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Indecent Proposals

We have been running this journal for 10 years now. We try to keep its quality as high as possible, publishing papers connected with at least two mathematical disciplines, one of which is rooted in discrete mathematics. We are still receiving too many papers that barely fit the scope of the journal, however, and we will address this in the near future.

But there is another matter that we would like to share with you today. Every so often we receive requests to do something that is not usual in our culture. We call these *indecent proposals*. Below we quote some of the must unusual of them, received from individuals we have never heard of before:

- "Dear Editor, I would like to become a member of the editorial board. Please find my CV enclosed."
- "Dear Editor, I do not understand why my paper has been rejected. Please consider the fact that I am willing to pay article processing charges if you reverse your ruling. You do not have to print the paper. It is sufficient that you put it on-line."
- "Dear Editor, We are looking for cooperation. We will secure 20 to 30 peer-reviewed papers that we would like to publish in your journal. We will pay 200 USD per paper."
- "Dear Editor, we are organising a conference on a charming little island. We would like to publish the proceedings as a special issue of your journal that we want to guest edit."

We consider any such unsolicited letters as *faux pas* and ignore them.

On the other hand, we would like to point out an important strategy we follow in our journal that might differ from the strategies of some other journals. We encourage authors to list in their cover letter some renowned mathematicians who are specialists in the field of their paper and could be regarded as objective referees. This is even more important if the topic of the submitted paper is not one with which our editors are familiar.

Of course, this might be considered an "indecent proposal" by some other journals! In practice it does not mean that we will choose referees from the list provided by the author. But such a list can be of considerable help to us in when we are seeking the right people to referee a paper, consistent with our stated aim of maintaining a high quality journal.

Dragan Marušič and Tomaž Pisanski

Editors In Chief

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Improved bounds for hypohamiltonian graphs*

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In loving memory of Ella.

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Abstract

A graph G is *hypohamiltonian* if G is non-hamiltonian and $G - v$ is hamiltonian for every $v \in V(G)$. In the following, every graph is assumed to be hypohamiltonian. Aldred, Wormald, and McKay gave a list of all graphs of order at most 17. In this article, we present an algorithm to generate all graphs of a given order and apply it to prove that there exist exactly 14 graphs of order 18 and 34 graphs of order 19. We also extend their results in the cubic case. Furthermore, we show that (i) the smallest graph of girth 6 has order 25, (ii) the smallest planar graph has order at least 23, (iii) the smallest cubic planar graph has order at least 54, and (iv) the smallest cubic planar graph of girth 5 with non-trivial automorphism group has order 78.

Keywords: Hamiltonian, hypohamiltonian, planar, girth, cubic graph, exhaustive generation.

Math. Subj. Class.: 05C10, 05C38, 05C45, 05C85

1 Introduction

Throughout this paper all graphs are undirected, finite, connected, and neither contain loops nor multiple edges, unless explicitly stated otherwise. A graph is *hamiltonian* if it contains a cycle visiting every vertex of the graph. Such a cycle or path is called *hamiltonian*. A graph G is *hypohamiltonian* if G is non-hamiltonian, and for every $v \in V(G)$ the graph $G - v$ is hamiltonian.

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We call a vertex *cubic* if it has degree 3, and a graph *cubic* if all of its vertices are cubic. Let G be a graph. We use $\deg(v)$ to denote the degree of a vertex v and $\Delta(G) = \max_{v \in V(G)} \deg(v)$. The *girth* of a graph is the length of its shortest cycle. A cycle of length k will be called a *k-cycle*. For $S \subset V(G)$, $G[S]$ shall denote the graph induced by S. A subgraph $G' = (V', E') \subset G = (V, E)$ is *spanning* if $V' = V$. For a set X, we denote by $|X|$ its cardinality. We refer to [14] for undefined notions.

The study of hypohamiltonian graphs was initiated in the early sixties by Sousselier [33], and Thomassen made numerous important contributions [34–38]; for further details, see the survey of Holton and Sheehan [21, Chapter 7] from 1993. For more recent results and new references not contained in the survey, we refer to the article of Jooyandeh, McKay, Östergård, Pettersson, and the second author [22].

In 1973, Chvátal showed [11] that if we choose n to be sufficiently large, then there exists a hypohamiltonian graph of order n. We now know that for every $n \geq 18$ there exists such a graph of order n, and that 18 is optimal, since Aldred, McKay, and Wormald showed that there is no hypohamiltonian graph on 17 vertices [2]. Their paper fully settled the question for which orders hypohamiltonian graphs exist and for which they do not exist. For more details, see [21, Chapter 7].

They also provide a complete list of hypohamiltonian graphs with at most 17 vertices. There are seven such graphs: exactly one for each of the orders 10 (the Petersen graph), 13, and 15, four of order 16 (among them Sousselier's graph), and none of order 17. Aldred, McKay, and Wormald [2] showed that there exist at least thirteen hypohamiltonian graphs with 18 vertices, but the exact number was unknown. In [25], McKay lists all known hypohamiltonian graphs up to 26 vertices (recall that the lists with 18 or more vertices may be incomplete). He also lists all cubic hypohamiltonian graphs up to 26 vertices as well as the cubic hypohamiltonian graphs with girth at least 5 and girth at least 6 on 28 and 30 vertices, respectively. In Section 2.3 we extend the results both for the general and cubic case.

The main contributions of this manuscript are: (i) an algorithm $\mathfrak{A}$ to generate all pairwise non-isomorphic hypohamiltonian graphs of a given order, (ii) the results of applying this algorithm, and (iii) an up-to-date overview of the best currently available lower and upper bounds on the order of the smallest hypohamiltonian graphs satisfying various additional properties, see Table 1. The algorithm $\mathfrak{A}$ is based on the algorithm of Aldred, McKay, and Wormald from [2], but is extended with several additional bounding criteria which speed it up substantially. Furthermore, $\mathfrak{A}$ also allows to generate planar hypohamiltonian graphs and hypohamiltonian graphs with a given lower bound on the girth far more efficiently.

We present $\mathfrak{A}$ in Section 2 and showcase the new complete lists of hypohamiltonian graphs we obtained with it. In Section 3 we illustrate how $\mathfrak{A}$ can be extended to generate planar hypohamiltonian graphs and show how we applied $\mathfrak{A}$ to improve the lower bounds on the order of the smallest planar hypohamiltonian graph. (In the following, unless stated otherwise, when we say that a graph is "smaller" or "the smallest", we always refer to its order.) Using the program *plantri* [9], we also give a new lower bound for the order of the smallest cubic planar hypohamiltonian graph. In an upcoming paper [15], we will adapt the approach used in the algorithm $\mathfrak{A}$ to generate *almost hypohamiltonian graphs* [43] efficiently. (A graph G is *almost hypohamiltonian*, if it is non-hamiltonian and there exists a vertex w such that $G - w$ is non-hamiltonian, but $G - v$ is hamiltonian for every vertex $v \neq w$.)

Table 1: Bounds for the order of the smallest hypohamiltonian graph with additional properties. The bold numbers are new bounds obtained in this manuscript; if an entry contains two lines, the upper line indicates the new bounds, while the lower line shows the previous bounds. The symbol "–" designates an impossible combination of properties and $a..b$ means that the number is at least a and at most b. $b = \infty$ signifies that no graph with the given properties is known.

girth	3	4	5	6	7	8	9
general	18	18	10	**25** 18..28	**28** 18..28	**36**..∞ 18..∞	**61**..∞ 18..∞
cubic	–	24	10	28	28	**50**..∞ 30..∞	**66**..∞ 58..∞
planar	**23**..240 18..240	**25**..40 18..40	45	–	–	–	–
planar & cubic	–	**54**..70 44..70	76	–	–	–	–

We now discuss the numbers given in Table 1 and start with the first row. For girth 3, Aldred, McKay, and Wormald [2] showed that there is no hypohamiltonian graph of girth 3 and order smaller than 18, and Collier and Schmeichel [12] showed already in 1977 that there exists such a graph on 18 vertices. For girth 4, the results of [2] imply that there is no such graph on fewer than 18 vertices, and the hypohamiltonian graph presented in Figure 1 (b) from Section 2.3 provides an example of order 18—this graph was given earlier and independently by McKay [25]. The third number is due to the Petersen graph, for which it is well-known that it is the smallest hypohamiltonian graph, see e.g. [19]. The smallest hypohamiltonian graph of girth 6 was obtained by the application of $\mathfrak{A}$ and is shown in Figure 2. For girth 7, Coxeter's graph provides the smallest example. Its minimality as well as the new lower bound for girth 8 follows from the application of $\mathfrak{A}$. The bound for girth 9 follows from an argument given at the end of the following paragraph. Note that, as Máčajová and Škoviera mention in [24], no hypohamiltonian graphs of girth greater than 7 are known, and Coxeter's graph is the only known cyclically 7-connected hypohamiltonian graph of girth 7.

Concerning the second row, Thomassen [38] showed that there exists a cubic hypohamiltonian graph of girth 4 and order 24. Petersen's graph is responsible for the second value, Isaacs' flower snark J_7 and Coxeter's graph give the upper bounds for girth 6 and 7, respectively. Through an exhaustive computer-search, McKay was able to determine the order of the smallest cubic hypohamiltonian graph of girth 4, 5, 6, and 7, establishing that the aforementioned graphs turned out to be the smallest of a fixed girth, see [25]. (Note that McKay does not state this explicitly, and that these results were verified independently by the first author.) We obtained the improved lower bounds for girth 8 and 9 through an exhaustive computer-search (see Section 2.3 for more details). Now let G be a hypohamiltonian graph of girth 9 containing a non-cubic vertex v. Then $\{w \in V(G) : d(v, w) \leq 4\}$, where $d(v, w)$ denotes the number of edges in a shortest path between vertices v and w, consists of pairwise different vertices, so $|V(G)| \geq 61$. (Recall that as is shown in Table 1, if G is a cubic hypohamiltonian graph of girth 9, then $|V(G)| \geq 66$.)

In the third row, the first upper bound is due to Thomassen, see [36], while the second one is due to Jooyandeh, McKay, Östergård, Pettersson, and the second author [22]. The previous best lower bounds were provided by [2]—although that paper does not address planarity—while the current best lower bounds are proven using $\mathfrak{A}$, see Section 3. In [22] it was also shown that there exists a planar hypohamiltonian graph of girth 5 on 45 vertices, and that there is no smaller such graph.

The upper bound for the smallest cubic planar hypohamiltonian graph of girth 4 was established by Araya and Wiener [3]. The best available lower bound prior to this paper can be found in the same article [3] and was 44. We improved this to 54 with the program *plantri* [9] as described in Section 3.3. Finally, McKay [28] recently proved that the order of the smallest cubic planar hypohamiltonian graph of girth 5 is 76.

In Table 1, we denote by "–" an impossible combination of properties. There are two arguments from which these impossibilities follow. Firstly, a cubic hypohamiltonian graph cannot contain triangles, as proven by Collier and Schmeichel [13]. Secondly, it follows from Euler's formula that a planar 3-connected graph—it is easy to see that every hypohamiltonian graph is 3-connected—has girth at most 5.

2 Generating hypohamiltonian graphs

2.1 Preparation

In this section we present our algorithm $\mathfrak{A}$ to generate all non-isomorphic hypohamiltonian graphs of a given order. $\mathfrak{A}$ is based on work of Aldred, McKay, and Wormald [2], but contains essential additional bounding criteria. It is easy to see that hypohamiltonian graphs are 3-connected and cyclically 4-connected.

We follow Aldred, McKay and Wormald [2] and say that a graph G is *hypocyclic* if for every $v \in V(G)$, the graph $G - v$ is hamiltonian. Hamiltonian hypocyclic graphs are usually called "1-hamiltonian" (see e.g. [10]), so the family of all hypocyclic graphs is the disjoint union of the families of all 1-hamiltonian and hypohamiltonian graphs.

We now present several lemmas with necessary conditions for a graph to be hypocyclic or hypohamiltonian. We then use a selection of these lemmas to prune the search in the generation algorithm. This selection, i.e. whether to use a certain lemma or not and the order in which these lemmas should be applied, is based on experimental evidence. The efficiency of the algorithm strongly depends on the strength of these pruning criteria.

To avoid confusion, we will generally use the same terminology as Aldred, McKay, and Wormald did in [2] (that is: e.g. type A, B, and C obstructions). Let G be a possibly disconnected graph. We will denote by $p(G)$ the minimum number of disjoint paths needed to cover all vertices of G, by $V_1(G)$ the vertices of degree 1 in G, and by $I(G)$ the set of all isolated vertices and all isolated edges of G. Put

$$k(G) = \begin{cases} 0 & \text{if } G \text{ is empty,} \\ \max\left\{1, \left\lceil \frac{|V_1|}{2} \right\rceil\right\} & \text{if } I(G) = \emptyset \text{ but } G \text{ is not empty,} \\ |I(G)| + k(G - I(G)) & \text{else.} \end{cases}$$

Lemma 2.1 (Aldred, McKay, and Wormald [2]). *Given a hypocyclic graph G, for any partition (W, X) of the vertices of G with $|W| > 1$ and $|X| > 1$, we have that*

$$p(G[W]) < |X| \quad \textit{and} \quad k(G[W]) < |X|.$$

Now consider a graph G containing a partition (W, X) of its vertices with $|W| > 1$ and $|X| > 1$. If $p(G[W]) \geq |X|$, then we call (W, X) a *type A obstruction*, and if $k(G[W]) \geq |X|$, then we speak of a *type B obstruction*. For efficiency reasons we only consider type A obstructions where $G[W]$ is a union of disjoint paths.

Lemma 2.2 (Aldred, McKay, and Wormald [2])**.** *Let G be a hypocyclic graph, and consider a partition (W, X) of the vertices of G with $|W| > 1$ and $|X| > 1$ such that W is an independent set. Furthermore, for some vertex $v \in X$, define n_1 and n_2 to be the number of vertices of $X - v$ joined to one or more than one vertex of W, respectively. Then we have $2n_2 + n_1 \geq 2|W|$ for every $v \in X$.*

If all assumptions of Lemma 2.2 are met and $2n_2 + n_1 < 2|W|$ for some $v \in X$, we call (W, X, v) a *type C obstruction*.

Intuitively, by a *good Y-edge* (for $Y \in \{A, B, C\}$) we mean an edge which works towards the destruction of a type Y obstruction. We will now formally define these good Y-edges.

We use Lemma 2.1 as follows. Assume G' is a hypohamiltonian graph and that G is a spanning subgraph of G' which contains a type A obstruction (W, X) (where $G[W]$ is a union of disjoint paths). Since G' is hypohamiltonian it cannot contain a type A obstruction, so there must be an edge e in $E(G') \setminus E(G)$ whose endpoints are in different components of $G[W]$ and for which at least one of the endpoints has degree at most one in $G[W]$. We call such an edge a *good A-edge* for (W, X).

Aldred, McKay, and Wormald [2] did use this obstruction, but they did not require these good A-edges to have an endpoint of degree at most one in $G[W]$ (which turns out to be far more restrictive). Similarly, a *good B-edge* for a type B obstruction (W, X) in G is a non-edge of G that joins two vertices of W where at least one of those vertices has degree at most one in $G[W]$. Finally, a *good C-edge* for a type C obstruction (W, X, v) in G is a non-edge e of G for which one of the two following conditions holds:

(i) Both endpoints of e are in W.

(ii) One endpoint of e is in W and the other endpoint is in $X - v$ and has at most one neighbour in W.

We leave the straightforward verification that this is the only way to destroy a type B/C obstruction to the reader. Likewise, it is elementary to see that every hypohamiltonian graph has minimum degree 3—we are mentioning this explicitly, since we will later make use of the fact that hypohamiltonian graphs do not contain vertices of degree 2—and that it is not bipartite. However, for every $k \geq 23$ there exists a hypohamiltonian graph containing the complete bipartite graph $K_{2k-44,2k-44}$, as proven by Thomassen [38].

Lemma 2.3 (Collier and Schmeichel [13])**.** *Let G be a hypohamiltonian graph containing a triangle T. Then every vertex of T has degree at least* 4*.*

A *diamond* is a K_4 minus an edge and the *central edge* of a diamond is the edge between the two cubic vertices.

Proposition 2.4. *Let G be a hypohamiltonian graph containing a diamond with vertices a, b, c, d and central edge ac. Then the degrees of a and c (in G) are at least* 5*.*

Proof. It follows from Lemma 2.3 that a is not cubic. Let a have degree 4. Since G is hypocyclic, $G - c$ contains a hamiltonian cycle $\mathfrak{h}$. $\mathfrak{h}$ must contain ab or ad (possibly both), say ab. But then $(\mathfrak{h} - ab) \cup acb$ is a hamiltonian cycle in G, a contradiction. □

Note that in Proposition 2.4, the edge bd may or may not be present in the graph. We have already mentioned that hypohamiltonian graphs are cyclically 4-connected. We can strengthen this in the following way.

Lemma 2.5. *One of the two components obtained when deleting a* 3*-edge-cut from a hypohamiltonian graph must be* K_1*.*

Proof. Consider a 3-edge-cut C in a hypohamiltonian graph G. $G - C$ has two components A and B with $|V(A)| \leq |V(B)|$. We put $C = \{a_1b_1, a_2b_2, a_3b_3\}$, where $a_i \in V(A)$ and $b_i \in V(B)$. Assume $A \neq K_1$. In this situation, since G is 3-connected, the elements of the set $\{a_1, a_2, a_3, b_1, b_2, b_3\}$ are pairwise distinct, as otherwise we would have a 2-cut.

Since G is hypohamiltonian, $G - b_3$ is hamiltonian, so there is a hamiltonian path $\mathfrak{p}_A$ in A with end-vertices a_1 and a_2. As $G - a_3$ is hamiltonian, there is a hamiltonian path $\mathfrak{p}_B$ in B with end-vertices b_1 and b_2. Now $\mathfrak{p}_A \cup \mathfrak{p}_B + a_1b_1 + a_2b_2$ is a hamiltonian cycle in G, a contradiction. □

Proposition 2.6. *Let G be a hypohamiltonian graph containing a* 3*-cut* $M = \{u, v, w\}$*.*

(i) We have $uv, vw, wu \notin E(G)$*.*

(ii) If M is not the neighbourhood of a vertex, then $\max_{x \in M} \deg(x) \geq 4$*.*

Proof. (i) Note that (i) was also already shown by Thomassen in [36], but here we give an alternative proof. Assume that $uv \in E(G)$. Since G is hypohamiltonian, there exists a hamiltonian cycle $\mathfrak{h}$ in $G - u$. Let A and B be the components of $G - M$ (we leave to the reader the easy proof that there are exactly two components in $G - M$) and put $\mathfrak{p}_A = \mathfrak{h} \cap G[V(A) \cup M]$.

Case 1: $A \neq K_1$ and $B \neq K_1$. Since M is a 3-cut, $\mathfrak{p}_A$ has end-vertices v and w. Analogously there exists a hamiltonian path $\mathfrak{p}_B$ in $G[V(B) \cup M]$ with end-vertices u and w. Now $\mathfrak{p}_A \cup \mathfrak{p}_B + uv$ is a hamiltonian cycle in G, a contradiction.

Case 2: $A = K_1$. We have $V(A) = \{a\}$, so $M = N(a)$. Now auv is a triangle containing the cubic vertex a, in contradiction to Lemma 2.3.

(ii) follows directly from Lemma 2.5. Note that the neighbourhood condition is necessary, since cubic hypohamiltonian graphs—such as the Petersen graph—do exist. □

Corollary 2.7. *In a cubic hypohamiltonian graph, every* 3*-cut must be the neighbourhood of a vertex.*

2.2 The enumeration algorithm

The pseudocode of the enumeration algorithm $\mathfrak{A}$ is given in Algorithm 1 and Algorithm 2.

In order to generate all hypohamiltonian graphs with n vertices we start from a graph G which consists of an $(n-1)$-cycle and an isolated vertex h (disjoint from the cycle), so $G - h$ is hamiltonian. Both in Algorithm 1 and Algorithm 2 we only add edges between existing vertices of the graph. So if a graph is hamiltonian, all graphs obtained from it will also be hamiltonian. Thus we can prune the search when a hamiltonian graph is constructed (cf. line 1 of Algorithm 2).

In Algorithm 1 we connect h to D vertices of the $(n-1)$-cycle in all possible ways and then perform Algorithm 2 on these graphs which will continue to recursively add edges without increasing the maximum degree of the graph.

It is essential for the efficiency of the algorithm that as few as possible edges are added (i.e. that as few as possible graphs are constructed), while still guaranteeing that all hypohamiltonian graphs are found by the algorithm. If a generated graph contains an obstruction for hypohamiltonicity, it clearly cannot be hypohamiltonian and hence we only add edges which destroy (or work towards the destruction of) that obstruction.

In the following theorem we show that this algorithm indeed finds all hypohamiltonian graphs.

Theorem 2.8. *If Algorithm 1 terminates, the list of graphs $\mathcal{H}$ outputted by the algorithm is the list of all hypohamiltonian graphs with n vertices.*

Proof. It follows from line 23 of Algorithm 2 that $\mathcal{H}$ only contains hypohamiltonian graphs. Now we will show that $\mathcal{H}$ indeed contains *all* hypohamiltonian graphs with n vertices.

Consider a hypohamiltonian graph G with n vertices. It follows from the definition of hypohamiltonicity that there is a spanning subgraph G_0 of G which consists of an $(n-1)$-cycle C and a vertex v disjoint from C which is connected to $\Delta(G)$ vertices of C. Since Algorithm 1 connects the vertex h with D vertices of an $(n-1)$-cycle in all possible ways, it will also construct a graph which is isomorphic to G_0.

We will now show by induction that Algorithm 2 produces a graph isomorphic to a spanning subgraph G with i edges for every $|E(G_0)| \leq i \leq |E(G)|$.

Assume this claim holds for some i with $|E(G_0)| \leq i \leq |E(G)| - 1$ and call the graph produced by Algorithm 2 which is isomorphic to a spanning subgraph of G with i edges G'.

Assume that G' contains a type A obstruction (W, X). By Lemma 2.1, G does not contain a type A obstruction, so there is a good A-edge e for (W, X) in $E(G) \setminus E(G')$. It follows from line 4 of Algorithm 2 that Construct($G' + e, D$) is called and $G' + e$ will be accepted by the algorithm since G is non-hamiltonian.

We omit the discussion of the cases where G' contains a type B or C obstruction (i.e. lines 18 and 10, respectively) as this is completely analogous.

So assume that G' does not contain a type A obstruction, but contains a vertex v of degree two (note that G' cannot contain vertices of degree less than two). Since a hypohamiltonian graph has minimum degree 3, there is an edge $e \in E(G) \setminus E(G')$ which contains v as an endpoint. It follows from line 8 of Algorithm 2 that Construct($G' + e, D$) is called.

The case where G' contains a cubic vertex which is part of a triangle (i.e. line 14) is completely analogous.

If none of the criteria is applicable, Algorithm 2 adds an edge e to G' in all possible ways (without increasing the maximum degree) and calls Construct($G' + e, D$) for each e. Since $|E(G')| < |E(G)|$, at least one of the graphs $G' + e$ will be a spanning subgraph of G with $i + 1$ edges. □

To make sure no isomorphic graphs are accepted, we use the program *nauty* [26, 30]. In principle more sophisticated isomorphism rejection techniques are known (such as the canonical construction path method [27]), but these methods are not compatible with the destruction of obstructions for hypohamiltonicity. Furthermore, isomorphism rejection is not a bottleneck in our implementation of this algorithm.

Algorithm 1 Generate all hypohamiltonian graphs with n vertices

let $\mathcal{H}$ be an empty list
let $G := C_{n-1} + h$
for all $3 \leq D \leq n-1$ **do**
  // Generate all hypohamiltonian graphs with $\Delta = D$
  for every way of connecting h of G with D vertices of the C_{n-1} **do**
    Call the resulting graph G'
    Construct(G', D) // i.e. perform Algorithm 2
  end for
end for
Output $\mathcal{H}$

Also note that we only have to perform the hypohamiltonicity test (which can be computationally very expensive) if the graph does not contain any obstructions for hypohamiltonicity (cf. line 23 of Algorithm 2). Therefore, the hypohamiltonicity test is not a bottleneck in the algorithm.

Since our algorithm only adds edges and never removes any vertices or edges, all graphs obtained by the algorithm from a graph with a g-cycle will have a cycle of length at most g. So in case we only want to generate hypohamiltonian graphs with a given lower bound k on the girth, we can prune the construction when a graph with a cycle with length less than k is constructed.

The order in which the bounding criteria of Algorithm 2 are tested is vital for the efficiency of the algorithm. By performing various extensive experiments, it turned out that the order in which the bounding criteria are listed in Algorithm 2 is the most efficient.

We also note that even though Aldred, McKay, and Wormald mentioned type C obstructions in their paper [2], they did not use them in their algorithm. However, our experimental results show that type C obstructions are significantly more helpful than e.g. type B obstructions.

