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The canonical coloring graph of trees and cycles*

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Abstract

For a graph G and an ordering of the vertices π, the set of canonical k-colorings of G under π is the set of non-isomorphic proper k-colorings of G that are lexicographically least under π. The canonical coloring graph $\mathrm{Can}_k^{\pi}(G)$ is the graph with vertex set the canonical colorings of G and two vertices are adjacent if the colorings differ in exactly one place. This is a natural variation of the color graph $\mathcal{C}_k(G)$ where all colorings are considered. We show that every graph has a canonical coloring graph which is disconnected; that trees have canonical coloring graphs that are Hamiltonian; and cycles have canonical coloring graphs that are connected.

Keywords: Graph coloring, Canonical coloring

Math. Subj. Class.: 05C15

1 Introduction

For a positive integer k and a graph G, the k-coloring graph of G, $\mathcal{C}_k(G)$, is the graph whose vertex set is the set of proper k vertex colorings of G. Two k colorings are joined by an edge if they differ in color on just one vertex of G. The basic question of whether $\mathcal{C}_k(G)$ is connected has recieved significant attention, see for example [1, 5, 6, 8]. In theoretical physics $\mathcal{C}_k(G)$ appears in the Glauber dynamics of an anti-ferromagnetic Potts model at zero temperature. In Glauber dynamics, the state space of the Markov chain is the set of all k-colorings of a graph. Markov processes also give a method to approximate the number of k-colorings of G. If $\mathcal{C}_k(G)$ is connected, the probabilities of moving between colorings in $\mathcal{C}_k(G)$ can be assigned so that the stationary distribution is uniform on all proper colorings. If the process converges fast enough (in polynomial time) then repeated sampling can be used to obtain a reasonable approximation of the total number of proper colorings. There is an extensive literature on when there is rapid mixing of $\mathcal{C}_k(G)$ see for example [5, 6, 7, 8].

*In memory of Michael Albertson, who provided primary colors in my graph theory toolbox.

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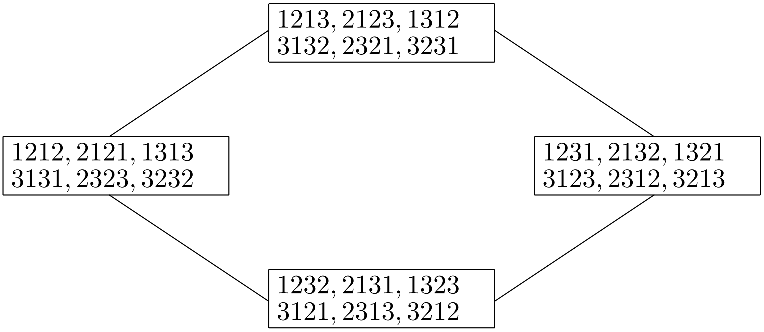


Figure 1: $I_3(P_4)$

Recently, several groups of graph theorists have given conditions which ensure the connectivity of $\mathcal{C}_k$ for classes of graphs. Moreover, they have shown that there is no such condition that depends only on the chromatic number $\chi(G)$.

Proposition 1.1 (Cereceda, van den Heuvel, and Johnson [1]). *There is no function $F(\chi)$, so that for all graphs G and integers $k > F(\chi(G))$, $\mathcal{C}_k(G)$ is connected.*

Recall the coloring number, $\mathrm{col}(G)$, is the smallest integer t for which there exists an ordering of the vertices $v_1, \dots v_n$, such that for all i the degree of v_i in the induced graph on $v_1, \dots v_i$ is less than t. That is, there exists an ordering of the vertices so that G can be greedily colored using t colors.

Theorem 1.2 (Dyer, Flaxman, Frieze, Vigoda [3]). *For any graph G and integer $k \geq \mathrm{col}(G) + 1$, $\mathcal{C}_k(G)$ is connected.*

Beyond whether $\mathcal{C}_k(G)$ is connected, MacGillivray and Choo have shown that for all graphs G, $\mathcal{C}_k(G)$ is Hamiltonian for sufficiently large k.

