Determining the elongation of a rod in different gravitational fields of a black hole

Luna Pestotnik Stres

Stres InŽeniring d.o.o

Tip publikacije: Monografska publikacija
Naslov: Determining the elongation of a rod in different gravitational fields of a black hole
Avtor: Luna Pestotnik Stres
Leto izdaje: 2025
Jezik: angleški
Država izdaje: Slovenija
Kraj izdaje: Tolmin
Založnik: Stres Inženiring d.o.o.
Format: PDF
URL: https://stres-apartments.eu/books/monografija_MAT.pdf

CIP:
Kataložni zapis o publikaciji (CIP) pripravili v Narodni in univerzitetni knjižnici v Ljubljani
COBISS.SI-ID 247658243
ISBN 978-961-93233-6-6 (PDF)

1. Introduction

Humans have been fascinated by the universe surrounding us for thousands of years. In ancient cultures, space was often viewed as a realm of heaven, with celestial objects frequently linked to gods, spirits, and myths. Throughout history, significant astronomical advancements have been achieved by some of the pioneers of astronomy, establishing the foundation for humanity to explore far beyond our planet [1]. The first theory proposing the existence of black holes—dense bodies from which even light cannot escape due to their strong gravity—was presented in the 18th century by the astronomer John Michell [2]. It was subsequently refined by Einstein's theory of general relativity, which predicted that any sufficiently compact mass could distort space and time to form a black hole. The theory also predicted the event horizon as the spherical outer boundary of a black hole where the gravitational influence of the black hole becomes so great that not even light is fast enough to escape it. [3]

As an astronomy enthusiast, I have pondered from a young age whether the exploration of black holes could be conducted from a closer perspective and whether humans could ever venture into one. As a child, my mother often said that such actions would be impossible, as humans would be "ripped apart" instantly. Eventually, I learned that this phenomenon is linked to the effects of gravitation.

In general, two contributions influence this process. Firstly, each object, suspended and secured by a hook extends under its weight, stretching until the gravitational force is balanced by the opposing force of the hook. This leads to the elongation of an object, which may have a temporary or permanent effect. The second contribution, however, arises from differences in the strength of the gravitational field throughout the body. According to Newton's law of gravitation $F = \frac{GMm}{r^2}$ [4], the gravitational field exerted by a body with mass M on an object of mass m decreases with the square of the distance r from the body's centre, where G is gravitational constant. An object positioned vertically in a gravitational field, therefore, experiences varying forces at both ends of the object as the distances from the mass generating the field differ. This impacts the extent of elongation, which, owing to decreasing r, is more pronounced on the lower side of the object (see a sketch in Fig. 1). The difference in the stretching of each part of the object reduces the effect of elongation due to the object's weight. In black holes, the gravitational field is exceptionally strong. Consequently, an object stretches in the direction of a black hole, a process called spaghettification. When the opposing force acting on the object exceeds the maximum possible force the material can sustain, the object fractures.

This basic knowledge and my mum's statement prompted the formation of the following question: Which type of black hole is the most suitable for observation in terms of the closest approach with a space probe and what is the significance of the changing field for diminishment of the effect of elongation?

However, as space probes are designed from many different materials, I simplified my inquiry. I focused on a steel rod with a length of $l = 4m$, which could theoretically represent a space probe.

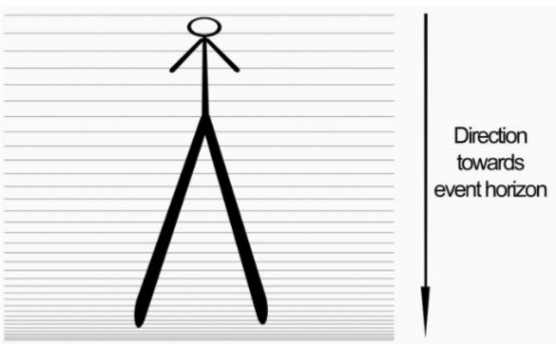


Figure 1: Spagettification of a human body in the black hole

Therefore, this exploration aims to investigate the elongation of a steel rod and its maximum length when positioned at the horizon of a black hole. The calculations will focus on black holes of different masses to determine where the stretching effects have the smallest impact. They will include three known black holes and two smaller hypothetical primordial black holes. The known black holes are classified as stellar-mass, intermediate-mass, and supermassive and the primordial black holes are the significantly smaller ones which, in theory, could have only been formed soon after the Big Bang [4].

Additionally, this investigation aims to determine the significance of the changing gravitational field on reducing the effect of gravitational stretching. The results will be derived by comparing equations for elongation and maximum length in both constant and changing gravitational fields.

Deformation of elastic bodies

When force is applied to a material, each object undergoes a deformation to a certain extent. The external forces can cause the object to shrink, elongate or even break apart. The scale of this deformation depends on the physical properties of a specific material [5]; however, each material can break when sufficient force is applied. The deformation process is usually described in terms of stress and strain. Stress (σ) is the cause of deformation and is defined as the force (F) acting per unit area (A) and is measured in SI unit Pascal.

$$\sigma = \frac{F}{A}. \tag{1}$$

It is specified as tensile stress when an object undergoes elongation. [6] Strain (ϵ), however, describes the ratio between the deformation (ΔL) and the original length (L) [7].

$$\epsilon = \frac{\Delta L}{L}. \tag{2}$$

The two quantities are connected, and their relationship changes depending on the stress applied. Since strain depends on the stress applied, the expected graph would include stress as an independent variable. However, as strain is usually the measured quantity and stress (σ) is calculated afterwards, a consensus for the graph showing the relation is that strain (ϵ) is the independent variable (Fig. 2).