2.3 Results

2.3.1 The general case

We implemented the algorithm $\mathfrak{A}$ in C and used it to generate all pairwise non-isomorphic hypohamiltonian graphs of a given order (with a given lower bound on the girth). Our implementation of this algorithm is called *GenHypohamiltonian*, and can be downloaded from [16].

Table 2 shows the counts of the complete lists hypohamiltonian graphs which were generated by our program. We generated all hypohamiltonian graphs up to 19 vertices and also went several steps further for hypohamiltonian graphs with a given lower bound on the girth. Recall that previously the complete lists of hypohamiltonian graphs were only known up to 17 vertices. For more information about the previous bounds and results, we refer to Table 1 from Section 1.

In [2] Aldred, McKay, and Wormald also produced a sample of 13 hypohamiltonian graphs with 18 vertices. It follows from our results that there are exactly 14 hypohamiltonian graphs with 18 vertices. These graphs are shown in Figure 1. The fourteenth graph which was not already known has girth 5 and is shown in Figure 1 (n). It has automorphism

Algorithm 2 Construct(Graph G, int D)

```
if G is non-hamiltonian AND not generated before then
    if G contains a type A obstruction (W, X) then
        for every good A-edge e ∉ E(G) for (W, X) for which Δ(G + e) = D do
            Construct(G + e, D)
        end for
    else if G contains a vertex v of degree 2 then
        for every edge e ∉ E(G) which contains v as an endpoint for which Δ(G + e) =
        D do
            Construct(G + e, D)
        end for
    else if G contains a type C obstruction (W, X, v) then
        for every good C-edge e ∉ E(G) for (W, X, v) for which Δ(G + e) = D do
            Construct(G + e, D)
        end for
    else if G contains a vertex v of degree 3 which is part of a triangle then
        for every edge e ∉ E(G) which contains v as an endpoint for which
                Δ(G + e) = D do
            Construct(G + e, D)
        end for
    else if G contains a type B obstruction (W, X) then
        for every good B-edge e ∉ E(G) for (W, X) for which Δ(G + e) = D do
            Construct(G + e, D)
        end for
    else
        if G is hypohamiltonian then
            add G to the list ℋ
        end if
        for every edge e ∉ E(G) for which Δ(G + e) = D do
            Construct(G + e, D)
        end for
    end if
end if
```

group size 36 and it has the largest group size among the hypohamiltonian graphs with 18 vertices. Using $\mathfrak{A}$, we also showed that there are exactly 34 hypohamiltonian graphs with 19 vertices. As can be seen from Table 2, all 34 of them have girth 5.

All graphs from Table 2 can also be downloaded from the *House of Graphs* [5] at `http://hog.grinvin.org/Hypohamiltonian` and also be inspected in the database of interesting graphs by searching for the keywords "hypohamiltonian * 2016".

Tables 3-5 list the running times of the algorithm. The column *"Max. nr. edges added"* denotes the maximum number of edges added by Algorithm 2 to a graph constructed by Algorithm 1 (i.e. the maximum number of recursive calls of *Construct()*).

The reported running times were obtained by executing our implementation of Algorithm 1 on an Intel Xeon CPU E5-2690 CPU at 2.90GHz. For the larger cases we did not include any running times in Tables 3-5 since these were executed on a heterogeneous

Table 2: The number of hypohamiltonian graphs. The columns with a header of the form $g \geq k$ contain the number of hypohamiltonian graphs with girth at least k. The counts of cases indicated with a '$\geq$' are possibly incomplete; all other cases are complete.

Order	# hypoham.	$g \geq 4$	$g \geq 5$	$g \geq 6$	$g \geq 7$	$g \geq 8$
$0-9$	0	0	0	0	0	0
10	1	1	1	0	0	0
11	0	0	0	0	0	0
12	0	0	0	0	0	0
13	1	1	1	0	0	0
14	0	0	0	0	0	0
15	1	1	1	0	0	0
16	4	4	4	0	0	0
17	0	0	0	0	0	0
18	14	13	8	0	0	0
19	34	34	34	0	0	0
20	?	≥ 98	4	0	0	0
21	?	?	85	0	0	0
22	?	?	420	0	0	0
23	?	?	85	0	0	0
24	?	?	2 530	0	0	0
25	?	?	?	1	0	0
26	?	?	?	0	0	0
27	?	?	?	?	0	0
28	?	?	?	≥ 2	1	0
29	?	?	?	?	0	0
30	?	?	?	?	0	0
$31-35$	?	?	?	?	?	0

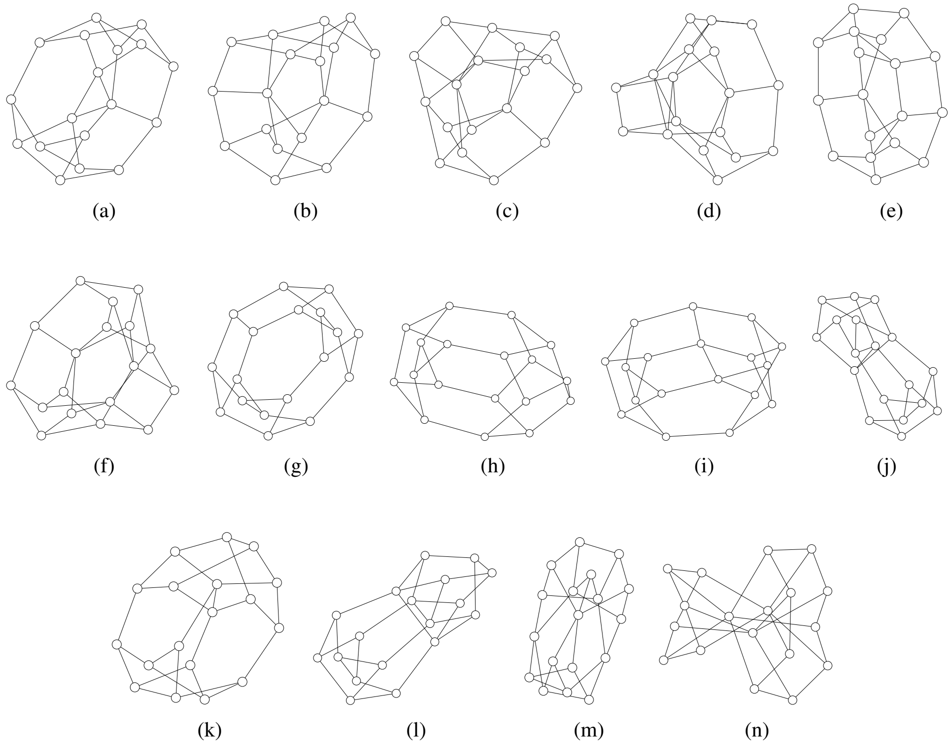


Figure 1: All 14 hypohamiltonian graphs of order 18. Graph (a) is the smallest hypohamiltonian graph of girth 3, while graphs (b)–(f) are the smallest hypohamiltonian graphs of girth 4.

cluster and the parallelisation also caused a significant overhead. However in each case we went as far as computationally possible (most of the largest cases took between 1 and 10 CPU years).

Since the running times and number of intermediate graphs generated by the algorithm grows that fast, it seems very unlikely that these bounds can be improved in the near future using only faster computers.

Starting from girth at least 7, the bottleneck is the case where the generated graphs have maximum degree 3 (so here we are generating cubic hypohamiltonian graphs). (Also for girth 6, the cubic case forms a significant part of the total running time.) Algorithm 1 can also be used to generate only cubic hypohamiltonian graphs (and we also did this for correctness testing, see Section 2.4). But here it is much more efficient to use a generator for cubic graphs with a given lower bound on the girth and testing if the generated graphs are hypohamiltonian as a filter. So for the generation of hypohamiltonian graphs with girth at least 6, we used Algorithm 1 only to construct hypohamiltonian graphs with maximum degree at least 4 and did the cubic case separately by using a generator for cubic graphs. More results on the cubic case can be found in Section 2.3.2.

Using Algorithm 1, we have also determined the smallest hypohamiltonian graph of girth 6. It has 25 vertices and is shown in Figure 2.

Table 3: Counts and generation times for hypohamiltonian graphs.

Order	# hypoham.	Time (s)	Increase	Max. nr. edges added
16	4	9		15
17	0	189	21.00	16
18	14	18 339	97.03	18
19	34			

Table 4: Counts and generation times for hypohamiltonian graphs with girth at least 4.

Order	# hypoham. $g \geq 4$	Time (s)	Increase	Max. nr. edges added
16	4	2		11
17	0	19	9.50	12
18	13	683	35.95	18
19	34	10 816	15.84	19

Table 5: Counts and generation times for hypohamiltonian graphs with girth at least 5.

Order	# hypoham. $g \geq 5$	Time (s)	Increase	Max. nr. edges added
17	0	1		8
18	8	9	9.00	9
19	34	81	9.00	10
20	4	1 125	13.89	11
21	85	11 470	10.20	12
22	420			

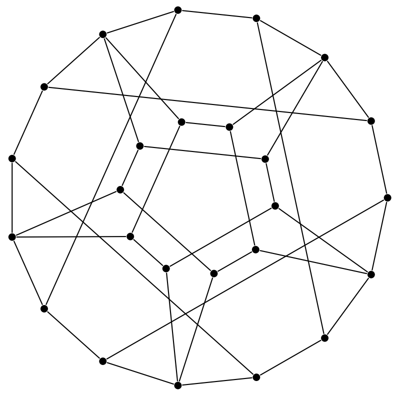


Figure 2: The smallest hypohamiltonian graph of girth 6. It has 25 vertices.

2.3.2 The cubic case

As already mentioned in the introduction, Aldred, McKay, and Wormald [2] determined all cubic hypohamiltonian graphs up to 26 vertices and all cubic hypohamiltonian graphs with girth at least 5 and girth at least 6 on 28 and 30 vertices, respectively. In Table 6 we extend these results. We used the program *snarkhunter* [6, 8] to generate all cubic graphs with girth at least k for $4 \leq k \leq 7$, the program *genreg* [31] for $k = 8$ and the program of McKay et al. [29] for $k = 9$. (Note that by Lemma 2.3 cubic hypohamiltonian graphs must have girth at least 4.)

For girth at least k for $k = 7, 8, 9$ we obtained the following results:

Theorem 2.9. *By generating all cubic graphs with a given lower bound on the girth and testing them for hamiltonicity we obtained the following:*

(i) The 28*-vertex Coxeter graph is the only non-hamiltonian cubic graph with girth* 7 *up to at least* 42 *vertices.*

(ii) The smallest non-hamiltonian cubic graph with girth 8 *has at least* 50 *vertices.*

(iii) The smallest non-hamiltonian cubic graph with girth 9 *has at least* 66 *vertices.*

Since hypohamiltonian graphs are non-hamiltonian, Theorem 2.9 also implies improved lower bounds for cubic hypohamiltonian graphs (see Table 1).

All hypohamiltonian graphs from Table 6 can also be downloaded from the *House of Graphs* [5] at `http://hog.grinvin.org/Hypohamiltonian`.

2.4 Correctness testing

To make sure that our implementation of Algorithm 1 did not contain any programming errors, we performed various correctness tests which we will describe in this section.

Previously, all hypohamiltonian graphs up to 17 vertices were known. We verified that our program yields exactly the same graphs. Aldred, McKay, and Wormald also produced a sample of 13 hypohamiltonian graphs with 18 vertices and a sample of 10 hypohamiltonian graphs with girth 5 and 22 vertices (see [25]). We verified that our program indeed also finds these graphs.

Table 6: Counts of hypohamiltonian graphs among cubic graphs. g stands for girth.

Order	$g \geq 4$	Non-ham. and $g \geq 4$	Hypoham.	Hypoham. and $g \geq 5$	Hypoham. and $g \geq 6$	Hypoham. and $g \geq 7$
10	6	1	1	1	0	0
12	22	0	0	0	0	0
14	110	2	0	0	0	0
16	792	8	0	0	0	0
18	7 805	59	2	2	0	0
20	97 546	425	1	1	0	0
22	1 435 720	3 862	3	3	0	0
24	23 780 814	41 293	1	0	0	0
26	432 757 568	518 159	100	96	0	0
28	8 542 471 494	7 398 734	52	34	2	1
30	181 492 137 812	117 963 348	202	139	1	0
32	4 127 077 143 862	2 069 516 990	304	28	0	0

Our program can also be restricted to generate cubic hypohamiltonian graphs. To find cubic hypohamiltonian graphs of larger orders it is actually much more efficient to use a generator for cubic graphs and then test the generated graphs for hypohamiltonicity as a filter. However we used our program to generate cubic hypohamiltonian graphs as a correctness test. We used it to generate all cubic hypohamiltonian graphs up to 22 vertices—note that these graphs must have girth at least 4 due to Lemma 2.3—and all cubic hypohamiltonian graphs with girth at least 5 up to 24 vertices. These results were in complete agreement with the known results for cubic graphs from Section 2.3.2.

Our routines for testing hamiltonicity and hypohamiltonicity were already extensively used and tested before (for example they were used in [7] to search for hypohamiltonian snarks). We also used multiple independent programs to test hamiltonicity and hypohamiltonicity—one of those programs was kindly provided to us by Gunnar Brinkmann—and in each case the results were in complete agreement.

Furthermore, our implementation of Algorithm 1 (i.e. the program *GenHypohamiltonian*) is released as open source software and the code can be downloaded and inspected at [16].

3 Generating planar hypohamiltonian graphs

In the early seventies, Chvátal [11] raised the problem whether *planar* hypohamiltonian graphs exist and Grünbaum conjectured that they do not exist [17]. In 1976, Thomassen [36] constructed infinitely many such graphs, the smallest among them having order 105. Subsequently, smaller planar hypohamiltonian graphs were found by Hatzel [18] (order 57), the second author and Zamfirescu [44] (order 48), Araya and Wiener [41] (order 42), and Jooyandeh, McKay, Östergård, Pettersson, and the second author [22] (order 40). The latter three graphs are shown in Figure 3. The 40-vertex example is the smallest known planar hypohamiltonian graph, together with other 24 graphs of the same order [22].

3.1 The general case

Jooyandeh, McKay, Östergård, Pettersson, and the second author [22] showed that the smallest planar hypohamiltonian graph of girth 5 has order 45, and that the graph with these properties is unique; see Figure 4.

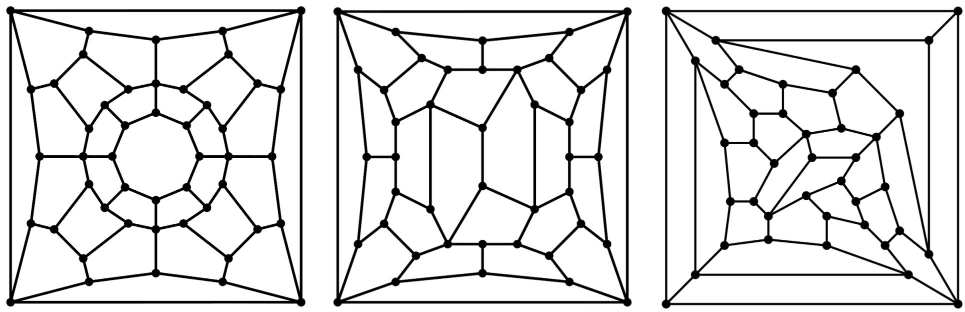


Figure 3: Planar hypohamiltonian graphs of order 48 [44], 42 [41], and 40 [22], respectively.

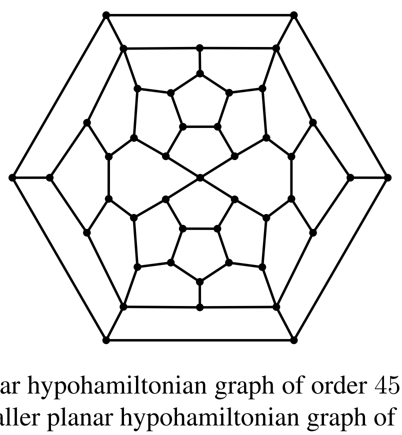


Figure 4: The unique planar hypohamiltonian graph of order 45 and girth 5. It was shown in [22] that there is no smaller planar hypohamiltonian graph of girth 5.

Since planar hypohamiltonian graphs have girth at most 5 (due to Euler's formula), the smallest planar hypohamiltonian graph must have girth either 3 or 4. Thomassen [35] proved that, rather surprisingly, hypohamiltonian graphs of girth 3 exist. In [36], Thomassen mentions how his approach from [35] can be applied to obtain a planar hypohamiltonian graph of girth 3. Using one of the aforementioned planar hypohamiltonian graphs of order 40 constructed in [22], one can obtain a planar hypohamiltonian graph of girth 3 and order 240. No smaller example is known.

Aldred, McKay, and Wormald [2] showed that the smallest planar hypohamiltonian has order at least 18. Up until now, 18 was also the best lower bound for the order of the smallest planar hypohamiltonian graph. Jooyandeh, McKay, Östergård, Pettersson, and the second author [22] recently improved the upper bound from 42 to 40. In [22], the authors emphasise that no extensive computer search had been carried out to increase the lower bound for the smallest planar hypohamiltonian graph. This was one of the principal motivations of the present work.

Since the algorithm for generating all hypohamiltonian graphs presented in Section 2 only adds edges and never removes any vertices or edges, all graphs obtained by the algorithm from a non-planar graph will remain non-planar. So in case we only want to generate planar hypohamiltonian graphs, we can prune the construction when a non-planar graph is constructed.

To this end we add a test for planarity on line 1 of Algorithm 2. We used Boyer and Myrvold's algorithm [4] to test if a graph is planar.

3.2 Additional properties of planar hypohamiltonian graphs

(a) Using a theorem of Whitney [39], Thomassen showed [38] that a planar hypohamiltonian graph does not contain a maximal planar graph G, where $G \neq K_3$.

(b) Let G be a planar hypohamiltonian graph. Let $\kappa(G)$, $\lambda(G)$, and $\delta(G)$ denote the vertex-connectivity, minimum degree, and edge-connectivity of G, respectively. Then $\kappa(G) = \lambda(G) = \delta(G) = 3$ (for a proof, see [22]).

We also present a result from [22] which restricts the family of polyhedra in which the smallest planar hypohamiltonian graph must reside. For further details, see [22]. In that article, the operation 4-*face deflater* $\mathcal{FD}_4$ is defined which squeezes a 4-face of a plane graph into a path of length 2. The inverse of this operation is called a 2-*path inflater* $\mathcal{PI}_2$, which expands a path of length 2 into a 4-face. Let $\mathcal{D}_5(f)$ be the set of all plane graphs with f faces and minimum degree at least 5. Let $G^\star$ denote the dual of a planar graph G, and put

$$\mathcal{M}^4_f(n) = \begin{cases} \{G^\star : G \in \mathcal{D}_5(n)\} & f = 0 \\ \bigcup_{G \in M^4_{f-1}(n-1)} \mathcal{PI}_2(G) & f > 0 \end{cases} \quad \text{and} \quad \mathcal{M}^4_f = \bigcup_n \mathcal{M}^4_f(n).$$

Theorem 3.1 (Jooyandeh et al. [22]). *Let G be the smallest planar hypohamiltonian graph. Then $G \notin \mathcal{M}^4_f$.*

We extended our algorithm from Section 2 to generate planar hypohamiltonian graphs and obtained the following results with it.

Theorem 3.2. *The smallest planar hypohamiltonian graph has at least* 23 *vertices.*

Theorem 3.3. *The smallest planar hypohamiltonian graph with girth at least* 4 *has at least* 27 *vertices.*

When we combine this with the known upper bounds, we get the following corollary.

Corollary 3.4. *Let* h *(*h_g*) denote the order of the smallest planar hypohamiltonian graph (of girth* g*). We have*

$$23 \leq h \leq 40, \quad 23 \leq h_3 \leq 240, \quad 27 \leq h_4 \leq 40, \quad \textit{and} \quad h_5 = 45.$$

The running times of our implementation of this algorithm restricted to planar graphs is given in Tables 7 and 8. For the larger cases we did not include any running times since these were executed on a heterogeneous cluster and the parallelisation also caused a non-negligible overhead. The column *"Max. nr. edges added"* denotes the maximum number of edges added by Algorithm 2 to a graph constructed by Algorithm 1.

Table 7: Counts and generation times for planar hypohamiltonian graphs.

Order	# hypoham.	Time (s)	Increase	Max. nr. edges added
16	0	4		9
17	0	35	8.75	11
18	0	235	6.71	14
19	0	1 245	5.30	16
20	0	13 517	10.86	17
21	0	109 294	8.09	19
22	0			

Table 8: Counts and generation times for planar hypohamiltonian graphs with girth at least 4.

Order	# hypoham. $g \geq 4$	Time (s)	Increase	Max. nr. edges added
16	0	2		6
17	0	11	5.50	7
18	0	35	3.18	8
19	0	231	6.60	10
20	0	1 649	7.14	10
21	0	9 545	5.79	12
22	0	53 253	5.58	12
23	0			
24	0			

3.3 The cubic case

Chvátal [11] asked in 1973 whether cubic planar hypohamiltonian graphs exist. His question was settled in 1981 by Thomassen [38], who constructed such graphs of order $94 + 4k$ for every $k \geq 0$. However, the following two questions raised in [21, Chapter 7] remained open: (i) Are there smaller cubic planar hypohamiltonian graphs? (ii) Does there exist a positive integer n_0 such that for every even $n \geq n_0$ there exists a cubic planar hypohamiltonian graph of order n? Araya and Wiener answered both of these questions affirmatively in [3]. Concerning (i), they showed that there exists a cubic planar hypohamiltonian graph of order 70. No smaller such graph is known. Regarding (ii), Araya and Wiener [3] showed that there exists a cubic planar hypohamiltonian graph of order n for every even $n \geq 86$. The second author [43] improved this result by showing that such graphs exist for every even $n \geq 74$.

Until recently, all known cubic planar hypohamiltonian graphs had girth 4. (Recall that by Lemma 2.3 cubic hypohamiltonian graphs must have girth at least 4). Due to a recent result of McKay [28], we now know that cubic planar hyohamiltonian graphs of girth 5 exist, and that the smallest ones have order 76. So the smallest cubic planar hypohamiltonian must have girth exactly 4.

From the results of Aldred, Bau, Holton, and McKay [1] it follows that there is no cubic planar hypohamiltonian graph on 42 or fewer vertices. (Completing the work of many researchers, Holton and McKay [20] showed that the order of the smallest non-hamiltonian cubic planar 3-connected graph is 38; one of the graphs realising this minimum is the famous Lederberg-Bosák-Barnette graph). Moreover, all 42-vertex graphs presented in [1] have exactly one face whose size is not congruent to 2 modulo 3, and it was already observed by Thomassen [34] that such a graph cannot be hypohamiltonian. Summarising, prior to this work we knew that the smallest planar hypohamiltonian graph has girth 4 and order at least 44 and at most 70.

3.4 Additional properties of cubic planar hypohamiltonian graphs

We now also mention obstructions specifically for cubic planar hypohamiltonian graphs. For the first obstruction below, we call a face F a *k-face* if $\text{size}(F) = k \bmod 3$. Let G be a cubic planar hypohamiltonian graph.

(a) Araya and Wiener [3] extended a remark of Thomassen [34] and showed that (i) G contains at least three non-2-faces, (ii) if G has exactly three non-2-faces, then these three non-2-faces do not have a common vertex, and (iii) two 1-faces or a 1-face and a 0-face cannot be adjacent.

(b) Kardoš [23] has recently proven Barnette's conjecture which states that every cubic planar 3-connected graph in which each face has size at most 6 is hamiltonian. Thus, G must contain a face of size at least 7.

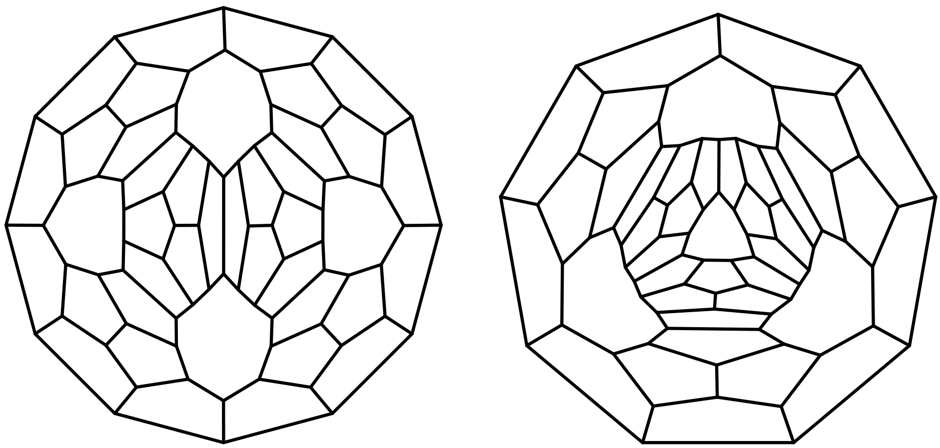
By using the program *plantri* [9] we generated all cubic planar cyclically 4-connected graphs with girth 4 up to 52 vertices and tested them for hypohamiltonicity. (Note that prior to our result, the best lower bound for the order of the smallest cubic planar hypohamiltonian graph was 44, see [3]). No hypohamiltonian graphs were found, so we have in summary the following.