Theorem 1.3 (Choo and MacGillivray [2]). *If $k \geq \mathrm{col}(G) + 2$, then $\mathcal{C}_k(G)$ is Hamiltonian.*

In this paper we consider only non-isomorphic colorings of a graph G and give some conditions for when the graph of such colorings is connected and Hamiltonian. Section 2 provides definitions and basic results. Sections 3 and 4 give results for trees and cycles respectively.

2 The canonical coloring graph

Two colorings of G are *isomorphic* if one results from permuting the names of the colors only (i.e, we do not consider automorphisms of the graph). They are *non-isomorphic* otherwise. It is natural to consider the variation of the coloring graph where the vertices correspond to isomorphism classes of colorings of G. For simplicity, we will refer to each vertex as a coloring of G (rather than an isomorphism class of colorings). There are several ways to define the edge set. This paper will mainly look at the canonical coloring graph defined below. Another possibility is the *isomorphic coloring graph*, $I_k(G)$, which is defined to be the graph with an edge between two colorings c, d if some representative of c

differs at exactly one vertex from some representative of d. Figure 1 shows $I_3(P_4)$, where P_4 is the path on 4 vertices.

Proposition 2.1. *If $\mathcal{C}_k(G)$ is connected then so is $I_k(G)$.*

Proof. There is natural graph homomorphism from $\mathcal{C}_k(G)$ to a multigraph which is $I_k(G)$ with the addition of loops at each vertex. □

On the other hand, $I_k(G)$ may be connected when $\mathcal{C}_k(G)$ is not. For example, $I_3(C_5)$ is also C_5 while $\mathcal{C}_3(C_5)$ is two disjoint 15 cycles.

For a graph G and an ordering of the vertices π, the set of *canonical k-colorings* of G under π is the set of non-isomorphic proper k-colorings of G that are lexicographically least under π. The *canonical coloring graph* $\mathrm{Can}_k^\pi(G)$ is the graph whose vertices are the canonical colorings of G where two vertices are adjacent if the colorings differ in exactly one place. The order π of vertices of G can result in different $\mathrm{Can}_k^\pi(G)$. For example, Figure 2 shows $\mathrm{Can}_3^{\pi_i}(P_4)$ for three different orderings of the vertices. In each case the path is formed by edges $\{(1,2),(2,3),(3,4)\}$. If the vertices are colored in the same order as the path, $\pi_1 = (1,2,3,4)$, the canonical coloring graph $\mathrm{Can}_3^{\pi_1}(P_4)$ is a path (Figure 2a). For $\pi_2 = (1,3,2,4)$, canonical coloring graph $\mathrm{Can}_3^{\pi_2}(P_4)$ is a pair of disjoint edges (Figure 2b). For $\pi_3 = (3,2,1,4)$, canonical coloring graph $\mathrm{Can}_3^{\pi_2}(P_4)$ is a cycle (Figure 2c). Thus properties of $\mathrm{Can}_k^\pi(G)$ may depend on the order π. Indeed, whenever G is not a complete graph there exists an order for which $\mathrm{Can}_k^\pi(G)$ is disconnected.

Theorem 2.2. *Let $G \neq K_n$ be a connected graph and $k \geq \chi(G)+1$. Then there exists an order π such that $\mathrm{Can}_k^\pi(G)$ is disconnected.*

