A_7 (degree 15)	$1, 12$	6
HS	$12, 72, 90$	7
Co_3	$112, 162$	6

Table 1: Third valencies and number of relations of ASTs from two-transitive groups.

3.1 S_k

Let $k \geq 3$. From [5], it is known that S_k is k-transitive. Remark 2.7 implies the associated AST has size $m = 4$. Thus, Lemma 2.5 describes the parameters of this AST.

Remark 3.1. Remark 2.8 ensures that the AST obtained from any group acting three-transitively on a set Ω with $k \geq 3$ points is equal to the AST obtained from the permutation group S_Ω acting on Ω. Such groups include $PGL(2,n)$ for n a prime power, $PSL(2,n)$ for n an even prime power, A_k for $k \geq 5$, and the Mathieu groups. Accordingly, we refrain from tabulating the parameters of the ASTs from these groups.

3.2 $P\Gamma L(k,n)$

Let $k \geq 3$ and n be a power of a prime. For each homogeneous vector $y = [y_1 : \cdots : y_k]^T \in PG(k-1,n)$, denote by y' the affine vector $\frac{1}{y_j}(y_1, y_2, \ldots, y_k)^T$, where j is the first coordinate such that $y_j \neq 0$. Further, denote by u, v, w, and x the respective elements $[1:0:\cdots:0]^T$, $[0:1:0:\cdots:0]^T$, $[1:1:0:\cdots:0]^T$, and $[0:0:1:0:\cdots:0]^T$ of $PG(k-1,n)$.

Direct computation reveals that the two-point stabilizer $P\Gamma L(k,n)_{u,v}$ consists of the maps $z \mapsto A\phi(z)$ for some $A \in GL(k,n)$ and $\phi \in Gal(GF(n))$. Further computations yield the two orbits of $P\Gamma L(k,n)_{u,v}$, namely $\{[1:c:0:\cdots:0]^T : c \neq 0\}$ of size $n-1$ and $\{[c_1:\cdots:c_n]^T : (c_3,\ldots,c_k) \neq (0,\ldots,0)\}$ of size $\frac{n^2(n^{k-2}-1)}{n-1}$.

We now obtain the intersection numbers of the AST $X = \{R_i\}_{i=0}^5$ from $P\Gamma L(k,n)$. For clarity, let $R^w = [(u,v,w)]$ and $R^x = [(u,v,x)]$. Correspondingly, let A^w and A^x denote the respective adjacency hypermatrices. We first prove the equality $A^xA^xA^x = \frac{(n^{k-2}-1)n^2}{n-1}A^w + (\frac{n^k-1}{n-1} - 3n)A^x$. Theorems 2.3 and 2.4 imply that it suffices to determine p^l_{ijk} with $R_i = R_j = R_k = R^x$ and $R_l \in \{R^w, R^x\}$. For this, we utilize the principal regularity condition from Definition 2.1. Initially, consider $R_l = R^w$ and $(u,v,w) \in R_l$. Remark 2.10 implies that $(u,v,z) \in [(u,v,x)]$ is equivalent to $z = [x_1 : \cdots : x_k]^T$ where $(x_3,\ldots,x_k) \neq (0,\ldots,0)$. Moreover, $(u,z,w) \in [(u,v,x)]$ is equivalent to the existence of $B \in GL(k,n)$ such that $B(u,v,x) = (u,z,w)$. Such a B is characterized by having nonzero multiples of u', z', and w' as its first three columns, so B exists provided these three are linearly independent. Similarly, $(z,v,w) \in [(u,v,x)]$ is equivalent to the existence of $A \in GL(k,n)$ such that $A(u,v,x) = (z,v,w)$. Such an A exists if and only if v', z' and w' are linearly independent. Thus, the z satisfying the above inclusions are those of the form $z = [x_1 : \cdots : x_k]^T$, where $(x_3,\ldots,x_k) \neq (0,\ldots,0)$. This yields $p^l_{ijk} = \frac{(n^{k-2}-1)n^2}{n-1}$ values for z when $R_l = R^w$. Similarly, $p^l_{ijk} = \frac{n^k-1}{n-1} - 3n$ when $R_l = R^x$. This proves the claim. The other intersection numbers are obtained similarly and are located in Table 2.

$A^wA^wA^w = (n-2)A^w$
$A^wA^wA^x = A^wA^xA^w = A^xA^wA^w = 0$
$A^wA^xA^x = A^xA^wA^x = A^xA^xA^w = (n-1)A^x$
$A^xA^xA^x = \frac{(n^{k-2}-1)n^2}{n-1}A^w + (\frac{n^k-1}{n-1} - 3n)A^x$
$I_1A^wA^w = (n-1)I_1$
$A^wI_2A^w = (n-1)I_2$
$A^wA^wI_3 = (n-1)I_3$
$I_1A^wA^x = I_1A^xA^w = 0$
$A^wI_2A^x = A^xI_2A^w = 0$
$A^wA^xI_3 = A^xA^wI_3 = 0$
$I_1A^xA^x = \frac{n^2(n^{k-2}-1)}{n-1}I_1$
$A^xI_2A^x = \frac{n^2(n^{k-2}-1)}{n-1}I_2$
$A^xA^xI_3 = \frac{n^2(n^{k-2}-1)}{n-1}I_3$

Table 2: Intersection numbers of ASTs from $P\Gamma L(k,n)$, $k \geq 3$.

Remark 3.2. Let $k \geq 3$ and n be a prime power.

(1) The ternary subalgebra spanned by the adjacency hypermatrices A^w and A^x of the AST from $P\Gamma L(k, n)$ is a commutative ternary algebra. Moreover, A^w spans its own ternary subalgebra.

(2) Let H be a subgroup of $P\Gamma L(k, n)$ containing $PSL(k, n)$. It can be verified that the orbits of $H_{u,v}$ are the same as the orbits of $P\Gamma L(k, n)_{u,v}$. By Lemma 2.9 and Remark 2.8, the ASTs obtained from H and $P\Gamma L(k, n)$ are equal.

3.3 $PSL(2, n)$

Let n be an an odd prime power and fix a quadratic nonresidue $\eta \in GF(n)$. Further, take $u = [1 : 0]^T$, $v = [0 : 1]^T$, $w = [1 : 1]^T$, and $s = [\eta : 1]^T$ from $PG(1, n)$. Lemma 4.3 of [10] states that the two-point stabilizer $PSL(k, n)_{u,v}$ has two equally-sized orbits. These are $\{[a : 1]^T : a$ a quadratic non-residue$\}$ and $\{[a : 1]^T : a$ a quadratic residue, $a \neq 0, 1\}$.

It follows that the AST obtained from $PSL(2, n)$ has two nontrivial relations $R^w = [(u, v, w)]$ and $R^s = [(u, v, s)]$. Letting A^w and A^s be the adjacency hypermatrices corresponding to R^w and R^s, we complete the list in [10, Theorem 4.6] of this AST's intersection numbers and tabulate them in Table 3. We illustrate only that $A^w A^s A^s = \frac{n-1}{4} A^w$ when $n \equiv 1 \pmod 4$ as computations of the other identities are similar. As with the prior subsection, it suffices to determine p^l_{ijk} when $R_i = R^w$, $R_j = R_k = R^s$, and $R_l \in \{A^w, A^s\}$. Taking $R_l = R^w$ and $(u, v, w) \in R_l$, it can be deduced from Lemma 4.3 of [10] that satisfying the inclusions $(z, v, w) \in R^w$, $(u, z, w) \in R^s$, and $(u, v, z) \in R^s$ is equivalent to $z = [c : 1]^T$, where c and $1 - c$ are quadratic non-residues. The number p^l_{ijk} of such c is the number of $x^2 \in GF(n)$ such that there is a y^2 with $\eta x^2 + \eta y^2 = 1$. This equation always has a solution (a_0, b_0) in $GF(n)$, since the sets $\{\eta a^2 : a \in GF(n)\}$ and $\{1 - \eta a^2 : a \in GF(n)\}$ each have $\frac{n+1}{2}$ elements and so must intersect. The solutions (x, y) are then given by $(a_0, b_0) + (a, b)t_{a,b}$ where $t_{a,b} = -2\frac{\eta a a_0 + \eta b b_0}{\eta a^2 + \eta b^2}$ and $(0, 0) \neq (a, b)$. Notice that the obtained solution from (a, b) is the same as the obtained solutions from any nonzero multiple of (a, b). Further, the parametrization fails if and only if $\eta a^2 = -\eta b^2$; i.e., if $\frac{a}{b}$ is a square root of -1. This yields a total of $n-1$ solutions (x, y) to $\eta x^2 + \eta y^2 = 1$. Since we only desire the number of x^2, this yields $p^l_{ijk} = \frac{n-1}{4}$. Similarly, $p^l_{ijk} = 0$ when $R_l = R^s$, completing the verification.

Remark 3.3. Let n be an odd prime power.

(1) The ternary subalgebra spanned by the adjacency matrices A^w and A^s of the AST from $PSL(2, n)$ is a commutative ternary algebra. Moreover, A^w and A^s each span their own ternary subalgebra when $n \equiv 1 \pmod 4$.

(2) From Table 3, the function interchanging A^w and A^s extends linearly to a ternary algebra automorphism of the ternary subalgebra spanned by A^w and A^s.

(3) In Theorem 4.6(ii) of [10], it is claimed that the AST obtained from $PSL(2, n)$ satisfies the following.

$$A^w A^s A_1 = \begin{cases} 0, & n \equiv 1 \pmod 4 \\ \frac{n-1}{2} A_1, & n \equiv 3 \pmod 4. \end{cases}$$

This cannot be the case, as Lemma 2.5 assures that $A^w A^s A_1$ is always 0.

$A^wA^wA^w = \begin{cases} \frac{n-5}{4}A^w, & n \equiv 1 \pmod 4 \\ \frac{n+1}{4}A^s, & n \equiv 3 \pmod 4. \end{cases}$
$A^wA^wA^s = A^wA^sA^w = A^sA^wA^w = \begin{cases} \frac{n-1}{4}A^s, & n \equiv 1 \pmod 4 \\ \frac{n-3}{4}A^w, & n \equiv 3 \pmod 4. \end{cases}$
$A^wA^sA^s = A^sA^wA^s = A^sA^sA^w = \begin{cases} \frac{n-1}{4}A^w, & n \equiv 1 \pmod 4 \\ \frac{n-3}{4}A^s, & n \equiv 3 \pmod 4. \end{cases}$
$A^sA^sA^s = \begin{cases} \frac{n-5}{4}A^s, & n \equiv 1 \pmod 4 \\ \frac{n+1}{4}A^w, & n \equiv 3 \pmod 4. \end{cases}$
$A_1A^wA^w = A_1A^sA^s = \begin{cases} \frac{n-1}{2}A_1, & n \equiv 1 \pmod 4 \\ 0, & n \equiv 3 \pmod 4. \end{cases}$
$A^wA_2A^w = A^sA_2A^s = \begin{cases} \frac{n-1}{2}A_2, & n \equiv 1 \pmod 4 \\ 0, & n \equiv 3 \pmod 4. \end{cases}$
$A^wA^wA_3 = A^sA^sA_3 = \begin{cases} \frac{n-1}{2}A_3, & n \equiv 1 \pmod 4 \\ 0, & n \equiv 3 \pmod 4. \end{cases}$
$A_1A^wA^s = A_1A^sA^w = \begin{cases} 0, & n \equiv 1 \pmod 4 \\ \frac{n-1}{2}A_1, & n \equiv 3 \pmod 4. \end{cases}$
$A^wA_2A^s = A^sA_2A^w = \begin{cases} 0, & n \equiv 1 \pmod 4 \\ \frac{n-1}{2}A_2, & n \equiv 3 \pmod 4. \end{cases}$
$A^wA^sA_3 = A^sA^wA_3 = \begin{cases} 0, & n \equiv 1 \pmod 4 \\ \frac{n-1}{2}A_3, & n \equiv 3 \pmod 4. \end{cases}$

Table 3: Intersection numbers of ASTs from $PSL(2,n)$, n odd.

3.4 $A\Gamma L(k, n)$

Let $k \geq 2$, $n = p^\alpha$ be a prime power, and $H = Gal(GF(n))$. Let $t = (0, 1, 0, \dots, 0)^T$ and $\vec{a} = (a, 0, \dots, 0)^T$ for each $a \in GF(n)$ be the given elements from the k-dimensional vector space over $GF(n)$. The elements of the two-point stabilizer $A\Gamma L(k, n)_{\vec{0},\vec{1}}$ are the transformations of the form $x \mapsto A\phi(x)$ for some $\phi \in H$ and $A \in GL(k, n)$. Applying these to the elements of the affine space yields two types of orbits. The orbits of the first type have the form $\{\overrightarrow{\phi(a)} : \phi \in H\}$ where $a \neq 0, 1$. The orbits of this type are in correspondence with the Galois conjugacy classes of H and have corresponding sizes $\deg_{GF(p)}(a)$. Letting κ be the number of orbits of the first type and letting H act on $GF(n)$, the Burnside Orbit Counting Theorem implies

$$\kappa + 2 = \frac{1}{|H|} \sum_{g \in H} |Fix(g)| = \frac{1}{\alpha} \sum_{\beta=1}^{\alpha} |Fix(x \mapsto x^{p^\beta})| = \frac{1}{\alpha} \sum_{\beta=1}^{\alpha} p^{\gcd(\alpha,\beta)}.$$

The addend 2 is due to 0 and 1 in $GF(n)$, which are stabilized by field automorphisms. The only orbit of the remaining type has size $n^k - n$ and consists of the vectors linearly independent from $\vec{1}$.

Let X be the AST obtained from the group $A\Gamma L(k, n)$. For each $a \in GF(n)$, denote by R^a the relation in X containing the triple $(\vec{0}, \vec{1}, \vec{a})$ and let A^a be the corresponding adjacency hypermatrix. Further, let R^* denote the relation $[(\vec{0}, \vec{1}, t)]$ and let A^* be the corresponding adjacency hypermatrix. Finally, fix any transversal T of the orbits of H on $GF(n) \setminus \{0, 1\}$ so that $\{R^a : a \in T\}$ is the set of nontrivial relations of X of the form $[(\vec{0}, \vec{1}, \vec{a})]$ for $a \neq 0, 1$.

We list the intersection numbers of X in Table 4. To verify the intersection numbers, take $a, b, c \notin \{0, 1\}$. We prove $A^a A^b A^c = \sum_{\ell \in T} p_\ell A^\ell$, where

$$p_\ell = |\{\phi(c) : \phi \in H, \text{ and } (\exists \psi, \tau \in H)((1-\phi(c))\tau(a) + \phi(c) = \psi(b)\phi(c) = \ell)\}|. \quad (3.1)$$

Indeed, let $R_i = R^a$, $R_j = R^b$, $R_k = R^c$ and $R^l \in \{R_a : a \in T\} \cup \{R^*\}$. We first locate $R_l = [(\vec{0}, \vec{1}, v)]$ such that p^l_{ijk} may be nonzero. Take $(\vec{0}, \vec{1}, v) \in R_l$. Direct computation reveals that $(\vec{0}, \vec{1}, z) \in [(\vec{0}, \vec{1}, \vec{c})]$ is equivalent to $z = \overrightarrow{\phi(c)}$ for some $\phi \in H$. Now, $(\vec{0}, z, v) \in (\vec{0}, \vec{1}, \vec{b})$ is equivalent to the existence of $B \in GL(k, n)$ and $\psi \in H$ such that $B(\vec{0}, \vec{1}, \overrightarrow{\psi(b)}) = (\vec{0}, z, v)$. Such $B \in GL(k, n)$ are characterized as those whose first column is z, with $v = B\overrightarrow{\psi(b)} = \psi(b)z$. Furthermore, $(z, \vec{1}, v) \in [(\vec{0}, \vec{1}, \vec{a})]$ is equivalent to the existence of $A \in GL(k, n)$ and $\tau \in H$ such that $A(\vec{0}, \vec{1}, \overrightarrow{\tau(a)}) + (z, z, z) = (z, \vec{1}, v)$. Such A are characterized by those whose first column is $\vec{1} - z$, with $v = \tau(a)(\vec{1} - z) + z$. These three conditions necessitate that $v = \tau(a)(\vec{1} - \overrightarrow{\phi(c)}) + \overrightarrow{\phi(c)} = \overrightarrow{\psi(b)\phi(c)}$ for some $\phi, \psi, \tau \in H$. Hence, for p^l_{ijk} to be nonzero, R_l must be of the form $R^\ell = [(\vec{0}, \vec{1}, \vec{\ell})]$, where $\psi(b)\phi(c) = \ell = \tau(a)(1 - \phi(c)) + \phi(c)$ for some $\phi, \psi, \tau \in H$. Fix such an ℓ via fixing such $\phi, \psi, \tau \in H$. With $R_l = R^\ell$, repeating the above reasoning yields Equation (3.1). All other intersection numbers are obtained similarly.

The next remark extends our results to other subgroups of $A\Gamma L(k, n)$ and the case where $k = 1$.

Remark 3.4. Let $k \geq 1$ and $n = p^\alpha$ be a prime power.

(1) The adjacency hypermatrices of the form A^a with $a \in T$ span a ternary algebra.

$A^aA^bA^c = \sum_{\ell\in T} p_\ell A^\ell$
$A^aA^bA^* = A^aA^*A^b = A^*A^aA^b = 0$
$A^aA^*A^* = A^*A^aA^* = A^*A^*A^a = p_*A^*$
$A^*A^*A^* = (n^k-3n+3)A^* + \sum_{a\in T}(n^k-n)A^a$
$I_1A^aA^b = p_1I_1$
$A^aI_2A^b = p_2I_2$
$A^aA^bI_3 = p_3I_3$
$I_1A^aA^* = I_1A^*A^a = 0$
$A^aI_2A^* = A^*I_2A^a = 0$
$A^aA^*I_3 = A^*A^aI_3 = 0$
$I_1A^*A^* = (n^k-n)I_1$
$A^*I_2A^* = (n^k-n)I_2$
$A^*A^*I_3 = (n^k-n)I_3$

$p_\ell = |\{\phi(c) : \phi \in H \text{ and } (\exists\psi,\tau \in H)((1-\phi(c))\tau(a)+\phi(c) = \psi(b)\phi(c) = \ell)\}|$
$p_* = |\{\tau(a) : \tau \in H\}| = \deg_{\mathrm{Fix}(H)}(a)$
$p_1 = |\{\psi(b) : \psi \in H \text{ and } (\exists\tau \in H)(\tau(a)\psi(b) = 1)\}|$
$p_2 = |\{\psi(b) : \psi \in H \text{ and } (\exists\tau \in H)(\tau(a)\psi(b) = \tau(a)+\psi(b)\}|$
$p_3 = |\{\psi(b) : \psi \in H \text{ and } (\exists\tau \in H)(\tau(a)+\psi(b) = 1)\}|$

Table 4: Intersection numbers of ASTs from $A\Gamma L(k,n)$, $k \geq 2$.

(2) When $k = 1$, the orbits of $A\Gamma L(1,n)_{0,1}$ will only be of the first type. In particular, there is no R^* but the third valencies and intersection numbers not involving R^* remain the same.

(3) The results extend to subgroups of $A\Gamma L(k,n)$ of the form $AGL(k,n) \rtimes K$, where K is any subgroup of $Gal(GF(n))$. In this case, $(AGL(k,n) \rtimes K)_{\vec{0},\vec{1}}$ retains two types of orbits. Orbits of the first type are of the form $\{\overrightarrow{\phi(a)} : \phi \in K\}$ with respective sizes $\deg_{\mathrm{Fix}(K)}(a)$. These are in correspondence with the Galois conjugacy classes of K. Similar computations reveal that there are $\kappa = -2 + \frac{\mathfrak{a}}{\alpha}\sum_{\beta=1}^{\frac{\alpha}{\mathfrak{a}}}(p^{\mathfrak{a}})^{\gcd(\frac{\alpha}{\mathfrak{a}},\beta)}$ orbits of the first type, where $p^{\mathfrak{a}}$ is the size of the fixed field $\mathrm{Fix}(K)$ of K. The set of vectors linearly independent from $\vec{1}$ remains the only orbit of the second type. Analogous computations verify that the intersection numbers of the AST from $AGL(k,n) \rtimes K$ are the same as those given in Table 4, provided any mention of H is replaced by K.

As a particular case of Remark 3.4(3), taking K to be the trivial automorphism group yields the parameters of the AST obtained from $AGL(k,n)$. Further, taking $k = 1$ corrects an error from [10].

Remark 3.5. In Proposition 4.7 of [10], it is claimed that the adjacency hypermatrices of the AST from $AGL(1,n)$ satisfy $A^bI_2A^c = I_2$ and $A^bA^cI_3 = I_3$ if $bc = 1$, and that both products are 0 if $bc \neq 1$. However, these are incorrect. Indeed, if we consider the AST obtained from $AGL(1,5)$, we obtain $A^3I_2A^4 = I_2$ and $A^3A^3I_3 = I_3$. The correct statements are given below.

$$A^aI_2A^b = \begin{cases} I_2, & \text{if } ab = a+b, \\ 0, & \text{otherwise.} \end{cases} \quad \text{and} \quad A^aA^bI_3 = \begin{cases} I_3, & \text{if } a+b = 1, \\ 0, & \text{otherwise.} \end{cases}$$

3.5 $Sz(2^{2k+1})$

Let $n = 2^{2k+1}$ for some $k \geq 0$. In [10, Proposition 4.8], the authors determined that the size of the AST obtained from the Suzuki group $Sz(n)$ is $n+5$. The proof was not given completely, so we include a proof here, thereby also obtaining the third valencies of the nontrivial relations.

Let σ be the automorphism $a \mapsto a^{2^{k+1}}$ of $GF(n)$, $f\colon (GF(n))^2 \to GF(n)$ be the function $(x,y) \mapsto f_{xy} = xy + \sigma(x)x^2 + \sigma(y)$, and Ω be the set

$$\Omega = \{(x, y, f(x,y)) : x, y \in GF(n)\} \cup \{\infty\}.$$

From [5], Ω has n^2+1 points and $Sz(n)$ acts two-transitively on Ω. By the same reference, the stabilizer of $(0,0,0)$ and ∞ is $Sz(n)_{0,\infty} = \{n_a : a \in GF(n) \setminus \{0\}\}$, where $n_a\colon \Omega \to \Omega$ is defined by

$$n_a\colon (x,y,z) \mapsto (ax, \sigma(a)ay, \sigma(a)a^2z).$$

We show that there are two types of orbits when the two-point stabilizer is applied to $\Omega \setminus \{0,\infty\}$. Orbits of the first type have representatives with a nonzero first coordinate while the orbit of the second type will have representatives with a zero first coordinate.

To see these, take any $(x, y, f_{xy}) \in \Omega$ with $x \neq 0$. Applying the elements n_a of $Sz(n)_{0,\infty}$ to (x, y, f_{xy}) yields $n-1$ distinct elements as a ranges over the nonzero elements of $GF(n)$. In particular, exactly one element of the orbit of (x, y, f_{xy}) has a first coordinate of 1. By fixing $x = 1$ and varying y, we obtain n orbits of the first type. If we instead consider an element $(0, y, f_{0,y})$ with $y \neq 0$, similar reasoning gives $n-1$ elements in its orbit and a unique representative of this orbit with second coordinate 1. However, this orbit must then contain all elements of $\Omega \setminus \{0,\infty\}$ with first coordinate 0; hence, there is only one orbit of the second type.

3.6 $Ree(3^{2k+1})$

Let $n = 3^{2k+1}$ for some $k \geq 0$, σ be the automorphism of $GF(n)$ given by $a \mapsto a^{3^{k+1}}$, and $f, g, h\colon (GF(n))^3 \to GF(n)$ be the functions given by

$$\begin{aligned}
f&\colon (x,y,z) \mapsto f_{xyz} = x^2y - xz + \sigma(y) - \sigma(x)x^3,\\
g&\colon (x,y,z) \mapsto g_{xyz} = \sigma(x)\sigma(y) - \sigma(z) + xy^2 + yz - (\sigma(x))^2x^3,\\
h&\colon (x,y,z) \mapsto h_{xyz} = x\sigma(z) - \sigma(x)xy + \sigma(x)x^3y + x^2y^2 - \sigma(y)y - z^2 + (\sigma(x))^2x^4.
\end{aligned}$$

The group $Ree(n)$ acts two-transitively on the set

$$\Omega = \{(x, y, z, f_{xyz}, g_{xyz}, h_{xyz}) : x, y, z \in GF(n)\} \cup \{\infty\}$$

of n^3+1 points [5]. By the same reference, the stabilizer of $(0,0,0,0,0,0)$ and ∞ is $Ree(n)_{0,\infty} = \{n_a : a \in GF(n) \setminus \{0\}\}$, where $n_a\colon \Omega \to \Omega$ is defined by

$$n_a\colon (x,y,z,r,s,t) \mapsto (ax, \sigma(a)ay, \sigma(a)a^2z, \sigma(a)a^3r, (\sigma(a))^2a^3s, (\sigma(a))^2a^4t).$$

Determining the orbits of $Ree(n)_{0,\infty}$ is approached similarly to the determination of the orbits of $Sz(n)_{0,\infty}$. This yields three types of orbits. There are n^2 orbits of the first kind, each with a unique representative of the form $(1, y, z, f_{1yz}, g_{1yz}, h_{1yz})$. There are n orbits of the second kind, each with a unique representative of the form

$(0, y, 1, f_{0y1}, g_{0y1}, h_{0y1})$. An orbit of either of these two types has size $n-1$. To obtain the remaining orbits, we reason as follows.

Take $y \neq 0$ and let $w = (0, y, 0, \sigma(y), 0, -\sigma(y)y)$. Suppose for $a, b \in GF(n) \setminus \{0\}$ that

$$\begin{aligned} n_a(w) &= (0, \sigma(a)ay, 0, \sigma(a)a^3\sigma(y), 0, (\sigma(a))^2a^4(-\sigma(y)y)) \\ &= (0, \sigma(b)by, 0, \sigma(b)b^3\sigma(y), 0, (\sigma(b))^2b^4(-\sigma(y)y)) = n_b(w). \end{aligned}$$

Using the middle equality and the second and fourth coordinates of the involved sextuples, we see that $\sigma(a)a = \sigma(b)b$, and $\sigma(a)a^3 = \sigma(b)b^3$. This tells us that $a = \pm b$. Hence, $n_a(w)$ takes $\frac{n-1}{2}$ distinct values as a ranges over the nonzero elements of $GF(n)$; that is, the orbit of w under $Ree(n)_{0,\infty}$ has $\frac{n-1}{2}$ elements. Since there are $n-1$ sextuples of the form $(0, y, 0, \sigma(y), 0, -\sigma(y)y)$ in Ω, there are two orbits of the third type. Considering possible values of y reveals that y either is or is not of the form $a\sigma(a) = a^{k+2}$ for some $a \neq 0$. Hence, taking $y = 1$ and $y = b$, where b is not a $(k+2)$-th power in $GF(n)$, yields the two orbits of the third type.

3.7 $PGU(3, n)$

For n a prime power, $PGU(3, n)$ is the group $GU(3, n)$ modulo its center. Here $GU(3, n)$ is the group of 3×3 invertible matrices over $GF(n^2)$ which preserve the Hermitian form

$$\varphi((u_1, u_2, u_3)^T, (v_1, v_2, v_3)^T) = u_1{v_3}^n + u_2{v_2}^n + u_3v_1^n$$

on the three-dimensional vector space over $GF(n^2)$. The group $PGU(3, n)$ acts two-transitively on

$$\Omega = \{\langle(1, 0, 0)^T\rangle\} \cup \{\langle(a, b, 1)^T\rangle : a + a^n + bb^n = 0,\ a, b \in GF(n^2)\},$$

the set of $n^3 + 1$ φ-isotropic lines [5]. Letting $E_1 = \langle(1, 0, 0)^T\rangle$ and $E_3 = \langle(0, 0, 1)^T\rangle$, [5] states that

$$GU(3, n)_{E_1,E_3} = \{Diag(c, d, c^{-n}) : c, d \in GF(n^2), dd^n = 1, c \neq 0\}.$$

By scaling each matrix in this set with the appropriate $\frac{1}{d}$, we see that $PGU(3, n)_{E_1,E_2}$ is a cyclic group isomorphic to C_{n^2-1}, each of whose elements is uniquely represented by a diagonal matrix $Diag(c, 1, c^{-n})$ with $c \neq 0$. Given $\langle(a, b, 1)^T\rangle \in \Omega$ with $a, b \neq 0$, the elements of its orbit under $PGU(3, n)_{E_1,E_2}$ are $\langle(ac^{n+1}, bc^n, 1)^T\rangle$, where $c \neq 0$. Since $x \mapsto x^n$ is a field automorphism, this orbit has size $n^2 - 1$ and contains a unique representative of the form $\langle(r, 1, 1)^T\rangle$ with $r + r^n + 1 = 0$. On the other hand, the elements of the orbit of $\langle(a, 0, 1)^T\rangle \in \Omega$, $a \neq 0$, are $\langle(ac^{n+1}, 0, 1)^T\rangle$, $c \neq 0$. The function $f_{n+1} \colon x \mapsto x^{n+1}$ is an endomorphism of $GF(n^2) \setminus \{0\} \cong C_{n^2-1}$ whose kernel consists of the $c \neq 0$ whose order divides $n+1$. This is the multiplicative subgroup of the field isomorphic to $C_{\gcd(n+1,n^2-1)} = C_{n+1}$. It follows that $\mathrm{Im(f_{n+1})} \cong \mathrm{C_{n-1}}$, so the orbit of $\langle(a, 0, 1)^T\rangle \in \Omega$ has $n-1$ elements.

We now determine how many of each type of orbit exists. First, we determine how many elements of Ω are of the form $\langle(s, 0, 1)^T\rangle$ for some $s \neq 0$. To be isotropic subspaces, $\langle(s, 0, 1)^T\rangle$ with $s \neq 0$ must satisfy $s + s^n = 0$ and $s \neq 0$. If n were even, this condition is equivalent to $s^{n-1} = 1$. These s form a subgroup of the multiplicative group of the field isomorphic to $C_{\gcd(n-1,n^2-1)} = C_{n-1}$. If n were odd, the conditions $s+s^n = 0$ and $s \neq 0$ are equivalent to $s^{n-1} = -1$. By reasoning as above, the set of s that satisfy this condition is $\operatorname{Ker}(f_{2(n-1)}) \setminus \operatorname{Ker}(f_{n-1})$ where $f_{n-1}\colon x \mapsto x^{n-1}$ and $f_{2(n-1)}\colon x \mapsto x^{2(n-1)}$ are the given endomorphisms of $GF(n)^2 \setminus \{0\}$. Thus, there are $n-1$ elements of Ω of the form $\langle (s, 0, 1)^T \rangle$ for some $s \neq 0$, regardless of whether or not n is even. Since an orbit of the second type has $n-1$ elements, there is only one such orbit. The remaining orbits must be of the first type, of which there are $\frac{n^3-1-(n-1)}{n^2-1} = n$.

Remark 3.6. Recall that $PGU(3, n) = PSU(3, n)$ whenever 3 does not divide $n + 1$ and that $PSU(3, n)$ is a proper subgroup of $PGU(3, n)$ otherwise [5]. Reasoning as above, each element of $PSU(3, n)_{E_1,E_3}$ is uniquely represented by a matrix of the form $Diag(c, 1, c^{-n})$, where c is in the order $\frac{n^2-1}{3}$ subgroup of $GF(n^2) \setminus \{0\}$. Similar computations then reveal that orbits of $PGU(3, n)_{E_1,E_2}$ of the form $[\langle (r, 1, 1)^T \rangle]$ will be a union of three equally sized orbits of $PSU(3, n)_{E_1,E_2}$. These three have representatives $\langle (a, 1, 1)^T \rangle$, $\langle (b, \alpha, 1)^T \rangle$, and $\langle (c, \alpha^2, 1)^T \rangle$ for some $a, b, c \in GF(q^2)$. Here, $\{1, \alpha, \alpha^2\}$ is a full set of coset representatives of the order $\frac{n^2-1}{3}$ subgroup of $GF(n^2)\setminus\{0\}$. Meanwhile, the orbit of $\langle (s, 0, 1)^T \rangle$ in $PGU(3, n)_{E_1,E_2}$ remains unchanged as an orbit of $PSU(3, n)_{E_1,E_2}$.

3.8 $Sp(2k, 2)$

The following description of the symplectic group uses [5, 8], and [12] as references. For $k \geq 2$, the symplectic group $Sp(2k, 2)$ is the subgroup of $GL(2k, 2)$ preserving the bilinear form

$$b : ((x_1, \ldots, x_k, y_1, \ldots, y_k)^T, (u_1, \ldots, u_k, v_1, \ldots, v_k)^T) \mapsto \sum_{i=1}^{k} (x_i v_i + u_i y_i)$$

on the $2k$-dimensional vector space V over $GF(2)$. The group $Sp(2k, 2)$ acts on the set Ω of quadratic forms q on V which satisfy $b(v, u) = q(v + u) - q(v) - q(u)$ for all $u, v \in V$. This action is given by $Aq(v) = q(A^{-1}v)$ for $q \in \Omega$, $v \in V$, and $A \in Sp(2k, 2)$. It is known that $Sp(2k, 2)$ has two orbits Ω^+ and Ω^- on Ω characterized by the Witt index. Those quadratic forms with Witt index k belong to Ω^+ while those with Witt index $k - 1$ belong to Ω^-. The actions of $Sp(2k, 2)$ on Ω^+ and Ω^- are the two-transitive actions of interest. We use the respective notations $Sp^+(2k, 2)$ and $Sp^-(2k, 2)$ when referring to the symplectic group with respect to these actions. Fix an $\varepsilon \in \{+, -\}$ and let $q \in \Omega^\varepsilon$. The point stabilizer of a quadratic form q in $Sp^\varepsilon(2k, 2)$ is the orthogonal group $O^\varepsilon(q)$. Notice that $O^\varepsilon(q)$ acts naturally on the $2k$-dimensional vector space V over $GF(2)$. In fact, the bijection $\tau\colon (x \mapsto q(x) + b(x, v)) \mapsto v$ from Ω to V is an isomorphism of $O^\varepsilon(q)$-sets. Under τ, the vector $0 \in V$ corresponds to the form $q \in \Omega^\varepsilon$ and the isotropic vectors of V with respect to q correspond to the other elements of Ω^ε.

We utilize τ to describe the orbits of a two-point stabilizer. Fix $k \geq 3$ and $\varepsilon \in \{+, -\}$. Take distinct quadratic forms q_1 and q_2 in Ω^ε. View Ω^ε as the set of q_1-isotropic vectors in V via the $O^\varepsilon(q_1)$-set isomorphism τ so that $\tau(q_1) = 0$ and $\tau(q_2) = v$ for some q_1-isotropic $v \in V$. Through this identification, we view the action of $Sp^\varepsilon(2k, 2)_{q_1,q_2} = O^\varepsilon(q_1)_{q_2}$ on $\Omega^\varepsilon \setminus \{q_1, q_2\}$ as the action on the q_1-isotropic vectors of $V \setminus \{0, v\}$. By Proposition 2(ii) of [7], one orbit consists of the nonzero q_1-isotropic vectors distinct from and orthogonal to v while the other orbit consists of the nonzero q_1-isotropic vectors not orthogonal to v. Proposition 1 in [13] then yields the respective sizes of these orbits, namely $2^{2k-2} + \varepsilon 2^{k-1} - 2$ and 2^{2k-2}.

Remark 3.7. The above reasoning also applies to the case when $k = 2$. Indeed, a two-point stabilizer of $Sp^{+}(4, 2)$ will have two orbits, both of size 4. However, $Sp^{-}(4, 2)$ occurs as a degenerate case since $Sp^{-}(4, 2)$ is three-transitive. In this case, any two-point stabilizer will have only one orbit, necessarily of size 4.

3.9 Sporadics

The sporadic two-transitive groups are the permutation representations of the Mathieu groups $M(n)$ of degree n (with $n \in \{11, 12, 22, 23, 24\}$), the projective group $PSL(2, 11)$ of degree 11, the Mathieu group $M(11)$ of degree 12, the alternating group A_7 of degree 15, the Higman-Sims group HS of degree 176, and the Conway group Co_3 of degree 276 [5]. Due to their higher transitivity, Example 2.5 and Remark 2.7 apply to the ASTs from the Mathieu groups. As such, we refrain from tabulating their parameters. The AST from HS is known to have seven relations [10]. Using GAP 4.11.1 [6], we provide the sizes and the third valencies of the ASTs from the other sporadic two-transitive groups in Table 1. For the ASTs with more than one nontrivial relation, we tabulate their intersection numbers in Tables 5, 6, 7, and 8. In particular, the subalgebra generated by the adjacency hypermatrices of the nontrivial relations of the AST obtained from any sporadic two-transitive group is commutative.

$p^1_{144} = 3$	$p^1_{155} = 6$	$p^2_{424} = 3$	$p^3_{443} = 3$	$p^5_{445} = 1$	$p^5_{454} = 1$
$p^4_{455} = 2$	$p^5_{455} = 1$	$p^2_{525} = 6$	$p^5_{544} = 1$	$p^4_{545} = 2$	$p^5_{545} = 1$
$p^3_{553} = 6$	$p^4_{554} = 2$	$p^5_{554} = 1$	$p^4_{555} = 2$	$p^5_{555} = 2$	

Table 5: Nonzero intersection numbers p^l_{ijk} of the AST from $PSL(2, 11) \leq M(24)$ when at most one of i, j, k is less than 4.

$p^1_{144} = 1$	$p^1_{155} = 12$	$p^2_{424} = 1$	$p^3_{443} = 1$	$p^5_{455} = 1$	$p^2_{525} = 12$
$p^5_{545} = 1$	$p^3_{553} = 12$	$p^5_{554} = 1$	$p^4_{555} = 12$	$p^5_{555} = 9$	

Table 6: Nonzero intersection numbers p^l_{ijk} of the AST from $A_7 \leq PSL(4, 2)$ when at most one of i, j, k is less than 4.

$p^1_{144} = 72$	$p^1_{155} = 90$	$p^1_{166} = 12$	$p^2_{424} = 72$	$p^3_{443} = 72$	$p^4_{444} = 20$
$p^5_{445} = 32$	$p^6_{445} = 30$	$p^5_{446} = 4$	$p^6_{446} = 6$	$p^5_{454} = 32$	$p^6_{454} = 30$
$p^4_{455} = 40$	$p^4_{456} = 5$	$p^5_{464} = 4$	$p^6_{464} = 6$	$p^4_{465} = 5$	$p^4_{466} = 1$
$p^2_{525} = 90$	$p^5_{544} = 32$	$p^6_{544} = 30$	$p^4_{545} = 40$	$p^4_{546} = 5$	$p^3_{553} = 90$
$p^4_{554} = 40$	$p^5_{555} = 41$	$p^6_{555} = 60$	$p^5_{556} = 8$	$p^4_{564} = 5$	$p^5_{565} = 8$
$p^2_{626} = 12$	$p^5_{644} = 4$	$p^6_{644} = 6$	$p^4_{645} = 5$	$p^4_{646} = 1$	$p^4_{654} = 5$
$p^5_{655} = 8$	$p^3_{663} = 12$	$p^4_{664} = 1$	$p^6_{666} = 5$		

Table 7: Nonzero intersection numbers p^l_{ijk} of the AST from HS when at most one of i, j, k is less than 4.

$p^1_{144} = 162$	$p^1_{155} = 112$	$p^2_{424} = 162$	$p^3_{443} = 162$	$p^4_{444} = 105$	$p^5_{445} = 81$
$p^5_{454} = 81$	$p^4_{455} = 56$	$p^2_{525} = 112$	$p^5_{544} = 81$	$p^4_{545} = 56$	$p^3_{553} = 112$
$p^4_{554} = 56$	$p^5_{555} = 30$				

Table 8: Nonzero intersection numbers p^l_{ijk} of the AST from Co_3 when at most one of i, j, k is less than 4.

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On a generalization of median graphs: k-median graphs*

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Abstract

Median graphs are connected graphs in which for all three vertices there is a unique vertex that belongs to shortest paths between each pair of these three vertices. To be more formal, a graph G is a median graph if, for all $\mu, u, v \in V(G)$, it holds that $|I(\mu, u) \cap I(\mu, v) \cap I(u, v)| = 1$ where $I(x, y)$ denotes the set of all vertices that lie on shortest paths connecting x and y.

In this paper we are interested in a natural generalization of median graphs, called k-median graphs. A graph G is a k-median graph, if there are k vertices $\mu_1, \dots, \mu_k \in V(G)$ such that, for all $u, v \in V(G)$, it holds that $|I(\mu_i, u) \cap I(\mu_i, v) \cap I(u, v)| = 1$, $1 \leq i \leq k$. By definition, every median graph with n vertices is an n-median graph. We provide several characterizations of k-median graphs that, in turn, are used to provide many novel characterizations of median graphs.

Keywords: Median graph, convexity, meshed and quadrangle property, modular, interval.

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1 Introduction

A median graph is a connected graph, in which, for each triple of vertices x, y, z there exists a unique vertex $\mathrm{med}(x, y, z)$, called the median, simultaneously lying on shortest paths between each pair of the triple [9]. Denoting with $I(u, v)$ the set of all vertices that

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lie on shortest paths connecting u and v and putting $I(x,y,z) := I(x,y) \cap I(x,z) \cap I(y,z)$, a graph is a median graph precisely if $|I(x,y,z)| = 1$ for all of its vertices x, y, z [34, 35]. Median graphs have been studied at least since the 1940's [1, 9, 37] and naturally arise in several fields of mathematics, for example, in algebra [5], metric graph theory [4], geometry [19] or the study of split systems [17, 22, 28]. Moreover, they have practical applications in areas such as social choice theory [2, 21], phylogenetics [16, 23, 28], and forensic science [39].

In this paper, we study a generalization of median graphs, that is, k-median graphs. A vertex μ in a graph G is called median-consistent (medico), if $|I(\mu,x,y)| = 1$ for all $x, y \in V(G)$. A graph G is a k-median graph if it contains (at least) k medico vertices. In other words, G is a k-median graph if the set W obtained from $V(G)$ by removal of all vertices x for which $|I(x,y,z)| \neq 1$ satisfies $|W| \geq k$. The existence of medico vertices has been studied in a work of Bandelt et al. [8] in the context of modular graphs, that is, graphs that satisfy $I(x,y,z) \neq \emptyset$ for all $x, y, z \in V(G)$. In [8], medico vertices were called "neutral" and the authors provided a simple characterization of medico vertices in modular graphs G in terms of intervals $I(x,y)$ and $K_{2,3}$ subgraphs (cf. [8, Proposition 5.5])

One may think of the integer k in k-median graphs as a measure on how "close" a given graph is to a median graph as every median graph G is a $|V(G)|$-median graph. Studying k-median graphs is also motivated by the following observations. In [16], the authors were interested in "representing" edge-colored graphs by rooted median graphs. To be more precise, given an edge-colored graph H the task is to find a vertex-colored median graph G with a distinguished vertex μ (called root in [16]) such that the color of the unique median $\mathrm{med}(\mu,x,y)$ corresponds to the color of the (non)edges $\{x,y\}$ in H for all $x, y \in V(H)$. The latter, in particular, generalizes the concept of so-called cographs [20], symbolic ultrametrics [10, 27, 30], or unp 2-structures [24, 29], that is, combinatorial objects that are represented by rooted vertex-colored trees. The idea of using median graphs instead of trees can be generalized even more by asking for an arbitrary vertex-colored graph G that contains a vertex μ for which the median $\mathrm{med}(\mu,x,y)$ is well-defined for all $x, y \in V(G)$ and the color of $\mathrm{med}(\mu,x,y)$ distinguishes between the colors of the underlying (non)edges $\{x,y\}$ in the edge-colored graph H. In this case, it is only required that G is a 1-median graph and thus, contains (at least) one medico vertex.

In this paper, we study the structural properties of k-median graphs which, in turn, lead also to novel characterizations of median graphs. By way of example, we show that median graphs are precisely those graphs G for which $I(x,y,z) \neq \emptyset$ and the subgraph induced by the vertices in $I(x,y,z)$ is connected for all $x, y, z \in V(G)$ (cf. Theorem 5.11). This paper is organized as follows. In Section 2, we introduce necessary notation and definitions. We then summarize our main results in Section 3. In Section 4, we establish basic structural properties and provide necessary conditions for k-median graphs. In Section 5 we provide characterizations of k-median that, in turn, result in plenty of new characterizations of median graphs. We finally show in Section 6 how k-median graphs and the Cartesian graph product are related. We complement these results by a simple program to verify if a given graph is a k-median graph and to compute the largest such integer k in the affirmative case. This straightforward algorithm is written in Python, hosted at GitHub [26]) and can be used to verify the examples.

2 Preliminaries

For a set A, we write $A^n := \times_{i=1}^n A$ for the n-fold Cartesian set product of A. All graphs $G = (V, E)$ considered here are undirected, simple and finite and have vertex set $V(G) := V$ and edge set $E(G) := E$. We put $|G| := |V|$ and $\|G\| := |E|$. We write $G[W]$ for the graph that is induced by the vertices in $W \subseteq V$. A graph G is *H-free* if G does not contain an induced subgraph isomorphic to H. For two graphs G and H, their *intersection* $G \cap H$ is defined as the graph $(V(G) \cap V(H), E(G) \cap E(H))$. A *complete graph* $K_{|V|} = (V, E)$ is a graph that satisfies $\{u, v\} \in E$ for all distinct $u, v \in V$. A graph $G = (V, E)$ is bipartite if its vertex set V can be partitioned into two non-empty sets V_1 and V_2 such that $\{x, y\} \in E$ implies $x \in V_i$ and $y \in V_j$, $i \neq j$. A bipartite graph $G = (V, E)$ with bipartition $V_1 \cupdot V_2 = V$ is *complete* and denoted by $K_{|V_1|,|V_2|}$, if $x \in V_i$ and $y \in V_j$, $i \neq j$ implies $\{x, y\} \in E$. A *cycle* C is a connected graph in which all vertices have degree 2. A cycle of length $n = |C|$ is denoted by C_n. A *hypercube* Q_n has vertex set $\{0, 1\}^n$ and vertices are adjacent if their coordinates differ in precisely one position. We write Q_n^- for the hypercube Q_n from which one vertex and its incident edges has been deleted.

All paths in $G = (V, E)$ are considered to be simple, that is, no vertex is traversed twice. The distance $d_G(u, v)$ between vertices u and v in a graph G is the length $\|P\|$ of a shortest path P connecting u and v. Note that the distance for all vertices is well-defined whenever G is connected and that $d_G(x, y) = d_G(y, x)$. For a subset $W \subseteq V$ we say that $w \in W$ is (among the vertices in W) *closest* to $v \in V$ if $d_G(v, w) = \min_{x \in W} d_G(v, x)$. We call paths connecting u and v also (u, v)-paths. A path on $n = |P|$ vertices is denoted by P_n. A subgraph H of G is *isometric* if $d_H(u, v) = d_G(u, v)$ for all $u, v \in V(H)$. A subgraph H of G is *convex* if for any $u, v \in V(H)$, all shortest (u, v)-paths belong to H. The *convex hull* of a subgraph H in G is the least convex subgraph of G that contains H. If the context is clear, we may write $d(\cdot, \cdot)$ instead of $d_G(\cdot, \cdot)$ for a given graph G.

For later reference, we provide the following two results.

Lemma 2.1. *For every edge $\{u, v\}$ in a connected graph G and every vertex $x \in V(G)$ it holds that $0 \leq |d_G(x, u) - d_G(x, v)| \leq 1$. Moreover, G is bipartite if and only if $d_G(x, u) \neq d_G(x, v)$ for all $x \in V(G)$ and $\{u, v\} \in E(G)$. In particular, if there is a vertex x such that $d_G(x, u) \neq d_G(x, v)$ for all $\{u, v\} \in E(G)$, then G is bipartite.*

Proof. The first statement is an easy consequence of the triangle inequality, that is, $d_G(x, u) \leq d_G(x, v) + d_G(u, v)$ implies $|d_G(x, u) - d_G(x, v)| \leq d_G(u, v) = 1$ for all edges $\{u, v\}$ and all vertices x. The second statement is equivalent to [38, Theorem 2.3]. Suppose now that there is a vertex x in $G = (V, E)$ such that $d_G(x, u) \neq d_G(x, v)$ for all edges $\{u, v\} \in E$. Let $V_1 := \{w \in V \mid d_G(x, w) \text{ is odd}\}$, $V_2 := \{w \in V \mid d_G(x, w) \text{ is even}\}$ and $\{u, v\} \in E$. Since $d_G(x, u) \neq d_G(x, v)$ and by the first statement, it holds $u \in V_i$ and $v \in V_j$, $i \neq j$. As this holds for all edges in E, the sets V_1, V_2 form a valid bipartition of G. □

Lemma 2.2. *Let $H \in \{K_{2,3}, C_6, Q_3^-\}$ be a subgraph of a bipartite graph G. Then, H is an induced subgraph of G if and only if H is an isometric subgraph of G.*

Proof. Note that every isometric subgraph H of any graph G must be induced. Hence, it suffices to show the *only-if* direction. Let H be an induced $K_{2,3}$. For any two vertices x, y in H it holds that $1 \leq d_H(x, y) \leq 2$. Hence, $d_G(x, y) < d_H(x, y)$ would imply that $d_G(x, y) = 1$ and $d_H(x, y) = 2$ in which case H would not be induced. Hence H is isometric. Let H be an induced C_6 or Q_3^- and thus, $d_H(x,y) \leq 3$ for any two vertices x, y in H. Again, $d_G(x,y) = 1$ and $d_H(x,y) > 1$ is not possible for any $x, y \in V(H)$ since, otherwise, H would not be induced. Assume, for contradiction, there are two vertices x, y in H with $d_G(x,y) < d_H(x,y)$. Taken the latter arguments together, $d_G(x,y) = 2$ and $d_H(x,y) = 3$ must hold. Let one of the (xy)-paths P in H contain the additional vertices u, v and edges $\{x,u\}$, $\{u,v\}$ and $\{v,y\}$. Let the path consisting of x, w, y and edges $\{x,w\}$ and $\{w,y\}$ be a shortest (x,y)-path P' in G. Observe first that $w \neq u$ and $w \neq v$ since, otherwise, H would not be induced. Hence, $\{x,u,v,y\}$ and $\{x,w,y\}$ intersect only in the vertices x and y. Consequently, $P \cup P'$ forms a C_5 in G. Hence, G is not bipartite; a contradiction. Therefore, $d_G(x,y) = d_H(x,y)$ for all vertices x, y in H and H is isometric. □

The *interval* between x and y is the set $I_G(x,y)$ of all vertices that lie on shortest (x,y)-paths. A vertex x is a *median* of a triple of vertices u, v and w if $d(u,x)+d(x,v) = d(u,v)$, $d(v,x)+d(x,w) = d(v,w)$ and $d(u,x)+d(x,w) = d(u,w)$. Equivalently, x is a median of u, v and w if $x \in I_G(u,v,w) \coloneqq I_G(u,v) \cap I_G(u,w) \cap I_G(v,w)$ [34]. If $I_G(u,v,w) = \{x\}$ consist of x only, we put $\mathrm{med}_G(u,v,w) \coloneqq x$ and say that $\mathrm{med}_G(u,v,w)$ *is well-defined*. A graph G is a *median graph* if, for every triple u, v and w of its vertices, $\mathrm{med}_G(u,v,w)$ is well-defined. [32, 36]. A graph is G *modular*, if $I_G(u,v,w) \neq \emptyset$ for all $u, v, w \in V$. One easily verifies that median graphs are modular and that modular graphs must be connected. Following [34], a graph G is called *interval-monotone* if, for all $u, v \in V$, $I(x,y) \subseteq I(u,v)$ for all $x, y \in I(u,v)$, i.e., the induced subgraph $G[I(u,v)]$ is convex. If the context is clear, we may write $I(\cdot,\cdot)$, resp., $I(\cdot,\cdot,\cdot)$ instead of $I_G(\cdot,\cdot)$, resp., $I_G(\cdot,\cdot,\cdot)$ for a given graph G.

For later reference, we summarize here the results Propositions 1.1.2(iii) and 1.1.3. established in [34].

Lemma 2.3. *Let G be a connected graph. Then, $x \in I_G(u,v)$ implies that $I_G(u,x) \subseteq I_G(u,v)$. Moreover, for any three vertices u, v, w of G there exists a vertex $z \in I_G(u,v) \cap I_G(u,w)$ such that $I_G(z,v) \cap I_G(z,w) = \{z\}$.*

In the upcoming proofs the following definition will play a particular role.

Definition 2.4 (distance-ℓ-static and meshed)**.** Let $G = (V,E)$ be a graph. A quartet $(x,z,y,w) \in V^4$ is *distance-ℓ-static* for some $\ell \geq 1$, if $d_G(x,w) = d_G(z,w) = \ell = d_G(y,w) - 1$ and $d_G(x,z) = 2$ with y a common neighbor of x and z.

G is called *meshed* if all distance-ℓ-static quartets $(x,z,y,w) \in V^4$ satisfy the *quadrangle property*: *there exists a common neighbor u of x and z with $d_G(u,w) = \ell - 1$.*

Meshed graphs play a central role in the characterization of median graphs [11, 32]. The vertices x, z, y, w in an distance-ℓ-static quartet (x,z,y,w) must, by definition, be pairwise distinct. Note that (x,z,y,w) is distance-ℓ-static precisely if (z,x,y,w) is distance-ℓ-static. The order of the remaining vertices in an distance-ℓ-static quartet (x,z,y,w) matters, since the last vertex w serves as "reference" to the vertex to which the distances are taken for x, y, z and the third vertex y implies that implies that x, y, z must induce a path P_3 with edges $\{x,y\}$ and $\{y,z\}$.

3 Main results

Consider a graph $G = (V,E)$ and let $W \subseteq V$ denote the set of vertices obtained from V by removing any three vertices u, v, w for which $\mathrm{med}_G(u,v,w)$ is not well-defined. In this

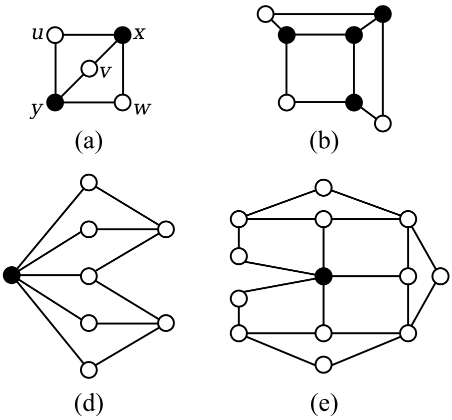

Figure 1: Shown are several k-median graphs. The respective medico vertices are highlighted as black vertices. Shown are (a) a $K_{2,3}$, (b) a Q_3^-, (c) a combination of both a $K_{2,3}$ and Q_3^-, (d) a Q_3^--free proper 1-median graph, (e) a $K_{2,3}$-free proper 1-median graph. (f) This graph serves as a generic example of a proper k-median graph for all $k \geq 2$.

case, W contains all vertices μ for which $\mathrm{med}_G(\mu, v, w)$ is well-defined for all $v, w \in V$ which gives rise to the following

Definition 3.1 (medico vertices and k-median graph)**.** A vertex $\mu \in V$ is *median-consistent (medico)* in a graph G if $\mathrm{med}_G(\mu, v, w)$ is well-defined for all distinct $v, w \in V(G) \setminus \{\mu\}$.