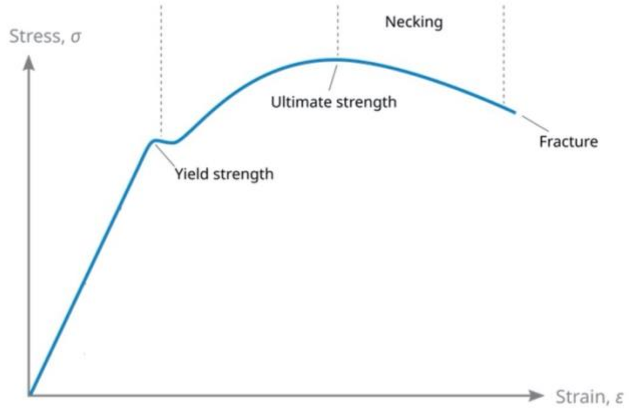


Figure 2: Stress-strain graph

Each object undergoes three stages when exposed to an external force. Until a certain point called yield strength, the stress changes with strain linearly, and the object returns to its original shape if the stress is removed. This region is described as elastic region. The ratio between the applied stress (σ) and resulting strain (ϵ) in this region is called Young's modulus (E), a mechanical property of the material that measures the tensile (elonged) or compressive stiffness of a material. It is calculated as:

$$E = \frac{\sigma}{\epsilon}. \tag{3}$$

After exiting the elastic region, the stress still increases with strain, but the relationship is not linear, and deformation occurs even when stress is nominally not decreasing any more. After reaching a point of ultimate strength, the object experiences necking, a process where the elongation only occurs at one location on the object. From that point onward, the force no longer increases, and the object begins to thin at the specific location – the neck. The strain, as defined in Eq. 1, still rises and the object eventually fractures.

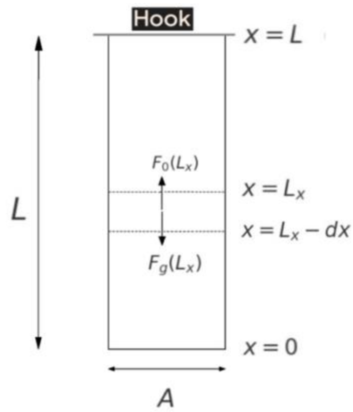
For this investigation, a linear approximation is used throughout the whole process of strain application resulting in elongation. All stages of the deformation are, therefore, considered as if they were in an elastic region and the rod fractures at the maximum σ that the material can endure (see Fig. 2) before fracturing. Moreover, a still rod is used to obtain calculations, the properties of which are summarised in Table 1.

Table 1: Physical properties of steel

	Density (ρ)	Maximum stress (σ) before fracturing	Young's modulus (E)
Units	kg/m^3	MPa	GPa
Steel	7860	400	200

Forces acting on an object

Figure 3: Forces acting on a segment of an attached object



In the multidimentional space, vector analysis of forces acting on an object would be needed, however, in the two dimentional up/down space of the suspended object, the forces are simplified from vectors to a one dimensional track of values within $\pm\infty$. Considering a small (infinitesimal) segment of the rod, the forces acting on the segment depend on its position within the rod. Each segment generally experiences a gravitational force, F_g, stretching the object, depending on the weight of the part of the rod below the segment in question and an equal. It also experiences an opposing force, F_o, resisting the change, $F_g = -F_o$, arising from the fact that the segment is being secured in place by the previous segment of the rod. Forces governing the segment and their directionality are depicted in Fig. 3. Elongation continues until the forces get into equilibrium. If the force applied is too big, the object fractures.

2. Exploration

Constant gravitational field

Some objects exert a gravitational field so weak that its changes due to distance are negligible. In such cases, we approximate the gravitational field to a constant. One such example is the planet Earth, the fundamental properties of which are summarised in Table 2 below.

Table 2: Properties of Earth

	Radius (r_0)	*Surface acceleration (g)*
Units	*Km*	*m/s^2*
Earth	*≈ 6378*	*≈ 9.81*

Let's consider a steel rod of length L and a cross-section A hanging from a hook, which is exposed to a constant gravitational field. The top of the rod experiences the largest F_g, as the weight of the whole rod is acting on it. Here, the opposing force is the force in the hook F_h. The bottom of the rod, however, experiences a force of zero since no weight is acting underneath it.

Considering a small rod segment positioned at $L_x - dx$ with a length dx, the gravitational force acting on it is denoted as $F_g(L_x)$. Therefore, the weight of the segment can be written as:

$$F_g(L_x) - F_g(L_x - dx) = dF_g. \tag{4}$$

The mass of this segment can be calculated as:

$$dm = \rho A dx\ , \tag{5}$$

where ρ is the density of the material, and m is the mass of the rod.

The gravitational force acting on this segment due to its weight can be defined as:

$$dF_g = dmg = \rho A g dx. \tag{6}$$

To evaluate the total weight acting on the segment (the rod underneath it and the weight of the segment itself), we integrate dF_g from the bottom, 0, to the top of the segment, L_x:

$$F_g(L_x) = \int_0^{L_x} dF_g = \int_0^{L_x} \rho \cdot A \cdot g dx = \rho \cdot A \cdot g \int_0^{L_x} dx$$

$$F_g(L_x) = \rho \cdot A \cdot g \cdot L_x. \tag{7}$$

This force alone is not responsible for the deformation of the rod. Both the opposing and gravitational force together can deform the object. The force in the hook resists the stretching of the rod, while gravitational force elongates it. Using the derived equation for F_g from Eq. 7, stress (σ) from Eq. 1 acting on the segment dx can now be written as:

$$\sigma(L_x) = \frac{F_g(L_x)}{A} = \frac{\rho \cdot A \cdot g \cdot L_x}{A} = \rho \cdot g \cdot L_x. \tag{8}$$