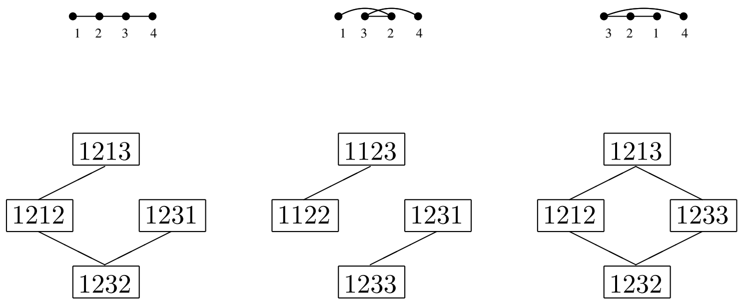
Proof. Observe that if G is connected on $n > 2$ vertices and not complete then it contains an induced P_3. Let π be an ordering of the vertices $(a,b,c,\dots)$ such that $(a,c),(b,c) \in E(G)$ and $(a,b) \notin E(G)$. Now every canonical coloring of G must begin either $112\dots$ or $123\dots$. Every coloring of the first kind differs in (at least) two places from every coloring of the second kind, thus as long as there is at least one coloring of each kind, the graph $\mathrm{Can}_k^\pi(G)$ is disconnected. Since there is a canonical proper coloring with $\chi(G)$ colors, at least one of coloring will be obtained with $\chi(G)$ colors. A proper coloring of the first kind is obtained by taking a proper coloring of G with exactly $\chi(G)$ colors, changing the color on vertices a and b to the $(\chi+1)$ color and recoloring to get the canonical coloring isomorphic to this. A proper coloring of the second kind is obtained by taking a proper coloring of G with exactly $\chi(G)$ colors, changing only the color on vertex b to the $(\chi+1)$ color and recoloring for the canonical coloring isomorphic to this. □

3 Trees

In this section we give conditions so that the canonical coloring graph of a tree has a Hamilton cycle.

Theorem 3.1. *For T a tree with $n \geq 4$ vertices, and $k \geq 3$ colors, there is an order π of the vertices such that $\mathrm{Can}_k^\pi(T)$ has a Hamilton cycle.*

Proof. The proof is in three parts. We first show the result for stars, then for trees using at least 4 colors, and finally for trees using exactly three colors.



(a) (b) (c)

Figure 2: P_4 and $\mathrm{Can}_3^{\pi}(P_4)$ for three different orderings of the vertices of P_4. The colors are listed in the order given by the permutation.

Part I: Stars. Let S_n be the star with $n \geq 4$ vertices and $k \geq 3$ colors, let π be an ordering of the vertices such that the root is first.

We first address the case where $k = 3$. It is straightforward to check if $n \leq 3$ $\mathrm{Can}_3^{\pi}(S_n)$ has a Hamilton path (though not a cycle). Assume now there is a Hamilton path on $\mathrm{Can}_3^{\pi}(S_{n-1})$, specifically colorings $c_1, c_2, \ldots, c_N$. Every coloring of S_n is obtained by taking a coloring c_i of S_{n-1} and choosing either color 2 or 3 for the nth vertex. Now $\mathrm{Can}_3^{\pi}(S_n) = Can_3^{\pi}(S_{n-1}) \bigotimes K_2$, the Cartesian product. Hence $\mathrm{Can}_3^{\pi}(S_n)$ has a Hamilton cycle if $\mathrm{Can}_3^{\pi}(S_{n-1})$ has a Hamilton path.

For $k > 3$, if $n = 1, 2, 3$, $\mathrm{Can}_k^{\pi}(S_n) = Can_3^{\pi}(S_n)$ has a Hamilton path. A Hamilton cycle on $\mathrm{Can}_k^{\pi}(S_4)$ is given by $1223, 1222, 1232, 1234, 1233$.

Assume there is a Hamilton cycle on $\mathrm{Can}_k^{\pi}(S_{n-1})$, specifically colorings $c_1, c_2, \ldots, c_N$. Choose c_N, c_1 so that they both use at least colors $1, 2, 3$. Every coloring of S_n is obtained by taking a coloring c_i of S_{n-1} and choosing a color from $2, 3, \ldots, r_i \leq k$ for the nth vertex, where $r_i = \min\{k, 1+$ largest color used in $c_i\}$. Let c_i^j denote the coloring of S_n that agrees with coloring c_i on the first $n-1$ vertices and is color j on the nth vertex. The following is a Hamilton cycle on all the colorings of $\mathrm{Can}_k^{\pi}(S_n)$:

$$\begin{array}{c}
c_1^4, c_1^5, \ldots, c_1^{r_1}, c_1^3, c_1^2, \\
c_2^2, c_2^4, c_2^5, \ldots, c_2^{r_2}, c_2^3, \\
c_3^3, c_3^4, c_3^5, \ldots c_3^{r_3}, c_3^2, \\
c_4^2, c_4^4, c_4^5, \ldots, c_4^{r_4}, c_4^3, \\
\ldots \\
< c_N^2, c_N^3 >, c_N^5, \ldots, c_N^{r_N}, c_N^4
\end{array}$$

where the order of $< c_N^2, c_N^3 >$ depends on the parity of N.