Theorem 3.5. *The smallest cubic planar hypohamiltonian graph has girth* 4*, at least* 54 *and at most* 70 *vertices.*

As mentioned earlier, McKay [28] recently showed that there exist no cubic planar hypohamiltonian graphs of girth 5 with less than 76 vertices, and exactly three such graphs of order 76. All three graphs have trivial automorphism group. In that paper the natural question is raised whether infinitely many such graphs exist. Using the program *plantri* [9] we generated all cubic planar cyclically 4-connected graphs with girth 5 with 78 vertices and tested them for hypohamiltonicity. This yielded exactly one such graph. Although we are not able to settle McKay's question, in the following theorem we make a first step.

Theorem 3.6. *There is exactly one cubic planar hypohamiltonian graph of order* 78 *and girth* 5*. This graph is shown in Figure 5. It is the smallest cubic planar hypohamiltonian graph of girth* 5 *with a non-trivial automorphism group and has* D_{3h} *symmetry (as an abstract group, this is the dihedral group of order* 12*).*

The graph from Theorem 3.6 can also be downloaded and inspected at the database of interesting graphs from the *House of Graphs* [5] by searching for the keywords "hypo-hamiltonian * D3h".



(a) (b)

Figure 5: The smallest cubic planar hypohamiltonian graph of girth 5 with a non-trivial automorphism group. It has 78 vertices and D_{3h} symmetry. Both Figure 5a and Figure 5b show different symmetries of the same graph.

4 Outlook

We would like to conclude with comments and open questions which might be worth pursuing as future work.

1. We have seen that the order of the smallest planar hypohamiltonian graph must lie between 23 and 40. Let us read "being planar" as "having crossing number 0". It is not difficult to show that the Petersen graph is the smallest hypohamiltonian graph with crossing number 2, see e.g. [42]. The second author showed in [42] that there exists a hypohamiltonian graph with crossing number 1 and order 46. Recently, Wiener [40] constructed a hypohamiltonian graph with crossing number 1 and order 36. This is the smallest example up to date—so we ask here: what is the order of the smallest hypohamiltonian graph with crossing number 1?

2. In the deep and technical paper [32], Sanders defines a graph G to be *almost hamiltonian* if every subset of $|V(G)| - 1$ vertices is contained in a cycle. Every hypocyclic

(and thus every hypohamiltonian) graph is almost hamiltonian, but the converse is not necessarily true: take a hamiltonian graph G in which there exists a vertex v such that $G - v$ is not hamiltonian. Sanders characterises almost hamiltonian graphs in terms of circuit injections and binary matroids (for the definitions, see [32]). Possibly an algorithmic implementation of Sanders' characterisation is worth pursuing.

3. Ad finem, we discuss the order of the smallest planar hypohamiltonian graph. In this article, we have increased the lower bound from 18 to 23, but there is still a considerable gap to 40, the best available upper bound [22]. As mentioned in [22], it would be somewhat surprising if every extremal graph would have trivial automorphism group—note that the smallest planar hypohamiltonian graphs we know of, the 40-vertex graphs from [22], all have only identity as automorphism. An exhaustive search for graphs with prescribed automorphisms might lead to smaller planar hypohamiltonian graphs.

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Improving upper bounds for the distinguishing index

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Abstract

The distinguishing index of a graph G, denoted by $D'(G)$, is the least number of colours in an edge colouring of G not preserved by any non-trivial automorphism. We characterize all connected graphs G with $D'(G) \geq \Delta(G)$. We show that $D'(G) \leq 2$ if G is a traceable graph of order at least seven, and $D'(G) \leq 3$ if G is either claw-free or 3-connected and planar. We also investigate the Nordhaus-Gaddum type relation: $2 \leq D'(G) + D'(\overline{G}) \leq \max\{\Delta(G), \Delta(\overline{G})\} + 2$ and we confirm it for some classes of graphs.

Keywords: Edge colouring, symmetry breaking in graph, distinguishing index, claw-free graph, planar graph.

Math. Subj. Class.: 05C05, 05C10, 05C15, 05C45

1 Introduction

We follow standard terminology and notation of graph theory (cf. [12]). In this paper, we consider general, i.e. not necessarily proper, edge colourings of graphs. Such a colouring f of a graph G *breaks an automorphism* $\varphi \in \mathrm{Aut}(G)$ if φ does not preserve colours of f. The *distinguishing index* $D'(G)$ of a graph G is the least number d such that G admits an edge colouring with d colours that breaks all non-trivial automorphisms (such a colouring is called a *distinguishing edge* d*-colouring*). Clearly, $D'(K_2)$ is not defined, so in this paper, a graph G is called *admissible* if neither G nor $\overline{G}$ contains K_2 as a connected component.

The definition of $D'(G)$ introduced by Kalinowski and Pilśniak in [17] was inspired by the *distinguishing number* $D(G)$ which was defined for general vertex colourings by Albertson and Collins [1]. Another concept is the *distinguishing chromatic number* $\chi_D(G)$

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introduced by Collins and Trenk [7] for proper vertex colourings. Both numbers, $D(G)$ and $\chi_D(G)$, have been intensively investigated by many authors in recent years [4, 5, 6, 9, 16].

Our investigation was motivated by the renowned result of Nordhaus-Gaddum [18] who proved in 1956 the following lower and upper bounds for the sum of the chromatic numbers of a graph and its complement (actually, the upper bound was first proved by Zykov [22] in 1949).

Theorem 1.1 ([18]). *If G is a graph of order n with the chromatic number $\chi(G)$, then*

$$2\sqrt{n} \le \chi(G) + \chi(\overline{G}) \le n + 1.$$

Since then, Nordhaus-Gaddum type bounds were obtained for many graph invariants. An exhaustive survey is given in [2]. Here, we adduce only those closely related to the topic of our paper.

In 1964, Vizing [20] considered proper edge colourings and he proved Nordhaus-Gaddum type bounds for the chromatic index of a graph.

Theorem 1.2 ([20]). *If G is a graph of order n with the chromatic index $\chi'(G)$, then*

$$n - 1 \le \chi'(G) + \chi'(\overline{G}) \le 2(n - 1).$$

In 2013, Collins and Trenk [8] proved Nordhaus-Gaddum type inequalities for the distinguishing chromatic number.

Theorem 1.3 ([8]). *For every graph of order n and distinguishing number $D(G)$ the following inequalities are satisfied*

$$2\sqrt{n} \le \chi_D(G) + \chi_D(\overline{G}) \le n + D(G).$$

Kalinowski and Pilśniak [17] also introduced a *distinguishing chromatic index* $\chi'_D(G)$ of a graph G as the least number of colours in a proper edge colouring that breaks all non-trivial automorphisms of G. They proved the following somewhat unexpected result.

Theorem 1.4 ([17]). *If G is a connected graph of order $n \ge 3$, then*

$$\chi'_D(G) \le \Delta(G) + 1$$

unless $G \in \{C_4, K_4, C_6, K_{3,3}\}$ when $\chi'_D(G) \le \Delta(G) + 2$.

The following Nordhaus-Gaddum type inequalities for the distinguishing chromatic index are the same as in Theorem 1.2 but we have to be more careful in the proof.

Theorem 1.5. *If G is an admissible graph of order $n \ge 3$, then*

$$n - 1 \le \chi'_D(G) + \chi'_D(\overline{G}) \le 2(n - 1)$$

with the only exception $K_{1,4}$.

Proof. Without loss of generality we may assume that G is connected. It can be easily checked that the conclusion holds if $G \in \{K_4, C_6, \overline{C_6}, K_{3,3}\}$. Otherwise, $\chi'_D(G) \le \Delta(G) + 1$. Suppose first that $\overline{G}$ is also connected. By Theorem 1.4,

$$\Delta(G) + \Delta(\overline{G}) \le \chi'_D(G) + \chi'_D(\overline{G}) \le \Delta(G) + \Delta(\overline{G}) + 2.$$

Clearly, $n-1 \leq \Delta(G) + \Delta(\overline{G}) \leq 2(n-2)$ since both G and $\overline{G}$ are connected.

Now, let $\overline{G}$ be disconnected (but admissible). If there are two nonisomorphic components of $\overline{G}$ of orders k_1 and k_2 such that $3 \leq k_1 \leq k_2$, then $\Delta(\overline{G}) \leq n - k_1 - 1 \leq n-4$, so $\chi'_D(\overline{G}) \leq n-2$. If $\overline{G}$ has $t \geq 2$ components isomorphic to a graph H of order at least three, then $\chi'_D(H) \leq \frac{n}{t} + 1$ as $\Delta(H) \leq \frac{n}{t} - 1$. Even if we wastefully add an extra colour for each additional copy of H, we get $\chi'_D(tH) \leq \frac{n}{t} + 1 + t - 1 = \frac{n}{t} + t \leq n-2$ unless $G = K_{3,3}$ but this we already checked.

To complete the proof it is enough to settle the case when $\overline{G}$ has only one component H of order at least three and some isolated vertices. Hence, $\Delta(H) \leq n-2$. It is easy to check that $\chi'_D(G) + \chi'_D(\overline{G}) \leq 2(n-1)$ for $H \in \{K_4, C_6, \overline{C_6}, K_{3,3}\}$ except for $H = K_4$ when $G = K_{1,4}$. Otherwise, $\chi'_D(\overline{G}) \leq n-1$ and the conclusion holds unless $|G| = |H| + 1$ and $\Delta(H) = n-2$. But then G has a unique vertex x of degree $n-1$ (hence, x is fixed by every automorphism of G) with a pendant edge. The graph $G - x$ has a distinguishing colouring with $n-1$ colours by Theorem 1.4 since $\Delta(G-x) \leq n-2$. It suffices to colour the pendant edge with a colour missing at x to see that $\chi'_D(G) \leq n-1$. □

Collins and Trenk observed in [8] that the Nordhaus-Gaddum type relation is trivial for the distinguishing number, as $D(G) + D(\overline{G}) = 2D(G)$ since $\mathrm{Aut}(\overline{G}) = \mathrm{Aut}(G)$ and every colouring of $V(G)$ breaking all non-trivial automorphisms of G also breaks those of $\overline{G}$.

In Section 4 we formulate and discuss the following conjecture.

Conjecture 1.6. *Let G be an admissible graph of order $n \geq 7$, and let $\Delta = \max\{\Delta(G), \Delta(\overline{G})\}$. Then*

$$2 \leq D'(G) + D'(\overline{G}) \leq \Delta + 2.$$

In Section 2 we characterize graphs G which need exactly $\Delta(G)$ colours to break all non-trivial automorphisms. In Section 3 we give upper bounds for the distinguishing index of traceable graphs, claw-free graphs, planar graphs and 2-connected graphs.

2 Improved general upper bound

In the sequel, we make use of some facts proved in [17].

Proposition 2.1 ([17]). *$D'(P_n) = 2$ for every $n \geq 3$.*

Proposition 2.2 ([17]). *$D'(C_n) = 3$ for $n \leq 5$, and $D'(C_n) = 2$ for $n \geq 6$.*

Proposition 2.3 ([17]). *$D'(K_n) = 3$ if $3 \leq n \leq 5$, and $D'(K_n) = 2$ if $n \geq 6$.*

Proposition 2.4 ([17]). *$D'(K_{3,3}) = 3$, and $D'(K_{n,n}) = 2$ if $n \geq 4$.*

By the well-known theorem of Jordan (cf. [12]), every finite tree T has either a central vertex or a central edge, which is fixed by every automorphism of T. In the proof of Theorem 2.8, which is the main result of this section, we use Lemma 2.5, a simple generalization of the theorem of Jordan. Recall that the *eccentricity* of a vertex v in a connected graph G is the number

$$\varepsilon_G(v) = \max\{d(v,u) \ : \ u \in V(G)\}.$$

The *center* of a graph G is the set $Z(G)$ of vertices with minimum eccentricity. Clearly, the center of G is setwise fixed by every automorphism $\varphi \in \mathrm{Aut}(G)$, i.e. $\varphi(v) \in Z(G)$ if $v \in Z(G)$. A proper subgraph H of G is called *pendant* if it has only one vertex adjacent to vertices outside H.

Lemma 2.5. *Let G be a connected graph such that every cycle is contained in a clique. Then the center of G is either a single vertex or a maximal clique.*

Proof. The claim is true if G is a clique K_k of order $k \geq 1$. Otherwise, $\kappa(G) = 1$, and each block of G is a clique of order at least two. We then modify the standard proof of the theorem of Jordan for trees. Let G^- be a graph obtained from G by deleting $k-1$ vertices of degree $k-1$ in every pendant clique K_k with $k \geq 2$. Clearly, $\varepsilon_{G^-}(v) = \varepsilon_G(v) - 1$ for each $v \in V(G^-)$. Consequently, $Z(G^-) = Z(G)$. We continue this process until only one clique K_k is left for some $k \geq 1$. This clique is maximal whenever $k \geq 2$. □

A *symmetric tree*, denoted by $T_{h,d}$, is a tree with a central vertex v_0, all leaves at the same distance h from v_0 and all vertices that are not leaves of equal degree d. A *bisymmetric tree*, denoted by $T''_{h,d}$, is a tree with a central edge e_0, all leaves at the same distance h from the edge e_0 and all vertices which are not leaves of equal degree d.

Theorem 2.6 ([17])**.** *If T is a tree of order $n \geq 3$, then $D'(T) \leq \Delta(T)$. Moreover, equality is achieved if and only if T is either a symmetric or a bisymmetric tree.*

For connected graphs in general there is the following upper bound for $D'(G)$.

Theorem 2.7 ([17])**.** *If G is a connected graph of order $n \geq 3$, then*

$$D'(G) \leq \Delta(G)$$

unless G is C_3, C_4 or C_5.

It follows for connected graphs that $D'(G) > \Delta(G)$ if and only if $D'(G) = \Delta(G) + 1$ and G is a cycle of length at most 5. The equality $D'(G) = \Delta(G)$ holds for cycles of length at least 6, for K_4, $K_{3,3}$ and for all symmetric or bisymmetric trees. Now, we show that $D'(G) < \Delta(G)$ for all other connected graphs. A *palette* of a vertex is the multiset of colours of edges incident to it.

Theorem 2.8. *Let G be a connected graph that is neither a symmetric nor a bisymmetric tree. If the maximum degree of G is at least 3, then*

$$D'(G) \leq \Delta(G) - 1$$

unless G is K_4 or $K_{3,3}$.

Proof. Denote $\Delta = \Delta(G)$. The conclusion holds for trees due to Theorem 2.6. Then assume that G contains a cycle. The general idea of the proof is the following. If G does not contain a cycle of length greater than three, then we define G' as an empty graph. Otherwise, we consecutively delete pendant trees and pendant triangles until we obtain a subgraph G'. Then, we construct an edge colouring f with $\Delta - 1$ colours stabilizing all vertices of G' by every automorphism preserving f. Finally, we colour pendant subtrees and pendant triangles to complete a distinguishing colouring with $\Delta - 1$ colours of the whole graph G.

If $\Delta(G') = 2$, then G' is a cycle C_p having a distinguishing colouring with $\Delta - 1$ colours unless $p \in \{4, 5\}$ and $\Delta = 3$. In this case, it can be easily checked that the graph G'_+ induced by C_p and the independent edges of G incident to C_p can always be coloured with two colours such that the vertices of C_p are fixed by every colour preserving automorphism. So we can assume that $\Delta(G') \geq 3$. If $G' \in \{K_4, K_{3,3}\}$, then $G' \neq G$ due to the assumption, hence $\Delta \geq 4$, so we can stabilize K_4 or $K_{3,3}$ with three colours.

Let $N_i(v)$ denote the *i-th sphere in* v, i.e. the set of vertices of distance i from the vertex v. Let x be a vertex with maximum degree in G'. We colour with 1 all edges incident with x. In our edge colouring f of the graph G', the vertex x will be the unique vertex of maximum degree with the monochromatic palette $\{1, \dots, 1\}$. Hence, x will be fixed by every automorphism φ preserving f. Consequently, φ maps each sphere $N_i(x)$ onto itself.

The first sphere $N_1(x)$ can be partitioned into subsets M_k, for $k = 0, \dots, \Delta - 1$, defined as

$$M_k = \{v \in N_1(x) \, : \, |N_1(v) \cap N_2(x)| = k\}.$$

Denote $M_k = \{v_1, \dots, v_{l_k}\}$. Thus, $l_0 + l_1 + \dots + l_{\Delta-1} = \Delta$.

We want to find a colouring f of the edges of $G'[N_1(x) \cup N_2(x)]$ and, if necessary, of some subsequent spheres, such that each vertex of $N_1(x) \cup N_2(x)$ is fixed by every automorphism preserving this colouring. To do this, we proceed in a number of steps M_k, for $k = 0, \dots, \Delta - 1$. In each step M_k, we find a colouring that fixes the vertices of M_k and their neighbours in $N_2(x)$.

Step $\mathbf{M_0}$. First we consider the case when the subgraph $G'[M_0]$ induced by the vertices of M_0 is connected. Observe that $\Delta(G'[M_0]) \leq \Delta - 1$ and, by Theorem 2.7, we can colour distinguishingly the edges of $G'[M_0]$ with $\Delta - 1$ colours, even if $G'[M_0]$ is a short cycle C_p with $3 \leq p \leq 5$. Indeed, if $G'[M_0] = C_3$ and $\Delta = 3$, then we would have $G = K_4$, but K_4 is excluded. Otherwise, $\Delta \geq 4$ and we can use a third colour in a short cycle C_p. It may happen that there exists a vertex $v \in M_0$ of degree Δ in G' (so $|M_0| = \Delta$) with a monochromatic palette $\{1, \dots, 1\}$ in a colouring of $G'[M_0]$ given by Theorem 2.7. In this case, either G is a complete graph K_n with $n \geq 5$ so $D'(K_n) \leq \Delta - 1$ by Proposition 2.3, or it is not difficult to see that there exists a colour c such that there is no vertex with all incident edges coloured with c; whence we can exchange c and 1 in this colouring of $G'[M_0]$.

Now, let $G'[M_0]$ be disconnected. Let $z_1, \dots, z_s$ be isolated vertices or end-vertices of isolated edges in $G'[M_0]$. Clearly, $s \leq \Delta - 1$ by the definition of G'. If $s = \Delta - 1$, then we colour with i every edge $z_i u$, where $u \in N_1(x) \setminus M_0$. Otherwise, we colour $z_i u$ with $i + 1$ for $i = 1, \dots, s$. Thus, we avoid a monochromatic palette of $\{1, \dots, 1\}$ at another vertex of maximum degree in G'.

We also have to distinguish all isomorphic components of $G'[M_0]$ of order greater than 2. Denote such a component by H and suppose that $G'[M_0]$ contains t components isomorphic to H, for some $t \geq 2$. Hence $t \leq \frac{\Delta}{3}$ and $\Delta(H) \leq \frac{\Delta}{t} - 1$. Therefore, we can choose distinct sets of $\frac{\Delta}{t}$ colours for every component since

$$\binom{\Delta - 1}{\frac{\Delta}{t}} \geq \binom{\Delta - 1}{3} \geq \frac{\Delta}{3} \geq t.$$

Thus each vertex of M_0 is fixed.

Step $\mathbf{M_1}$. For every $i = 1, \dots, l_1$, we colour the edge $v_i u$, where $u \in N_2(x)$, with a distinct colour from $\{1, \dots, \Delta - 1\}$. This is impossible only if $l_1 = \Delta$, when we have to have two vertices $a, b \in M_1$ with the same colour of edges aa' and bb', where a' and b' are neighbours of a and b in $N_2(x)$, respectively. If $G'[M_1]$ contains an edge e, then we colour it with 1, and all other edges of $G'[M_1]$ with 2. Then we choose exactly one of the vertices a, b incident to e. We proceed analogously when $G'[N_2(x)]$ contains an edge. Then all vertices of M_1 are fixed unless $l_1 = \Delta$ and neither $G'[N_1(x)]$ nor $G'[N_2(x)]$ contains an edge.

If $|N_2(x)| = 1$, then G' is isomorphic to $K_{2,\Delta}$. It is easy to see that $D'(K_{2,\Delta}) \leq \Delta - 1$ for $\Delta \geq 3$ (for $\Delta \geq 4$ this immediately follows from Lemma 3.1 and Corollary 3.8). If $2 \leq |N_2(x)| \leq \Delta - 1$, then choosing a and b such that a' has at least two neighbours in $N_1(x)$ and $b' \neq a'$ yields a colouring fixing $N_1(x) \cup N_2(x)$.

Suppose $|N_2(x)| = \Delta$. If there is a vertex $v \in N_2(x)$ with less than $\Delta - 1$ neighbours in $N_3(x)$, then we choose a such that $a' = v$, and it suffices to reserve a unique set of colours for the edges between a' and $N_3(x)$.

Hence, assume that every vertex of $N_2(x)$ has $\Delta - 1$ neighbours in $N_3(x)$. We select two vertices $a, b \in M_1$ and assume that the colours of the edges aa' and bb' are the same. Next, we implement the following Procedure SUBTREES (a, b), which we also use in subsequent steps.

Procedure SUBTREES (a, b)

We are given two vertices $a, b \in N_1(x)$ such that each their neighbour in $N_2(x)$ is adjacent to $\Delta - 1$ vertices of $N_3(x)$.

Let T_a be a maximal subtree of the graph $G'[\{a\} \cup \bigcup_{i \geq 2} N_i(x)]$, rooted at a, such that all leaves of T_a belong to the same sphere $N_{l-1}(x)$ and each vertex of $V(T_a) \cap N_{i-1}(x)$ has $\Delta - 1$ neighbours in $N_i(x)$ for $i = 3, \ldots, l$. Thus $l \geq 3$. Define a graph

$$\widetilde{T_a} = G'[\bigcup_{v \in V(T_a) \setminus \{a\}} N(v)],$$

i.e. $\widetilde{T}_a$ is a graph obtained from T_a by adding all edges incident with the leaves of T_a. Analogously, we define a tree T_b and a graph $\widetilde{T}_b$. Observe that the trees T_a and T_b are disjoint and non-empty.

The edges incident to the roots a and b are already coloured. For every other vertex of T_a and T_b, we colour its incident edges going to the next sphere with distinct colours from $\{1, \ldots, \Delta - 1\}$. Thus we obtain an edge colouring f. The only automorphism of T_a (as well as of T_b) preserving f is the identity. The vertex x will be fixed by every colour preserving automorphism φ. Consequently, φ maps $\widetilde{T}_a$ onto $\widetilde{T}_b$ whenever $\varphi(a) = b$. Thus, if $\widetilde{T}_a$ and $\widetilde{T}_b$ are not isomorphic, then f distinguishes all vertices in $V(T_a) \cup V(T_b)$. Hence, assume that the rooted graphs $\widetilde{T}_a$ and $\widetilde{T}_b$ are isomorphic. Observe that there exists exactly one non-trivial isomorphism $\psi_0 \colon V(T_a) \to V(T_b)$ preserving f since each vertex in T_a has a distinct coloured path from the root a.

Denote $W_l = (V(\widetilde{T}_a) \cup V(\widetilde{T}_b)) \cap N_l(x)$. By our choice of G', all vertices in W_l are of degree at least two in G'. It follows that one of the following three cases has to hold.

Case 1. There exist vertices in W_l adjacent to more than one vertex of W_{l-1}. Then we modify f by colouring again all edges between such vertices and W_{l-1} in order to break any possible permutation of W_l. A permutation of a set $L \subseteq W_l$ can be extended to an automorphism of G' that fixes all leaves of $\widetilde{T}_a \cup \widetilde{T}_b$ only if every vertex from L have the same set of neighbours $U = \{u_1, \ldots, u_d\}$ in W_{l-1}. Such a set L contains at most $\Delta - 1$ leaves since the number of edges joining U to W_l equals $d(\Delta - 1)$. Every permutation of L will be broken whenever for every vertex $w \in L$ the multiset of colours of the edges $wu_1, \ldots, wu_d$ will be distinct. Clearly, $d \leq \Delta$. There are $\binom{\Delta+d-2}{d}$ such possible multisets of $\Delta - 1$ colours. Clearly, $\binom{\Delta+d-2}{d} - 1 \geq \Delta - 1$ for $\Delta \geq 3$ and $d \geq 2$. We can exclude a

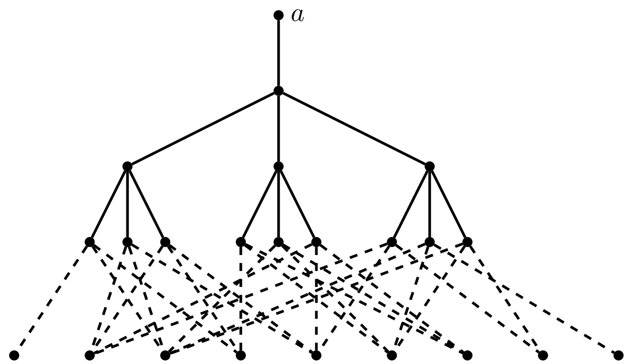

Figure 1: An example of the subgraph $\widetilde{T}_a$ for $\Delta = 4$ and $l = 4$. The edges of $\widetilde{T}_a$ between W_3 and W_4 that do not belong to the tree T_a are dashed.

rainbow multiset $P = \{1, \ldots, d\}$ (or an almost rainbow multiset $P = \{1, \ldots, \Delta-1, \Delta-1\}$ if $d = \Delta$) and we still have enough multisets to colour the edges incident with vertices of L. Moreover, for $d = \Delta$ we can also exclude a monochromatic palette $\{1, \ldots, 1\}$ since $\binom{2\Delta-2}{\Delta} - 2 \geq \Delta - 1$ for $\Delta \geq 3$.