This equation will later be used to obtain the maximum length of the rod. Moreover, total elongation of the rod can also be calculated. This is done by connecting the Young's modulus (E) from Eq. 3, with the Eq. 8, so that the strain is expressed as:

$$\epsilon(L_x) = \frac{\sigma(L_x)}{E} = \frac{\rho \cdot g \cdot L_x}{E}. \tag{9}$$

Using Eq. 2 for strain, the total elongation of a rod for a certain length L_x is calculated. We evaluate the elongation of a small segment dL_x starting at $L_x - dx$ as:

$$d(\Delta L_x) = \epsilon(L_x) \cdot dL_x = \frac{\rho\, g\, L_x}{E} dL_x. \tag{10}$$

The segments over the whole rod have to be added to obtain the total elongation of the rod. To achieve that, we integrate both sides from $x = 0$ to $x = L$. Using the fundamental theorem of calculus, left hand side is now ΔL_x.

$$\Delta L_x = \int_0^L \frac{\rho \cdot g \cdot L_x}{E}\, dL_x. \tag{11}$$

ΔL_x now becomes $\varDelta L$, as the expression does not refer to some length on the rod but instead the total length L of the rod. The expression is calculated by first excluding the constants from the integral and evaluating the rest according to the integration rules.

$$\Delta L = \frac{\rho \cdot g}{E} \int_0^L L_x\, dL_x = \frac{\rho \cdot g}{E} \cdot \left[\frac{L_x^2}{2}\right]_0^L = \frac{\rho \cdot g \cdot L^2}{2E}. \tag{12}$$

The total deformation (elongation) of the rod is therefore:

$$\Delta L = \frac{\rho \cdot g \cdot L^2}{2E}. \tag{13}$$

These findings can now be used to illustrate the effect of the object's weight on its deformation when the change in gravitational force along the length of the rod is not a considerable factor. Using data from Table 1, Table 2 and Eq. 13, the elongation of a steel rod, $L = 4\ m$, positioned on a hook in a uniform gravitational field of Earth is calculated:

$$\Delta L = \frac{\rho g L^2}{2E} = \frac{7860 \cdot 9.81 \cdot 4^2}{2 \cdot 200 \cdot 10^9} \approx 3.1\ 10^{-6} m. \tag{14}$$

Although this change is minimal compared to the length of the rod, it is still evidential proof that deformation occurs due to the object's weight.

The maximum length of a steel rod before breaking can also be determined using Eq. 8, where $L_x = L_{max}$:

$$\sigma_{max}(L_{max}) = \rho \cdot g \cdot L_{max}$$

$$L_{max} = \frac{\sigma_{max}(L_{max})}{\rho \cdot g} = \frac{400 \cdot 10^6}{7860 \cdot 9.81} \approx 5188\text{m}. \tag{15}$$

Again, as the effect of elongation is almost negligible, a steel rod would have to exceed a very large length to fracture due to its own weight.

If we consider the exact steel rod in the same uniform field but falling freely, it can be observed that the only force acting on the rod is a force of gravity F_g. The opposing force in the hook equals 0, because the rod is not attached to support but instead falls freely. Therefore, there is no stress in the rod itself, the total deformation of the rod is $\Delta L = 0m$, and the steel rod would not break at any length due to elongation.

Gravitational field changing inversely with the square of the distance

The constant field serves merely as an approximation for situations where the changes in gravitation can be disregarded. However, in many instances, such an approximation may not feasible as the gravitational force may vary significantly along the length of the object, influencing the behaviour of the objects subjected to gravitational fields. So, the precise equation of a changing field, varying inversely with the square of the distance from the centre of the generating object, is applied hereafter. The equation is derived from the Newton's law, which has been defined in introduction.

$$F = mg = \frac{GMm}{r^2}. \tag{16}$$

Here a variable g can be defined in terms of surface gravitation g_0 of the mass M, radius r_0 of the surface of the mass M and distance x from this surface.

$$g = \frac{GM}{r^2} \Rightarrow g_0 = \frac{GM}{{r_0}^2} \Rightarrow \mathrm{GM} = g_0 {r_0}^2 \Rightarrow g = \frac{g_0 {r_0}^2}{r^2} \Rightarrow g = \frac{g_0}{\left(\frac{r}{r_0}\right)^2} \Rightarrow g = \frac{g_0}{\left(1 + \frac{x}{r_0}\right)^2}.$$

We define $\frac{1}{r_0} = \beta$. Therefore, the varying gravity $g(x)$ can be expressed as:

$$g(x) = \frac{g_0}{(1 + \beta x)^2}\ . \tag{17}$$

In this case, contrary to the case with the constant gravitational field, the top of the object experiences the smallest Fg, as g is diminishing with $x > 0$. Each segment of the suspended object bounded by a hook, therefore, experiences different forces. This difference causes the object to elongate differently depending on the location x on the object. It is important to note that x in the case of a changing field is the same as L_x in the constant field, as the lower part of the object is positioned at r_0. Therefore variable x is used for a more concise notation.