Part II: Trees with $k > 3$ colors. Order the vertices $v_1, \ldots v_n$ so each vertex v_i is a leaf of the tree induced by $v_1, \ldots v_i$. For $n = 4$, the case where T is a star was considered above. The other tree on 4 vertices is a path. A possible ordering of the vertices and the resulting Hamilton cycle for $\mathrm{Can}_k^{\pi}(P_4)$ is shown in Figure 2(c). Again we proceed by induction on n. Let $T' = T - v_n$. By assumption $\mathrm{Can}_k^{\pi}(T')$ has a Hamilton cycle $c_1, c_2, \ldots, c_N$. We may take c_1 to be the unique coloring that uses only two colors. For any i, c_i can be extended to a coloring of T by coloring v_n. Let F_i be the set of colors for v_n compatible with c_i and let $\hat{c}_i$ be the set of colorings of T that agree with c_i on T'. All colorings of T are obtained this way. Vertex v_n can receive any color already used except the one used on its unique neighbor v_q. If only $r < k$ colors have been used in c_i then color $r+1$ can be used as well. Thus $|F_1| = 2$ and $|F_i| \geq 3$ for $i > 1$.

As in the part I of the proof we will proceed by using all colorings of $\hat{c}_1$ followed by those of $\hat{c}_2$ etc., returning to $\hat{c}_1$. Note that for all i the induced graph on $\hat{c}_i$ is complete. Thus we merely need to ensure there is an edge between $\hat{c}_{i-1}$ and $\hat{c}_i$ that uses a different vertex in $\hat{c}_i$ than some edge between $\hat{c}_i$ and $\hat{c}_{i+1}$.

If c_i and c_{i+1} use the same color for vertex v_q then $|F_i \cap F_{i+1}| = \min(|F_i|, |F_{i+1}|)$. If c_i and c_{i+1} are different on vertex v_q then $|F_i \cap F_{i+1}| = \min(|F_i|, |F_{i+1}|) - 1$. Thus $|F_i \cap F_{i+1}| \geq 2$ except possibly when c_i, c_{i+1} differ on vertex v_q and additionally either $i = 1$ or $i = N$ (the case $|F_N \cap F_1|$). Since c_1 uses just colors $1, 2$, the unique coloring that differs from c_1 only on v_q must assign v_q color 3. So either c_N or c_2 must differ from c_1 on a vertex other than v_q. Thus at least one of $|F_N \cap F_1|$ and $|F_1 \cap F_2|$ will have 2 elements and we can select an edge from each of $\hat{c_N}\hat{c}_1$ and $\hat{c_1}\hat{c}_2$ that use different vertices in $\hat{c}_1$.

Part III Trees using exactly 3 colors. A more involved proof is necessary for the case when exactly 3 colors are used to color the tree. The proof is similar to that used by MacGillvray and Choo in [2] to show $\mathcal{C}_3(T)$ is Hamiltonian for T not a star.

The proof is by induction on $n = |V(T)|$. Observe that for $n \leq 4$, $\mathrm{Can}_3^{\pi}(T)$ has a Hamilton path. If T is a star, then $\mathrm{Can}_3^{\pi}(T)$ has a Hamilton cycle by part I of the proof. Otherwise, consider $T' = T - \{u, v\}$ for some pair of leaves u, v, that do not have a common neighbor. Such a pair must exist since T is not a star. If $n > 4$ the inductive hypothesis gives that $\mathrm{Can}_3^{\pi}(T')$ has a Hamilton path $c_0, c_1, \ldots, c_{N-1}$.