A graph is a *k-median graph* if it has $k \geq 1$ medico vertices. Moreover, a k-median graph is a *proper* k-median graph if it has precisely k medico vertices.

Note that if a graph G is not connected, then $I_G(x, y, z) = \emptyset$ for any two vertices x, y in one connected component and z in another connected component. Hence, disconnected graphs cannot contain any medico vertex. We summarize this finding and other simple results in

Observation 3.2. *Let G be a graph.*

- *If G contains a medico vertex μ, then G is connected.*
- *For every $k \geq 1$, there is a proper k-median graph (cf.* Figure 1(d,e,f)*).*
- *Every k-median graph is an ℓ-median graph for all $\ell \in \{1, \dots, k\}$.*
- *A graph $G = (V, E)$ is a* median graph *if and only if G is $|V|$-median graph.*

The following result shows that there are no proper k-median graphs G with $k \geq |V|-2$ and provides a new although rather simple characterization of median graphs.

Proposition 3.3. *A graph $G = (V, E)$ is a median graph if and only if G is a $(|V| - 1)$- or a $(|V| - 2)$-median graph. Hence, there is no proper $(|V| - 1)$- or $(|V| - 2)$-median graph.*

Proof. If $G = (V, E)$ is median graph, then the last two statements in Observation 3.2 imply that G is a $(|V| - 1)$- or $(|V| - 2)$-median graph. By contraposition, assume that G is not a median graph. In this case, there is at least one vertex x that is not a medico vertex of G. Thus, there are two vertices y and z such that x, y and z are pairwise distinct and $\mathrm{med}(x, y, z)$ is not well-defined. Hence, neither y nor z can be a medico vertex. Hence, G is either no k-median graph at all or, if G is a k-median graph, then $k \leq |V| - 3$ which implies that $k \notin \{|V| - 1, |V| - 2\}$. □

In contrast to median graphs, k-median graphs may contain an induced $K_{2,3}$ (cf. Figure 1). Moreover, median graphs are characterized as those graphs G for which the convex hull of every isometric cycle C is a hypercube (cf. [32, Theorem 5]). As a Q_3^- is a 4-median graph that contains an isometric cycle C_6, the latter property is, in general, not satisfied for k-median graphs.

We give now a brief overview of our main results. In Section 4, we provide several necessary conditions and show, among other results, that every k-median is bipartite, $K_{3,3}$-free and that every edge $e \in E(C)$ of every cycle C of G is also part of an induced C_4 in G. Moreover, we characterize the existence of induced $K_{2,3}$s in k-median graphs G and show, in addition, that G contains a Q_3^- precisely if it contains an induced C_6. We then continue in Section 5 with a generalization of convex subgraphs.

Definition 3.4 (v-convex)**.** Let G be a graph and $v \in V(G)$. A subgraph H of G is *v-convex* if $v \in V(H)$ and every shortest path connecting v and x in G is also contained in H, for all $x \in V(H)$.

Note that v-convex subgraphs H of G satisfy $I_G(v, x) \subseteq V(H)$ for all $x \in V(H)$. Moreover, v-convex subgraphs are connected but not necessarily induced, isometric or convex, see Figure 2 for an example. We then derive

Characterization 1 (Theorem 5.4)**.** *$G = (V, E)$ is a k-median graph if and only if there k vertices $\mu_1, \ldots, \mu_k \in V$ such that μ_i is a medico vertex in every μ_i-convex subgraph of G, $1 \leq i \leq k$.*

This, in turn, yields a well-known characterization of median graphs stating that G is a median graph if and only if every convex subgraph of G is a median graph, see e.g. [34]. We then make frequent use of the following conditions (C0), (C1) and (C2).

Definition 3.5. Let $G = (V, E)$ be a graph and $u \in V$. Then, G satisfies

(C0) with respect to u if: For all $v, w \in V$ it holds that

$$I_G(u, v) \cap I_G(v, w) = \{v\} \text{ implies } v \in I_G(u, w).$$

(C1) with respect to u if: For all $v, w \in V$ it holds that

$$G[I_G(u, v, w)] \text{ contains at least one vertex and is connected.}$$

(C2) with respect to u if: For all $v, w \in V$ it holds that

$$G[I_G(u, v, w)] \text{ contains an edge whenever it contains more than one vertex.}$$

Note that G satisfies (C0) with respect to u if and only if $I_G(u, v) \cap I_G(v, w) = \{v\}$ implies that $d_G(u, v) + d_G(v, w) = d_G(u, w)$ for all $v, w \in V$. Moreover, if G satisfies (C1) with respect to some $u \in V$, then G must be connected as, otherwise, $I_G(u, v, w) = \emptyset$ for u and v being in distinct connected components. One easily verifies that every median graph satisfies (C0), (C1) and (C2) with respect to each of its vertices. A simple example of a graph that does neither satisfy (C0) nor (C1) with respect to some vertex u is an induced C_6. To see this, let $G \simeq C_6$ and v and w be the vertices in G that have distance 2 to u. In this

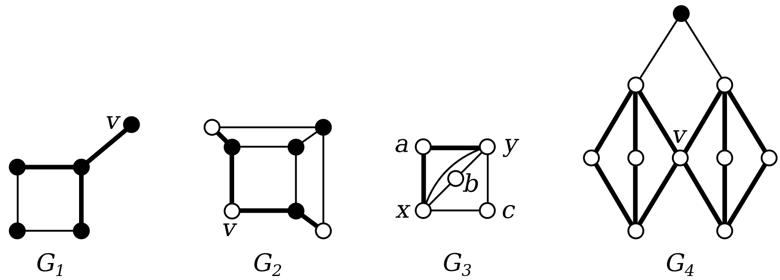

Figure 2: Shown are four graphs G_1, G_2, G_3 and G_4. Medico vertices are highlighted as black vertices and subgraphs H_i of G_i, $1 \leq i \leq 4$, are highlighted by thick edges. All H_i are v-convex subgraphs of G_i but not convex. Since G_1 is a median graph and v a medico vertex of G_1, H_1 is isometric and thus, induced (cf. Lemma 5.2). In contrast, v is not a medico vertex in the graph G_2. Although H_2 is v-convex and induced, it is not isometric. The graph G_3 does not contain any medico vertices. The subgraph H_3 of G_3 is a-convex, but not induced. Adding the edge $\{x, y\}$ to H_3 would yield another a-convex subgraph that is also convex. Note that $I := I_{G_3}(a, b, c) = \{x, y\}$. Hence, $G_3[I] \simeq K_2$ consists of the edge $\{x, y\}$ and is thus, connected and non-empty. The v-convex subgraph H_4 of G_4 is not a 1-median graph. According to Lemma 5.1, v-convex subgraphs with specified vertex set are uniquely determined in bipartite graphs.

case, $G[I(u, v, w)]$ does not contain any vertices and $I_G(u, v) \cap I_G(v, w) = \{v\}$ although the unique shortest path connecting u and w does not contain v, that is, $v \notin I_G(u, w)$. In contrast, $G \simeq C_6$ trivially satisfies (C2) with respect to all of its vertices, since $|I(u, v, w)|$ is either empty or contains precisely one element for all $u, v, w \in V$ Consider now the graph $G := G_3$ in Figure 2. The set $I_G(a, b, c) = \{x, y\}$ is the only one among the other $I_G(a, \cdot, \cdot)$ that contains more than one vertex. Moreover, $G[I_G(a, b, c)]$ contains the edge $\{x, y\}$ and is, thus, connected. Hence G satisfies (C2) with respect to a. However, $I_G(a, x, y) = \emptyset$ which implies that G does not satisfy (C1) with respect to a. Moreover, G does not satisfy (C0) with respect to a, since $I_G(a, x) \cap I_G(a, y) = \{a\}$ but $a \notin I_G(x, y)$.

We provide first a characterization that is based on Condition (C1).

Characterization 2 (Theorem 5.8)**.** *A graph G is a k-median graph if and only if G satisfies* (C1) *with respect to $\mu_1, \ldots, \mu_k \in V(G)$. In this case, the vertices $\mu_1, \ldots, \mu_k \in V$ are medico vertices of G.*

Mulder provided the following characterization of median graphs.

Theorem 3.6 ([34, Theorem 3.1.7])**.** *A graph G is a median graph if and only if G is connected, interval-monotone and satisfies* (C0) *with respect to all of its vertices.*

By definition, k-median graphs are connected and, as shall we see later in Lemma 5.9, always satisfy (C0) with respect to k of its vertices. However, k-median graphs are, in general, not interval-monotone. The simplest example is possibly the 2-median graph $K_{2,3}$ as shown in Figure 1(a) with bipartition $\{x, y\} \cup \{u, v, w\}$. One observes that $I(u, v) = \{u, v, x, y\}$ and, since $w \in I(x, y)$, it holds that $I(x, y) \not\subseteq I(u, v)$. Hence, the subgraph induced by $I(u, v)$ is not convex. This begs the question to what extent Theorem 3.6

can be generalized to cover the properties of k-median graphs. As it turns out, interval-monotonicity can be replaced by Condition (C2) together with bipartiteness.

Characterization 3 (Theorem 5.10)**.** *A graph G is a k-median graph if and only if G is connected, bipartite and satisfies (C0) and (C2) with respect to $\mu_1, \dots, \mu_k \in V(G)$. In this case, the vertices $\mu_1, \dots, \mu_k \in V$ are medico vertices of G.*

The latter results naturally translate into novel characterizations of median graphs.

Characterization 4 (Theorems 5.4, 5.11 & 5.12)**.** *For every graph G, the following statements are equivalent.*

(1) *G is a median graph.*

(2) *Every $v \in V(G)$ is a medico vertex in every v-convex subgraph of G.*

(3) *G satisfies (C1) with respect to all of its vertices.*

(4) *G is connected, bipartite and satisfies (C0) and (C2) with respect to all of its vertices.*

(5) *The graph $G(u, v, w) := G(u, v) \cap G(u, w) \cap G(v, w)$ is not empty and connected for all $u, v, w \in V$ where $G(x, y)$ denotes the subgraph of G with $V(G(x, y)) = I_G(x, y)$ and where $E(G(x, y))$ consists precisely of all edges that lie on the shortest (x, y)-paths, $x, y \in V$.*

We complement these results by showing in Section 6 how k-median graphs and the Cartesian graph product are related.

4 Necessary conditions and subgraphs

We provide in this section several properties that must be satisfied by every k-median graph. We start with considering distances of medico vertices to adjacent vertices and to vertices on isometric cycles.

Lemma 4.1. *Let $G = (V, E)$ be a k-median graph and μ be a medico vertex in G. Then $|d_G(\mu, u) - d_G(\mu, v)| = 1$ for all edges $\{u, v\} \in E$.*

Proof. Let $G = (V, E)$ be a k-median graph with medico vertex μ and $\{u, v\} \in E$ be an edge. If $u = \mu$ or $v = \mu$, then the statement is vacuously true. Hence, suppose that $u, v \neq \mu$. Since $\{u, v\} \in E$, we have $I_G(u, v) = \{u, v\}$. Moreover, since μ is a medico vertex in G it must hold $|I_G(\mu, u) \cap I_G(\mu, v) \cap I_G(u, v)| = 1$. Hence, we may assume, without loss of generality, that $\mathrm{med}(\mu, u, v) = u$. Then, by the definition of medians, $d_G(\mu, v) = d_G(\mu, u) + d_G(u, v) = d_G(\mu, u) + 1$ and, therefore, $d_G(\mu, v) - d_G(\mu, u) = 1$. □

Lemma 2.1 and 4.1 imply

Proposition 4.2. *Every k-median graph is bipartite.*

As a consequence of Lemma 2.2 and Proposition 4.2 we obtain

Observation 4.3. *For $K_{2,3}$, C_6 and Q_3^- subgraphs of a k-median graph the terms "induced" and "isometric" can be used interchangeably.*

Lemma 4.4. *Let $G = (V, E)$ be a k-median graph and μ be a medico vertex in G. Moreover, suppose that G contains an induced cycle C. Let $v \in V(C)$ be a vertex in C that is closest to μ with $K := d_G(v, \mu) = \min_{w \in V(C)} d_G(w, \mu)$. Furthermore, let $u \in V(C)$ and put $i := d_C(u, v)$. Then,*

$$d_G(\mu, u) \in \begin{cases} \{K+1, K+3, \dots, K+i\} & \text{if } i \text{ is odd} \\ \{K, K+2, \dots, K+i\} & \text{if } i \text{ is even.} \end{cases}$$

Proof. Let $G = (V, E)$ be a k-median graph, μ be a medico vertex in G and C be an induced cycle in G. Moreover, let $v \in V(C)$ be a vertex in C that is closest to μ and put $K := d_G(v, \mu)$. Now consider a vertex u for which $d_C(u, v) = i \geq 0$. Note that $i \leq \frac{|V(C)|}{2}$.

We proceed now by induction on the length $d_C(u, v) = i$. If $i = 0$, we have $u = v$ and thus, trivially, $d_G(u, \mu) = K$ and i is even. If $i = 1$, then $\{u, v\} \in E$. Since v is a vertex in C that is closest to μ, we have $d_G(\mu, u) \geq d_G(\mu, v) = K$. This together with Lemma 4.1 implies that $d_G(\mu, u) - d_G(\mu, v) = 1$ and thus, $d_G(\mu, u) \in \{K + 1\}$ and i is odd.

Assume, now that the statement is true for all i with $0 \leq i < \frac{|V(C)|}{2}$. Let $u \in V(C)$ be a vertex with $d_C(u, v) = i + 1$. As we already verified the cases $d_C(u, v) \in \{0, 1\}$, we may assume that $d_C(u, v) \geq 2$. By Proposition 4.2, G is bipartite and thus, $|C| \geq 4$. Hence, such a vertex $u \in V(C)$ with $d_C(u, v) \geq 2$ indeed exists. Since $d_C(u, v) \geq 2$, there is a vertex $u' \in V(C)$ that satisfies $d_C(v, u') = i$ and $\{u', u\} \in E$. If $d_C(u, v)$ is odd, $d_C(u', v)$ must be even and we obtain, by induction hypothesis, $d_G(\mu, u') \in \{K, K+2, \dots, K+i\}$. By Lemma 4.1, $|d_G(\mu, u) - d_G(\mu, u')| = 1$. Note that the choice of v and $d_G(v, \mu) = K$ implies that $d_G(\mu, u) = K - 1$ is not possible. The latter three arguments now imply that $d_G(\mu, u) \in \{K+1, K+3, \dots, K+(i+1)\}$. By similar arguments, if $d_C(u, v)$ is even, $d_C(u', v)$ must be odd and $d_G(\mu, u') \in \{K+1, K+3, \dots, K+i\}$ which together with Lemma 4.1 implies that $d_G(\mu, u) \in \{K, K+2, K+4, \dots, K+(i+1)\}$. □

We provide now a mild generalization of Proposition 5.5 in [8] which states that every *modular* k-median graph must be $K_{3,3}$-free.

Lemma 4.5. *Every k-median graph is $K_{3,3}$-free.*

Proof. Let $G = (V, E)$ be a k-median graph and μ be a medico vertex in G. Assume, for contradiction, G contains an induced $K_{3,3} \subseteq G$ whose vertex set is partitioned into $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, y_3\}$ and all edges of this $K_{3,3}$ are of the form $\{x_i, y_j\}$, $1 \leq i, j \leq 3$.

Let $v \in X \cup Y$ be one of the vertices that is closest to μ in G, i.e., $d_G(\mu, v) = \min_{w \in X \cup Y} d_G(\mu, w) = K \geq 0$. Since the $K_{3,3}$ is induced, we can assume without loss of generality that $v = x_1$. Note that x_1, x_2, y_1, y_2 induce a C_4. Now, we can apply Lemma 4.4 and conclude that $d_G(\mu, y_1) = d_G(\mu, y_2) = K + 1$ and $d_G(\mu, x_2) \in \{K, K + 2\}$.

Assume, for contradiction, that $d_G(\mu, x_2) = K$. Note that $x_1, x_2 \in I_G(y_1, y_2)$ since the $K_{3,3}$ is induced. Moreover, one can extend any shortest path from μ to x_1 (which has length K) by adding the edge $\{x_1, y_1\}$ to obtain a shortest path from μ to y_1 of length $K + 1$ and thus, $x_1 \in I_G(\mu, y_1)$. By similar arguments, we obtain $x_2 \in I_G(\mu, y_1)$ as well as $x_1, x_2 \in I_G(\mu, y_2)$. In summary, $x_1, x_2 \in I_G(\mu, y_1) \cap I_G(\mu, y_2) \cap I_G(y_1, y_2)$. Hence, $\operatorname{med}(\mu, y_1, y_2)$ is not well-defined; a contradiction.

Thus, $d_G(\mu, x_2) = K+2$ must hold and, by analogous arguments, $d_G(\mu, x_3) = K+2$. By similar arguments as before and since $d_G(\mu, y_1) = d_G(\mu, y_2) = K+1$, we can conclude that $y_1, y_2 \in I_G(\mu, x_2) \cap I_G(\mu, x_3) \cap I_G(x_2, x_3)$. Hence, $\text{med}(\mu, x_2, x_3)$ is not well-defined; a contradiction. □

All median graphs are bipartite and meshed [32]. Bipartite meshed graphs are precisely the modular graphs [18, Lemma 2.8]. However, since the three non-medico vertices u, v, w in a Q_3^- satisfy $I_G(u, v, w) = \emptyset$, it follows that a Q_3^- is not modular and therefore, not meshed. Moreover, since a Q_3^- is a 4-median graph, there are k-median graphs that are not meshed. In the upcoming proofs it will, therefore, be convenient to use a generalization of meshed graphs defined as follows.

Definition 4.6 (μ-meshed)**.** A graph G is *μ-meshed* if $\mu \in V(G)$ and all distance-ℓ-static quartets $(x, z, y, \mu) \in V^4$ satisfy the quadrangle property.

As we shall see next, distance-ℓ-static quadruples (x, z, y, μ) of G can be characterized in terms of induced C_4s containing x, y and z provided that μ is a medico vertex in G. As a consequence, k-median graphs are μ-meshed for all medico vertices μ. This, in turn, can be used to show that every edge of a cycle in a k-median graphs is also part of an induced C_4. The following lemma provides a characterization of distance-ℓ-static quartets in k-median graphs. Interestingly, this results shows that, in every C_4 of a k-median graph, two opposite vertices have both distance $\ell_\mu + 1$ while the other two opposite vertices have distance ℓ_μ, resp., $\ell_\mu + 2$ to any medico vertex μ where ℓ_μ is the smallest distance to μ among the vertices in this C_4.

Lemma 4.7. *Let G be a k-median graph and μ be some medico vertex in G. Then, the following two statements are equivalent.*

(1) *The quartet $(x, z, y, \mu) \in V^4$ is distance-ℓ-static.*

(2) *There is an induced C_4 in G with vertices x, z, y, u and edges $\{x, y\}, \{y, z\}, \{z, u\}$, $\{u, x\}$ where u is a vertex that is closest to μ among the vertices x, y, z, u.*

*In particular, every k-median graph G is μ-*meshed *for every medico vertex μ of G.*

Proof. Suppose that μ is a medico vertex in the k-median graph G. Assume first that the quartet $(x, z, y, \mu) \in V^4$ is distance-ℓ-static. Hence, for some $\ell \geq 1$, we have $d_G(x, \mu) = d_G(z, \mu) = \ell = d_G(y, \mu) - 1$ and $d_G(x, z) = 2$ with y a common neighbor of x and z. This, in particular, implies that x, y, z, μ are pairwise distinct and $\{x, y\}, \{y, z\} \subseteq E(G)$ and $\{x, z\} \notin E(G)$. Since μ is a medico vertex in G, $\text{med}_G(\mu, x, y)$ and $\text{med}_G(\mu, x, z)$ are well-defined. Put $u \coloneqq \text{med}_G(\mu, x, z)$. We claim that $u \neq x$ and $u \neq z$. Assume, for contradiction, that $u = x$. Hence, x lies on a shortest between μ and z and, in particular, $d_G(\mu, z) = d_G(\mu, x) + d_G(x, z) > \ell$; a contradiction. Hence, $u \neq x$. By similar arguments, $u \neq z$ must hold. Since $d_G(x, z) = 2$ and $u = \text{med}_G(\mu, x, z)$, we have by definition of medians that $d_G(x, z) = d_G(x, u) + d_G(u, z) = 2$. This together with $u \neq x, z$ implies that $\{u, x\}, \{u, z\} \in E(G)$. Furthermore, $u = \text{med}_G(\mu, x, z)$ implies $d_G(\mu, x) = d_G(\mu, u) + d_G(u, x)$. Since $d_G(\mu, x) = \ell$ and $d_G(u, x) = 1$ we obtain $d_G(\mu, u) = \ell - 1$. Since $d_G(\mu, u) = \ell - 1$ and $d_G(y, \mu) = \ell + 1$ we have $u \neq y$. Hence, (x, z, y, μ) satisfies the quadrangle property. As the latter arguments hold for any distance-ℓ-static quadruple (x, z, y, μ), G is μ-meshed for all medico vertices μ in G. Moreover, Lemma 2.1 implies that u and y cannot be adjacent. Now, $\{x, y\}, \{y, z\}, \{x, u\}, \{z, u\} \in E(G)$ but $\{x, z\}, \{u, y\} \notin E(G)$ implies that x, y, z, u induce a C_4.

Assume now that there is an induced C_4 in G with vertices x, z, y, u and edges $\{x, y\}, \{y, z\}, \{z, u\}, \{u, x\}$ and suppose, that u is closest to μ in G, i.e., $d_G(\mu, u) = \min_{v\in\{x,y,z,u\}} d_G(v, \mu)$. Suppose that $d_G(\mu, u) = \ell - 1$ for some $\ell \geq 1$. This and the fact that $\{z, u\}$ and $\{u, x\}$ are edges in G implies together with Lemma 4.1 that $d_G(z, \mu), d_G(x, \mu) \in \{\ell, \ell - 2\}$. Since, however, $d_G(\mu, u) = \ell - 1$ and u is closest to μ it must hold that $d_G(z, \mu) = d_G(x, \mu) = \ell$. Furthermore, $\{x, y\} \in E$ and $d_G(x, \mu) = \ell$ together with Lemma 4.1 implies that $d_G(y, \mu) \in \{\ell - 1, \ell + 1\}$. Assume, for contradiction, that $d_G(y, \mu) = \ell - 1$. In this case, $d_G(x, \mu) = \ell$ and $\{u, x\} \in E$ implies that there is a shortest (μ, x)-path containing u and thus, $u \in I_G(\mu, x)$. By similar arguments, $y \in I_G(\mu, x)$ as well as $u, y \in I_G(\mu, z)$ must hold. Moreover, since the C_4 in G is induced by the vertices x, z, y, u, we have $d_G(x, z) = 2$ and, by definition of the edges in this C_4, we have $u, y \in I_G(x, z)$. But then $u, y \in I_G(\mu, x) \cap I_G(\mu, z) \cap I_G(x, z)$, which implies that $\mathrm{med}_G(\mu, x, z)$ is not well-defined; a contradiction to μ being a medico vertex in G. Hence, $d_G(y, \mu) = \ell + 1$ must hold. It is now easy to verify that $(x, z, y, \mu) \in V^4$ is distance-ℓ-static. □

Lemma 4.8. *Let G be a k-median graph, μ be a medico vertex and C be a cycle in G. Then, there are edges $\{x, y\}, \{y, z\} \in E(C)$ such that (x, z, y, μ) is distance-ℓ-static for some $\ell \geq 1$.*

Proof. Assume, for contradiction, that (x, z, y, μ) is not distance-ℓ-static for all $\{x, y\}, \{y, z\} \in E(C)$ and any $\ell \geq 1$. Note that G bipartite, so $d_G(x, z) = 2$ for all $\{x, y\}, \{y, z\} \in E(C)$. Let the vertices of C be labeled such that $y_1, \dots, y_{|C|}$ and $\{y_1, y_2\}, \{y_2, y_3\}, \dots, \{y_{|C|}, y_1\} \in E(C)$. By Lemma 4.1, $|d_G(\mu, y_1) - d_G(\mu, y_2)| = 1$ and we can assume without loss of generality that $d_G(\mu, y_1) = k$ and $d_G(\mu, y_2) = k + 1$ for some $k \geq 0$. Again, by Lemma 4.1, $|d_G(\mu, y_2) - d_G(\mu, y_3)| = 1$ and so $d_G(\mu, y_2) \in \{k, k + 2\}$. Since there is no distance-ℓ-static quadruple using three consecutive vertices in C, $d_G(\mu, y_3) = k + 2$ must hold. Repeating the latter arguments, yields $d_G(\mu, y_{|C|}) = k + |C| - 1$. Since $\{y_1, y_{|C|}\} \in E(G)$ and by Lemma 4.1, $1 = d_G(\mu, y_{|C|}) - d_G(\mu, y_1) = k + |C| - 1 - k = |C| - 1$ which is only possible if $|C| = 2$; a contradiction. □

Corollary 4.9. *A k-median graph cannot contain convex cycles C_n, $n > 4$.*

Proof. Let C be a cycle in G with $n > 4$ vertices. Lemma 4.8 implies that there are edges $\{x, y\}, \{y, z\} \in E(C)$ such that (x, z, y, μ) is distance-ℓ-static for some $\ell \geq 1$. By Lemma 4.7, there is an induced C_4 in G with edges $\{z, u\}, \{u, x\}$. Hence, there are two shortest (x, z)-paths. Since $|C| > 4$, only one them is located on C. Thus, C is not a convex subgraph of G. □

Proposition 4.10. *Let G be a k-median graph. Then, every edge $e \in E(C)$ of every cycle C of G is an edge of an induced C_4.*

Proof. Let $G = (V, E)$ be a k-median graph and μ be a medico vertex in G. Assume that $e = \{x, y\}$ is an edge of some cycle in G. Among all cycles that contain e, let C be one of the cycles of smallest length. Hence C must be induced. Assume for contradiction, that $|C| > 4$. Thus, $|C| \geq 6$ since G is bipartite. By Lemma 4.1, $|d_G(\mu, x) - d_G(\mu, y)| = 1$ and we may assume without loss of generality that $d_G(\mu, x) < d_G(\mu, y)$. Let $\{y, z\} \in E(C)$ be the edge in C that is incident to y and where $z \neq x$. If (x, z, y, μ) is distance-ℓ-static, then by Lemma 4.7, the vertices x, y, z and thus, the edge $\{x, y\}$ is contained in a common induced C_4. Hence, suppose that (x, z, y, μ) is not distance-ℓ-static. This together with $x \neq z$, $d_G(\mu, x) < d_G(\mu, y)$ and Lemma 4.1 implies that $d_G(\mu, x) < d_G(\mu, y) < d_G(\mu, z)$. Now, label the vertices in C with $v_0, \dots, v_{|C|-1}$ such that $v_0 := x$, $v_1 := y$, $v_2 := z$ and $E(C) = \{\{v_0, v_1\}, \{v_1, v_2\}, \dots, \{v_{|C|-1}, v_0\}\}$. In the following, we denote with $V_j := \{v_0, \dots, v_j\}$ the set of vertices of C up to the index j. By Lemma 4.8, C contains a distance-ℓ-static quartet (a, c, b, μ), i.e., in particular, $d_G(\mu, a) < d_G(\mu, b)$ and $d_G(\mu, b) > d_G(\mu, c)$. Hence, we can choose the smallest index $i \geq 1$ for which $d_G(\mu, v_i) = \ell < d_G(\mu, v_{i+1})$ and $d_G(\mu, v_{i+1}) > d_G(\mu, v_{i+2})$ is satisfied. Since $d_G(\mu, x) < d_G(\mu, y)$ and by choice of i, we have $d_G(\mu, v_j) < d_G(\mu, v_{j+1})$ for all $j \in \{0, \dots, i\}$. Lemma 4.1 implies that $d_G(\mu, v_i) = d_G(\mu, v_{i+2}) = \ell$ and $d_G(\mu, v_{i+1}) = \ell + 1$. Since v_i and v_{i+2} are distinct vertices, we have $\ell \geq 1$. Moreover, since G is bipartite, $d_G(v_i, v_{i+2}) = 2$ must hold. In summary, $(v_i, v_{i+2}, v_{i+1}, \mu)$ is distance-ℓ-static. By Lemma 4.7, there exists an induced C_4 containing the edges $\{v_i, v_{i+1}\}$ and $\{v_{i+1}, v_{i+2}\}$ and an additional vertex u such that $d_G(\mu, u) = \ell - 1$ and, in particular, $\{u, v_i\}, \{u, v_{i+2}\} \in E(G)$. Note that if $u \in V(C)$, then C is either not induced or $i = 1$ and $u = v_0$ must holds in which case $C \simeq C_4$. Hence, $u \notin V(C)$ and, in particular, $u \notin V_i$. By choice of i, $d_G(\mu, v_{i-1}) < d_G(\mu, v_i) = \ell$ which together with Lemma 4.1 implies that $d_G(\mu, v_{i-1}) = \ell - 1$. Moreover, since G is bipartite and $\{u, v_i\}, \{v_i, v_{i-1}\} \in E(G)$, we have $d_G(v_{i-1}, u) = 2$, Note that $\ell - 1 \geq 1$ as otherwise, $d_G(\mu, u) = d_G(\mu, v_{i-1}) = \ell - 1 = 0$ would imply that $u = \mu = v_{i-1}$; a contradiction to u and v_{i-1} being distinct. In summary, (v_{i-1}, u, v_i, μ) is distance-$(\ell - 1)$-static. By Lemma 4.7, there is an induced C_4 containing the edges $\{v_{i-1}, v_i\}$ and $\{v_i, u\}$ and an additional vertex u' with $d_G(\mu, u') = \ell - 2$ as well as the edges $\{v_{i-1}, u'\}$ and $\{u, u'\}$. If $i = 1$, then we found an induced C_4 containing $\{v_0, v_1\} = \{x, y\}$; contradicting to the choice of C. Thus, $i \geq 2$ must hold. We claim that $u' \notin V_{i-2}$. To see this, assume, for contradiction, that $u' \in V_{i-2}$. Suppose first that $u' = v_0$ and consider the cycle C' traversing $u', v_1, \dots, v_i, u, u'$ of length $|C'| = i + 2$. If $|C| = i + 2$, then $v_{i+2} = v_0$. Hence, $u' = v_0$ implies $v_{i+2} = u'$ and thus there is an edge $\{v_{i-1}, u'\} = \{v_{i-1}, v_{i+2}\}$ and C is not induced; a contradiction. Hence, $|C| > i + 2$ must hold. But then C' is shorter than C and contains the edge $\{u', v_1\} = \{x, y\}$; a contradiction to the choice of C. Thus, $u' = v_j \in V_{i-2} \setminus \{v_0\}$ must hold. Now consider the cycle C'' traversing $v_0, v_1, \dots, v_{j-1}, u', u, v_{i+2}, v_{i+3} \dots, v_{|C|-1}, v_0$ ($v_1 = u'$ or $v_1 = v_{j-1}$ may possible). Note that $d_C(v_j, v_{i+2}) \geq 4$ since $j \leq i - 2$. Moreover, we have $d_G(v_j, v_{i+2}) = 2$ since there is a path with edges $\{u', u\} = \{v_j, u\}$, and $\{u, v_{i+2}\}$ in G. Hence, C'' is shorter than C and contains $\{v_0, v_1\} = \{x, y\}$; again a contradiction. Hence, we found an induced C_4 with edges $\{v_{i-1}, v_i\}$, $\{v_i, u\}$, $\{u, u'\}$, $\{u', v_{i-1}\}$ with $u \notin V_{i-1}$ and $u' \notin V_{i-2}$. Now, we can repeat the arguments on u', v_{i-1}, v_{i-2} and find an induced C_4 with edges $\{v_{i-2}, v_{i-1}\}$, $\{v_{i-1}, u'\}$, $\{u', u''\}$, $\{u'', v_{i-2}\}$ with $u' \notin V_{i-2}, u'' \notin V_{i-3}$. By induction and by repeating the latter arguments, one shows that all edges $\{v_0, v_1\} = \{x, y\}$, $\{v_1, v_2\}$, ..., $\{v_i, v_{i+1}\}$ are located in some induced C_4; a contradiction to the choice C. Hence, any smallest cycle containing $\{x, y\}$ must be an induced C_4. □

Lemma 4.11. *Let G be a k-median graph with medico vertex μ. If G contains two distance-ℓ-static quartets $(x, z, y, \mu), (x, z', y, \mu) \in V^4$, then G contains an induced $K_{2,3}$ or Q_3^-.*

Proof. Let μ be a medico vertex in $G = (V, E)$ and $(x, z, y, \mu), (x, z', y, \mu) \in V^4$ be distance-ℓ-static quartets. Hence, we have $d_G(x, \mu) = d_G(z, \mu) = \ell = d_G(y, \mu) - 1$ and $d_G(x, z) = 2$ with y a common neighbor of x and z as well as $d_G(x, \mu) = d_G(z', \mu) = \ell$ and $d_G(x, z') = 2$ with y a common neighbor of x and z'. Thus, y is a common neighbor of z and z'. Since G is bipartite, $d_G(z, z') = 2$. Moreover, $d_G(z, \mu) = d_G(z', \mu) = \ell = d_G(y, \mu) - 1$. Thus, $(z, z', y, \mu) \in V^4$ is distance-ℓ-static.

By Lemma 4.7, there is an induced C_4 in G with edges $\{x, y\}, \{y, z\}, \{z, u\}, \{u, x\}$ and an induced C_4 in G with edges $\{x, y\}, \{y, z'\}, \{z', u'\}, \{u', x\}$. If $u = u'$, then G contains an induced $K_{2,3}$ with bipartition $\{x, z\} \cup \{u, y, z'\}$. Hence, assume that $u \neq u'$. Since $(z, z', y, \mu) \in V^4$ is distance-ℓ-static, there is an induced C_4 in G with $\{z, y\}, \{y, z'\}, \{z', u''\}, \{u'', z\}$. Again if $u'' = u$ or $u'' = u'$, there is an induced $K_{2,3}$. If u, u', u'' are pairwise distinct, then x, z, z', y, u, u', u'' induce a Q_3^-. □

Proposition 4.12. *Let G be a Q_3^--free k-median graph with medico vertex μ. Then, G contains an induced $K_{2,3}$ if and only if G contains two distance-ℓ-static quartets (x, z, y, μ), $(x, z', y, \mu) \in V^4$.*

Proof. The *if* direction follows from Lemma 4.11 and the fact that G is Q_3^--free. For the *only-if* direction suppose that $G = (V, E)$ contains an induced $K_{2,3}$ with bipartition $\{x, x'\} \cup \{y, y', y''\}$. Let $v \in \{x, x', y, y', y''\}$ be one of the vertices that is closest μ and suppose that $d_G(\mu, v) = \ell - 1$ for some $\ell \geq 1$. Assume, for contradiction, that $v = y$. Since y is closest to μ and $\{y, x\}, \{y, x'\} \in E$, Lemma 4.1 implies that $d_G(\mu, x) = d_G(\mu, x') = \ell$. Since $\{y', x\}, \{y', x'\} \in E$ and by Lemma 4.1, we have $d_G(\mu, y') \in \{\ell - 1, \ell + 1\}$. In case $d_G(\mu, y') = \ell - 1$, on easily verifies that $y, y' \in I_G(\mu, x, x')$ which is not possible, since $\mathrm{med}_G(\mu, x, x')$ is well-defined. But then, $d_G(\mu, y') = \ell + 1$ and $x, x' \in I_G(\mu, y, y')$ which yields the desired contradiction. Therefore, $v \notin \{y, y', y''\}$ must hold. We may assume without loss of generality that $v = x$. In this case, x is closest to μ and $\{x, y\}, \{x, y'\}, \{x, y''\} \in E$ together with Lemma 4.1 implies that $d_G(\mu, y) = d_G(\mu, y') = d_G(\mu, y'') = \ell$. Again by Lemma 4.1 and since x' is adjacent to y, y' and y'', we have $d_G(\mu, x') \in \{\ell - 1, \ell + 1\}$. If, however, $d_G(\mu, x') = \ell - 1$, then $x, x' \in I_G(\mu, y, y')$ which is not possible, since $\mathrm{med}_G(\mu, y, y')$ is well-defined. Hence, we have $d_G(\mu, x') = \ell + 1$. Thus, G contains distance-ℓ-static quartets (y, y', x', μ) and (y, y'', x', μ). □

Median graphs are characterized as those graph in which the convex hull of every isometric cycle is a hypercube. This property is, in general, not shared by k-median graphs. By way of example, the convex hull of the isometric "outer" cycle C_6 of the graph G in Figure 1(c) coincides with G and contains an induced Q_3^- and a $K_{2,3}$ but is, obviously, not a hypercube. However, the existence of isometric (or equivalently, induced) cycles on six vertices in a k-median graph G is a clear indicator for isometric (or equivalently, induced) Q_3^-s in G.

Proposition 4.13. *A k-median graph G contains an induced Q_3^- if and only if G contains an induced C_6.*

Proof. One easily verifies that any induced $Q_3^- \subseteq G$ contains an induced C_6 (cf. Figure 1(b)). Suppose now that $G = (V, E)$ contains an induced C_6, call it C. Let $\delta \colon V \to \mathbb{N}$ be the map that assigns to each vertex its distance to μ, i.e., $\delta(v) = d_G(\mu, v)$. We assume that $V(C) = \{v_1, \dots, v_6\}$ and $E(C) = \{\{v_1, v_2\}, \dots \{v_5, v_6\}, \{v_6, v_1\}\}$. Without loss of generality assume that v_1 is a vertex in C that is closest to μ and that $\delta(v_1) = \ell - 1$ for some $\ell \geq 1$. When traversing C from v_1 to v_2 ... to v_6 we obtain the ordered tuple $\Delta(C) := (\delta(v_1), \delta(v_2) \dots, \delta(v_6))$. We denote by $\Delta^*(C)$ the tuple $(\delta(v_1), \delta(v_6), \dots, \delta(v_2))$, i.e., the tuple obtained by traversing C from v_1 to v_6 ... to v_2.

Full enumeration of all possible distances of the vertices in C to μ and using Lemma 4.1, shows that $\Delta(C)$ is always of one of the form

$\Delta_1 \coloneqq (\ell-1, \ell, \ell-1, \ell, \ell-1, \ell)$;
$\Delta_2 \coloneqq (\ell-1, \ell, \ell-1, \ell, \ell+1, \ell)$;
$\Delta_2^* \coloneqq (\ell-1, \ell, \ell+1, \ell, \ell-1, \ell)$;
$\Delta_3 \coloneqq (\ell-1, \ell, \ell+1, \ell, \ell+1, \ell)$; or
$\Delta_4 \coloneqq (\ell-1, \ell, \ell+1, \ell+2, \ell+1, \ell)$.

Suppose first that $\Delta(C) = \Delta_1 = (\ell-1, \ell, \ell-1, \ell, \ell-1, \ell)$. Then, (v_3, v_5, v_4, μ) is a distance-$(\ell-1)$-static quartet. By Lemma 4.7, there is an induced C_4 in G with vertices v_3, v_4, v_5, u and edges $\{v_3, v_4\}, \{v_4, v_5\}, \{v_5, u\}, \{u, v_3\}$. In particular, $\delta(u) = \ell-2$. Since C is induced, $u \notin V(C)$. If u is adjacent to v_1, then G contains an induced Q_3^- and we are done. Hence, suppose that $\{u, v_1\} \notin E$. Now consider the distance-$(\ell-1)$-static quartet (v_1, v_5, v_6, μ). Again there is an induced C_4 containing v_1, v_5, v_6 and u' with $\delta(u') = \ell-2$. If $u' = u$ or u is adjacent to v_3, then G contains an induced Q_3^- and we are done. Hence, assume $u \neq u'$ and that $\{u', v_3\} \notin E$. Analogously, for the distance-$(\ell-1)$-static quartet (v_1, v_3, v_2, μ), Again there is an induced C_4 containing v_1, v_2, v_3 and u'' with $\delta(u'') = \ell-2$. If $u'' = u$ or $u'' = u'$ or u'' is adjacent to v_5, then G contains an induced Q_3^- and we are done. Hence, we can assume that u, u' and u'' are pairwise distinct and that $\{u', v_5\} \notin E$. Note that $\delta(u) = \delta(u') = \delta(u'') = \ell-2$. Moreover, by the latter arguments, the cycle traversing $v_1, u'', v_3, u, v_5, u'$ is an induced C_6, call it C'. Moreover, $\Delta(C') = (\ell-1, \ell-2, \ell-1, \ell-2, \ell-1, \ell-2)$. Now we can repeat the latter arguments on C' to obtain either an induced Q_3^- or an induced C_6, call it C'', such that $\Delta(C'') = (\ell-3, \ell-2, \ell-3, \ell-2, \ell-3, \ell-2)$. Since G is finite, the latter process must terminate, i.e., in one of the steps we cannot find a further induced cycle $\tilde{C}$ on six vertices, as the distances in $\Delta(\tilde{C})$ become closer and closer to μ. Thus, G must contain an induced Q_3^-.

Assume now that $\Delta(C) = \Delta_2 = (\ell-1, \ell, \ell-1, \ell, \ell+1, \ell)$. In this case, (v_4, v_6, v_5, μ) is a distance-ℓ-static quartet. Hence, there is an induced C_4 in G with vertices v_4, v_5, v_6, u. In particular, $\delta(u) = \ell-1$ and, since C is induced, $u \notin V(C)$. If $\{u, v_2\} \in E$, then G contains a Q_3^-. Otherwise, if $\{u, v_2\} \notin E$, then G contains an C_6, called C', that is induced by $v_1, v_2, v_3, v_4, u, v_6$ and for which $\Delta(C') = \Delta_1$. As already shown, this implies that G contains a Q_3^-. The case $\Delta(C) = \Delta_2^*$ is shown analogously.

Assume now that $\Delta(C) = \Delta_3 = (\ell-1, \ell, \ell+1, \ell, \ell+1, \ell)$ In this case, (v_4, v_6, v_5, μ) is a distance-ℓ-static quartet and there is an induced C_4 in G with vertices v_4, v_5, v_6, u and we have $\delta(u) = \ell-1$. Since C is induced, $u \notin V(C)$. If $\{u, v_2\} \in E$, then G contains a Q_3^-. Otherwise, the cycle C' along the vertices $v_1, v_2, v_3, v_4, u, v_6$ is an induced C_6 and satisfies $\Delta(C') = (\ell-1, \ell, \ell+1, \ell, \ell-1, \ell)$ and we can reuse the arguments as in case Δ_2^* to conclude that G contains an induced Q_3^-.

Finally suppose that $\Delta(C) = \Delta_4 = (\ell-1, \ell, \ell+1, \ell+2, \ell+1, \ell)$. In this case, (v_3, v_5, v_4, μ) is a distance-$(\ell+1)$-static quartet and there is an induced C_4 in G with vertices v_3, v_4, v_5, u. and we have $\delta(u) = \ell$. Since C is induced, $u \notin V(C)$. If $\{u, v_1\} \in E$, then G contains a Q_3^-. Otherwise, the cycle C' along the vertices $v_1, v_2, v_3, u, v_5, v_6$ is an induced C_6 and satisfies $\Delta(C') = (\ell-1, \ell, \ell+1, \ell, \ell+1, \ell) = \Delta_3$ and we can reuse the arguments as in case Δ_3 to conclude that G contains an induced Q_3^-. □

Partial cubes are graphs that have an isometric embedding into a hypercube. A cycle C_6 is a partial cube. Hence, Proposition 4.13 implies that there are partial cubes that are not k-median graphs. Although every induced C_6 in a k-median graph indicates the existence

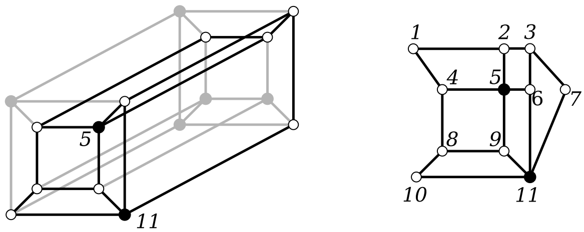

Figure 3: A $K_{2,3}$-free proper 2-median graph G (right). Its medico vertices 5 and 11 are highlighted as black vertices. G is an isometric subgraph of the hypercube Q_4 and its isometric embedding is shown left. According to Proposition 4.13, each of its isometric cycles of length six (induced by $4, 5, 6, 8, 10, 11$, resp., $2, 3, 5, 7, 9, 11$) are contained in a Q_3^- subgraph of G. There is an isometric C_8 in G induced by $1, 2, 3, 4, 7, 8, 10, 11$ whose convex hull is G. This C_8 is not contained in an induced Q_4^- in G.

of a Q_3^-, a similar result does not necessarily hold for larger isometric cycles, see Figure 3 for an example.

Figure 1 shows that there are k-median graphs that contain induced Q_3^-s or $K_{2,3}$s. In fact, we have neither found examples nor clear arguments that shows that there are $K_{2,3}$- and Q_3^--free k-median graphs (except the median graphs whose isometric cycles are C_4s) or $K_{2,3}$-free k-median graphs that are not partial cubes. This, in particular, poses the following open problems.

Problem 1. *Does every k-median graph that is not a median graph contain an induced Q_3^- or $K_{2,3}$?*

Corollary 5.16 shows that every modular k-median graph that is not a median graph must contain an induced $K_{2,3}$.

Problem 2. *Is every $K_{2,3}$-free k-median graph an (isometric) subgraph of a hypercube?*

5 Characterization of k-median graphs

5.1 Convexity

We start here with results utilizing a generalized notion of convexity. To recall, a subgraph $H \subseteq G$ is v-convex, if $v \in V(H)$ and every shortest path connecting v and x in G is also contained in H, for all $x \in V(H)$. Note that v-convex subgraphs on the same vertex sets do not necessarily share the same edge set and thus may differ from each other. By way of example, in a K_3 a v-convex subgraph on three vertices could either be a path P_3 or the graph K_3. See Figure 2 for further examples. Nevertheless, uniqueness for v-convex subgraphs holds in case the underlying graph is bipartite.

Lemma 5.1. *For every bipartite graph G, every v-convex subgraph is induced. Thus, for all $W \subseteq V(G)$ and all $v \in W$, any two v-convex subgraphs of G with vertex set W are identical.*

Proof. Let G be a bipartite graph and H be a v-convex subgraph of G. Let $\{x, y\} \in E(G)$ be an edge with $x, y \in V(H)$. By v-convexity of H, we have $d_G(v, x) = d_H(v, x) = k$

and $d_G(v,y) = d_H(v,y) = k'$. Since $H \subseteq G$ is bipartite and connected, we can apply Lemma 2.1 to conclude that $|k-k'| = 1$ must hold. We may assume without loss of generality that $k' = k+1$. Consider a shortest (v,x)-path P in G. Since $\|P\| = k < d_G(v,y)$, the vertex y cannot be located on P. Thus, one can extend P to a (v,y)-path P' in G by adding the $\{x,y\}$. By construction, $\|P'\| = k+1$ and so P' is a shortest (v,y)-path in G. Since H is v-convex, we have $P' \subseteq H$ and so $\{x,y\} \in E(H)$. Hence, H is an induced subgraph of G. □

Although v-convex subgraphs are not necessarily convex (cf. Figure 2), they are isometric subgraphs in 1-median graphs whenever v is a medico vertex in G.

Lemma 5.2. *Let G be a k-median graph, $\mu \in V(G)$ be some medico vertex in G and H be a μ-convex subgraph of G. Then, H is an isometric subgraph of G. In particular, H is a 1-median graph with medico vertex μ that satisfies $\mathrm{med}_H(\mu,x,y) = \mathrm{med}_G(\mu,x,y)$ for all $x,y \in V(H)$.*

Proof. Let G be a k-median graph and $\mu \in V(G)$ be some medico vertex in G and H be a μ-convex subgraph H of G. We start with showing that H is an isometric subgraph of G, that is, $d_H(x,y) = d_G(x,y)$ for all $x,y \in V(H)$. To this end, let $x,y \in V(H)$ be chosen arbitrarily. If $x = y$, the statement is vacuously true. Hence, assume that $x \neq y$. Suppose first that $x = \mu$. Since H is a subgraph of G, we have $d_H(\mu,y) \not< d_G(\mu,y)$. Moreover, μ-convexity of H implies that all shortest paths between μ and y are also in H and thus, $d_H(\mu,y) = d_G(\mu,y)$. Since y was chosen arbitrarily, we have $d_H(\mu,v) = d_G(\mu,v)$ for all $v \in V(H)$.

Assume now that x and y are both distinct from μ. Since μ is a medico vertex in G, $z \coloneqq \mathrm{med}_G(\mu,x,y)$ is well-defined. We show first that $d_H(z,x) = d_G(z,x)$. Clearly, if $x = z$, the statement is vacuously true. Thus, assume that $x \neq z$. Since $d_H(\mu,v) = d_G(\mu,v)$ for all $v \in V(H)$, it holds that $d_H(\mu,x) = d_G(\mu,x)$. This and μ-convexity of H implies that every shortest (μ,x)-path in G is also contained in H and, in particular, a shortest (μ,x)-path in H. As z is the unique median between μ, x and y in G, z is located on some shortest (μ,x)-path P in G that is, by the latter arguments, also a shortest (μ,x)-path in H. This, in particular, implies that $z \in I_H(\mu,x)$. Consider the subpath P' of P that connects z and x. Since P is a shortest path in H, resp., G, we can conclude that P' is shortest path in H, resp., G between z and x. Hence, $d_H(z,x) = d_G(z,x)$. By similar arguments, $z \in I_H(\mu,y)$ and $d_H(z,y) = d_G(z,y)$ must hold. By the triangle inequality, $d_H(x,z) + d_H(z,y) \geq d_H(x,y)$. Since $d_H(z,x) = d_G(z,x)$ and $d_H(z,y) = d_G(z,y)$, we can conclude that $d_G(x,z)+d_G(z,y) = d_H(x,z)+d_H(z,y)$. Moreover, $z = \mathrm{med}_G(\mu,x,y)$ and the fact that H is a subgraph of G implies that $d_H(x,y) \geq d_G(x,y) = d_G(x,z) + d_G(z,y)$. Taking the latter three arguments together yields

$$d_H(x,y) \geq d_G(x,y) = d_G(x,z) + d_G(z,y) = d_H(x,z) + d_H(z,y) \geq d_H(x,y).$$

Hence, $d_H(x,y) = d_G(x,y)$ as claimed and $z \in I_H(x,y)$. Therefore, H is an isometric subgraph of G. In particular, $z = \mathrm{med}_G(\mu,x,y) \in I_H(\mu,x) \cap I_H(\mu,y) \cap I_H(x,y) = I_H(\mu,x,y)$ must hold.

It remains to show that H is a 1-median graph with medico vertex μ. By the latter argument, $z \in I_H(\mu,x,y)$. We continue with showing that $|I_H(\mu,x,y)| = 1$. Assume, for contradiction, that $|I_H(\mu,x,y)| \geq 2$ and let $z' \in I_H(\mu,x,y) \setminus \{z\}$. Hence, z' is located on some shortest (μ, x)-path P, some shortest (μ, y)-path P' and some shortest (x, y)-path P'' in H. Since H is isometric, we have $\|P\| = d_G(\mu, x)$, $\|P'\| = d_G(\mu, y)$ and $\|P''\| = d_G(x, y)$. This and the fact that H is a subgraph of G implies that P, P' and P'' are shortest (μ, x)-paths, shortest (μ, y)-paths and shortest (x, y)-paths in G. But then $z' \in I_G(\mu, x) \cap I_G(\mu, y) \cap I_G(x, y)$ must hold which implies that $\mathrm{med}_G(\mu, x, y)$ is not well-defined; a contradiction. In summary, H is a 1-median graph with medico vertex μ. In particular, $\{z\} = I_H(\mu, x, y)$ holds and thus, $\mathrm{med}_H(\mu, x, y) = \mathrm{med}_G(\mu, x, y)$ for all $x, y \in V(H)$. □

Lemma 5.2 holds, in general, only for v-convex subgraphs with v being a medico vertex. By way of example, the graph G_2 in Figure 2 is a k-median graph but the v-convex subgraphs for the non-medico vertex is not isometric. Even more, the graph G_4 in Figure 2 shows that v-convex subgraphs of k-median graphs are not necessarily 1-median graphs. Lemma 5.2 can be used to obtain the following

Proposition 5.3. *For every graph G and every vertex $\mu \in V(G)$, the following statements are equivalent.*

(1) *μ is a medico vertex in G.*

(2) *μ is a medico vertex in every μ-convex subgraph of G.*

(3) *μ is a medico vertex in every convex subgraph of G that contains μ.*

Proof. Let $\mu \in V(G)$ be a medico vertex in G. Suppose that H is a μ-convex subgraph of G. By Lemma 5.2, μ is a medico vertex in H. Hence, Statement (1) implies (2). Moreover, if μ is a medico vertex in every μ-convex subgraph of G. then μ is a medico vertex in every convex subgraph H of G that contains μ, since a convex subgraph H is, in particular, μ-convex. Hence, Statement (2) implies (3). Statement (3) implies (1), since G is trivially a convex subgraph of G that contains μ. □

The definition of k-median graphs and median graphs together with Proposition 5.3 immediately implies

Theorem 5.4. *For every graph G the following statements are equivalent.*

(1) *G is a k-median graph.*

(2) *There are k vertices $\mu_1, \dots, \mu_k \in V$ such that μ_i is a medico vertex in every μ_i-convex subgraph of G, $1 \leq i \leq k$.*

(3) *There are k vertices $\mu_1, \dots, \mu_k \in V$ such that μ_i is a medico vertex in every convex subgraph of G that contains μ_i, $1 \leq i \leq k$.*

In particular, G is a median graph if and only if v is a medico vertex in every v-convex subgraph of G, for all $v \in V(G)$.