Here, the Eq. 5 still applies. The force Fg acting on a segment can again be calculated using Eq. 6. However, for this derivation, the changes in gravitation from Eq. 17 over the object's length have to be taken into account. The differential expression for Fg can, therefore, be written as:

$$dF_g = d\big(mg(x)\big) = d(\rho \cdot A \cdot g(x) \cdot x) = \mathrm{d}\left(\rho \cdot A \cdot \frac{g_0}{(1 + \beta x)^2} \cdot x\right). \tag{18}$$

The equation can further be rearranged by expanding and obtaining the derivative, which is calculated with respect to x, as distance is the independent variable influencing the magnitude of Fg. The constants $\rho A g_0$ are excluded from the derivation as their value is not influenced by x.

$$dF_g = \rho A g_0 ((1 + \beta x)^{-2} \cdot x)' dx. \tag{19}$$

Applying the product rule for differentiation, $\frac{d}{dx}\, m(x)\, g(x) = \; g'(x)\, m(x) + \; g(x)\, m'(x)$ we differentiate $((1+\beta x)^{-2} \cdot x)$:

$$
\begin{aligned}
dF_g &= \rho A g_0(((1+\beta x)^{-2})'x + x'(1+\beta x)^{-2})dx = \\
&= \rho A g_0(-2\beta x(1+\beta x)^{-3} + (1+\beta x)^{-2})dx = \\
&= \rho A g_0((1+\beta x)^{-3}(-2\beta x + 1 + \beta x))dx
\end{aligned}
$$

$$
\mathrm{d}F_g = \rho A g_0 \frac{1-\beta x}{(1+\beta x)^3} dx. \tag{20}
$$

By dividing both sides by dx, the rate of change of Fg with respect to distance x is obtained.

$$
\frac{dF_g}{dx} = \rho A g_0 \frac{1-\beta x}{(1+\beta x)^3}. \tag{21}
$$

In order to determine the force F_g acting on the total length of the rod, $\frac{dF_g}{dx}$ is integrated over the whole rod. We integrate from $x = 0$ to L, with respect to x, where x represents the variable of integration along the length of the rod.

$$
F_g = \int_0^L \frac{dF_g}{dx} dx = \rho A g_0 \int_0^L \frac{1-\beta x}{\left(1+\beta x\right)^3} dx. \tag{22}
$$

This equation can further be simplified by substituting βx with y. Consequently $x = \frac{y}{\beta}$ and $dx = \frac{dy}{\beta}$. As β is a constant, it can be factored out from the equation. Furthermore, the limits of integration also have to be adjusted as we now integrate with respect to y. As we still integrate over the entire length of the rod, the limits in terms of x correspond to the following y: when $x = 0, y = 0$ when $x = L, y = \beta L$:

$$
F_g = \frac{\rho A g_0}{\beta} \int_0^{\beta L} \frac{1-y}{(1+y)^3} dy. \tag{23}
$$

To simplify the derivation, only the part including the integrals is considered in the next step. The integral is separated following the rule for the addition of integrals $(\int \big(f(x) + g(x)\big)dx = \int f(x)dx + \int g(x)dx)$.

$$\int_0^{\beta L} \frac{1}{(1+y)^3} dy - \int_0^{\beta L} \frac{y}{(1+y)^3} dy. \tag{24}$$

For a clear representation, each of the terms is integrated separately. We first consider the first term, where integration by substitution is used. v and dv are defined as follows:

$$v = 1 + y \quad \text{and} \quad dv = dy. \tag{25}$$

The defined variables are substituted into the integral in order to obtain the result. The limits of integration change accordingly when $y = 0, v = 1$, and $y = \beta L,\ v = 1 + \beta L$. After the integral is evaluated, the original notation is reinstated for consistency.

$$\int_0^{\beta L} \frac{1}{(1+y)^3} = \int_1^{1+\beta L} v^{-3}\, dv = \left[-\frac{1}{2} \cdot v^{-2}\right]_1^{1+\beta L} = \left[-\frac{1}{2} \cdot (1+y)^{-2}\right]_0^{\beta L} =$$

$$= -\frac{1}{2} \cdot (1+\beta L)^{-2} + \frac{1}{2}. \tag{26}$$

The second term consists of a product of two different functions. Therefore, integration by parts is suitable for use in this case. New variables u and v are defined as:

$$u = y \quad \text{and} \quad dv = \frac{1}{(1+y)^3}\, dy. \tag{27}$$

$$du = dy \quad \text{and} \quad v = -\frac{1}{2(1+y)^2}. \tag{28}$$

Equation $\int u\, dv = uv - \int v\, du$ is applied to integrate the expression by parts. After the final expression is obtained, all the variables are replaced by the original terms. The calculation proceeds as follows:

$$\int_0^{\beta L} \frac{y}{(1+y)^3} dy = \left[-\frac{y}{2(1+y)^2}\right]_0^{\beta L} + \int_0^{\beta L} \frac{1}{2(1+y)^2} dy =$$

$$= \left[-\frac{y}{2(1+y)^2}\right]_0^{\beta L} + \left[-\frac{1}{2(1+y)}\right]_0^{\beta L} = -\frac{\beta L}{2(1+\beta L)^2} - \frac{1}{2(1+\beta L)} + \frac{1}{2} \tag{29}$$

The solutions from each of the terms Eq. 26 and Eq. 29 are combined in the original Eq. 22. The total gravitational force on the rod over its whole length can now be defined as:

$$F_g = \rho A g_0 \int_0^L \frac{1-\beta x}{(1+\beta x)^{-3}} dx =$$

$$= \frac{\rho A g_0}{\beta}\left(\left(-\frac{1}{2\cdot(1+\beta L)^2}+\frac{1}{2}\right)+\left(-\frac{\beta L}{2(1+\beta L)^2}-\frac{1}{2(1+\beta L)}+\frac{1}{2}\right)\right) \tag{30}$$

$$F_g = \frac{\rho A g_0}{\beta}\frac{L\beta}{1+L\beta} = \frac{\rho A L g_0}{1+L\beta}. \tag{31}$$

The relationship between Fg and $\rho A L g_0$ can be established through known quantities ρ, A, L and g_0. However, the expression in the denominator $1 + L\beta$ reduces the Fg with increasing distance from mass M's surface ($\beta = \frac{1}{r}$) and with the increasing lenght of the suspended object under inspection. Moreover, we see again as in Eq. 31, that when L is small, $Fg \approx \rho A L g_0$. The result coincides with the result for gravitational force in a constant field, observed in Eq. 7.