Order the vertices of T with u, v last and so the order on T' is the one that gave the Hamilton path $c_0, c_1, \ldots, c_{N-1}$. Let $\hat{c}_i$ be the induced subgraph of $\mathrm{Can}_3^{\pi}(T)$ whose vertex set is the set of canonical 3-colorings of T that agree with c_i on T'. Clearly, $\hat{c}_i$ is a 4 cycle. The Hamilton cycle in $\mathrm{Can}_3^{\pi}(T)$ will be constructed so it passes through $\hat{c}_0, \hat{c}_1, \ldots \hat{c}_{N-2}, \hat{c}_{N-1}, \hat{c}_{N-2}, \ldots \hat{c}_2, \hat{c}_1, \hat{c}_0$, in that order.

Let u_0 and v_0 denote the unique vertices adjacent to $u, v \in T$ respectively. Let $[\hat{c}_i, \hat{c}_{i+1}]$ denote the subgraph consisting of the edges between $\hat{c}_i$ and $\hat{c}_{i+1}$ in $\mathrm{Can}_3^{\pi}(T)$ and the vertices they are incident to. Recall that the colorings c_i and c_{i+1} differ on exactly one vertex, say x. There are 3 cases.

Case 1: $x \notin \{u_0, v_0\}$. Then there is a perfect matching between the 4 vertices in $\hat{c}_i$ and $\hat{c}_{i+1}$.

Case 2: $x = u_0$. In this case there will be two edges between the vertices in $\hat{c}_i$ and $\hat{c}_{i+1}$. We describe the subgraph induced by the vertices in $\hat{c}_i \cup \hat{c}_{i+1}$ in this case. Without loss of generality, let the 2 colors available for use on u be $\{1, 3\}$ in $\hat{c}_i$ and $\{2, 3\}$ in $\hat{c}_{i+1}$; and the two colors for use on v be $\{1, 2\}$ in both $\hat{c}_i$ and $\hat{c}_{i+1}$. The 8 vertices in $\hat{c}_i \cup \hat{c}_{i+1}$ are labeled by the colorings on T' (c_i or c_{i+1}) followed by the color on u and the color on v. Thus the 10 edges will be $\{(c_i11, c_i12), (c_i12, c_i32), (c_i32, c_i31), (c_i31, c_i11)\}$ in $\hat{c}_i$, $\{(c_{i+1}31, c_{i+1}21), (c_{i+1}21, c_{i+1}22), (c_{i+1}22, c_{i+1}32), (c_{i+1}32, c_{i+1}31)\}$ in $\hat{c}_{i+1}$, and $\{(c_i32, c_{i+1}32), (c_i31, c_{i+1}31)\}$ in $E([\hat{c}_i, \hat{c}_{i+1}])$.

Case 3: $x = v_0$ is similar. Again there are exactly two edges in $E([\hat{c}_i, \hat{c}_{i+1}])$. The vertices they are incident to in $\hat{c}_i$ are adjacent, and the vertices in $\hat{c}_{i+1}$ are adjacent.

Next, for each i we construct the two pieces of the cycle P_i, Q_i that use all vertices of $\hat{c}_i$ such that $P_0 P_1 P_2 \ldots P_{N-1} Q_{N-1} \ldots Q_2 Q_1$ is the desired Hamilton cycle.

Step 0. Construct P_0. Say the 4-cycle $\hat{c}_0$ is $\alpha_0, \beta_0, \gamma_0, \delta_0$. Assume without loss of generality that α_0, δ_0 are adjacent to vertices in $\hat{c}_1$, call these vertices in $\hat{c}_1$ β_1, γ_1 respectively. Then define $P_0 = \alpha_0, \beta_0, \gamma_0, \delta_0$.

Step i. β_i, γ_i have been defined previously as vertices in $\hat{c}_i$ adjacent to vertices in $\hat{c}_{i-1}$. There are several cases. In each case P_i will begin with γ_i and Q_i will end with β_i.

Case uu. Suppose the colorings c_{i-1}, c_i, c_{i+1} differ only by color change on u_0 (respectively, only on v_0). In this case u (respectively, v) must take on all 3 colors, so that $V([\hat{c}_{i-1}, \hat{c}_i]) \cap V([\hat{c}_i, \hat{c}_{i+1}]) = \emptyset$. Label α_i, δ_i such that the 4-cycle that is $\hat{c}_i$ is $\alpha_i, \beta_i, \gamma_i, \delta_i$. Now α_i, δ_i are the vertices adjacent to vertices in $\hat{c}_{i+1}$, which we now label $\beta_{i+1}, \gamma_{i+1}$ respectively. Define $P_i = \gamma_i, \delta_i$ and $Q_i = \alpha_i, \beta_i$. Figure 3(Top) gives an example of this case.