Corollary 5.5. *Let G be a 1-median graph with μ being some medico vertex in G. Then, for all $v \in V(G)$, the subgraph $G[I_G(\mu, v)]$ induced by the interval $I_G(\mu, v)$ is a μ-convex, isometric subgraph of G and a 1-median graph with medico vertex μ.*

Proof. Let G be a 1-median graph with μ being some medico vertex in G and let $v \in V(G)$. Put $I := I_G(\mu, v)$. By definition, all shortest paths in G between μ and v are contained in $G[I]$. Let $w \in I$. By Lemma 2.3, $I(\mu, w) \subseteq I$. Thus, all shortest (μ, w)-paths are in $G[I]$ for all $w \in I$. Hence, I is μ-convex. The second statement now readily follows from Lemma 5.2. □

Note that the converse of Corollary 5.5 is, in general, not satisfied. To see this, consider the graph $G \simeq K_{2,3}$ with bipartition $V(G) = X \cup Y$ where $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2, y_3\}$. Put $\mu := y_1$. We have $I(\mu, y_1) = \{y_1\}$ and $I(\mu, y_i) = \{\mu, x_1, x_2, y_i\}$ in case $i \in \{2, 3\}$. In addition, $I(\mu, x_i) = \{\mu, x_i\}$, $i \in \{1, 2\}$. One easily observes that for each of these intervals I, the induced subgraph $G[I]$ is μ-convex, isometric and a 1-median graph with medico vertex μ. However, since $I(\mu, y_2, y_3) = \{x_1, x_2\}$, μ is not a medico vertex of G.

The following result is well-known (cf. [34]) and the latter results allow us to establish a simple alternative proof.

Corollary 5.6. *G is a median graph if and only if every convex subgraph of G is a median graph.*

Proof. The *if* direction follows from the simple fact that G is a convex subgraph of G. For the *only-if* direction, suppose that G is a median graph and H is some convex subgraph of G. Since H is convex, H is v-convex for all $v \in V(H)$. By Theorem 5.4, v is a medico vertex in H for all $v \in V(H)$. Hence, H is a $|V(H)|$-median graph and, therefore, a median graph. □

It remains, however, an open question if a similar result holds for k-median graphs as well.

Problem 3. *Is every convex subgraph of a k-median graph a 1-median graph?*

An affirmative answer to Problem 3 will be provided in Proposition 5.19 for modular k-median graphs.

5.2 Conditions (C0), (C1) and (C2)

We provide now two further characterizations of k-median graphs. To this end, we prove first

Lemma 5.7. *If a graph G satisfies (C1) with respect to some vertex $u \in V(G)$, then $d_G(u, a) \neq d_G(u, b)$ for all edges $\{a, b\} \in E(G)$ and, thus, G is bipartite.*

Proof. By contraposition, suppose that u is a vertex in G for which there is an edge $\{a, b\} \in E(G)$ such that $d_G(u, a) = d_G(u, b)$. Since $\{a, b\} \in E(G)$, it holds that $I_G(a, b) = \{a, b\}$. We have $b \notin I_G(u, a)$, since otherwise, $d_G(u, a) = d_G(u, b) + d_G(a, b) \neq d_G(u, b)$; a contradiction. By similar arguments, $a \notin I_G(u, b)$. Hence, $I_G(u, a, b) = \emptyset$ and thus, G does not satisfy (C1) with respect to u. Lemma 2.1, finally, implies that G is bipartite. □

Theorem 5.8. *A graph $G = (V, E)$ is a k-median graph if and only if G satisfies (C1) with respect to $\mu_1, \ldots, \mu_k \in V$. In this case, the vertices $\mu_1, \ldots, \mu_k \in V$ are medico vertices of G.*

Proof. Let $G = (V, E)$ be a graph, $v, w \in V$ be chosen arbitrarily and put $I := I_G(\mu, v, w)$. Suppose that μ is a medico vertex in G. By Proposition 4.2, G is bipartite. Moreover, since μ is a medico vertex, we have $|I| = 1$. Thus, $G[I] \simeq K_1$ is non-empty and connected. Since v, w where chosen arbitrarily, G satisfies (C1) with respect to μ.

Assume now that G satisfies (C1) with respect to $\mu \in \{\mu_1, \ldots, \mu_k\}$. Hence G must be connected. Since $G[I]$ is not the empty graph, we have $I \neq \emptyset$. Assume, for contradiction, that $|I| > 1$. Since G satisfies (C1) with respect to μ, $G[I]$ is connected. Thus, there are $a, b \in I$ such that $\{a, b\} \in E(G)$. In the following, distances $d(\cdot,\cdot) = d_G(\cdot,\cdot)$ are taken with respect to G. By Lemma 5.7, $d(\mu, a) \neq d(\mu, b)$ and Lemma 2.1 implies that $d(\mu, a)$ and $d(\mu, b)$ differ by exactly one, say $d(\mu, b) = d(\mu, a) + 1$. Since $a, b \in I(\mu, v)$, it holds that $d(\mu, v) = d(\mu, a) + d(a, v) = d(\mu, b) + d(b, v)$. Thus, $d(b, v) = d(a, v) + d(\mu, a) - d(\mu, b) = d(a, v) - 1$. By similar arguments and since, $a, b \in I_G(\mu, w)$, we obtain $d(b, w) = d(a, w) - 1$. The latter two arguments together with $a, b \in I_G(v, w)$ imply that $d(v, w) = d(v, b) + d(b, w) = d(v, a) + d(a, w) - 2 = d(v, w) - 2$; a contradiction. Therefore, $|I| = 1$ must hold. Since v, w where chosen arbitrarily, $|I_G(\mu, v, w)| = 1$ for all $v, w \in V$. Consequently, μ is a medico vertex of G and G is a k-median graph. □

By Theorem 3.6, median graphs are interval-monotone and always satisfy (C0) with respect to all of its vertices. As argued in Section 3, k-median graphs are not necessarily interval-monotone. However, they always satisfy (C0) with respect to to its medico vertices.

Lemma 5.9. *G satisfies (C0) with respect to μ for every medico vertex $\mu \in V(G)$.*

Proof. Let μ be a medico vertex of $G = (V, E)$. Suppose there are vertices $v, w \in V$ such that $I(\mu, v) \cap I(v, w) = \{v\}$. Since μ is a medico vertex in G, it must hold that $|I(\mu, v) \cap I(v, w) \cap I(\mu, w)| = 1$ and thus, $\mathrm{med}_G(\mu, v, w) = v$. Consequently, $v \in I(\mu, w)$ and G satisfies (C0) with respect to μ. □

Theorem 3.6 and Lemma 5.9 beg the question to what extent Theorem 3.6 can be generalized to cover the properties of k-median graphs. The next results provides an answer.

Theorem 5.10. *A graph G is a k-median graph if and only if G is connected, bipartite and satisfies (C0) and (C2) with respect to $\mu_1, \ldots, \mu_k \in V(G)$. In this case, the vertices $\mu_1, \ldots, \mu_k \in V$ are medico vertices of G.*

Proof. For the *only-if* direction, assume first that $G = (V, E)$ is a k-median graph. By Proposition 4.2, G is bipartite. Let $\mu \in V$ be one of its (at least k) medico vertices. By Lemma 5.9, G satisfies (C0) with respect to μ. Furthermore, since μ is a medico vertex in G, we have $G[I(\mu, v, w)] \simeq K_1$ for all $u, v \in V$. Therefore, G trivially satisfies (C2) with respect to μ.

For the *if* direction, assume now that G is connected, bipartite and satisfies (C0) and (C2) with respect to $\mu_1, \ldots, \mu_k$. Let $\mu \in \{\mu_1, \ldots, \mu_k\}$. We show first that $I(\mu, v, w) \neq \emptyset$ for all $v, w \in V$. Since G is connected, we can apply Lemma 2.3 to conclude that there exists a vertex $z \in I(\mu, v) \cap I(v, w)$ such that $I(\mu, z) \cap I(z, w) = \{z\}$ for all $v, w \in V$. Since G satisfies (C0) with respect to μ, we have $z \in I(\mu, w)$ and so, $I(\mu, v, w) \neq \emptyset$ for all $v, w \in V$. Assume now, for contradiction, that μ is not a medico vertex in G. Hence, there are vertices v, w such that $\mathrm{med}_G(\mu, v, w)$ is not well-defined. By the latter arguments, $|I(\mu, v, w)| > 1$. Since G satisfies (C2) with respect to μ, there is an edge $\{a, b\} \in E(G)$ with $a, b \in I(\mu, v, w)$. Bipartiteness of G together with 2.1 implies that $d_G(\mu, a)$

and $d_G(\mu, b)$ differ by exactly one, say $d_G(\mu, b) = d_G(\mu, a) + 1$. Now we can apply exactly the same arguments as in the proof of Theorem 5.8 to conclude that $d_G(v, w) = d_G(v, w) - 2$ and obtain the desired contradiction. Hence, $|I\mu, v, w)| = 1$ for all $v, w \in V$. Consequently, μ is a medico vertex of G and G is a k-median graph. □

The latter results have direct implications for median graphs.

Theorem 5.11. *For every graph G, the following statements are equivalent.*

(1) *G is a median graph.*

(2) *G satisfies (C1) with respect to all of its vertices.*

(3) *G is connected, bipartite and satisfies (C0) and (C2) with respect to all of its vertices.*

Proof. The equivalence between (1) and (2) follows from Theorem 5.8 and the fact that in median graphs all vertices are medico vertices. Similarly, Theorem 5.10 provides the equivalence between (1) and (3). □

We discussed the latter results, in particular Theorem 5.11(2), with our friend and colleague Wilfried Imrich. As he pointed out, the induced subgraphs $G[I_G(u, v, w)]$ in Condition (C1) can be replaced by "intersections of graphs" to obtain an alternative characterization of median graphs. To be more precise, let $G(u, v)$ be the subgraph of G with $V(G(u, v)) = I_G(u, v)$ and where $E(G(u, v))$ consists precisely of all edges that lie on the shortest (u, v)-paths. Moreover, put $G(u, v, w) := G(u, v) \cap G(u, w) \cap G(v, w)$ for any $u, v, w \in V(G)$. On easily verifies that $V(G(u, v, w)) = I_G(u, v, w)$ and $G(u, v, w) \subseteq G[I_G(u, v, w)]$. The difference between $G[I_G(u, v, w)]$ and $G(u, v, w)$ is illustrated in Figure 2. In this example, for the graph G'', we have $I := I_{G''}(a, b, c) = \{x, y\}$. The induced subgraph $G''[I] \simeq K_2$ is connected and non-empty as it consists of the edge $\{x, y\}$. In contrast, $G''(a, b, c)$ is disconnected and consists of the vertices x and y only. Based on this idea, we obtain

Theorem 5.12. *G is a median graph if and only if $G(u, v, w)$ is not empty and connected for all $u, v, w \in V(G)$.*

Proof. If $G = (V, E)$ is a median graph, then $V(G(u, v, w)) = I_G(u, v, w)$ implies that $|V(G(u, v, w))| = 1$ and thus, $G(u, v, w) \simeq K_1$ for all $u, v, w \in V$. Thus, $G(u, v, w)$ is not empty and connected.

Assume now that $G(u, v, w)$ is not empty and connected for all $u, v, w \in V(G)$. Since $G[I_G(u, v, w)]$ and $G(u, v, w)$ have the same vertex sets and since $G(u, v, w) \subseteq G[I_G(u, v, w)]$ it follows that G satisfies (C1) with respect to all of its vertices. Theorem 5.11 implies that G is a median graph. □

5.3 Modular graphs

For the sake of completeness, we provide here known results established by Bandelt et al. for the special case of modular graphs and some of their consequences.

Theorem 5.13 ([8, Proposition 5.5])**.** *A modular graph G is a k-median graph with medico vertices $\mu_1, \dots, \mu_k$ if and only if the following statement is satisfied: If u, v are vertices that have degree three in an induced subgraph $K_{2,3} \subseteq G$, then $u \in I_G(\mu, v)$ or $v \in I_G(\mu, u)$ for all $\mu \in \{\mu_1, \dots, \mu_k\}$.*

Since Theorem 5.13 is always satisfied for $K_{2,3}$-free graphs, it implies

Corollary 5.14. *Every modular $K_{2,3}$-free graph G is a k-median graph for all $k \in \{1, \dots, |V(G)|\}$.*

This and the fact that median graphs are modular and $K_{2,3}$-free implies

Corollary 5.15 ([32, Theorem 3])**.** *A graph G is a median graph if and only if G is modular and $K_{2,3}$-free.*

As a simple consequence of Corollary 5.15, we obtain the following structural result which partially answer the question raised in Problem 1.

Corollary 5.16. *Every modular graph and thus, in particular every modular k-median graph, that is not a median graph must contain an induced $K_{2,3}$.*

Not all k-median graphs are modular. By way of example, a Q_3^- is a 4-median graph but not modular. To see this, consider the three non-medico vertices x, y, z in a Q_3^- (cf. Figure 1(b)). On easily verifies that $I(x, y, z) = \emptyset$ must hold. 1-median graphs that are modular are precisely the meshed graphs. To prove this, we first note that in [18] weakly modular graphs have been defined as meshed graphs that satisfy in addition a so-called triangle property. This triangle property is trivially satisfied in bipartite graphs. Hence, in bipartite graphs the terms meshed and weakly modular are equivalent. With this in hand, we can rephrase Lemma 2.8 in [18] (see also [8, Proposition 1.7]) as

Lemma 5.17. *A graph G is modular if and only if G is connected, bipartite and meshed.*

Corollary 5.18. *Let G be a k-median graph. Then, G is modular if and only if G is meshed.*

Proof. If G is modular, then Lemma 5.17 implies that G is meshed. Conversely, suppose that G is meshed. Since G is a k-median graph, it must be connected. Moreover, Proposition 4.2 implies that G is bipartite. By Lemma 5.17, G is modular. □

We provide now a partial answer to the question raised in Problem 3.

Proposition 5.19. *Every convex subgraph of a modular k-median graph is a 1-median graph.*

Proof. Let $G = (V, E)$ be a modular k-median graph, μ be a medico vertex of G and H be a convex subgraph of G. Assume first that $\mu \in V(H)$. Since H is convex, H is μ-convex. By Proposition 5.3, μ is a medico vertex in H. Thus, H is a 1-median graph.

Suppose now that $\mu \notin V(H)$ for any medico vertex μ of G. Let $v \in V(H)$ be a vertex that is closest to μ among the vertices in H and let $x, y \in V(H)$ be chosen arbitrarily. Since G is modular, we have $I_G(v, x, y) \neq \emptyset$. Let P be a shortest (v, x)-path in H (and thus, in G). Let $w \neq v$ be a vertex in P. If $d_G(\mu, w) + d_G(w, v) = d_G(\mu, v)$, then w would be closer to μ than v. Hence, $d_G(\mu, w) + d_G(w, v) > d_G(\mu, v)$ must hold and, therefore, $w \notin I_G(\mu, v)$ for all $w \in V(P) \setminus \{v\}$ and every shortest (v, x)-path P. By similar arguments, none of the vertices on a shortest (v, y)-path can be contained in $I_G(\mu, v)$ except vertex v. Hence, $I_G(\mu, v) \cap I_G(v, x) = I_G(\mu, v) \cap I_G(v, y) = \{v\}$. Since G is a k-median graph, Theorem 5.10 implies that G satisfies (C0) with respect to μ. Taken the latter two arguments together shows that $v \in I_G(\mu, x) \cap I_G(\mu, y)$. Lemma 2.3 implies that $I_G(v, x) \subseteq I_G(\mu, x)$ and $I_G(v, y) \subseteq I_G(\mu, y)$. Hence, $I_G(v, x, y) \subseteq I_G(\mu, x, y)$.

Since μ is a medico vertex, $I_G(\mu, x, y) = \{z\}$ for some $z \in V$. The latter two arguments and the fact that $I_G(v, x, y) \neq \emptyset$ imply that $I_G(v, x, y) = \{z\}$. Since z is located on a shortest (x, y)-path in G and since H is convex it follows that $z \in V(H)$. As the latter arguments hold for all vertices $x, y \in V(H)$, that is, $\mathrm{med}_G(v, x, y) = \mathrm{med}_H(v, x, y)$ for all $x, y \in V(H)$, it follows that H is a 1-median graph with medico vertex v. □

Note that medico vertices in a convex subgraph of a k-median graph G are not necessarily medico vertices of G. By way of example, each induced $K_{2,3}$ subgraph of the graph G_4 in Figure 2 is convex and, at the same time, a 2-median graph. However, none of the medico vertices within these $K_{2,3}$s are medico vertices of G_4.

6 The Cartesian graph product

Graph products are commonly employed constructions to assess how graph invariants behave concerning their respective values in the factors and vice versa. We consider the Cartesian product $G \Box H$ that has as vertex set the Cartesian set product $V(G) \times V(H)$ and where two vertices $(g, h), (g', h') \in V(G) \times V(H)$ are adjacent if either (i) $\{g, g'\} \in E(G)$ and $h = h'$ or (ii) $g = g'$ and $\{h, h'\} \in E(H)$. A well-known invariant that "behaves well" in factors and the resulting Cartesian product is the distance between vertices.

Proposition 6.1 ([25, Proposition 5.1])**.** *If $(g, h), (g', h') \in V(G \Box H)$, then $d_{G \Box H}((g, h), (g', h')) = d_G(g, g') + d_H(h, h')$.*

Since distances "behave well", one may ask if medians and medico vertices in products can be constructed from those of the factors and vice versa.

Proposition 6.2. *Let G and H be two graphs. Then, $a \in V(G)$ is a medico vertex in G and $u \in V(H)$ is a medico vertex in H if and only if (a, u) is a medico vertex in $G \Box H$.*

Proof. Assume that a is a medico vertex in G and u is a medico vertex in H. Let (b, v), $(c, w) \in V(G \Box H) \setminus \{(a, u)\}$ be chosen arbitrarily. In order to prove that (a, u) is a medico vertex in $G \Box H$ we must verify that $\mathrm{med}_{G \Box H}((a, u), (b, v), (c, w))$ is well-defined. By assumption, $x := \mathrm{med}_G(a, b, c)$ as well as $y := \mathrm{med}_H(u, v, w)$ are well-defined. In what follows, we show that $(x, y) = \mathrm{med}_{G \Box H}((a, u), (b, v), (c, w))$ holds. Observe first that $d_G(a, x) + d_G(b, x) = d_G(a, b)$ and $d_H(u, y) + d_H(v, y) = d_H(u, v)$ holds. This together with Proposition 6.1 implies that

$$\begin{aligned} d_{G \Box H}((a, u), (x, y)) + d_{G \Box H}((x, y), (b, v)) \\ &= d_G(a, x) + d_H(u, y) + d_G(b, x) + d_H(v, y) \\ &= d_G(a, b) + d_H(u, v) \\ &= d_{G \Box H}((a, u), (b, v)). \end{aligned}$$

Similarly, we can prove that $d_{G \Box H}((a, u), (x, y)) + d_{G \Box H}((x, y), (c, w)) = d_{G \Box H}((a, u), (c, w))$ and $d_{G \Box H}((b, v), (x, y)) + d_{G \Box H}((x, y), (c, w)) = d_{G \Box H}((b, v), (c, w))$. Therefore, (x, y) is a median of the vertices $(a, u), (b, v)$ and (c, w) in $G \Box H$. It remains to show that (x, y) is the only median of these vertices. Assume, for contradiction, that there exists another median (x', y') for $(a, u), (b, v), (c, w)$ in $G \Box H$. Since $G \Box H \simeq H \Box G$, we can assume without loss of generality that $x \neq x'$. Hence, we obtain

$$\begin{aligned} d_{G\Box H}((a,u),(x,y)) + d_{G\Box H}((x,y),(b,v)) \\ &= d_{G\Box H}((a,u),(b,v)) \\ &= d_{G\Box H}((a,u),(x',y')) + d_{G\Box H}((x',y'),(b,v)). \end{aligned}$$

This together with Proposition 6.1 implies that

$$d_G(a,x)+d_H(u,y)+d_G(b,x)+d_H(v,y) = d_G(a,x')+d_H(u,y')+d_G(b,x')+d_H(v,y').$$

Since $x \coloneqq \mathrm{med}_G(a,b,c)$ and $y \coloneqq \mathrm{med}_H(u,v,w)$, we obtain

$$d_G(a,b) + d_H(u,v) = d_G(a,x') + d_H(u,y') + d_G(b,x') + d_H(v,y'). \tag{6.1}$$

Note that $d_G(a,b) \leq d_G(a,x') + d_G(b,x')$ and $d_H(u,v) \leq d_H(u,y') + d_H(v,y')$. Hence, Equation (6.1) is true only in case that $d_G(a,b) = d_G(a,x') + d_G(b,x')$ and $d_H(u,v) = d_H(u,y') + d_H(v,y')$. By similar arguments, one can show that $d_G(b,c) = d_G(b,x')+d_G(c,x')$ and $d_G(a,c) = d_G(a,x')+d_G(c,x')$. Thus, x' is a median of the vertices a,b,c in G; a contradiction to $x \neq x'$ and a being a medico vertex of G. In summary, (a,u) is a medico vertex in $G\Box H$.

Conversely, assume that (a,u) is a medico vertex in $G\Box H$. We need to show that a is a medico vertex in G and u is a medico vertex in H. Let $b,c \in V(G) \setminus \{a\}$ and consider the three vertices $(a,u),(b,u)$ and (c,u) in $G\Box H$. Since (a,u) is a medico vertex in $G\Box H$, $(x,y) \coloneqq \mathrm{med}_{G\Box H}((a,u),(b,u),(c,u))$ is well-defined. We continue with showing that x is a median of a,b,c in G. Observe that

$$\begin{aligned} d_G(a,x) + d_G(x,b) &= d_{G\Box H}((a,u),(x,y)) + d_{G\Box H}((x,y),(b,u)) - 2d_H(u,y) \\ &= d_{G\Box H}((a,u),(b,u)) - 2d_H(u,y), \text{ since } (x,y) \text{ is the median} \\ &= d_G(a,b) - 2d_H(u,y) \end{aligned}$$

Since $d_G(a,x) + d_G(x,b) \geq d_G(a,b)$ is always satisfied, the latter equation implies that $d_H(u,y) = 0$ holds. Thus $d_G(a,x) + d_G(x,b) = d_G(a,b)$. In a similar way, one shows that $d_G(b,x) + d_G(x,c) = d_G(b,c)$ and $d_G(a,x) + d_G(x,c) = d_G(a,c)$. Hence, x is a median of the vertices a,b,c in G. Assume, for contradiction, there exists a median $x' \neq x$ for (a,b,c) in G. We show that, in this case, (x',u) is a median for $(a,u),(b,u),(c,u)$ in $G\Box H$. By Proposition 6.1, we have

$$\begin{aligned} d_{G\Box H}((a,u)(x',u)) + d_{G\Box H}((x',u)(b,u)) \\ &= d_G(a,x') + d_H(u,u) + d_G(x',b) + d_H(u,u) \\ &= d_G(a,b), \text{ since } x' \text{ is a median of } (a,b,c) \text{ in } G \\ &= d_{G\Box H}((a,u),(b,u)) \end{aligned}$$

Similarly, one shows that $d_{G\Box H}((a,u)(x',u)) + d_{G\Box H}((x',u)(c,u)) = d_{G\Box H}((a,u),(c,u))$ and $d_{G\Box H}((b,u)(x',u)) + d_{G\Box H}((x',u)(c,u)) = d_{G\Box H}((b,u),(c,u))$. Hence (x',u) is a median for $(a,u),(b,u),(c,u)$ in $G\Box H$. This together with the facts that $(x,u) \neq (x',u)$ and that $(x,u),(x',u)$ are both medians of $(a,u),(b,u),(c,u)$ in $G\Box H$ yields a contradiction to $\mathrm{med}_{G\Box H}((a,u),(b,u),(c,u))$ being well-defined. In summary, $x' = \mathrm{med}_G(a,b,c)$ is well-defined for a and any two vertices b,c in G which implies that a is a medico vertex G. By similar arguments and since the Cartesian product is commutative, y is a medico vertrex in H. $\Box$

Theorem 6.3. *Let G and H be graphs. If G is a proper k-median graph and H a proper l-median graph, then $G\Box H$ is a proper $(k \cdot l)$-median graph.*

If $G\Box H$ is a proper k-median graph, then there is a factorization $k = k_1 \cdot k_2$ such that G is a proper k_1-median graph and H a proper k_2-median graph.

Proof. Let G be a proper k-median graph and H be a proper l-median graph. Furthermore, let $X \subseteq V(G)$ and $Y \subseteq V(H)$ be the set of all medico vertices in G and H, respectively. By Proposition 6.2, the vertices in $X \times Y$ are precisely the medico vertices $G\Box H$. Since, $|X \times Y| = |X||Y| = k \cdot l$ it follows that $G\Box H$ is a proper $(k \cdot l)$-median graph.

Conversely, assume that $G\Box H$ is a proper k-median graph. Let $W \subseteq V(G) \times V(H)$ be the set of all medico vertices in $G\Box H$ and thus, $k = |W|$. Moreover, let $X = \{a \mid (a, u) \in W\}$ and $Y = \{u \mid (a, u) \in W\}$. Since $W \neq \emptyset$ it follows that neither X nor Y is empty. Proposition 6.2 implies that every $a \in X$ is a medico vertex in G and every $u \in Y$ is a medico vertex in H. Moreover, there cannot be more medico in G than provided by X since, otherwise, there is a set $X' \subseteq V(G)$ of medico vertices in G with $X \subsetneq X'$ in which case Proposition 6.2 implies that $X' \times Y$ is a set of medico vertices in $G\Box H$; a contradiction to $W \subseteq X \times Y \subsetneq X' \times Y$ being the set of all medico vertices in $G\Box H$. Thus, X and, by similar arguments, Y is the set of all medico vertices in G and H, respectively. This together with $W = X \times Y$ and, thus $k = |W| = |X||Y|$, implies that G is a proper $|X|$-median graph and H a proper $|Y|$-median graph. □

As an immediate consequence of Theorem 6.3 we obtain

Theorem 6.4. *$G\Box H$ is a median graph if and only if G and H are median graphs.*

7 Summary and outlook

In this contribution, we considered k-median graphs as a natural generalization of median graphs. We provided several characterizations of k-median graphs based on a generalization of convexity (υ-convexity) as well as three simple conditions (C0), (C1) and (C2). These results, in turn, imply several novel characterizations of median graphs in terms of the structure of subgraphs based on three vertices and the respective shortest paths and intervals between them. A simple tool written in python to verify if a given graph is a k-median graph and to compute the largest such integer k in the affirmative case is provided at GitHub [26].

In the last decades, dozens of interesting characterizations of median graphs have been established and we refer to [32] for an excellent overview. It would be of interest to see in more detail how other characterizations and results are linked to the structure of k-median graphs. In particular, we want to understand in more detail how $K_{2,3}$-free k-median graphs are linked to the structure of hypercubes and median graphs (cf. Problem 2). Moreover, does every k-median graph that is not a median graph contain an induced $K_{2,3}$ or Q_3^- (Problem 1)? Are convex subgraphs of k-median graphs 1-median graphs (Problem 3)?

Furthermore, distance-ℓ-static quartets (x, z, y, μ) in k-median graphs determine C_4s induced by x, z, y and an additional vertex u that is closer to the respective medico vertex μ. That is, repeated application of distance-ℓ-static quartets to reconstruct C_4s must terminate at a certain point and provides valuable information about induced C_4s. This begs the question to what extent a k-median graph can be reconstructed from its distance-ℓ-static quartets along large isometric cycles. Even more, which type of k-median graphs can be uniquely determined by large isometric cycles and the resulting distance-ℓ-static quartets?

In this paper, we provided simple results showing how to obtain a k-median graph from other k-median graphs using the Cartesian graph product. However, the possibly most prominent characterization established by Martyn Mulder [34, 36] states that every median graph can be obtained from a single vertex graph K_1 by a so-called convex expansion procedure. While it is possibly a relative easy task to show that k-median graphs are closed under the convex expansion procedure, it remains an open question if every k-median graph can be obtained by convex expansions applied to a particular set of starting graphs. Another interesting graph operation is the so-called *amalgamation*, which is related to generalizations of median graphs such as quasi-median graphs, pseudo-median graphs, and weakly median graphs. These operations, under various constraints, can be uses to generate all of these type of graphs [3, 6, 7]. It would be interesting for future research to explore whether and how k-median graphs can also be generated by this operation. Moreover, what is the connection of k-median graphs to other generalizations or subclasses of median graphs, see e.g. [7, 12, 13, 14, 15, 31, 33, 40]?

So far, we have considered k-median graphs without making additional assumptions on the integer k. For future research, it would be of interest to investigate the structure of *proper* k-median graphs for a fixed k. Moreover, what are the requirements such that certain graph modification operations (e.g. edge-deletion, edge-addition or contraction of edges) preserve the property of a graph being a k-median graph?

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Mutual-visibility problems in Kneser and Johnson graphs*

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Abstract

Let G be a connected graph and $\mathcal{X} \subseteq V(G)$. By definition, two vertices u and v are $\mathcal{X}$-visible in G if there exists a shortest u, v-path with all internal vertices being outside of the set $\mathcal{X}$. The largest size of $\mathcal{X}$ such that any two vertices of G (resp. any two vertices from $\mathcal{X}$) are $\mathcal{X}$-visible is the total mutual-visibility number (resp. the mutual-visibility number) of G.

In this paper, we determine the total mutual-visibility number of Kneser graphs, bipartite Kneser graphs, and Johnson graphs. The formulas proved for Kneser, and bipartite Kneser graphs are related to the size of transversal-critical uniform hypergraphs, while the total mutual-visibility number of Johnson graphs is equal to a hypergraph Turán number. Exact values or estimations for the mutual-visibility number over these graph classes are also established.

Keywords: Mutual-visibility set, total mutual-visibility set, Kneser graph, bipartite Kneser graph, Johnson graph, Turán-type problem, covering design.

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1 Introduction

Several studies have been conducted to address network visibility problems, both from theoretical and practical points of view. From the practical perspective, many papers focus on robot navigation problems (see, e.g., [1, 5, 12, 24]). From the theoretical perspective, the mutual-visibility notion in graphs was proposed by Di Stefano [13] and then investigated in several graph classes [2, 6, 7, 9, 10, 19]. As a natural extension, Cicerone *et al.* [5] proposed the notion of total mutual-visibility to provide a "visibility" property to all the network nodes instead of only the nodes where robots are located. Total mutual-visibility sets have been investigated recently in product graphs [20, 26] and Hamming graphs [4]. It is also shown that both the mutual-visibility number and the total mutual-visibility number are NP-complete to compute in general [8, 13]. Very recently, some new variants of the mutual-visibility notion were defined and discussed for different graph classes [3, 8, 11].

For a connected graph G with vertex set $V(G)$ and edge set $E(G)$, let $\mathcal{X}$ be a subset of $V(G)$. Two vertices u and v from $V(G)$ are *$\mathcal{X}$-visible* if there exists a shortest u,v-path such that no internal vertex belongs to $\mathcal{X}$. That is, u and v are $\mathcal{X}$-visible if there exists a shortest u,v-path P in G such that $V(P) \cap \mathcal{X} \subseteq \{u,v\}$. If any two vertices from $\mathcal{X}$ are $\mathcal{X}$-visible then the set $\mathcal{X}$ is a *mutual-visibility set*, while if any two vertices from $V(G)$ are $\mathcal{X}$-visible, then the set $\mathcal{X}$ is a *total mutual-visibility set* in G. The cardinality of a largest mutual-visibility (resp. total mutual-visibility) set in G is defined to be the *mutual-visibility number* (resp. *total mutual-visibility number*), $\mu(G)$ (resp. $\mu_{\rm t}(G)$), of G. By definition, $\mu(G) \geq \mu_{\rm t}(G)$ holds for every (connected) graph G.

It has been proved that the mutual-visibility problems are connected to several classical combinatorial problems which are extremely difficult to solve. To determine the mutual-visibility numbers for Cartesian products of two complete graphs is equivalent to the famous problem of Zarankievicz [9]. The total mutual-visibility number of Hamming graphs and line graphs of complete graphs can be expressed as Turán numbers for graphs or hypergraphs [4, 11].

In this manuscript, we prove exact formulas for the total mutual-visibility numbers of Kneser graphs, bipartite Kneser graphs, and Johnson graphs. For Kneser and bipartite Kneser graphs, these formulas show a direct connection to the minimum size of covering designs. For Johnson graphs, we prove that $\mu_{\rm t}(G)$ equals a hypergraph Turán number. Along the way, we also obtain exact values or estimations for the mutual-visibility number over these graph classes.

1.1 Terminology

This paper considers only simple undirected graphs that are connected. The *distance* between two vertices $u, v \in V(G)$, denoted by $\mathrm{dist}(u,v)$, is the length of a shortest path between u and v. The maximum distance between any two vertices in G is the *diameter* of G and is denoted by $\mathrm{diam}(G)$. Throughout the paper, $[n]$ stands for the set $\{1,\dots,n\}$ for every positive integer n. For graph theory concepts not defined here, we refer the reader to the book [29].

A *hypergraph* $\mathcal{H}$ is a set system over the underlying vertex set $V(\mathcal{H})$. We assume, as usual, that the edge set $E(\mathcal{H})$ does not contain the empty set that is $E(\mathcal{H}) \subseteq 2^{V(\mathcal{H})} \setminus \{\emptyset\}$. A hypergraph is *$k$-uniform*, if every (hyper)edge $e \in E(\mathcal{H})$ consists of exactly k vertices. Then 2-uniform hypergraphs correspond to graphs without loops. A k-uniform hypergraph is often called a *k-graph*, while an edge $e \in E(\mathcal{H})$ is a *k-edge* if $|e| = k$. The *degree* of a vertex $v \in V(\mathcal{H})$ is the number of edges incident to v, and v is called *an isolated vertex* (isolate) if its degree is 0. For a set $Y \subseteq V(\mathcal{H})$, the *induced subhypergraph* contains those edges of $\mathcal{H}$ that are fully contained in Y.

A set $T \subseteq V(\mathcal{H})$ is a *transversal* in $\mathcal{H}$ if $e \cap T \neq \emptyset$ for every edge $e \in E(\mathcal{H})$. The minimum cardinality of a transversal is the *transversal number* $\tau(\mathcal{H})$ of the hypergraph.[1]

Let n and k be integers such that $n \geq k \geq 1$. The *Kneser graph* $KG(n,k)$ is the graph with vertices representing the k-subsets of the *ground set* $[n]$, and where two vertices A and B are adjacent if and only if $A \cap B = \emptyset$ holds for the corresponding sets. Formally, the vertex set is $V(KG(n,k)) = \binom{[n]}{k}$ and the edge set is

$$E(KG(n,k)) = \{\{A,B\} : A, B \in V(KG(n,k)) \text{ and } A \cap B = \emptyset\}.$$

This family of graphs was introduced in 1955 by M. Kneser. If $n < 2k$, then $KG(n,k)$ consists of $\binom{n}{k}$ isolated vertices, and if $n = 2k$, then $KG(n,k)$ is the union of disjoint copies of K_2. Moreover, $KG(n,1)$ is the complete graph on n vertices. In this paper, we consider only connected Kneser graphs. Thus, to avoid some trivial cases, we assume that $n \geq 2k+1$ and $k \geq 2$ for a Kneser graph $KG(n,k)$.

The vertex set of a *bipartite Kneser graph* $H(n,k)$ consists of vertices representing the k-element and $(n-k)$-element subsets of the ground set $[n]$. Two vertices u and v are adjacent if one of the represented sets is a subset of the other. By definition, $H(n,k)$ is bipartite for every $n > 2k \geq 2$. The partite classes correspond to $\binom{[n]}{k}$ and $\binom{[n]}{n-k}$, respectively. As $H(n,k) \cong H(n,n-k)$ and $H(2k,k)$ consists of k copies of K_2-components, we will assume $n \geq 2k+1 \geq 5$ while studying mutual-visibility problems in bipartite Kneser graphs.

The *Johnson graph* $J(n,k)$ is the graph with vertex set consisting of all the k-subsets of the ground set $[n]$ and the edge set is

$$E(J(n,k)) = \{\{A,B\} : A, B \in V(J(n,k)) \text{ and } |A \cap B| = k-1\}.$$

Observe that, by definition, $J(n,2)$ corresponds to the line graph of the complete graph K_n. It is well-known that $J(n,k)$ is isomorphic to $J(n,n-k)$ and that $J(n,1)$ is isomorphic to the complete graph on n vertices. To avoid these cases, in this paper, we deal with Johnson graphs $J(n,k)$ where $n \geq k+2 \geq 4$.

When we consider a graph G that is a Kneser, bipartite Kneser, or a Johnson graph, the vertices will be denoted by italic capital letters, and *the same notation will be used for a vertex of* G *and the represented subset of* $[n]$. For a subset $\mathcal{S} \subseteq V(G)$, we define the *underlying hypergraph* $\mathcal{F}(\mathcal{S})$ on the ground set $[n]$ that contains edges corresponding to the sets represented by the vertices in $\mathcal{S}$. That is, if G and $\mathcal{S} \subseteq V(G)$ are given,

$$E(\mathcal{F}(\mathcal{S})) = \{S \subseteq [n] : S \in \mathcal{S}\}.$$

If G is a Kneser graph $KG(n,k)$ or a Johnson graph $J(n,k)$, then $\mathcal{F}(\mathcal{S})$ is a k-graph for every nonempty vertex set $\mathcal{S}$.

Turán-type problems form a central area in extremal graph and hypergraph theory. Typically, in a hypergraph Turán problem two integers n, k and a k-graph $\mathcal{F}$ are given. We want to determine the maximum number of edges in a k-graph $\mathcal{H}$ on an n-element vertex

[1] Transversals are often called vertex covers, especially in the context of graphs. However, in this paper, we use the term "cover" in a different sense (as a subset relation between sets) and keep the term "transversal" as defined here.

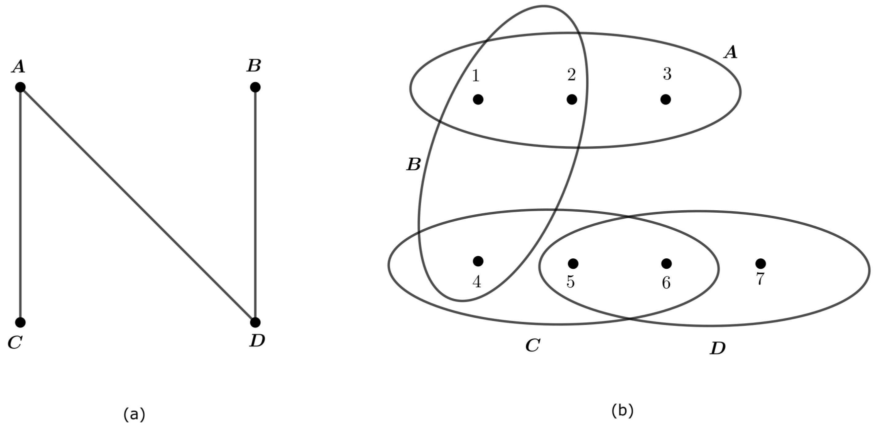

Figure 1: Figure (a) shows the subgraph of $K(7,3)$ induced by $\mathcal{S} = \{A, B, C, D\}$ where vertices A, B, C, D represent the sets $\{1,2,3\}, \{1,2,4\}, \{4,5,6\}, \{5,6,7\}$, respectively. In figure (b) the underlying hypergraph $\mathcal{F}(\mathcal{S})$ is depicted where $E(\mathcal{F}(\mathcal{S})) = \{A, B, C, D\}$.

set such that $\mathcal{H}$ contains no subhypergraph isomorphic to $\mathcal{F}$. This maximum number is denoted by $\mathrm{ex}_k(n, \mathcal{F})$. If $k = 2$ and $\mathcal{F}$ is a graph, we omit the index 2 and write simply $\mathrm{ex}(n, \mathcal{F})$. Most of the hypergraph Turán problems seem extremely difficult. Even the tight asymptotics for $\mathrm{ex}_k(n, \mathcal{F})$ as $n \to \infty$ might be hard to identify. For more results on the subject, we refer the reader to [18] and [14, Chapter 5].

For three positive integers with $n \geq k \geq t$, an (n, k, t) *covering design* is a pair $(X, \mathcal{B})$, where X is a set of n elements and $\mathcal{B}$ is a collection of k-element subsets of X (blocks) such that every t-element subset of X is a subset of at least one block from $\mathcal{B}$. The minimum number of blocks that may occur in an (n, k, t) covering design is the *covering number* $C(n, k, t)$. By double-countation, we obtain that $\binom{k}{t} C(n, k, t) \geq \binom{n}{t}$ is always true. Further, this inequality holds with equality if and only if a Steiner system $S(t, k, n)$ exists (see [16]). This fact shows that determining the values $C(n, k, t)$ is a challenging problem in general. If $(X, \mathcal{B})$ is an (n, k, t) covering design and $\mathcal{H} = (X, \mathcal{B}^c)$ is the $(n-k)$-uniform hypergraph containing the complements of the blocks $B \in \mathcal{B}$, then no t-element subset of X intersects every edge of $\mathcal{H}$. Then, we have $\tau(\mathcal{H}) > t$. As this implication also holds in the other direction, we may conclude that $C(n, n-k, t)$ is the minimum number of edges in a k-uniform hypergraph with n vertices and transversal number $t + 1$. This correlation was first remarked by D.B. West (see Chapter 7.1 in [17]).

1.2 Results and structure of the paper

We start our study in Section 2 by proving a lemma on the values of the covering number $C(n, k, t)$ under specific conditions.

In Section 3, the total mutual-visibility numbers of Kneser graphs $KG(n, k)$ are determined. While we prove explicit formulas in terms of n and k for the cases when $n \leq 3k-1$ or $2k^2 \leq n$, the formula for the middle range contains the constant $C(n, n-k, 2k-1)$.

In Section 4, we determine the mutual-visibility number of $KG(n, k)$ for $n \geq 7k - 5$ and estimate the value for the remaining cases.

In Section 5, the total mutual-visibility numbers of bipartite Kneser graphs $H(n,k)$ are determined. We also provide a lower bound for the mutual-visibility number $\mu(H(n,k))$.

Section 6 is devoted to the study of Johnson graphs. The main theorem establishes equality between $\mu_t(J(n,k))$ and a hypergraph Turán number for all nontrivial cases. The exact mutual-visibility number is proved for $J(n,2)$, and it is shown that $\mu(J(n,k))$ is sandwiched between two Turán numbers in general.

In the last section, we answer a question from the recent manuscript [11] by determining the exact values for the mutual-visibility numbers of Kneser graphs $KG(n,2)$. Further, we give some remarks and formulate open problems related to the topic of this paper.

2 A lemma on the covering number

As a preparation for the proofs in Sections 3 and 4, here we mainly study the covering number. First, we prove a short technical lemma.

Lemma 2.1. *If n and k are two positive integers and $k < n < 2k$, then*

$$\binom{n}{k} > 2\binom{n-1}{k}.$$

Proof. The condition $2k > n$ implies $\frac{n}{n-k} > 2$ and hence,

$$\binom{n}{k} = \frac{n}{n-k}\binom{n-1}{k} > 2\binom{n-1}{k}$$

holds. □

Now, we concentrate on the covering numbers given in the form $C(n, n-k, 2k-1)$. To avoid trivial or non-defined cases, we will assume that $n \geq 3k$ and $k \geq 2$. Under these conditions, to simplify our notation, we set

$$C^*(n,k) = C(n, n-k, 2k-1)$$

that is the minimum number of edges in a k-uniform hypergraph of order n with transversal number $2k$. Recall that $\tau(\mathcal{H})$ denotes the transversal number of hypergraph $\mathcal{H}$.

Lemma 2.2. *Let n and k be integers.*

(i) *If $k \geq 2$ and $n \geq 2k^2$, then $C^*(n,k) = 2k$.*

(ii) *If $k \geq 3$ and $n \geq 7k-5$, then $C^*(n,k) \leq 2\binom{2k-3}{k} + 6$.*

(iii) *If $k \geq 2$ and $n \geq 2k^2 + k$, then $C(n, n-k, 2k) = 2k+1$.*

Proof. To prove (i), we note that having $\tau = 2k$ needs at least $2k$ edges. Thus, $C^*(n,k) \geq 2k$. Moreover, the condition $n \geq 2k^2$ allows us to construct a k-uniform hypergraph on n vertices with $2k$ pairwise disjoint edges. As its transversal number equals $2k$, the minimum number of edges in such a hypergraph is at most $2k$. Therefore, $C^*(n,k) = 2k$.

To prove (ii) we consider a hypergraph $\mathcal{H}_{n,k}$ built from the following types of components.

- A k-uniform generalized triangle $\mathcal{H}_1^{(k)}$ has $n_1^{(k)} = k + \lceil \frac{k}{2} \rceil$ vertices partitioned into three sets V_1, V_2, V_3 such that $|V_1| = \lfloor \frac{k}{2} \rfloor$ and $|V_2| = |V_3| = \lceil \frac{k}{2} \rceil$. If k is even, we simply put
$$E(\mathcal{H}_1^{(k)}) = \{V_1 \cup V_2, V_1 \cup V_3, V_2 \cup V_3\}.$$
If k is odd, we specify a vertex $x_3 \in V_3$ and replace the edge $V_2 \cup V_3$ with the k-edge $V_2 \cup (V_3 \setminus \{x_3\})$ in the definition of $E(\mathcal{H}_1^{(k)})$. Then, $\mathcal{H}_1^{(k)}$ has $m_1^{(k)} = 3$ edges and $\tau(\mathcal{H}_1^{(k)}) = 2$ for all $k \geq 2$.

- A complete k uniform hypergraph $\mathcal{K}_{2k-3}^{(k)}$ is defined on $2k-3$ vertices such that all k-element subsets of the vertex set are hyperedges in $\mathcal{K}_{2k-3}^{(k)}$. Then, the size of this hypergraph is $m_2^{(k)} = \binom{2k-3}{k}$ and
$$\tau(\mathcal{K}_{2k-3}^{(k)}) = (2k-3) - (k-1) = k-2.$$

Now, to define the hypergraph $\mathcal{H}_{n,k}$, take two disjoint copies of $\mathcal{H}_1^{(k)}$ and two disjoint copies of $\mathcal{K}_{2k-3}^{(k)}$. They together cover
$$2\left(k + \left\lceil \frac{k}{2} \right\rceil\right) + 2(2k-3) \leq 7k - 5$$
vertices. The remaining vertices (if any) remain isolates. The transversal number of $\mathcal{H}_{n,k}$ is
$$\tau(\mathcal{H}_{n,k}) = 2 \cdot 2 + 2(k-2) = 2k.$$
By definition of the covering number, we may conclude that
$$C^*(n,k) \leq |E(\mathcal{H}_{n,k})| = 2\binom{2k-3}{k} + 6.$$

The proof for part (iii) is similar to that for (i). Since $C(n, n-k, 2k)$ is the minimum number of edges in a k-uniform hypergraph on n vertices having transversal number at least $2k+1$, at least $2k+1$ edges are needed. On the other hand, $n \geq 2k^2 + k$ vertices are enough to take $2k+1$ pairwise vertex-disjoint k-edges and achieve $\tau \geq 2k+1$. □

3 Total mutual-visibility in Kneser graphs

The main result of this section is Theorem 3.2 that determines the total mutual-visibility number of Kneser graphs for all nontrivial cases. In its proof, we will use the following lemma.

Lemma 3.1. *Let $\mathcal{X}$ be a set of vertices of the Kneser graph $KG(n,k)$ and $\overline{\mathcal{X}} = V(KG(n,k)) \setminus \mathcal{X}$. For every $n \geq 3k-1$, the set $\mathcal{X}$ is a total mutual-visibility set in $KG(n,k)$ if and only if $\tau(\mathcal{F}(\overline{\mathcal{X}})) \geq 2k$.*

Proof. If $n \geq 3k-1$, then $\operatorname{diam}(KG(n,k)) = 2$ (see [28]). Suppose first that $\tau(\mathcal{F}(\overline{\mathcal{X}})) \geq 2k$ and consider two arbitrary vertices A, B from the Kneser graph $G = KG(n,k)$. If A and B are adjacent, they are $\mathcal{X}$-visible. Otherwise $\operatorname{dist}(A,B) = 2$, and the k-sets A and B are intersecting. Thus $|A \cup B| \leq 2k-1 < \tau(\mathcal{F}(\overline{\mathcal{X}}))$, and the set $A \cup B$ cannot intersect all edges of $\mathcal{F}(\overline{\mathcal{X}})$. Let us choose such an edge Y with $Y \cap (A \cup B) = \emptyset$ from $\mathcal{F}(\overline{\mathcal{X}})$. The corresponding vertex Y belongs to $\overline{\mathcal{X}}$ in G. As we have $Y \cap A = Y \cap B = \emptyset$, the vertex Y is a common neighbor of A and B. We conclude that A and B are $\mathcal{X}$-visible, and hence, $\mathcal{X}$ is a total mutual-visibility set in G.

To prove the other direction, we suppose that $\tau(\mathcal{F}(\overline{\mathcal{X}})) \leq 2k-1$. Let us choose $2k-1$ elements, say $a_1, \dots, a_{k-1}, b_1, \dots, b_{k-1}, c$, from the ground set $[n]$ such that they together intersect all edges of $\mathcal{F}(\overline{\mathcal{X}})$. Consider the two nonadjacent vertices of G corresponding to the sets $A = \{a_1, \dots, a_{k-1}, c\}$ and $B = \{b_1, \dots, b_{k-1}, c\}$. Since $A \cup B$ intersects every k-edge from $\mathcal{F}(\overline{\mathcal{X}})$, the vertices A and B in G have no common neighbor from $\overline{\mathcal{X}}$. As $\mathrm{dist}(A, B) = 2$ in G, we may conclude that A and B are not $\mathcal{X}$-visible and, therefore $\mathcal{X}$ is not a total mutual-visibility set. □

Theorem 3.2. *If n and k are integers such that $n \geq 2k+1$ and $k \geq 2$, then*

$$\mu_t(KG(n,k)) = \begin{cases} 0 & \text{if } 2k+1 \leq n \leq 3k-1, \\ \binom{n}{k} - C^*(n,k) & \text{if } 3k \leq n \leq 2k^2 - 1, \\ \binom{n}{k} - 2k & \text{if } 2k^2 \leq n. \end{cases}$$

Proof. First, suppose that $2k+1 \leq n \leq 3k-1$ and a total mutual-visibility set $\mathcal{X}$ in $G = KG(n,k)$ contains at least one vertex A. Let $\overline{A}$ denote the $(n-k)$-element set $[n] \setminus A$. As $k+1 \leq |\overline{A}| \leq 2k-1$, we can always find two k-element subsets B and C such that $B \cap C \neq \emptyset$ and $B \cup C = \overline{A}$. The corresponding vertices B and C are nonadjacent in G, and only vertex A is adjacent to both. Consequently, BAC is the unique shortest B,C-path in G. Since $A \in \mathcal{X}$, the vertices B and C are not $\mathcal{X}$-visible, and $\mathcal{X}$ is not a total mutual visibility set in G. This contradiction proves $\mu_t(G) = 0$ for the range $2k+1 \leq n \leq 3k-1$.

Now, suppose that $n \geq 3k$. By Lemma 3.1, $\tau(\mathcal{F}(\overline{\mathcal{X}})) \geq 2k$ is exactly the property expected from the k-graph $\mathcal{F}(\overline{\mathcal{X}})$ to ensure that $\mathcal{X}$ is a total mutual-visibility set. Thus $C^*(n,k)$ is the minimum number of k-edges in $\mathcal{F}(\overline{\mathcal{X}})$ that is the minimum number of vertices in $\overline{\mathcal{X}}$ when $\mathcal{X}$ is a total mutual-visibility set in G. Therefore $\mu_t(G) = \binom{n}{k} - C^*(n,k)$, if $n \geq 3k$.

As $2k^2 \geq 3k$, the formula $\mu_t(G) = \binom{n}{k} - C^*(n,k)$ remains valid under the condition $n \geq 2k^2$. Moreover, in this case we have $C^*(n,k) = 2k$ according to Lemma 2.2(i). This finishes the proof of the theorem. □

4 Mutual-visibility number of Kneser graphs

Here we study the mutual-visibility number of Kneser graphs. Theorem 4.1 shows that $\mu(KG(n,k))$ equals $\mu_t(KG(n,k))$ if $n \geq 7k-5$, while Proposition 4.2 states a lower bound for the range $2.5k \leq n \leq 7k-4$.

Theorem 4.1. *If n and k are two integers such that $n \geq 7k-5$ and $k \geq 2$, then*

$$\mu(KG(n,k)) = \binom{n}{k} - C^*(n,k).$$

In particular, if $n \geq 2k^2$ and $k \geq 2$, then

$$\mu(KG(n,k)) = \binom{n}{k} - 2k.$$

Proof. Since $\mu(KG(n,k)) \geq \mu_t(KG(n,k))$ holds by definition, Theorem 3.2 directly implies $\mu(KG(n,k)) \geq \binom{n}{k} - C^*(n,k)$ when $n \geq 7k-5 \geq 3k$. Therefore, it is enough to prove that $\mu(KG(n,k)) \leq \binom{n}{k} - C^*(n,k)$ when $n \geq 7k-5$.

Let k and n be integers so that $n \geq 7k-5$ and denote the graph $KG(n,k)$ by G. Recall that $\mathrm{diam}(G) = 2$ under the given conditions. Suppose for a contradiction that $\mathcal{X}$ is a mutual-visibility set in G and $|\overline{\mathcal{X}}| \leq C^*(n,k) - 1$. It follows that $\tau(\mathcal{F}(\overline{\mathcal{X}})) \leq 2k-1$. By Lemma 2.2(ii), if $k \geq 3$, the number of k-edges in $\mathcal{F}(\overline{\mathcal{X}})$, that is denoted by m and equals $|\overline{\mathcal{X}}|$, satisfies

$$m \leq 2\binom{2k-3}{k} + 5.$$

Our aim is to find two k-element subsets A and B of $[n]$ such that

($\star$) $A, B \notin E(\mathcal{F}(\overline{\mathcal{X}}))$, A intersects B, and $A \cup B$ is a transversal in $\mathcal{F}(\overline{\mathcal{X}})$.

Then, in the graph G, the vertices A and B belong to $\mathcal{X}$ and have no common neighbor in $\overline{\mathcal{X}}$. As A and B are nonadjacent and $\mathrm{diam}(G) = 2$, there is no shortest A, B-path to ensure the mutual-visibility. This will give the contradiction and proves that $|\overline{\mathcal{X}}| \geq C^*(n,k)$ when $\mathcal{X}$ is a mutual-visibility set in G. We consider three cases in the continuation.