Combining the procedures from Eq. 8, 9 and 10, we determine the elongation of one segment on the rod with variable gravitational acceleration.

$$d\Delta L = \frac{F_g}{AE}dx = \frac{\rho A \frac{x g_0}{1+x\beta}}{AE}dx = \frac{\rho}{E}\frac{g_0 x}{(1+x\beta)}dx \tag{32}$$

We find the total elongation by summing all the segments. This is done using integration with limits $x = 0$ to $x = L$:

$$\Delta L = \int_0^L d\Delta L = \frac{\rho g_0}{E}\int_0^L \frac{x}{(1+x\beta)}dx \tag{33}$$

As the expression does not have a direct antiderivative that would solve the, the method of integration by substitution is used. We define new variable *u* and rearrange its definition to obtain the following expressions:

$$u = 1 + \beta x \Rightarrow du = \beta dx \Rightarrow dx = \frac{du}{\beta} \Rightarrow x = \frac{u-1}{\beta} \tag{34}$$

The obtained expressions are substituted into Eq. 33. The expression is now integrated with respect to *du*, and the limits of integration change accordingly: when $x = 0, u = 1$, when $x =$

$L, u = 1 + \beta L$. Moreover β is exluded from the integration, as its value is a constant and remains unaffected. The resulting expression becomes:

$$\Delta L = \frac{\rho g_0}{E} \int_1^{1+\beta L} \frac{\frac{u-1}{\beta}}{u} \cdot \frac{du}{\beta} = \frac{\rho g_0}{E\beta^2} \int_1^{1+\beta L} \frac{u-1}{u} du$$

$$\Delta L = \frac{\rho g_0}{E\beta^2} \int_1^{1+\beta L} \left(1 - \frac{1}{u}\right) du \quad (35)$$

$$\Delta L = \frac{\rho g_0}{E\beta^2} |u - \ln u|_1^{1+\beta L} \quad (36)$$

We evaluate this expression using fundamental theorem of calculus $\int_a^b f'(x)dx = f(b) - f(a)$ to obtain the total elongation of the rod:

$$\Delta L = \frac{\rho g_0}{E\beta^2} ((1 + \beta L) - \ln(1 + \beta L) - 1 + \ln 1)$$

$$\Delta L = \frac{\rho g_0}{E\beta^2} (\beta L - \ln(1 + \beta L)) \quad (37)$$

Now replacing β with $\frac{1}{r_0}$ the total elongation of the string exposed to gravitational field changing inversely with square of the distance is:

$$\Delta L = \frac{\rho g_0 {r_0}^2}{E} \left(\frac{L}{r_0} - \ln\left(1 + \frac{L}{r_0}\right)\right) \quad (38)$$

It is interesting to see that for large distances r_0 the Eq. 38 converges to Eq. 14 . In this case the term $\frac{L}{r_0}$ is small and approaches 0. We can use the Taylor's series expansion for small x and write $ln(1 + x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots$ and use the result to show that the $\frac{L}{r_0} - ln\left(1 + \frac{L}{r_0}\right)$ approximately equals $\frac{L}{r_0} - \frac{L}{r_0} + 2\left(\frac{L}{r_0}\right)^2 - \cdots \approx 2\left(\frac{L}{r_0}\right)^2$ and hence $\Delta L = \frac{\rho g L^2}{2E}$.

To put the obtained results additionally into perspective, we can use Eq. 38 to calculate the elongation of a $4m$ steel rod suspended on a hook in the Earth's atmosphere. The equation requires data from Table 1 and Table 2.

$$\Delta L = \frac{7860 \times 9.81 \times 64378^2 \times 10^6}{200 \times 10^9}\left(\frac{4}{64378 \times 10^3} - \ln\left(1 + \frac{4}{64378 \times 10^3}\right)\right)$$

$$\Delta L = 3.08 \times 10^{-6}\, m \tag{39}$$

This result aligns with the expectations, as the value obtained from Eq. 14 is only marginally higher, larger by only 0.02 or 0,6 %. This proves that the changing gravitational field diminishes the effect of elongation. Moreover, we can also calculate the length at which the suspended rod would fracture, by combining, expanding and rearranging Eq. 1 and Eq. 31, where β is replaced by $\frac{1}{r_0}$:

$$\sigma = \frac{F_g}{A} = \frac{\rho A L g_0}{(1 + L\beta)A} = \frac{\rho L g_0}{(1 + L\beta)}$$

$$\sigma(1 + L\beta) = \rho L g_0$$

$$\sigma + \sigma L\beta = \rho L g_0$$

$$\sigma = \rho L g_0 - \sigma L\beta$$

$$L_{\max} = \frac{\sigma}{\rho g_0 - \sigma\beta} = \frac{\sigma}{\rho g_0 - \frac{\sigma}{r_0}} = \frac{400 \times 10^6}{7860 \times 9.81 - \frac{400 \times 10^6}{6378 \times 10^3}}$$

$$L_{\max} = 5192m \tag{40}$$

It can be observed that the maximum length in comparison to the maximum length calculated in Eq. 15 for a constant field, is by $4m$ longer. The result is expected, as the field changes and the gravitational force at $x = 5192m$ is smaller than at $x = 0$. Therefore, less force acts upon the object of the equivalent length, and it can stretch more. This means that an object can reach greater lengths before reaching the critical point where it fractures, because the strain is smaller.