Case uv. Suppose the colorings c_{i-1}, c_i, c_{i+1} differ by color change first on u_0 and then on v_0. In this case $V([\hat{c}_{i-1}, \hat{c}_i]) \cup V([\hat{c}_i, \hat{c}_{i+1}])$ use only three different vertices of $\hat{c}_i$. If $V([\hat{c}_{i-1}, \hat{c}_i]) \cap V([\hat{c}_i, \hat{c}_{i+1}]) = \gamma_i$ then label the rest of the 4 cycle in $\hat{c}_i$ so that δ_i is also adjacent to a vertex in $\hat{c}_{i+1}$. Next label the vertices adjacent to γ_i, δ_i in $\hat{c}_{i+1}$ as $\beta_{i+1}, \gamma_{i+1}$ respectively. Now define $P_i = \gamma_i$ and $Q_i = \delta_i, \alpha_i, \beta_i$. Figure 3(Bottom) gives an example of this case.

Similarly, if $V([\hat{c}_{i-1}, \hat{c}_i]) \cap V([\hat{c}_i, \hat{c}_{i+1}]) = \beta_i$ then label so that $\beta_{i+1}, \gamma_{i+1} \in \hat{c}_{i+1}$ are the vertices adjacent to α_i, β_i respectively. Now define $P_i = \gamma_i, \delta_i, \alpha_i$ and $Q_i = \beta_i$.

Case ux. Suppose the colorings c_{i-1}, c_i, c_{i+1} differ first by a color change on u_0 and then on some $x \notin \{u_0, v_0\}$. Label α_i, δ_i such that the 4-cycle that is $\hat{c}_i$ is $\alpha_i, \beta_i, \gamma_i, \delta_i$. Now α_i, δ_i are adjacent to vertices in $\hat{c}_{i+1}$, which we now label $\beta_{i+1}, \gamma_{i+1}$ respectively. Define $P_i = \gamma_i, \delta_i$ and $Q_i = \alpha_i, \beta_i$.

Case xx. Suppose the colorings c_{i-1}, c_i, c_{i+1} differ only on vertice(s) not $\{u_0, v_0\}$. As in the other cases, label α_i, δ_i such that the 4-cycle that is $\hat{c}_i$ is $\alpha_i, \beta_i, \gamma_i, \delta_i$. Now α_i, δ_i are adjacent to vertices in $\hat{c}_{i+1}$, which we now label $\beta_{i+1}, \gamma_{i+1}$ respectively. Define $P_i = \gamma_i, \delta_i$ and $Q_i = \alpha_i, \beta_i$.

Cases vu, xu, xv and vx are similar to those above. □

4 Cycles

Another simple class of graphs to consider is C_n, the cycle on n vertices. We prove that there exists an order such that $\mathrm{Can}_k^\pi(C_n)$ is connected for most n, k. While lengthy computer calculations indicate $\mathrm{Can}_k^\pi(C_n)$ is Hamiltonian for $n > 5$, a concise proof is elusive.

Theorem 4.1. *For C_n a cycle with $n \geq 3$, vertices, and $k \geq 4$ colors, there is an order π of the vertices such that $\mathrm{Can}_k^\pi(C_n)$ is connected. Further, $\mathrm{Can}_3^\pi(C_4)$ and $\mathrm{Can}_3^\pi(C_5)$ are connected for some π but $\mathrm{Can}_3^\pi(C_n)$ is disconnected for all π, for $n > 5$.*

Proof. The proof is in two parts.