Case 1. $k \geq 3$ and $\tau(\mathcal{F}(\overline{\mathcal{X}})) = 2k-1$.
Let T be a minimum transversal in $\mathcal{F}(\overline{\mathcal{X}})$ and denote by $\mathcal{T}$ the subhypergraph induced by T in $\mathcal{F}(\overline{\mathcal{X}})$. Then $|T| = 2k-1$. By the minimality of T, the set $T \setminus \{u\}$ cannot be a transversal if $u \in T$. Equivalently, every vertex $u \in T$ is incident to a 'private' edge e_u from $\mathcal{F}(\overline{\mathcal{X}})$ such that the intersection $e_u \cap T$ contains only the vertex u. For the number of hyperedges in $\mathcal{T}$, which is denoted by $m_{\mathcal{T}}$, we therefore have

$$m_{\mathcal{T}} \leq m - (2k-1) \leq m - 5 \leq 2\binom{2k-3}{k}.$$

To estimate the average vertex degree $\overline{d}_{\mathcal{T}}$ in the induced subhypergraph $\mathcal{T}$, we apply Lemma 2.1:

$$\begin{aligned}\overline{d}_{\mathcal{T}} = \frac{k\, m_{\mathcal{T}}}{2k-1} &\leq 2\binom{2k-3}{k}\frac{k}{2k-1} \\ &< \frac{1}{2}\binom{2k-1}{k}\frac{k}{2k-1} \\ &= \frac{1}{2}\binom{2k-2}{k-1}. \end{aligned} \tag{4.1}$$

The upper bound implies that there exists a vertex $z \in T$ being incident to less than $\frac{1}{2}\binom{2k-2}{k-1}$ hyperedges in $\mathcal{T}$. Consider now the pairs $\{A', B'\}$ where A' and B' are disjoint $(k-1)$-element subsets of $T \setminus \{z\}$. Every $(k-1)$-element subset appears in exactly one such pair and there are $\frac{1}{2}\binom{2k-2}{k-1}$ pairs. By the pigeonhole principle, there exists a pair $\{A', B'\}$ such that neither $A = A' \cup \{z\}$ nor $B = B' \cup \{z\}$ is an edge in $\mathcal{T}$. Consequently, the k-sets A and B are not edges in $\mathcal{F}(\overline{\mathcal{X}})$, they intersect in z, and the union $T = A \cup B$ is a transversal in $\mathcal{F}(\overline{\mathcal{X}})$. We may infer that A and B satisfy ($\star$) and the corresponding vertices $A, B \in \mathcal{X}$ are not $\mathcal{X}$-visible in G. This contradiction proves $\mu(KG(n,k)) \leq \binom{n}{k} - C^*(n,k)$.

Case 2. $k \geq 3$ and $\tau(\mathcal{F}(\overline{\mathcal{X}})) \leq 2k-2$.
First, suppose that there is an isolated vertex z in $\mathcal{F}(\overline{\mathcal{X}})$. Then there exists a (not necessarily minimum) transversal T in $\mathcal{F}(\overline{\mathcal{X}})$ such that $|T| = 2k-2$ and $z \notin T$. For an arbitrary $(k-1)$-element subset A' of T and for the complement $B' = T \setminus A'$, the sets $A = A' \cup \{z\}$ and $B = B' \cup \{z\}$ satisfy the three properties in $(\star)$ and give the required contradiction.
In the other case, there is no isolated vertex in $\mathcal{F}(\overline{\mathcal{X}})$. Let T be an arbitrary $(2k-1)$-element transversal of this hypergraph and denote by $m_{\mathcal{T}}$ the number of hyperedges in the subhypergraph $\mathcal{T}$ induced by T. For the

$$n - |T| \geq (7k-5) - (2k-1) = 5k - 4 > 5(k-1)$$

non-isolated vertices outside T, we need at least 6 hyperedges to cover them all in $\mathcal{F}(\overline{\mathcal{X}})$. It implies $m_{\mathcal{T}} \leq m - 6 < 2\binom{2k-3}{k}$. In the continuation, we can proceed as in Case 1; that is, we estimate the average degree in $\mathcal{T}$ according to (4.1) and choose a vertex $z \in T$ of minimum degree. Considering the pairs of disjoint $(k-1)$-element subsets of $T \setminus \{z\}$, we may conclude that there exist k-sets A and B satisfying $(\star)$. The contradiction proves $\mu(KG(n,k)) \leq \binom{n}{k} - C^*(n,k)$.

Case 3. $k = 2$.
If $k = 2$, the condition $n \geq 7k-5$ gives $n \geq 9$ and implies $n > 2k^2 = 8$. Hence, by Lemma 2.2(i), $C^*(n,2) = 4$ also holds. By our supposition, $\mathcal{X}$ is a mutual visibility set in G and $\mathcal{F}(\overline{\mathcal{X}})$ contains $m \leq C^*(n,2) - 1 = 3$ edges. As $n \geq 9$, there exist isolated vertices. If $\tau(\mathcal{F}(\overline{\mathcal{X}})) = 3$, then we have three vertex-disjoint hyperedges in $\mathcal{F}(\overline{\mathcal{X}})$ and, for a minimum transversal T, it is easy to choose two 2-element sets $A \subset T$ and $B \subset T$ satisfying $(\star)$. If $\tau(\mathcal{F}(\overline{\mathcal{X}})) \leq 2$, we take a 2-element transversal $\{a,b\}$ and a third vertex c that is isolated in $\mathcal{F}(\overline{\mathcal{X}})$. The sets $A = \{a,c\}$, $B = \{b,c\}$ satisfy $(\star)$. This gives the desired contradiction and proves the formula for $k = 2$.

Concerning the second part of the statement, if $k \geq 3$ and $n \geq 2k^2$, then $n \geq 7k-5$ also holds. Thus the formula $\mu(KG(n,k)) = \binom{n}{k} - C^*(n,k)$ and Lemma 2.2(i) directly imply $\mu(KG(n,k)) = \binom{n}{k} - 2k$. The same is true if $k = 2$ and $n \geq 9$. The only remaining case is the graph $KG(8,2)$. Here, a mutual-visibility set $\mathcal{X}$ with more than $\binom{8}{2} - 4$ vertices defines the 2-uniform underlying hypergraph $\mathcal{F}(\overline{\mathcal{X}})$ with $n = 8$ vertices and at most three edges. As in Case 3 of the proof, it is easy to exhibit two non-edges in $\mathcal{F}(\overline{\mathcal{X}})$ that satisfy $(\star)$ and prove that $\mathcal{X}$ is not a mutual visibility set in $KG(8,2)$. □

The mutual-visibility notion is related to the general position problem in graphs. A subset S of vertices in a connected graph G is a *general position set* if no triple of vertices from S lie on a common shortest path in G. Equivalently, for every three different vertices $u, v, z \in S$, the strict inequality $\mathrm{dist}(u,v) + \mathrm{dist}(v,z) > \mathrm{dist}(u,z)$ holds. The general position problem [15, 21] is to find a largest general position set of G and the order of such a set is the *general position number* $\mathrm{gp}(G)$. By definition, every general position set is a mutual visibility set and therefore, $\mu(G) \geq \mathrm{gp}(G)$ holds for every connected graph G.

Proposition 4.2. *If n and k are integers satisfying $2.5k - 0.5 \leq n \leq 7k-5$ and $k \geq 2$, then*

$$\mu(KG(n,k)) \geq \binom{n-1}{k-1}.$$

Proof. The results from [15] and [23] yield that $\operatorname{gp}(KG(n,k)) = \binom{n-1}{k-1}$ holds when $n \geq 2.5k - 0.5$. Then, the relation $\mu(KG(n,k)) \geq \operatorname{gp}(KG(n,k))$ directly implies the statement. □

5 Bipartite Kneser graphs

In this section we study the total mutual-visibility number and mutual-visibility number of bipartite Kneser graphs. Since a k-element subset $S_1 \subseteq [n]$ is a subset of an $(n-k)$-element subset $S_2 \subseteq [n]$ if and only if $[n] \setminus S_2 \subseteq [n] \setminus S_1$, bipartite Kneser graphs satisfy the following property:

Observation 5.1. Let G be the bipartite Kneser graph $H(n,k)$ such that the vertices in the partite classes $\mathcal{P}_1$ and $\mathcal{P}_2$ represent the k-element and $(n-k)$-element subsets of $[n]$, respectively. Let ϕ be the function that assigns to each vertex $S \in V(H(n,k))$ the vertex S^c which represents the complement set $S^c = [n] \setminus S$. Then, ϕ is an automorhism of $H(n,k)$ that maps $\mathcal{P}_1$ into $\mathcal{P}_2$.

It follows from the observation that any property proved for $\mathcal{P}_1$ holds for $\mathcal{P}_2$, and vice versa. Remark that bipartite Kneser graphs satisfy the much stronger property of symmetry i.e., they are vertex- and edge-transitive graphs [25].

In the main result of this section, we will use again the constant $C(n,k,t)$ that is the minimum size of a covering design with the given parameters.

Theorem 5.2. *If n and k are integers such that $n \geq 2k+1$, then*

$$\mu_{\mathrm{t}}(H(n,k)) = \begin{cases} 0 & \text{if } 2k+1 \leq n \leq 3k, \\ 2\binom{n}{k} - 2\,C(n, n-k, 2k) & \text{if } 3k+1 \leq n < 2k^2+k, \\ 2\binom{n}{k} - 4k - 2 & \text{if } 2k^2+k \leq n. \end{cases}$$

Proof. Consider a bipartite Kneser graph $H(n,k)$ with parameters satisfying $n \geq 2k+1$, and let $\mathcal{P}_1$ and $\mathcal{P}_2$ denote the two partite classes of $H(n,k)$ such that $\mathcal{P}_1$ consists of the vertices representing the k-element subsets of $[n]$. Let $\mathcal{X}$ be a total mutual visibility set in $H(n,k)$.

We first assume that $n \leq 3k$ and show that $\mathcal{X} \cap \mathcal{P}_1 = \mathcal{X} \cap \mathcal{P}_2 = \emptyset$. Note that $n \leq 3k$ implies $k + 2(n-2k) \leq n$ and then, for every vertex $A \in \mathcal{P}_1$, we can choose two $(n-k)$-element sets B_1 and B_2 such that $B_1 \cap B_2 = A$. Vertex A is clearly the only common neighbor of B_1 and B_2 in $H(n,k)$. The $\mathcal{X}$-visibility of B_1 and B_2 therefore implies $A \notin \mathcal{X}$. By Observation 5.1, we infer that the same is true for the vertices in $\mathcal{P}_2$ and therefore, $\mathcal{X} = \emptyset$. It proves $\mu_{\mathrm{t}}(H(n,k)) = 0$ for the case $n \leq 3k$.

Assume now that $n \geq 3k+1$ and show that the family $\mathcal{P}_2 \cap \overline{\mathcal{X}}$ of $(n-k)$-element sets covers all $(2k)$-element subsets of $[n]$. Suppose to the contrary that no $B \in \mathcal{P}_2 \cap \overline{\mathcal{X}}$ covers the $(2k)$-element set $S \subseteq [n]$. Split S into two disjoint k-sets A_1 and A_2. Since $n-k > 2k$, there exist $(n-k)$-element subsets of $[n]$ that cover both A_1 and A_2. However, by our supposition, none of the vertices corresponding to these covering sets belong to $\mathcal{P}_2 \cap \overline{\mathcal{X}}$. Equivalently, in $H(n,k)$, $\operatorname{dist}(A_1, A_2) = 2$ but none of the common neighbors belong to $\overline{\mathcal{X}}$. This contradicts the $\mathcal{X}$-visibility of A_1 and A_2. We infer that the $(n-k)$-sets in $\mathcal{P}_2 \cap \overline{\mathcal{X}}$ cover all $(2k)$-subsets of $[n]$ and therefore, $|\mathcal{P}_2 \cap \overline{\mathcal{X}}| \geq C(n, n-k, 2k)$. By

Observation 5.1, $|\mathcal{P}_1 \cap \overline{\mathcal{X}}| \geq C(n, n-k, 2k)$ follows and we may conclude

$$\mu_t(H(n,k)) \leq 2\binom{n}{k} - 2\,C(n, n-k, 2k).$$

To prove the other direction, let $\mathcal{B}$ be a family of $(n-k)$-element subsets of $[n]$ that cover all $(2k)$-element subsets. We also suppose that $\mathcal{B}$ is minimum that is, $|\mathcal{B}| = C(n, n-k, 2k)$ and use the same notation $\mathcal{B}$ for the set of corresponding vertices in $\mathcal{P}_2$. Moreover, let

$$\mathcal{A} = \{A \subseteq [n] : [n] \setminus A \in \mathcal{B}\} = \phi(\mathcal{B}),$$

where ϕ is the automorphism from Observation 5.1. We are going to prove that the vertex set $\mathcal{X} = \overline{\mathcal{A} \cup \mathcal{B}}$ forms a total mutual-visibility set in $H(n,k)$.

Take two vertices A_1 and A_2 from the partite class $\mathcal{P}_1$ of $H(n.k)$. Since $|A_1 \cup A_2| \leq 2k$, the union is covered by an $(n-k)$-element set $B \in \mathcal{B}$. The corresponding vertex $B \in \overline{\mathcal{X}}$ is a common neighbor of A_1 and A_2 and consequently A_1 and A_2 are $\mathcal{X}$-visible. By Observation 5.1 and by the symmetry of our definitions for $\mathcal{A}$ and $\mathcal{B}$, the same is true for any two vertices $B_1, B_2 \in \mathcal{P}_2$.

The last case to check is when $A \in \mathcal{P}_1$ and $B \in \mathcal{P}_2$. If they are adjacent, the $\mathcal{X}$-visibility follows. If they are nonadjacent vertices, the distance is at least 3 (in fact, it is exactly 3). We may choose an arbitrary vertex Y from $\mathcal{P}_1$ that is different from A. As we have already seen, there exists a vertex $B_1 \in \mathcal{P}_2 \cap \overline{\mathcal{X}}$ which is a common neighbor of A and Y. Similarly, there is a vertex A_1 in $\mathcal{P}_1 \cap \overline{\mathcal{X}}$ which is a common neighbor of B and B_1. We may infer that AB_1A_1B is a shortest A,B-path and both internal vertices belong to $\overline{\mathcal{X}}$. Therefore, A and B are $\mathcal{X}$-visible. It finishes the proof for $\mathcal{X} = \overline{\mathcal{A} \cup \mathcal{B}}$ being a total mutual-visibility set in $H(n,k)$. We may conclude that

$$\mu_t(H(n,k)) \geq |\mathcal{X}| = 2\binom{n}{k} - |\mathcal{A} \cup \mathcal{B}| = 2\binom{n}{k} - 2\,C(n, n-k, 2k).$$

This proves the desired formula for $n \geq 3k+1$.

If $n \geq 2k^2 + k$, the equality $\mu_t(H(n,k)) = 2\binom{n}{k} - 2\,C(n, n-k, 2k)$ still remains valid, and together with Lemma 2.2(iii) they yield the formula stated for the last case. □

Concerning the mutual-visibility number of bipartite Kneser graphs, we propose two simple lower bounds.

Proposition 5.3. *If n and k are integers such that $n \geq 3k+1$, then*

$$\mu(H(n,k)) \geq \max\left\{\binom{n}{k},\, 2\binom{n}{k} - 2\,C(n, n-k, 2k)\right\}.$$

Proof. By definition, $\mu(G) \geq \mu_t(G)$ holds for every graph G. Theorem 5.2 then implies $\mu(H(n,k)) \geq 2\binom{n}{k} - 2\,C(n, n-k, 2k)$. The inequality $\mu(H(n,k)) \geq \binom{n}{k}$ follows from the fact that $\mathcal{X} = \mathcal{P}_1$ is a mutual-visibility set if $n \geq 3k+1$. Indeed, as $\mathrm{dist}(A_1, A_2) = 2$ holds for every two vertices A_1, A_2 from $\mathcal{P}_1 = \mathcal{X}$, each shortest path between them contains only one internal vertex, and this vertex is from $\mathcal{P}_2 = \overline{\mathcal{X}}$. □

6 Johnson graphs

In this section, we determine the total mutual-visibility number and estimate the mutual visibility number of Johnson graphs in terms of graph and hypergraph Turán numbers. Before stating the main results, we define the notion of *k-uniform suspension*. For a graph G and an integer $k \geq 2$, the k-uniform suspension of G, denoted by $\mathcal{G}^{k+}$, is obtained from G by taking $k-2$ new vertices and adding them to every edge of the graph. Formally, the vertex set is $V(\mathcal{G}^{k+}) = V(G) \cup Y$, where Y is a set of vertices with $|Y| = k-2$ and $V(G) \cap Y = \emptyset$. The edge set is

$$E(\mathcal{G}^{k+}) = \{Y \cup e : e \in E(G)\}.$$

In our results, we will refer to the hypergraphs $\mathcal{C}_4^{k+}$ and $\mathcal{K}_4^{k+}$ which are the k-uniform suspensions of C_4 and K_4, respectively.[2] The $(k+2)$-element vertex set for both of them is given as $Y \cup \{z_1, z_2, z_3, z_4\}$, and the two edge sets are specified as

$$E(\mathcal{C}_4^{k+}) = \{Y \cup \{z_i, z_{i+1}\}\} : i \in [3]\} \cup \{Y \cup \{z_4, z_1\}\}, \quad (6.1)$$

$$E(\mathcal{K}_4^{k+}) = \{Y \cup \{z_i, z_j\}\} : i, j \in [4], \;\; i < j\}. \quad (6.2)$$

The following tight asymptotic result was proved by Mubayi:

Theorem 6.1 ([22]). *For every $k \geq 2$,*

$$\mathrm{ex}_k(n, \mathcal{C}_4^{k+}) = (1 + o(1)) \frac{1}{k!}\, n^{k-0.5}.$$

The main theorem of this section establishes a direct connection between total mutual-visibility numbers of Johnson graphs and hypergraph Turán numbers $\mathrm{ex}_k(n, \mathcal{C}_4^{k+})$. For $k = 2$, the Johnson graph $J(n,2)$ is the line graph of the complete graph K_n. Therefore, Theorem 3.7 in [11] has already established the equality $\mu_{\mathrm{t}}(J(n,2)) = \mathrm{ex}(n, C_4)$. Here we use a different method to approach the general problem.

Theorem 6.2. *It holds for every two integers n and k satisfying $n \geq k+2$ and $k \geq 2$ that*

$$\mu_{\mathrm{t}}(J(n,k)) = \mathrm{ex}_k(n, \mathcal{C}_4^{k+})$$

Proof. Assume that $\mathcal{X}$ is a total mutual visibility set of maximum cardinality in $J(n,k)$. We first prove that the k-uniform underlying hypergraph $\mathcal{F}(\mathcal{X})$ is $\mathcal{C}_4^{k+}$-free. Suppose to the contrary that $\mathcal{C}_4^{k+}$ is a subhypergraph in $\mathcal{F}(\mathcal{X})$. Naming the vertices in this subhypergraph as in (6.1), let us consider the k-element sets $A_1 = Y \cup \{z_1, z_3\}$ and $A_2 = Y \cup \{z_2, z_4\}$. The corresponding vertices A_1 and A_2 are at distance $k - |Y| = 2$ in $J(n,k)$. Further, A_1 and A_2 have exactly four common neighbors in $J(n,k)$ and these neighbors correspond to the four edges in the subhypergraph $\mathcal{C}_4^{k+}$. By the supposition, the corresponding four vertices all belong to $\mathcal{X}$ and therefore, A_1 and A_2 are not $\mathcal{X}$-visible. This contradiction proves that $\mathcal{F}(\mathcal{X})$ is $\mathcal{C}_4^{k+}$-free and, since the size of $\mathcal{X}$ equals the number of edges in $\mathcal{F}(\mathcal{X})$, we have

$$\mu_{\mathrm{t}}(J(n,k)) = |\mathcal{X}| \leq \mathrm{ex}_k(n, \mathcal{C}_4^{k+}).$$

To prove the other direction, we suppose that $\mathcal{F}(\mathcal{X})$ is $\mathcal{C}_4^{k+}$-free and prove that every two vertices A and B are $\mathcal{X}$-visible in $J(n,k)$. We proceed by induction on the distance of

[2]Note that $\mathcal{C}_4^{k+}$ corresponds to the complete k-partite k-graph $\mathcal{K}^{(k)}_{1,\dots,1,2,2}$.

A and B. If $\mathrm{dist}(A,B)=1$, then the statement clearly holds. Suppose that $A\cap B=D$ holds for the corresponding k-sets and then $\mathrm{dist}(A,B)=k-|D|\geq 2$. Let us choose four different vertices $z_1,z_2\in A\setminus D$ and $w_1,w_2\in B\setminus D$. Since $\mathcal{F}(\mathcal{X})$ is $\mathcal{C}_4^{k+}$-free, there are two indices i and j such that $A'=A\setminus\{z_i\}\cup\{w_j\}$ is not an edge in $\mathcal{F}(\mathcal{X})$. Then, for the corresponding vertex, $A'\notin\mathcal{X}$. As $\mathrm{dist}(A',B)=\mathrm{dist}(A,B)-1$, we can now apply the hypothesis for A' and B. As there is a shortest path between A' and B such that all internal vertices belong to $\overline{\mathcal{X}}$ and A' itself is from $\overline{\mathcal{X}}$, we can construct a shortest path between A and B that contains no internal vertex from $\mathcal{X}$. This proves that A and B are $\mathcal{X}$-visible and we can conclude that $\mathcal{X}$ is a total mutual-visibility set if $\mathcal{F}(\mathcal{X})$ is $\mathcal{C}_4^{k+}$-free.

By choosing a k-uniform $\mathcal{C}_4^{k+}$-free hypergraph with maximum number of edges, the corresponding vertex set $\mathcal{X}$ will be a total mutual-visibility set in $J(n,k)$. This proves $\mu_{\mathrm{t}}(J(n,k))\geq \mathrm{ex}_k(n,\mathcal{C}_4^{k+})$ and finishes the proof of the theorem. □

By Theorems 6.1 and 6.2, we may conclude the following asymptotically tight result:

Corollary 6.3. *For every fixed $k\geq 2$,*

$$\mu_{\mathrm{t}}(J(n,k))=(1+o(1))\frac{1}{k!}\,n^{k-0.5}.$$

Finally, we turn to the problem of mutual-visibility in Johnson graphs. Here, we can give an exact formula for $k=2$ and $k=n-2$ (that was proved in [11] with a different approach), while establishing lower and upper bounds for the remaining cases.

Theorem 6.4. *Let n and k be integers.*

(i) *If $k\geq 2$ and $n\geq k+2$, then*

$$\mathrm{ex}_k(n,\mathcal{C}_4^{k+})\leq \mu(J(n,k))\leq \mathrm{ex}_k(n,\mathcal{K}_4^{k+}). \tag{6.3}$$

(ii) *If $n\geq 4$, then*

$$\mu(J(n,2))=\mu(J(n,n-2))=\left\lfloor\frac{n^2}{3}\right\rfloor. \tag{6.4}$$

Proof. By definitions of the parameters, $\mu(G)\geq\mu_{\mathrm{t}}(G)$ holds for every graph G. Theorem 6.2 then directly implies $\mathrm{ex}_k(n,\mathcal{C}_4^{k+})\leq\mu(J(n,k))$.

To show the upper bound in (6.3), we prove that the k-uniform underlying hypergraph $\mathcal{F}(\mathcal{X})$ is $\mathcal{K}_4^{k+}$-free whenever $\mathcal{X}$ is a mutual visibility set in $J(n,k)$. Indeed, let us suppose that $\mathcal{F}(\mathcal{X})$ contains a subhypergraph on the vertex set $Y\cup\{z_1,\dots,z_4\}$ that is isomorphic to $\mathcal{K}_4^{k+}$. We name the vertices according to (6.2). Then, by the definition of underlying hypergraph $\mathcal{F}(\mathcal{X})$, the vertices of $J(n,k)$ that correspond to the k-sets $Y\cup\{z_1,z_2\}$ and $Y\cup\{z_3,z_4\}$ belong to $\mathcal{X}$. Their distance equals 2 and all the four common neighbors are from $\mathcal{X}$ as well. Thus, the two vertices $Y\cup\{z_1,z_2\}$ and $Y\cup\{z_3,z_4\}$ from $\mathcal{X}$ are not $\mathcal{X}$-visible, and $\mathcal{X}$ is not a mutual visibility set in $J(n,k)$. We may conclude that $\mathcal{F}(\mathcal{X})$ is $\mathcal{K}_4^{k+}$-free for every mutual visibility set $\mathcal{X}$ of $J(n,k)$ and consequently, $\mu(J(n,k))\leq \mathrm{ex}_k(n,\mathcal{K}_4^{k+})$ is valid.

To prove (6.4), we first recall the well-known facts that $\mathrm{ex}(n,K_4)=\lfloor n^2/3\rfloor$ [27] and that $J(n,2)\cong J(n,n-2)$ holds for every $n\geq 4$. It is enough then to prove the equality $\mu(J(n,2))=\mathrm{ex}(n,K_4)$. As $\mathcal{K}_4^{2+}$ is the graph K_4, part (i) provides the upper bound $\mu(J(n,2))\leq\mathrm{ex}(n,K_4)$.

Consider a set $\mathcal{X} \subseteq V(J(n,2))$ so that the 2-uniform underlying hypergraph $\mathcal{F}(\mathcal{X})$ is K_4-free. For every two vertices A and B from $\mathcal{X}$, they are either adjacent and $\mathcal{X}$-visible or $\mathrm{dist}(A,B) = 2$. In the latter case, the corresponding 2-element sets are disjoint, say $A = \{a_1, a_2\}$ and $B = \{b_1, b_2\}$. Since $\mathcal{F}(\mathcal{X})$ is K_4-free, there is a 2-element set $\{a_i, b_j\}$ with $i, j \in [2]$ that is not an edge in $\mathcal{F}(\mathcal{X})$. Thus $C = \{a_i, b_j\}$ is in $\overline{\mathcal{X}}$ and A, B are $\mathcal{X}$-visible via the path ACB. Consequently, a mutual-visibility set $\mathcal{X}$ can be chosen in $J(n,2)$ such that $\mathcal{F}(\mathcal{X})$ is an extremal K_4-free graph having $\mathrm{ex}(n, K_4)$ edges. We conclude $\mathrm{ex}(n, K_4) \leq \mu(J(n,2))$, which finishes the proof of (6.4). □

7 Conclusion

In this paper, we determined the total mutual-visibility numbers for all Kneser graphs, bipartite Kneser graphs, and Johnson graphs. Our formulas gave easily computable values for some ranges, but we had to include invariants from extremal combinatorics in other cases. Computing these invariants (namely hypergraph Turán numbers of suspensions and covering numbers from design theory) is considered an extremely challenging problem.

We also studied the mutual-visibility numbers over these three graph classes and obtained exact results for all Kneser graphs $KG(n,k)$ with $n \geq 7k-5$ and $k \geq 2$ (see Theorem 4.1). For the remaining cases, we gave estimations (see Propositions 4.2, 5.3, and Theorem 6.4), the exact determination is posed here as an open problem.

Problem 7.1. Determine the mutual visibility number of the Kneser graph $KG(n,k)$ if $3 \leq k$ and $2k+1 \leq n \leq 7k-6$.

Problem 7.2. Determine the mutual visibility number of the bipartite Kneser graph $H(n,k)$ if $k \geq 2$ and $n \geq 2k+1$.

Problem 7.3. Determine the mutual visibility number of the Johnson graph $H(n,k)$ if $k \geq 3$ and $n \geq k+2$.

For every graph G, the relation $\mu(G) \geq \mu_{\mathrm{t}}(G)$ holds by definition. If G satisfies $\mu(G) = \mu_{\mathrm{t}}(G)$, we call it a *$(\mu, \mu_{\mathrm{t}})$-graph*. Our Theorems 3.2 and 4.1 imply that all Kneser graphs $KG(n,k)$ with $k \geq 2$ and $n \geq 7k-5$ are (μ, μ_{t})-graphs. Further, by the same theorems, $K(8,2)$ is also a (μ, μ_{t})-graph.

Cicerone *et al.* [8] introduced further parameters on mutual-visibility. Given a graph G and a vertex set $\mathcal{X}$, we say that $\mathcal{X}$ is a *dual mutual-visibility set*, if every two vertices from $\mathcal{X}$ and every two vertices from $\overline{\mathcal{X}}$ are $\mathcal{X}$-visible. The maximum size of a dual mutual-visibility set in G is the *dual mutual-visibility number* $\mu_{\mathrm{d}}(G)$. Similarly, if $\mathcal{X}$-visibility is required for every two vertices u, v when at least one of them is from $\mathcal{X}$, then $\mathcal{X}$ is an *outer mutual-visibility set* and the maximum size of such a set is denoted by $\mu_{\mathrm{o}}(G)$. It holds by definition that

$$\mu_{\mathrm{t}}(G) \leq \mu_{\mathrm{d}}(G) \leq \mu(G) \qquad \text{and} \qquad \mu_{\mathrm{t}}(G) \leq \mu_{\mathrm{o}}(G) \leq \mu(G). \tag{7.1}$$

It follows that the four parameters are equal for each (μ, μ_{t})-graph.

As a consequence of (7.1), Theorems 3.2 and 4.1, it is easy to determine the four mutual-visibility parameters for Kneser graphs $KG(n,2)$ with $n \geq 8$. By doing so, we answer a question of Cicerone *et al.* posed in [11].

Proposition 7.4. *It holds for every integer n with $n \geq 8$ that*

$$\mu(KG(n,2)) = \mu_{\mathrm{d}}(KG(n,2)) = \mu_{\mathrm{o}}(KG(n,2)) = \mu_{\mathrm{t}}(KG(n,2)) = \binom{n}{2} - 4.$$

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How far are ternary words from shuffle squares?*

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Abstract

A shuffle square is a word which consists of two identical and disjoint, but possibly intertwining subwords. For every natural number n, we construct a ternary word of length n which requires a removal of $\Omega(\sqrt{n})$ of its letters in order to become a shuffle square.

Keywords: Words over finite alphabet, twins in words, shuffle squares.

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1 Introduction

Given a finite set A, called the *alphabet*, and a positive integer $n \geq 1$, a *word of length n over A* is any sequence $s \in A^n$. We will be denoting the length of s by $|s|$. Given a word $s = s_1, \dots, s_n$, by a *subword* of s we mean any subsequence $s' = s_{i_1}, \dots, s_{i_t}$, where $1 \leq i_1 < \cdots < i_t \leq n$. The set of positions $\{i_1, \dots, i_t\}$ is called the *support* of s' and is denoted by $supp(s')$. If the support consists of consecutive integers, then we call such a subword a *block* [1]. We write $s = s's''$ and say that s is the *concatenation* of s' and s''

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[1]This terminology may differ from that used in previous papers. For instance, in [1] blocks are called factors, while in [3] blocks are called subwords and subwords – subsequences.

if s' and s'' are blocks of s with $supp(s') = [1, \dots, i-1]$ and $supp(s'') = [i, \dots, n]$ for some $i \geq 1$ (for $i = 1$, we assume that $s' = \emptyset$). This definition can be naturally iterated to concatenations of more than two words.

One of the most interesting features of ternary words concerns the presence of squares. A word s is a *square* if it is a concatenation of two identical words s' and s''. For instance, $s = 101100\ 101100$ is a square. Trivially, every binary word of length at least 4 contains (as a block) one of the squares 11, 00, 1010, or 0101. It is therefore quite surprising that, as proved by Thue [9] more than a hundred years ago, there exist ternary words of every length which are square-free (a word is square-free if none of its blocks is a square).

In this note, however, we are interested in a more relaxed notion. A word s is a *shuffle square* if it contains two identical subwords s' and s'' such that $supp(s')$ and $supp(s'')$ form a partition of $supp(s)$. For instance, the binary word $s = \underline{1}\ \underline{0}\ \overline{1}\ \overline{0}\ \underline{1}\ \underline{1}\ \overline{1}\ \underline{0}\ \underline{0}\ \overline{1}\ \overline{0}\ \overline{0}$ is a shuffle square, where $s' = s'' = 101100$ with supports $supp(s') = \{1, 2, 5, 6, 8, 9\}$ and $supp(s'') = \{3, 4, 7, 10, 11, 12\}$ (underlined and overlined, resp.). Shuffle squares have attracted some attention, see [8, Section 3.2] for a survey focusing on the avoidance (Thue-type) problems and their computational complexity. Here we consider a somewhat different problem.

Clearly, in shuffle squares every letter must be present an even number of times, though this is by far not sufficient. For instance, $s = 1001$ is not a shuffle square. A question emerges how close a given word is from a shuffle square and what is the worst case. One way to quantify this question is in terms of largest twins. *Twins* in a word s are any two disjoint, identical subwords of s. The length of twins is defined as the length of either of them. For instance, in $s = 01220212$ there are twins of length 3, (both) equal to 012, or 022, but there are no twins of length 4 which is equivalent to s not being a shuffle square.

Given a word s, let $f(s)$ be the length of the longest twins in s, e.g., $f(01220212) = 3$, as demonstrated above. To measure the distance of a word s of length n from a shuffle square we now introduce a new parameter $g(s) = n - 2f(s)$. Thus, $g(s)$ is the smallest number of elements to be removed from s in order to turn it into a shuffle square. E.g., $g(01220212) = 8 - 6 = 2$.

Since we are interested in the worst case, for $n \geq 1$ and $k \geq 2$ we define $g(n, k)$ as the maximum of $g(s)$ over all words s of length n over a k-element alphabet. Note that $g(n, k)$ is increasing as a function of k and, trivially, $g(n, k) \geq 1$, as there is always a word with an odd number of occurrences of a letter.

This parameter, or its equivalent version $f(n, k) = \frac{1}{2}(n - g(n, k))$, was studied in [1, 3, 4, 5]. It was shown by Axenovich, Person, and Puzynina in [1] that $g(n, 2) = o(n)$, more specifically, $g(n, 2) = O\left(n((\log\log n)^2/\log n)^{1/3}\right)$, and, on the other hand, a construction was given showing that $g(n, 2) = \Omega(\log n)$. Recently, it turned out that Boris Bukh in an unpublished note from 2013 ([2]) modified their construction and indicated how to get a better bound.

Theorem 1.1 ([2]). *We have* $g(n, 2) = \Omega(n^{1/3})$.

For completeness we provide a proof of Theorem 1.1 in Section 3.

For $k \geq 4$, Bukh and Zhou in [3, Theorems 2 and 4] proved bounds $c_k n \leq g(n, k) \leq d_k n$ for some constants $0 < c_k < d_k < 1$, in particular, $0.014n \leq g(n, 4) \leq 0.493n$ (all linear (in n) lower bound are non-constructive).

For ternary sequences ($k = 3$), we only know that $g(n,3) \leq 0.32n$ ([3]) and, as a consequence of Theorem 1.1, that $g(n,3) \geq g(n,2) = \Omega(n^{1/3})$. It remains wide open if $g(n,3) = o(n)$. In this paper we improve the lower bound.

Theorem 1.2. *We have* $g(n,3) = \Omega(\sqrt{n})$.

The main body of this note is devoted to proving Theorems 1.1 and 1.2. In the proofs we do not try to optimize the constants hidden in the $\Omega(\cdot)$ terms. In the last section we discuss various versions of the problem and state open problems.

2 Constructions

Our binary construction needed in the proof of Theorem 1.1 is inspired by the one used in [1], but based on the one used in [2]. The new idea behind Theorem 1.2 is to concatenate the binary words constructed in [2] and mix them with blocks of the third letter in an alternating fashion. Throughout we assume that our binary alphabet is $\{0,1\}$ and the ternary one is $\{0,1,2\}$.

Both constructions consist of two steps: first we construct words whose length is an integer n of a special form, then we extend the construction to all values of n. By z^m we mean the word of length m with all elements equal to letter z. For instance, $0^4 = 0000$. For two subwords s', s'' of s by $s' \cap s''$ we mean the subword of s with support $supp(s' \cap s'') = supp(s') \cap supp(s'')$.

2.1 Binary words

Fix an *even* integer $j \geq 2$ and define binary words

$$s_2 = 1^5 0^3 \quad \text{and} \quad s_j = 1^{j^2+1} 0^{(j-1)j+1} \cdots 0^{j+1} \quad \text{for} \quad j \geq 4.$$

For instance, $s_4 = 1^{17}0^{13}1^9 0^5$ and $s_6 = 1^{37}0^{31}1^{25}0^{19}1^{13}0^7$. Note that for all $j \geq 2$

$$\ell_j := |s_j| = \sum_{i=1}^{j}(ij+1) = j\binom{j+1}{2} + j. \tag{2.1}$$

Given $n \geq 1$, let $j := j_n$ be the unique even integer such that

$$\ell_{j-2} < n \leq \ell_j$$

(here we assume that $\ell_0 = 0$) and let u_n be the suffix of s_j of length n.

It can be easily checked that for $j \geq 3$

$$\ell_j - n \leq \ell_j - \ell_{j-2} - 1 = 3j^2 - 4j + 3 < [j^2+1] + [(j-1)j+1] + [(j-2)j+1].$$

Thus, for $n > \ell_2 = 8$, i.e., when $j := j_n \geq 4$, the word u_n contains the entire suffix $0^{(j-3)j+1} \cdots 0^{j+1}$ of s_j preceded by a suffix of $1^{j^2+1}0^{(j-1)j+1}1^{(j-2)j+1}$ of a suitable length. (For $n \leq 8$, u_n is just a suffix of s_2.) For instance, as $\ell_4 = 44$ and $\ell_6 = 132$, we have $u_5 = 11000$, $u_9 = 1^4 0^5$, $u_{20} = 0^6 1^9 0^5$, $u_{32} = 1^5 0^{13} 1^9 0^5$, $u_{45} = 1^6 0^{19} 1^{13} 0^7$, and $u_{100} = 1^5 0^{31} 1^{25} 0^{19} 1^{13} 0^7$.

2.2 Ternary words

Assume that k is even, set $k' = 2\lfloor k/4 \rfloor$, and for each $k \geq 4$ define a ternary word

$$t_k = 2^{\ell_k} s_k \, 2^{\ell_{k-2}} s_{k-2} \cdots 2^{\ell_{k'}} s_{k'},$$

where the word s_k and its length ℓ_k have been defined in the previous subsection. For instance, as $\ell_2 = 8$, $\ell_4 = 44$, $\ell_6 = 132$, and $\ell_8 = 296$, we have

$$t_4 = 2^{44} s_4 2^8 s_2, \quad t_6 = 2^{132} s_6 2^{44} s_4 2^8 s_2 = 2^{132} s_6 t_4, \quad \text{and} \quad t_8 = 2^{296} s_8 2^{132} s_6 2^{44} s_4.$$

(Note that $t_8 \neq 2^{296} s_8 t_6$, as for $k = 8$, we have $k' = 4$.)

By (2.1), setting $\tau_k = |t_k|$ for $k \geq 4$,

$$\begin{aligned} \tau_k &= 2 \sum_{j=k'/2}^{k/2} \ell_{2j} = \sum_{j=k'/2}^{k/2} \big((2j)^2(2j+1) + 4j\big) \\ &= \sum_{j=k'/2}^{k/2} (8j^3 + 4j^2 + 4j) < \frac{k}{2}\big(k^3 + k^2 + 2k\big) < k^4. \end{aligned} \tag{2.2}$$

Given $n \geq 1$, let $k := k_n$ be the unique even integer such that

$$\tau_{k-2} < n \leq \tau_k$$

(here we assume that $\tau_2 = 0$) and let v_n be the suffix of t_k of length n. Note that for $k \geq 6$

$$\tau_k - n < \tau_k - \tau_{k-2} = 2 \sum_{j=\lfloor k/4 \rfloor}^{k/2} \ell_{2j} - 2 \sum_{j=\lfloor (k-2)/4 \rfloor}^{k/2-1} \ell_{2j} \leq 2\ell_k,$$

so, for $n > \tau_4 = 104$, the word v_n contains the entire suffix $2^{\ell_{k-2}} s_{k-2} \cdots 2^{\ell_{k'}} s_{k'}$ of t_k preceded by a suffix of $2^{\ell_k} s_k$ of a suitable length. (For $n \leq 104$, v_n is just a suffix of t_4.) For instance, as $\tau_4 = 104$, $\tau_6 = 368$, and $\tau_8 = 944$, while, recall, $s_2 = 1^5 0^3$, $s_4 = 1^{17} 0^{13} 1^9 0^5$, $s_6 = 1^{37} 0^{31} 1^{25} 0^{19} 1^{13} 0^7$, we have $v_5 = 11000$, $v_{13} = 2^5 s_2$, $v_{17} = 02^8 s_2$, $v_{80} = 2^{20} s_4 2^8 s_2$, $v_{111} = 0^7 2^{44} s_4 2^8 s_2$, $v_{234} = 1^{35} 0^{31} 1^{25} 0^{19} 1^{13} 0^7 2^{44} s_4 2^8 s_2$, and $v_{900} = 2^{252} s_8 2^{132} s_6 2^{44} s_4$.

3 Proofs of Theorems 1.1 and 1.2

In both proofs we employ a special type of twins. Given a word $w = (w_1 \cdots w_n)$, twins $x = (w_{i_1} \cdots w_{i_t})$ and $y = (w_{j_1} \cdots w_{j_m})$ are *monotone* if for all $k = 1, \dots, t$ we have $i_k < j_k$. Thus, for monotone twins their order matters and we will designate them as an ordered pair (x, y). The crucial, though trivial, observation is that if there are twins of length m in a word w, then there are also monotone twins of length m (it suffices to swap elements between the twins which are in the wrong order). For instance, twins (x, y) shown here (x - underlined, y - overlined):

$$\underline{1}\ \underline{0}\ 1\ \overline{1}\ \overline{0}\ \overline{1}\ 1\ \underline{1}\ \overline{0}\ \underline{0}$$

are not monotone as the third element of x is to the right of the third element of y and the same happens with the fourth elements. However, by swapping these pairs, one can easily "correct" the twins to become monotone:

$$\underline{1}\ \underline{0}\ 1\ \overline{1}\ \overline{0}\ \underline{1}\ 1\ \overline{1}\ \underline{0}\ \overline{0}.$$

3.1 Proof of Theorem 1.1

We first prove a pivotal statement about a richer class of binary words than just u_n's. It generalizes the construction from [2] and its proof employs the idea from there. For positive integers $\lambda_1, \dots, \lambda_b$, set

$$w := w(\lambda_b, \dots, \lambda_1) = 1^{\lambda_b}\ 0^{\lambda_{b-1}}, \cdots,\ (*)^{\lambda_1}$$

where $* = 1$ if b is odd and $* = 0$ otherwise. Thus w consists of alternating runs of ones and zeros of lengths $\lambda_b, \dots, \lambda_1$.

Lemma 3.1. *Fix integers $r \geq 0$ and $b \geq 1$, and let positive integers $\lambda_1, \dots, \lambda_{b-1}$ be odd and such that $\lambda_{b-1} > \lambda_{b-2} > \cdots > \lambda_1$. Further, let $\delta = \min_{2 \leq i \leq b-1}(\lambda_i - \lambda_{i-1})$. Then*

$$g(w(r, \lambda_{b-1}, \dots, \lambda_1)) \geq \min\{b-1, \delta\}.$$

Proof. Note that w can be represented as a concatenation of maximal unary blocks

$$w = w_b w_{b-1} \cdots w_1 = 1^r\ 0^{\lambda_{b-1}} \cdots (*)^{\lambda_1}.$$

Consider monotone twins (x, y) in w. If, for all i, $y \cap w_i \neq \emptyset$, then it follows that, for all i, $|x \cap w_i| = |y \cap w_i|$. Indeed, as y consists of b runs, both x and y must intersect every unary block (and, being twins, in equal quantities). Thus, owing to the oddity of the λ_i's for all $i = 1, \dots, b-1$, we get $g(w) \geq b-1$. Otherwise, let $j = \min\{i : y_i \cap w_i = \emptyset\}$. Note that if $j = b$, then also $x \cap w_b = \emptyset$ and consequently $|x \cap w_i| = |y \cap w_i| \neq 0$ for all $i = 1, \dots, b-1$ which leads to the same conclusion.

Assume thus that $j \leq b-1$ and consider two subcases. If $|x \cap w_j| \leq \lambda_j - \delta$, then $g(w) \geq \delta$ and we are done. If, on the other hand, $|x \cap w_j| \geq \lambda_j - \delta$, then y should make up this deficit of at least $\lambda_j - \delta$ elements of w_j from a *single* block w_i, for some $i \leq j-2$ (here we use the definition of j). But $|w_i| = \lambda_i \leq \lambda_{j-2} \leq \lambda_j - 2\delta$, a contradiction. □

The following immediate corollary of Lemma 3.1 will be needed in the proof of Theorem 1.2. Recall that the word s_j have been defined in Section 2.

Corollary 3.2. *For every even $j \geq 2$ we have $g(s_j) \geq j$.*

Proof. Apply Lemma 3.1 to $w(0, j^2+1, (j-1)j, \dots, j+1, 0)) = s_j$ with $r = 0, b = j+1$ and $\delta = j$. This is possible, because $\lambda_i = ij + 1$, $i = 1, \dots, b-1$, are odd and form an increasing sequence. □

Proof of Theorem 1.1. For every n, consider the word u_n constructed in Subsection 2.1. No matter how large $\ell_j - n$ is, u_n can be written as $w(\lambda_{b'}, \dots, \lambda_1)$ where $j - 2 \leq b' \leq j$ and $\lambda_i = ij + 1$ for $i = 1, \dots, b' - 1$, while $\lambda_{b'}$ is some integer. Then, observing that all λ_i's are odd and form an increasing sequence, by Lemma 3.1 with $r = \lambda_{b'}$, we have $g(u_n) \geq \min\{b' - 1, j\} \geq j - 3$. As $n \leq \ell_j \leq j^3$ for $j \geq 2$, this yields $g(n, 2) \geq n^{1/3} - 3$. □

3.2 Proof of Theorem 1.2

For every n, consider the word v_n constructed in Subsection 2.2. As explained therein, v_n looks like t_k with some prefix of length $\tau_k - n < 2\ell_k$ removed. Thus,

$$v_n = \begin{cases} 2^r s_k\ 2^{\ell_{k-2}} s_{k-2} \cdots 2^{\ell_{k'}} s_{k'} & \text{if} \quad \tau_k - n < \ell_k \\ s'_k\ 2^{\ell_{k-2}} s_{k-2} \cdots 2^{\ell_{k'}} s_{k'} & \text{if} \quad \tau_k - n \geq \ell_k, \end{cases}$$

where $r = \ell_k - \tau_k + n > 0$ and s'_k is a suitable suffix of s_k. To unify notation, for each even $k' \leq j \leq k-2$, let

$$s^*_j = \begin{cases} 2^{\ell_j} & \text{if} \quad * = (2) \\ s_j & \text{if} \quad * = (0,1) \end{cases} \qquad \text{and} \qquad s^*_k = \begin{cases} 2^r & \text{if} \quad * = (2) \text{ and } \tau_k - n < \ell_k \\ \emptyset & \text{if} \quad * = (2) \text{ and } \tau_k - n \geq \ell_k \\ s'_k & \text{if} \quad * = (0,1). \end{cases}$$

Note that, for each n, if $s^{(2)}_k \neq \emptyset$, then $s^{(0,1)}_k = s_k$. In this notation the word v_n can be written as

$$v_n = s^{(2)}_k \, s^{(0,1)}_k \, s^{(2)}_{k-2} \, s^{(0,1)}_{k-2} \cdots s^{(2)}_{k'} \, s^{(0,1)}_{k'}.$$

Let (x, y) be monotone twins in v_n. We are going to show that $g(v_n) \geq k^2/8$.

Case I: Assume that $y \cap s^*_j \neq \emptyset$ for all blocks s^*_j. Then x and y must take equally from each s^*_j and, consequently, within each $s^{(0,1)}_j$ they must form twins. Hence, by Corollary 3.2, recalling that $k \geq 4$,

$$g(v_n) \geq \sum_{i=k'/2}^{k/2-1} g(s_{2i}) \geq \sum_{i=k'/2}^{k/2-1} 2i = \frac{k}{2}\left(\frac{k}{2} - 1\right) - \frac{k'}{2}\left(\frac{k'}{2} - 1\right) \geq \frac{k^2}{4} - \frac{k}{2} - \frac{k^2}{16} + \frac{k}{4} > \frac{k^2}{8}.$$

Case II: Let the rightmost block skipped by twin y be s^*_j. This means that $y \cap s^*_j = \emptyset$, while $y \cap s^*_i \neq \emptyset$ for all $i < j$ and both values of $*$. In addition, if the rightmost block skipped by y is $s^{(2)}_j$, then also $y \cap s^{(0,1)}_j \neq \emptyset$.

Observe that if $\tau_k - n < \ell_k$ and $s^*_j = s^{(2)}_k$, then x has to skip $s^{(2)}_k$ as well and we are back in the previous case. A similar situation occurs when $\tau_k - n \geq \ell_k$ and $s^*_j = s^{(0,1)}_k$. Otherwise, assume that $j \leq k-1$ and that twin x leaves out less than $k^2/8$ letters of s^*_j, that is, $|x \cap s^*_j| > \ell_k - k^2/8$ (as otherwise we are done). Then y has to make this deficit up from a *single* block s^*_i for some $i \leq j-2$ and with the same value of $*$. (This is because, by the definition of s^*_j, y cannot skip any further block.) In particular, it implies that $j \geq k' + 2$. However, the deficit to be made up is greater than $\ell_j - k^2/8 > \ell_{j-2} > \ell_i$ for all $i < j-2$, as

$$\ell_j - \ell_{j-2} = 3j^2 - 4j + 4 \geq 2j^2 > k^2/2 > k^2/8,$$

which is a contradiction. Hence, we proved that $g(v_n) \geq k^2/8$.

Recalling (cf. (2.2)) that $n < \tau_k < k^4$, we conclude that

$$g(v_n) \geq \frac{k^2}{8} > \frac{\sqrt{n}}{8}. \qquad \square$$

4 Concluding remarks

4.1 Multiple twins

It is tempting to generalize the notions and results discussed earlier to r-tuplets. For $r \geq 2$, an *r-tuplet* in a word s is a set of r disjoint and identical subwords of s. The length of an r-tuplet is measured by the length of just one of the r subwords forming it. Of course, 2-tuplets are just twins defined earlier, while 3-tuplets are traditionally called *triplets*.

Following [8], we define a *shuffle r-power* as any word s which contains r disjoint subwords whose supports form a partition of $supp(s)$ (so, these subwords form an r-tuplet in s). For instance, $\underline{1}\ \underline{0}\ \bar{1}\ \underline{0}\ \bar{0}\ \hat{1}\ \bar{0}\ \underline{1}\ \hat{0}\ \bar{1}\ \hat{0}\ \hat{1}$ is a shuffle 3-power, as it consists of three copies of 1001 as indicated.

Let $f_r(s)$ be the length of the longest r-tuplet in s and $g_r(s) = n - rf_r(s)$. Finally, let $f_r(n,k)$ be the minimum of $f_r(s)$ and $g_r(n,k)$ – the maximum of $g_r(s)$ over all words s of length n over a k-element alphabet. So, $g_r(n,k) = n - rf_r(n,k)$ measures the distance of such words from being shuffle r-powers in the worst case. Note that, again, it is an increasing function of k (as a function of r it is not monotone, though $f_r(n,k)$ is).

The function $f_r(n,k)$ was studied in [1]. It was proved there (again, we quote those results in terms of $g_r(n,k)$) that, for $2 \le k \le r$, $g_r(n,k) = o(n)$, while for all $k, r \ge 2$, a construction yielded $g_r(n,k) = \Omega(\log n)$. By amending that construction in a similar way as in Theorem 1.2, we can improve the logarithmic lower bound to $g_r(n,k) \ge g_r(n,3) = \Omega(\log^2 n)$. Moreover, for $k > r$ a general sufficient condition for linearity in n was given. In particular, it follows that $g_3(n,6) \ge 0.019n$ and $g_4(n,7) \ge 0.012n$, both bounds non-constructive (see next subsection for more on that).

It seems not unlikely that the more refined (random) methods from [3], used to establish the lower bound on $g_2(n,4)$, can be extended to yield $g_r(n,k) = \Theta(n)$ for some pairs of k and r not covered by the linear bound in [1]. In particular, we suspect that $g_3(n,5) = \Theta(n)$.

4.2 Random words

So far we have been interested in the worst case parameters $g(n,k)$ and, more generally, $g_r(n,k)$. Let us now consider the average case, that is, $g(R_k(n))$, where $R_k(n)$ is a *random* word drawn uniformly at random from all k^n words of length n over a k-element alphabet. Equivalently, $R_k(n)$ is an outcome of n independent tosses of a fair k-sides die. It was proved in [4] that, with probability tending to 1 as $n \to \infty$ (or a.a.s., for short), we have $g(R_3(n)) \le 0.178n$, much better than the worst case bound $g(n,3) \le 0.32n$ from [3] quoted in the introduction.

As mentioned earlier, all linear lower bounds on $g(n,k)$ from [1] and [3] were obtained via the probabilistic method, that is, by showing that $R_k(n)$ a.a.s satisfies them. In fact, in both papers the expected number $\mu_2(m)$ of twins of length $m = \alpha n$ was estimated. In [1] it was a simple estimate easily generalized to r-tuplets:

$$\mu_r(m) \le \frac{n! k^{-(r-1)m}}{m!^r (n-rm)!} \le \frac{n^n k^{-(r-1)m}}{m^{rm}(n-rm)^{n-rm}} = \left(\frac{(1-r\alpha)^{1-r\alpha}}{k^{(r-1)\alpha}\alpha^{r\alpha}} \right)^n \to 0$$

if the expression under the n-th power is a constant strictly smaller than 1. Since, obviously, $m \le n/r$, as a heuristic, we may try α's which are smaller than but very close to $1/r$. Then the quantity in the numerator becomes very close to 1 and can be ignored. It follows that a sufficient condition for the whole fraction to be less than one (with a suitably chosen α) is that $k^{(r-1)/r} > r$. For small values of r and k this holds whenever $k \ge r+3$. With a little help of a calculator, one can check that, in particular, the triples $(r = 2, k = 5, \alpha = 0.49)$, $(r = 3, k = 6, \alpha = 0.327)$ as well as $(r = 4, k = 7, \alpha = 0.247)$ all do the job, yielding a.a.s. respective linear lower bounds on $g(R_5(n))$, $g_3(R_6(n))$, and $g_4(R_7(n))$, and consequently worst-case lower bounds, that is, bounds on $g(n,5)$, $g_3(n,6)$, and $g_4(n,7)$.

In [3] the estimate was much more refined giving the improvement $f(R_5(n)) \leq 0.48n$, but also a new result $f(R_4(n)) \leq 0.493n$, both holding a.a.s. Unfortunately, due to its complexity, this argument does not seem to generalize to r-tuplets easily. It is realistic, however, to hope that with some additional effort it could yield $g_3(R_5(n)) = \Theta(n)$ a.a.s.