Schwarzschild radius

Table 3: Some physical properties and relations to be used in further calculations

G (gravitational constant)	*c (speed of light)*	*Kinetic energy*	*Gravitational potential energy*
$6.6743 \times \frac{10^{-11} m^3}{kg\, s^2}$	$3 \times 10^8 m/s$	$\frac{mv^2}{2}$	$\frac{GMm}{r_0}$

Previous calculations prove that the deformation of an object depends on the material properties as well as on the strength of the gravitational field and the distance of the object from the centre of the mass M producing this field. The field exibited on the object can also be expressed with the mass M's radius and the distance of the object from this radius. Therefore, to calculate the elongation of the object exposed to the gravitational field generated by black holes, their radiuses (Schwarzschield's radiuses) need to be defined. Black holes are one of the densest bodies in the universe [8] and generate almost infinite gravitational fields from which even light cannot escape. This is helpful in determining their radiuses (horizons), which will denote the last point at which light will be detained, and therefore the border of the black hole. We also use physical properties and relations shown in Table 3. The escape velocity (velocity needed for an object to escape from the black hole) can be equated to the speed of light $v = c$. As the object wants to move further away, it increases its potential energy. Therefore, we can denote that the kinetic energy of this object equals the gravitational potential energy. [9]

$$\frac{mc^2}{2} = \frac{GMm}{r_0} \quad , \tag{41}$$

where G is again the gravitational constant (now of the black hole with mass M) and m is the mass of the object to escape the black hole.

By rearranging this equation, we can obtain the Schwarzschild's radius:

$$r_0 = \frac{2GM}{c^2} \tag{42}$$

Calculations for a black hole

For the calculations of a black hole, we consider five examples (Table 4), each belonging to a specific type of black hole, depending on their size. Three of those (XTE J1650-500, HLX-1, TON 618) are real life examples of black holes, while data for the two primordial black holes (one of which would be of a mass M of the planet Mars) is hypothetical, and has not yet been found in the universe.

Table 4: Considered black holes, their masses and surface gravity

Type	***Example***	***Mass (kg)***
Primordial	*Hypothetical*	10^{12}
Planet as primordial	*Mars*	6.41×10^{23}
Stellar-mass	*XTE J1650-500*	7.5×10^{30}
Intermediate-mass	*HLX-1*	2.0×10^{34}
Supermassive	*TON 618*	1.31×10^{41}

To complete calculations for deformation in the environment of those black holes, the Schwarzchield's radiuses first have to be defined by Eq. 42 and using Table 4. Afterwards, using Eq. 16, surface gravitation is calculated. Lastly, Table 1, Table 3 and the result for gravitation and radiuses are used for calculations of total elongation of a suspended $4m$ steel rod at rest, bounded by a hook. The length at which the rod would break when exposed to either a constant field, equal to the surface gravity or to gravitational field changing inversely with square of a distance is also calculated. The calculations are obtained using Eq. 13,15, 38, and 40. An example of calculation for a hypothetical primordial black hole is presented below.

Determination of Schwarzschields radius:

$$r_0 = \frac{2GM}{c^2} = \frac{2 \cdot 6.6743 \cdot 10^{-11} \cdot 10^{12}}{9 \cdot 10^{16}} \approx 1.48 \cdot 10^{-15}\, m \tag{43}$$

Calculation of surface gravitation:

$$g = \frac{GM}{r^2} = \frac{6.6743 \cdot 10^{-11} \cdot 10^{12}}{(1.48 \cdot 10^{-15})^2} \approx 3.05 \cdot 10^{31} \tag{44}$$

Calculating total elongation for constant approximation of gravitational field using Eq.13:

$$\Delta L = \frac{\rho \cdot g \cdot L^2}{2E} = \frac{7860 \cdot 3.05 \cdot 10^{31} \cdot 4^2}{2 \cdot 200 \cdot 10^9} \approx 9.59 \cdot 10^{24}\, m \tag{45}$$

Calculating the maximum length of the rod before breaking in a field with a constant approximation using Eq. 15:

$$L_{max} = \frac{\sigma(L)}{\rho \cdot g} = \frac{400 \cdot 10^6}{7860 \cdot 3.05 \cdot 10^{31}} \approx 1.67 \cdot 10^{-27} m \tag{46}$$

Calculating total elongation in a changing field using Eq. 38:

$$\Delta L = \frac{\rho g_0 {r_0}^2}{E}\left(\frac{L}{r_0} - \ln\left(1+\frac{L}{r_0}\right)\right) =$$

$$= \frac{7860\ \cdot\ 3.05 \cdot 10^{31} \cdot (1.48 \cdot 10^{-15})^2}{200 \cdot 10^9}\left(\frac{4}{1.48 \times 10^{-15}} - \ln\left(1+\frac{4}{1.48 \times 10^{-15}}\right)\right) \approx 7.10 \cdot 10^9 \quad (47)$$

Calculating the maximum length of the rod before breaking in a changing field using Eq. 40:

$$L_{max} = \frac{\sigma}{\rho g_0 - \frac{\sigma}{r_0}} = \frac{400\ \cdot 10^6}{7860\ \cdot 3.05 \cdot 10^{31} - \frac{400\ \cdot 10^6}{1.48 \cdot 10^{-15}}} \approx 1.67\ \cdot 10^{-27} m \qquad (48)$$

The results of the calculations for the other examples from Table 4 are displayed in a Table 5.