We partition the set W_l into maximal subsets L with the same set of neighbours and assign suitable multisets of colours to each set L. We thus obtain a colouring fixing all vertices from W_l unless ψ_0 can be extended to an isomorphism $\widetilde{\psi_0}$ of $\widetilde{T}_a$ onto $\widetilde{T}_b$ preserving this colouring. To break every such possible extension $\widetilde{\psi_0}$, it suffices to assign the excluded multiset P to one vertex of one set L.

Case 2. Every vertex in W_l has only one neighbour in W_{l-1} and the set of edges $F = E(G'[W_l])$ is non-empty. Then we colour one edge of F with 1, and all other edges in F with 2. This colouring fixes all vertices of $\widetilde{T}_a$ and $\widetilde{T}_a$ unless all edges in F are of the form $w\widetilde{\psi_0}(w)$, where $w\widetilde{\psi_0}(w)$ is one of possible extensions of ψ_0 to an isomorphism of $\widetilde{T}_a$ onto $\widetilde{T}_b$. In such a case, we choose one edge $ww' \in F$ and exchange colours of the edge wu, where $u \in W_{l-1}$, with another edge between u and W_l.

Case 3. Every vertex in W_l has only one neighbour in W_{l-1} and no neighbours in W_l. By the maximality of the trees T_a and T_b and the definition of G', each vertex in W_l has at least one neighbour in $N_{l+1}(x)$ and there exists a vertex $w_0 \in W_l$ with $s < \Delta - 1$ neighbours $y_1, \ldots, y_s \in N_{l+1}(x)$. We colour each edge $w_0 y_j$ with colour $j + 1$ for $j = 1, \ldots, s$. Next, for every vertex $w \in W_l$, we colour the set of edges between w and $N_{l+1}(x)$ with a set of $\Delta - 1$ colours excluding the set $\{2, \ldots, s + 1\}$.

We thus obtained a colouring f of the edges of $G'[V(\widetilde{T}_a) \cup V(\widetilde{T}_b)]$, and the edges incident to W_l in Case 3, fixing all vertices of $\widetilde{T}_a$ and $\widetilde{T}_b$.

———— **End of Procedure** SUBTREES (a, b) ————

Step $\mathbf{M_2}$. For every $i = 1, \ldots, l_2$, we colour the edges $v_i u_i^1$, $v_i u_i^2$ where $\{u_i^1, u_i^2\} \subseteq N_2(x)$, with distinct sets of colours from among $\binom{\Delta-1}{2}$ sets. This is impossible only in the following three cases (in each case, we can assume that neither $G'[N_1(x)]$ nor $G'[N_2(x)]$ contains an edge, otherwise we could construct a distinguishing colouring f of $G'[N_1(x) \cup N_2(x)]$ analogously as in step M_1):

a) $l_2 = \Delta = 4$. If there exist two vertices a and b in M_2 such that $N(a) \cap N(b) \cap N_2(x) \neq \emptyset$, then we colour with 2 both edges incident with b, and for the remaining vertices in M_2 we have distinct sets of colours from among $\binom{3}{2}$ sets. If for every two vertices $a, b \in M_2$, the set $N(a) \cap N(b) \cap N_2(x)$ is empty, then two vertices a and b are assign the same pair of distinct colours, and we can distinguish them in next spheres using the procedure SUBTREES (a, b).

b) $l_2 = \Delta - 1$ and $\Delta = 3$. Let $M_2 = \{a, b\}$. If $N(a) \cap N(b) \cap N_2(x) \neq \emptyset$, then we colour edges incident with a with colours 1 and 2, and both edges incident with b with 2. If the set $N(a) \cap N(b) \cap N_2(x)$ is empty, then a and b get the same pair of distinct colours and we can distinguish them in next spheres by the procedure SUBTREES (a, b).

c) $l_2 = \Delta = 3$. Let $M_2 = \{a, b, c\}$. If for two vertices of M_2, say a and b, the set $N(a) \cap N(b) \cap N_2(x)$ is non-empty, then we can colour with 2 both edges incident with b and we colour edges incident with the remaining vertices of M_2 with a couple $\{1, 2\}$. It is not difficult to verify that this way, for every configuration of neighbours of M_2, we can obtain colouring fixing the vertices of $N_1(x) \cup N_2(x)$ unless $|N(a) \cap N(b) \cap N(c) \cap N_2(x)| = 2$. But then $G' = G = K_{3,3}$, contrary to the assumption. If every vertex of $N_2(x)$ is adjacent only to one vertex of M_2, then the pairs of edges incident to a and b are assign the same pair of colours $\{1, 2\}$, and we distinguish them using the procedure SUBTREES (a, b). Both edges cu^1, cu^2 incident with c are coloured with 2, and to distinguish them, we split c into two vertices c^1 and c^2, each joined by an edge coloured with 2 to u^1 and u^2, respectively, and apply the procedure SUBTREES (c^1, c^2).

Step $\mathbf{M_k}$, for $\mathbf{k \geq 3}$. For every $i = 1, \ldots, l_k$, we colour the edges between v_i and $N_2(x)$ with distinct sets of k colours from among $\binom{\Delta-1}{k}$ sets. It is always possible whenever $\binom{\Delta-1}{k} \geq l_k$. This inequality does not hold only in two cases:

a) $k = \Delta - 2$ and $l_k = \Delta$. In this case we define a colouring with $\Delta - 1$ colours like in step M_2 a). Namely, if either a vertex of M_k or its neighbour in $N_2(x)$ is adjacent to a vertex in the same sphere, then we can define a colouring fixing all these vertices analogously as in step M_1 and step M_2. Also, if there are two vertices $a, b \in M_{\Delta-2}$ with a common neighbour in $N_2(x)$, we can assign the same palette to a and b as in the previous steps. Otherwise, two vertices $a, b \in M_{\Delta-2}$ are assign the same palette of $\Delta - 2$ colours and we distinguish them using Procedure SUBTREES (a, b).

b) $k = \Delta - 1$ and $l_k \geq 2$. Hence, $\Delta \geq 4$. For every $i = 1, \ldots, l_k$, the set of edges between $v_i \in M_{\Delta-1}$ and $N_2(x)$ will be assign a distinct multiset P^i of colours from the set $\{1, \ldots, \Delta - 1\}$, where only colour i appears twice. Moreover, one vertex can assign a rainbow palette $\{1, \ldots, \Delta - 1\}$. Thus every vertex of $M_{\Delta-1}$ will have a distinct palette, and hence will be stabilized. To stabilize the two vertices of $N_2(x)$ joined to v_i by edges of colour i, we examine the vertices $v_1, \ldots, v_{\Delta-1}$ of $M_{\Delta-1}$ in the following order.

First, we consider each vertex v_i that have a neighbour $w_i \in N_2(x)$ with at least one but at most $\Delta - 2$ neighbours in $N_3(x)$. We choose another neighbour $w_i' \in N_2(x)$ of v_i and assign two distinct sets of colours for the edges going to $N_3(x)$ from w_i and w_i', respectively. We colour the edges $v_i w_i$ and $v_i w_i'$ with the same colour i. Thus all neighbours of v_i are stabilized.

In the next stage, we consider every vertex v_i with every neighbour in $N_2(x)$ adjacent to $\Delta - 1$ vertices of $N_3(x)$. We colour the set of edges between v_i and $N_2(x)$ with the palette P^i, where two edges v_iu_1, v_iu_2 are coloured with i. Then we delete v_i and introduce two vertices v_i^1, v_i^2 and edges $v_i^1u_1$ and $v_i^2u_2$ coloured with i. Then we use the procedure SUBTREES (v_i^1, v_i^2) to stabilize u_1 and u_2.

Further, we consider each vertex v_i with a neighbour $w_i \in N_2(x)$ incident to an edge w_iu, where $u \in N_2(x)$. First, we look for such an edge w_iu, which is already coloured. If there is no such edge, we take an uncoloured w_iu and colour it with colour 3. In both cases, we put colour i on the edge v_iw_i and another edge v_iw with $w \neq u$. After we examine each such vertex v_i, we colour with 2 all remaining edges contained in $N_2(x)$.

Finally, we are left with at most Δ vertices v_i such that every neighbour of v_i is adjacent only to (at least two) vertices of $N_1(x)$. We take a first such vertex v_i and assign colour i to two its incident edges v_iw_i and v_iw_i'. Thus all neighbours of v_i are stabilized unless common neighbours of w_i and w_i' were not considered yet. Then we take such a neighbour v_j and colour its incident edges with the palette P^j such that the edges v_jw_i and v_jw_i' have distinct colours. We repeat this procedure until only one vertex of $M_{\Delta-1}$ is left. We put a rainbow palette $\{1, \ldots, \Delta - 1\}$ on its incident edges.

After we accomplish steps $M_0, \ldots, M_{\Delta-1}$, we colour all uncoloured edges in subgraphs $G'[N_1(x)]$ and $G'[N_2(x)]$ with 2. Each vertex of $N_1(x) \cup N_2(x)$ is now fixed by every automorphism preserving our colouring f of edges of $G'[\{x\} \cup N_1(x) \cup N_2(x)]$, and of some edges between next spheres, if the procedure SUBTREES was used.

Then we recursively colour all yet uncoloured edges incident to consecutive spheres $N_i(x)$ as follows: for $v \in N_i(x)$, $i \geq 2$, we colour all edges vu, where $u \in N_{i+1}(x)$, with distinct colours from $\{1, \ldots, \Delta - 1\}$. This is always possible since every vertex of $N_i(x)$ has at most $\Delta - 1$ neighbours in $N_{i+1}(x)$. Finally, we colour all uncoloured edges with end-vertices in the same sphere with 2. Hence, all vertices of G' are fixed by any automorphism preserving our colouring f. It is also easily seen that the already coloured edges can save their colours. Moreover, it is not difficult to observe that x is the unique vertex of maximum degree with a monochromatic palette $\{1, \ldots, 1\}$. Thus, the whole subgraph G' (or G'_+) is fixed.

To end the proof, we colour pendant trees and triangles deleted from G at the beginning. First assume that G' is not empty. Let $N_i(G')$, for $i \geq 0$, be the set of vertices of distance i from G'. Then we recursively colour the edges incident to consecutive spheres $N_i(G')$ in the following way: for $v \in N_i(G')$, $i \geq 0$, we colour all edges vu, where $u \in N_{i+1}(G')$, with distinct colours from $\{1, \ldots, \Delta - 1\}$ and the remaining edges incident to v, contained in $N_i(x)$, with 2. Hence, all vertices of G will be fixed by any automorphism preserving our colouring f.

If G' is empty, then we start with the centre $Z(G)$ that is setwise fixed by every automorphism. It follows from Lemma 2.5 that $Z(G)$ either induces K_3, or K_2 (not contained in K_3), or K_1. Let first $Z(G)$ induce a triangle K_3. If $\Delta = 3$, then we stabilize $Z(G)$ by colouring with two colours all edges incident with vertices of $Z(G)$. When $\Delta \geq 4$, we can colour the edges of the triangle $Z(G)$ with three colours. Next, we recursively colour edges incident to subsequent spheres $N_i(Z(G))$ with $\Delta - 1$ colours.

If $Z(G)$ is an edge e, then $G - e$ has two components. We distinguish each of them by colouring subsequent spheres $N_i(Z(G))$ with $\Delta - 1$ colours. If the components are isomorphic, then by assumption, each of them has a triangle. We colour two edges of these triangles contained in a sphere $N_i(Z(G))$, for some $i \geq 2$, with two distinct colours.

Finally, let $Z(G)$ be a single vertex z. Hence, $G - z$ has $q \geq 2$ components, each joined to z by one or two edges. If $q < \Delta$, then we can easily colour distinguishingly the edges incident with subsequent spheres $N_i(z)$, $i \geq 0$, with $\Delta - 1$ colours. If $q = \Delta$, then we choose two components of $G - z$, at least one of them with a triangle, and colour their two edges incident with z with the same colour. Then we distinguish these two components by an edge of the triangle. □

3 Some classes of graphs

A graph G is called *asymmetric* if its automorphism group is trivial. Then obviously $D'(G) = 1$.

We say that a graph G is *almost spanned* by a subgraph H (not necessarily connected) if $G - v$ is spanned by H for some $v \in V(G)$. The following observation will play a crucial role in this section.

Lemma 3.1. *If a graph G is spanned or almost spanned by a subgraph H, then*

$$D'(G) \leq D'(H) + 1.$$

Proof. We colour the edges of H with colours $1, \ldots, D'(H)$, and all other edges of G with an additional colour 0. If φ is an automorphism of G preserving this colouring, then $\varphi(x) = x$, for each $x \in V(H)$. Moreover, if H is a spanning subgraph of $G - v$, then also $\varphi(v) = v$. Therefore, φ is the identity. □

3.1 Traceable graphs

Recall that a graph is *traceable* if it contains a Hamiltonian path.

Theorem 3.2. *If G is a traceable graph of order $n \geq 7$, then $D'(G) \leq 2$.*

Proof. Let $P_n = v_1 v_2 \ldots v_n$ be a Hamiltonian path of G. If $G = P_n$, then the conclusion follows from Proposition 2.1. If G is isomorphic to $P_n + v_1 v_3$, then we colour the edge $v_1 v_3$ with 1, and all other edges with 2 breaking all non-trivial automorphisms of G. So suppose that G contains an edge $v_i v_j$ distinct from $v_1 v_3$ and $v_{n-2} v_n$ with $i < j - 1$. Without loss of generality we may assume that $i - 1 \leq n - j$ (otherwise we reverse the labeling). It is easy to see that at least one of the graphs $P_n + v_i v_j - v_{j-1} v_j$, $P_n + v_i v_j - v_{j-1}$ or $P_n + v_i v_j - v_n$ is an asymmetric spanning or almost spanning subgraph of G for any $n \geq 7$. The conclusion follows from Lemma 3.1. □

The assumption $n \geq 7$ is substantial in Theorem 3.2 as $D'(K_{3,3}) = 3$.

3.2 Claw-free graphs

A $K_{1,3}$-free graph, called also *a claw-free graph*, is a graph containing no copy of $K_{1,3}$ as an induced subgraph. Claw-free graphs have numerous applications, e.g., in operations research and scheduling theory. For a survey of claw-free graphs and their applications consult [10].

A *k-tree* of a connected graph is its spanning tree with maximum degree at most k. Win [21] investigated spanning trees in 1-tough graphs and proved the following result.

Theorem 3.3 ([21]). *A 2-connected claw-free graph has a 3-tree.*

We use this result to give an upper bound for the distinguishing number of claw-free graphs.

Theorem 3.4. *If G is a connected claw-free graph, then $D'(G) \leq 3$.*

Proof. Assume first that G is 2-connected. By Theorem 3.3, G contains a 3-tree T. By Theorem 2.6, we have $D'(T) \leq 2$ if T is neither symmetric nor bisymmetric tree. In such a case, $D'(G) \leq 3$ by Lemma 3.1.

Let T be a symmetric tree $T_{h,3}$. Denote a central vertex of T by x and its neighbours by a, b, c. Since G is a claw-free graph, there exists in G at least one edge, say bc, in the neighbourhood of x in T. Define a subgraph $\widetilde{T} = T + bc$. We colour bc, xa and xb with 1, and xc with 2. Thus all vertices a, b, c, x are fixed by every non-trivial automorphism of $\widetilde{T}$. We now colour the remaining edges in $\widetilde{T}$ starting from the edges incident to a, b, c in such a way that two uncoloured adjacent edges obtain two different colours 1 and 2. This 2-colouring breaks all non-trivial automorphisms of $\widetilde{T}$. Hence, $D'(G) \leq 3$ by Lemma 3.1.

Let T be a bisymmetric tree $T''_{h,3}$. Denote a central edge by xy and its neighbours by a, b adjacent to x, and c, d adjacent to y. We colour xy, xa and yc with 1, and xb and yd with 2. Since G is claw-free, there exists in G either at least one of the edges by, cx (or symmetrically dx or ay) or both ab and cd. We define a subgraph $\widetilde{T}$ obtained from the tree T by adding either one of the edges by, cx (or symmetrically, dx or ay) or both ab and cd. In the first case we colour by or cx (or symmetrically, dx or ay) with 1, in the second case we colour ab with 1 and cd with 2. Now all vertices a, b, c, d, x, y are fixed by every non-trivial automorphism of $\widetilde{T}$. We then colour the remaining edges of $\widetilde{T}$ as above, and we obtain the claim.

If a graph G is not 2-connected, then its graph of blocks and cut-vertices is a path, since G is claw-free. We colour every block according to the rules described above. Then to break all non-trivial automorphisms of G, it is enough to break a possible automorphism $\psi \in \mathrm{Aut}(G)$ that exchanges two terminal blocks. Let z be a cut-vertex that belongs to a terminal block B_0. It follows that z and its neighbours in B_0 induce a clique K of order $k \geq 2$. We have three colours in our disposal, so it is easily seen that we can permute the colours to obtain a nonisomorphic colouring of K, thus breaking ψ. □

The theorem is sharp for graphs of order at most 5. We conjecture that the distinguishing index of claw-free graphs of order big enough is 2.

3.3 Planar graphs

First, recall that by the famous Theorem of Tutte [19], every 4-connected planar graph G is Hamiltonian. Hence, its distinguishing index is at most 2, by Theorem 3.2, whenever $|G| \geq 7$. A similar result as for claw-free graphs we obtain for 3-connected planar graphs. In the proof, we use the following result of Barnette about spanning trees of such graphs.

Theorem 3.5 ([3]). *Every 3-connected planar graph has a 3-tree.*

Using a similar method as in the proof of Theorem 3.4, we obtain the following.

Theorem 3.6. *If G is 3-connected planar graph, then $D'(G) \leq 3$.*

Proof. Let T be a 3-tree of G. It follows from Theorem 2.6 that $D'(T) \leq 2$ and hence, $D'(G) \leq 3$ by Lemma 3.1, if T is neither a symmetric nor a bisymmetric tree.

Let then T be a symmetric tree $T_{h,3}$. Denote the central vertex by x, and by T_a, T_b and T_c the connected components of $T - x$ which are trees rooted at the neighbours a, b, c of a vertex x, respectively. Since G is 3-connected, there exist an edge e between T_a and T_b in G. Consider a spanning subgraph $\widetilde{T} = T + e$. Then we colour xa and xc with 1, and xb with 2, and extend this colouring as in the proof of Theorem 3.4 to a colouring of $\widetilde{T}$ breaking all non-trivial automorphisms of $\widetilde{T}$ (the colour of e is irrelevant). Consequently, $D'(G) \leq 3$ by Lemma 3.1.

If T is a bisymmetric tree $T''_{h,3}$ with the central edge xy, then we can add to T one edge in a subtree of $T - xy$ rooted at x, and such a graph can be easily distinguished by two colours. Again, our claim follows from Lemma 3.1. □

3.4 2-connected graphs

For a 2-connected planar graph G, the distinguishing index may attain $1 + \left\lceil \sqrt{\Delta(G)} \right\rceil$ as it is shown by the complete bipartite graph $K_{2,q}$ with $q = r^2$ for a positive integer r. In this case, $D'(K_{2,q}) = r + 1$ as it follows from the result obtained independently by Fisher and Isaak [11] and by Imrich, Jerebic and Klavžar [14]. They proved the following theorem. Actually, they formulated it for the distinguishing number $D(K_p \Box K_q)$ of the Cartesian product of complete graphs, but $D'(K_{p,q}) = D(K_p \Box K_q)$.

Theorem 3.7 ([11, 14]). *Let p, q, d be integers such that $d \geq 2$ and $(d-1)^p < q \leq d^p$. Then*

$$D'(K_{p,q}) = \begin{cases} d, & \text{if } \ q \leq d^p - \lceil \log_d p \rceil - 1, \\ d+1, & \text{if } \ q \geq d^p - \lceil \log_d p \rceil + 1. \end{cases}$$

If $q = d^p - \lceil \log_d p \rceil$ then the distinguishing index $D'(K_{p,q})$ is either d or $d+1$ and can be computed recursively in $O(\log^*(q))$ time.

In the next section, we make use of the following immediate corollary.

Corollary 3.8. *If $p \leq q$, then $D'(K_{p,q}) \leq \lceil \sqrt[p]{q} \rceil + 1$.*

In the proof of Proposition 3.10 we also make use of an earlier result of Imrich and Klavžar [15] which is a slightly weaker version of Theorem 3.7 for $d = 2$.

Theorem 3.9 ([15]). *If $2 \leq p \leq q \leq 2^p - p + 1$, then $D'(K_{p,q}) = 2$.*

Proposition 3.10. *If $p \leq q \leq 2^p - p + 1$ and $p + q \geq 7$, then there exists a distinguishing edge 2-colouring of $K_{p,q}$ such that the edges in one of colours induce a connected spanning or almost spanning, asymmetric subgraph of $K_{p,q}$.*

Proof. The assumptions imply that $p \geq 3$, and $D'(K_{p,q}) = 2$ by Theorem 3.9. Let P and Q be the two sets of bipartition of $K_{p,q}$ with $|P| = p$ and $|Q| = q$. If $p = q$, then $p \geq 4$, and there exists a spanning asymmetric tree of $K_{p,p}$ (see [17]). If $p < q \leq 2^p - p + 1$, then for the proof of Theorem 3.9, Imrich and Klavžar in [15] constructed a distinguishing vertex 2-colouring of $K_p \Box K_q$ that corresponds to a distinguishing edge 2-colouring f of $K_{p,q}$, where a colouring of vertices in a K_q-layer can be represented by a sequence from $\{1, 2\}^q$ and it corresponds to a colouring of edges incident to a vertex in P (the same is true for K_p-layers and vertices in Q). We wish to show that this colouring yields a connected asymmetric subgraph of $K_{p,q}$ which is spanning or almost spanning.

First assume that $q = 2^p - p + 1$. In the coloring f, every vertex in P has distinct positive number of edges coloured with 1, and there exists a vertex v_1 with all incident edges coloured with 1. Moreover, distinct vertices from Q have distinct sets of neighbours joined by edges coloured with 1, and there exists a vertex, say v_2, with all incident edges coloured with 2. Let S be a subgraph induced by edges coloured with 1. Then S is an almost spanning subgraph since v_2 is the only vertex outside S. The graph S is connected because v_1 is adjacent to every vertex in Q, and every vertex in P is joined to a vertex in Q by an edge coloured with 1. Moreover, S is also asymmetric since f breaks all non-trivial automorphisms of $K_{p,q}$ and any automorphism interchanging some parts of the sets P and Q does not preserve distances in S.

Following [15] for $p < q < 2^p - p + 1$, we exclude a relevant number of such pairs of sequences of colours that the sum of them is a sequence $(3, \ldots, 3)$. Additionally, if both q and p are odd, we exclude the sequence $(0, \ldots, 0)$. Again, we obtain a connected spanning (or almost spanning) asymmetric subgraph S of $K_{p,q}$ induced by the edges coloured with 1. □

Proposition 3.10 and Lemma 3.1 immediately imply the following.

Corollary 3.11. *If a graph G of order at least 7 is spanned by $K_{p,q}$ and $p \le q \le 2^p - p + 1$, then $D'(G) \le 2$.*

In general, for 2-connected graphs we conjecture that the complete bipartite graph K_{2,r^2} is the worst case, i.e. attains the highest value of the distinguishing index.

Conjecture 3.12. *If G is a 2-connected graph, then*

$$D'(G) \le 1 + \left\lceil \sqrt{\Delta(G)} \right\rceil .$$

4 Nordhaus-Gaddum inequalities for D'

In this section, we discuss Conjecture 1.6, formulated at the end of Introduction, stating that

$$2 \le D'(G) + D'(\overline{G}) \le \Delta + 2$$

for every admissible graph G of order $n \ge 7$, where $\Delta = \max\{\Delta(G), \Delta(\overline{G})\}$.