In the random context, even the binary case is still interesting. As was proved in [1] via the regularity lemma for words, $g(n, 2) = O(n((\log\log n)^2/\log n)^{1/3})$, that is, *all* binary words of length n are this close to shuffle squares. It was observed in [4] that, as a consequence of Chernoff's bound, a.a.s. $g(R_2(n)) = O((\log n)^{1/2}n^{2/3})$. In [5], He, Huang, Nam, and Thaper, cleverly analysing a simple greedy algorithm, improved that bound further by showing that a.a.s $g(R_2(n))/\sqrt{n} \to \infty$ arbitrarily slowly.

The same authors also studied the probability that $R_2(n)$, n even, *is* a shuffle square and showed that it is at least $\binom{n}{n/2}2^{-n} = \Theta(n^{-1/2})$. Of course, this is equivalent to estimating the number of shuffle squares among all such words from below by $\binom{n}{n/2}$. Clearly, this probability has to be at most $1/2 + o(1)$, as about half of all binary words have an odd number of 0's. Conjecture 4.3 below, if true, would imply that, in fact, it is asymptotic to $1/2$.

4.3 Open problems and conjectures

We believe that the lower bounds on $g(n, 2)$ from Theorem 1.1 and on $g(n, 3)$ from Theorem 1.2 can be improved to, respectively, $\Omega(\sqrt{n})$ and $\Omega(n^{2/3})$. A natural candidate for a construction yielding the former improvement seems to be the word $w = w(\lambda_1, \lambda_2, \dots, \lambda_k)$ where the λ_i's are consecutive *odd* numbers. Once it is established that $g(w) = \Omega(k)$, the latter bound will follow by mixing w with blocks of twos in a similar manner as in the proof of Theorem 1.2.

Below, we list several other questions about the deterministic case.

Problem 4.1. Recall that $g_r(n, k)$ equals the smallest number of elements to be removed in the worst case from a word of length n over a k-element alphabet to turn it into a shuffle r-power. For $r = 2$, the subscript r is dropped.

- Decide if $g(n, 3) = o(n)$.
- Determine the order of magnitude of $g(n, 2)$ and $g(n, 3)$.
- All of the above for $g_r(n, 2)$ and $g_r(n, 3)$, where $r \geq 3$.
- Estimate $g_3(n, 4)$.
- Decide if $g_3(n, 5) = \Theta(n)$.

Next, let us mention a problem and a conjecture in the random context.

Problem 4.2. Does $g(R_k(n)) = g(n, k)(1 + o(1))$, as $n \to \infty$, holds a.a.s. for all $k \geq 2$?

An analog of Problem 4.2 comparing $f(R_k(n))$ with $f(n, k)$ was stated in [4]. It is true for $k = 2$ by the result in [1].

Conjecture 4.3 ([5])**.** *The random binary word $R_2(n)$, n even, conditioned on having an even number of 0's (and so an even number of 1's), is a.a.s. a shuffle square.*

As was observed in [5], if true, this conjecture would imply that, unconditionally, the probability that $R_2(n)$, n even, is a shuffle square converges to $1/2$. Also, it would follow that a.a.s. $g(R_2(n)) \leq 2$, regardless of the parity of n. Indeed, for n even, just flip the last letter to adjust the parity if necessary, while for n odd just drop the last letter to get into the even case.

Returning to the worst case analysis, one may replace the parameter $g(s)$ by $h(s)$, the smallest number of insertions of new elements into a word s to obtain a shuffle square. For instance, $h(1001) = 2$, as 110011 is a shuffle square. More generally, one may define $\ell(s)$ as the smallest Levenshtein (edit) distance of a word s from any shuffle square. The Levenshtein distance between two words is the minimum number of single-character edits (insertions, deletions, or substitutions) required to change one word into the other (see [7]).

Problem 4.4. Estimate $h(n,k) = \max h(s)$ and $\ell(n,k) = \max \ell(s)$, where the maxima are taken over all words s of length n over a k-element alphabet.

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Coverings of general digraphs

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Abstract

A unified theory of covering projections of graphs and digraphs is presented as one theory by considering coverings of general digraphs, where multiple directed and undirected edges together with oriented and unoriented loops and semiedges, are allowed. It transpires that coverings of general digraphs can display certain pathological behaviour since the naturally defined projections of their underlying graphs may not be coverings in the usual topological sense. Consequently, homotopy does not always lift, although the unique walk lifting property still holds. Yet, it is still possible to grasp such coverings algebraically in terms of the action of the fundamental monoid. This action is permutational and has certain nice properties that monoid actions in general do not have. As a consequence, such projections can be studied combinatorially in terms of voltages. The problem of isomorphism and equivalence, and in particular, the problem of lifting automorphisms, is treated in depth. All known results about covering projections of graphs are simple corollaries of just three general theorems.

Keywords: Mixed graph, general digraph, graphoid, dart, covering projection, voltage, homotopy, monoid action, lifting automorphisms.

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1 Introduction

In graph theory, covering space techniques became a prominent object of study in the early '70 with the work of Alpert and Gross [19], who realized that the methods behind the final solution of the Heawood Map Colour Problem by Ringel and Youngs in 1968, see [48], were actually covering projections of graph embeddings in closed surfaces. Soon after, Gross [18] introduced a combinatorial description of regular coverings of graphs in terms of the so called voltages. These ideas have been further developed by Ezzel [14], and Gross and Tucker, see [20] for an extensive list of relevant references.

An early topic regarding graph coverings was studying the symmetries of covering graphs in terms of symmetries of the base graphs. This is the problem of lifting automorphisms, an exceptional instance of a more general topological problem of lifting continuous maps $Y \to X$ into a base space to a continuous map $Y \to \tilde{X}$ into the covering space, which sprang in topology out of the necessity of studying multi-valued functions. However, it appears that the specific problem of lifting automorphisms became a more lively area of research in the combinatorial context of graph covers, where the primary incentive was the construction and characterization of graphs with a high degree of symmetry. Two seminal results in this direction appeared in a paper of Djoković [10] and in Biggs' monograph [4].

Further development along these lines took place in the second part of '80 with the work of Hoffmann [21], Biggs [5], and Škoviera [52]. A more general concept of a graph bundle, introduced by Pisanski and Vrabec [42], was later considered by Kwak and Lee [29].

In the '90, a number of papers by different authors appeared, dealing both with abstract graphs as well as with graph embeddings in surfaces: Hofmeister [22, 23, 24], Malnič and Marušič [32], Archdeacon, Gvozdjak, Richter and Širáň [2, 3], Nedela and Škoviera [41], and Du, Marušič and Waller [12]. The first systematic treatment of lifting automorphisms along graph covering projections was given by Malnič, Nedela and Škoviera [36]. In the second part of '90, investigations related to graph covers developed in other directions as well. For instance, Seifter and Trofimov [49] studied covering projections in connection with infinite graphs of polynomal growth while Kratochvíl, Proskurowski and Telle [27] studied complexity issues of covering recognition.

In the last twenty years, covering space techniques became even more popular, as there is an almost explosive growth of papers mainly discussing symmetry properties, both of abstract and of embedded graphs. J. Širáň [51] considered cyclic covers of map automorphisms in terms of certain $\mathbb{Z}$-modules. Malnič, Marušič and Potočnik [33] developed a method for treating regular abelian covers via a linear representation of automorphisms on the homology group. Around the same time, a number of papers dealing with construction and classification of highly symmetric graphs appeared. From a long list of authors we mention Conder, Du, Feng, Jones, Kuzman, Ma, Malnič, Potočnik, Požar, Širan, and Wang, among others. See [1, 7, 25, 26, 28, 30, 31, 34, 35, 37, 38, 39, 43, 44, 45, 46, 47, 53, 54]. There are also some references regarding covers of digraphs. The first is probably due to Dörfler, Harary, and Malle [11]. Other papers that appeared from '90 on were mainly concerned with studying the spectral properties of digraphs, see for instance [8, 9].

In order to treat symmetry properties of digraphs in a similar fashion as with graphs, there is the need to consider covers of graphs and covers of digraphs as one theory. It therefore seems reasonable to set up a unifying theory of covers of "general" digraphs, where the adjacencies between vertices are allowed to be multiple oriented and/or unoriented links, oriented and/or unoriented loops, and/or semiedges. This is the main objective of this paper. Working with general digraphs comes at a certain price, though, which is mainly related to a formal treatment of homotopy. The problem is that homotopy does not always lift since coverings of general digraphs can be pathological in the sense that the naturally defined projections of the underlying graphs are not covering projections in the topological sense.

As long as the authors of this paper can tell, such a consistent theory has not appeared so far. However, a different formalization of such "general" digraphs has been recently set up by Fiala and Sefrtova [15], although with a completey different scope. The results of the present paper were presented by the first author at the Bled conference in 2019 [31].

Finally, we remark that the name "general digraph" does not mean much. In the literature there is well established concept of a "mixed graph", which, however, does not include semiedges and often also not multiple edges. It is for this reason that we propose to name such a "general digraph" a *graphoid* in the future. See [40].

2 General digraphs

A *general digraph* $X = (V, D, \mathrm{bd}, ^{-1})$ consists of a set of *vertices* $V_X = V$, a set of *darts* $D_X = D$, and a function $\mathrm{bd}\colon D \to V \times V$ (bd stands for "border") that assignes to each dart $x \in D$ an ordered pair of (not necessarily distinct) vertices (u, v), its *initial vertex* $\mathrm{beg}\, x = u$ and its *terminal vertex* $\mathrm{end}\, x = v$. This is indicated by writing $x\colon u \to v$. A dart of the form $x\colon u \to u$ is called a *loop*. Note that several distinct darts $u \to v$ or $u \to u$ are allowed. Additionally, there is a partial involution $^{-1}$ acting on a subset of darts, $x \mapsto x^{-1}$, such that $x^{-1}\colon v \to u$ whenever $x\colon u \to v$. A pair of mutually inverse darts x, x^{-1} is called an *edge*. A pair of distinct mutually inverse loops $x, x^{-1}\colon u \to u$ is an *unoriented loop* while a self inverse loop $x = x^{-1}\colon u \to u$ is called a *semiedge*. A dart without an inverse (that is, the involution $^{-1}$ is not defined on this dart) is either a *directed edge* (a *directed link*) or a *directed loop*. See Figure 1 for an illustration of these concepts.

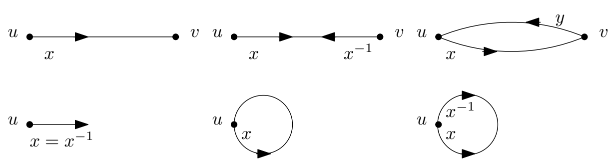

Figure 1: Top: a dart $x\colon u \to v$, a pair of mutually inverse darts $x\colon u \to v$ and $x^{-1}\colon v \to u$, a pair of mutually non-inverse darts $x\colon u \to v$ and $y\colon v \to u$. Bottom: a semiedge $x^{-1} = x\colon u \to u$, a directed loop $x\colon u \to u$, and an undirected loop consisting of distinct mutually inverse $x\colon u \to u$ and $x^{-1}\colon u \to u$.

A general digraph in which every dart has an inverse is a *graph* possibly with multiple edges, multiple unoriented loops and multiple semiedges, and is called a *genuine digraph* (or just a *digraph* for short) whenever no dart has an inverse. Adjoining a formal inverse to each dart without an inverse results in a graph $\underline{X}$, the *underlying graph* of the general digraph X. For instance, the underlying graph of a directed edge $x\colon u \to v$ and of an undirected edge formed by $x\colon u \to v$ and $x^{-1}\colon v \to u$ is K_2; the underlying graph of a pair of mutually noninverse darts $x\colon u \to v$ and $y\colon v \to u$ is C_2. A general digraph $G' = (V', D', \mathrm{bd}', ^{-1'})$ is a *sub-digraph* if $V' \subseteq V$, $D' \subseteq D$, bd' and $^{-1'}$ are restrictions of bd and $^{-1}$ to D'. The *span* of a general digraph X is the genuine digraph $\mathrm{sp}(X)$ that has the same set of darts and set of vertices as X, the same functions beg and end while the involution $^{-1}$ is the empty function. An edge in X becomes two “oppositely” directed links in $\mathrm{sp}(X)$, an undirected loop becomes a pair of directed loops while a semiedge becomes a directed loop. Note that $\mathrm{sp}(X)$ is not a subdigraph of X. Occasionally we will also need the spanning general digraph X^+ of *preferred orientation* arising from X by taking into its dart-set all darts from X that have no inverse, exactly one of the darts from each pair of distinct darts that are inverse to each other, and all semiedges. Clearly, $\underline{X} = \underline{X^+}$.

Let X and Y be general digraphs. A *homomorphism* $h\colon Y \to X$ maps the vertex- and the dart-set of Y to the vertex- and the dart-set of X, respectively, such that whenever $y\colon u \to v$ is a dart in Y, then $h(y)\colon h(u) \to h(v)$ is a dart in X, and moreover, invertible darts are mapped to invertible darts such that $h(y^{-1})\colon h(v) \to h(u)$ is the inverse of $h(y)$ whenever $y^{-1}\colon v \to u$ exists. In other words, $\mathrm{beg}\, h(y) = h(\mathrm{beg}\, y)$, $\mathrm{end}\, h(y) = h(\mathrm{end}\, y)$, and $h(y^{-1}) = h(y)^{-1}$. General digraphs and homomorphisms form a category. Injective, surjective and bijective homomorphisms are standardly referred to as *monomorphisms, epimorphisms, isomorphisms*, and *automorphisms*, respectively.

3 Walks and homotopy in general digraphs

3.1 Walks

Let $X = (V, D, \mathrm{bd}, ^{-1})$ be a general digraph. The intuitive idea of a walk in X is a sequence of ”neighbouring” vertices in X corresponding to traversals of X from a vertex to the next vertex in the sequence, which is ”neighbouring” to it along a dart. To this end, let x^+ indicate the traversal of the dart x from $\mathrm{beg}\, x$ to $\mathrm{end}\, x$, and let x^- indicate the traversal of x from $\mathrm{end}\, x$ to $\mathrm{beg}\, x$. We define $\mathrm{beg}\, x^+ = \mathrm{beg}\, x$ and $\mathrm{end}\, x^+ = \mathrm{end}\, x$ while $\mathrm{beg}\, x^- = \mathrm{end}\, x$ and $\mathrm{end}\, x^- = \mathrm{beg}\, x$. The *signed darts* x^+ and x^- are also called *arcs*. Note that “opposite” arcs are distinct by definition, $x^+ \neq x^-$, even if this is a semiedge. If x is a dart, then x^+ indicates traversal of x in its “natural” positive orientation whereas for loops and semiedges the distinction between x^+ and x^- is completely artificial. If $x\colon v \to v$ is directed loop, then $v\, x^+\, v$ and $v\, x^-\, v$ indicate traversals in opposite directions (although we cannot tell which is which). If $x\colon v \to v$ is an undirected loop, then $v\, x^+\, v$ and $v\,(x^{-1})^-\, v$ indicate traversals in the same direction, although represented by different darts. To consider traversals $v\, x^+\, v$ and $v\, x^-\, v$ of a semiedge $x\colon v \to v$ distinct may seem awkward since it appears that when traversing a semiedge the intuitive concept of ”direction” hardly has any meaning. But we will see that such a distinction is necessary.

A *walk of length* $n \geq 1$ is a sequence $W = u\, x_1^{\epsilon_1}\, x_2^{\epsilon_2}\, \ldots\, x_n^{\epsilon_n}\, v$, where $x_1, \ldots, x_n \in D$ and $\epsilon_j = \pm$, such that $\mathrm{beg}\, x_1^{\epsilon_1} = u$, $\mathrm{end}\, x_n^{\epsilon_n} = v$, and for each index $j = 1, \ldots, n-1$ $(n \geq 2)$ we have

$$\mathrm{end}\, x_j^{\epsilon_j} = \mathrm{beg}\, x_{j+1}^{\epsilon_{j+1}}.$$

Occasionally the *endvertices* of a walk are omitted by only giving the sequence of arcs for simplicity. In particular, walks of length 1 are essentially arcs. The sequence vv is a walk of length 0, also called the *trivial walk* at v and usually abbreviated to v. The vertex $\mathrm{beg}\, W = u$ is the *initial vertex* of a walk W while $\mathrm{end}\, W = v$ is its *terminal vertex*. This is often denoted $W\colon u \to v$. The walk

$$W^- = v\, x_n^{-\epsilon_n}\, \ldots\, x_1^{-\epsilon_1}\, u,$$

where $-\epsilon = +$ if $\epsilon = -$ and $-\epsilon = -$ otherwise, is the *opposite walk* to W. A walk $W\colon v \to v$ is a *closed walk at* v. A general digraph is *connected* if for each pair of vertices $u, v \in V$ there exists a walk $u \to v$. This is often referred to as *weak connectivity* in the literature, and coincides with the connectivity of the underlying graph in the usual (topological) sense. Connectivity may often be assumed without loosing much on the generality.

If $W_1\colon u \to v$ and $W_2\colon v \to w$ are walks, then $W_1 \cdot W_2 = W_1W_2\colon u \to w$ is the *concatenated walk* obtained by juxtaposition of the two sequences (and omitting the "middle" vertex v). The set of all closed walks at some vertex v_0 (usually referred to as the "base vertex") equipped with concatenation as the operation is a monoid with involution $W \mapsto W^-$. This is the *fundamental monoid* at v_0, which we denote by $\Pi(X, v_0)$. The fundamental monoid of X at v_0 is really the fundamental monoid of the connected component to which v_0 belongs. It therefore often makes sense to tacitly assume X to be connected.

Two walks $W, W'\colon u \to v$ of the same length are *congruent* if, intuitively speaking, W and W' traverse the same "underlying edges in the same direction" at each step. This is written $W \equiv W'$. More precisely, let $W = u\, x_1^{\epsilon_1}\, x_2^{\epsilon_2}\, \ldots\, x_n^{\epsilon_n}\, v$ and $W' = u\, y_1^{\epsilon_1'}\, y_2^{\epsilon_2'}\, \ldots\, y_n^{\epsilon_n'}\, v$. Then $W \equiv W'$ whenever for each index $j = 1, 2, \ldots, n$ we have

$$y_j^{\epsilon_j'} = x_j^{\epsilon_j} \quad \text{or} \quad y_j^{\epsilon_j'} = (x_j^{-1})^{-\epsilon_j}.$$

In particular, let $x\colon u \to v$ be a dart and $x^{-1}\colon v \to u$ its inverse. Then $v\,(x^{-1})^+\,u \equiv v\,x^-\,u$ and $u\,(x^{-1})^-\,v \equiv u\,x^+\,v$. The same holds for an undirected loop $x\colon u \to u$, that is, $u\,(x^{-1})^+\,u \equiv u\,x^-\,u$ and $u\,(x^{-1})^-\,u \equiv u\,x^+\,u$. Note that $u\,x^+\,u \not\equiv u\,x^-\,u$. With a semiedge, however, we have $u\,x^+\,u = u\,(x^{-1})^+\,u \equiv u\,x^-\,u$. Thus, traversals $u\,x^+\,u$ and $u\,x^-\,u$ of a semiedge are congruent, yet distinct by definition. It should be clear that congruence of walks is an equivalence relation.

The traversals

$$x^\epsilon x^{-\epsilon} \text{ and } x^\epsilon (x^{-1})^\epsilon \text{ (if } x^{-1} \text{ exists)}$$

are called *forth-back traversals* at a dart x. They correspond to traveling forth and back along the dart x, or forth and back along the dart x and its inverse. Erasing such a traversal,

$$x^\epsilon x^{-\epsilon} \to \text{empty string} \quad \text{or} \quad x^\epsilon (x^{-1})^\epsilon \to \text{empty string},$$

that is, substituting the forth-back traversal with the trivial walk, is called an *elementary reduction* at x. Specifically, consider a walk

$$W = u\, x_1^{\epsilon_1}\, \ldots\, x_{j-1}^{\epsilon_{j-1}} \cdot x_j^{\epsilon_j}\, x_{j+1}^{\epsilon_{j+1}} \cdot x_{j+2}^{\epsilon_{j+2}}\, \ldots\, x_n^{\epsilon_n}\, v,$$

where $x_j^{\epsilon_j}\, x_{j+1}^{\epsilon_{j+1}}$ is a forth-back traversal. The elementary reduction at index j in W is denoted T_j, and results in the walk

$$W' = u\, x_1^{\epsilon_1}\, \ldots x_{j-1}^{\epsilon_{j-1}}\, x_{j+2}^{\epsilon_{j+2}}\, \ldots x_n^{\epsilon_n}\, v.$$

A *reduction* is a sequence of elementary reductions. A walk admitting no further reductions is *reduced*. Walks that can be reduced to a trivial walk are *contractible*. One would expect that each walk has a unique associated reduced walk. But this impression is false. For instance, the walk $u\,x^+\,x^-\,(x^{-1})^-\,v$ has two reduced walks, $u\,(x^{-1})^-\,v$ and $u\,x^+\,v$. These reduced walks are congruent – an observation that holds in general, but requires a rigorous although a somewhat tedious proof.

Proposition 3.1. *Reduced walks of congruent walks are themselves congruent. In particular, any two reduced walks of a given walk are congruent.*

Proof. Let $W_1 \equiv W_2$ be congruent walks. We first show that after performing an arbitrary elementary reduction to W_1 and an elementary reduction to W_2, then either the resulting walks are congruent, or there is a further pair of elementary reductions that make the resulting walks congruent. Write

$$W_1 = P \cdot x_i^{\epsilon_i}\, x_{i+1}^{\epsilon_{i+1}} \cdot Q \cdot x_j^{\epsilon_j}\, x_{j+1}^{\epsilon_{+1}} \cdot R$$
$$W_2 = P' \cdot y_i^{\epsilon_i'}\, y_{i+1}^{\epsilon_{i+1}'} \cdot Q' \cdot y_j^{\epsilon_j'}\, y_{j+1}^{\epsilon_{+1}'} \cdot R'.$$

Since $W_1 \equiv W_2$ we have $P \equiv P'$, $x_i^{\epsilon_i}\, x_{i+1}^{\epsilon_{i+1}} \equiv y_i^{\epsilon_i'}\, y_{i+1}^{\epsilon_{i+1}'}$, $Q \equiv Q'$, $x_j^{\epsilon_j}\, x_{j+1}^{\epsilon_{+1}} \equiv y_j^{\epsilon_j'}\, y_{j+1}^{\epsilon_{+1}'}$, and $R \equiv R'$. We may assume that T_i is applied to W_1 and T_j to W_2. Then

$$T_i W_1 = P \cdot Q \cdot x_j^{\epsilon_j}\, x_{j+1}^{\epsilon_{+1}} \cdot R$$

$$T_j W_2 = P' \cdot y_i^{\epsilon_i'}\, y_{i+1}^{\epsilon_{i+1}'} \cdot Q' \cdot R'.$$

We may assume $j \geq i$. There are three cases to consider. If $j = i$, then $T_i W_1 = P \cdot R$ while $T_j W_2 = P' \cdot R'$, and so $T_i W_1 \equiv T_j W_2$. If $j \geq i+2$, then $T_j T_i W_1 = P \cdot Q \cdot R$ while $T_i T_j W_2 = P' \cdot Q' \cdot R'$, and so $T_j T_i W_1 \equiv T_i T_j W_2$ It remains to consider the case when the two reductions overlap, $j = i+1$. Then the walks W_1 and W_2 take the following forms:

$$W_1 = P \cdot x_i^{\epsilon_i}\, x_{i+1}^{\epsilon_{i+1}}\, x_{i+2}^{\epsilon_{j+2}} \cdot R$$
$$W_2 = P' \cdot y_i^{\epsilon_i'}\, y_{i+1}^{\epsilon_{i+1}'}\, y_{i+2}^{\epsilon_{j+2}'} \cdot R'.$$

Now we have $T_i W_1 = P \cdot x_{i+2}^{\epsilon_{j+2}} \cdot R$ and $T_j W_2 = P' \cdot y_i^{\epsilon_i'} \cdot R'$. To finish the proof of the claim we need to see that

$$x_{i+2}^{\epsilon_{j+2}} \equiv y_i^{\epsilon_i'}.$$

But this is straightforward if we take into account that W_1 and W_2 are congruent, and that there exists a reduction at position i in W_1 and a reduction at position $j = i+1$ in W_2. Indeed, suppose that $y_i^{\epsilon_i'} \in \{y^+, (y^{-1})^-\}$. By congruence we have $x_i^{\epsilon_i} \in \{y^+, (y^{-1})^-\}$, and reducibility then implies $x_{i+1}^{\epsilon_{i+1}} \in \{y^-, (y^{-1})^+\}$. By congruence we further have $y_{i+1}^{\epsilon_{i+1}'} \in \{y^-, (y^{-1})^+\}$, and then $y_{i+2}^{\epsilon_{i+2}'} \in \{y^+, (y^{-1})^-\}$ by reduciblity. By congruence we finally obtain $x_{i+2}^{\epsilon_{i+2}} \in \{y^+, (y^{-1})^-\}$. In a similar fashion, if $y_i^{\epsilon_i'} \in \{y^-, (y^{-1})^+\}$ then $x_{i+2}^{\epsilon_{i+2}} \in \{y^-, (y^{-1})^+\}$. We conclude that $x_{i+2}^{\epsilon_{j+2}} \equiv y_i^{\epsilon_i'}$, as required.

To prove the proposition we now proceed by induction on the length of the walk. For the trival walks there is nothing to prove. Consider two congruent walks W_1 and W_2 of length n, and let us inductively assume that the proposition holds for any two congruent shorther walks.

Consider the reductions $W_1 \to T_i W_1 \to \bar{W}_1$ and $W_2 \to T_j W_2 \to \bar{W}_2$, where the walks $\bar{W}_1$ and $\bar{W}_2$ are reduced, and T_i and T_j are elementary reductions. By the first part of the proof there are reductions $W_1 \to T_i W_1 \to W_1'$ and $W_2 \to T_j W_2 \to W_2'$ such that $W_1' \equiv W_2'$. Denote by $\bar{W}_1'$ and $\bar{W}_2'$ two arbitrary reduced walks of W_1' and W_2'. By induction hypothesis we have $\bar{W}_1' \equiv \bar{W}_2'$. Now $\bar{W}_1$ and $\bar{W}_1'$ are reduced walks of $T_i W_1$ and by induction hypothesis we have $\bar{W}_1 \equiv \bar{W}'_1$. Similarly, $\bar{W}'_2 \equiv \bar{W}_2$. Since congruence is an equivalence relation we finally obtain $\bar{W}_1 \equiv \bar{W}_2$, and the proof is complete. □

3.2 Homotopy

Two walks with congruent reduced walks are called *homotopic*. *Homotopy* in a general digraph is an equivalence relation with equivalence classes (the *homotopy classes*) denoted $[W]$. It clearly corresponds to the usual notion of homotopy in the underlying graph. Restricted to the fundamental monoid $\Pi(X, v_0)$, the set of homotopy classes forms the *fundamental group* $\pi(X, v_0)$. The homotopy class $1 = [v_0]$ is the trivial group element, and $[W]^{-1} = [W^-]$. Note that a walk $v_0\, x^\epsilon\, v_0$, where x is a semiedge, is an element of order 2 in $\pi(X, v_0)$.

As X is tacitly assumed connected, a minimal generating set for $\pi(X, v_0)$ can be constructed by taking the homotopy classes of *fundamental closed walks* at v_0 with respect to an arbitrary spanning tree T in X^+. The number of generators is known as the *Betti number*, and is equal to $\beta(X) = \beta(X^+) = |D_{X^+}| - |V_X| + 1 = r(X^+) + s(X)$, where $r(X^+)$ is the number of cotree directed links and directed loops in X^+ while $s(X)$ is the number of semiedges. Also, $\beta(X)$ is exactly the Betti number $\beta(\underline{X})$ of the underlying graph. The group $\pi(X, v_0)$ is isomorphic to the free product of $r(X)$ copies of $\mathbb{Z}$ (corresponding to generators of infinite order) and $s(X)$ copies of $\mathbb{Z}_2$ (corresponding to semiedges). All these basic facts can be proved similarly as for graphs, except that they require a bit of extra technicalities, which we omit.

3.3 Weak homotopy

As we have seen, homotopy in a general digraph exactly corresponds to the usual notion of homotopy in its underlying graph. But there is a weaker notion than homotopy. Two walks $W, W' \colon u \to v$ in a connected general digraph X are called *weakly homotopic* if the homotopic changes that transform W to W' are performed exclusively by deletion and/or insertion of subwalks of the form $x^\epsilon x^{-\epsilon}$. This clearly is an equivalence relation. The equivalence classes $[W]_w$ are called the *weak homotopy classes*. Note that each such class contains a unique weakly reduced walk. If x is a semiedge, then the walk $x^\epsilon x^\epsilon$ is not weakly homotopic to the trivial walk since $x^\epsilon x^\epsilon \neq x^\epsilon x^{-\epsilon}$ because $x^\epsilon \neq x^{-\epsilon}$ by definition. The weak homotopy classes of closed walks at a vertex v_0 naturally constitute a group, the *weak fundamental group* $\pi_w(X, v_0)$. Clearly, there is a natural monoid epimorphism $\Pi(X, v_0) \to \pi_w(X, v_0)$, and since weakly homotopic walks are homotopic, there is also a group epimorphism $\pi_w(X, v_0) \to \pi(X, v_0)$.

Homotopy and weak homotopy coincide in genuine digraphs. Since a general digraph and its span have the same set of vertices and darts, walks in X and $\mathrm{sp}(X)$ are the same, and moreover, weak homotopy in X exactly corresponds to homotopy in $\mathrm{sp}(X)$. Thus,

$$\pi_w(X, v_0) = \pi(\mathrm{sp}(X), v_0) = \pi_w(\mathrm{sp}(X), v_0).$$

Note that if x is a semiedge, $x^\epsilon x^\epsilon$ corresponds to traversing the respective directed loop in $\mathrm{sp}(X)$ twice in the same direction while $x^\epsilon x^{-\epsilon}$ corresponds to traversing this loop in opposite directions.

Clearly, $\pi_w(X, v_0)$ of a finite general digraph is finitely generated. For X connected, a minimal generating set is formed by the *weak fundamental closed walks* at v_0 defined by the cotree darts relative to a *genuine spanning tree* T in X arising from T in $\mathrm{sp}(X)$ (or X^+). The corresponding *weak Betti number* is $\beta^w(X) = |D_X| - |V_X| + 1 = \beta(\mathrm{sp}(X))$. Each generator has infinite order, hence $\pi_w(X, v_0)$ is a free group of rank $\beta^w(X)$.

An isomorphism maps homotopy classes bijectively to homotopy classes, and maps weak homotopy classes to weak homotopy classes. Thus, an isomorphism $h\colon X \to X'$ naturally induces an isomorpism that maps

$$h_*\colon \pi(X, v_0) \to \pi(X', h(v_0)) \text{ and } h_*\colon \pi_w(X, v_0) \to \pi_w(X', h(v_0)).$$

4 Coverings of general digraphs

4.1 The concept

We now come to the main concept of this paper. A homomorphism $\wp\colon \tilde{X} \to X$ of general digraphs is a *covering* of X if the following two conditions are satisfied:

(1) The mapping $\wp$ is an epimorphism, that is, it maps the set of vertices of $\tilde{X}$ onto the set of vertices of X, and it maps the set of darts of $\tilde{X}$ onto the set of darts in X.

(2) For each vertex $\tilde{u}$ in $\tilde{X}$, the set of darts with initial vertex $\tilde{u}$ is mapped bijectively onto the set of darts with initial vertex $u = \wp(\tilde{u})$, and the set of darts with terminal vertex $\tilde{u}$ is mapped bijectively onto the set of darts with terminal vertex $u = \wp(\tilde{u})$.

Moreover, we shall be assuming that the *base general digraph* X is connected, for if $\wp\colon \tilde{X} \to X$ is a covering, then an appropriate restriction mapping part of $\tilde{X}$ onto a connected component of X is also a covering. We could as well require that $\tilde{X}$ be connected, in which case we sometimes say that the *covering* $\tilde{X} \to X$ *is connected*. This additional assumption will become significant in later sections.

For a vertex v and a dart x in X, the preimage $\mathrm{fib}_v = \wp^{-1}(v)$ is called the *vertex-fibre* over v (or just the *fibre* for short), and $\wp^{-1}(x)$ is the *dart-fibre* over x. From conditions (1) and (2) it follows that for each dart $x\colon u \to v$ in X the dart-fibre is a directed perfect matching of darts from fib_u to fib_v. This establishes a bijection

$$\tau_x\colon \mathrm{fib}_u \to \mathrm{fib}_v.$$

It follows by induction on the length of the walk that whenever X is connected, all fibres have equal cardinality. If this cardinality is n, then the covering is n*-fold*. Another important consequence is that each walk $W\colon u \to v$ lifts to a unique walk $\tilde{W}_{\tilde{u}}$ that projects to W and starts at an arbitrarily given vertex $\tilde{u} \in \mathrm{fib}_u$. This is the *unique walk lifting property*. For convenience we denote the terminal vertex of the walk $\tilde{W}_{\tilde{u}}$ by

$$\tilde{u} \centerdot W.$$

This is in fact the *walk-action* since the trivial walks act trivially, and $(\tilde{u} \centerdot W_1) \centerdot W_2 = \tilde{u} \centerdot W_1 W_2$ holds for all walks $W_1\colon u \to v$ and $W_2\colon v \to w$.

The above definition of a covering seems to be a sensible definition as it respects the ”local isomorphism” property. However, coverings as defined above may have rather unexpected properties. In particular, one would expect that a covering $\tilde{X} \to X$ induces a covering of the underlying graphs $\underline{\tilde{X}} \to \underline{X}$, consistent with the topological interpretation. But this is not the case in general. We present three simple examples.

Example 4.1. There is a covering of a directed 4-cycle onto a complete graph on 2 vertices, $\vec{C}_4 \to K_2$. Similarly, two directed loops $2\vec{C}_1$ attached at a common vertex project as a covering onto an undirected loop C_1. As a third example, consider the covering $\vec{C}_3 \to s_1$ of a directed 3-cycle onto the semistar s_1 with one semiedge. In all these cases the corresponding induced maps $C_4 \to K_2$, $2C_1 \to C_1$, and $C_3 \to s_1$, are definitely not covering projections.

Example $\vec{C}_3 \to s_1$ illustrates the need to consider the positive and the negative traversals of a semiedge distinct. Indeed, the positive traversal of s_1 lifts to a walk consistent with the natural orientation of $\vec{C}_3$ while the negative one lifts to a walk that goes against it.

The reason behind these anomalies is that a dart without an inverse was mapped to a dart that had an inverse. We define a covering projection $\wp\colon \tilde{X} \to X$ of general digraphs *homogenuous* whenever no dart in $\tilde{X}$ without an inverse is mapped to a dart that has an inverse. The lift of a semiedge is a collection of edges and/or semiedges, a 1-factor on the corresponding fibre. Coverings of genuine digraphs are homogenuous. As for coverings of graphs, in order for the covering objects to be graphs as well we have to require that the respective projections are homogeneous. Thus, by considering homogenuous coverings of general digraphs we indeed cover the most important cases of covering projections in the frame of a unified theory. The following result reveals why the above definition makes sense.

Proposition 4.2. *Let $\wp\colon \tilde{X} \to X$ be an onto homomorphism of general digraphs. Then the associated homomorphism $\underline{\wp}\colon \underline{\tilde{X}} \to \underline{X}$ of underlying graphs is a covering projection if and only if $\wp\colon \tilde{X} \to X$ is a homogeneous covering.*

Proof. Suppose first that $\wp\colon \tilde{X} \to X$ is a homogeneous covering. Take an arbitrary vertex $\tilde{u}$ in $\tilde{X}$, and let $u = \wp(\tilde{u})$. Denote the following subsets of darts in X and $\tilde{X}$ that either start or terminate in u and $\tilde{u}$, respectively:

$$I^+(u) = \{x \mid \mathrm{beg}\, x = u,\ x^{-1} \text{ exists}\}, \quad N^+(u) = \{x \mid \mathrm{beg}\, x = u,\ x^{-1} \text{ does not exist}\},$$
$$I^-(u) = \{x \mid \mathrm{end}\, x = u,\ x^{-1} \text{ exists}\}, \quad N^-(u) = \{x \mid \mathrm{end}\, x = u,\ x^{-1} \text{ does not exist}\},$$

$$I^+(\tilde{u}) = \{\tilde{x} \mid \mathrm{beg}\, \tilde{x} = \tilde{u},\ \tilde{x}^{-1} \text{ exists}\}, \quad N^+(\tilde{u}) = \{\tilde{x} \mid \mathrm{beg}\, \tilde{x} = \tilde{u},\ \tilde{x}^{-1} \text{ does not exist}\},$$
$$I^-(\tilde{u}) = \{\tilde{x} \mid \mathrm{end}\, \tilde{x} = \tilde{u},\ \tilde{x}^{-1} \text{ exists}\}, \quad N^-(\tilde{u}) = \{\tilde{x} \mid \mathrm{end}\, \tilde{x} = \tilde{u},\ \tilde{x}^{-1} \text{ does not exist}\}.$$

Note that $\wp(I^+(\tilde{u})) \subseteq I^+(u)$ and $\wp(I^-(\tilde{u})) \subseteq I^-(u)$ since $\wp$ is a homomorphism. Also, we have $\wp(N^+(\tilde{u})) \subseteq N^+(u)$ and $\wp(N^-(\tilde{u})) \subseteq N^-(u)$ since $\wp$ is homogeneous. As $\wp$ is a covering projection the restrictions $\wp\colon I^+(\tilde{u}) \cup N^+(\tilde{u}) \to I^+(u) \cup N^+(u)$ and $\wp\colon I^-(\tilde{u}) \cup N^-(\tilde{u}) \to I^-(u) \cup N^-(u)$ are bijections. Hence the restrictions

$$\wp\colon I^+(\tilde{u}) \to I^+(u) \text{ and } \wp\colon N^+(\tilde{u}) \to N^+(u)$$
$$\wp\colon I^-(\tilde{u}) \to I^-(u) \text{ and } \wp\colon N^-(\tilde{u}) \to N^-(u)$$

are bijections as well.

Now let us extend the two general digraphs to underlying graphs $\underline{X}$ and $\underline{\tilde{X}}$ by adjoining additional inverse darts. Denote by $N^+_{-1}(u)$ the added inverses to $N^+(u)$ and by $N^-_{-1}(u)$ the added inverses to $N^-(u)$. Similarly, let $N^+_{-1}(\tilde{u})$ be the added inverses to $N^+(\tilde{u})$ and $N^-_{-1}(\tilde{u})$ the added inverses to $N^-(\tilde{u})$. Obviously, in $\underline{X}$ and $\underline{\tilde{X}}$ we have

$$\begin{aligned}\{x \mid \operatorname{beg} x = u\} &= I^+(u) \cup N^+(u) \cup N^-_{-1}(u)\\ \{x \mid \operatorname{end} x = u\} &= I^-(u) \cup N^-(u) \cup N^+_{-1}(u)\end{aligned}$$

$$\begin{aligned}\{\tilde{x} \mid \operatorname{beg} \tilde{x} = \tilde{u}\} &= I^+(\tilde{u}) \cup N^+(\tilde{u}) \cup N^-_{-1}(\tilde{u})\\ \{\tilde{x} \mid \operatorname{end} \tilde{x} = \tilde{u}\} &= I^-(\tilde{u}) \cup N^-(\tilde{u}) \cup N^+_{-1}(\tilde{u}).\end{aligned}$$

Since $|N^-_{-1}(\tilde{u})| = |N^-(\tilde{u})| = |N^-(u)| = |N^-_{-1}(u)|$ and $|N^+_{-1}(\tilde{u})| = |N^+(\tilde{u})| = |N^+(u)| = |N^+_{-1}(u)|$ we can extend $\wp$ to a bijective mapping $\underline{\wp}\colon \underline{\tilde{X}} \to \underline{X}$ by sending $N^-_{-1}(\tilde{u}) \to N^-_{-1}(u)$ and $N^+_{-1}(\tilde{u}) \to N^+_{-1}(u)$ in such way that $\underline{\wp}$ commutes with $^{-1}$; hence $\underline{\wp}$ is a homomorphism, in fact, a covering projection of the corresponding underlying graphs.

For the converse, let $\wp\colon \tilde{X} \to X$ be an onto homomorphism such that $\underline{\wp}\colon \underline{\tilde{X}} \to \underline{X}$ is a covering projection. Let $\tilde{u}$ be an arbitrary vertex in $\tilde{X}$, and let $u = \wp(\tilde{u})$. As $\wp$ is an onto homomorphism, the restrictions

$$\wp\colon I^+(\tilde{u}) \cup N^+(\tilde{u}) \to I^+(u) \cup N^+(u) \text{ and } \wp\colon I^-(\tilde{u}) \cup N^-(\tilde{u}) \to I^-(u) \cup N^-(u)$$

are onto mappings. Now the extended mapping must send the set of added inverses onto the set of added inverses, $\underline{\wp}\colon N^-_{-1}(\tilde{u}) \to N^-_{-1}(u)$ and $\underline{\wp}\colon N^+_{-1}(\tilde{u}) \to N^+_{-1}(u)$. But as $\underline{\wp}$ is a covering projection, the restrictions

$$\underline{\wp}\colon I^+(\tilde{u}) \cup N^+(\tilde{u}) \cup N^-_{-1}(\tilde{u}) \to I^+(u) \cup N^+(u) \cup N^-_{-1}(u)$$

$$\underline{\wp}\colon I^-(\tilde{u}) \cup N^-(\tilde{u}) \cup N^+_{-1}(\tilde{u}) \to I^-(u) \cup N^-(u) \cup N^+_{-1}(u)$$

are bijections. It follows that $\underline{\wp}\colon N^-_{-1}(\tilde{u}) \to N^-_{-1}(u)$ and $N^+_{-1}(\tilde{u}) \to N^+_{-1}(u)$ are bijections, and so $\wp\colon I^+(\tilde{u}) \cup N^+(\tilde{u}) \to I^+(u) \cup N^+(u)$ and $\wp\colon I^-(\tilde{u}) \cup N^-(\tilde{u}) \to I^-(u) \cup N^-(u)$ are bijections as well. Thus, $\wp$ is a covering projection of digraphs. Moreover, since $\wp$ commutes with beg and end, we have that $\wp(I^+(\tilde{u})) \subseteq I^+(u)$ and $\wp(I^-(\tilde{u})) \subseteq I^-(u)$. Consequently,

$$\wp\colon I^+(\tilde{u}) \to I^+(u) \text{ and } \wp\colon I^-(\tilde{u}) \to I^-(u),$$

$$\wp\colon N^+(\tilde{u}) \to N^+(u) \text{ and } \wp\colon N^-(\tilde{u}) \to N^-(u)$$

are bijections, that is, no dart without an inverse is mapped to a dart that has an inverse. The covering $\wp\colon \tilde{X} \to X$ is homogeneous. □

Although restricting our considerations to homogenuous coverings is rather inviting, we shall not confine exclusively to such coverings. Yet certain restrictions, weaker than homogeneity, are in some sense necessary. The reason is that with the most general type of coverings their structural properties and the combinatorial reconstruction cannot be studied in a sufficiently meaningful manner, see Theorem 4.6. To this end, we shall be considering coverings that we call inverse-consistent. A covering projection is *inverse-consistent* whenever two darts $\tilde{u} \to \tilde{v}$ and $\tilde{v} \to \tilde{u}$ in $\tilde{X}$ must form a pair of inverse darts provided that they project to a pair of inverse darts in X. Note that a homogeneous covering is inverse-consistent, but the converse does not hold. See Remark 4.3 below.

Remark 4.3. The projection $\vec{C}_4 \to K_2$ is inverse-consistent but not homogeneous. Next, observe that an inverse-consistent 1-fold covering is homogeneous, and moreover, an isomorphism. However, a 1-fold covering need not be an isomorphism as it need not be inverse-consistent. A typical example is $\vec{C}_2 \to K_2$.

4.2 Morphisms of covering projections

Let $\wp\colon \tilde{X} \to X$ and $\wp'\colon \tilde{X}' \to X'$ be covering projections of general digraphs. A *morphism of covering projections* $\wp \to \wp'$ is a pair of general digraph homomorphisms $\alpha\colon X \to X'$ and $\tilde{\alpha}\colon \tilde{X} \to \tilde{X}'$ such that the following diagram is commutative:

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{\alpha}} & \tilde{X}' \\ {\scriptstyle \wp}\downarrow & & \downarrow{\scriptstyle \wp'} \\ X & \xrightarrow[\alpha]{} & X' \end{array}$$

This is denoted $(\alpha, \tilde{\alpha})\colon \wp \to \wp'$. We say that α *lifts along* $\wp$ *and* $\wp'$ to $\tilde{\alpha}$, and that $\tilde{\alpha}$ *projects along* $\wp$ and $\wp'$ to α. If in the above diagram α and $\tilde{\alpha}$ are isomorphisms, then $(\alpha, \tilde{\alpha})\colon \wp \to \wp'$ is an *isomorphism of covering projections*. This is a standard concept that formalizes the intuitive notion of "coverings that are structuraly the same".

(1) Perhaps the most important case that is met in practice is the class $\mathcal{C}(X)$ of all covering projections $\tilde{X} \to X$ of a given finite connected general digraph X. We would like to classify all (connected) coverings in $\mathcal{C}(X)$ up to isomorphism, and to study properties of these coverings in terms of properties of X. This problem can be solved, succesfully to some extent, at the theoretical level. In concrete cases, however, it becomes an extremely difficult computational task.

A somewhat easier problem is to classify coverings in $\mathcal{C}(X)$ up to equivalence. We say that $\wp\colon \tilde{X} \to X$ and $\wp'\colon \tilde{X}' \to X$ are *equivalent* whenever there exists an isomorphism of the form $(\mathrm{id}, \tilde{\alpha})\colon \wp \to \wp'$. At this point we would like to emphasise that it is wrong to think of coverings without specifying the actual projection. For it can happen that $\tilde{X}$ projects as a covering onto X in non-isomorphc ways; it could also happen that two equivalent coverings $\wp\colon \tilde{X} \to X$ and $\wp'\colon \tilde{X} \to X$ have completely different fibres.

(2) Another important issue when studying $\mathcal{C}(X)$ is composition and decomposition of coverings. First note that a composition of two covering projections is a covering projection. Moreover, the composition of inverse-consistent coverings is inverse consistent, and the composition of homogeneous coverings is homogeneous. However, a composition of a homogeneous covering and a non inverse-consistent one can be inverse-consistent. For example, $\vec{C}_4 \to \vec{C}_2$ is homogeneous while $\vec{C}_2 \to K_2$ is not inverse-consistent. Yet their composition $\vec{C}_4 \to K_2$ is inverse-consistent.

With decomposition we are faced in certain sense with a dual problem. Given covering projections $\wp\colon \tilde{X} \to X$ and $\wp'\colon \tilde{X}' \to X$, does there exist an onto homomorphism $q\colon \tilde{X} \to \tilde{X}'$ such that the folowing diagram

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\;q\;} & \tilde{X}' \\ {\scriptstyle\wp}\big\downarrow & & \big\downarrow{\scriptstyle\wp'} \\ X & \xrightarrow[\mathrm{id}]{} & X \end{array}$$

is commutative? If this is the case, then we say that $\wp$ *has a decomposition via* $\wp'$. Note that q must be a covering projection as well, and moreover, inverse-consistent whenever $\wp$ is inverse-consistent. However, since our focus is on inverse-conistent coverings we should take into account the fact that an inverse-consistent covering might decompose via a non-inverse-consistent one.

(3) Last but not least there is the problem of studying symmetry properties of the cover $\tilde{X}$ in terms of symmetry properties of X. This boils down to a question of *lifting automorphisms*: given an automorphism $\alpha\colon X \to X$, is there an automorphism $\tilde{\alpha}\colon \tilde{X} \to \tilde{X}$ such that $(\alpha, \tilde{\alpha})\colon \wp \to \wp$ is a self-isomorphism, or an *automorphism*, of $\wp$? The problem is a special case of (1) above, however, the point of view here is different.

Rather than individual automorphisms we lift or project groups of automorphisms. A group $G \leq \mathrm{Aut}(X)$ *lifts along* $\wp\colon \tilde{X} \to X$ if each automorphism from G has a lift (it is enough to require that some generating set lifts). The respective covering is called *G-admissible*. The collection $\tilde{G}$ of all lifts of all $\alpha \in G$ constitutes a group, and the associated mapping $\wp^*\colon \tilde{G} \to G$ defined by $\tilde{\alpha} \to \alpha$ is a group epimorphism. The lifted group $\tilde{G}$ is the largest group in $\mathrm{Aut}(\tilde{X})$ that projects along $\wp$ to G.

An important kind of an automorphism of $\wp$ is *self-equivalence*, an automorphism of the form $(\mathrm{id}, \tilde{\alpha})\colon \wp \to \wp$. In the literature, the respective $\tilde{\alpha}$ is known as a *covering transformation* or a *deck transformation*, see for instance [36]. All covering transformations are clearly all the lifts of the identity automorphism, and hence constitute the *group of covering transformations*, denoted $\mathrm{CT}_\wp = \widetilde{\mathrm{id}}$. This group is the kernel of the epimorphism $\wp^*\colon \tilde{G} \to G$. In particular, the set of all lifts of $\alpha \in G$ forms a coset of $\mathrm{CT}_\wp$ within $\tilde{G}$.

This puts forward the following problem: if a group lifts, determine the isomorphism class of the lifted group, or more precisely, study the extension $\mathrm{CT}_\wp \to \tilde{G} \to G$. Note that if the covering projections $\wp$ and $\wp'$ are equivalent, then $\mathrm{CT}_\wp$ and $\mathrm{CT}_{\wp'}$ are isomorphic and their actions are isomorphic; this holds also for the lifted groups $\tilde{G}$ and $\tilde{G}'$ of G along $\wp$ and $\wp'$, respectively. Moreover, the extensions $\mathrm{CT}_{\wp'} \to \tilde{G}' \to G$ and $\mathrm{CT}_\wp \to \tilde{G} \to G$ are isomorphic. This enables us to study the problem of lifting automorphisms and the extension problem in a purely combinatorial setting.

4.3 Action of the fundamental monoid

The structural properties of a given covering projection $\wp\colon \tilde{X} \to X$ are usually studied in terms of the action of the fundamental group $\pi(X, v_0)$ on the *base fibre* fib_{v_0} induced by the unique walk lifting – in order to transfer inherently topological questions to algebraic ones. However, as we shall see, in our context this holds only for homogeneous coverings. For non-homogeneous ones we need to consider the action of the weak homotopy group

$\pi_w(X, v_0)$ instead. But actually, all relevant implications of these actions are a consequence of a more general theorem involving the action of the fundamental monoid $\Pi(X, v_0)$ on fib_{v_0} via the unique walk lifting. We call the action of $\Pi(X, v_0)$ *permutational*, because the representation homomorphism $\chi\colon \Pi(X, v_0) \to \mathrm{Fun}(\mathrm{fib}_{v_0})$ is in fact a homomorphism into the right symmetric group

$$\chi\colon \Pi(X, v_0) \to \mathrm{Sym}_r \mathrm{fib}_{v_0}.$$

Remark 4.4. Note that permutational monoid actions have certain nice properties that monoid actions generally do not have. In a bit wider context, let M be a monoid with involution $s \mapsto s^*$ and Γ a group (considered as a monoid with involution $g \mapsto g^{-1}$).

(1) If $q\colon M \to \Gamma$ is a monoid homomorphism, then one can define the *algebraic kernel* $\mathrm{Alg\,Ker\,q} = \{\mathrm{s} \in \mathrm{M} \mid \mathrm{q(s)} = 1\}$. This determines the *kernel relation* $s \equiv_q t \Leftrightarrow q(s) = q(t)$ in the sense that $s \equiv_q t \Leftrightarrow st^* \in \mathrm{Alg\,Ker\,q}$.

(2) As a consequence, let M and M' be isomorphic monoids, and $q\colon M \to \Gamma$ and $q'\colon M' \to \Gamma'$ epimorphisms. Then a monoid isomorphism $\phi\colon M \to M'$ projects along q, q' to $\phi^{\#}\colon \Gamma \to \Gamma'$ if and only if ϕ takes $\mathrm{Alg\,Ker\,q}$ isomorphically onto $\mathrm{Alg\,Ker\,q'}$.

(3) Similarly, an epimorphism $\phi\colon M \to \Gamma'$ factorizes through an epimorphism $q\colon M \to \Gamma$ such that $\phi = \phi^{\#} q$ if and only if $\mathrm{Alg\,Ker\,q} \subseteq \mathrm{Alg\,Ker}\,\phi$.

(4) Let M act permutationally (on the right) on a set. If $q\colon M \to \Gamma$ is a monoid epimorphism, then there exists an induced action of Γ such that $x \,.\, s = x \cdot q(s)$ if and only if $\mathrm{Alg\,Ker\,q}$ is contained in the algebraic kernel $\mathrm{Alg\,Ker}\,\chi$ for the permutation representation χ of M. Moreover, the orbits of M and Γ coincide, and q maps the stabilizers of M onto the stabilizers of Γ.

(5) Let isomorphic monoids M and M' with involution act permutationally (on the right) on sets F and F' of equal cardinality. A monoid isomorphism $\phi\colon M \to M'$ is called *admissible* if it "extends" to an isomorphism of actions in the sense that there is a bijection $\tau\colon F \to F'$ such that $\tau(j \cdot s) = \tau(j) \cdot \phi(s)$.

(6) Further, let $q\colon M \to \Gamma$ and $q'\colon M' \to \Gamma'$ be epimorphisms, and let the actions of Γ and Γ' be induced by the actions of M and M', respectively. Suppose now that a monoid isomorphism $\phi\colon M \to M'$ projects along q, q' to $\phi^{\#}\colon \Gamma \to \Gamma'$. Then ϕ is admissible for the action of M and M' if and only if $\phi^{\#}$ is admissible for the induced actions of Γ and Γ'.

Coming back to fundamental monoids, observe that the actions of $\Pi(X, u)$ and $\Pi(X, v)$ are isomorphic whenever X is connected. Indeed, let $P\colon u \to v$ be an arbitrary walk. Then $\tilde{u} \mapsto \tilde{u} \,.\, P$ is a bijection $\mathrm{fib}_u \to \mathrm{fib}_v$, and $W \mapsto P^- W P$ is an isomorphism $\Pi(X, u) \to \Pi(X, v)$. The equation $(\tilde{u} \,.\, W) \,.\, P = (\tilde{u} \,.\, P) \,.\, P^- W P$ says that the two mappings establish an isomorphism of actions.