Table 5: Results of calculations for the black hole examples of Table 4

Type	*Mass (kg)*	*Schwarzschild radius (m)*	*Surface gravity (g m/s²)*	*ΔL (m) (constant field)*	*ΔL (m) (changing field)*	*L_{max}(m) (changing field)*	*L_{max}(m) (const. field)*
Primordial	10^{12}	1.48×10^{-15}	3.03×10^{31}	9.54×10^{24}	7.07×10^{9}	1.68×10^{-27}	1.68×10^{-27}
Mars	6.41×10^{23}	9.51×10^{-4}	4.73×10^{19}	1.49×10^{13}	7.06×10^{9}	1.08×10^{-15}	1.08×10^{-15}
XTE J1650-500	7.5×10^{30}	1.11×10^{4}	4.05×10^{12}	1.27×10^{6}	1.27×10^{6}	1.26×10^{-8}	1.26×10^{-8}
HLX-1	2.0×10^{34}	2.97×10^{7}	1.52×10^{9}	4.77×10^{2}	4.76×10^{2}	3.35×10^{-5}	3.35×10^{-5}
TON 618	1.31×10^{41}	1.94×10^{14}	2.32×10^{2}	7.28×10^{-5}	-2.17×10^{2}	2.20×10^{2}	2.20×10^{2}

Firstly, it can be observed that with increasing mass M, Schwarzschild's radius also increases. A significant change can be observed between the smallest and the biggest radius, ranging from $10^{-15} m$ to $10^{14} m$. Moreover, surface gravitation ranges from twenty times the Earths gravitation, to the order of 10^{31}. The values of the maximum length of an object (the rod) before fracturing do not differ for constant and changing fields when shown to three significant figures. This suggests that the contribution from the varying gravitational field is negligible. Moreover, an increase in the maximum length before object fractures is observed with increasing radius. The graph is plotted in Fig. 4.

A linear relationship between the two variables is observed, confirming that the maximum length increases linearly with increasing black hole's mass M.

Moreover, data for total rod elongation can also be analysed for the constant and varying gravitational field. Graphs for both examples are presented in Fig. 5.

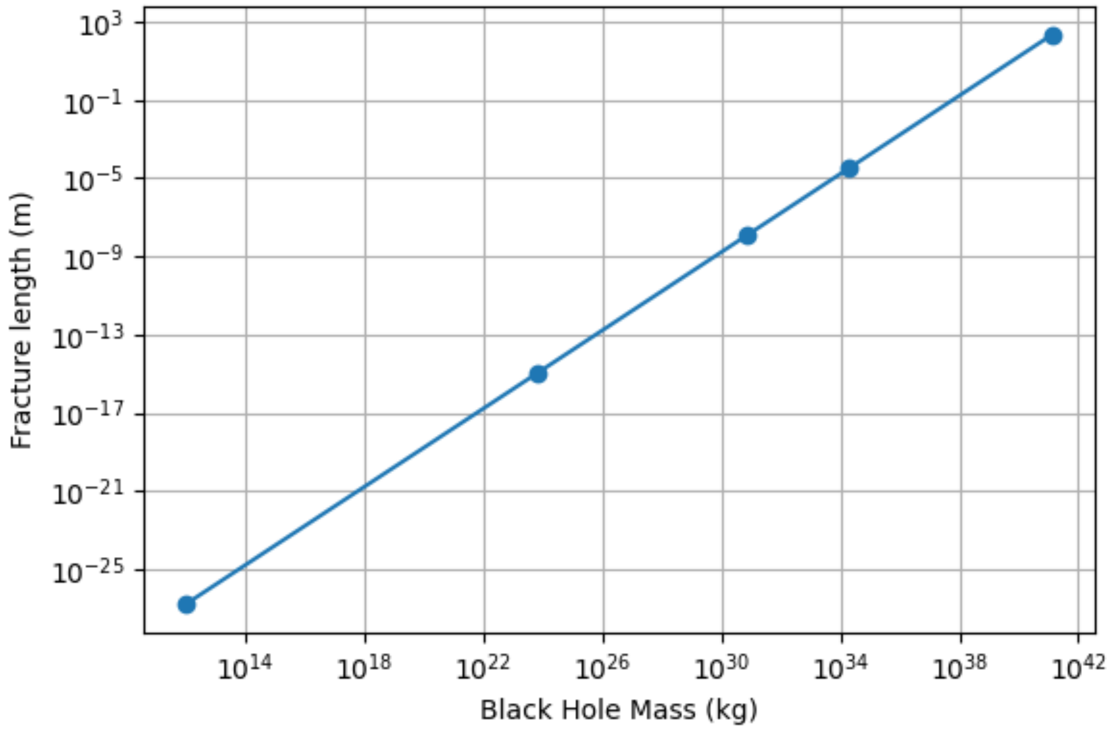


Figure 4: Maximum length versus black hole's mass M

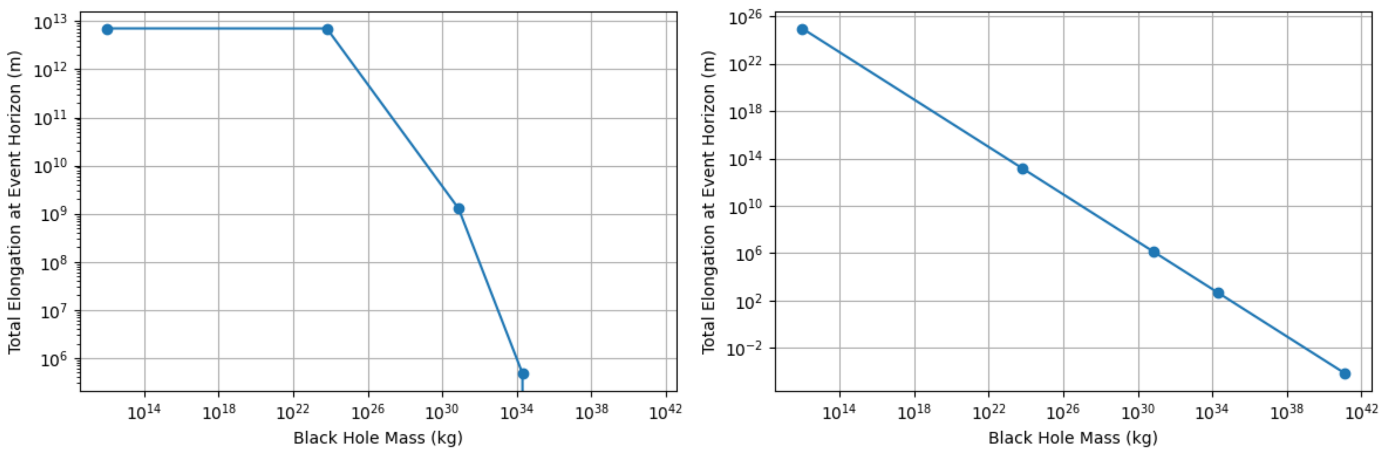


Figure 5: Total elongation of the rod for changing (left) and constant (right) gravitational field versus black hole's mass M