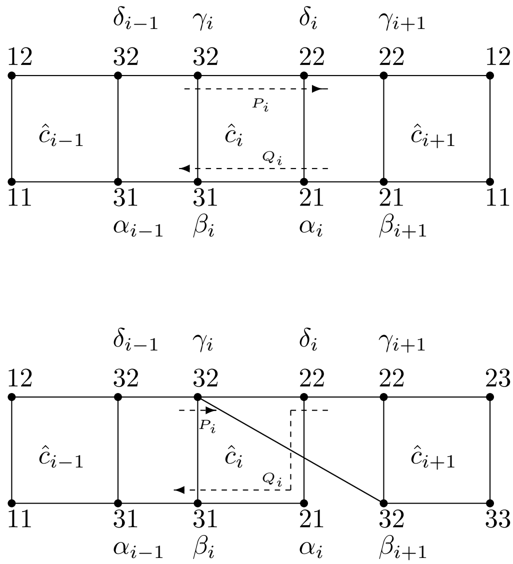


Figure 3: Both figures show possible subgraphs induced by $\hat{c}_{i-1} \cup \hat{c}_i \cup \hat{c}_{i+1}$. The numbers (eg 12) give colors on u, v in the underlying graph. The subscripted letters correspond to the notation developed in Step 0 and i of the proof. The dashed lines represent the paths P_i and Q_i. Top: The colorings c_{i-1}, c_i, c_{i+1} differ only by color change on u_0. Bottom: The colorings c_{i-1}, c_i, c_{i+1} differ by color change first on u_0 and then v_0.

Part I: $k = 3$**.** It is easy to check that $\mathrm{Can}_3^\pi(C_4)$ and $\mathrm{Can}_3^\pi(C_5)$ are connected for many π (for example, let the cycle be $(1,2,3,4,5)$ and color in the order given by $\pi = (1,4,2,3,5)$). We show that for $n > 5$ $\mathrm{Can}_3^\pi(C_n)$ is disconnected. In a 3-colored cycle, the only vertices that can change color are those whose neighbors in the cycle are both the same color. Thus, for $n > 5$, when $n = 3t$ the coloring $[1,2,3,1,2,3,\ldots,1,2,3]$ (where the vertices are listed in the usual order), will be an isolated vertex under any ordering π. For $n = 3t+2$ consider first $\mathcal{C}_3(C_n)$, all colorings of C_n in the standard order. Define L to be the set of colorings which for some choice of initial vertex are contiguous substrings of $1,2,3,1,2,3,\ldots 1,2,3$. Colorings in L are only adjacent to other colorings in L. For example, consider $[1,2,3,1,2,3,\ldots,1,2,3,1,2]$ in the usual order. The only colorings this is adjacent to in $\mathcal{C}_3(C_n)$, whether canonical or not, are $[\mathbf{3},2,3,1,2,3,\ldots,1,2,3,1,2]$ and $[1,2,3,1,2,3,\ldots,1,2,3,1,\mathbf{3}]$, where the bolded number is the changed color. Each of these is also a coloring in L, the first by considering the second vertex as the "initial vertex" and the second by considering the last vertex as the initial vertex. Thus any coloring not in L must be in a different connected component of $\mathcal{C}_3(C_n)$. Since at least one coloring of L is in $\mathrm{Can}_3^\pi(C_{3t+2})$ for any ordering π, $\mathrm{Can}_3^\pi(C_{3t+2})$ must also be disconnected.

For $n = 3t+1$ things are slightly more complex. Again consider first the regular coloring graph $\mathcal{C}_3(C_n)$. Define M to be the set of colorings consisting two disjoint contiguous substrings of $[1,2,3,1,2,3,\ldots,1,2,3]$. Colorings in M are only adjacent to other such colorings in $\mathcal{C}_3(C_n)$. Without loss of generality suppose the substrings meet as follow $[\ldots,3,1,2,3,2,3,1,\ldots]$. This is only adjacent to $[\ldots,3,1,2,\mathbf{1},2,3,1,\ldots]$ and $[\ldots,3,1,2,3,\mathbf{1},3,1,\ldots]$ and two more similar colorings at the other end of where the two strings meet. In each case the new coloring simply increases the length of one substring of $[1,2,3,1,2,3,1,2,3,\ldots,1,2,3]$ and decreases the other. Under any ordering π there will be at least one coloring of type M when considered in the usual order, and at least one coloring not of type M, so $\mathrm{Can}_3^\pi(C_{3t+1})$ must also be disconnected.