The left-hand inequality is obvious. Indeed, if a graph G is asymmetric, then so is $\overline{G}$. Thus we are only interested in the right-hand inequality $D'(G) + D'(\overline{G}) \le \Delta + 2$. Note also that at least one of the graphs G and $\overline{G}$ is connected.

The bound $\Delta + 2$ cannot be improved. To see this, consider a star $K_{1,n-1}$ of any order $n \ge 7$. As $\overline{K_{1,n-1}}$ is a disjoint union of a complete graph K_{n-1} and an isolated vertex, it follows from Proposition 2.3 that $D'(\overline{K_{1,n-1}}) = 2$. Therefore, $D'(K_{1,n-1}) + D'(\overline{K_{1,n-1}}) = n - 1 + 2 = \Delta + 2$.

If T is a tree, then $\Delta(T)$ can be much smaller than $\Delta = \Delta(\overline{T}) = n - 1$. However, the following holds.

Proposition 4.1. *If T is a tree of order $n \ge 7$, then*

$$D'(T) + D'(\overline{T}) \le \Delta(T) + 2.$$

Proof. As it was shown above, the conclusion holds for stars. If T is not a star, then $D'(\overline{T}) \leq 2$ by Lemma 3.1. Indeed, as it was proved by Hedetniemi et al. in [13], a complete graph K_n contains edge disjoint copies of any two trees of order n distinct from a star $K_{1,n-1}$. Thus, the complement $\overline{T}$ contains a spanning asymmetric tree. By Theorem 2.6, we have the inequality $D'(T) + D'(\overline{T}) \leq \Delta(T) + 2$. □

This fact emboldened us to formulate the following stronger conjecture.

Conjecture 4.2. *Every connected admissible graph G of order $n \geq 7$ satisfies the inequality*

$$D'(G) + D'(\overline{G}) \leq \Delta(G) + 2.$$

Now we show that Conjecture 1.6 holds not only for trees, but also for some other classes of graphs. To do this we use the following fact.

Theorem 4.3. *Let G be a connected admissible graph of order $n \geq 7$. If either G or every connected component of $\overline{G}$ has the distinguishing index at most 3, then*

$$D'(G) + D'(\overline{G}) \leq \Delta + 2,$$

where $\Delta = \max\{\Delta(G), \Delta(\overline{G})\}$.

Proof. Our claim is true for trees by Proposition 4.1. Observe also, that it is true if G is a path or a cycle of order at least 7 since its complement $\overline{G}$ is Hamiltonian, and $D'(G) + D'(\overline{G}) \leq 4$. So, now we can assume that $\Delta(G) \geq 3$ and neither G nor $\overline{G}$ is a tree. We consider two cases.

Case A. Every component H of $\overline{G}$ satisfies $D'(H) \leq 3$.
Then $D'(G) \leq \Delta(G) - 1$ by Theorem 2.8, and if $\overline{G}$ is connected, then our claim holds. Assume now that $\overline{G}$ is disconnected. Then G is spanned by $K_{p,q}$ with $p \leq q$ and $\Delta \geq q$, where $p + q = |V(G)|$. Suppose that the graph $\overline{G}$ has t isomorphic components. If we had a distinct set of three colours for every component, then $D'(\overline{G}) \leq \lceil \sqrt[3]{6t} \rceil$. We then consider two cases:

a) If $q \leq 2^p - p + 1$, then $D'(G) = 2$ by Corollary 3.11. Moreover, we then have at most $\frac{n}{3}$ components of $\overline{G}$, so $D'(\overline{G}) \leq \lceil \sqrt[3]{2n} \rceil$. And we can easily see that

$$\lceil \sqrt[3]{2n} \rceil + 2 \leq \frac{n}{2} + 2$$

for every $n \geq 4$.

b) If $q \geq 2^p - p + 1$, then there exists a big component (of order q) in $\overline{G}$ and we can assume that $t \leq \frac{p}{3}$ remaining components are isomorphic. In this case, by assumptions we have $p \leq \lceil \log_2(q + p - 1) \rceil$, therefore

$$D'(\overline{G}) \leq \lceil \sqrt[3]{6t} \rceil \leq \sqrt[3]{2 \lceil \log_2(q + p - 1) \rceil}.$$

On the other hand, $D'(G) \leq \lceil \sqrt[p]{q} \rceil + 2$ by Corollary 3.8 and Theorem 3.1. Then it is not difficult to check that for $q \geq 2^p - p + 1$

$$\sqrt[3]{2 \lceil \log_2(q + p - 1) \rceil} + \lceil \sqrt[p]{q} \rceil + 2 \leq q + 2$$

what finishes the proof in Case A.

Case B. $D'(G) \leq 3$.
If graph $\overline{G}$ is connected, then the claim follows immediately from Theorem 2.7 whenever $D'(G) = 2$ or $D'(\overline{G}) = 2$, and it follows from Theorem 2.8 if $D'(G) = 3$. Assume now that $\overline{G}$ has $t \geq 2$ components. Then $\Delta \geq \frac{n}{2}$ and, in the worst case, all components of $\overline{G}$ are isomorphic. Observe that maximal degree of every component is at most $\frac{n}{t} - 1$. If we assign one extra colour to every component, then we need at most $\frac{n}{t} - 1 + (t-1)$ colours to distinguish $\overline{G}$. Hence, if

$$\frac{n}{t} + t \leq \frac{n}{2} - 1,$$

then $D'(\overline{G}) \leq \Delta - 1$, and our claim is true. The above inequality holds unless $t = 2$.

If there exist two isomorphic components in $\overline{G}$, then $D'(G) \leq 2$ due to Corollary 3.11 since G is spanned by $K_{\frac{n}{2},\frac{n}{2}}$. Then $D'(\overline{G}) \leq \frac{n}{2}$, and finally $D'(G) + D'(\overline{G}) \leq \frac{n}{2} + 2$. □

Now we can formulate some consequences of Theorem 4.3 and suitable results proved in Section 3.

Corollary 4.4. *Let G be an admissible graph of order $n \geq 7$. If G satisfies at least one of the following conditions:*

i) G is a traceable graph, or

ii) G is a claw-free graph, or

iii) G is a triangle-free graph, or

iv) G is a 3-connected planar graph,

then

$$D'(G) + D'(\overline{G}) \leq \Delta + 2,$$

where $\Delta = \max\{\Delta(G), \Delta(\overline{G})\}$.

Proof. It suffices to apply Theorem 4.3 together with Theorem 3.2, Theorem 3.4 and Theorem 3.6, respectively. Observe also that if the girth of a graph G is at least 4, i.e., G is triangle-free, then its complement $\overline{G}$ is claw-free. □

Finally, it has to be noted that there exist graphs of order less than 7 such that the right-hand inequality in Conjecture 1.6 is not satisfied. For example, for the graph $K_{3,3}$ we have $D'(K_{3,3}) = 3$, $D'(\overline{K_{3,3}}) = D'(2K_3) = 4$ and $\Delta = 3$, hence $D'(K_{3,3}) + D'(\overline{K_{3,3}}) = \Delta + 4$. Also, $D'(C_5) + D'(\overline{C_5}) = 3 + 3 = \Delta + 4$, and $D'(K_{1,i}) + D'(\overline{K_{1,i}}) = \Delta + 3$ for $i = 3, 4, 5$.

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Large circulant graphs of fixed diameter and arbitrary degree

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Abstract

We consider the degree-diameter problem for undirected and directed circulant graphs. To date, attempts to generate families of large circulant graphs of arbitrary degree for a given diameter have concentrated mainly on the diameter 2 case. We present a direct product construction yielding improved bounds for small diameters and introduce a new general technique for "stitching" together circulant graphs which enables us to improve the current best known asymptotic orders for every diameter. As an application, we use our constructions in the directed case to obtain upper bounds on the minimum size of a subset A of a cyclic group of order n such that the k-fold sumset kA is equal to the whole group. We also present a revised table of largest known circulant graphs of small degree and diameter.

Keywords: Degree-diameter problem, Cayley graphs, circulant graphs, sumsets.

Math. Subj. Class.: 05C25, 05C35

1 Introduction

The goal of the *degree-diameter problem* is to identify the largest possible number $n(d, k)$ of vertices in a graph having diameter k and maximum degree d. This paper considers the problem for the restricted category of circulant graphs, which we view as Cayley graphs of cyclic groups. We consider both undirected and directed versions of the problem in this paper. For a history and more complete summary of the degree-diameter problem, see the survey paper by Miller and Širáň [5].

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All groups considered in this paper are Abelian (indeed cyclic) and hence we use additive notation for the group operation. With this convention we define a *Cayley graph* as follows.

Let G be an Abelian group and $S \subseteq G$ a subset such that $0 \notin S$. Then the *Cayley graph* $\mathrm{Cay}(G, S)$ has the elements of G as its vertex set and each vertex g has an edge to $g + s$ for each $s \in S$. The following properties of $\mathrm{Cay}(G, S)$ are immediate from the definition:

- $\mathrm{Cay}(G, S)$ has order $|G|$ and is a regular graph of degree $|S|$.
- $\mathrm{Cay}(G, S)$ has diameter at most k if and only if every element of G can be expressed as a sum of no more than k elements of S.
- $\mathrm{Cay}(G, S)$ is an undirected graph if $S = -S$; otherwise it is a directed graph.

A *circulant graph* is a Cayley graph of a cyclic group, and we use these terms interchangeably.

Throughout the paper we use the following notation:

- $CC(d, k)$ is the largest order of an undirected circulant graph with degree d and diameter k.
- $DCC(d, k)$ is the largest order of a directed circulant graph with degree d and diameter k.

For a given diameter k, we are interested in determining the asymptotics of $CC(d, k)$ and $DCC(d, k)$ as the degree d tends to infinity. We make use of the following limits:

- $L_C^-(k) = \liminf_{d\to\infty} CC(d, k)/d^k$; $L_C^+(k) = \limsup_{d\to\infty} CC(d, k)/d^k$.
- $L_D^-(k) = \liminf_{d\to\infty} DCC(d, k)/d^k$; $L_D^+(k) = \limsup_{d\to\infty} DCC(d, k)/d^k$.

We begin with some trivial bounds on L^- and L^+. The following asymptotic upper bound is easily obtained; see for example the survey paper [5]:

Observation 1.1 (Trivial upper bound)**.** $L_C^+(k) \le L_D^+(k) \le \frac{1}{k!}$.

For a lower bound, consider $\mathbb{Z}_{r^k}$ with generators $\{hr^\ell : |h| \le \lfloor \frac{r}{2} \rfloor, 0 \le \ell < k\}$:

Observation 1.2 (Trivial lower bound)**.** $L_D^-(k) \ge L_C^-(k) \ge \frac{1}{k^k}$.

In this paper, we present constructions which yield, for each $k \ge 2$, lower bounds on $L_C^-(k)$ and $L_D^-(k)$ that are greater than the trivial $1/k^k$ bound. No such bounds were known previously. Our results include the following (see Corollary 4.6):

- For any diameter $k \ge 2$ and any degree d large enough, $CC(d, k) > \left(1.14775\,\frac{d}{k}\right)^k$.
- For any diameter k that is a multiple of 5 or sufficiently large, and any degree d large enough, $CC(d, k) > \left(1.20431\,\frac{d}{k}\right)^k$.
- For any diameter $k \ge 2$ and any degree d large enough, $DCC(d, k) > \left(1.22474\,\frac{d}{k}\right)^k$.
- For any diameter k that is a multiple of 6 or sufficiently large, and any degree d large enough, $DCC(d, k) > \left(1.27378\,\frac{d}{k}\right)^k$.

We also deduce a result concerning sumsets covering $\mathbb{Z}_n$, and use our techniques to construct a revised table of the largest known circulant graphs of small degree and diameter.

For larger diameters, the trivial bounds become numerically small, and the ratio between the upper and lower bound becomes arbitrarily large. Therefore, in order more easily to assess the success of our constructions, we make use of the following measure which records improvement over the trivial lower bound.

Let $R^-_C(k) = kL^-_C(k)^{1/k}$, and define $R^+_C(k)$, $R^-_D(k)$ and $R^+_D(k)$ analogously. Thus, $R^-_C(k) \geq 1$, with equality if the trivial lower bound is approached asymptotically for large degrees. For each k, these R values thus provide a useful indication of the success of our constructions in exceeding the trivial lower bound. In Section 4, we show how to construct a cyclic Cayley graph from two smaller ones in such a way that the R values are preserved.

The R values are bounded above by $R_{\max}(k) = k(k!)^{-1/k}$. Using the asymptotic version of Stirling's approximation, $\log k! \sim k \log k - k$, we see that as the diameter tends to infinity,

$$1 \;\leq\; \liminf_{k\to\infty} R^-_C(k) \;\leq\; \liminf_{k\to\infty} R^+_C(k) \;\leq\; e,$$

and similarly for $R^-_D(k)$ and $R^+_D(k)$.

In the next section, we extend a result of Vetrík [7] to deduce new lower bounds for $L^-_C(2)$ and $R^-_C(2)$. In Section 3, we describe a direct product construction and use it to build large cyclic Cayley graphs of small diameter and arbitrarily large degree. We also prove that this construction is unable to yield values that exceed the trivial lower bound for large diameter. However, in Section 4, we demonstrate a method of building a circulant graph by "stitching" together two smaller ones, and show how the application of this method to the constructions from Section 3 enables us to exceed the trivial lower bound for every diameter.

Section 5 contains an application of our constructions to obtain upper bounds on the minimum size of a set $A \subseteq \mathbb{Z}_n$ such that the k-fold sumset kA is equal to $\mathbb{Z}_n$. We conclude, in Section 6, by presenting a revised table of the largest known circulant graphs of small degree and diameter, including a number of new largest orders resulting from our constructions.

2 Diameter 2 bounds for all large degrees

Much of the study of this problem to date has concentrated on the diameter 2 undirected case. In this instance, the trivial lower bound on $L^-_C(2)$ is 1/4 and the trivial upper bound on $L^+_C(2)$ is 1/2. Vetrík [7] (building on Macbeth, Šiagiová & Širáň [4]) presents a construction that proves that $L^+_C(2) \geq \frac{13}{36} \approx 0.36111$, and thus $R^+_C(2) > 1.20185$.

In this section, we begin by extending this result to yield bounds for $L^-_C(2)$ and $R^-_C(2)$. This argument can also be found in Lewis [3]. We reproduce it here for completeness, since we make use of the resulting bounds below.

Vetrík's theorem applies only to values of the degree d of the form $6p-2$, where p is a prime such that $p \neq 13, p \not\equiv 1 \pmod{13}$. We extend this result to give a slightly weaker bound valid for all sufficiently large values of d. The strategy is as follows:

- Given a value of d, we select the largest prime p in the allowable congruence classes such that $6p-2 \leq d$.
- We construct the graph of Vetrík [7] using this value of p.

- We add generators to the Vetrík construction (and hence edges to the Cayley graph) to obtain a new graph of degree d which still has diameter 2.

Note that the graphs in the Vetrík construction always have even order and hence we may obtain an odd degree d by adding the unique element of order 2 to the generator set.

Success of this method relies on being able to find a prime p sufficiently close to the optimal value so that we need only add asymptotically few edges to our graph. We use recent results of Cullinan & Hajir [1] following Ramaré & Rumely [6].

Lemma 2.1 (Cullinan & Hajir [1], Ramaré & Rumely [6])**.** *Let* $\delta = 0.004049$. *For any* $x_0 \geq 10^{100}$ *there exists a prime* $p \equiv 2 \pmod{13}$ *in the interval* $[x_0, x_0 + \delta x_0]$.

Proof. We use the method of Cullinan and Hajir [1, Theorem 1]. This method begins by using the tables of Ramaré and Rumely [6] to find a value ϵ corresponding to $k = 13, a = 2, x_0 = 10^{100}$. Following the proof of Cullinan and Hajir [1, Theorem 1], if $\delta > \frac{2\epsilon}{1-\epsilon}$ it follows that there must exist a prime $p \equiv 2 \pmod{13}$ in the interval $[x_0, x_0 + \delta x_0]$. From Table 1, Ramaré and Rumely [6] we find $\epsilon = 0.002020$ and hence $\delta = 0.004049$ will suffice. □

Our improved bound for circulant graphs of diameter 2 follows:

Theorem 2.2 (see [3, Theorem 6])**.** $L_C^-(2) > 0.35820$, *and hence* $R_C^-(2) > 1.19700$.

Proof. Let $\delta = 0.004049$ and let $d > 10^{101}$. We seek the largest prime $p \equiv 2 \pmod{13}$ such that $6p - 2 \leq d$. By the result of Lemma 2.1, there exists such a p with $p \geq (d + 2)/6(1 + \delta)$. Let $d' = 6p - 2$. Then by the result of Vetrík [7] we can construct a circulant graph of degree d', diameter 2 and order $n = \frac{13}{36}(d' + 2)(d' - 4)$. We can add $d - d'$ generators to this construction to obtain a graph of degree d and diameter 2, with the same order n.

Since $n = \frac{13}{36}d^2/(1 + \delta)^2 + O(d) \approx 0.358204d^2 + O(d)$ the result follows. □

3 Direct product constructions for small diameters

In this section, we construct large undirected circulant graphs of diameters $k = 3, 4, 5$ and arbitrary large degree. We also construct large directed circulant graphs of diameters $k = 2, \ldots, 9$ and arbitrary large degree. We then prove that the approach used is insufficient to yield values that exceed the trivial lower bound for large diameter.

3.1 Preliminaries

The diameter 2 constructions of Macbeth, Šiagiová & Širáň and of Vetrík both have the form $F_p^+ \times F_p^* \times \mathbb{Z}_w$ for some fixed w and variable p, where F_p^+ and F_p^* are the additive and multiplicative groups of the Galois field $GF(p)$. Thus the first two components of their constructions are very tightly coupled, and this coupling is a key to their success. However, a significant limitation of this method is that it is only applicable in the diameter 2 case.

In contrast, the constructions considered here have components that are as loosely coupled as possible. For diameter k, they have the form $\mathbb{Z}_{r_1} \times \mathbb{Z}_{r_2} \times \ldots \times \mathbb{Z}_{r_k} \times \mathbb{Z}_w$ for some fixed w and variable pairwise coprime r_i. This gives us greater flexibility, especially in terms of the diameters we can achieve. The price for this is that we lose the inherent structure of the finite field, which consequently places limits on the bounds we can achieve.

The constructions in this section make use of the following result concerning the representation of each element of the cyclic group $\mathbb{T} = \mathbb{Z}_r \times \mathbb{Z}_s$ (r and s coprime) as the sum of a small multiple of the element $(1, 1)$ and a small multiple of another element (u, v). It can be helpful to envisage $\mathbb{T}$ as a group of vectors on the $r \times s$ discrete torus.

Lemma 3.1. *Let u, d, s and m be positive integers with $s > 1$ and coprime to md. Let $v = u + d$. Suppose $s \geq mv(u - 1)$. Then, for every element (x, y) of $\mathbb{T} = \mathbb{Z}_{s+md} \times \mathbb{Z}_s$, there exist nonnegative integers $h < s + mv$ and $\ell < s - m(u - 1)$ such that $(x, y) = h(1, 1) + \ell(u, v)$.*

Observe that the construction ensures that $(s + mv)(1, 1) = m(u, v)$. Figure 1 illustrates the case with parameters $u = 2$, $v = 5$, $s = 11$, $m = 2$.

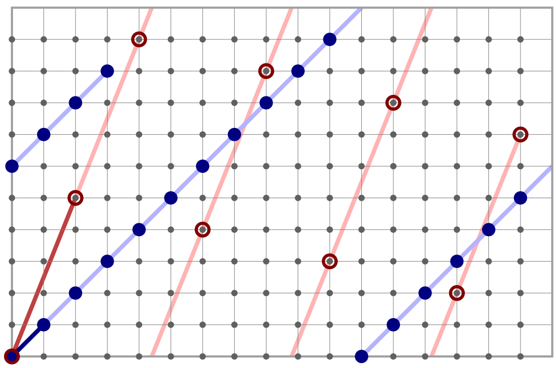

Figure 1: Every element of $\mathbb{Z}_{17} \times \mathbb{Z}_{11}$ is the sum of one of the 21 solid elements and one of the 9 circled elements.

Proof. Let $t = s - m(u - 1)$. Since s is coprime to md, $(1, 1)$ generates $\mathbb{T}$. Hence, it suffices to show that, in the list $(0, 0)$, $(1, 1)$, $(2, 2)$, $\ldots$, the gaps between members of $\{\ell(u, v) : 0 \leq \ell < t\}$ are not "too large".

Specifically, we need to show that, for each nonnegative $\ell < t$, there is some positive $h' \leq s + mv$ and nonnegative $\ell' < t$ such that $\ell(u, v) + h'(1, 1) = \ell'(u, v)$.

There are two cases. If $\ell < t - m$, then we can take $h' = s + mv$ and $\ell' = \ell + m$:

$$\begin{aligned} \ell(u, v) + (s + mv)(1, 1) &= (\ell u + s + mu + md,\ \ell v + s + mv) \\ &= (\ell u + mu,\ \ell v + mv) \\ &= (\ell + m)(u, v). \end{aligned}$$

If $\ell \geq t - m$, then we can take $h' = muv$ and $\ell' = \ell + m - t = \ell + mu - s$:

$$\begin{aligned} \ell(u, v) + muv(1, 1) &= (\ell u + mu^2 + mud,\ \ell v + muv) \\ &= (\ell u + mu^2 + mud - u(s + md),\ \ell v + muv - vs) \\ &= (\ell + mu - s)(u, v). \end{aligned}$$

The requirement that $muv \leq s + mv$ is clearly equivalent to the condition on s in the statement of the lemma. □

In our direct product constructions, we make use of Lemma 3.1 via the following crucial lemma. Our strategy is to construct a cyclic group of the form $\mathbb{T} = \mathbb{Z}_{r_1} \times \mathbb{Z}_{r_2} \times \ldots \times \mathbb{Z}_{r_k}$ such that $r_1 > r_2 > \ldots > r_k > 1$, and for each pair $\mathbb{Z}_{r_i} \times \mathbb{Z}_{r_j}$ for $i < j$ we will bring Lemma 3.1 to bear. In the notation of that lemma we will set $u = i$, $d = j - i$, $s = r_j$, $s + md = r_i$. The conditions of Lemma 3.2 below are designed to ensure that for each pair i, j we can find $m = m_{i,j}$ to make this work.

Lemma 3.2. *Let $k > 1$ and let $r_1 > r_2 > \ldots > r_k$ be pairwise coprime integers greater than 1 such that r_i is coprime to i for all $1 \leq i \leq k$. Suppose that for each i, j with $1 \leq i < j \leq k$ there exists a positive integer $m_{i,j}$ such that:*

- $r_i - r_j = m_{i,j}(j - i)$
- $r_j \geq m_{i,j}(i - 1)j$.

Let $\mathbb{T} = \mathbb{Z}_{r_1} \times \mathbb{Z}_{r_2} \times \ldots \times \mathbb{Z}_{r_k}$. Let $\mathbf{o} = (1, 1, \ldots, 1)$, $\mathbf{u} = (1, 2, \ldots, k)$ and, for each i, $\mathbf{e}_i = (0, \ldots, 1, \ldots, 0)$ be elements of $\mathbb{T}$, where only the ith coordinate of $\mathbf{e}_i$ is 1, and let the set A consist of these $k + 2$ elements.

Let $c_{\mathbf{o}} = \max_{i<j}(r_j + j m_{i,j})$, $c_{\mathbf{u}} = r_1$, and for each i, $c_{\mathbf{e}_i} = r_i$.

Then, for every element $\mathbf{x}$ of $\mathbb{T}$ and every k-element subset S of A, there exist nonnegative integers $h_{\mathbf{s}} < c_{\mathbf{s}}$ for each $\mathbf{s} \in S$, such that $\mathbf{x} = \sum_{\mathbf{s}} h_{\mathbf{s}} \mathbf{s}$.

Proof. There are four cases. If S contains neither $\mathbf{o}$ nor $\mathbf{u}$, the result follows trivially.

If S contains $\mathbf{o}$ but not $\mathbf{u}$, omitting $\mathbf{e}_i$, then we can choose $h_{\mathbf{o}}$ to be the ith coordinate of $\mathbf{x}$. Note that, as required, $c_{\mathbf{o}} \geq r_2 + 2(r_1 - r_2) = r_1 + (r_1 - r_2) > r_i$ for all i.

If S contains $\mathbf{u}$ but not $\mathbf{o}$, omitting $\mathbf{e}_i$, then, since i and r_i are coprime, we can choose $h_{\mathbf{u}}$ such that $i h_{\mathbf{u}} \pmod{r_i}$ is the ith coordinate of $\mathbf{x}$.