Theorem 4.5. *Let $\wp\colon \tilde{X} \to X$ be a covering projection of general digraphs, where X is connected. Then:*

(i) *The connected components of $\tilde{X}$ are in bijective correspondence with the orbits of the action of $\Pi(X, v_0)$ on the fibre fib_{v_0} via unique walk lifting. In particular, $\tilde{X}$ is connected if and only if the action of $\Pi(X, v_0)$ is transitive.*

(ii) *Let $\tilde{v} \in \mathrm{fib}_{v_0}$. Then $\wp\colon \Pi(\tilde{X}, \tilde{v}) \to \Pi(X, v_0)$ is a monomorphism, and the stabilizer of the action of $\Pi(X, v_0)$, which consists precisely of all those closed walks at v_0 that lift as closed walks at $\tilde{v}$, is equal to $\Pi(X, v_0)_{\tilde{v}} = \wp(\Pi(\tilde{X}, \tilde{v}))$.*

Proof. To prove (i), suppose that $\tilde{X}$ is connected, and let $\tilde{v} \in \mathrm{fib}_{v_0}$ be an arbitrary vertex. If $\tilde{u} \in \mathrm{fib}_{v_0}$ is another vertex, then there exists a walk $\tilde{P}\colon \tilde{v} \to \tilde{u}$ in $\tilde{X}$. Let P be its projection in X. Clearly, $P \in \Pi(X, v_0)$, and $\tilde{u} = \tilde{v} \centerdot P$. Hence the action of $\Pi(X, v_0)$ is transitive.

Conversely, suppose that the action of $\Pi(X, v_0)$ is transitive, and let $\tilde{v} \in \mathrm{fib}_{v_0}$ be an arbitrarily chosen base vertex in $\tilde{X}$. Further, let $\tilde{u}$ be an arbitrary vertex in $\tilde{X}$. As X is connected there exists a walk $Q\colon u \to v_0$, where $u = \wp(\tilde{u})$. By unique walk lifting we have $\tilde{u} \centerdot Q = \tilde{w} \in \mathrm{fib}_{v_0}$. Since the action of $\Pi(X, v_0)$ is transitive on fib_{v_0}, there exists a closed walk P at v_0 in X such that $\tilde{v} \centerdot P = \tilde{w}$. Thus, $\tilde{u} = \tilde{v} \centerdot PQ^{-}$. This is clearly enough to guarantee that $\tilde{X}$ is connected.

If the covering digraph is not connected, then fib_{v_0} is a disjoint union of subsets of fib_{v_0} such that each of these subsets consists precisely of those vertices in fib_{v_0} that belong to the same connected component. By the above, $\Pi(X, v_0)$ acts transitively on each of these subsets, and so it is clear that the connected components are in bijective correspondence with the orbits of the action of $\Pi(X, v_0)$. This proves (i).

In order to prove (ii), since the concatenation of two walks projects to the concatenation of the projected walks, there is an associated homomorphism $\wp\colon \Pi(\tilde{X}, \tilde{v}) \to \Pi(X, v_0)$ of fundamental monoids, where $\tilde{v} \in \mathrm{fib}_{v_0}$. It is clearly injective, by unique walk lifting. Of course $W \in \Pi(X, v_0)$ belongs to the stabilizer of $\tilde{v}$ if and only if the lift $\tilde{W}_{\tilde{v}}$ is a closed walk at $\tilde{v}$, and so the stabilizer is equal to $\wp(\Pi(\tilde{X}, \tilde{v}))$. This proves (ii). □

The next result states, roughly speaking, that the structural properties of the covering are closely related to the action of the fundamental monoid – in the sense that isomorphisms of coverings correspond to isomorphisms of monoid actions – however, as long as both projections are inverse-consistent. By abuse of notation we use the same symbol $\centerdot$ for different walk actions (instead of $\centerdot$ and, say, $\centerdot'$). This should cause no confusion.

Theorem 4.6. *Let X be a connected general digraph. Then the inverse-consistent covering projections $\wp\colon \tilde{X} \to X$ and $\wp'\colon \tilde{X}' \to X$ are isomorphic if and only if there exists an automorphism $\alpha\colon X \to X$ and a bijection $\tau\colon \mathrm{fib}_{v_0} \to \mathrm{fib}'_{\alpha(v_0)}$ such that (τ, α) maps the action of $\Pi(X, v_0)$ on fib_{v_0} isomorphically onto the action of $\Pi(X, \alpha(v_0))$ on $\mathrm{fib}'_{\alpha(v_0)}$. In particular, $\wp\colon \tilde{X} \to X$ and $\wp'\colon \tilde{X}' \to X$ are equivalent if and only if the actions of $\Pi(X, v_0)$ on fib_{v_0} and fib'_{v_0} are equivalent.*

Proof. Suppose first that there is an isomorphism $(\alpha, \tilde{\alpha})\colon \wp \to \wp'$ of covering projections. Take $v_0' = \alpha(v_0)$. Then α maps closed walks at v_0 bijectively to closed walks at v_0', and this induces an isomorphism of fundamental monoids $\alpha\colon \Pi(X, v_0) \to \Pi(X, v_0')$. Additionally, the restriction $\tau = \tilde{\alpha}|_{\mathrm{fib}_{v_0}}$ is a bijection $\mathrm{fib}_{v_0} \to \mathrm{fib}'_{v_0'}$. It follows that if W is a closed walk at v_0, the lifted walk starting at $\tilde{v} \in \mathrm{fib}_{v_0}$ is mapped to the walk that starts at $\tau(\tilde{v})$ and projects to $\alpha(W)$. In other words, we have $\tau(\tilde{v} \centerdot W) = \tau(\tilde{v}) \centerdot \alpha(W)$, and so the actions of fundamental monoids are isomorphic. Note that for the proof in this direction we do not need the assumption that the projections be inverse-consistent.

Additionally, observe that $\tilde{\alpha}$ is uniquely determined by the mapping of the "base fibre" $\tau\colon \mathrm{fib}_{v_0} \to \mathrm{fib}_{\alpha(v_0)}$. Namely, take an arbitrary vertex $\tilde{u}$, and let $P\colon u = \wp(\tilde{u}) \to v_0$ be an arbitrary walk. Then $\tau(\tilde{u} \centerdot P) = \tilde{\alpha}(\tilde{u}) \centerdot \alpha(P)$. Consequently, if $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ are two lifts of α that induce the same bijection τ, then $\tilde{\alpha}_1(\tilde{u}) \centerdot \alpha(P) = \tilde{\alpha}_2(\tilde{u}) \centerdot \alpha(P)$. Hence $\tilde{\alpha}_1(\tilde{u}) = \tilde{\alpha}_2(\tilde{u})$. As $\tilde{u}$ was arbitrary we have $\tilde{\alpha}_1 = \tilde{\alpha}_2$, as required.

Conversely, suppose that an automorphism $\alpha\colon X \to X$ is admissible for the actions of $\Pi(X, v_0)$ on fib_{v_0} and $\Pi(X, v_0')$ on $\mathrm{fib}'_{v_0'}$, where $v_0' = \alpha(v_0)$: there exists a bijection $\tau\colon \mathrm{fib}_{v_0} \to \mathrm{fib}'_{v_0'}$ such that $\tau(\tilde{v} \centerdot W) = \tau(\tilde{v}) \centerdot \alpha(W)$ for all $\tilde{v} \in \mathrm{fib}_{v_0}$ and $W \in \Pi(X, v_0)$.

Define a mapping $\tilde{\alpha}\colon \tilde{X} \to \tilde{X}'$ as follows. For an arbitrary vertex $\tilde{u}$ in $\tilde{X}$ let $u = \wp(\tilde{u})$, and let $P\colon u \to v_0$ be an arbitrary walk. Then set

$$\tilde{\alpha}(\tilde{u}) = \tau(\tilde{u} \centerdot P) \centerdot \alpha(P)^{-1}. \tag{4.1}$$

So $\wp'\tilde{\alpha} = \alpha\,\wp$ on vertices by construction. As for the mapping of darts, let $\tilde{x}$ be an arbitrary dart in $\tilde{X}$ and $x = \wp(\tilde{x})$. The image $\tilde{\alpha}(\tilde{x})$ is set to be the unique dart that starts at $\tilde{\alpha}(\mathrm{beg}\,\tilde{x})$ and projects to $\alpha(x)$. Thus, $\wp'\,\tilde{\alpha} = \alpha\,\wp$ holds on darts by construction as well.

We now need to prove several things: that the mapping of vertices is well defined, that it is a bijection (which immediately implies that the mapping of darts is well defined and a bijection as well), and that $\tilde{\alpha}$ is a general digraph isomorphism; that $(\alpha, \tilde{\alpha})$ is an isomorphism of covering projections then follows by construction.

To prove that $\tilde{\alpha}$ is well defined on the set of vertices, let $Q\colon u \to v_0$ be another arbitrary walk. Then $\tau(\tilde{u} \centerdot Q) = \tau((\tilde{u} \centerdot P) \centerdot P^- Q) = \tau(\tilde{u} \centerdot P) \centerdot \alpha(P^- Q)$. Hence

$$\begin{aligned}
\tau(\tilde{u} \centerdot Q) \centerdot \alpha(Q)^{-1} &= (\tau(\tilde{u} \centerdot P) \centerdot \alpha(P^- Q)) \centerdot \alpha(Q)^{-1}\\
&= \tau(\tilde{u} \centerdot P) \centerdot \alpha(P^- Q)\alpha(Q)^{-1}\\
&= \tau(\tilde{u} \centerdot P) \centerdot \alpha(P^-)\alpha(Q)\alpha(Q)^{-1}\\
&= \tau(\tilde{u} \centerdot P) \centerdot \alpha(P)^{-1}.
\end{aligned}$$

Therefore the mapping $\tilde{\alpha}$ is well defined on the set of vertices as it does not depend on the choice of a walk $u \to v_0$. As for the mapping of darts, let $\tilde{x}$ be an arbitrary dart, and let $x = \wp(\tilde{x})$. Since the image $\alpha(x)$ of x is uniquely defined, we set $\tilde{\alpha}(\tilde{x})$ to be the unique dart that starts at $\tilde{\alpha}(\mathrm{beg}\,\tilde{x})$ and projects to $\alpha(x)$. The image $\tilde{\alpha}(\tilde{x})$ is well defined, and $\tilde{\alpha}$ is a bijection on the set of darts as well.

Moreover, $\tilde{\alpha}$ is a digraph isomorphism. Indeed. By construction we have $\mathrm{beg}\,\tilde{\alpha}(\tilde{x}) = \tilde{\alpha}(\mathrm{beg}\,\tilde{x})$. To prove that $\mathrm{end}\,\tilde{\alpha}(\tilde{x}) = \tilde{\alpha}(\mathrm{end}\,\tilde{x})$, let $\mathrm{beg}\,\tilde{x} = \tilde{u}$ and $\mathrm{end}\,\tilde{x} = \tilde{w}$, let $u = \wp(\tilde{u})$ and $w = \wp(\tilde{w})$, and let $x = \wp(\tilde{x})$. Then, for a walk $P\colon u \to v_0$ we have

$$\begin{aligned}
\tilde{\alpha}(\tilde{w}) &= \tau(\tilde{w} \centerdot x^- \cdot P) \centerdot \alpha(x^- \cdot P)^{-1}\\
&= \tau(\tilde{u} \centerdot P) \centerdot \alpha(P)^{-1} \cdot \alpha(x^-)^{-1}\\
&= \tilde{\alpha}(\tilde{u}) \centerdot \alpha(x^-)^{-1}\\
&= \tilde{\alpha}(\tilde{u}) \centerdot \alpha(x)^{+}\\
&= \mathrm{end}\,\tilde{\alpha}(\tilde{x}).
\end{aligned}$$

Note that the last line in the above computation follows because $\tilde{\alpha}(\tilde{u}) = \tilde{\alpha}(\text{beg}\,\tilde{x}) = \text{beg}\,\tilde{\alpha}(\tilde{x})$ and $\wp'\tilde{\alpha}(\tilde{x}) = \alpha(\wp(\tilde{x})) = \alpha(x)$. This shows that $\tilde{\alpha}(\text{end}\,\tilde{x}) = \text{end}\,\tilde{\alpha}(\tilde{x})$.

To finish the proof we need to see that $\tilde{\alpha}$ and $\tilde{\alpha}^{-1}$ commute with the partial involution. At this point we do need the assumption that $\wp$ and $\wp'$ be inverse-consistent. Let $\tilde{x}$ and $\tilde{y}$ be a pair of inverse darts in $\tilde{X}$, so $x = \wp(\tilde{x})$, $y = \wp(\tilde{y})$ and consequently, $\alpha(x)$ and $\alpha(y)$ are inverse darts as well. Now $\text{beg}\,\tilde{\alpha}(\tilde{x}) = \text{end}\,\tilde{\alpha}(\tilde{y})$ and $\text{end}\,\tilde{\alpha}(\tilde{x}) = \text{beg}\,\tilde{\alpha}(\tilde{y})$ because $\text{beg}\,\tilde{x} = \text{end}\,\tilde{y}$ and $\text{end}\,\tilde{x} = \text{beg}\,\tilde{y}$, and since $\tilde{\alpha}$ commutes with beg and end. Therefore, $\tilde{\alpha}(\tilde{x})$ and $\tilde{\alpha}(\tilde{y})$ are inverse darts because $\wp'$ is inverse-consistent. It follows that $\tilde{\alpha}$ commutes with the partial involution $^{-1}$. The same reasoning applies to $\tilde{\alpha}^{-1}$ because $\tilde{\alpha}^{-1}$ also commutes with beg and end, and the projection $\wp$ is assumed inverse-consistent.

Finally, by construction we have that $\alpha\,\wp = \wp'\,\tilde{\alpha}$, and so the pair $(\alpha, \tilde{\alpha})\colon \wp \to \wp'$ is an isomorphism of covering projections. This proves the first claim in the theorem. The second claim is just a special case with $\alpha = \text{id}$, and the proof follows. □

Remark 4.7. Theorem 4.6 is quite fundamental. Although the nesessary and sufficent condition for the existence of an isomorphism $(\alpha, \tilde{\alpha})\colon \wp \to \wp'$ is expressed in a rather elegant form, testing separately for each automorphism $\alpha\colon X \to X$ whether there is a bijection $\tau\colon \text{fib}_{v_0} \to \text{fib}'_{\alpha(v_0)}$ such that $\tau(\tilde{v}\,.\,W) = \tau(\tilde{v})\,.\,\alpha(W)$ holds for all closed walks at v_0 is inherently difficult. As it is currently stated, the problem is not even finite. But it can be made finite. A number of specific theorems dealing with this problem are just corollaries to Theorem 4.6, mostly geared at facilitating its application in concrete cases. But the level of computational difficulty does not disappear.

Let now make another assumption: connectivity of the covering. Assuming connectivity of a base general digraph enabled us to study coverings via fundamental monoids, at least the inverse-consistent ones. By requiring connectivity of the respective covering we also do not loose (much) on generality, however, we gain a lot.

Then the fundamental monoid acts transitively, which has a number of significant consequences. For instance, similarly as with transitive actions of groups the existence of an isomorphism between coverings can be tested by considering the stabilizer of just one vertex. This follows from Theorem 4.6 and the fact that the actions of the fundamental monoids are permutational. See the remarks in Subsection 4.3.

Corollary 4.8. *Let $\wp\colon \tilde{X} \to X$ and $\wp'\colon \tilde{X}' \to X$ be connected inverse-consistent covering projections. Further, let $\tilde{v} \in \text{fib}_v$ in $\tilde{X}$ and $\tilde{u} \in \text{fib}'_u$ in $\tilde{X}'$. Then there exists an isomorphism $(\alpha, \tilde{\alpha})\colon \wp \to \wp'$ such that $u = \alpha(v)$ and $\tilde{u} = \tilde{\alpha}(\tilde{v})$ if and only if α maps the stabilizer $\Pi(X, v)_{\tilde{v}}$ isomorphically onto the stabilizer $\Pi(X, u)_{\tilde{u}}$. In particular, $\wp$ and $\wp'$ are equivalent if and only if $\Pi(X, v)_{\tilde{v}} = \Pi(X, u)_{\tilde{u}}$, for some $\tilde{v} \in \text{fib}_v$ and $\tilde{u} \in \text{fib}'_v$.*

As a consequence of the unique walk lifting, connectivity of the covering trivially implies the following proposition.

Proposition 4.9. *Let $\wp\colon \tilde{X} \to X$ be a covering projection of connected general digraphs. Then the action of $\text{CT}(\wp)$ is semiregular. Additionally, each lift of an automorphism, if it exists, is uniquely determined by the mapping of just one vertex.*

4.4 Action of the weak fundamental group

Proposition 4.10. *Let $\wp\colon \tilde{X} \to X$ be a covering projection. If two walks $W, W'\colon u \to v$ in X are weakly homotopic, then their lifts $\tilde{W}_{\tilde{u}}$ and $\tilde{W}'_{\tilde{u}}$ are weakly homotopic.*

Proof. Consider an arbitrary walk $W\colon u \to v$ in X. Recall that the walk WW^- acts trivially, that is, $\tilde{u} \cdot WW^- = \tilde{u}$, for $\tilde{u} \in \mathrm{fib}_u$. As an immediate consequence it follows that transforming W by a series of deletions and/or insertions of subwalks of the form $x^\epsilon\, x^{-\epsilon}$ reflects in the same series of transformations in any of its lifts $\tilde{W}$. □

If $W, W'\colon u \to v$ are weakly homotopic, then $\tilde{u} \,\centerdot\, W = \tilde{u} \,\centerdot\, W'$. Hence there is a well defined action of the weak homotopy group $\pi_w(X, v_0)$ on fib_{v_0}. In fact, $\pi_w(X, v_0)$ acts "in the same way" as $\Pi(X, v_0)$, and can be viewed as induced via the natural epimorphism $\Pi(X, v_0) \to \pi_w(X, v_0)$. Also, if $\alpha\colon X \to X'$ is an isomorphism, then weakly homotopic walks are mapped to weakly homotopic walks, and the induced isomorphism $\alpha_*\colon \pi_w(X, v_0) \to \pi_w(X', \alpha(v_0))$ makes the following diagram

$$\begin{array}{ccc} \Pi(X, v_0) & \xrightarrow{\ \alpha\ } & \Pi(X', \alpha(v_0)) \\ \big\downarrow & & \big\downarrow \\ \pi_w(X, v_0) & \xrightarrow[\ \alpha_*\ ]{} & \pi_w(X', \alpha(v_0)) \end{array}$$

commutative. Note that α is admissible for the action of the fundamental monoids if and only if α_* is admissible for the action of the weak fundamental groups. That is, there exists a bijection $\tau\colon \mathrm{fib}_{v_0} \to \mathrm{fib}'_{\alpha(v_0)}$ such that (τ, α) is an isomorphism of monoid actions if and only if (τ, α_*) is an isomorphism of actions of the weak fundamental groups. Also, orbits and stabilizers of weak fundamental groups and fundamental monoids correspond nicely. We remark again that the above facts hold because the actions of the fundamental monoids are permutational. See the remarks in Subsection 4.3. Consequently, all the results of Subsection 4.3 can be restated in terms of weak homotopy groups. Details are left the reader.

On the one hand, at the conceptual level we have chosen to give proofs using monoids simply because the proofs appear to be technically easier. On the other hand, replacing monoids with groups is of interest because weak fundamental groups are finitely generated whereas the fundamental monoids are not. This is relevant for doing computations in concrete examples. In particular, to see whether $\alpha_*\colon \pi_w(X, v_0) \to \pi_w(X', \alpha(v_0))$ "extends" to an isomorphism of actions we need a bijection $\tau\colon \mathrm{fib}_{v_0} \to \mathrm{fib}'_{\alpha(v_0)}$ such that

$$\tau(\tilde{v} \,\centerdot\, [W]_w) = \tau(\tilde{v}) \,\centerdot\, \alpha_*([W]_w)$$

holds. Such a system of "action equations" can be further simplified by only considering the generators of $\pi_w(X, v_0)$, in which case the above lifting condition eventually becomes finite. Technically we only need to consider the weak fundamental closed walks at v_0. These can be constructed using a genuine spanning tree arising from a spanning tree in $\mathrm{sp}(X)$.

4.5 Action of the fundamental group

Proposition 4.11. *Let $\wp\colon \tilde{X} \to X$ be a covering projection. Then homotopic walks $u \to v$ lift to homotopic walks at $\tilde{u} \in \mathrm{fib}_u$ if and only if $\wp$ is homogenous.*

Proof. We first show that congruent walk lift to congruent walks if an only if $\wp$ is homogeneous.

Indeed. Let congruent walks lift to congruent walks. Clearly, it is enough to assume the claim for walks of length 1. Consider a walk $x^+ \colon u \to v$ in X, and let $\tilde{x}^+ \colon \tilde{u} \to \tilde{v}$ in $\tilde{X}$ be its lift at $\tilde{u} \in \mathrm{fib}_u$. The opposite walk $x^- \colon v \to u$ lifts to $\tilde{x}^- \colon \tilde{v} \to \tilde{u}$. If the dart x has an inverse, let $\tilde{y}^+ \colon \tilde{v} \to \tilde{w}$ be the lift of $(x^{-1})^+ \colon v \to u$ at $\tilde{v}$. Now, $(x^{-1})^+ \colon v \to u \equiv x^- \colon v \to u$ implies $\tilde{y}^+ \colon \tilde{v} \to \tilde{w} \equiv \tilde{x}^- \colon \tilde{v} \to \tilde{u}$, hence $\tilde{w} = \tilde{u}$ and $\tilde{y}$ must be the inverse of $\tilde{x}$. The proof is analogous if a walk in X is of the form $x^- \colon u \to v$. It lifts to $\tilde{x}^- \colon \tilde{u} \to \tilde{v}$ and its inverse $x^+ \colon v \to u$ to $\tilde{x}^+ \colon \tilde{v} \to \tilde{u}$. Let $\tilde{y}^+ \colon \tilde{u} \to \tilde{w}$ be the lift of $(x^{-1})^+ \colon u \to v$ at $\tilde{u}$. Since $(x^{-1})^+ \colon u \to v \equiv x^- \colon u \to v$ it follows that $\tilde{y}^+ \colon \tilde{u} \to \tilde{w} \equiv \tilde{x}^- \colon \tilde{u} \to \tilde{v}$. Thus, $\tilde{w} = \tilde{v}$ and $\tilde{y} = \tilde{x}^{-1}$.

This works nicely with links and loops. Some care is needed if $x^+ \colon u \to u$ is a traversal of a semiedge, but the above reasoning goes through as well. Let $\tilde{x}^+ \colon \tilde{u} \to \tilde{v}$ and $\tilde{y}^+ \colon \tilde{v} \to \tilde{w}$ be the lifts of a semiedge $x^+ \colon u \to u$. As $\tilde{x}^- \colon \tilde{v} \to \tilde{u}$ is the lift of $x^- \colon u \to u \equiv x^+ \colon u \to u$ we must have that $\tilde{y}^+ \colon \tilde{v} \to \tilde{w} \equiv \tilde{x}^- \colon \tilde{v} \to \tilde{u}$, which again implies $\tilde{w} = \tilde{u}$ and $\tilde{y} = \tilde{x}^{-1}$. We conclude that no dart in $\tilde{X}$ without an inverse is mapped to an dart in X that has an inverse. Hence $\wp$ is homogeneous by definition.

For the converse, let two congruent walks lift to walks that are not congruent. Again, it is enough to consider two congruent walks of length 1, say $x^+ \colon u \to v \equiv (x^{-1})^- \colon u \to v$, that lift to non-congruent walks $\tilde{x}^+ \colon \tilde{u} \to \tilde{v}$ and $\tilde{y}^- \colon \tilde{w} \to \tilde{v}$. Then $\tilde{x}$ has no inverse, for otherwise $(\tilde{x}^{-1})^- \colon \tilde{u} \to \tilde{v}$ is mapped to $(x^{-1})^- \colon u \to v$; so $\tilde{y} = \tilde{x}^{-1}$, $\tilde{w} = \tilde{u}$, and $\tilde{x}^+ \colon \tilde{u} \to \tilde{v} \equiv \tilde{y}^- \colon \tilde{u} \to \tilde{v}$, a contradiction.

To finish the proof, let $W, W' \colon u \to v$ be homotopic walks, and let $\tilde{W}_{\tilde{u}}$ and $W'_{\tilde{u}}$ be their lifts at $\tilde{u} \in \mathrm{fib}_u$. The reduced walks $\bar{W}$ and $\bar{W}'$ are congruent by the definition of homotopy. When performing a reduction in X we can clearly do the analogous reductions in $\tilde{X}$ as long as we cancel out inverses – not so, however, if a reduction step includes congruence. That is possible, as we have shown above, if and only if darts that have no inverse are only mapped to darts that have no inverse. Consequently, homotopy does lift if and only if the projection $\wp$ is homogeneous. □

By Proposition 4.11, there exists an action of the fundamental group $\pi(X, v_0)$ by unique walk lifting. In turn, for homogeneous coverings, all results of Subsection 4.3 can be reformulated in terms of fundametal groups, just like in the context of graphs or in topology. We leave this to the reader.

Additionally, the next proposition might be of interest with inverse-consistent coverings that are not homogeneous since the covering between the corresponding spans is always homogeneous, and we can work with the fundamental group of the span.

Proposition 4.12. *Let $\wp \colon \tilde{X} \to X$ and $\wp' \colon \tilde{X}' \to X$ be inverse-consistent covering projections of a connected general digraph X. Then an automorphism $\alpha \colon X \to X$ lifts to an isomorphism $\tilde{X} \to \tilde{X}'$ if and only if $\alpha \colon \mathrm{sp}(X) \to \mathrm{sp}(X)$ lifts to an isomorphism $\mathrm{sp}(\tilde{X}) \to \mathrm{sp}(\tilde{X}')$. In particular, inverse-consistent coverings $\tilde{X} \to X$ and $\tilde{X}' \to X$ are equivalent if and only if $\mathrm{sp}(\tilde{X}) \to \mathrm{sp}(X)$ and $\mathrm{sp}(\tilde{X}') \to \mathrm{sp}(X)$ are equivalent.*

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{\alpha}} & \tilde{X}' \\ {\scriptstyle\wp}\downarrow & & \downarrow{\scriptstyle\wp'} \\ X & \xrightarrow[\alpha]{} & X \end{array} \qquad\qquad \begin{array}{ccc} \mathrm{sp}(\tilde{X}) & \xrightarrow{\tilde{\alpha}} & \mathrm{sp}(\tilde{X}') \\ {\scriptstyle\wp}\downarrow & & \downarrow{\scriptstyle\wp'} \\ \mathrm{sp}(X) & \xrightarrow[\alpha]{} & \mathrm{sp}(X) \end{array}$$

Proof. The projection $\mathrm{sp}(X) \to X$ is a 1-fold covering, and not inverse-consistent unless $\mathrm{sp}(X) = X$. Thus, an automorphism $\alpha\colon X \to X$ clearly lifts to $\alpha\colon \mathrm{sp}(X) \to \mathrm{sp}(X)$. It is also evident that if α lifts to $\tilde{\alpha}\colon \tilde{X} \to \tilde{X}'$, then $\tilde{\alpha}\colon \mathrm{sp}(\tilde{X}) \to \mathrm{sp}(\tilde{X})$ is the lift of $\alpha\colon \mathrm{sp}(X) \to \mathrm{sp}(X)$.

For the converse, suppose that $\alpha\colon \mathrm{sp}(X) \to \mathrm{sp}(X)$ lifts to $\tilde{\alpha}\colon \mathrm{sp}(\tilde{X}) \to \mathrm{sp}(\tilde{X}')$. Then $\tilde{\alpha}$ is a bijection $\tilde{X} \to \tilde{X}'$ on vertices and darts that commutes with the functions beg and end. What we must show is that $\tilde{\alpha}$ and $\tilde{\alpha}^{-1}$ commute with the involution $^{-1}$. To this end we need the assumption that the coverings are inverse-consistent. So let $\tilde{x}$ and $\tilde{y}$ be inverse darts in $\tilde{X}$, and let $x = \wp(\tilde{x})$ and $y = \wp(\tilde{y})$ be their projections, which are inverse to each other. Then $\alpha(x)$ and $\alpha(y)$ are also inverses, and are the projections of $\tilde{\alpha}(\tilde{x})$ and $\tilde{\alpha}(\tilde{y})$. Since $\mathrm{beg}\,\tilde{x} = \mathrm{end}\,\tilde{y}$ and $\mathrm{end}\,\tilde{x} = \mathrm{beg}\,\tilde{y}$ we have $\mathrm{beg}\,\tilde{\alpha}(\tilde{x}) = \mathrm{end}\,\tilde{\alpha}(\tilde{y})$ and $\mathrm{end}\,\tilde{\alpha}(\tilde{x}) = \mathrm{beg}\,\tilde{\alpha}(\tilde{y})$. But then $\tilde{\alpha}(\tilde{x})$ and $\tilde{\alpha}(\tilde{y})$ are inverses because $\wp'$ is inverse-consistent. This shows that $\tilde{\alpha}$ commutes with $^{-1}$. The same holds for $\tilde{\alpha}^{-1}$, only that this time we need that $\wp$ is inverse-consistent.

The last claim in the proposition is a trivial consequence of what we have just proved. □

4.6 Combinatorialization by voltages

Let $X = (V, D, \mathrm{bd}, ^{-1})$ be a general digraph, Γ a group acting on the right on a *labelling set* F, with $\chi\colon \Gamma \to \mathrm{Sym}_r F$ denoting the corresponding permutation representation. The group action is denoted $j \cdot g$, and the corresponding induced permutation is $(j)\chi_g = j^{\chi_g} = j \cdot g$. The group Γ is called the *voltage group* while F is the *abstract fibre*. Further, let $\zeta\colon D \to \Gamma$ be a *voltage function* that assigns to each dart $x \in D$ its *voltage* $\zeta_x \in \Gamma$. The *derived general digraph* $X \times_{\Gamma,\zeta} F$ has vertex set $V \times F$ and $D \times F$ as the dart-set. The functions beg and end are defined by

$$\mathrm{beg}(x, j) = (\mathrm{beg}\,x, j) \text{ and } \mathrm{end}(x, j) = (\mathrm{end}\,x, j \cdot \zeta_x),$$

while the partial involution is defined for darts (x, j) for which x^{-1} exists and $j \cdot \zeta_x \zeta_{x^{-1}} = j$. Then $(x, j)^{-1} = (x^{-1}, j \cdot \zeta_x)$. The following result is easily proved and is left to the reader.

Proposition 4.13. *The mapping $\wp_{\Gamma,\zeta}\colon X \times_{\Gamma,\zeta} F \to X$ given projections onto the first coordinate $(u, j) \mapsto u$ and $(x, j) \mapsto x$ is an inverse-consistent covering projection. Moreover, if we require that $\zeta_{x^{-1}} = \zeta_x^{-1}$ (this is ment to denote the inverse group element, not the inverse function) holds whenever x^{-1} exists, then the corresponding covering is homogeneous.*

The mapping $\wp_{\Gamma,\zeta}$ is called the *derived covering projection*. So we have a nice combinatorial way for constructing inverse-consistent coverings. Some special cases are readily at hand.

- First, take the right symmetric group $\Gamma = \mathrm{Sym}_r F$ and its natural action on F. Such voltages are known as *permutation voltages*. Usually we take $F = [n] = \{1, 2, \ldots, n\}$. To simplify the notation we write $\wp_\zeta\colon X \times_\zeta [n] \to X$. See Figure 2.

- Second, let Γ act by right multiplication on the set of right cosets $F = \Delta|\Gamma$ of some subgroup $\Delta \leq \Gamma$. The corresponding voltages in Γ are the *coset voltages*, known as *relative voltages*. See Figure 3.

- Third, by taking Δ to be trivial group we obtain the *regular voltages*, where the group Γ acts regularly on $F = \Gamma$ by right multiplication on itself. This kind of voltages are also known as *ordinary* or *Cayley voltages*. To simplify the notation we write $\wp_\zeta \colon X \times_\zeta \Gamma \to X$. See Figure 4.

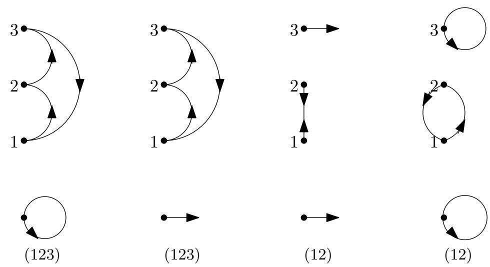

Figure 2: Permutation voltages.

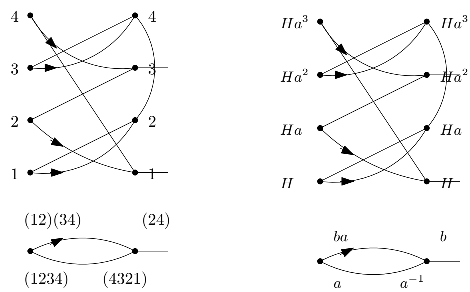

Figure 3: Permutation and coset voltages. The coset action is that of the dihedral group $D_4 = \langle a, b \mid a^4 = (ab)^2 = 1 \rangle$ acting on the right cosets of the subgroup $H = \langle b \rangle$. For simplicity, pairs of inverse darts are not shown.

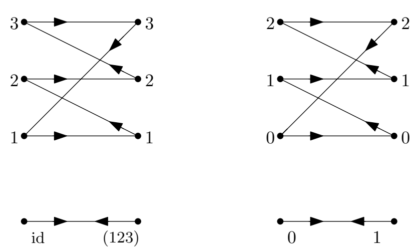

Figure 4: Permutation and regular voltages. The regular action is that of $\mathbb{Z}_3$ acting on itself.

Theorem 4.14. *Let $\wp\colon \tilde{X} \to X$ be an inverse-consistent covering projection. Then there exists a derived covering $\wp_\zeta\colon X \times_{\Gamma,\zeta} F \to X$ such that the following diagram*

$$\begin{array}{ccc} \tilde{X} & \longrightarrow & X \times_{\Gamma,\zeta} F \\ {\scriptstyle\wp}\big\downarrow & & \big\downarrow{\scriptstyle\wp_{\Gamma,\zeta}} \\ X & \xrightarrow[\text{id}]{} & X. \end{array}$$

is commutative. Moreover, if the projection is homogeneous, then we may assume that $\zeta_{x^{-1}} = \zeta_x^{-1}$ holds whenever a dart x has an inverse.

The formal proof of Theorem 4.14 is easy although a bit tedious. All what one needs to do is the following. Arbitrarily label the fibres by a labelling set F and then take, for each dart $x\colon u \to v$, the permutation $\zeta_x\colon F \to F$ induced by the bijection $\tau_x\colon \mathrm{fib}_u \to \mathrm{fib}_v$ to obtain the corresponding permutation voltage by which we can reconstruct the respective dart-fibre.

In constructing covering projections as in Proposition 4.13 we had all the freedom to choose the voltage action arbitrarily. So how much freedom do we have in choosing the voltage action when reconstructing a given covering? Quite a lot, although there are restrictions since the permutation representation of the action must comply with the permutation voltage action that reconstructs the cover as described above. An alternative choice is to use coset voltages, particularly when the covering is connected. Explicit conversion between permutation voltages and coset voltages is left to the reader, see Figure 3. In contrast, the regular voltages can only be used with special kind of covering projections, called regular coverings. For a detailed treatment of regular covers of general digraphs see [40].

Moreover, observe that a covering can be reconstructed by non-isomorphic voltage actions. But of course, if $(r,\phi)\colon (F,\Gamma) \to (F',\Gamma')$, $r(j\cdot g) = \tau(j)\cdot\phi(g)$, is an isomorphism of actions, then ζ and $\zeta' = \phi\zeta$ give rise to equivalent coverings.

In view of Theorem 4.14 we can now indeed confirm that all relevant problems regarding inverse-consistent coverings can be without loss of generality studied combinatorially in terms of voltages. To rephrase all results of previous sections in terms of voltages we first extend the voltage function $\zeta\colon D \to \Gamma$ to a function defined on walks. The *voltage of a walk W*, denoted $\zeta(W)$ or ζ_W, is defined recursively as follows:

$$\zeta(u) = 1,$$
$$\zeta_{x^+} = \zeta_x, \qquad \zeta_{x^-} = \zeta_x^{-1},$$
$$\zeta_{WW'} = \zeta_W \zeta_{W'}.$$

By induction on the length of the walk it follows that opposite walks receive inverse voltages, $\zeta_{W^-} = \zeta_W^{-1}$, and that any two weakly homotopic walks have the same voltage. We also note the following. Let T be a genuine spanning tree in X. Two walks $W, W'\colon u \to v$ are *homotopic within T* if the homotopic changes transforming W to W' are performed exclusively on the darts of T. Let $W \in \Pi(X, v_0)$ traverse (in this order) the signed darts $x_1^{\epsilon_1}, x_2^{\epsilon_2}, \ldots, x_n^{\epsilon_n}$ not in T, and let $W_1, W_2, \ldots, W_n$ be the respective (weak) fundamental closed walks defined by T and these cotree darts. Then W is homotopic within T to $W_1 W_2 \cdots W_n$. In particular, W and $W_1 W_2 \cdots W_n$ have the same voltage since they are weakly homotopic. Thus,

$$\zeta_W = \zeta_{W_1} \zeta_{W_2} \cdots \zeta_{W_n}.$$

Next, in the covering $X \times_{\Gamma,\zeta} F \to X$ the walk action on fibres via unique walk lifting is "simulated" by the voltage action. Specifically, a walk $W\colon u \to v$ lifts by the rule

$$(u, j) \centerdot W = (v, j \cdot \zeta_W).$$

The mapping $\zeta^{v_0}\colon \Pi(X, v_0) \to \Gamma$, $W \mapsto \zeta_W$, defines a monoid homomorphism into the voltage group. Its image $\Gamma^{v_0} \leq \Gamma$ is called the *local voltage group* at v_0. The local voltage group can be seen as the image of the group homomophism (denoted by the same letter for simplicity)

$$\zeta^{v_0}\colon \pi_w(X, v_0) \to \Gamma^{v_0}.$$

If the voltage function satisfies $\zeta_{x^{-1}} = \zeta_x^{-1}$ (whenever x^{-1} exists), then two homotopic walks have the same voltage. As this can be assumed when the covering is homogeneous, we can see the local voltage group as the image of the group homomorphism $\zeta^{v_0}\colon \pi(X, v_0) \to \Gamma^{v_0}$. We prefer to look at Γ^{v_0} as the image of the $\pi_w(X, v_0)$ or $\pi(X, v_0)$ since these groups, in contrast with $\Pi(X, v_0)$, are finitely generated. Since $\zeta_W = \zeta_{W_1}\zeta_{W_2}\cdots\zeta_{W_n}$, the local group Γ^{v_0} is generated by the voltages of the weak fundamental closed walks at v_0, or the fundamental closed walks at v_0 whenever the covering is homogeneous. We summarize the above discussion in Proposition 4.15.

Proposition 4.15. *The abstract action of Γ^{v_0} (as a subaction of Γ) is equivalent to the action of Γ^{v_0} induced by the action of $\Pi(X, v_0)$ via unique walk lifting, or rather, to the action induced by $\pi_w(X, v_0)$. For a homogeneous covering it is equivalent to the action induced by $\pi(X, v_0)$.*

Since the local group simulates the action of the fundamental monoid we can conveniently rephrase previous results about orbits, stabilizers, and isomorphism of derived covering projections in terms of voltages. We first rephrase Theorem 4.5 dealing with the number of components and the stabilizers of the walk action.

Corollary 4.16. *Let $\wp_{\Gamma,\zeta}\colon X \times_{\Gamma,\zeta} F \to X$ be a derived covering projection. Then:*

(i) *The connected components of the covering digraph are in bijective correspondence with the orbits of Γ^{v_0} in its action on F. In particular, the covering is connected if and only if Γ^{v_0} acts transitively.*

(ii) *Closed walks at the vertex (v_0, j) are in bijective correspondence with closed walks at v_0 whose voltages belong to the stabilizer $\Gamma_j^{v_0}$, in short: $W \in \Pi(X, v_0)_{(v_0,j)} \Leftrightarrow \zeta_W \in \Gamma_j^{v_0}$.*

Corollary 4.16 has a number of specific results for particular voltage actions, with permutation and regular voltage actions as the most interesting cases. For instance:

- In the case of permutation voltages the covering $\wp_\zeta\colon X \times_\zeta [n] \to X$ is connected if and only if the local group is a transitive subgroup of $\mathrm{Sym}_r\,[n]$. Algorithms for finding stabilizers of permutation groups are well known, see [50].
- Consider a regular voltage action. Then a covering $\wp_\zeta\colon X \times_\zeta \Gamma \to X$ is connected if and only if the local group is equal to the voltage group, $\Gamma^{v_0} = \Gamma$. This is evident since no proper subgrup of a regular group can act transitively. Stated differently, the covering is connected if and only if the voltage group is generated by the voltages assigned to the weak fundamental closed walks at v_0.

- Suppose that a covering constructed via regular voltages is disconnected. Then the local group $\Gamma^{v_0} < \Gamma$ acts intransitively with trivial stabilizers. Thus, the orbits are all of the same size, and so their number is equal to the index $[\Gamma : \Gamma^{v_0}]$. In fact, the orbits are precisely the left cosets of Γ^{v_0} in Γ, and correspond to the connected components of the covering.

As for the isomorphism problem dealt with in Theorem 4.6, recall that inverse-consistent covering projections are isomorphic if and only if an automorphism of the base general digraph gives rise to an isomorphism of the actions of the fundamental monoids. In view of Proposition 4.15, the actions of $\Pi(X, v_0)$ can be substituted by the actions of the local groups on the abstract fibres. Thus, the necessary and sufficient lifting condition can be expressed by a finite system of action equations over the local voltage groups. By abuse of notation, the two voltage actions below are indicated by the same dot sign for simplicity.

Corollary 4.17. *Let $\wp_{\Gamma,\zeta}\colon X \times_{\Gamma,\zeta} F \to X$ and $\wp_{\Gamma',\zeta'}\colon X \times_{\Gamma',\zeta'} F' \to X$ be derived covering projections of a connected general digraph X. Then an automorphism $\alpha\colon X \to X$ lifts to an isomorphism $(\alpha, \tilde{\alpha})\colon \wp_{\Gamma,\zeta} \to \wp_{\Gamma',\zeta'}$ if and only there exists a bijection $\tau\colon F \to F'$ that satisfies the following system of action equations*

$$\tau(j \cdot \zeta_W) = \tau(j) \cdot \zeta'_{\alpha(W)}, \tag{4.2}$$

where W runs through the weak fundamental closed walks at v_0 (which can be replaced by fundamental closed walks when the covering is homogeneous).

Note that the voltage actions of Γ on F and Γ' on F' need not be isomorphic. It is only important that the local groups act isomorphically. But of course, in most cases we take two voltage functions ζ and ζ' with respect to the same voltage action of $\Gamma = \Gamma'$ on $F = F'$.

A bijection $\tau\colon F \to F'$ satisfying (4.2) describes the mapping of the base fibre $\tilde{\alpha}\colon \mathrm{fib}_{v_0} \mapsto \mathrm{fib}'_{\alpha(v_0)}$, $(v_0, j) \mapsto (\alpha(v_0), \tau(j))$. By Theorem 4.6, this uniquely determines the action of $\tilde{\alpha}$ on other vertices. Indeed, let $\tilde{\alpha}\colon (u, j) \mapsto (\alpha(u), \tau_u(j))$. If $P\colon u \to v_0$ is an arbitrary walk, then (4.1) rewrites as

$$\tau_u(j) = \tau(j \cdot \zeta_P) \cdot \zeta^{-1}_{\alpha(P)}. \tag{4.3}$$

From Corollary 4.17 a number of special results follow easily. We briefly mention a few.

- In the case of permutation voltages, consider the covering projections $\wp_\zeta\colon X \times_\zeta [n] \to X$ and $\wp_{\zeta'}\colon X \times_{\zeta'} [n] \to X$ defined by two permutation voltage functions ζ and ζ'. Then the necessary and sufficient lifting condition (4.2) rewrites as a system

$$\zeta'_{\alpha(W)} = \tau^{-1} \zeta_W \, \tau$$

  of permutation equations in the right symmetric group $\mathrm{Sym}_r\,[n]$, where W runs through the (weak) fundamental closed walks at v_0. The problem is known as the *simultaneous conjugacy problem*. This problem arises naturally in many areas of mathematics, see for instance [13, 17], as well as in other fields like computer science, chemistry, and even biology – because it is related to an important problem of deciding whether two objects from a given class of objects are structuraly equivalent.

- In particular, the covering projections $\wp_\zeta \colon X \times_\zeta [n] \to X$ and $\wp_{\zeta'} \colon X \times_{\zeta'} [n] \to X$ are equivalent if and only if $\zeta'_W = \tau^{-1}\zeta_W\,\tau$ holds for all (weak) fundamental closed walks at v_0. Hence the corresponding local groups are conjugate subgroups of $\mathrm{Sym}_r\,[n]$. But note that the condition $\zeta'_W = \tau^{-1}\zeta_W\,\tau$ requires much more than merely the requirement that the local groups be conjugate. There is a simple $O(n^2)$-time algorithm for solving the simultaneous conjugacy problem in $\mathrm{Sym}_r\,[n]$, first described by Fontet in 1977 [16]. See also [21, 55]. The first subquadratic algorithm was recently given in [6], although the problem whether there is an algorithm with running time $n^{2-\epsilon}$, for some $\epsilon > 0$, remains open.

- Still in particular, the group of covering transformations of $\wp_\zeta \colon X \times_\zeta [n] \to X$ is isomorphic to the centralizer, within $\mathrm{Sym}_r\,F$, of the local group. This is clear since an arbitrary covering transformation is uniquely defined by a permutation τ satisfying the system of permutation equations $\zeta_W = \tau^{-1}\zeta_W\,\tau$. A fast algorithm for computing the centralizer of a subgroup in a symmetric group has recently been developed by Požar [47].

- Let us compute $\mathrm{CT}_\wp$ in the case of regular voltages, $\wp_\zeta \colon X \times_\zeta \Gamma \to X$. Then (4.2) rewrites as $\tau(a\zeta_W) = \tau(a)\zeta_W$, for all $a \in \Gamma$ and $\zeta_W \in \Gamma^{v_0}$, in fact

$$\tau(ag) = \tau(a)g, \qquad \text{for all } a \in \Gamma \text{ and } g \in \Gamma^{v_0}. \tag{4.4}$$

If we assume, in addition, that the covering is connected, then $\Gamma^{v_0} = \Gamma$. The permutation $\tau \in \mathrm{Sym}_r\,\Gamma$ is uniquely defined by the mapping of a single point, and each covering transformation is uniquely determined by one such τ. Taking $a = 1$ we get $\tau(g) = \tau(1)g$, for all $g \in \Gamma$. Consequently, each covering transformation is uniquely defined by $\tau(1) \in \Gamma$. Hence $\mathrm{CT}_\wp$ is isomorphic to Γ, and its action is the action on Γ by left multiplication (while the voltage action is action by right multiplication). More precisely, for an arbitrary vertex u and a dart x, each covering transformation $\tilde{\mathrm{id}}_c$, where $c \in \Gamma$, acts on vertices and darts as

$$\tilde{\mathrm{id}}_c(u, g) = (u, cg) \text{ and } \tilde{\mathrm{id}}_c(x, g) = (x, cg),$$

respectively. This follows from (4.3) by taking into account that between any pair of vertices in X there exists a walk with trivial voltage. Indeed, take an arbitrary walk $P \colon u \to v_0$. Since $\Gamma^{v_0} = \Gamma$, there exists a closed walk W at v_0 with $\zeta_W = \zeta_P$, and so $Q = PW^- \colon u \to v_0$ has trivial voltage.

- Consider the regular voltage action again, where the derived covering $\wp_\zeta \colon X \times_\zeta \Gamma \to X$ is disconnected. Then $\Gamma^{v_0} < \Gamma$ acts semiregularly with the left cosets of Γ^{v_0} as orbits. A function $\tau \colon \Gamma \to \Gamma$ satisfying (4.4) maps a coset $a\Gamma^{v_0}$ bijectively onto $\tau(a)\Gamma^{v_0}$. Thus, in order for τ to be a bijection $\Gamma \to \Gamma$ we do the following. Let $[\Gamma : \Gamma^{v_0}] = k$. For fixed coset representatives $a_1, a_2, \ldots, a_k$ choose $\tau(a_1), \tau(a_2), \ldots, \tau(a_k) \in \Gamma$ in such a way that $\tau(a_j) \in a_{\pi(j)}\Gamma^{v_0}$, where π permutes the index set $[k] = \{1, 2, \ldots, k\}$. Then $\tau \colon \Gamma \to \Gamma$ is indeed a bijection since it permutes the left cosets. Clearly, all solutions of the respective system are obtained through all possible choices of π and $\tau(a_j)$, $j = 1, 2, \ldots, k$. There are $k!|\Gamma^{v_0}|^k$ solutions, which constitute a group that can be identified as the wreath product $\Gamma^{v_0} \wr \mathrm{Sym}\,[k]$. Moreover, this group is isomorphic to $\mathrm{CT}_\wp$.

Yet another approach in treating isomorphisms of coverings is based on replacing admissible isomorphsms between monoids (weak fundamental groups, fundamental groups) by admissible isomorphisms between quotient objects, that is, local groups. However, this approach requires some care since passing to "smaller monoids (groups) acting in the same way", information might be lost. See the remarks in Subsection 4.3.

In our present context we need to substitute an isomorphism $\alpha\colon \Pi(X, v_0) \to \Pi(X, \alpha(v_0))$ from Theorem 4.6 by an isomorphism between the local groups $\alpha^{\#_{v_0}}\colon \Gamma^{v_0} \to \Gamma'^{\alpha(v_0)}$ as shown in the following commutative diagram:

$$\begin{array}{ccc} \Pi(X, v_0) & \xrightarrow{\alpha} & \Pi(X, \alpha(v_0)) \\ \zeta^{v_0}\downarrow & & \downarrow \zeta'^{\alpha(v_0)} \\ \Gamma^{v_0} & \xrightarrow[\alpha^{\#_{v_0}}]{} & \Gamma'^{\alpha(v_0)}. \end{array}$$

Because the fundamental monoids act permutationally, α projects to $\alpha^{\#_{v_0}}$ if and only if α maps the algebraic kernel $\operatorname{Alg\,Ker} \zeta^{\mathrm{v}_0}$ isomorphically onto $\operatorname{Alg\,Ker} \zeta'^{\mathrm{v}_0}$. This condition is expressed by requiring that $\zeta_W = 1 \Leftrightarrow \zeta'_{\alpha(W)} = 1$ holds for all closed walks at v_0. Of course, in the above diagram we can substitute the fundamental monoids with the weak fundamental groups (or with fundamental groups in case of homogeneous coverings). The isomorphism $\alpha^{\#_{v_0}}$ does not change. Again, this is preferable in concrete cases for computational reasons. For simplicity we state the next result in terms of monoids.

Corollary 4.18. *Let $\wp_{\Gamma,\zeta}\colon X \times_{\Gamma,\zeta} F \to X$ and $\wp_{\Gamma',\zeta'}\colon X \times_{\Gamma',\zeta'} F' \to X$ be derived covering projections of a connected general digraph X. Further, let $\alpha\colon X \to X$ be an automorphism. Then the following holds.*

(i) *Suppose that $\alpha\colon \Pi(X, v_0) \to \Pi(X, \alpha(v_0))$ projects to an isomorphism $\alpha^{\#_{v_0}}\colon \Gamma^{v_0} \to \Gamma'^{\alpha(v_0)}$. Then α lifts to an isomorphism of covering projections if and only if $\alpha^{\#_{v_0}}$ is admissible for the action of local groups.*

(ii) *Suppose that the local groups act faithfully. Then α lifts to an isomorphism of covering projections if and only if α projects to an admissible isomorphism $\alpha^{\#_{v_0}}\colon \Gamma^{v_0} \to \Gamma'^{\alpha(v_0)}$.*

The reader should observe the subtle diference in assumptions in (i) and (ii). In (ii) we do not need to assume in advance that α projects. Of course, if α does project to an admissible isomorphism $\alpha^{\#_{v_0}}$, then α is admissible by (i). Conversely, if α is admissible, then it maps the kernel of the first action isomorphically onto the kernel of the second action. Hence α projects, and so $\alpha^{\#_{v_0}}$ is admissible, again by (i). Corollary 4.18 has several useful consequences in cases when the voltage actions or isomorphisms are of special kind.

- In the case of permutation voltages, consider the covering projections $\wp_\zeta\colon X \times_\zeta [n] \to X$ and $\wp_{\zeta'}\colon X \times_{\zeta'} [n] \to X$. The fact that α projects rewrites as $\alpha^{\#_{v_0}}(\zeta_W) = \zeta'_{\alpha(W)}$. With $\alpha^{\#_{v_0}}$ in hand, the condition $\tau(j \cdot g) = \tau(j) \cdot \alpha^{\#_{v_0}}(g)$ for the admissibility of $\alpha^{\#_{v_0}}$ rewrites as $\alpha^{\#_{v_0}}(g) = \tau^{-1} g \tau$, for $g \in \Gamma^{v_0}$.
- In the case of regular voltages, take $\wp_\zeta\colon X \times_\zeta \Gamma \to X$ and $\wp_{\zeta'}\colon X \times_{\zeta'} \Gamma \to X$ where the voltage action is by right multiplication on itself. Additionally, let both

coverings have the same number of connected components. Then an automorphism α lifts if and only if $\alpha\colon \Pi(X, v_0) \to \Pi(X', \alpha(v_0))$ projects, that is, if and only if the equivalence

$$\zeta_{\alpha(W)} = 1 \;\Leftrightarrow\; \zeta'_{\alpha(W)} = 1$$

holds for all closed walks W at v_0. This condition alone is necessary and sufficient since the projected isomorphism between the local groups is always admissible – because the local groups act semiregularly and with the same number of orbits. The above necessary and sufficient condition is sometimes referred to as the *basic lifting lemma*.

- In the same setting as above, let us assume for simplicity that the coverings are connected. Then two such coverings are equivalent if and only if there is an automorphism $\phi\colon \Gamma \to \Gamma$ such that

$$\zeta'^{v_0} = \phi\,\zeta^{v_0}.$$

This is an alternative way of saying that the identity automorphism projects to an automorphism of the voltage group. The claim in one direction has already been mentioned at the beginning of this subsection.

4.7 Concluding remarks

In this paper we considered the problem of isomorphism of covering projections of a given general digraph X, and in particular, the problem of lifting automorphisms. However, in order to better understand the class $\mathcal{C}(X)$ it is necessary to understand the decomposition of projections in $\mathcal{C}(X)$ via other members in $\mathcal{C}(X)$. This includes the Existence thereom for covering projections and the construction of the Universal covering. Along these lines, much more can be said if we restrict to the so called regular coverings, and to abelian regular coverings in particular. All this questions are dealt with in a separate paper [40].