Both graphs show that the total elongation of the rod decreases with increasing mass of a black hole. This is an expected result coinciding with the results of the calculations for maximum length (Figure 4). The smaller the object generating the gravitational field is, the stronger the forces acting on a body in such a field are, resulting in extensive stretching. Interestingly, the elongation decreases linearly in a constant field and non-linearly in the case of a changing field.

In the integration, elongations much smaller than the length L were assumed. However, the elongation in the presence of the small black holes are much bigger than the length of the object itself. This means that the results has no physical meaning and that the rod breaks much before the object reaches the event horizon. Moreover, the results for stellar and intermediate-mass black holes coincide with the calculations for a constant field, suggesting that, again, the change in the gravitation over the length of the rod does not have a significant influence on the rod's behaviour. Interestingly, when calculating the results for the most massive black holes, the numerical error due to 16 bit representation of the real numbers in the calculator resulted in the negative difference of the linear and logarithmic term in the Eq. 38.

3. Conclusion

This investigation gives a fascinating insight into the behaviour of objects at the event horizon of a black hole. The aim of this exploration was first to determine what types of black holes are the most suitable candidates for exploration in terms of elongation of the space probes leading to their fracture. As a good approximation a rigid steel rod was used. The effect of the changing gravitational field was evaluated, and compared to the case where the gravitational field strenght is the same at upper and lower end of the rod. Firstly, it can be seen in Figure 4 that the maximum length of the rod in a black hole increases linearly with the black hole's mass. The calculation showed that the maximum length of an objects positioned in heavy black holes is greater. This is a consequence of a large distance r between the black hole center and the object influencing the gravitation acceleration as $g = \frac{GM}{r^2}$. Therefore, if the distance is smaller, the small value of the denominator increases the gravitational force and results in a more significant stretch. In conclusion, larger black holes would be best for exploration, as they could be approached closest to the event horizon.

Secondly, by comparing calculations for constant and changing fields, it has been discovered that the effect of changing gravitation between the lower and upper end of the object is negligible and that the most significant contribution to an object's elongation is its weight. A significant difference in the calculated elongation is observed for low-weight black holes. Note however that in both cases the calculated elongations cannot be realized physically as it has only been assumed that the elongations are small compared to the length of the object.

4. Evaluation

While the investigation has provided some important insights, it is essential to note the limited accuracy of the results. Firstly, a linear dependence between the strain and stress in the whole elongation region has been assumed. While this is a useful approximation for calculation, it might not be that accurate as the stress does not change linearly with strain after exiting the elastic region. Secondly, as already noted above, it has been assumed that the elongation is much smaller compared to the instantaneous integration length of the rod. Had this assumption not been used, the calculations would be much more complex and might not be analytically solvable.

Interestingly, the numerical error due to the limited accuracy of real numbers in the computer results in evidently wrong negative result, although the analytical expresion is positive.

5. Sources

[1] Space exploration. (n.d).) Smithsonian Books. [internet]. Available from: https://www.smithsonianbooks.com/store/aviation-military-history/the-smithsonian-history-of-space-exploration-from-the-ancient-world-to-the-extraterrestrial-future/. [Cited on December 26, 2024]

[2] The future of black hole exploration. (2024). Cosmofluencer. [internet]. Available from: https://cosmofluencer.com/the-future-of-black-hole-exploration/. [Cited on January 3, 2025]

[3] Black hole. (2024). Wikipedia. [internet]. Available from: https://en.wikipedia.org/wiki/Black_hole. [Cited on December 26, 2024]

[4] Newton's law of universal gravitation. (2024) Wikipedia. [internet]. Available from: https://en.wikipedia.org/wiki/Newton%27s_law_of_universal_gravitation. [Cited on December 26, 2024]

[5] Physical properties of material. (n.d.). Savree. [internet]. Available from: https://www.savree.com/en/encyclopedia/physical-properties-of-material. [Cited on October 15, 2024]

[6] What is stress formula? (2024). Geeks for Geeks. [internet]. Available from: https://www.geeksforgeeks.org/what-is-the-stress-formula/. [Cited on: October 10, 2024]

[7] Mechanics of Slender Structures (n.d.). Boston University of Mechanical Engineering. [internet]. Available from: https://www.bu.edu/moss/mechanics-of-materials-strain/. [Cited on November 13, 2024]

[8] Density of black holes. (n.d.). LibreText Chemistry. [internet]. Available from: https://chem.libretexts.org/Ancillary_Materials/Exemplars_and_Case_Studies/Exemplars/Physics_and_Astronomy/Density_of_Black_Holes. [Cited on January 13, 2025]

[9] Kinetic energy of an object falliing into an event horizon. (2025). PhysicsForums. [internet]. Available from: https://www.physicsforums.com/threads/kinetic-energy-of-an-object-falling-into-an-event-horizon.982919/#google_vignette. [Cited on January 3, 2025]