Finally, if S contains both $\mathbf{o}$ and $\mathbf{u}$, omitting $\mathbf{e}_i$ and $\mathbf{e}_j$, then we can choose $h_{\mathbf{o}}$ and $h_{\mathbf{u}}$ by applying Lemma 3.1 to $\mathbb{Z}_{r_i} \times \mathbb{Z}_{r_j}$ with $(u, v) = (i, j)$. □

We note that the conditions of Lemma 3.2 imply that at most one of the r_i can be even, and if $k \geq 4$ then all r_i must be odd.

3.2 Undirected constructions

We can use Lemma 3.2 to construct undirected circulant graphs of any diameter by means of the following theorem:

Theorem 3.3. *Let w and k be positive integers and suppose that there exist sets B and T of positive integers with the following properties:*

- *$B = \{b_1, \ldots, b_{k+2}\}$ has cardinality $k + 2$ and the property that every element of $\mathbb{Z}_w$ can be expressed as the sum of exactly k distinct elements of $B \cup -B$, no two of which are inverses.*
- *$T = \{r_1, r_2, \ldots, r_k\}$ has cardinality k and the properties that all its elements are coprime to w, and satisfies the requirements of Lemma 3.2, i.e. for each $i < j$:*

  (a) $r_i > r_j$

  (b) $\gcd(r_i, r_j) = 1$

  (c) $\gcd(r_i, i) = 1$

(d) There is a positive integer $m_{i,j}$ such that equalities $r_i - r_j = m_{i,j}(j-i)$ and $r_j \geq m_{i,j}(i-1)j$ hold.

Let $c_\mathbf{o} = \max_{i<j}(r_j + jm_{i,j})$ and $c_\mathbf{u} = r_1$ as in Lemma 3.2.

Then there exists an undirected circulant graph of order $w\prod_{i=1}^{k} r_i$, degree at most $2\left(\sum_{i=1}^{k} r_i + c_o + c_u\right)$ and diameter k.

Proof. Let $\mathbb{T} = \mathbb{Z}_{r_1} \times \mathbb{Z}_{r_2} \times \ldots \times \mathbb{Z}_{r_k} \times \mathbb{Z}_w$. Then $\mathbb{T}$ is a cyclic group since all its factors have coprime orders.

Let X be the generating set consisting of the following elements:

- $(x, 0, 0, \ldots, 0, \pm b_1)$, $x \in \mathbb{Z}_{r_1}$
- $(0, x, 0, \ldots, 0, \pm b_2)$, $x \in \mathbb{Z}_{r_2}$

  $\vdots$
- $(0, 0, \ldots, 0, x, \pm b_k)$, $x \in \mathbb{Z}_{r_k}$
- $\pm(x, x, \ldots, x, x, b_{k+1})$, $0 \leq x < c_\mathbf{o}$
- $\pm(x, 2x, \ldots, (k-1)x, kx, b_{k+2})$, $0 \leq x < c_\mathbf{u}$

Then by construction and by Lemma 3.2, every element of $\mathbb{T}$ is the sum of at most k elements of X. Since $|\mathbb{T}| = w\prod_{i=1}^{k} r_i$ and $|X| = 2\left(\sum_{i=1}^{k} r_i + c_o + c_u\right)$, the result follows. □

For small diameters this technique results in the following asymptotic bounds:

Theorem 3.4. *For diameters $k = 3, 4, 5$, we have the following lower bounds:*

(a) $L_C^+(3) \geq \frac{57}{1000}$ and $L_C^-(3) \geq \frac{7}{125}$, so $R_C^+(3) > 1.15455$ and $R_C^-(3) > 1.14775$.

(b) $L_C^+(4) \geq L_C^-(4) \geq \frac{25}{3456}$, so $R_C^+(4) \geq R_C^-(4) > 1.16654$.

(c) $L_C^+(5) \geq L_C^-(5) \geq \frac{109}{134456}$, so $R_C^+(5) \geq R_C^-(5) > 1.20431$.

Proof. Given a diameter k, the strategy is to find an optimal value of w which admits a set B satisfying the conditions of Theorem 3.3. We then seek an infinite family of positive integers q and a set $\Delta = \{\delta_1, \delta_2, \ldots, \delta_{k-1}\}$ such that for each of our values of q, the set $T = \{q, q-\delta_1, \ldots, q-\delta_{k-1}\}$ satisfies the conditions of the theorem. We illustrate for $k = 3$.

To prove *(a)* we take $w = 57$ and $B = \{1, 2, 7, 8, 27\}$. It is easily checked that every element of $\mathbb{Z}_{57}$ is the sum of three distinct elements of $B \cup -B$, no two of which are inverses. Now we let $\Delta = \{4, 6\}$. For any $q \geq 17, q \equiv 5 \pmod 6, q \not\equiv 0, 4, 6 \pmod{19}$ it is straightforward to verify that the set $T = \{q, q-4, q-6\}$ satisfies the conditions of Theorem 3.3. In the notation of Lemma 3.2, we have $c_\mathbf{o} = q + 4$.

Taking a generating set X as defined in Theorem 3.3 we may construct a circulant graph of diameter 3, degree $d = |X| = 10q - 12$ and order $57q(q-4)(q-6) = \frac{57}{1000}(d+12)(d-28)(d-48)$.

We can do this for an infinite number of values of q, and hence for an infinite number of values of $d = 10q - 12$ we have

$$CC(d, 3) \geq \frac{57}{1000}(d+12)(d-28)(d-48).$$

This yields $L_C^+(3) \geq \frac{57}{1000}$. Now we need to consider $L_C^-(3)$. The strategy will be to try to add "few" edges to our graphs to cover all possible degrees. Observe that we can use this construction for any $q \equiv 17 \pmod{114}$ and hence for any $d \equiv 158 \pmod{1140}$. Given any arbitrary *even* degree d, we can therefore find some d' no smaller than $d - 1140$ for which the construction works. We can then add $d - d'$ generators to our graph to obtain a graph of the same order, degree d and diameter 3.

However our graphs always have odd order, and so we are unable to obtain an odd degree graph by this method. To get round this problem we may use $w = 56$, $B = \{1, 2, 7, 14, 15\}$, $\Delta = \{2, 4\}$ and $c_{\mathbf{o}} = q + 2$. Again it is easy to check that the relevant conditions are satisfied for any $q \geq 15$ such that $q \equiv 3, 5 \pmod 6$ and $q \equiv 1, 3, 5, 6 \pmod 7$. Then for $d = 10q - 8$ we can construct a graph of order $\frac{7}{25}(d+8)(d-12)(d-32)$, degree d and diameter 3. We can do this for any $q \equiv 15 \pmod{42}$ and hence for any $d \equiv 142 \pmod{420}$. So given any arbitrary degree d, we can therefore find some d' no smaller than $d - 420$ for which the construction works, and then add $d - d'$ generators to our graph to obtain a graph of the same order and diameter 3. (Since our graphs now have even order it is possible to add an odd number of generators.) Since the number of added generators is bounded above (by 419), the order of the graph is $\frac{7}{125}d^3 + O(d^2)$. Result *(a)* for $L_C^-(3)$ follows.

For *(b)* and *(c)* we adopt a similar method, except that in both cases the graphs used have even order and so our bounds on L^+ and L^- are equal. For brevity we show only the relevant sets in the construction, summarised as follows:

(b) ($k = 4$) – Take $w = 150$, $B = \{1, 7, 16, 26, 41, 61\}$ and $\Delta = \{6, 8, 12\}$ so $c_{\mathbf{o}} = q + 6$. Then for $q \geq 49$, $q \equiv 19 \pmod{30}$ and $d = 12q - 40$, we have

$$CC(d, 4) \geq \frac{25}{3456}(d + 40)(d - 32)(d - 56)(d - 104).$$

(c) ($k = 5$) – Take $w = 436$, $B = \{1, 15, 43, 48, 77, 109, 152\}$ and $\Delta = \{0, 4, 10, 12, 16\}$ so $c_{\mathbf{o}} = q + 8$. Then for $q \geq 77$, $q \equiv 5 \pmod 6$, $q \not\equiv 0, 1 \pmod 5$, $q \not\equiv 0, 4, 10, 12, 16 \pmod{109}$ and $d = 14q - 68$, we have

$$CC(d, 5) \geq \frac{109}{134456}(d + 68)(d + 12)(d - 72)(d - 100)(d - 156).$$

□

3.3 Directed constructions

An analogous method yields directed circulant graphs via the following theorem:

Theorem 3.5. *Let w and k be positive integers and suppose that there exist sets B and T of non-negative integers with the following properties:*

- *$B = \{0, b_2, \ldots, b_{k+2}\}$ has cardinality $k + 2$ and the property that every element of $\mathbb{Z}_w$ can be expressed as the sum of exactly k distinct elements of B.*
- *$T = \{r_1, r_2, \ldots, r_k\}$ has cardinality k and the properties that all its elements are coprime to w, and it satisfies the requirements of Lemma 3.2, i.e. for each $i < j$:*

  (a) $r_i > r_j$

(b) $\gcd(r_i, r_j) = 1$

(c) $\gcd(r_i, i) = 1$

(d) *There is a positive integer* $m_{i,j}$ *such that equalities* $r_i - r_j = m_{i,j}(j - i)$ *and* $r_j \geq m_{i,j}(i-1)j$ *hold.*

Let $c_{\mathbf{o}} = \max_{i<j}(r_j + jm_{i,j})$ *and* $c_{\mathbf{u}} = r_1$ *as in Lemma 3.2.*

Then we may construct a directed circulant graph of order $w \prod_{i=1}^{k} r_i$*, degree at most* $\sum_{i=1}^{k} r_i + c_o + c_u - 1$ *and diameter* k*.*

Proof. Let $\mathbb{T} = \mathbb{Z}_{r_1} \times \mathbb{Z}_{r_2} \times \ldots \times \mathbb{Z}_{r_k} \times \mathbb{Z}_w$. Then $\mathbb{T}$ is a cyclic group since all its factors have coprime orders.

Let X be the generating set consisting of the following elements:

- $(x, 0, 0, \ldots, 0, 0)$, $x \in \mathbb{Z}_{r_1} \setminus \{0\}$
- $(0, x, 0, \ldots, 0, b_2)$, $x \in \mathbb{Z}_{r_2}$

  $\vdots$

- $(0, 0, \ldots, 0, x, b_k)$, $x \in \mathbb{Z}_{r_k}$
- $(x, x, \ldots, x, x, b_{k+1})$, $0 \leq x < c_{\mathbf{o}}$
- $(x, 2x, \ldots, (k-1)x, kx, b_{k+2})$, $0 \leq x < c_{\mathbf{u}}$

Then by construction and by Lemma 3.2, every element of $\mathbb{T}$ is the sum of at most k elements of X. Since $|\mathbb{T}| = w \prod_{i=1}^{k} r_i$ and $|X| = \sum_{i=1}^{k} r_i + c_o + c_u - 1$, the result follows. □

For small diameters this technique results in the following asymptotic bounds:

Theorem 3.6. *For diameters* $k = 2, \ldots, 9$*, we have the following lower bounds on* $L_D^-(k)$ *and* $R_D^-(k)$*:*

(a) $L_D^-(2) \geq \frac{3}{8}$*, so* $R_D^-(2) > 1.22474$*.*

(b) $L_D^-(3) \geq \frac{9}{125}$*, so* $R_D^-(3) > 1.24805$*.*

(c) $L_D^-(4) \geq \frac{13}{1296}$*, so* $R_D^-(4) > 1.26588$*.*

(d) $L_D^-(5) \geq \frac{17}{16807}$*, so* $R_D^-(5) > 1.25881$*.*

(e) $L_D^-(6) \geq \frac{3}{32768}$*, so* $R_D^-(6) > 1.27378$*.*

(f) $L_D^-(7) \geq \frac{10}{1594323}$*, so* $R_D^-(7) > 1.26436$*.*

(g) $L_D^-(8) \geq \frac{9}{25000000}$*, so* $R_D^-(8) > 1.25206$*.*

(h) $L_D^-(9) \geq \frac{42}{2357947691}$*, so* $R_D^-(9) > 1.23939$*.*

Proof. The method is exactly the same as the proof of Theorem 3.4 and we summarise as follows:

(a) ($k = 2$) – Take $w = 6$, $B = \{0, 1, 2, 4\}$ and $\Delta = \{2\}$ so $c_{\mathbf{o}} = q + 2$. Then for $q \geq 7$, $q \equiv 1 \pmod 6$ and $d = 4q - 1$, we have

$$DCC(d, 2) \geq \frac{3}{8}(d + 1)(d - 7).$$

(b) ($k = 3$) – Take $w = 9$, $B = \{0, 1, 2, 3, 6\}$ and $\Delta = \{4, 6\}$ so $c_{\mathbf{o}} = q + 4$. Then for $q \geq 17$, $q \equiv 5 \pmod 6$ and $d = 5q - 7$, we have

$$DCC(d, 3) \geq \frac{9}{125}(d + 7)(d - 13)(d - 23).$$

(c) ($k = 4$) – Take $w = 13$, $B = \{0, 1, 3, 5, 7, 8\}$ and $\Delta = \{2, 4, 6\}$ so $c_{\mathbf{o}} = q + 2$. Then for $q \geq 23$, $q \equiv 5 \pmod 6$, $q \not\equiv 0, 2, 4, 6 \pmod{13}$ and $d = 6q - 11$, we have

$$DCC(d, 4) \geq \frac{13}{1296}(d + 11)(d - 1)(d - 13)(d - 25).$$

(d) ($k = 5$) – Take $w = 17$, $B = \{0, 1, 2, 3, 4, 8, 13\}$ and $\Delta = \{4, 10, 12, 16\}$ so $c_{\mathbf{o}} = q + 8$. Then for $q \geq 77$, $q \equiv 5 \pmod 6$, $q \not\equiv 0, 1 \pmod 5$, $q \not\equiv 0, 4, 10, 12, 16 \pmod{17}$ and $d = 7q - 35$, we have

$$DCC(d, 5) \geq \frac{17}{16807}(d + 35)(d + 7)(d - 35)(d - 49)(d - 77).$$

(e) ($k = 6$) – Take $w = 24$, $B = \{0, 1, 2, 4, 8, 13, 18, 22\}$ and $\Delta = \{6, 12, 18, 24, 30\}$ so $c_{\mathbf{o}} = q + 6$. Then for $q \geq 181$, $q \equiv 1, 5 \pmod 6$, $q \not\equiv 0, 4 \pmod 5$ and $d = 8q - 85$, we have

$$DCC(d, 6) \geq \frac{3}{32768}(d + 85)(d + 37)(d - 11)(d - 59)(d - 107)(d - 155).$$

(f) ($k = 7$) – Take $w = 30$, $B = \{0, 1, 2, 6, 9, 12, 16, 17, 18\}$ and $\Delta = \{0, 2, 6, 18, 20, 30, 42\}$ so $c_{\mathbf{o}} = q + 42$. Then for $q \geq 529$, $q \equiv 1 \pmod 6$, $q \equiv 4 \pmod 5$, $q \not\equiv 0, 2, 6 \pmod 7$, $q \not\equiv 9 \pmod{11}$ and $d = 9q - 77$, we have

$$DCC(d, 7) \geq \frac{10}{1594323}(d + 77)(d + 59)(d + 23)(d - 85)(d - 103)(d - 193)(d - 301).$$

(g) ($k = 8$) – Take $w = 36$, $B = \{0, 1, 2, 3, 6, 12, 19, 20, 27, 33\}$ and $\Delta = \{0, 6, 12, 18, 24, 30, 36, 42\}$ so $c_{\mathbf{o}} = q + 6$. Then for $q \geq 353$, $q \equiv 1, 5 \pmod 6$, $q \equiv 3 \pmod 5$, $q \not\equiv 0, 1 \pmod 7$ and $d = 10q - 163$, we have

$$DCC(d, 8) \geq \frac{9}{25000000}(d + 163)(d + 103)(d + 43)(d - 17)$$
$$(d - 77)(d - 137)(d - 197)(d - 257).$$

(h) ($k = 9$) – Take $w = 42$, $B = \{0, 1, 2, 3, 4, 9, 16, 20, 26, 30, 37\}$ and $\Delta = \{0, 2, 6, 12, 20, 30, 42, 56, 72\}$ so $c_{\mathbf{o}} = q + 72$. Then for $q \geq 1093$, $q \equiv 1 \pmod 6$, $q \equiv 3, 4 \pmod 5$, $q \equiv 1, 3, 4 \pmod 7$, $q \not\equiv 1, 6, 9 \pmod{11}$, $q \not\equiv 4, 7 \pmod{13}$ and $d = 11q - 169$, we have

$$DCC(d, 9) \geq \frac{42}{2357947691}(d + 169)(d + 147)(d + 103)(d + 37)(d - 51)$$
$$(d - 161)(d - 293)(d - 447)(d - 623).$$

□

3.4 Limitations

In [3], Lewis showed that an analogous class of constructions using finite fields to create graphs of diameter 2 is limited by the bound $L_C^-(2) \leq \frac{3}{8}$. The constructions in this section have a similar limitation:

Observation 3.7. *Let k be a positive integer. The direct product constructions of Theorems 3.3 and 3.5 can never yield a lower bound on $L_C^-(k)$ or $L_D^-(k)$ that exceeds $\frac{k+1}{2(k+2)^{k-1}}$.*

Proof. First we consider the undirected case. Suppose the requirements of Theorem 3.3 hold and for each $i = 1, \ldots, k$, we have $r_i = q - a_i$, where $a_1 < a_2 < \ldots < a_k$. Let $\mathbb{T} = \mathbb{Z}_{q-a_1} \times \ldots \times \mathbb{Z}_{q-a_k} \times \mathbb{Z}_w$ and X be its generating set as in the proof of Theorem 3.3.

Since every element of $\mathbb{Z}_w$ is a sum of k distinct elements of B, no pair of which are inverses, we must have $w \leq \binom{k+2}{k} 2^k = (k+1)(k+2)2^{k-1}$.

By the requirements of Lemma 3.2, for any $i < j$, we have $m_{i,j} \leq r_i - r_j$ and $c_{\mathbf{o}} = \max_{i<j}(r_j + jm_{i,j})$. Hence, since $r_i = q - a_i$, we have $m_{i,j} \leq a_k - a_1$, and so $c_{\mathbf{o}} \leq q + ka_k$.

Thus X is the generating set for a Cayley graph on $\mathbb{T}$ with diameter k, degree d no greater than $2(k+2)q - 2\sum_{i=1}^{k} a_i + 2ka_k - 2a_1$, and order $n = w(q-a_1)(q-a_2)\ldots(q-a_k)$.

Hence, $n = \frac{w}{(2(k+2))^k} d^k + O(d^{k-1}) \leq \frac{(k+1)(k+2)2^{k-1}}{(2(k+2))^k} d^k + O(d^{k-1}) = \frac{k+1}{2(k+2)^{k-1}} d^k + O(d^{k-1})$, as required.

The directed case is analogous. We follow Theorem 3.5 and its proof. In this case, every element of $\mathbb{Z}_w$ is the sum of k distinct elements of B, so $w \leq \binom{k+2}{k} = (k+1)(k+2)/2$, and X is the generating set for a Cayley graph on $\mathbb{T}$ with diameter k, degree $d \leq (k+2)q - \sum_{i=1}^{k} a_i + ka_k - a_1 - 1$, and order $n = w(q-a_1)(q-a_2)\ldots(q-a_k)$.

Hence, $n = \frac{w}{(k+2)^k} d^k + O(d^{k-1}) \leq \frac{(k+1)(k+2)}{2(k+2)^k} d^k + O(d^{k-1}) = \frac{k+1}{2(k+2)^{k-1}} d^k + O(d^{k-1})$. □

Observe that, in the limit,

$$\lim_{k\to\infty} k \left(\frac{k+1}{2(k+2)^{k-1}} \right)^{1/k} = 1.$$

As a consequence, these direct product constructions themselves can never yield an improvement on the trivial lower bound for the limiting value of $R_C^-(k)$ or $R_D^-(k)$. However, it is possible to combine graphs of small diameter to produce larger graphs in such a way that we can improve on the trivial lower bound in the limit as the diameter increases. The next section introduces this idea.

4 A general graph product construction

The following theorem gives a simple way to combine two cyclic Cayley graphs to obtain a third cyclic Cayley graph. It is valid in both the directed and undirected cases.

Theorem 4.1. *Let G_1 and G_2 be two cyclic Cayley graphs of diameters k_1 and k_2, orders n_1 and n_2, and degrees d_1 and d_2 respectively. In the case of undirected graphs where d_1 and d_2 are both odd let $\delta = 1$, otherwise $\delta = 0$. In the directed case let $\delta = 0$ always. Then there exists a cyclic Cayley graph with diameter $k_1 + k_2$, degree at most $d_1 + d_2 + \delta$, and order $n_1 n_2$.*

Proof. Let S_1 be the connection set of G_1 so that $|S_1| = d_1$ and similarly for G_2. For convenience we consider each S_i to consist of elements within the interval $(-n_i/2, n_i/2]$. Let G be the cyclic group $\mathbb{Z}_{n_1 n_2}$ and consider the connection set $S' = n_2 S_1 \cup S_2$. Then $|S'| \leq n_1 + n_2$.

We now construct a connection set S for the group G such that the Cayley graph $\mathrm{Cay}(G, S)$ has diameter $k_1 + k_2$. In the directed case we may simply take $S = S'$. In the undirected case we need to ensure that $S = -S$. If at least one of d_1, d_2 is even we may assume without loss of generality that d_2 is even and then we may again let $S = S'$ and $S = -S$ by construction.

It remains to consider the undirected case when d_1 and d_2 are both odd (the case $\delta = 1$). In that case we know $n_2/2 \in S_2 \subset S'$ and we let $S = S' \cup \{-n/2\}$ so that $S = -S$.

It is then clear that the Cayley graph $\mathrm{Cay}(G, S)$ has degree at most $d_1 + d_2 + \delta$, diameter $k_1 + k_2$ and order $n_1 n_2$. ◻

We can use this "stitching" construction to obtain lower bounds on our L and R values for large diameters, given values for smaller diameters.

Corollary 4.2. *If $L(k)$ is one of $L^-_C(k)$, $L^+_C(k)$, $L^-_D(k)$ or $L^+_D(k)$ and $R(k)$ is one of $R^-_C(k)$, $R^+_C(k)$, $R^-_D(k)$ or $R^+_D(k)$, then*

(a) $$L(k_1 + k_2) \geq \frac{L(k_1)L(k_2)k_1^{k_1}k_2^{k_2}}{(k_1 + k_2)^{k_1 + k_2}},$$

(b) $$R(k_1 + k_2) \geq \left(R(k_1)^{k_1} R(k_2)^{k_2}\right)^{\frac{1}{k_1 + k_2}}.$$

Proof. (a) Let $d > 1$. For $i = 1, 2$ we may construct graphs Γ_i of diameter k_i, degree $k_i d$ and order $L(k_i)(k_i d)^{k_i} + o(d^{k_i})$. Theorem 4.1 yields a product graph of diameter $k_1 + k_2$, degree at most $(k_1 + k_2)d + 1$ and order $L(k_1)L(k_2)k_1^{k_1}k_2^{k_2}d^{k_1+k_2} + o(d^{k_1+k_2})$.

Part *(b)* follows by straightforward algebraic manipulation. ◻

In particular, we note that the stitching construction of Theorem 4.1 preserves lower bounds on the R values: $R(mk) \geq R(k)$ for every positive integer m.

We may use this idea to obtain better bounds for some particular diameters; for example we may improve on the undirected diameter 4 construction in Theorem 3.4:

Corollary 4.3.

(a) $L^+_C(4) \geq \frac{169}{20736} \approx 0.0081501$*, and hence* $R^+_C(4) > 1.20185$.

(b) $L^-_C(4) > 0.0080194$*, and hence* $R^-_C(4) > 1.19700$.

Proof. For statement *(a)* we note $R^+_C(2) \geq \frac{13}{36}$ from Vetrík [7] and apply Corollary 4.2 with $k_1 = k_2 = 2$. For *(b)* we use the same method starting with Theorem 2.2. ◻

The stitching process of Theorem 4.1 can be iterated to produce a construction for any desired diameter, and Corollary 4.2 then gives us a lower bound for the R values for that diameter. We illustrate the results for small diameter k in Table 1. As an indicator of progress we show also the largest possible value of R for a particular k, given by $R_{\max}(k) = k(k!)^{-1/k}$.

It is worth noting that the method of Corollary 4.2 may be used to produce values of R which are larger than those achievable from the direct product constructions of Section 3.

Table 1: The best R values for diameter $k \leq 9$.