Part II: $k \geq 4$ **colors.** For $k \geq 4$ colors, the cases $n = 3,4$ are easily verified. For $n \geq 5$, Theorem 3.1 gives a method for finding a Hamilton cycle in $\mathrm{Can}_k^\pi(P_{n-1})$, where P_n is the path on n vertices. Let the colorings of P_{n-1} be $c_1, c_2, \ldots c_N$ listed in the order they give a Hamilton cycle and so that c_1 is the unique coloring that uses only 2 colors.

For any i, c_i can be extended to a coloring of C_n by coloring v_n. Let F_i be the set of colors for v_n compatible with c_i and let $\hat{c}_i$ be the set of colorings of C_n. All colorings of C_n are obtained this way. Vertex v_n can receive any color already used except the ones used on its neighbors v_1 and v_{n-1}. If only $r < k$ colors have been used in c_i then color $r+1$ can be used as well. Thus $|F_i| \geq 2$ for $i > 1$. Consider first the case where $i \neq 1, N$. If c_i and c_{i+1} are the same on vertices v_1, v_{n-1} then $|F_i \cap F_{i+1}| = \min(|F_i|, |F_{i+1}|) \geq 2$. If c_i and c_{i+1} are different on one of the vertices v_1, v_{n-1} then $|F_i \cap F_{i+1}| = \min(|F_i|, |F_{i+1}|) - 1 \geq 1$.

For $i = 1$ or $i = N$ consider the cases that n is odd or even separately. If $n-1$ is odd the Hamilton cycle constructed on $\mathrm{Can}_k^\pi(P_{n-1})$ will contain the segment consisting of colorings $c_N = [1,2,1,2,1,\ldots,1,3,1]; c_1 = [1,2,1,2,1,\ldots,1,2,1]; c_2 = [1,2,1,2,1,\ldots,1,2,3]$. Here $|F_N \cap F_1| = |\{2,3\}| = 2 > 0$; and $|F_1 \cap F_2| = |\{2\}| > 0$. So there is at least one edge between $\hat{c}_N$ and $\hat{c}_1$ and also between $\hat{c}_1$ and $\hat{c}_2$. If $n-1$ is even the Hamilton cycle constructed on $\mathrm{Can}_k^\pi(P_{n-1})$ will contain the the segment consisting of colorings $c_N = [1,2,1,2,1,\ldots,2,1,3]$; $c_1 = [1,2,1,2,1,\ldots,2,1,2]$; $c_2 = [1,2,1,2,1,\ldots,2,3,2]$. In this case $F_N = \{2,4\}$; $F_1 = \{3\}$, $F_2 = \{3,4\}$ so $F_N \cap F_1 = \emptyset$. But since $F_1 \cap F_2 = \{3\}$ there is one edge between $\hat{c}_1$ and $\hat{c}_2$ and $\mathrm{Can}_k^\pi(C_n)$ is connected. □

Note that the method in this proof is not enough to guarantee a Hamilton Cycle in $\mathrm{Can}_k^\pi(C_n)$ as it is possible that $|F_{i-1} \cap F_i| = |F_i \cap F_{i+1}| = 1$, and each of $|F_{i-1}| = |F_i| = |F_{i+1}| = 2$. This occurs when each of c_{i-1}, c_i, c_{i+1} uses exactly colors $\{1, 2, 3\}$ and the colors on (v_1, v_{n-1}) are $(1, 3), (1, 2), (3, 2)$ in colorings c_{i-1}, c_i, c_{i+1} respectively. In this case $F_{i-1} \cap F_i = F_i \cap F_{i+1} = \{4\}$, while the possibilities for v_n are $F_{i-1} = \{2, 4\}, F_i = \{3, 4\}, F_{i+1} = \{1, 4\}$.

It would be interesting to have broad results paralleling Theorems 1.2 and 1.3 for the cannonical coloring graph. Indeed, for all G does there exists k, π such that $\mathrm{Can}_k^\pi(G)$ is connected? In a forthcoming paper the author shows connectivity for some classes of graphs (see [4]).

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