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On the wreath product of signed and gain graphs and its spectrum*

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Abstract

We introduce a notion of wreath product of two gain graphs (Γ_1, ψ_1, G_1) and (Γ_2, ψ_2, G_2), producing a gain graph over the direct product group $G_2^{|V_{\Gamma_1}|} \times G_1$, whose underlying graph is the classical wreath product of graphs $\Gamma_1 \wr \Gamma_2$. By composition with a suitable group homomorphism, our construction produces a signed graph when the two factors are signed graphs. We prove that the wreath product is stable under switching isomorphism. By using group representations, we are able to perform spectral computations on the wreath product: in particular, we determine its largest and its smallest eigenvalue, and we give a description of the spectrum when the first factor is a complex unit complete balanced or antibalanced gain graph, and the second factor is circulant. Finally, when G_1 is a group of permutations of the vertex set of the first factor, and the group G_2 is abelian, we give an alternative definition producing a gain graph over the group wreath product $G_1 \wr G_2$, which turns out to be stable under switching equivalence of the second factor, when the first factor is balanced.

Keywords: Gain graph, signed graph, wreath product of graphs, wreath product of groups, circulant gain graph, mixed Kronecker product, π-spectrum.

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1 Introduction

Graphs are ubiquitous objects in several branches of pure and applied mathematics. Their study represents a central issue in combinatorics and algebra, where one is interested in their topological, spectral, combinatorial properties, but they also have a number of applications in probability, physics, mathematical chemistry, computer science, social sciences, since they are discrete structures allowing to model a huge variety of interactions.

The construction of new graphs by composition of smaller graphs is a very natural procedure, and it was largely developed in literature, both for its theoretical interest and for its applications. Many papers in the last decades have been devoted to graph compositions: among them, the Cartesian product, the direct (or tensor) product, the strong product, the lexicographic product (see [27] and reference therein). Observe that the Cartesian, the direct, and the strong products can be regarded as particular cases of a more general construction, known as NEPS (*non-complete extended p-sum*), which was introduced by Cvetković and Lučić in [16]. More recently, the zig-zag product was introduced [34], in order to produce expanders of constant degree and arbitrary size; in [20], its connections with random walks have been investigated. In all these cases, the vertex set of the resulting graph is the Cartesian product of the vertex sets of the factor graphs.

This is not the case for the wreath product of graphs, which produces a graph much larger than the factor graphs. This construction is nowadays largely studied, and some generalizations have been introduced, for instance, in [21]. Several degree-based and distance-based invariants have been investigated for the wreath product of graphs in [10, 11, 45]. It is worth mentioning that the wreath product has also interesting connections with geometric group theory and probability, via the notions of Lamplighter group and Lamplighter random walk (see, e.g., [26, 36]). In [18] a matrix operation known as *wreath product of matrices* has been introduced. It is a matrix analogue of the wreath product of graphs, since it provides the adjacency matrix of the wreath product of two graphs, when applied to the adjacency matrices of the factors. Although an explicit formula for its spectrum is not known in general, several efforts have been made to derive the spectrum when the first factor is a complete graph and the second factor is a circulant graph [3, 4, 22, 30].

It is a remarkable fact that many graph compositions have an algebraic counterpart in group theory. For instance, it turns out that, when applied to the Cayley graphs of two groups, suitable graph compositions provide the Cayley graph of an appropriate product of such groups [1, 21].

In this paper, a notion of wreath product for signed and gain graphs is introduced. A gain graph (see, for instance, [41]) consists of a triple (Γ, ψ, G), where $\Gamma = (V_\Gamma, E_\Gamma)$ is a graph, called the underlying graph, G is an arbitrary group, and ψ is a map, called the gain function, which associates with every oriented edge an element of G, in such a way that the inverse element is associated with the opposite oriented edge. Signed graphs, introduced by Harary in [28], are a special instance of the gain graphs, obtained when the gain group is $\mathbb{T}_2 = \{\pm 1\}$, and the gain function σ is called a signature in this case (see, for instance, [40, 44]); another remarkable class is obtained when the gain group is the group $\mathbb{T}$ of complex numbers of module 1, and the gain graph in this case is said to be a complex unit gain graph. The reader is referred to [42, 43] for a comprehensive and periodically updated bibliography and glossary on signed and gain graphs.

A Hermitian adjacency matrix can be naturally associated with a signed graph, as well as with a complex unit gain graph, so that a spectral analysis can be developed [32, 44]. In the more general case of a gain graph over an arbitrary group G, the most natural adjacency matrix has entries in the group algebra of G. In [9], an approach via representation theory has been adopted, allowing to associate with such a matrix a complex Hermitian matrix, by using a (unitary) representation of G, and perform spectral computations. The notion of represented spectrum and cospectrality has been further investigated in [12].

Some compositions, also in connection with the study of the balance property and the spectral analysis, have been introduced and studied for signed graphs [7, 29, 31, 37, 38]. See also [6], where the two existing notions in literature of lexicographic product of signed graphs have been extended to $\mathbb{T}$-gain graphs. In [25], the NEPS of signed graphs has been introduced; a generalization of this construction to the cases of complex unit gain graphs and general gain graphs was given in [2, 13]. Despite the extensive literature on the topic, until now, as far as we know, a definition for a wreath product of gain graphs, or even just signed graphs, has not been provided, as pointed out in [5, Section 3.7].

Signed graphs and, more generally, gain graphs, are usually studied up to an equivalence relation called switching isomorphism (see Definition 2.4 and the discussion below). For this reason, when dealing with compositions of signed and gain graphs, one is interested in defining a construction which preserves such equivalence classes.

In this paper, we offer different possibilities for such a construction, all having the classical wreath product of graphs as underlying graph:

1. a general construction associating with two gain graphs (Γ_1, ψ_1, G_1) and (Γ_2, ψ_2, G_2) a new gain graph over the group $G_2^{|V_{\Gamma_1}|} \times G_1$;

2. a signed specialization producing a signed graph as a wreath product of two signed graphs;

3. a gain graph over the wreath product group $G_1 \wr G_2$.

In Section 2, we recall some preliminary notions about the wreath product of classical graphs, and about signed and gain graphs, their adjacency matrices and their spectra. In Section 3, we define the wreath product of two gain graphs (Γ_1, ψ_1, G_1) and (Γ_2, ψ_2, G_2) (Definition 3.1), which produces a gain graph over the group $G_2^{|V_{\Gamma_1}|} \times G_1$, and we show that this construction is stable with respect to the switching isomorphism (Theorem 3.3). By introducing in Definition 3.2 the composition with a possible homomorphism ϕ, we show that with a suitable choice of ϕ, our construction associates with two signed factor graphs $(\Gamma_1, \sigma_1, \mathbb{T}_2)$ and $(\Gamma_2, \sigma_2, \mathbb{T}_2)$ a new signed graph. The corresponding adjacency matrices are described in Proposition 3.6 and in Corollary 3.7.

In Section 4 we perform our spectral investigation of the wreath product of (Γ_1, ψ_1, G_1) and (Γ_2, ψ_2, G_2) with respect to the representation $\pi_2^{\otimes^{|V_{\Gamma_1}|}} \otimes \pi_1$, where π_i is a representation of G_i. In particular, Section 4.2 is devoted to the study of the spectrum of a wreath product of circulant gain graphs (see Definition 4.6), and in Theorem 4.8 we apply a reduction argument to explicitly determine the spectrum of a wreath product whose first factor is a complete (balanced or antibalanced) graph, and the second factor is circulant and endowed with a unitary 1-dimensional representation. We conclude our paper by considering, in Section 5, the special case of two gain groups $G_1 \leq \mathrm{Sym}(|V_{\Gamma_1}|)$ and G_2 abelian, where we are able to define a compound gain graph over the group wreath product $G_1 \wr G_2$, and we prove (Theorem 5.2) that such a construction is stable under switching equivalence of the second factor, when the first factor is balanced.

2 Preliminaries

In this preliminary section we recall the classical definition of the wreath product of graphs and the description of its adjacency matrix, together with some basic notions about gain graphs and their spectra with respect to a representation of the gain group.

2.1 The wreath product of graphs and its adjacency matrix

Let $\Gamma = (V_\Gamma, E_\Gamma)$ be a finite, simple, undirected graph, with vertex set V_Γ and edge set E_Γ, so that E_Γ consists of unordered pairs of type $\{u, v\}$, with $u, v \in V_\Gamma$. We write $u \sim v$ if $\{u, v\} \in E_\Gamma$, and we say that the vertices u and v are *adjacent* in Γ. For any $v \in V_\Gamma$, we denote by $\deg v$ the *degree* of v, that is, the number of vertices which are adjacent to v. A *walk* W of length k in Γ is an ordered list of $k+1$ vertices $v_0, v_1, \ldots, v_k$ such that $v_i \sim v_{i+1}$. The walk W is closed if $v_0 = v_k$.

Let $|V_\Gamma| = n$ and fix an ordering $V_\Gamma = \{v_1, \ldots, v_n\}$. The *adjacency matrix* A_Γ of Γ with respect to this ordering is defined as:

$$(A_\Gamma)_{i,j} := \begin{cases} 1 & \text{if } v_i \sim v_j \\ 0 & \text{otherwise.} \end{cases}$$

Since A_Γ is symmetric by definition, its spectrum is real, and it is called the *adjacency spectrum* (or spectrum for short) of Γ.

We denote by $M_{m\times n}(\mathbb{C})$ the set of matrices with m rows and n columns with entries in the complex field; we write $M_n(\mathbb{C})$ when $m = n$ and we denote by I_n the identity matrix in $M_n(\mathbb{C})$. Now let $A = (a_{ij}) \in M_{m\times n}(\mathbb{C})$ and $B = (b_{hk}) \in M_{p\times q}(\mathbb{C})$. The *Kronecker product* $A \otimes B$ is defined as the $mp \times nq$ matrix

$$A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{pmatrix}.$$

For $A \in M_n(\mathbb{C})$, we use the notation $A^{\otimes^i}$ for the i-th iterated Kronecker product of A with itself, and we put $A^{\otimes^0} = 1 \in M_1(\mathbb{C})$. Notice that, given $A \in M_m(\mathbb{C})$ and $B \in M_n(\mathbb{C})$, it is known (see, e.g., [35, Proposition 1]) that there exist two permutation matrices P_n^m, P_m^n, satisfying $P_n^m = (P_m^n)^{-1}$ such that $A \otimes B = P_m^n(B \otimes A)P_n^m$. See also the paper [17], where permutation matrices allowing to rearrange in an arbitrary order a Kronecker product of matrices are constructed.

Next we recall the classical notion of Cartesian product of graphs [27].

Definition 2.1. Let $\Gamma_1 = (V_{\Gamma_1}, E_{\Gamma_1})$ and $\Gamma_2 = (V_{\Gamma_2}, E_{\Gamma_2})$ be two finite graphs. The *Cartesian product* $\Gamma_1 \square \Gamma_2$ is the graph with vertex set $V_{\Gamma_1} \times V_{\Gamma_2}$, where two vertices (u_1, u_2) and (v_1, v_2) are adjacent if:

(1) either $u_1 = v_1$ in Γ_1 and $u_2 \sim v_2$ in Γ_2;

(2) or $u_1 \sim v_1$ in Γ_1 and $u_2 = v_2$ in Γ_2.

The Cartesian product of graphs is associative. In general, given the graphs $\Gamma_1 = (V_{\Gamma_1}, E_{\Gamma_1}), \ldots, \Gamma_n = (V_{\Gamma_n}, E_{\Gamma_n})$, their Cartesian product $\Gamma_1 \square \cdots \square \Gamma_n$ can be defined, and it follows from the definition that, if A_{Γ_i} is the adjacency matrix of Γ_i, for each $i = 1, \ldots, n$, then

$$A_{\Gamma_1 \square \cdots \square \Gamma_n} = \sum_{i=1}^{n} I_{|V_{\Gamma_1}|} \otimes \cdots \otimes I_{|V_{\Gamma_{i-1}}|} \otimes A_{\Gamma_i} \otimes I_{|V_{\Gamma_{i+1}}|} \otimes \cdots \otimes I_{|V_{\Gamma_n}|}.$$

Therefore (see, for instance, [15]), if $\lambda_{i,1}, \ldots, \lambda_{i,|V_{\Gamma_i}|}$ are the eigenvalues of A_{Γ_i}, then the spectrum of $A_{\Gamma_1 \square \cdots \square \Gamma_n}$ consists of the values $\sum_{i=1}^{n} \lambda_{i,j_i}$, where $j_i \in \{1, \ldots, |V_{\Gamma_i}|\}$. In particular, an eigenvector associated with the eigenvalue $\sum_{i=1}^{n} \lambda_{i,j_i}$ is obtained as $\mathbf{u}_1 \otimes \cdots \otimes \mathbf{u}_n$, where $\mathbf{u}_i$ is an eigenvector of A_{Γ_i} of eigenvalue λ_{i,j_i}. We denote by $\Gamma^{\square^n}$ the n-th iterated Cartesian product of Γ with itself.

We pass now to recall the definition of the *wreath product* $\Gamma_1 \wr \Gamma_2$ of two graphs $\Gamma_1 = (V_{\Gamma_1}, E_{\Gamma_1})$ and $\Gamma_2 = (V_{\Gamma_2}, E_{\Gamma_2})$, which will be crucial throughout the paper.

Definition 2.2. Let $\Gamma_1 = (V_{\Gamma_1}, E_{\Gamma_1})$ and $\Gamma_2 = (V_{\Gamma_2}, E_{\Gamma_2})$ be two graphs, with $|V_{\Gamma_1}| = n$ and $|V_{\Gamma_2}| = m$, with $V_{\Gamma_1} = \{v_1, v_2, \ldots, v_n\}$. The wreath product $\Gamma_1 \wr \Gamma_2$ is the graph with vertex set

$$V_{\Gamma_2}^n \times V_{\Gamma_1} = \{(w_{v_1}, \ldots, w_{v_n}, v) : w_{v_i} \in V_{\Gamma_2}, v \in V_{\Gamma_1}\},$$

where two vertices $(w_{v_1}, \ldots, w_{v_n}, v)$ and $(w'_{v_1}, \ldots, w'_{v_n}, v')$ are adjacent if

(1) *(edge of type I)* either $v = v' = v_i$, $w_{v_i} \sim w'_{v_i}$ in Γ_2 and $w_{v_j} = w'_{v_j}$ if $j \neq i$;

(2) *(edge of type II)* or $w_{v_j} = w'_{v_j}$ for each $j = 1, \ldots, n$ and $v \sim v'$ in Γ_1.

Note that $\Gamma_1 \wr \Gamma_2$ has nm^n vertices. Moreover, if Γ_1 is regular of degree d_1 and Γ_2 is regular of degree d_2, then $\Gamma_1 \wr \Gamma_2$ is regular of degree $d_1 + d_2$. It follows from the definition that $\Gamma_1 \wr \Gamma_2$ can be regarded as a subgraph of the Cartesian product $\Gamma_2^{\square^n} \square \Gamma_1$.

In [18, Definition 2.1] the *wreath product of matrices* is defined as follows.

Definition 2.3. Let $A \in M_n(\mathbb{C})$ and $B \in M_m(\mathbb{C})$. For each $i = 1, \ldots, n$, let $C_i \in M_n(\mathbb{C})$ be the matrix such that

$$(C_i)_{h,k} = \begin{cases} 1 & \text{if } h = k = i \\ 0 & \text{otherwise.} \end{cases}$$

Then $A \wr B$ is the square matrix of size nm^n defined as

$$A \wr B = I_m^{\otimes^n} \otimes A + \sum_{i=1}^{n} I_m^{\otimes^{i-1}} \otimes B \otimes I_m^{\otimes^{n-i}} \otimes C_i.$$

It follows from [18, Theorem 3.2] that, if A_{Γ_1} (resp. A_{Γ_2}) is the adjacency matrix of Γ_1 (resp. of Γ_2), then the adjacency matrix $A_{\Gamma_1 \wr \Gamma_2}$ is

$$A_{\Gamma_1} \wr A_{\Gamma_2} = I_m^{\otimes^n} \otimes A_{\Gamma_1} + \sum_{i=1}^{n} I_m^{\otimes^{i-1}} \otimes A_{\Gamma_2} \otimes I_m^{\otimes^{n-i}} \otimes C_i.$$

2.2 Basic notions on gain graphs

Let G be a group, with unit element 1_G, and let $\Gamma = (V_\Gamma, E_\Gamma)$ be a graph. For every edge $\{u, v\} \in E_\Gamma$, we consider two ordered pairs (u, v) and (v, u), that we call oriented edges. Let us define a map ψ on the set of all oriented edges of Γ, with values in G, such that

$$\psi(u, v) = \psi(v, u)^{-1}.$$

The triple (Γ, ψ, G) is said to be a *G-gain graph* (or a gain graph on G) and ψ is said to be a *gain function* (or G-gain function). The graph Γ is called the *underlying graph* of the gain graph (Γ, ψ, G), and the group G is called the *gain group* of (Γ, ψ, G).

Let (Γ, ψ, G) be a gain graph and let $W\colon v_0, v_1, \dots, v_k$ be a walk of length k in Γ. The gain of W is defined as $\psi(W) := \psi(v_0, v_1)\psi(v_1, v_2)\cdots\psi(v_{k-1}, v_k)$. For a walk W of length 0, we put $\psi(W) = 1_G$. The gain graph (Γ, ψ, G) is said to be *balanced* if $\psi(W) = 1_G$ for every closed walk W. A fundamental concept in the theory of gain graphs, inherited from the theory of signed graphs, is the notion of *switching equivalence*.

Definition 2.4. Two gain functions ψ_1 and ψ_2 on the same underlying graph Γ are switching equivalent, and we write $\psi_1 \sim \psi_2$, if there exists a *switching function* $f\colon V_\Gamma \to G$ such that

$$\psi_2(u, v) = f(u)^{-1}\psi_1(u, v)f(v),$$

for each pair of adjacent vertices u and v.

Two gain graphs (Γ_1, ψ_1, G) and (Γ_2, ψ_2, G) are said to be *switching isomorphic* if there exists a graph isomorphism $\phi\colon V_{\Gamma_1} \to V_{\Gamma_2}$ such that $\psi_1 \sim (\psi_2 \circ \phi)$, with $(\psi_2 \circ \phi)(u, v) = \psi_2(\phi(u), \phi(v))$ for every oriented edge (u, v) of Γ_1.

An element $s \in G$ such that $s^2 = 1_G$ is called an *involution*, and we denote by the bold letter $\mathbf{s}$ the constant gain function such that $\mathbf{s}(u, v) = s$ for any $u, v \in V_\Gamma$ such that $u \sim v$. Given an involution $s \in G$, we say that (Γ, ψ, G) is s-balanced if $\psi \sim \mathbf{s}$, so that a G-gain graph is 1_G-balanced if and only if it is balanced (see [41, Lemma 5.3]), and a $\mathbb{T}$-gain graph (in particular a signed graph) is said to be *antibalanced* if it is -1-balanced (see, for instance, [42]).

Remark 2.5. More generally, if ψ_1 and ψ_2 are G-gain functions on Γ such that $\psi_1 \sim \psi_2$ then for any closed walk W the group elements $\psi_1(W)$ and $\psi_2(W)$ are conjugated. On the other hand, if $\psi_1(W) = \psi_2(W)$ for any closed walk W then $\psi_1 \sim \psi_2$. This last condition is sufficient but not necessary for switching equivalence (see [12, Theorem 3.3]), unless the group G is abelian, in which case $\psi_1 \sim \psi_2$ if and only if $\psi_1(W) = \psi_2(W)$ for any closed walk W (see also [33]).

Let $\mathbb{C}G$ be the group algebra consisting of all finite $\mathbb{C}$-linear combinations of elements of G. An element $f \in \mathbb{C}G$ can be expressed as $f = \sum_{x \in G} f_x x$, with $f_x \in \mathbb{C}$, where the set $\{x \in G : f_x \neq 0\}$ is finite. The product in $\mathbb{C}G$ is defined as:

$$\left(\sum_{x \in G} f_x x\right) \cdot \left(\sum_{y \in G} h_y y\right) := \sum_{x, y \in G} f_x h_y \, xy \qquad \text{for each } f, h \in \mathbb{C}G.$$

An involution * on $\mathbb{C}G$ is defined as $f^* := \sum_{x \in G} \overline{f_x} x^{-1}$, where $\overline{f_x}$ denotes the complex conjugate of f_x. Clearly, there is a natural embedding of G in $\mathbb{C}G$ and we simply write 0 to indicate the zero element of $\mathbb{C}G$ such that $f_x = 0$ for each $x \in G$.

Consider now the $\mathbb{C}$-vector space $M_{n\times m}(\mathbb{C}G)$ consisting of the $n \times m$ matrices with entries in $\mathbb{C}G$. For a given $A \in M_{n\times m}(\mathbb{C}G)$, we define the matrix $A^* \in M_{m\times n}(\mathbb{C}G)$ as

$$(A^*)_{i,j} = (A_{j,i})^*,$$

where the * on the right is the involution of $\mathbb{C}G$. Moreover, one can define the product of $A \in M_{n\times m}(\mathbb{C}G)$ and $B \in M_{m\times q}(\mathbb{C}G)$ as:

$$(AB)_{i,j} = \sum_{l=1}^{m} A_{i,l}B_{l,j},$$

where the product on the right is that of $\mathbb{C}G$. Therefore, the space $M_n(\mathbb{C}G)$ is also an algebra with involution *. For more details see [8, 9].

Given $A \in M_{m\times n}(\mathbb{C}G)$ and $B \in M_{p\times q}(\mathbb{C}G)$, the *Kronecker product* $A \otimes B$ (see [13]) is defined as the matrix in $M_{mp\times nq}(\mathbb{C}G)$ having the block structure

$$\begin{pmatrix} a_{11}\cdot B & a_{12}\cdot B & \cdots & a_{1n}\cdot B \\ a_{21}\cdot B & a_{22}\cdot B & \cdots & a_{2n}\cdot B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}\cdot B & a_{m2}\cdot B & \cdots & a_{mn}\cdot B \end{pmatrix} \tag{2.1}$$

where $a_{ij}\cdot B$ is the matrix in $M_{p\times q}(\mathbb{C}G)$ whose (r,s)-entry is $a_{ij}\cdot b_{rs}$.

The description of the algebra $M_n(\mathbb{C}G)$ allows us to define the adjacency matrix of a G-gain graph (Γ,ψ,G) for a general group G.

Definition 2.6. Let $V_\Gamma = \{v_1, v_2, \dots, v_n\}$ be an ordering of V_Γ. The *adjacency matrix* of (Γ,ψ,G) is the matrix $A_{\Gamma,\psi} \in M_n(\mathbb{C}G)$ defined by

$$(A_{\Gamma,\psi})_{i,j} = \begin{cases} \psi(v_i,v_j) & \text{if } v_i \sim v_j \\ 0 & \text{otherwise.} \end{cases}$$

We will often use the notation A_ψ when the underlying graph Γ will be clear from the context.

Remark 2.7. If an ordering of V_Γ is fixed, two gain functions ψ_1 and ψ_2 are equal if and only if $A_{\Gamma,\psi_1} = A_{\Gamma,\psi_2}$. Moreover, $\psi_1 \sim \psi_2$ if and only if A_{Γ,ψ_1} and A_{Γ,ψ_2} are conjugated by a diagonal matrix with diagonal entries in G (see [9, Theorem 4.1]). Finally, observe that $A_{\Gamma,\psi} \in M_n(\mathbb{C}G)$ is by definition Hermitian in the group algebra sense, that is $A^*_{\Gamma,\psi} = A_{\Gamma,\psi}$.

We are going to introduce, in the next section, a notion of wreath product for two gain graphs (Γ_1,ψ_1,G_1) and (Γ_2,ψ_2,G_2). In order to define a convenient adjacency matrix for the resulting gain graph, we need to recall from [13] the definition of *mixed Kronecker product* $\otimes_M$ of group algebra valued matrices.

We start by considering the Kronecker product of elements of two possibly distinct group algebras: let $\sum_{x\in G_1} f_x x \in \mathbb{C}G_1$ and $\sum_{y\in G_2} h_y y \in \mathbb{C}G_2$, then

$$\left(\sum_{x\in G_1} f_x x\right) \otimes \left(\sum_{y\in G_2} h_y y\right) = \sum_{(x,y)\in G_1\times G_2} f_x h_y (x,y) \in \mathbb{C}(G_1\times G_2).$$

Definition 2.8. Let G_1 and G_2 be two groups. Given $A \in M_{m\times n}(\mathbb{C}G_1)$ and $B \in M_{p\times q}(\mathbb{C}G_2)$, their *mixed Kronecker product* is the matrix $A \otimes_M B \in M_{mp\times nq}(\mathbb{C}(G_1 \times G_2))$ having the following block structure:

$$\begin{pmatrix} a_{11} \otimes_M B & a_{12} \otimes_M B & \cdots & a_{1n} \otimes_M B \\ a_{21} \otimes_M B & a_{22} \otimes_M B & \cdots & a_{2n} \otimes_M B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} \otimes_M B & a_{m2} \otimes_M B & \cdots & a_{mn} \otimes_M B \end{pmatrix}$$

where $a_{ij} \otimes_M B$ is the matrix in $M_{p\times q}(\mathbb{C}(G_1 \times G_2))$ whose (r, s)-entry is $a_{ij} \otimes b_{rs}$.

As an example, if $I_3 \in M_3(\mathbb{C}G)$ is the diagonal matrix whose diagonal entries are equal to 1_G, then $I_3 \otimes_M I_3$ is the 9×9 diagonal matrix with $(1_G, 1_G)$ on the main diagonal and 0 elsewhere. We will use the notation $A^{\otimes_M^n}$ to denote the n-th iterated mixed Kronecker product of A with itself.

2.3 Spectrum of a gain graph with respect to a representation

Let G be a group. A representation π of G of *degree* k (we write $\deg \pi = k$) is a group homomorphism $\pi\colon G \to GL_k(\mathbb{C})$, where $GL_k(\mathbb{C})$ is the group of all invertible matrices in $M_k(\mathbb{C})$. Let $U_k(\mathbb{C}) = \{M \in GL_k(\mathbb{C}) : M^{-1} = M^*\}$ be the subgroup of $GL_k(\mathbb{C})$ consisting of unitary matrices. The representation π is *unitary* if $\pi(g) \in U_k(\mathbb{C})$, for every $g \in G$. Equivalently, π can be regarded as a group homomorphism from G to the general linear group $GL(V)$ of a k-dimensional complex vector space V or, in the unitary case, to the unitary group $U(V)$. Two representations π and π' of degree k of G are equivalent, and we write $\pi \sim \pi'$, if there exists a matrix $S \in GL_k(\mathbb{C})$ such that $\pi'(g) = S^{-1}\pi(g)S$ for every $g \in G$. It is known that, for a finite group G, each equivalence class contains a unitary representative.

We denote by $\pi_0\colon G \to \mathbb{C}$ the *trivial representation* of G, which is defined by $\pi_0(g) = 1$ for every $g \in G$ and is unitary. When G is a subgroup of $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, one has the *identical representation* $\pi_{id}\colon G \to GL_1(\mathbb{C})$, defined as $\pi_{id}(z) = z$. For further information and a more general discussion on group representation theory we refer, for instance, to [24].

A representation $\pi\colon G \to GL_k(\mathbb{C})$ can be linearly extended to $M_{n\times m}(\mathbb{C}G)$. We keep the notation

$$\pi\colon M_{n\times m}(\mathbb{C}G) \to M_{nk\times mk}(\mathbb{C}).$$

In words, the matrix $\pi(A)$ is obtained from A by replacing each occurrence of $g \in G$ with the block $\pi(g) \in GL_k(\mathbb{C})$, and each 0 with the zero matrix of size $k \times k$. In particular, the extension $\pi\colon M_n(\mathbb{C}G) \to M_{nk}(\mathbb{C})$ for square matrices, when π is unitary, is a homomorphism of algebras such that $\pi(A^*) = \pi(A)^*$ (see [8]).

Throughout the paper, we will use the notion of tensor product of two representations $\pi\colon G \to GL(U)$ and $\tau\colon H \to GL(V)$ of two groups G and H, which gives a representation of the direct product $G \times H$ on $U \otimes V$ defined by $(\pi \otimes \tau)(g, h)(u \otimes v) = \pi(g)(u) \otimes \tau(h)(v)$, for all $g \in G, h \in H, u \in U, v \in V$. Notice that $\deg(\pi \otimes \tau) = \deg \pi \cdot \deg \tau$. We will use the notation $\pi^{\otimes^i}$ for the i-th iterated tensor product of π with itself.

Given a gain graph (Γ, ψ, G), with adjacency matrix $A_{\Gamma,\psi} \in M_n(\mathbb{C}G)$, we will refer to $\pi(A_{\Gamma,\psi}) \in M_{nk}(\mathbb{C})$ as the *represented adjacency matrix* of (Γ, ψ, G) with respect to the representation π. If π is unitary, then $\pi(A)$ is Hermitian, and we say that its spectrum is the *π-spectrum of* (Γ, ψ, G). Notice that, in this framework, the classical adjacency matrix of a complex unit gain graph (and in particular, signed graph) is nothing but its represented adjacency matrix with respect to π_{id}, and the classical spectrum for a complex unit gain graph is its π_{id}-spectrum. In analogy to the cases of signed and complex unit gain graphs, the cospectrality with respect to a representation π is invariant under switching equivalence and switching isomorphism.

3 Wreath product of gain graphs

In this section we introduce a notion of wreath product of two gain graphs (Γ_1, ψ_1, G_1) and (Γ_2, ψ_2, G_2). As we will see, the resulting gain function $\psi_1 \wr \psi_2$ is a function taking values in the group

$$\underbrace{G_2 \times \cdots \times G_2}_{|V_{\Gamma_1}| \text{ times}} \times G_1.$$

Moreover, by considering the composition with a group homomorphism

$$\phi : \underbrace{G_2 \times \cdots \times G_2}_{|V_{\Gamma_1}| \text{ times}} \times G_1 \longrightarrow H,$$

we can obtain a gain graph over the group H. We will also write G^n to denote the n-th iterated direct product of the group G with itself. The definition and properties of these gain functions, together with the associated adjacency matrices, are given in Sections 3.1 and 3.2. As highlighted in Section 3.3, if $(\Gamma_1, \sigma_1, \mathbb{T}_2)$ and $(\Gamma_2, \sigma_2, \mathbb{T}_2)$ are two signed graphs, a particular choice of the homomorphism ϕ allows to obtain a resulting graph which is still a signed graph.

3.1 Definition and switching stability property

Definition 3.1. Let (Γ_1, ψ_1, G_1) and (Γ_2, ψ_2, G_2) be two gain graphs. The gain function $\psi_1 \wr \psi_2$ on the graph $\Gamma_1 \wr \Gamma_2$ is defined as follows: for an oriented edge of type I

$$\psi_1 \wr \psi_2 \begin{pmatrix} w_{v_1} & w_{v_2} & \dots & w_{v_i} & \dots & w_{v_n} & v_i \\ w_{v_1} & w_{v_2} & \dots & w'_{v_i} & \dots & w_{v_n} & v_i \end{pmatrix}$$
$$:= (1_{G_2}, \dots, \underbrace{\psi_2(w_{v_i}, w'_{v_i})}_{i\text{-th place}}, \dots, 1_{G_2}, 1_{G_1})$$

and for an oriented edge of type II

$$\psi_1 \wr \psi_2 \begin{pmatrix} w_{v_1} & w_{v_2} & \dots & w_{v_n} & v_i \\ w_{v_1} & w_{v_2} & \dots & w_{v_n} & v_j \end{pmatrix} := (1_{G_2}, \dots, 1_{G_2}, \psi_1(v_i, v_j)).$$

Definition 3.2. Let (Γ_1, ψ_1, G_1) and (Γ_2, ψ_2, G_2) be two gain graphs and let $\phi\colon G_2^{|V_{\Gamma_1}|} \times G_1 \longrightarrow H$ be a group homomorphism. Then the gain function $\psi_1 \wr_\phi \psi_2$ is the H-gain function on $\Gamma_1 \wr \Gamma_2$ defined as the composition $\phi \circ (\psi_1 \wr \psi_2)$.

Theorem 3.3. *For each $i = 1, 2$, let (Γ_i, ψ_i, G_i) and $(\Gamma'_i, \psi'_i, G_i)$ be two switching isomorphic gain graphs via the isomorphism $\Phi_i \colon \Gamma_i \longrightarrow \Gamma'_i$. Then $(\Gamma_1 \wr \Gamma_2, \psi_1 \wr \psi_2, G_2^{|V_{\Gamma_1}|} \times G_1)$ and $(\Gamma'_1 \wr \Gamma'_2, \psi'_1 \wr \psi'_2, G_2^{|V_{\Gamma_1}|} \times G_1)$ are switching isomorphic. In particular, if $\Gamma_i = \Gamma'_i$ and $\psi_i \sim \psi'_i$, then $\psi_1 \wr \psi_2 \sim \psi'_1 \wr \psi'_2$. Finally, if $\phi \colon G_2^{|V_{\Gamma_1}|} \times G_1 \longrightarrow H$ is a homomorphism, then the same properties also hold for $(\Gamma_1 \wr \Gamma_2, \psi_1 \wr_\phi \psi_2, H)$ and $(\Gamma'_1 \wr \Gamma'_2, \psi'_1 \wr_\phi \psi'_2, H)$.*

Proof. For $i = 1, 2$, let $\omega_i \colon V_{\Gamma_i} \longrightarrow G_i$ be a switching function such that

$$\psi_i(x, y) = \omega_i(x)\psi'_i(\Phi_i(x), \Phi_i(y))\omega_i(y)^{-1}$$

for each $x \sim y$ in V_{Γ_i}. Denote by Φ the isomorphism between $\Gamma_1 \wr \Gamma_2$ and $\Gamma'_1 \wr \Gamma'_2$ defined for each vertex $(w_{v_1}, \dots, w_{v_n}, v_i)$ as

$$\Phi(w_{v_1}, \dots, w_{v_n}, v_i) = (\Phi_2(w_{v_1}), \dots, \Phi_2(w_{v_n}), \Phi_1(v_i)).$$

Let $\Omega \colon V_{\Gamma_2}^{|V_{\Gamma_1}|} \times V_{\Gamma_1} \longrightarrow G_2^{|V_{\Gamma_1}|} \times G_1$ be the function defined as

$$\Omega(w_{v_1}, \dots, w_{v_n}, v_i) = (\omega_2(w_{v_1}), \dots, \omega_2(w_{v_n}), \omega_1(v_i)).$$

Then for an oriented edge $e_j = \begin{pmatrix} w_{v_1} & \dots & w_{v_j} & \dots & w_{v_n} & v_j \\ w_{v_1} & \dots & w'_{v_j} & \dots & w_{v_n} & v_j \end{pmatrix}$ of type I, we obtain

$$\begin{aligned} &\psi_1 \wr \psi_2(e_j) \\ &= (1_{G_2}, \dots, \psi_2(w_{v_j}, w'_{v_j}), \dots, 1_{G_2}, 1_{G_1}) \\ &= (1_{G_2}, \dots, \omega_2(w_{v_j}) \cdot \psi'_2(\Phi_2(w_{v_j}), \Phi_2(w'_{v_j})) \cdot \omega_2(w'_{v_j})^{-1}, \dots, 1_{G_2}, 1_{G_1}) \\ &= \Omega(w_{v_1}, \dots w_{v_j}, \dots, w_{v_n}, v_j) \cdot \psi'_1 \wr \psi'_2(\Phi(e_j)) \cdot \Omega(w_{v_1}, \dots w'_{v_j}, \dots, w_{v_n}, v_j)^{-1} \end{aligned} \tag{3.1}$$

and for and oriented edge $e_{rs} = \begin{pmatrix} w_{v_1} & \dots & w_{v_n} & v_r \\ w_{v_1} & \dots & w_{v_n} & v_s \end{pmatrix}$ of type II, we have

$$\begin{aligned} \psi_1 \wr \psi_2(e_{rs}) &= (1_{G_2}, \dots, 1_{G_2}, \psi_1(v_r, v_s)) \\ &= (1_{G_2}, \dots, 1_{G_2}, \omega_1(v_r)\psi'_1(\Phi_1(v_r), \Phi_1(v_s))\omega_1(v_s)^{-1}) \\ &= \Omega(w_{v_1}, \dots, w_{v_n}, v_r) \cdot \psi'_1 \wr \psi'_2(\Phi(e_{rs})) \cdot \Omega(w_{v_1}, \dots, w_{v_n}, v_s)^{-1}. \end{aligned} \tag{3.2}$$

This proves the statement for $\psi_1 \wr \psi_2$ and $\psi'_1 \wr \psi'_2$. In order to conclude the proof it is sufficient to apply the homomorphism ϕ to Equation (3.1) and Equation (3.2), and we obtain that the gain graphs $(\Gamma_1 \wr \Gamma_2, \psi_1 \wr_\phi \psi_2, H)$ and $(\Gamma'_1 \wr \Gamma'_2, \psi'_1 \wr_\phi \psi'_2, H)$ are switching isomorphic. □

Remark 3.4. In Section 2 we said that the wreath product $\Gamma_1 \wr \Gamma_2$ is actually a subgraph of the Cartesian product $\Gamma_2^{\square^n} \square \Gamma_1$. With the theory developed in [13] in the more general context of a NEPS of gain graphs, it is possible to construct a gain function on this Cartesian product starting from the gain functions ψ_1 and ψ_2 on Γ_1 and Γ_2, respectively: it can be verified that the restriction of this gain function to $\Gamma_1 \wr \Gamma_2$ is precisely the gain function $\psi_1 \wr \psi_2$ of Definition 3.1.

3.2 The adjacency matrix of a wreath product

In order to describe the adjacency matrix of the gain graphs $(\Gamma_1 \wr \Gamma_2, \psi_1 \wr \psi_2, G_2^{|V_{\Gamma_1}|} \times G_1)$ and $(\Gamma_1 \wr \Gamma_2, \psi_1 \wr_\phi \psi_2, H)$, we need to introduce a notion of wreath product of matrices with entries in group algebras.

Definition 3.5. Let G_1 and G_2 be two groups. Let $A \in M_n(\mathbb{C}G_1)$ and $B \in M_m(\mathbb{C}G_2)$. The *mixed wreath product* of A and B is the matrix $A \wr_M B$ in $M_{nm^n}(\mathbb{C}(G_2^n \times G_1))$ defined as

$$A \wr_M B := I_m^{\otimes_M^n} \otimes_M A + \sum_{j=1}^n I_m^{\otimes_M^{j-1}} \otimes_M B \otimes_M I_m^{\otimes_M^{n-j}} \otimes_M C_j,$$

where $\otimes_M$ is the mixed Kronecker product of Definition 2.8, $I_m \in M_m(\mathbb{C}G_2)$ is the diagonal matrix whose diagonal entries are equal to 1_{G_2}, and $C_j \in M_n(\mathbb{C}G_1)$ is the diagonal matrix with a unique nonzero entry, equal to 1_{G_1}, occurring in position j.

Proposition 3.6. *Let (Γ_1, ψ_1, G_1) and (Γ_2, ψ_2, G_2) be two gain graphs. For each $i = 1, 2$, fix an ordering on V_{Γ_i} and let A_{ψ_i} be the adjacency matrix with respect to this ordering. Then, with respect to the lexicographic order induced on $V_{\Gamma_1 \wr \Gamma_2}$, the adjacency matrix $A_{\psi_1 \wr \psi_2}$ of $\Gamma_1 \wr \Gamma_2$ associated with $\psi_1 \wr \psi_2$ is $A_{\psi_1} \wr_M A_{\psi_2}$.*

Proof. Let $n = |V_{\Gamma_1}|$ and $m = |V_{\Gamma_2}|$. Notice that, by [13, Theorem 3.7], the adjacency matrix for the gain function Ψ on the Cartesian product $\Gamma_2^{\square^n} \square \Gamma_1$ induced by ψ_1 and ψ_2 is

$$\begin{aligned} A_\Psi &= I_m^{\otimes_M^n} \otimes_M A_{\psi_1} + \sum_{j=1}^n I_m^{\otimes_M^{j-1}} \otimes_M A_{\psi_2} \otimes_M I_m^{\otimes_M^{n-j}} \otimes_M I_n \\ &= I_m^{\otimes_M^n} \otimes_M A_{\psi_1} + \sum_{j=1}^n \mathcal{N}_j \otimes_M I_n, \end{aligned} \tag{3.3}$$

where $\mathcal{N}_j := I_m^{\otimes_M^{j-1}} \otimes_M A_{\psi_2} \otimes_M I_m^{\otimes_M^{n-j}}$. Since A_Ψ is an adjacency matrix, for each position indexed by pairs of elements in $V_{\Gamma_2}^n \times V_{\Gamma_1}$ at most one of the summands in Equation (3.3) has a nonzero entry in that position.

By Remark 3.4 we have to show that $A_{\psi_1} \wr_M A_{\psi_2}$ can be obtained from A_Ψ by replacing with 0 the entries in positions associated with pairs of adjacent vertices in the Cartesian product that are not adjacent in $\Gamma_1 \wr \Gamma_2$.

If $v_i \sim v_j \in V_{\Gamma_1}$ then for every $\overline{x} \in V_{\Gamma_2}^n$ we have that $(\overline{x}, v_i)$ and $(\overline{x}, v_j)$ in $V_{\Gamma_2}^n \times V_{\Gamma_1}$ are adjacent vertices both in the Cartesian product and in the wreath product, since they are indeed the endpoints of an edge of type II. In both cases, the gain is $(1_{G_2}, \dots, 1_{G_2}, \psi_1(v_i, v_j))$, that is equal to $1_{G_2}^{\otimes^n} \otimes (A_{\psi_1})_{i,j}$ in $\mathbb{C}(G_2^n \times G_1)$. The first summand of both A_Ψ and $A_{\psi_1} \wr_M A_{\psi_2}$ is in fact $I_m^{\otimes_M^n} \otimes_M A_{\psi_1}$, that is exactly the matrix having $1_{G_2}^{\otimes^n} \otimes (A_{\psi_1})_{i,j}$ as entry in position $((\overline{x}, v_i), (\overline{x}, v_j))$.

The remaining oriented edges of the Cartesian product are of the form

$$e = \begin{pmatrix} (\overline{x}, v_i) \\ (\overline{y}, v_i) \end{pmatrix} = \begin{pmatrix} x_{v_1} & x_{v_2} & \dots & x_{v_j} & \dots & x_{v_n} & v_i \\ x_{v_1} & x_{v_2} & \dots & y_{v_j} & \dots & x_{v_n} & v_i \end{pmatrix} \tag{3.4}$$

with $x_{v_j} \sim y_{v_j}$ in Γ_2, so that:

$$\Psi(e) = (1_{G_2}, \dots, \underbrace{\psi_2(x_{v_j}, y_{v_j})}_{j\text{-th place}}, \dots, 1_{G_2}, 1_{G_1}) = 1_{G_2}^{\otimes^{j-1}} \otimes \psi_2(x_{v_j}, y_{v_j}) \otimes 1_{G_2}^{\otimes^{n-j}} \otimes 1_{G_1} \tag{3.5}$$

that is exactly the entry of $\mathcal{N}_j \otimes_M I_n$ in position $((\overline{x}, v_i), (\overline{y}, v_i))$, for all $i \in \{1, 2, \dots, n\}$. The matrix $\mathcal{N}_j \otimes_M C_j$ is instead the matrix whose entry in position $((\overline{x}, v_i), (\overline{y}, v_i))$ is the term in Equation (3.5) if and only if $i = j$, and it is zero otherwise. The statement follows by observing that the oriented edge in Equation (3.4) is an oriented edge (of type I) also for $\Gamma_1 \wr \Gamma_2$ if and only if $i = j$. □

Let $\phi \colon G_2^n \times G_1 \longrightarrow H$ be a group homomorphism. Notice that, given $(g_1, \dots, g_n, h)$ in $G_2^n \times G_1$, the homomorphism ϕ can be factorized as

$$\phi(g_1, \dots, g_n, h) = \phi_1(g_1) \cdot \phi_2(g_2) \cdots \phi_n(g_n) \cdot \phi_{n+1}(h) \tag{3.6}$$

where, for each $i = 1, \dots, n$, the homomorphism $\phi_i \colon G_2 \longrightarrow H$ is defined by

$$\phi_i(g_i) = \phi(1_{G_2}, \dots, \underbrace{g_i}_{i\text{-th place}}, \dots, 1_{G_2}, 1_{G_1}), \qquad \text{for each } g_i \in G_2,$$

and $\phi_{n+1} \colon G_1 \longrightarrow H$ is the homomorphism defined as $\phi_{n+1}(h) = \phi(1_{G_2}, \dots, 1_{G_2}, h)$, for each $h \in G_1$.

Given $A = (a_{ij})$ with entries in $\mathbb{C}(G_2^n \times G_1)$, we denote by $\phi(A)$ the matrix with entries $\phi(a_{ij}) \in \mathbb{C}H$, so that ϕ can also be regarded as an algebra homomorphism.

Assume that $A = A_1 \otimes_M A_2 \otimes_M \cdots \otimes_M A_n \otimes_M B$ where A_j are matrices with entries in $\mathbb{C}G_2$, B is a matrix with entries in $\mathbb{C}G_1$ and $\otimes_M$ is the mixed Kronecker product introduced in Definition 2.8. Then it follows from Equation (3.6) that

$$\phi(A_1 \otimes_M A_2 \otimes_M \cdots \otimes_M A_n \otimes_M B) = \phi_1(A_1) \otimes \phi_2(A_2) \otimes \cdots \otimes \phi_n(A_n) \otimes \phi_{n+1}(B),$$

where $\otimes$ is the Kronecker product of matrices with entries in the same group algebra $\mathbb{C}H$, defined in Equation (2.1).

Corollary 3.7. *Let* (Γ_1, ψ_1, G_1) *and* (Γ_2, ψ_2, G_2) *be two gain graphs. Let*

$$\phi = (\phi_1, \dots, \phi_{n+1}) : G_2^{|V_{\Gamma_1}|} \times G_1 \longrightarrow H$$

be a group homomorphism. Then, with respect to the ordering in $V_{\Gamma_1 \wr \Gamma_2}$ *given in* Proposition 3.6, *the adjacency matrix of* $\Gamma_1 \wr \Gamma_2$ *associated with the gain function* $\psi_1 \wr_\phi \psi_2$ *is*

$$A_{\psi_1 \wr_\phi \psi_2} = I_m^{\otimes^n} \otimes \phi_{n+1}(A_{\psi_1}) + \sum_{j=1}^{n} I_m^{\otimes^{j-1}} \otimes \phi_j(A_{\psi_2}) \otimes I_m^{\otimes^{n-j}} \otimes C_j.$$

Proof. It is sufficient to apply ϕ to each entry of $A_{\psi_1} \wr_M A_{\psi_2}$. □

3.3 The case of signed graphs

Let $(\Gamma_1, \sigma_1, \mathbb{T}_2)$ and $(\Gamma_2, \sigma_2, \mathbb{T}_2)$ be two signed graphs. Since we have $G_1 = G_2 = \mathbb{T}_2 = \{\pm 1\}$ in this case, we will simply write 1 instead of 1_{G_1} and 1_{G_2}.

Put $|V_{\Gamma_1}| = n_1$ and $|V_{\Gamma_2}| = n_2$, and consider the homomorphism

$$\begin{aligned} \phi \colon \mathbb{T}_2^{n_1} \times \mathbb{T}_2 &\longrightarrow \quad \mathbb{T}_2 \\ (g_1, \ldots, g_{n_1}, g) &\longmapsto g_1 \cdots g_{n_1} \cdot g, \end{aligned} \tag{3.7}$$

as in [13, Section 4.2]. Then, from Definitions 3.1 and 3.2, we obtain that the wreath product of the signed graphs $(\Gamma_1, \sigma_1, \mathbb{T}_2)$ and $(\Gamma_2, \sigma_2, \mathbb{T}_2)$ is a signed graph whose underlying graph is $\Gamma_1 \wr \Gamma_2$, with signature $\sigma_1 \wr_\phi \sigma_2$, that we will shortly denote by $\sigma_1 \wr \sigma_2$ to stress the fact that the resulting graph is a signed graph, such that:

- for an edge of type I one has:

$$\sigma_1 \wr \sigma_2 \begin{pmatrix} w_{v_1} & w_{v_2} & \ldots & w_{v_i} & \ldots & w_{v_{n_1}} & v_i \\ w_{v_1} & w_{v_2} & \ldots & w'_{v_i} & \ldots & w_{v_{n_1}} & v_i \end{pmatrix} = \sigma_2(w_{v_i}, w'_{v_i});$$

- for an edge of type II one has:

$$\sigma_1 \wr \sigma_2 \begin{pmatrix} w_{v_1} & w_{v_2} & \ldots & w_{v_{n_1}} & v_i \\ w_{v_1} & w_{v_2} & \ldots & w_{v_{n_1}} & v_j \end{pmatrix} = \sigma_1(v_i, v_j).$$

Moreover, it follows from Theorem 3.3 that the signed graph $(\Gamma_1 \wr \Gamma_2, \sigma_1 \wr \sigma_2, \mathbb{T}_2)$ is switching equivalent to the signed graph $(\Gamma_1 \wr \Gamma_2, \sigma'_1 \wr \sigma'_2, \mathbb{T}_2)$, if one has $\sigma_1 \sim \sigma'_1$ and $\sigma_2 \sim \sigma'_2$.

Remark 3.8. Observe that, anytime (Γ_1, ψ_1, G) and (Γ_2, ψ_2, G) are two gain graphs and G is abelian, the map $\phi \colon G^{n_1} \times G \to G$ defined as in Equation (3.7) is indeed a homomorphism, and the associated $\psi_1 \wr_\phi \psi_2$ is in fact a gain function taking values in G. In general, this is not true when G is not abelian (see [13]).

Finally, if A_{σ_1} is the signed adjacency matrix of $(\Gamma_1, \sigma_1, \mathbb{T}_2)$ and A_{σ_2} is the signed adjacency matrix of $(\Gamma_2, \sigma_2, \mathbb{T}_2)$, then the signed adjacency matrix $A_{\sigma_1 \wr \sigma_2}$ of $(\Gamma_1 \wr \Gamma_2, \sigma_1 \wr \sigma_2, \mathbb{T}_2)$ is given by the classical wreath product of matrices (see Definition 2.3):

$$A_{\sigma_1} \wr A_{\sigma_2} = I_{n_2}^{\otimes^{n_1}} \otimes A_{\sigma_1} + \sum_{i=1}^{n_1} I_{n_2}^{\otimes^{i-1}} \otimes A_{\sigma_2} \otimes I_{n_2}^{\otimes^{n_1-i}} \otimes C_i.$$

Example 3.9. Let K_n denote the complete graph on n vertices. Consider, as a first factor, the signed graph $(K_2, \sigma, \mathbb{T}_2)$, where σ is the trivial signature. Take, as a second factor, the signed graph $(K_5, \tau, \mathbb{T}_2)$ depicted in Figure 1, with the convention that the line joining two vertices u and v is solid when $\tau(u, v) = 1$ and dashed when $\tau(u, v) = -1$. In particular, the signed adjacency matrices of $(K_2, \sigma, \mathbb{T}_2)$ and $(K_5, \tau, \mathbb{T}_2)$ are:

$$A_\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad A_\tau = \begin{pmatrix} 0 & 1 & -1 & -1 & 1 \\ 1 & 0 & 1 & -1 & -1 \\ -1 & 1 & 0 & 1 & -1 \\ -1 & -1 & 1 & 0 & 1 \\ 1 & -1 & -1 & 1 & 0 \end{pmatrix}.$$

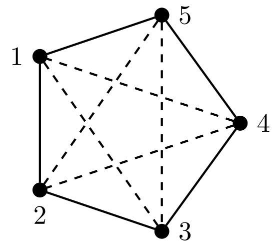

Figure 1: The signed graph $(K_5, \tau, \mathbb{T}_2)$ of Example 3.9.

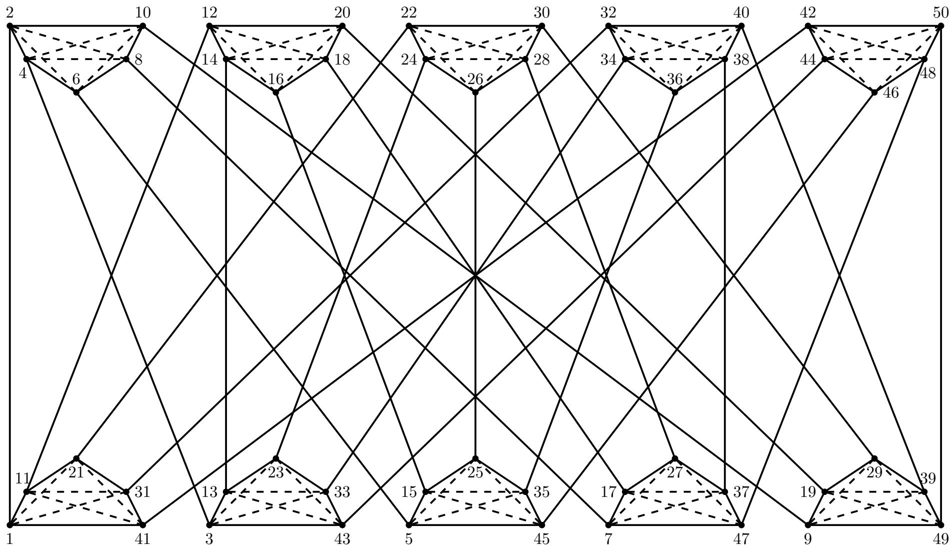

Figure 2: The signed graph $(K_2 \wr K_5, \sigma \wr \tau, \mathbb{T}_2)$ of Example 3.9.

The signed graph $(K_2 \wr K_5, \sigma \wr \tau, \mathbb{T}_2)$ is depicted in Figure 2, and its adjacency matrix is

$$A_{\sigma \wr \tau} = I_5^{\otimes^2} \otimes A_\sigma + A_\tau \otimes I_5 \otimes C_1 + I_5 \otimes A_\tau \otimes C_2,$$

with $C_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $C_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. The spectral analysis of this signed graph will be developed in Example 4.11.

4 Spectral analysis of the wreath product of gain graphs

In this section, we develop our investigation on the spectrum of a wreath product of gain graphs. We start with some general results in Section 4.1, then we focus our attention on the case of a circulant second factor graph in Section 4.2.

4.1 Spectral properties of a general wreath product of gain graphs

We start by giving the following lemma about the so-called Weyl's inequalities (see, for instance, [23, Theorem 1, page 157]), which will be used in the proof of Theorem 4.2. Notice that, by [18, Proposition 2.3], the wreath product of two Hermitian matrices gives a Hermitian matrix.