	Diameter (k)							
	2	3	4	5	6	7	8	9
$R_{\max}(k) \approx$	1.41421	1.65096	1.80720	1.91926	2.00415	2.07100	2.12520	2.17016
$R^+_C(k) >$	1.20185^a	1.15455^d	1.20185^c	1.20431^d	1.20185^f	1.20360^f	1.20185^f	1.20321^f
$R^-_C(k) >$	1.19700^b	1.14775^d	1.19700^c	1.20431^d	1.19700^f	1.20222^f	1.19700^f	1.20105^f
$R^-_D(k) >$	1.22474^e	1.24805^e	1.26588^e	1.25881^e	1.27378^e	1.26436^e	1.26588^f	1.26514^f

a. Vetrík [7]; b. Theorem 2.2; c. Corollary 4.3; d. Theorem 3.4; e. Theorem 3.6; f. Corollary 4.2

For example, the limitations noted in Observation 3.7 show that the maximum possible value of $R^-_D(10)$ we could achieve using Theorem 3.5 is approximately 1.26699. However, combining the results for diameters 4 and 6 in Table 1 yields $R^-_D(10) > 1.27061$.

Next we use our previous results to show that R is well-behaved in the limit.

Theorem 4.4. *Let $L(k)$ be one of $L^-_C(k)$, $L^+_C(k)$, $L^-_D(k)$ or $L^+_D(k)$, and let $R(k) = k\,L(k)^{1/k}$. The limit $R = \lim_{k\to\infty} R(k)$ exists and is equal to* $\sup R(k)$.

Proof. $R(k)$ is bounded above (by e), so $s = \sup R(k)$ is finite. Hence, given $\varepsilon > 0$, we can choose k so that $s - R(k) < \varepsilon/2$. By Corollary 4.2 (b), $R(mk) \geq R(k)$ for every positive integer m. Moreover, for any fixed $j < k$, since $R(j) \geq 1$, we have $R(mk+j) \geq R(k)^{mk/(mk+j)} \geq R(k)^{m/(m+1)}$, which, by choosing m large enough, can be made to differ from $R(k)$ by no more than $\varepsilon/2$. □

Corollary 4.5.

(a) $\lim_{k\to\infty} R^-_C(k) \geq \dfrac{5 \times 109^{1/5}}{7 \times 2^{3/5}} > 1.20431$

(b) $\lim_{k\to\infty} R^-_D(k) \geq \dfrac{3^{7/6}}{2^{3/2}} > 1.27378$

Proof. We choose the largest entry in the relevant row in Table 1. For *(a)* we know from Theorem 3.4 that $L^-_C(5) \geq \frac{109}{2^3 \times 7^5}$. For *(b)* we know from Theorem 3.6 that $L^-_D(6) \geq \frac{3}{2^{15}}$. □

We conclude this section by using the foregoing to derive new lower bounds for the maximum possible orders of circulant graphs of given diameter and sufficiently large degree.

Corollary 4.6.

(a) For any diameter $k \geq 2$ and any degree d large enough, $CC(d,k) > \left(1.14775\,\frac{d}{k}\right)^k$.

(b) For any diameter k that is a multiple of 5 or sufficiently large, and any degree d large enough, $CC(d,k) > \left(1.20431\,\frac{d}{k}\right)^k$.

(c) For any diameter $k \geq 2$ and any degree d large enough, $DCC(d,k) > \left(1.22474\,\frac{d}{k}\right)^k$.

(d) For any diameter k that is a multiple of 6 or sufficiently large, and any degree d large enough, $DCC(d,k) > \left(1.27378\,\frac{d}{k}\right)^k$.

Proof.

(a) From Theorem 4.4 and Corollary 4.2, we know that $R_C^-(k)$ always exceeds the smallest value in the R_C^- row of Table 1, which is 1.14775.

(b) For k a multiple of 5, we know from Theorem 3.4 and Corollary 4.2 that $R_C^-(k) > 1.20431$. The result for sufficiently large k follows from Corollary 4.5.

(c) and (d) follow by using similar logic in the directed case.

□

These represent significant improvements over the trivial bound of $\left(\frac{d}{k}\right)^k$.

5 Sumsets covering $\mathbb{Z}_n$

Our constructions of directed circulant graphs can be used to obtain an upper bound on the minimum size, $SS(n,k)$, of a set $A \subset \mathbb{Z}_n$ for which the sumset

$$kA = \underbrace{A + A + \ldots + A}_{k} = \mathbb{Z}_n.$$

The trivial bound is $SS(n,k) \leq kn^{1/k}$ which follows in the same way as the trivial lower bound for the directed circulant graph (see Observation 1.2). Improvements to this trivial bound do not appear to have been investigated in the literature.

The idea is that, given $S \subseteq \mathbb{Z}_n$ such that $\mathrm{Cay}(\mathbb{Z}_n, S)$ has diameter k, if we let $A = S \cup \{0\}$ then $kA = \mathbb{Z}_n$. Our constructions thus enable us to bound $SS(n,k)$ for fixed k and infinitely many values of n. For example, if we let $L_S^-(k) = \liminf_{n\to\infty} SS(n,k)/n^{1/k}$, then the following new result for $k = 2$ follows from Theorem 3.6 (a):

Corollary 5.1. $L_S^-(2) \leq \sqrt{\frac{8}{3}} \approx 1.63299$.

More generally, Corollary 4.5 shows that for large enough k and infinitely many values of n, $SS(n,k)$ is at least 21 percent smaller than the trivial bound:

Corollary 5.2. $\lim_{k\to\infty} k^{-1} L_S^-(k) \leq \frac{2^{3/2}}{3^{7/6}} \approx 0.78506$.

6 Largest graphs of small degree and diameter

We can use the construction of Theorem 4.1 to obtain large undirected circulant graphs for small degrees and diameters. Recently in [2], Feria-Puron, Pérez-Rosés and Ryan published a table of largest known circulant graphs with degree up to 16 and diameter up to 10. Their method uses a construction based on graph Cartesian products which is somewhat similar to ours. In contrast, however, Theorem 4.1 does not in general result in a graph isomorphic to the Cartesian product of the constituents. Furthermore, our construction does not require the constituent graph orders to be coprime, which allows more graphs to be constructed.

Using Theorem 4.1 allowed us to improve many of the entries in the published table. However, at the same time we developed a computer search which allows us to find circulant graphs of given degree, diameter and order. It turns out that this search is able to find larger graphs (at least in the range $d \leq 16, k \leq 10$) than the Theorem 4.1 method. We therefore present a much improved table of largest known circulant graphs based on the outputs of this search.

In Table 2, we show the largest known circulant graphs of degree $d \leq 16$ and diameter $k \leq 10$. In Table 3 we give a reduced generating set for each new record largest graph found by the search. The computer search has been completed as an exhaustive search in the diameter 2 case up to degree 23, and these results are included in Table 3 for completeness.

Table 2: Largest known circulant graphs of degree $d \leq 16$ and diameter $k \leq 10$.

$d \setminus k$	1	2	3	4	5	6	7	8	9	10
2	3	5	7	9	11	13	15	17	19	21
3	4	8	12	16	20	24	28	32	36	40
4	5	13	25	41	61	85	113	145	181	221
5	6	16	36	64	100	144	196	256	324	400
6	7	21	55	117	203	333	515	737	1027	1393
7	8	26	76	160	308	536	828	1232	1764	2392
8	9	35	104	248	528	984	1712	2768	4280	6320
9	10	42	130	320	700	1416	2548	4304	6804	10320
10	11	51	177	457	1099	$2380^\dagger$	$4551^\dagger$	$8288^\dagger$	$14099^\dagger$	$22805^\dagger$
11	12	56	210	576	$1428^\dagger$	$3200^\dagger$	$6652^\dagger$	$12416^\dagger$	$21572^\dagger$	$35880^\dagger$
12	13	67	275	$819^\dagger$	$2040^\dagger$	$4283^\dagger$	$8828^\dagger$	$16439^\dagger$	$29308^\dagger$	$51154^\dagger$
13	14	80	312	$970^\dagger$	$2548^\dagger$	$5598^\dagger$	$12176^\dagger$	$22198^\dagger$	$40720^\dagger$	$72608^\dagger$
14	15	90	381	$1229^\dagger$	$3244^\dagger$	$7815^\dagger$	$17389^\dagger$	$35929^\dagger$	$71748^\dagger$	$126109^\dagger$
15	16	96	448	$1420^\dagger$	$3980^\dagger$	$9860^\dagger$	$22584^\dagger$	$48408^\dagger$	$93804^\dagger$	$177302^\dagger$
16	17	112	$518^\dagger$	$1717^\dagger$	$5024^\dagger$	$13380^\dagger$	$32731^\dagger$	$71731^\dagger$	$148385^\dagger$	$298105^\dagger$

† new record largest value

Table 3: Largest circulant graphs of small degree d and diameter k found by computer search.

d	k	Order	Generators
6	2	21*	$1, 2, 8$
6	3	55*	$1, 5, 21$
6	4	117*	$1, 16, 22$
6	5	203*	$1, 7, 57$
6	6	333*	$1, 9, 73$
6	7	515*	$1, 46, 56$
6	8	737*	$1, 11, 133$
6	9	1027*	$1, 13, 157$
6	10	1393*	$1, 92, 106$
7	2	26*	$1, 2, 8$
7	3	76*	$1, 27, 31$
7	4	160*	$1, 5, 31$
7	5	308*	$1, 7, 43$
7	6	536*	$1, 231, 239$
7	7	828*	$1, 9, 91$
7	8	1232*	$1, 11, 111$
7	9	1764*	$1, 803, 815$
7	10	2392*	$1, 13, 183$

Continues on next page

Table 3 – continued from previous page

d	k	Order	Generators
8	2	35*	$1, 6, 7, 10$
8	3	104*	$1, 16, 20, 27$
8	4	248*	$1, 61, 72, 76$
8	5	528*	$1, 89, 156, 162$
8	6	984*	$1, 163, 348, 354$
8	7	1712*	$1, 215, 608, 616$
8	8	2768	$1, 345, 1072, 1080$
8	9	4280	$1, 429, 1660, 1670$
8	10	6320	$1, 631, 2580, 2590$
9	2	42*	$1, 5, 14, 17$
9	3	130*	$1, 8, 14, 47$
9	4	320*	$1, 15, 25, 83$
9	5	700*	$1, 5, 197, 223$
9	6	1416	$1, 7, 575, 611$
9	7	2548	$1, 7, 521, 571$
9	8	4304	$1, 9, 1855, 1919$
9	9	6804	$1, 9, 1849, 1931$
9	10	10320	$1, 11, 4599, 4699$
10	2	51*	$1, 2, 10, 16, 23$
10	3	177*	$1, 12, 19, 27, 87$
10	4	457*	$1, 20, 130, 147, 191$
10	5	1099*	$1, 53, 207, 272, 536$
10	6	2380	$1, 555, 860, 951, 970$
10	7	4551	$1, 739, 1178, 1295, 1301$
10	8	8288	$1, 987, 2367, 2534, 3528$
10	9	14099	$1, 1440, 3660, 3668, 6247$
10	10	22805	$1, 218, 1970, 6819, 6827$
11	2	56*	$1, 2, 10, 15, 22$
11	3	210*	$1, 49, 59, 84, 89$
11	4	576*	$1, 9, 75, 155, 179$
11	5	1428	$1, 169, 285, 289, 387$
11	6	3200	$1, 259, 325, 329, 1229$
11	7	6652	$1, 107, 647, 2235, 2769$
11	8	12416	$1, 145, 863, 4163, 5177$
11	9	21572	$1, 663, 6257, 10003, 10011$
11	10	35880	$1, 2209, 5127, 5135, 12537$
12	2	67*	$1, 2, 3, 13, 21, 30$
12	3	275*	$1, 16, 19, 29, 86, 110$
12	4	819	$7, 26, 119, 143, 377, 385$
12	5	2040	$1, 20, 24, 152, 511, 628$
12	6	4283	$1, 19, 100, 431, 874, 1028$
12	7	8828	$1, 29, 420, 741, 2727, 3185$
12	8	16439	$1, 151, 840, 1278, 2182, 2913$
12	9	29308	$1, 219, 1011, 1509, 6948, 8506$
12	10	51154	$1, 39, 1378, 3775, 5447, 24629$
13	2	80*	$1, 3, 9, 20, 25, 33$
13	3	312*	$1, 14, 74, 77, 130, 138$
13	4	970	$1, 23, 40, 76, 172, 395$
13	5	2548	$1, 117, 121, 391, 481, 1101$
13	6	5598	$1, 12, 216, 450, 1204, 2708$
13	7	12176	$1, 45, 454, 1120, 1632, 1899$
13	8	22198	$1, 156, 1166, 2362, 5999, 9756$
13	9	40720	$1, 242, 3091, 4615, 5162, 13571$
13	10	72608	$1, 259, 4815, 8501, 8623, 23023$
14	2	90*	$1, 4, 10, 17, 26, 29, 41$
14	3	381*	$1, 11, 103, 120, 155, 161, 187$
14	4	1229	$1, 8, 105, 148, 160, 379, 502$

Continues on next page

Table 3 – continued from previous page

d	k	Order	Generators
14	5	3244	$1, 108, 244, 506, 709, 920, 1252$
14	6	7815	$1, 197, 460, 696, 975, 2164, 3032$
14	7	17389	$1, 123, 955, 1683, 1772, 2399, 8362$
14	8	35929	$1, 796, 1088, 3082, 3814, 13947, 14721$
14	9	71748	$1, 1223, 3156, 4147, 5439, 11841, 25120$
14	10	126109	$1, 503, 4548, 7762, 9210, 9234, 49414$
15	2	96*	$1, 2, 3, 14, 21, 31, 39$
15	3	448*	$1, 10, 127, 150, 176, 189, 217$
15	4	1420	$1, 20, 111, 196, 264, 340, 343$
15	5	3980	$1, 264, 300, 382, 668, 774, 1437$
15	6	9860	$1, 438, 805, 1131, 1255, 3041, 3254$
15	7	22584	$1, 1396, 2226, 2309, 2329, 4582, 9436$
15	8	48408	$1, 472, 2421, 3827, 4885, 5114, 12628$
15	9	93804	$1, 3304, 4679, 9140, 10144, 10160, 13845$
15	10	177302	$1, 2193, 8578, 18202, 23704, 23716, 54925$
16	2	112*	$1, 4, 10, 17, 29, 36, 45, 52$
16	3	518	$1, 8, 36, 46, 75, 133, 183, 247$
16	4	1717	$1, 46, 144, 272, 297, 480, 582, 601$
16	5	5024	$1, 380, 451, 811, 1093, 1202, 1492, 1677$
16	6	13380	$1, 395, 567, 1238, 1420, 1544, 2526, 4580$
16	7	32731	$1, 316, 1150, 1797, 2909, 4460, 4836, 16047$
16	8	71731	$1, 749, 4314, 7798, 10918, 11338, 11471, 25094$
16	9	148385	$1, 6094, 6964, 10683, 11704, 14274, 14332, 54076$
16	10	298105	$1, 5860, 11313, 15833, 21207, 26491, 26722, 99924$
17	2	130*	$1, 7, 26, 37, 47, 49, 52, 61$
18	2	138*	$1, 9, 12, 15, 22, 42, 27, 51, 68$
19	2	156*	$1, 15, 21, 23, 26, 33, 52, 61, 65$
20	2	171*	$1, 11, 31, 36, 37, 50, 54, 47, 65, 81$
21	2	192*	$1, 3, 15, 23, 32, 51, 57, 64, 85, 91$
22	2	210*	$2, 7, 12, 18, 32, 35, 63, 70, 78, 91, 92$
23	2	216*	$1, 3, 5, 17, 27, 36, 43, 57, 72, 83, 95$

* proven extremal

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Uniformly dissociated graphs*

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Abstract

A set D of vertices in a graph G is called a dissociation set if every vertex in D has at most one neighbor in D. We call a graph G uniformly dissociated if all maximal dissociation sets are of the same cardinality. Characterizations of uniformly dissociated graphs with small cardinalities of dissociation sets are proven; in particular, the graphs in which all maximal dissociation sets are of cardinality 2 are the complete graphs on at least two vertices from which possibly a matching is removed, while the graphs in which all maximal dissociation sets are of cardinality 3 are the complements of the K_4-free geodetic graphs with diameter 2. A general construction by which any graph can be embedded as an induced subgraph of a uniformly dissociated graph is also presented. In the main result we characterize uniformly dissociated graphs with girth at least 7 to be either isomorphic to C_7, or obtainable from an arbitrary graph H with girth at least 7 by identifying each vertex of H with a leaf of a copy of P_3.

Keywords: Dissociation number, well-covered graphs, girth, Moore graph, polarity graph.

Math. Subj. Class.: 05C69, 05C70, 05C75

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1 Introduction

A set D of vertices in a graph G is called a *dissociation set* if the subgraph induced by vertices of D has maximum degree at most 1. The cardinality of a maximum dissociation set D in a graph G is called the *dissociation number* of G, and is denoted by $diss(G)$. The dissociation number was introduced by Papadimitriou and Yannakakis [14] in relation with the complexity of the so-called restricted spanning tree problem. Another closely related concept is the *k-path vertex cover*, which was introduced in [5] and studied in several papers [4, 10]; the corresponding invariant, the *k-path vertex cover number* of an arbitrary graph G, is denoted by $\psi_k(G)$. As it turns out, dissociation sets are complements of 3-path vertex covers of G, and so the following relation holds:

$$diss(G) = |V(G)| - \psi_3(G),$$

where $\psi_3(G)$ is the size of a minimum 3-path vertex cover. The decision version of the k-path vertex cover number is NP-complete [5], moreover, in the case $k = 3$ it is NP-complete even in bipartite graphs which are C_4-free and have maximum degree 3 [2]; cf. also [13] for further strengthening of this result and [12] for an approximation algorithm.

Are there any graphs in which the dissociation number is easily computable? The approach taken in this paper will be similar to the one related to well-covered graphs, as introduced by Plummer in 1970 [15]. These are the graphs in which every maximal independent set of vertices is of the same size, and hence maximum. Whereas determining the independence number of an arbitrary graph is also NP-complete, it is easy for a well-covered graph since a greedy algorithm will produce the desired result. One approach to deciding if a graph is well-covered has been to restrict the girth [7]. We shall employ that technique in this paper and characterize the graphs of girth 7 or more in which every maximal dissociation set is maximum. Such an approach has been used also on other similar problems, notably the limited packing problem [8] and equipackable graphs [9].

We say that a graph G is a *uniformly dissociated graph* if all maximal dissociation sets are of the same size; in other words, every maximal dissociation set in G is of cardinality $diss(G)$. In particular, this implies that a greedy algorithm, in which vertices are being added to the set, taking care that a newly added vertex is adjacent to at most one vertex of degree 0 and to no vertex of degree 1 in the subgraph induced by the previously added vertices, at the end always gives a dissociation set of maximum cardinality.

The paper is organized as follows. In the next section we study the uniformly dissociated graphs whose maximal dissociation sets are of cardinalities 1, 2 or 3. For the latter class of graphs we present two characterizations, one of which states that they are precisely the complements of the K_4-free geodetic graphs with diameter 2 (geodetic graphs with diameter 2 have been studied in several papers, and in the triangle-free case coincide with the well-known Moore graphs; graphs in this class that have triangles include another known family—the polarity graphs). In Section 3 we introduce the concept of extendable vertices with respect to uniformly dissociated graphs, by following a similar approach as is known for building bigger well-covered graphs using extendable vertices with respect to the well-covered notion. We prove that from an arbitrary graph G by attaching an extendable vertex of a uniformly dissociated graph to each vertex of G one obtains a uniformly dissociated graph. Section 4 contains our main result, a characterization of uniformly dissociated graphs with girth at least 7. Notably, they are precisely the graphs of which each connected component is either isomorphic to C_7, or can be obtained from an arbitrary connected graph H with girth at least 7, by identifying each vertex of H with a leaf of a copy of P_3.

We conclude this section by presenting the notation used throughout the paper.

Let G be a graph and $S \subset V(G)$. We write $G[S]$ for the subgraph of G induced by S and write $G - S$ for the subgraph of G induced by the set $V(G) \setminus S$. On the other hand, if $F \subset E(G)$, then $G - F$ is the subgraph of G obtained from G by removing the edges of F. Let $N_G(v)$ denote the *(open) neighborhood* in G of a vertex v, while $N_G[v] = N_G(v) \cup \{v\}$ is its *closed neighborhood* in G. When the graph G is clear from the context we omit the subscript. If $S \subset V(G)$, then $N_G[S] = \bigcup_{v \in S} N_G[v]$. The *degree* of a vertex v is defined to be $|N_G(v)|$. We call a vertex of degree 1 a *leaf*, while the neighbor of a leaf will be called a *stem*. A *matching* M in a graph G is a set of edges in G having the property that no two edges in M have a common endvertex. Given a matching M in G, we denote by $V(M)$ the set of endvertices of edges from M. Recall that a matching M is an *induced* matching if the only edges in $G[V(M)]$ are the edges in M itself. We denote the cardinality of the largest independent set of vertices by $\alpha(G)$. The *girth*, $g(G)$, is the length of a shortest cycle in G. Given a graph G, the *complement* of G is the graph $\bar{G}$ that has the same vertex set as G, while the edge set of $\bar{G}$ is the complement of the edge set of G.

2 Classes of uniformly dissociated graphs

Let $\mathcal{D}_k$ be the set of uniformly dissociated graphs G such that $diss(G) = k$. Suppose that $G \in \mathcal{D}_k$ and that H is an induced subgraph of G. Since any dissociation set of H is also a dissociation set of G, it follows that $diss(H) \leq k$. However, it need not be the case that $H \in \mathcal{D}_k$. For example, the path P_4 is an induced subgraph of C_5 and $C_5 \in \mathcal{D}_3$, but P_4 has maximal dissociation sets of orders 2 and 3.

Clearly $\mathcal{D}_1 = \{K_1\}$. In fact, K_1 is the only graph with dissociation number 1. Consider now the class $\mathcal{D}_2$. Since the only maximal dissociation set of order 1 in a graph is an isolated vertex, we see that a graph has dissociation number 2 if and only if it belongs to the class $\mathcal{D}_2$. It is also clear that complete graphs K_n, for $n \geq 2$, are in the class, because any pair of (adjacent) vertices forms a maximal dissociation set. Furthermore, if a matching M is removed from K_n, then every set consisting of a vertex is extended to a maximal dissociation set, consisting either of two adjacent or two non-adjacent vertices. We claim that these graphs are precisely all the graphs from $\mathcal{D}_2$. Suppose that G is not in the class of graphs obtained from complete graphs K_n with $n \geq 2$, by removing a (possibly empty) matching M from G. In this case G contains a vertex, say x, which is not adjacent to two vertices from G, say y and z. It is clear that $\{x, y, z\}$ is a (not necessarily maximal) dissociation set, and hence G is not in $\mathcal{D}_2$. We have proved the following statement.

Observation 2.1. $\mathcal{D}_2 = \{K_n - M \mid n \geq 2, M$ *a (possibly empty) matching in* $K_n\}$.

Note that the path P_3 is one of the graphs from $\mathcal{D}_2$. In particular, as we will see in Theorem 3.2, this graph can be used in constructing infinite families of uniformly dissociated trees.

Next, we present two characterizations of the graphs in $\mathcal{D}_3$. The following lemma will be used in several proofs in the paper.

Lemma 2.2. *Let G be a nontrivial uniformly dissociated graph and M an induced matching in G. If $2|M| < k$ and $G \in \mathcal{D}_k$, then $G - N[V(M)] \in \mathcal{D}_{k-2|M|}$.*

Proof. Assume that $G \in \mathcal{D}_k$. Let M be an induced matching in G and assume that $2|M| < k$. Let S_1 and S_2 be any maximal dissociation sets of $G - N[V(M)]$. It is clear that $V(M) \cup S_1$ and $V(M) \cup S_2$ are maximal dissociation sets of G, and consequently $2|M| + |S_1| = k = 2|M| + S_2$. This implies that $|S_1| = |S_2|$, and therefore $G - N[V(M)] \in \mathcal{D}_{k-2|M|}$. □

Theorem 2.3. *A graph G with at least one edge is in $\mathcal{D}_3$ if and only if*

(1) for every $xy \in E(G)$ we have $|V(G) \setminus N[\{x, y\}]| = 1$; and

(2) for every $uv \notin E(G)$ we have $|V(G) \setminus N[\{u, v\}]| \leq 1$, and if $\{u, v\}$ is a maximal independent set of G, then $N(u) \neq N(v)$.

Proof. Suppose that G is a uniformly dissociated graph with $diss(G) = 3$. That means that regardless of how we build a maximal dissociation set we end up with 3 vertices in it. Let $xy \in E(G)$. By Lemma 2.2, $G - N[\{x, y\}] \in \mathcal{D}_1$, which implies property (1), because $\mathcal{D}_1$ contains only K_1. Suppose u and v are two non-adjacent vertices. If $|V(G) \setminus N[\{u, v\}]| > 1$, then there exists a dissociation set, consisting of u, v, and two vertices from $V(G) \setminus N[\{u, v\}]$, a contradiction with $diss(G) = 3$. This proves that $|V(G) \setminus N[\{u, v\}]| \leq 1$. Now, assume that $\{u, v\}$ is a maximal independent set of G. If $N(u) = N(v)$, then $\{u, v\}$ is a maximal dissociation set, a contradiction, which completes the proof of one direction.