Lemma 4.1 (Weyl's inequalities)**.** *Let $M, H, P \in M_n(\mathbb{C})$ be Hermitian matrices such that $M = H + P$. Let M have eigenvalues $\mu_1 \geq \ldots \geq \mu_n$; let H have eigenvalues $\nu_1 \geq \ldots \geq \nu_n$, and let P have eigenvalues $\rho_1 \geq \ldots \geq \rho_n$. Then, for each $i = 1, \ldots, n$:*

$$\mu_i \geq \nu_i + \rho_n \text{ and } \mu_i \leq \nu_i + \rho_1.$$

Theorem 4.2. *Let $A \in M_n(\mathbb{C})$ and $B \in M_m(\mathbb{C})$ be two Hermitian matrices, with eigenvalues $\alpha_1 \geq \ldots \geq \alpha_n$ and $\beta_1 \geq \ldots \geq \beta_m$, respectively. Let $\mu_1 \geq \ldots \geq \mu_{nm^n}$ be the eigenvalues of $A \wr B$. Then*

$$\mu_1 = \alpha_1 + \beta_1 \text{ and } \mu_{nm^n} = \alpha_n + \beta_m$$

and, for each $j = 1, \ldots, nm^n$,

$$\alpha_{\lfloor \frac{j-1}{m^n} \rfloor + 1} + \beta_m \leq \mu_j \leq \alpha_{\lfloor \frac{j-1}{m^n} \rfloor + 1} + \beta_1. \tag{4.1}$$

Proof. The proof is a consequence of Weyl's inequalities. As a first step, we will prove the inequalities Equation (4.1). Observe that

$$A \wr B = \mathcal{A} + \mathcal{D},$$

where $\mathcal{A} = I_m^{\otimes^n} \otimes A$ and $\mathcal{D} = \sum_{j=1}^n I_m^{\otimes^{j-1}} \otimes B \otimes I_m^{\otimes^{n-j}} \otimes C_j$. It is clear that the spectrum of $\mathcal{A}$ consists of m^n copies of the spectrum of A. Let us focus our attention on the spectrum of $\mathcal{D}$.

Let $\mathcal{B} = \{v_1, \ldots, v_m\}$ be a basis of eigenvectors of B, where v_i has associated eigenvalue β_i, and let $\mathcal{B}' = \{e_1, \ldots, e_n\}$ be the standard basis of $\mathbb{C}^n$. Then, the basis $\mathcal{B}'' = \{v_{i_1} \otimes \cdots \otimes v_{i_n} \otimes e_j | i_k = 1, \ldots, m; j = 1, \ldots, n\}$ of $\mathbb{C}^{nm^n}$ is a basis of eigenvectors of $\mathcal{D}$. Indeed

$$\begin{aligned}
\mathcal{D}(v_{i_1} \otimes \cdots \otimes v_{i_n} \otimes e_j) &= \left(\sum_{k=1}^n I_m^{\otimes^{k-1}} \otimes B \otimes I_m^{\otimes^{n-k}} \otimes C_k \right) (v_{i_1} \otimes \cdots \otimes v_{i_n} \otimes e_j) \\
&= \sum_{k=1}^n I_m v_{i_1} \otimes \cdots \otimes B v_{i_k} \otimes \cdots \otimes I_m v_{i_n} \otimes C_k e_j \\
&= \sum_{k=1}^n \beta_{i_k} \delta_j^k (v_{i_1} \otimes \cdots \otimes v_{i_n} \otimes e_j) \\
&= \beta_{i_j} (v_{i_1} \otimes \cdots \otimes v_{i_n} \otimes e_j),
\end{aligned}$$

where δ_j^k is the Kronecker delta. This implies that the spectrum of $\mathcal{D}$ consists of nm^{n-1} copies of the spectrum of B. Then, by the Weyl's inequalities we get the inequalities in Equation (4.1): in particular, for $j = 1$ and $j = nm^n$, we obtain $\mu_1 \leq \alpha_1 + \beta_1$, and $\mu_{nm^n} \geq \alpha_n + \beta_m$.

Now we have to prove that $\mu_1 \geq \alpha_1 + \beta_1$ and $\mu_{nm^n} \leq \alpha_n + \beta_m$. We will use again the Weyl's inequalities. Put

$$\mathcal{C} = I_m^{\otimes^n} \otimes A + \sum_{j=1}^{n} I_m^{\otimes^{j-1}} \otimes B \otimes I_m^{\otimes^{n-j}} \otimes I_n.$$

Then the eigenvalues of $\mathcal{C}$ have the form

$$c_{j_1 \ldots j_n i} := \beta_{j_1} + \cdots + \beta_{j_n} + \alpha_i,$$

with $j_k = 1, \ldots, m$ and $i = 1, \ldots, n$. Now put

$$\mathcal{R} := \mathcal{C} - A \wr B = \sum_{j=1}^{n} I_m^{\otimes^{j-1}} \otimes B \otimes I_m^{\otimes^{n-j}} \otimes \hat{C}_j,$$

where $\hat{C}_j = I_n - C_j$. One has:

$$\begin{aligned}
\mathcal{R}(v_{j_1} \otimes \cdots \otimes v_{j_n} \otimes e_i) &:= \left(\sum_{r=1}^{n} I_m^{\otimes^{r-1}} \otimes B \otimes I_m^{\otimes^{n-r}} \otimes \hat{C}_r \right) v_{j_1} \otimes \cdots \otimes v_{j_n} \otimes e_i \\
&= \sum_{r=1}^{n} I_m v_{j_1} \otimes \cdots \otimes B v_{j_r} \otimes \cdots \otimes I_m v_{j_n} \otimes \hat{C}_r e_i \\
&= \left(\sum_{r=1}^{n} \beta_{j_r} (1 - \delta_r^i) \right) v_{j_1} \otimes \cdots \otimes v_{j_n} \otimes e_i \\
&= \left(\sum_{r=1, r \neq i}^{n} \beta_{j_r} \right) v_{j_1} \otimes \cdots \otimes v_{j_n} \otimes e_i.
\end{aligned}$$

Notice that the largest eigenvalue of $\mathcal{C}$ is $\alpha_1 + n\beta_1$ and the smallest one is $\alpha_n + n\beta_m$. On the other hand, the largest eigenvalue of $-\mathcal{R}$ is $-(n-1)\beta_m$ and the smallest one is $-(n-1)\beta_1$. Then it is sufficient to observe that $A \wr B = \mathcal{C} - \mathcal{R}$ and by applying the Weyl's inequalities we obtain

$$\begin{aligned}
\alpha_1 + n\beta_1 - (n-1)\beta_1 &\leq \mu_1 \\
\alpha_1 + \beta_1 &\leq \mu_1
\end{aligned}$$

and

$$\begin{aligned}
\mu_{nm^n} &\leq \alpha_n + n\beta_m - (n-1)\beta_m \\
\mu_{nm^n} &\leq \alpha_n + \beta_m.
\end{aligned}$$

□

Remark 4.3. Let π_1 (resp. π_2) be a representation of the group G_1 (resp. G_2) of degree d_1 (resp. d_2). We denote by $\pi_1^{n\pi_2}$ the representation $\pi_2^{\otimes^n} \otimes \pi_1$, where $\pi_2^{\otimes^n}$ is the n-th iterated tensor product of π_2 with itself, regarded as a representation of G_2^n (see Section 2.3). Notice that, if (Γ_1, ψ_1, G_1) and (Γ_2, ψ_2, G_2) are two gain graphs, the matrices $\pi_1^{n\pi_2}(A_{\psi_1 \wr \psi_2})$ (of size $(md_2)^n nd_1$) and $\pi_1(A) \wr \pi_2(B)$ do not coincide, and they have in general a different size. However, in Proposition 4.4, we are able to determine the extremal values of the $\pi_1^{n\pi_2}$-spectrum.

Proposition 4.4. *Let (Γ_1, ψ_1, G_1) and (Γ_2, ψ_2, G_2) be two gain graphs, with $|V_{\Gamma_1}| = n$ and $|V_{\Gamma_2}| = m$. For each $i = 1, 2$, let π_i be a representation of G_i of degree d_i. Let $\alpha_1 \geq \ldots \geq \alpha_{nd_1}$ be the eigenvalues of $\pi_1(A_{\psi_1})$ and let $\beta_1 \geq \ldots \geq \beta_{md_2}$ be the eigenvalues of $\pi_2(A_{\psi_2})$. Then, if $\mu_1 \geq \ldots \geq \mu_{nm^n d_1 d_2^n}$ are the eigenvalues of $\pi_1^{n\pi_2}(A_{\psi_1 \wr \psi_2})$, one has:*

$$\mu_1 = \alpha_1 + \beta_1 \ \text{ and } \ \mu_{nm^n d_1 d_2^n} = \alpha_{nd_1} + \beta_{md_2}.$$

Proof. The proof follows the same strategy of the proof of Theorem 4.2, by applying the Weyl's inequalities to the matrix

$$I_{md_2}^{\otimes^n} \otimes \pi_1(A_{\psi_1}) + \sum_{j=1}^{n} I_{md_2}^{\otimes^{j-1}} \otimes \pi_2(A_{\psi_2}) \otimes I_{md_2}^{\otimes^{n-j}} \otimes C_j \otimes I_{d_1},$$

which is similar to the matrix $\pi_1^{n\pi_2}(A_{\psi_1} \wr_M A_{\psi_2})$, according to [13, Lemma 3.9]. □

Proposition 4.5. *Let $(K_n, \psi_1, \mathbb{T})$ be a balanced $\mathbb{T}$-gain graph. Let (Γ_2, ψ_2, G_2) be a gain graph on m vertices, and let $\lambda_1 \geq \ldots \geq \lambda_m$ be its π_2-spectrum, where π_2 is a unitary representation of G_2 with* $\deg \pi_2 = 1$*. Let π_{id} be the identical representation of $\mathbb{T}$.*

Then, for each $j = 1, \ldots, m$, the values

$$n - 1 + \lambda_j \ \text{ and } \ -1 + \lambda_j$$

are in the $\pi_{id}^{n\pi_2}$-spectrum of $(K_n \wr \Gamma_2, \psi_1 \wr \psi_2, G_2^n \times \mathbb{T})$.

Proof. By virtue of Theorem 3.3 and by the balance of $(K_n, \psi_1, \mathbb{T})$, we can assume $\psi_1 = \mathbf{1}$ and then $A_{\psi_1} = A_{K_n} = J_n - I_n$, where $J_n \in M_n(\mathbb{C})$ is the matrix whose entries are all equal to 1. Moreover, since $\deg \pi_2 = 1$ and π_{id} is the identical representation, we have in this case

$$\pi_{id}^{n\pi_2}(A_{\psi_1 \wr \psi_2}) = A_{\psi_1} \wr \pi_2(A_{\psi_2}) = A_{K_n} \wr \pi_2(A_{\psi_2}).$$

Therefore, in order to prove the statement, it is enough to prove that, given a Hermitian matrix $B \in M_m(\mathbb{C})$ with eigenvalues $\lambda_1, \ldots, \lambda_m$, the values $n - 1 + \lambda_j$ and $-1 + \lambda_j$ belong to the spectrum of $A_{K_n} \wr B = (J_n - I_n) \wr B$, for each $j = 1, \ldots, m$.

Let $\mathbf{j} \in \mathbb{C}^n$ be the vector with all entries equal to 1 and, for each $j = 1, \ldots, m$, let w_j be an eigenvector of eigenvalue λ_j of B.

Then, it is not difficult to show that $w_j^{\otimes^n} \otimes \mathbf{j}$ is an eigenvector of $A_{K_n} \wr B$ with eigenvalue $n - 1 + \lambda_j$.

Now, let $\{e_1, \ldots, e_n\}$ be the standard basis of $\mathbb{C}^n$. A direct computation shows that the vector $w_j^{\otimes^n} \otimes (e_r - e_s)$ is an eigenvector of $A_{K_n} \wr B$ with eigenvalue $-1 + \lambda_j$, for all $r \neq s$. This concludes the proof. □

4.2 Spectra of wreath products of circulant graphs

In this section, we specialize our spectral investigation to the case of a wreath product whose second factor graph is circulant. We start with a definition.

Definition 4.6. Let G be a group and let $B = (b_{ij})_{i,j=0,\dots,m-1} \in M_m(\mathbb{C}G)$. Then B is *circulant* if $b_{ij} = b_{i+1,j+1}$ for each $i, j = 0, \dots, m-1$, where the sums in the indices must be taken modulo m. Therefore, B is a matrix of type

$$B = \begin{pmatrix} b_0 & b_1 & b_2 & \cdots & b_{m-2} & b_{m-1} \\ b_{m-1} & b_0 & b_1 & \cdots & b_{m-3} & b_{m-2} \\ b_{m-2} & b_{m-1} & b_0 & \cdots & b_{m-4} & b_{m-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b_2 & b_3 & b_4 & \cdots & b_0 & b_1 \\ b_1 & b_2 & b_3 & \cdots & b_{m-1} & b_0 \end{pmatrix} \quad \text{with } b_i \in \mathbb{C}G. \tag{4.2}$$

A gain graph (Γ, ψ, G) is *circulant* if it admits a circulant adjacency matrix.

Observe that, if (Γ, ψ, G) is a circulant gain graph, then its adjacency matrix has the form of Equation (4.2) and it satisfies $b_0 = 0$ and $b_i = b^*_{m-i}$ for each $i = 1, \dots, m-1$.

Moreover, a circulant gain graph in the sense of Definition 4.6 has a circulant underlying graph (in the classical sense), which is isomorphic to $Cay(\mathbb{Z}_m, S)$ for some m and S, where $Cay(\mathbb{Z}_m, S)$ is the Cayley graph (see [19]) of the abelian group $\mathbb{Z}_m$ with respect to the symmetric generating subset $S \subseteq \mathbb{Z}_m \setminus \{0\}$. Moreover, observe that $Cay(\mathbb{Z}_m, S)$ is a regular graph of degree $|S|$, and the regularity degree is odd if and only if m is even and the involution $m/2$ belongs to S.

The next theorem can be regarded as a represented version of [18, Theorem 5.1], and it provides a reduction method to determine the spectrum of the matrix $A \wr_M B$ (see Definition 3.5) with respect to the representation $\pi_1^{n\pi_2}$, under the assumption that B is circulant.

Theorem 4.7. *For each $j = 1, 2$, let G_j be a group and let π_j a unitary representation of G_j, with $\deg \pi_j = d_j$. Let $A \in M_n(\mathbb{C}G_1)$ such that $A = A^*$. Let $B \in M_m(\mathbb{C}G_2)$ be a circulant matrix as in* Equation (4.2) *such that $B = B^*$ and $b_0 = 0$. Then the $\pi_1^{n\pi_2}$-spectrum of $A \wr_M B$ is the union of the spectra of the m^n matrices of size $nd_1d_2^n$ defined as*

$$\overline{M}^{i_1 i_2 \dots i_n} := I_{d_2}^{\otimes^n} \otimes \pi_1(A) + \sum_{t=1}^{n} \sum_{h=1}^{m-1} \rho^{h i_t} I_{d_2}^{\otimes^{t-1}} \otimes \pi_2(b_h) \otimes I_{d_2}^{\otimes^{n-t}} \otimes \tilde{C}_t,$$

where $i_t \in \{0, 1, \dots, m-1\}$ for every $t = 1, \dots n$; $\rho = \exp\left(\frac{2\pi i}{m}\right)$ and $\tilde{C}_t = C_t \otimes I_{d_1}$.

Proof. First we observe that, by virtue of [13, Lemma 3.9], there exists a permutation matrix P such that

$$\begin{aligned} &\pi_1^{n\pi_2}(A \wr_M B) \\ &\quad = (\pi_2^{\otimes^n} \otimes \pi_1)\left(I_m^{\otimes_M^n} \otimes_M A + \sum_{j=1}^{n} I_m^{\otimes_M^{j-1}} \otimes_M B \otimes_M I_m^{\otimes_M^{n-j}} \otimes_M C_j \right) \\ &\quad = P\left(\pi_2(I_m)^{\otimes^n} \otimes \pi_1(A) + \sum_{j=1}^{n} \pi_2(I_m)^{\otimes^{j-1}} \otimes \pi_2(B) \otimes \pi_2(I_m)^{\otimes^{n-j}} \otimes \pi_1(C_j) \right) P^{-1} \\ &\quad = P\left(I_{md_2}^{\otimes^n} \otimes \pi_1(A) + \sum_{j=1}^{n} I_{md_2}^{\otimes^{j-1}} \otimes \pi_2(B) \otimes I_{md_2}^{\otimes^{n-j}} \otimes \tilde{C}_j \right) P^{-1}. \end{aligned}$$

Therefore, to study the $\pi_1^{n\pi_2}$-spectrum of $A \wr_M B$ is equivalent to study the spectrum of the matrix

$$M := I_{md_2}^{\otimes^n} \otimes \pi_1(A) + \sum_{j=1}^{n} I_{md_2}^{\otimes^{j-1}} \otimes \pi_2(B) \otimes I_{md_2}^{\otimes^{n-j}} \otimes \tilde{C}_j.$$

Notice that M is not, in general, the wreath product $\pi_1(A) \wr \pi_2(B)$. However, it can be easily seen that M is a block circulant matrix of the form

$$M = \begin{pmatrix} M_0 & M_1 & M_2 & \cdots & M_{m-2} & M_{m-1} \\ M_{m-1} & M_0 & M_1 & \cdots & M_{m-3} & M_{m-2} \\ M_{m-2} & M_{m-1} & M_0 & \cdots & M_{m-4} & M_{m-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ M_2 & M_3 & M_4 & \cdots & M_0 & M_1 \\ M_1 & M_2 & M_3 & \cdots & M_{m-1} & M_0 \end{pmatrix}$$

whose blocks are the matrices of size $nd_1d_2^n m^{n-1}$ defined as:

$$M_0 = I_{d_2} \otimes I_{md_2}^{\otimes^{n-1}} \otimes \pi_1(A) + \sum_{j=2}^{n} I_{d_2} \otimes I_{md_2}^{\otimes^{j-2}} \otimes \pi_2(B) \otimes I_{md_2}^{\otimes^{n-j}} \otimes \tilde{C}_j$$

and

$$M_i = \pi_2(b_i) \otimes I_{md_2}^{n-1} \otimes \tilde{C}_1, \qquad \text{with } i = 1, \ldots, m-1$$

(recall that $\pi_2(b_0)$ is the zero matrix of size d_2). Notice that, since π_1 and π_2 are unitary representations, then

$$M_i = \pi_2(b_i) \otimes I_{md_2}^{n-1} \otimes \tilde{C}_1 = \pi_2(b_{m-i})^* \otimes (I_{md_2}^{n-1})^* \otimes \tilde{C}_1^* = M_{m-i}^*$$

for each i and M_0 is Hermitian. Therefore, by arguing as in [39, Section 2.3], we obtain that the spectrum of M is the union of the spectra of the m matrices

$$\tilde{M}^{i_1} := \sum_{h=0}^{m-1} \rho^{hi_1} M_h$$

where $i_1 \in \{0, \ldots, m-1\}$. Now, by conjugating with suitable permutation matrices, it can be seen that $\tilde{M}^{i_1}$ is similar to (and so cospectral with) the matrix

$$\overline{M}^{i_1} := \sum_{h=0}^{m-1} \rho^{hi_1} \overline{M}_h,$$

with

$$\overline{M}_0 = I_{md_2}^{\otimes^{n-1}} \otimes I_{d_2} \otimes \pi_1(A) + \sum_{j=2}^{n} I_{md_2}^{\otimes^{j-2}} \otimes \pi_2(B) \otimes I_{md_2}^{\otimes^{n-j}} \otimes I_{d_2} \otimes \tilde{C}_j$$

and

$$\overline{M}_i = I_{md_2}^{n-1} \otimes \pi_2(b_i) \otimes \tilde{C}_1, \qquad i = 1, \ldots, m-1.$$

As a consequence, we obtain that $\overline{M}^{i_1}$ is itself a block circulant matrix of the form

$$\overline{M}^{i_1} = \begin{pmatrix} \overline{M}_0^{i_1} & \overline{M}_1^{i_1} & \overline{M}_2^{i_1} & \cdots & \overline{M}_{m-2}^{i_1} & \overline{M}_{m-1}^{i_1} \\ \overline{M}_{m-1}^{i_1} & \overline{M}_0^{i_1} & \overline{M}_1^{i_1} & \cdots & \overline{M}_{m-3}^{i_1} & \overline{M}_{m-2}^{i_1} \\ \overline{M}_{m-2}^{i_1} & \overline{M}_{m-1}^{i_1} & \overline{M}_0^{i_1} & \cdots & \overline{M}_{m-4}^{i_1} & \overline{M}_{m-3}^{i_1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \overline{M}_2^{i_1} & \overline{M}_3^{i_1} & \overline{M}_4^{i_1} & \cdots & \overline{M}_0^{i_1} & \overline{M}_1^{i_1} \\ \overline{M}_1^{i_1} & \overline{M}_2^{i_1} & \overline{M}_3^{i_1} & \cdots & \overline{M}_{m-1}^{i_1} & \overline{M}_0^{i_1} \end{pmatrix}$$

with

$$\begin{aligned} \overline{M}_0^{i_1} &= I_{d_2} \otimes I_{md_2}^{\otimes^{n-2}} \otimes I_{d_2} \otimes \pi_1(A) \\ &+ \sum_{j=3}^{n} I_{d_2} \otimes I_{md_2}^{\otimes^{j-3}} \otimes \pi_2(B) \otimes I_{md_2}^{\otimes^{n-j}} \otimes I_{d_2} \otimes \tilde{C}_j \\ &+ \sum_{h=1}^{m-1} \rho^{hi_1} (I_{d_2} \otimes I_{md_2}^{\otimes^{n-2}} \otimes \pi_2(b_h) \otimes \tilde{C}_1) \end{aligned}$$

and

$$\overline{M}_i^{i_1} = \pi_2(b_i) \otimes I_{md_2}^{n-2} \otimes I_{d_2} \otimes \tilde{C}_2, \qquad i = 1, \ldots, m-1.$$

By construction, we still have that $\overline{M}_0^{i_1}$ is Hermitian and that $\overline{M}_i^{i_1} = (\overline{M}_{m-i}^{i_1})^*$. More generally, if we iterate this reduction argument k times, we get a block-circulant matrix $\overline{M}^{i_1 i_2 \ldots i_k}$ with blocks

$$\begin{aligned} \overline{M}_0^{i_1 i_2 \ldots i_k} &= I_{d_2} \otimes I_{md_2}^{\otimes^{n-k-1}} \otimes I_{d_2}^{\otimes^k} \otimes \pi_1(A) \\ &+ \sum_{j=k+2}^{n} I_{d_2} \otimes I_{md_2}^{\otimes^{j-k-2}} \otimes \pi_2(B) \otimes I_{md_2}^{\otimes^{n-j}} \otimes I_{d_2}^{\otimes^k} \otimes \tilde{C}_j \\ &+ \sum_{t=1}^{k} \sum_{h=1}^{m-1} \rho^{hi_t} (I_{d_2} \otimes I_{md_2}^{\otimes^{n-k-1}} \otimes I_{d_2}^{\otimes^{t-1}} \otimes \pi_2(b_h) \otimes I_{d_2}^{k-t} \otimes \tilde{C}_t) \end{aligned}$$

and

$$\overline{M}_i^{i_1 i_2 \ldots i_k} = \pi_2(b_i) \otimes I_{md_2}^{n-k-1} \otimes I_{d_2}^{k} \otimes \tilde{C}_{k+1}, \qquad i = 1, \ldots, m-1,$$

with $\overline{M}_0^{i_1 i_2 \ldots i_k}$ Hermitian and $\overline{M}_i^{i_1 i_2 \ldots i_k} = (\overline{M}_{m-i}^{i_1 i_2 \ldots i_k})^*$. Then the claim is proved for $k = n$. □

In the next theorem, which generalizes Theorems 2.3 and 2.4 of [30], we compute the $\pi_1^{n\pi_2}$-spectrum of the wreath product of a complex unit complete graph, which is supposed to be balanced or antibalanced, with a circulant gain graph, under the assumption that π_1 is the identical representation of $\mathbb{T}$ and π_2 is a unitary representation of degree 1.

To do that, we need to recall the definition of elementary symmetric polynomial. Let $x_1, x_2, \ldots, x_n$ be n variables. We put $e_0(x_1, \ldots, x_n) = 1$ and, for each $j = 1, \ldots, n$, the *elementary symmetric polynomial* $e_j(x_1, \ldots, x_n)$ is defined as

$$e_j(x_1, \ldots, x_n) = \sum_{1 \leq i_1 < \ldots < i_j \leq n} \prod_{k=1}^{j} x_{i_k}.$$

It is known that $e_j(x_1,\dots,x_n)$ equals the coefficient of x^{n-j} in the polynomial $\prod_{i=1}^n(x+x_i)$.

Theorem 4.8. *Let $(K_n,\psi_1,\mathbb{T})$ be a complex unit gain graph, with $\psi_1\sim\pm\mathbf{1}$, and let (Γ_2,ψ_C,G_2) be a circulant gain graph with underlying graph isomorphic to $Cay(\mathbb{Z}_m,S)$, with $S=\{s_l\}_l\subseteq\mathbb{Z}_m\setminus\{0\}$. In particular A_{ψ_C} is as in* Equation (4.2)*, with $\psi_C(i,i+s_l)=b_{s_l}$ for each $s_l\in S$ and $b_j=0$ for each $j\notin S$.*

Let π_2 be a unitary representation of G_2 such that $\deg\pi_2=1$. Then the $\pi_{id}^{n\pi_2}$-spectrum of the adjacency matrix of $(K_n\wr\Gamma_2,\psi_1\wr\psi_C,G_2^n\times\mathbb{T})$ is the union of the m^n partial spectra, consisting of the zeros of the following m^n polynomials of degree n in the variable λ defined, for each choice of the multi-index $(i_1,\dots,i_n)\in\{0,1,\dots,m-1\}^n$, as

$$\lambda^n+\sum_{j=1}^n(e_j(x_1,\dots,x_n)\mp(n-j+1)e_{j-1}(x_1,\dots,x_n))\lambda^{n-j}\qquad(4.3)$$

with

- $x_t=\pm1-2\mathrm{Re}(\sum_{l=1}^k\pi_2(b_{s_l})\rho^{s_li_t})-\pi_2\left(b_{\frac{m}{2}}\right)\rho^{\frac{m}{2}i_t}$ *if* $|S|=2k+1$*;*
- $x_t=\pm1-2\mathrm{Re}(\sum_{l=1}^k\pi_2(b_{s_l})\rho^{s_li_t})$ *if* $|S|=2k$*,*

for every $t=1,\dots,n$, with $\rho=\exp\left(\frac{2\pi i}{m}\right)$.

Remark 4.9. Notice that, if $|S|=2k+1$, then m must be even and $\pi_2\left(b_{\frac{m}{2}}\right)\rho^{\frac{m}{2}i_t}\in\{\pm1\}$.

Proof. It follows from Theorem 4.7 that the $\pi_{id}^{n\pi_2}$-spectrum of $(K_n\wr\Gamma_2,\psi_1\wr\psi_C,G_2^n\times\mathbb{T})$ is the union of the spectra of the matrices

$$\overline{M}^{i_1i_2\dots i_n}=\pi_{id}(A_{\psi_1})+\sum_{t=1}^n\sum_{h=1}^{m-1}\rho^{hi_t}\pi_2(b_h)C_t.$$

Since $\psi_1\sim\pm\mathbf{1}$ then, without loss of generality (see Theorem 3.3) we can assume that $\pi_{id}(A_{\psi_1})=\pm(J_n-I_n)$. Observe that, since $\pi_2(b_{m-s_l})\rho^{(m-s_l)i_t}=\overline{\pi_2(b_{s_l})\rho^{s_li_t}}$, then

$$\pi_2(b_{s_l})\rho^{s_li_t}+\pi_2(b_{m-s_l})\rho^{(m-s_l)i_t}=2\mathrm{Re}(\pi_2(b_{s_l})\rho^{s_li_t}).$$

Therefore, if $|S|=2k+1$, we have

$$\begin{aligned}\overline{M}^{i_1i_2\dots i_n}&=\pm(J_n-I_n)+2\sum_{t=1}^n\left(\sum_{l=1}^k\mathrm{Re}(\pi_2(b_{s_l})\rho^{s_li_t})+\pi_2\left(b_{\frac{m}{2}}\right)\rho^{\frac{m}{2}i_t}\right)C_t\\&=\pm(J_n-I_n)+2\sum_{t=1}^n\left(\mathrm{Re}\left(\sum_{l=1}^k\pi_2(b_{s_l})\rho^{s_li_t}\right)+\pi_2\left(b_{\frac{m}{2}}\right)\rho^{\frac{m}{2}i_t}\right)C_t\\&=\pm J_n+Q,\end{aligned}$$

where Q is the diagonal matrix whose t-th entry is

$$\mp1+2\mathrm{Re}\left(\sum_{l=1}^k\pi_2(b_{s_l})\rho^{s_li_t}+\pi_2\left(b_{\frac{m}{2}}\right)\rho^{\frac{m}{2}i_t}\right).$$

On the other hand, if $|S| = 2k$, then

$$\overline{M}^{i_1 i_2 \ldots i_n} = \pm J_n + Q,$$

where Q is the diagonal matrix whose t-th entry is $\mp 1 + 2\mathrm{Re}\left(\sum_{l=1}^{k} \pi_2(b_{s_l})\rho^{s_l i_t}\right)$. Hence

$$\begin{aligned}\det(\lambda I_n - \overline{M}^{i_1 i_2 \ldots i_n}) &= \det(\lambda I_n \mp J_n - Q)\\ &= \det((\lambda I_n - Q)(I_n \mp (\lambda I_n - Q)^{-1} J_n))\\ &= \det(\lambda I_n - Q) \cdot \det(I_n \mp (\lambda I_n - Q)^{-1} J_n).\end{aligned}$$

Notice that, if $|S| = 2k+1$, then

$$\det(\lambda I_n - Q) = \prod_{t=1}^{n}\left(\lambda \pm 1 - 2\mathrm{Re}\left(\sum_{l=1}^{k}\pi_2(b_{s_l})\rho^{s_l i_t}\right) - \pi_2\left(b_{\frac{m}{2}}\right)\rho^{\frac{m}{2} i_t}\right) = \prod_{t=1}^{n}(\lambda + x_t),$$

and similarly if $|S| = 2k$, then

$$\det(\lambda I_n - Q) = \prod_{t=1}^{n}\left(\lambda \pm 1 - 2\mathrm{Re}\left(\sum_{l=1}^{k}\pi_2(b_{s_l})\rho^{s_l i_t}\right)\right) = \prod_{t=1}^{n}(\lambda + x_t).$$

Now, it can be easily seen that $I_n \mp (\lambda I_n - Q)^{-1} J_n$ has $n-1$ eigenvalues equal to 1 and one eigenvalue equal to $1 \mp \sum_{t=1}^{n} \frac{1}{\lambda + x_t}$. Hence, we obtain:

$$\begin{aligned}\det\left(\lambda I_n - \overline{M}^{i_1 i_2 \ldots i_n}\right) &= \prod_{t=1}^{n}(\lambda + x_t) \cdot \left(1 \mp \sum_{t=1}^{n}\frac{1}{\lambda + x_t}\right)\\ &= \prod_{t=1}^{n}(\lambda + x_t) \mp \sum_{t=1}^{n}\frac{\prod_{j=1}^{n}(\lambda + x_j)}{\lambda + x_t}.\end{aligned}$$

One can conclude the proof by using an inductive argument as in [4, Theorem 3.18]. □

Remark 4.10. The spectral results of Theorem 4.8 can be applied to the case of a wreath product of two signed graphs. In fact, if $(K_n, \psi_1, \mathbb{T}_2)$ and $(\Gamma_2, \psi_C, \mathbb{T}_2)$ are signed graphs and both π_1 and π_2 are the identical representations, then the $\pi_1^{n\pi_2}$-spectrum of $(K_n \wr \Gamma_2, \psi_1 \wr \psi_C, \mathbb{T}_2^n \times \mathbb{T}_2)$ coincides with the spectrum of the signed graph obtained via the wreath product of $(K_n, \psi_1, \mathbb{T}_2)$ and $(\Gamma_2, \psi_C, \mathbb{T}_2)$ by using the construction of Section 3.3.

We compute the spectrum of the signed wreath product constructed in Example 3.9.

Example 4.11. Let us choose as a first factor the signed graph $(K_2, \sigma, \mathbb{T}_2)$, where σ is any signature. Since in this case we have $n = 2$, Equation (4.3) becomes:

$$\lambda^2 + (x_1 + x_2 \mp 2)\lambda + (x_1 x_2 \mp (x_1 + x_2)). \tag{4.4}$$

A direct computation shows that, if (Γ_2, ψ_C, G_2) is any second factor circulant graph, the eigenvalues of the adjacency matrix $A_{\sigma \wr \psi_C}$ are

$$\frac{-x_1 - x_2 \pm 2 \pm \sqrt{(x_1 - x_2)^2 + 4}}{2},$$

where x_1 and x_2 are described in the statement of Theorem 4.8. One can check that the signatures $\sigma = +\mathbf{1}$ and $\sigma = -\mathbf{1}$ provide the same spectrum, according to the fact that the signed graph $(K_2, \sigma, \mathbb{T}_2)$ is simultaneously balanced and antibalanced.

Now, let us specialize our example to the signed graphs $(K_2, \sigma, \mathbb{T}_2)$ and $(K_5, \tau, \mathbb{T}_2)$ of Example 3.9, so that $\sigma = +\mathbf{1}$ and Γ_2 is the complete graph K_5 with the signature τ depicted in Figure 1.

The spectrum of the adjacency matrix $A_{\sigma \wr \tau}$ can be explicitly computed by using the results of Theorem 4.8, where:

- $n = 2$ and $\psi_1 = \sigma$;
- $m = 5$, $S = \{1, 2, 3, 4\}$, $\psi_C = \tau$, $\rho = \exp(2\pi i/5)$, $b_0 = 0$, $b_1 = b_4 = 1$, $b_2 = b_3 = -1$;
- both π_1 and π_2 are the identical representations of $\mathbb{T}_2$.

For each choice of the multi-index $(i_1, i_2) \in \{0, 1, \dots, 4\}^2$ (see statement of Theorem 4.8), a direct computation provides the polynomials of degree 2 of Equation (4.4), listed in Table 1.

Table 1: Spectral analysis of $A_{\sigma \wr \tau}$.

Polynomial	(i_1, i_2)
$\lambda^2 - 1$	$(0, 0)$
$\lambda^2 - \sqrt{5}\lambda - 1$	$(0,1), (0,4), (1,0), (4,0)$
$\lambda^2 + \sqrt{5}\lambda - 1$	$(0,2), (0,3), (2,0), (3,0)$
$\lambda^2 - 2\sqrt{5}\lambda + 4$	$(1,1), (1,4), (4,1), (4,4)$
$\lambda^2 + 2\sqrt{5}\lambda + 4$	$(2,2), (2,3), (3,2), (3,3)$
$\lambda^2 - 6$	$(1,2), (1,3), (2,1), (2,4), (3,1), (3,4), (4,2), (4,3)$

Therefore, the spectrum of $A_{\sigma \wr \tau}$ is

$$\left\{ \pm 1, \left(\frac{\sqrt{5} \pm 3}{2} \right)^{(4)}, \left(\frac{-\sqrt{5} \pm 3}{2} \right)^{(4)}, \left(\sqrt{5} \pm 1 \right)^{(4)}, \left(-\sqrt{5} \pm 1 \right)^{(4)}, \left(\pm\sqrt{6} \right)^{(8)} \right\}.$$

5 A gain function on wreath product of groups

Let $\Gamma_1 = (V_{\Gamma_1}, E_{\Gamma_1})$ and $\Gamma_2 = (V_{\Gamma_2}, E_{\Gamma_2})$ be two simple graphs, and let ψ_1 be a G_1-gain function on Γ_1, with $G_1 \leq \mathrm{Sym}(|V_{\Gamma_1}|)$, where $\mathrm{Sym}(n)$ denotes the symmetric group on n elements, for all $n \geq 1$. Notice that, when we write the product $\sigma\tau$ of two permutations of $\mathrm{Sym}(n)$, we let the rightmost permutation act first. Finally, let ψ_2 be a G_2-gain function on Γ_2, where G_2 is an abelian group. Let $|V_{\Gamma_1}| = n$ and let us fix an ordering $V_{\Gamma_1} = \{v_1, \dots, v_n\}$.

Our aim in this section is to define a gain function on the wreath product $\Gamma_1 \wr \Gamma_2$ with values in the group wreath product $G_1 \wr G_2$. For this reason, we start by recalling the classical definition of the wreath product $G_1 \wr G_2$ of G_1 with G_2 (see, for instance, the monograph [14]).

The group $G_1 \leq \mathrm{Sym}(|V_{\Gamma_1}|)$ acts on the set $\{1, \dots, n\}$, and then G_1 acts on G_2^n by shift, that is, for every $\sigma \in G_1$ and $x \colon \{1, \dots, n\} \to G_2$, the element x^σ satisfies

$x^\sigma(i) = x(\sigma^{-1}i)$, or equivalently $(x_1, x_2, \ldots, x_n)^\sigma = (x_{\sigma^{-1}1}, x_{\sigma^{-1}2}, \ldots, x_{\sigma^{-1}n})$. Then $G_1 \wr G_2 = G_2^n \rtimes G_1$ with respect to this action, so that the multiplication of two elements $(y_1, \ldots, y_n)\sigma$ and $(x_1, \ldots, x_n)\tau$ of $G_1 \wr G_2$ is

$$(y_1, \ldots, y_n)\sigma(x_1, \ldots, x_n)\tau = (y_1 x_{\sigma^{-1}1}, \ldots, y_n x_{\sigma^{-1}n})\sigma\tau,$$

with $x_i, y_i \in G_2$ for all $i = 1, \ldots, n$, and $\sigma, \tau \in G_1 \leq \mathrm{Sym}(n)$.

For any $a \in G_2$ and $i \in \{1, \ldots, n\}$, let us put

$$[a, i] = (1_{G_2}, \ldots, \underbrace{a}_{i\text{-th place}}, \ldots, 1_{G_2})1_{G_1} \in G_1 \wr G_2.$$

We shortly denote by σ the element $(1_{G_2}, \ldots, 1_{G_2})\sigma$ of $G_1 \wr G_2$. With these notations, since G_2 is supposed to be abelian, we have:

$$\sigma[a, i] = [a, \sigma i]\sigma; \qquad [a, i][b, j] = [b, j][a, i]. \tag{5.1}$$

In the next definition we introduce a gain function $\psi_1 \wr\!\wr \psi_2$ on $\Gamma_1 \wr \Gamma_2$, taking values in the group wreath product $G_1 \wr G_2$.

Definition 5.1. Let (Γ_1, ψ_1, G_1) and (Γ_2, ψ_2, G_2) be two gain graphs. The gain function $\psi_1 \wr\!\wr \psi_2$ on $\Gamma_1 \wr \Gamma_2$ is defined as follows. Given an oriented edge of type I:

$$\psi_1 \wr\!\wr \psi_2 \begin{pmatrix} w_{v_1} & w_{v_2} & \ldots & w_{v_i} & \ldots & w_{v_n} & v_i \\ w_{v_1} & w_{v_2} & \ldots & w'_{v_i} & \ldots & w_{v_n} & v_i \end{pmatrix} = [\psi_2(w_{v_i}, w'_{v_i}), i] \in G_1 \wr G_2;$$

given an oriented edge of type II:

$$\psi_1 \wr\!\wr \psi_2 \begin{pmatrix} w_{v_1} & w_{v_2} & \ldots & w_{v_n} & v_i \\ w_{v_1} & w_{v_2} & \ldots & w_{v_n} & v_j \end{pmatrix} = \psi_1(v_i, v_j) \in G_1 \wr G_2.$$

Given a vertex $(w_{v_1}, w_{v_2}, \ldots, w_{v_n}, v_j)$ of $\Gamma_1 \wr \Gamma_2$, we denote by π_i, for $i = 0, 1, \ldots, n$, the projection such that

$$\pi_0((w_{v_1}, w_{v_2}, \ldots, w_{v_n}, v_j)) = v_j; \qquad \pi_i((w_{v_1}, w_{v_2}, \ldots, w_{v_n}, v_j)) = w_{v_i},$$

for all $i = 1, \ldots, n$.

Given a walk W in $\Gamma_1 \wr \Gamma_2$, the projection $\pi_i(W)$ can be naturally defined by concatenation. In particular, observe that if W is closed, then $\pi_i(W)$ induces a closed walk in Γ_2, for each $i = 1, \ldots, n$, as well as $\pi_0(W)$ induces a closed walk in Γ_1 (by possibly removing consecutive repetitions of a same vertex).

Theorem 5.2. *Let (Γ_1, ψ_1, G_1) and (Γ_2, ψ_2, G_2) be two gain graphs, with $G_1 \leq \mathrm{Sym}(|V_{\Gamma_1}|)$ and G_2 abelian. If ψ_1 is balanced and $\psi_2 \sim \psi'_2$, then $\psi_1 \wr\!\wr \psi_2 \sim \psi_1 \wr\!\wr \psi'_2$.*

Proof. If we prove that $\psi_1 \wr\!\wr \psi_2(W) = \psi_1 \wr\!\wr \psi'_2(W)$, for any closed walk W in $\Gamma_1 \wr \Gamma_2$, then the thesis follows by Remark 2.5. If in W either all edges are of type I or all edges are of type II, then one can easily check that $\psi_1 \wr\!\wr \psi_2(W) = \psi_1 \wr\!\wr \psi'_2(W)$. In the general case, we can write W as a composition of walks

$$W = W_1 W'_1 W_2 W'_2 \cdots W_k W'_k$$

with possibly W_1 or W'_k of length 0 and such that edges involved in each W_i are of type I and edges involved in each W'_i are of type II.

More precisely, for every $j = 1, \ldots, k$ there exists $i_j \in \{1, \ldots, n\}$ such that each vertex v in W_j is such that $\pi_0(v) = v_{i_j}$. In particular, $\psi_1 \wr \psi_2(W_j) = [\psi_2(\pi_{i_j}(W_j)), i_j]$. On the other hand, W'_j consists of edges of type II such that $\pi_0(W'_j)$ is a walk from v_{i_j} to $v_{i_{j+1}}$ in Γ_1, so that $\psi_1 \wr \psi_2(W'_j)$ is just the permutation $\psi_1(\pi_0(W'_j))$. Thus

$$\begin{aligned} \psi_1 \wr \psi_2(W) &= [a_1, i_1]\tau_1[a_2, i_2]\tau_2 \cdots [a_k, i_k]\tau_k \\ \psi_1 \wr \psi'_2(W) &= [a'_1, i_1]\tau_1[a'_2, i_2]\tau_2 \cdots [a'_k, i_k]\tau_k \end{aligned} \tag{5.2}$$

where $a_j = \psi_2(\pi_{i_j}(W_j))$, $a'_j = \psi'_2(\pi_{i_j}(W_j))$ and $\tau_j = \psi_1(\pi_0(W'_j))$. Observing that $\tau_1\tau_2 \cdots \tau_k$ is the gain of the closed walk associated to $\pi_0(W)$, one has

$$\tau_1\tau_2 \cdots \tau_k = 1_{G_1}. \tag{5.3}$$

Moreover, for every $i \in \{1, \ldots, n\}$, one has

$$\pi_i(W) = \prod_{j: i_j = i} \pi_i(W_j).$$

The walk $\pi_i(W)$ is closed in Γ_2; moreover, since G_2 is abelian and $\psi_2 \sim \psi'_2$ by hypothesis, by Remark 2.5 we have $\psi_2(\pi_i(W)) = \psi'_2(\pi_i(W))$. Therefore

$$\prod_{j: i_j = i} a_j = \prod_{j: i_j = i} a'_j \quad \text{and then} \quad \prod_{j: i_j = i} [a_j, i] = \prod_{j: i_j = i} [a'_j, i]. \tag{5.4}$$

By combining Equations (5.1), (5.2) and (5.3), one has

$$\begin{aligned} \psi_1 \wr \psi_2(W) &= [a_1, i_1][a_2, \tau_1 i_2][a_3, \tau_1\tau_2 i_3] \cdots [a_k, \tau_1 \cdots \tau_{k-1} i_k] \\ \psi_1 \wr \psi'_2(W) &= [a'_1, i_1][a'_2, \tau_1 i_2][a'_3, \tau_1\tau_2 i_3] \cdots [a'_k, \tau_1 \cdots \tau_{k-1} i_k]. \end{aligned} \tag{5.5}$$

Now we are going to prove that if $t, s \in \{1, \ldots, k\}$ are such that $i_s = i_t = i$, then also $\tau_1 \cdots \tau_{s-1} i_s = \tau_1 \cdots \tau_{t-1} i_t$, that is, in the product in Equation (5.5) the contributions a_t and a_s (and then a'_t and a'_s) appear in the same coordinate. Suppose $t > s$. Then $\tau_s\tau_{s+1} \cdots \tau_{t-1}$ is the gain of a walk in Γ_1 starting and ending at v_i, and then $\tau_s\tau_{s+1} \cdots \tau_{t-1} = 1_{G_1}$ by the balance of ψ_1. It follows that $\tau_1 \cdots \tau_{s-1}\tau_s\tau_{s+1} \cdots \tau_{t-1} i = \tau_1 \cdots \tau_{s-1} i$. This implies that we can define a map

$$f\colon \{1, \ldots, n\} \longrightarrow \{1, \ldots, n\}$$

such that $f(i_s) = \tau_1 \cdots \tau_{s-1} i_s$ and Equation (5.5) can be rewritten as:

$$\begin{aligned} \psi_1 \wr \psi_2(W) &= [a_1, f(i_1)][a_2, f(i_2)][a_3, f(i_3)] \cdots [a_k, f(i_k)] \\ \psi_1 \wr \psi'_2(W) &= [a'_1, f(i_1)][a'_2, f(i_2)][a'_3, f(i_3)] \cdots [a'_k, f(i_k)]. \end{aligned} \tag{5.6}$$

In words, every time the walk $\pi_0(W)$ returns to a vertex, the gain multiplication occurs in the same coordinate. Finally, by combining Equation (5.4) with Equation (5.6), one obtains

$$\begin{aligned} \psi_1 \wr \psi_2(W) &= \prod_{h=1}^{n} \prod_{\substack{j: \\ f(i_j) = h}} [a_j, f(i_j)] = \prod_{h=1}^{n} \prod_{\substack{i: \\ f(i) = h}} \prod_{j:\, i_j = i} [a_j, i] \\ &= \prod_{h=1}^{n} \prod_{\substack{i: \\ f(i) = h}} \prod_{j:\, i_j = i} [a'_j, i] = \psi_1 \wr \psi'_2(W). \end{aligned}$$

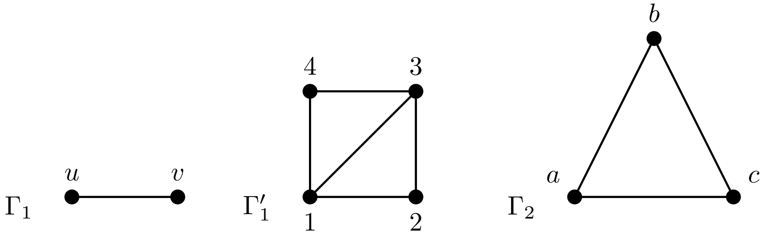

Figure 3: The graphs Γ_1, Γ'_1 and Γ_2 of Examples 5.4, 5.5 and 5.6.

□

Corollary 5.3. *Let* (Γ_1, ψ_1, G_1) *be a gain graph, with* $G_1 \leq \mathrm{Sym}(|V_{\Gamma_1}|)$ *and let* (Γ_2, ψ_2, G_2) *be a gain graph, with* G_2 *abelian. If* ψ_1 *and* ψ_2 *are balanced, then* $\psi_1 \wr \psi_2$ *is balanced.*

Proof. By the balance of (Γ_2, ψ_2, G_2) we have $\psi_2 \sim \mathbf{1}_{G_2}$. Moreover, it follows from Theorem 5.2 that $\psi_1 \wr \psi_2 \sim \psi_1 \wr \mathbf{1}_{G_2}$, so that it is enough to prove the claim by assuming $\psi_2 = \mathbf{1}_{G_2}$. Now, this assumption implies that the gain of a closed walk W in $\Gamma_1 \wr \Gamma_2$ is just $\psi_1(\pi_0(W)) \in G_1$, considered as an element of $G_1 \wr G_2$ (see proof of Theorem 5.2). The claim follows since also (Γ_1, ψ_1, G_1) is balanced. □

In summary, with respect to the gain function introduced in Definition 5.1, the product of balanced gain graphs is balanced; moreover, if the first graph is fixed and balanced, replacing the second factor with a switching equivalent graph provides a gain function on the wreath product belonging to the same switching equivalence class.

However, it is not true that the replacement of the first factor with a switching equivalent graph provides a gain graph in the same switching equivalence class for $\Gamma_1 \wr \Gamma_2$, as shown in Example 5.4. Moreover, the conclusions of Theorem 5.2 and Corollary 5.3 do not hold if the group G_2 is not abelian, as shown in Example 5.5. Finally, the switching equivalence class of the product is not preserved with the assumption (analogous to those in Theorem 5.2 but with roles exchanged between the first and second factor) that the second factor is balanced, see Example 5.6.

Example 5.4. Let Γ_1 and Γ_2 be the graphs depicted in Figure 3. Consider $G_1 = \mathrm{Sym}(2) \cong \mathbb{T}_2$: we have two gain functions ψ_1 and ψ'_1, such that $\psi_1(u,v) = 1$ and $\psi'_1(u,v) = -1$. On the other hand, let G_2 be a nontrivial group and let ψ_2 be such that $\psi_2(a,b)\psi_2(b,c)\psi_2(c,a) = x \in G_2$, with $x \neq 1_{G_2}$. Consider the following closed walk W in $\Gamma_1 \wr \Gamma_2$:

$$(a,a,u), (b,a,u), (c,a,u), (a,a,u), (a,a,v), (a,b,v), (a,c,v), (a,a,v), (a,a,u).$$

One can easily check that $\psi_1 \wr \psi_2(W) = (x,x)$ but $\psi'_1 \wr \psi_2(W) = (x^2, 1)$. As a consequence, by Remark 2.5, one has that $\psi_1 \wr \psi_2 \nsim \psi'_1 \wr \psi_2$, since (x,x) and $(x^2,1)$ are not conjugated in $G_1 \wr G_2$.

Example 5.5. Let Γ_1 and Γ_2 be the graphs depicted in Figure 3. Consider $G_1 = \mathrm{Sym}(2) \cong \mathbb{T}_2$ and the gain function ψ_1 on Γ_1 such that $\psi_1(u,v) = -1$. Let $G_2 = Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ be the Quaternion group; consider the gain function ψ_2 on Γ_2 such that

$$\psi_2(a,b) = i, \quad \psi_2(b,c) = j, \quad \psi_2(c,a) = -k.$$

Notice that $(\Gamma_1, \psi_1, \mathbb{T}_2)$ and (Γ_2, ψ_2, Q_8) are both balanced; however, one can explicitly check that the gain of the closed walk

$$(a,a,u),(b,a,u),(b,a,v),(b,b,v),(b,b,u),(c,b,u),(c,b,v),\\(c,c,v),(c,c,u),(a,c,u),(a,c,v),(a,a,v),(a,a,u)$$

in $(\Gamma_1 \wr \Gamma_2, \psi_1 \wr\!\wr \psi_2, \mathbb{T}_2 \wr Q_8)$ is nontrivial, so that the wreath product is not balanced.

Example 5.6. Let $\Gamma_1' = (V_{\Gamma_1'}, E_{\Gamma_1'})$ and $\Gamma_2 = (V_{\Gamma_2}, E_{\Gamma_2})$ be the graphs depicted in Figure 3. Let us consider on Γ_1' the following $\mathrm{Sym}(4)$-gain functions ψ_1 and ψ_1', where we use the cycle notation:

$$\begin{cases}\psi_1(1,2) = (12)\\ \psi_1(2,3) = \psi_1(1,4) = \psi_1(1,3) = id\\ \psi_1(3,4) = (123)\end{cases} \qquad \begin{cases}\psi_1'(1,2) = (23)\\ \psi_1'(2,3) = \psi_1'(1,4) = \psi_1'(1,3) = id\\ \psi_1'(3,4) = (132).\end{cases}$$

The gain functions ψ_1 and ψ_1' are switching equivalent; indeed, for each x, y in $V_{\Gamma_1'}$, we have $\psi_1'(x,y) = (13)\psi_1(x,y)(13)$. Consider the $\mathbb{T}$-gain function ψ_2 on Γ_2 such that

$$\psi_2(a,b) = \psi_2(b,c) = \psi_2(c,a) = \exp\left(\frac{2\pi i}{3}\right) =: \alpha.$$

Notice that $(\Gamma_2, \psi_2, \mathbb{T})$ is balanced. However, we are going to show that $\psi_1 \wr\!\wr \psi_2 \nsim \psi_1' \wr\!\wr \psi_2$. Let W be the closed walk in $\Gamma_1' \wr \Gamma_2$ defined as follows:

$$(a,a,a,a,3),(a,a,b,a,3),(a,a,b,a,1),(a,a,b,a,2),(a,a,b,a,3),\\(a,a,c,a,3),(a,a,c,a,4),(a,a,c,a,1),(a,a,c,a,3),(a,a,a,a,3).$$

One can check that

$$\psi_1 \wr\!\wr \psi_2(W) = [\alpha,3](12)[\alpha,3](123)[\alpha,3] = (1,\alpha,\alpha^2,1)(23)$$

and

$$\psi_1' \wr\!\wr \psi_2(W) = [\alpha,3](23)[\alpha,3](132)[\alpha,3] = (1,\alpha,\alpha^2,1)(12).$$

So, if we are able to show that there is no $\sigma \in \mathrm{Sym}(4)$ such that

$$\begin{aligned}(1,\alpha,\alpha^2,1)(23) &= \sigma^{-1}(x^{-1},y^{-1},z^{-1},w^{-1})(1,\alpha,\alpha^2,1)(12)(x,y,z,w)\sigma\\ &= \sigma^{-1}(x^{-1}y, y^{-1}\alpha x, \alpha^2, 1)(12)\sigma,\end{aligned}$$

then we obtain $\psi_1 \wr\!\wr \psi_2 \nsim \psi_1' \wr\!\wr \psi_2$ (Remark 2.5). Now, if such a permutation σ exists, then it must fix 3, corresponding to the position of the only element equal to α^2; on the other hand, σ has also to satisfy $\sigma^{-1}(12)\sigma = (23)$, that is impossible.

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