For the converse, assume that G satisfies properties (1) and (2). Consider any maximal dissociation set S of G. If S contains two adjacent vertices, then property (1) shows that S contains exactly three elements. Otherwise, S consists of an independent set of vertices, which is by (1) of size at least 2 (we can use (1), since G has an edge). Let u and v belong to S, C be the set of common neighbors of u and v, $A = N(u) \setminus N(v)$, and $B = N(v) \setminus N(u)$.

By (2), $|V(G) \setminus N[\{u, v\}]| \leq 1$; so first consider the case that $G - N[\{u, v\}] = \{w\}$. Note that (1) ensures that each vertex, say x, in A must be adjacent to every vertex in B, since if not, $G - N[\{u, x\}]$ is not isomorphic to K_1. Also observe that w must be adjacent to all vertices of A (resp. B). Suppose that w is not adjacent to u', where $u' \in A$. Then $G - N[\{u, u'\}]$ contains v and w, which contradicts (1) (we derive a similar contradiction, if $v' \in B$ is not adjacent to w). Now, note that since u and v belong to the independent set S no vertex in $A \cup B \cup C$ does. Because S is maximal, we infer that $S = \{u, v, w\}$. Finally, consider the case when $|V(G) \setminus N[\{u, v\}]| = 0$. This means that $\{u, v\}$ is a maximal independent set, and using property (2) we see that $\{u, v\}$ is not a maximal dissociation set and $|S| = 3$. □

Now, we present another characterization of the graphs from $\mathcal{D}_3$. If a graph G belongs to $\mathcal{D}_3$, then in its complement, which we denote by H, every pair of vertices that are non-adjacent have exactly one common neighbor (using condition (1) of Theorem 2.3 expressed in the complement of G). Condition (2) of the theorem expressed in H is that for every pair u and v of vertices that are adjacent in H there is at most one common neighbor of u and v. In other words, any edge of H belongs to at most one triangle. Hence H is diamond-free, and the second part of condition (2) implies that either u or v must have some other neighbor, which readily implies that H must be connected.

The described conditions for the graph H are equivalent to the definition of the so-called geodetic graphs with diameter 2 that are diamond-free. (Recall that a graph is *geodetic*, if between any pair of vertices there is a unique shortest path.) Since in geodetic graphs any cycle on 4 vertices lies in the complete graph on the same 4 vertices, we derive the following characterization of graphs from $\mathcal{D}_3$.

Theorem 2.4. *A graph G is in $\mathcal{D}_3$ if and only if its complement $\bar{G}$ is a connected K_4-free geodetic graph with diameter 2.*

Geodetic graphs with diameter 2 were studied by Stemple [16], (see also the monograph [6], where these graphs were further classified) who proved in [16, Result II] that triangle-free geodetic graphs with diameter 2 are precisely the Moore graphs with diameter 2 (and girth 5). There are three known graphs of this type – C_5, the Petersen graph and the Hoffman-Singleton graph, which is a 7-regular graph on 50 vertices. It is one of the big open problems, whether there exist other Moore graphs. As the analysis shows, the only possible candidates for other Moore graphs are regular with degree 57 on 3250 vertices. If there exists such a Moore graph, it might not be unique. Note that the complement of any such graph (if it exists) is in $\mathcal{D}_3$.

The complement of a graph from $\mathcal{D}_3$ cannot have any 4-cycle as a subgraph, because the existence of an induced C_4 or a diamond contradicts the characteristic property of geodetic graphs, and K_4 is also forbidden. Now, if one forbids 4-cycles as subgraphs in a graph of diameter 2, then any two vertices that are not adjacent have exactly one common neighbor. Therefore, these are exactly the geodetic graphs with diameter 2, that is, the complements of graphs from $\mathcal{D}_3$. Bondy, Erdős, and Fajtlowicz characterized in [3] the graphs with diameter 2 that have no 4-cycles as the graphs H that fall into three different families:

(i) $\Delta(H) = |V(H)| - 1$ and H has no 4-cycles,

(ii) H is a Moore graph,

(iii) H is a polarity graph.

The first family are the graphs having a universal vertex, and all other vertices have degree at most 2. Clearly, the complement of any such graph is the disjoint union of a graph from $\mathcal{D}_2$ and K_1. While Moore graphs are well-known, let us focus on the third family – polarity graphs. The study of these graphs started in the context of projective geometries by Kantor [11], and they were later considered in several papers. See the recent study [1]. For a formal definition of polarity graphs we present some notions from finite geometries.

Let $\mathcal{P}$ and $\mathcal{L}$ be disjoint, finite sets, and let $\mathcal{I} \subset \mathcal{P} \times \mathcal{L}$. The triple $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ is called a *finite geometry*, elements of $\mathcal{P}$ are called *points*, while elements of $\mathcal{L}$ are *lines*. A *polarity* of the geometry is a bijection from $\mathcal{P} \cup \mathcal{L}$ to $\mathcal{P} \cup \mathcal{L}$ that sends points to lines, sends lines to points, is an involution, and respects the incidence structure. Given a finite geometry $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ and a polarity π, the *polarity graph* G_π is the graph with vertex set $V(G_\pi) = \mathcal{P}$, and $pq \in E(G_\pi)$ whenever p and q are points such that $(p, \pi(q)) \in \mathcal{I}$.

Alternatively, for any prime power q, let $PG(2, q)$ denote the standard projective geometry over the Galois field $GF(q)$, where points are represented by projective triples, see [1] for details. The vertex set of the corresponding *polarity graph* consists of $(q^2 + q + 1)$ points of $PG(2, q)$, which are adjacent whenever the corresponding triples are orthogonal. In particular, for any prime power q there exists a (unique) polarity graph, which readily implies that there are also infinitely many graphs in $\mathcal{D}_3$. From the result of Bondy, Erdős, and Fajtlowicz [3] and our discussion we derive another characterization of these graphs.

Corollary 2.5. *A graph G is in $\mathcal{D}_3$ if and only if either G is the disjoint union of a graph from $\mathcal{D}_2$ and the $\mathcal{D}_1$-graph, or G is the complement of a Moore graph, or G is the complement of a polarity graph.*

In order to present some small examples of connected graphs in $\mathcal{D}_3$ we performed a structural analysis of these graphs, which results in the following proposition, the proof of which is omitted.

Proposition 2.6. *Let G be a connected graph in $\mathcal{D}_3$ having minimum degree k.*

(1) If $k \leq 2$, then $G = C_5$.

(2) If $k \geq 3$ and v is a vertex in G such that $\deg(v) = k$, then the open neighborhood of v partitions into ℓ subsets $S_1, \ldots, S_\ell$ such that $|S_i| = m$ for all i, $k = \ell m$, and $m+1 \leq \ell \leq m+2$. In addition, $B = V(G) \setminus N[v] = \{b_1, \ldots, b_\ell\}$, $N(b_i) \cap S_i = \emptyset$ and b_i is adjacent to every vertex in S_j for $j \neq i$. The subgraphs $G[B], G[S_1], \ldots, G[S_\ell]$ all belong to $\mathcal{D}_2$.

In the case that $G[B]$ is a complete graph we are able to deduce that each $G[S_i]$ is also a complete graph. Indeed, let $\delta(G[B]) = \ell - 1$, let $1 \leq i \leq \ell$ and let s be any vertex in S_i. For $i \neq j$, $|V(G) \setminus N[\{s, b_j\}]| = 1$ and hence s is adjacent to exactly $m - 1$ vertices in S_j. If e denotes the number of neighbors of s in $G[S_i]$, then

$$m\ell = k \leq \deg(s) = 1 + e + (\ell - 1) + (\ell - 1)(m - 1)\,.$$

From this it follows that $e = m - 1$, and we see that $G[S_i]$ is a complete subgraph.

When $k \geq 3$, $\ell > m$, and $k = \ell m$, it follows that $\ell \geq 3$. Next we find all graphs in $\mathcal{D}_3$ with $\ell = 3$. Note that in this case m is either 1 or 2. Let $A = N(v)$ where v is a vertex of minimum degree as in the statement of Proposition 2.6(2).

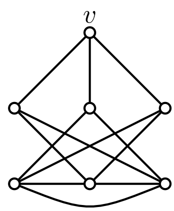

Figure 1: The only graph in $\mathcal{D}_3$ with $\ell = 3$ and $m = 1$.

Suppose first that $m = 1$. In this case $k = \ell$ and the subgraph $G[A]$ is isomorphic to the complement of $G[B]$. Since $G[B] \in \mathcal{D}_2$, it follows that the maximum degree of $G[A]$ is at most 1. If A is an independent set, then we get that G is isomorphic to the graph in Figure 1. On the other hand, if $\Delta(G[A]) = 1$, then G is isomorphic to the graph in Figure 2, which is, in turn, isomorphic to the graph in Figure 1.

Next suppose that $m = 2$, and hence $\ell = m + 1 = 3$. As above, $G[B] = K_3$ and $G[S_i] = K_2$ for $1 \leq i \leq 3$. As it turns out, the only possibility that yields a graph from $\mathcal{D}_3$ is that the subgraph $G[A]$ is isomorphic to $K_2 \square K_3$; we derive that G is the graph in Figure 3. (Note that it is the complement of the Petersen graph.)

Stemple proved [16, Result X] that the order of a geodetic graph H with diameter 2, which has triangles but no complete subgraphs of order 4, is $\Delta^2 - \Delta + 1$, where Δ is maximum degree of H. Note that Δ is equal to the maximum number of non-neighbors of vertices in G from $\mathcal{D}_3$, which is, by the construction from Proposition 2.6, equal to ℓ. Hence $\ell(m + 1) = \Delta(\Delta - 1)$. We deduce that unless the complement of G is triangle-free (and thus a Moore graph), we have $\ell = m + 2$. For $\ell = m + 2 = 3$ this is exactly

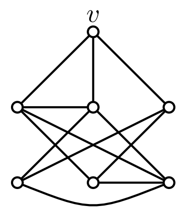

Figure 2: A graph isomorphic to the one in Figure 1.

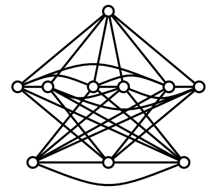

Figure 3: The only graph in $\mathcal{D}_3$ with $\ell = 3$ and $m = 2$.

the graph in Figure 1. When $\ell = m + 2 = 4$ we have the graph in Figure 4. As in the description of the connected graphs in $\mathcal{D}_3$ from above, the vertex v is adjacent to all vertices in $S = S_1 \cup S_2 \cup S_3 \cup S_4$. For $1 \leq i \leq 4$, b_i is adjacent to every vertex in $S - S_i$. The subgraph induced by B is a complete graph of order 4 with the matching edges $b_1 b_2$ and $b_3 b_4$ removed. This graph, $G[B]$, is in $\mathcal{D}_2$.

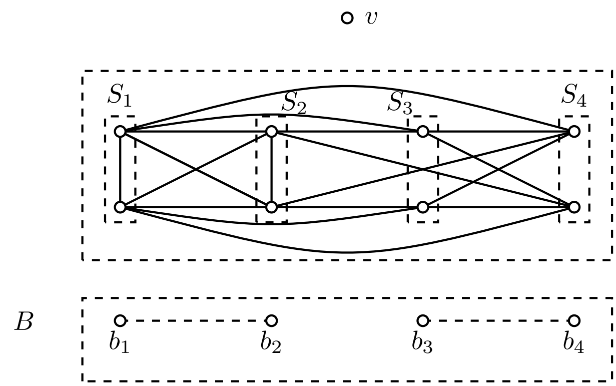

Figure 4: Graph in $\mathcal{D}_3$ of order 13.

Let us only mention that the path P_6 and the cycle C_7 belong to $\mathcal{D}_4$, while a special family of graphs in $\mathcal{D}_{2k}$, where $k \geq 3$, will be presented in the next section.

3 Extendable vertices

The term extendable vertices of graphs was coined in the context of well-covered graphs, where such vertices were used as attachment vertices to build bigger graphs from smaller well-covered building blocks [7]. We will use a similar approach, and introduce extendable vertices in the context of uniformly dissociated graphs.

Let G be a uniformly dissociated graph with $diss(G) = k$. We say that $x \in V(G)$ is $\mathcal{D}_k$*-extendable*, if the following two properties hold:

(i) $(G - x) \in \mathcal{D}_k$ and

(ii) $(G - N[x]) \in \mathcal{D}_{k-1}$.

Since in this paper we use only this version of extendability, we will often simplify the wording by calling $\mathcal{D}_k$-extendable vertices just *extendable vertices*. It is clear that the only vertex of K_1 (which is the only graph of $\mathcal{D}_1$) is not extendable in the above sense. On the other hand, it is easy to verify that all graphs from $\mathcal{D}_2$, except the complete graphs, contain an extendable vertex.

Proposition 3.1. *Let G be any graph in $\mathcal{D}_2$, and G not a complete graph. Any vertex that is not universal in G, is $\mathcal{D}_2$-extendable.*

The application of this concept in constructing large families of uniformly dissociated graphs is presented in the next result.

Theorem 3.2. *Let G be an arbitrary graph, having vertices denoted by $x_1, \dots, x_n$; let $G_1, \dots, G_n$ be (not necessarily different) uniformly dissociated graphs, each having an extendable vertex. If G^* is obtained from G by identifying x_i with an extendable vertex of G_i for all $i \in \{1, \dots, n\}$, then G^* is a uniformly dissociated graph.*

The proof of the theorem follows directly from the construction of G^* and the definition of extendable vertices. In particular, Theorem 3.2 shows that every graph is an induced subgraph of a uniformly dissociated graph. (Since non-complete graphs in $\mathcal{D}_2$ form an infinite family, every graph is an induced subgraph of infinitely many uniformly dissociated graphs.)

In the rest of this section, we shed some more light on the uniformly dissociated graphs, (not) having extendable vertices.

Proposition 3.3. *No vertex of a connected graph from $\mathcal{D}_3$ is extendable.*

Proof. Let G be a connected graph in $\mathcal{D}_3$, and assume that $w \in V(G)$ is an extendable vertex of G. If there exists an edge $xy \in E(G)$ such that w is adjacent to neither x nor to y, then by property (1) from Theorem 2.3, we infer that $\{x, y\}$ is a maximal dissociation set of $G - w$, a contradiction with $G - w \in \mathcal{D}_3$. Hence $G - N[w]$ does not contain any edge, which implies that $deg_G(w) \geq |V(G)| - 3$ (for otherwise $V(G) \setminus N(w)$ would be an independent set of cardinality at least 4). Now, if $V(G) \setminus N[w]$ consisted of only one vertex, say y, then w and a neighbor of y would form a maximal dissociation set of G of size 2, again a contradiction. This implies that there exist exactly two vertices in the complement of $N[w]$, and let us denote them by y and z.

If y and z had a common neighbor x, then again we derive a contradiction with $G \in \mathcal{D}_3$ (because $\{w, x\}$ would be a maximal dissociation set of G). This implies that $N(y) \cap N(z) = \emptyset$, and each of $N(y)$ and $N(z)$ is non-empty, since G is connected. If there exists a vertex $a \in N(w)$ such that $\{y, z\} \cap N(a) = \emptyset$, then $\{y, z\} \subseteq V(G) \setminus N[\{w, a\}]$, which contradicts property (1) of Theorem 2.3. Thus $N(y), N(z)$ is a partition of $N(w)$. Now, if there exists $y' \in N(y)$ and $z' \in N(z)$ such that $y'z' \notin E(G)$, then $\{y, y', z, z'\}$ is a dissociation set of G of cardinality 4, a contradiction. Otherwise, the set $\{y', z'\}$, where $y' \in N(y)$ and $z' \in N(z)$, is a maximal dissociation set of G of cardinality 2, which is the final contradiction, showing that w is not an extendable vertex of G. □

There are many $\mathcal{D}_k$-extendable vertices, where k is an even number; in fact, any vertex in the construction of a graph G^* from Theorem 3.2, which corresponds to a vertex from the initial graph G, is extendable. On the other hand, we know of no example of a connected $\mathcal{D}_k$-extendable vertex for k being odd. More precisely, we know that there are no $\mathcal{D}_3$-extendable vertices in connected graphs, and, in addition, we do not know if any connected $\mathcal{D}_{2\ell+1}$-extendable graphs exist, when $\ell > 1$. Therefore we pose the following question.

Question 3.4. Are there any connected graphs in $\mathcal{D}_k$, where k is an odd number greater than 3? If there are, does there exist a $\mathcal{D}_k$-extendable vertex for some such k.

It would be interesting to know, if any connected graphs in $\mathcal{D}_{2t+1}$, for $t > 1$ exist, also because they would present a natural common extension of the classes of Moore graphs with diameter 2 and polarity graphs.

4 Uniformly dissociated graphs with girth at least 7

Suppose that each of the graphs $G_1, \dots, G_n$ is isomorphic to P_3. The construction given in Theorem 3.2 presents a large family of uniformly dissociated graphs, each of which has many leaves (in fact, a third of the vertices have degree 1). Note that in these graphs each neighbor of a leaf has degree 2, and is in particular adjacent to only one leaf. This latter property holds in all uniformly dissociated graphs that have minimum degree 1 and order at least 4, as the following lemma shows.

Lemma 4.1. *Let G be a connected uniformly dissociated graph on more than three vertices. If x is a stem, then it has exactly one leaf as a neighbor.*

Proof. Let G be a connected uniformly dissociated graph with $|V(G)| > 3$. For the purposes of reaching a contradiction, let us assume that there exists a vertex x, which is adjacent to more than one leaf. Let $x_1, \dots, x_k$, where $k \geq 2$, be the leaves adjacent to x. If G is the star $K_{1,k}$, then $\{x, x_1\}$ is a maximal dissociation set of size 2, and $\{x_1, \dots, x_k\}$ a maximal dissociation set of size k, where $k \geq 3$, because G has at least 4 vertices. Hence G is not uniformly dissociated.

If G is not a star, then there exists a neighbor y of x, which is not a leaf. Let S be a maximal dissociation set that contains vertices x and y (such a set always exists, because we can start a greedy procedure of obtaining a dissociation set by picking the endvertices of the edge xy). Note that the leaves $x_1, \dots, x_k$ are not in S, and, moreover, x and y are the only vertices from $N[\{x, y\}]$ that are in S. Let $S' = S \setminus \{x, y\}$. Clearly, S' is a (maximal) dissociation set of $G - N[\{x, y\}]$. Now, let $\overline{S}$ be the set $S' \cup \{y, x_1, \dots, x_k\}$. Note that $\overline{S}$ is a dissociation set of G (not necessarily maximal), and $|\overline{S}| \geq |S'| + 3 > |S|$. Since $\overline{S}$ lies in a maximal dissociation set, we derive that G is not a uniformly dissociated graph, a contradiction, which shows that G contains no vertex adjacent to more than one leaf. □

Lemma 4.2. *If G is a uniformly dissociated graph of order at least 3, then no two stems of G are adjacent.*

Proof. Let $G \in \mathcal{D}_m$ for some $m \geq 2$. If $|V(G)| = 3$, then G does not have two stems, so we may assume that G is of order greater than 3. Now, if $m = 2$, then G is isomorphic to a complete graph from which a (possibly empty) matching is removed (by Observation 2.1). Hence G has no leaves, and consequently also no stems. We may thus assume that G is a graph of order greater than 3, and $G \in \mathcal{D}_m$, for $m \geq 3$.

Assume that G has two stems u and v that are adjacent. Let us denote by x and y the leaves that are adjacent to u and v, respectively. By Lemma 4.1 each stem is adjacent to exactly one leaf. Let D_1 be a maximal dissociation set that contains vertices u and v. By Lemma 2.2, as u and v form a vertex set of a trivial induced matching in G, we have $G - N[\{u, v\}] \in \mathcal{D}_{m-2}$. Now, note that $D_2 = D_1 \cup \{x, y\} \setminus \{v\}$ is a dissociation set of G, which is not necessarily maximal. Hence, there exists a maximal dissociation set in G that contains D_2 and is of cardinality at least $m + 1$, a contradiction with $G \in \mathcal{D}_m$. □

In the rest of this section we restrict ourselves to graphs with girth at least 7.

Lemma 4.3. *If G is a uniformly dissociated graph with $g(G) \geq 7$, then no two stems of G are at distance 2.*

Proof. Let G be a uniformly dissociated graph, that is $G \in \mathcal{D}_m$ for some $m \geq 2$, and let $g(G) \geq 7$. Assume that G has two stems v and w that are at distance 2, and let u be their common neighbor. Let us denote by x and y the leaves that are adjacent to v and w, respectively (by Lemma 4.1 each stem has only one leaf).

Denote by $z_1, \ldots, z_p$ the neighbors of w, different from u, and note that they are not stems and not leaves, by Lemma 4.2. Hence each of them has a neighbor, and let us denote them by $x_{1,1}, \ldots, x_{1,j_1}, \ldots, x_{p,1} \ldots, x_{p,j_p}$, where $x_{i,j}$ are the neighbors of z_i for all $i \in \{1, \ldots, p\}$. Since $x_{i,j}$ are not leaves, each of them has another neighbor, and let us denote the neighbors of $x_{i,j}$ by $y_{i,j,1}, \ldots, y_{i,j,k_{i,j}}$ for all $i \in \{1, \ldots, p\}$, $j \in \{1, \ldots, j_i\}$. Now, we build an induced matching M, consisted of edges $x_{i,j}y_{i,j,k}$ in the following way. As long as this is possible, for each z_i choose a j from $\{1, \ldots, j_i\}$, and add an edge $x_{i,j}y_{i,j,k}$ to M, so that it does not destroy the property of M being an induced matching. Note that since the girth is at least 7, the only possibility for destroying the property of M being an induced matching is that some vertex $y_{i,j,k}$ is adjacent to a vertex $y_{i',j',k'}$, which is already in $V(M)$. More precisely, the procedure can end before an edge $x_{i,j}y_{i,j,k}$ has been added to M for all z_i, only if for some z_i and for all of its neighbors $x_{i,j}$ all of their neighbors $y_{i,j,k}$ cannot be chosen, because each of them is adjacent to some $y_{i',j',k'}$ that is an endvertex of an edge from M. In this case, by using Lemma 2.2, we infer that since M is an induced matching in G, and $2|M| < m$, we have $G - N[V(M)] \in \mathcal{D}_{m-2|M|}$. Now, this implies that all neighbors of $x_{i,1}, \ldots, x_{i,j_i}$ (except for z_i) are in $N[V(M)]$ and thus in $G - N[V(M)]$ all $x_{i,1}, \ldots, x_{i,j_i}$ are leaves. Hence z_i is a stem in $G - N[V(M)]$ and is adjacent to w, which is also a stem in $G - N[V(M)]$. Now, this is a contradiction with Lemma 4.2, because $G - N[V(M)]$ is a uniformly dissociated graph with two adjacent stems.

Hence, the only possibility is that the procedure of building an induced matching M consisted from edges $x_{i,j}y_{i,j,k}$ ends, so that for each z_i we have chosen one edge $x_{i,j}y_{i,j,k}$ to belong to M. Since M is an induced matching and $2|M| < m$, we have $G - N[V(M)] \in \mathcal{D}_{m-2|M|}$ by Lemma 2.2. Note that in $G - N[V(M)]$, w is a stem of degree 2 (adjacent only to u and the leaf y), and v also belongs to $G - N[V(M)]$ because an edge between v and any $y_{i,j,k}$ in G would imply the existence of a 6-cycle. Now, let D_1 be a maximal dissociation set of $G - N[V(M)]$, which contains v, u and y, and let $D_2 = D_1 \cup \{x, w\} \setminus \{u\}$. Clearly, D_2 is a dissociation set (not necessarily maximal) of cardinality $|D_1| + 1$, which is a contradiction with $G - N[V(M)]$ being uniformly dissociated. The proof is complete. □

Lemma 4.4. *If G is a uniformly dissociated graph with $g(G) \geq 7$, then for each stem v, $\deg(v) = 2$.*

Proof. Let $G \in \mathcal{D}_m$ for some $m \geq 2$, and assume that v is a stem adjacent to the leaf x, and v has at least two other neighbors, which we denote by w and w'. Now, we use a similar idea as in the proof of Lemma 4.3.

Denote by $z_1, \ldots, z_p$ the neighbors of w, different from v, which are not stems and not leaves, by Lemma 4.2. Hence each of them has a neighbor, and let us denote them by $x_{1,1}, \ldots, x_{1,j_1}, \ldots, x_{p,1}, \ldots, x_{p,j_p}$, where $x_{i,j}$ are the neighbors of z_i for all $i \in \{1, \ldots, p\}$, $j \in \{1, \ldots, j_i\}$. Since $x_{i,j}$ are not leaves, each of them has another neighbor, and let us denote the neighbors of $x_{i,j}$ by $y_{i,j,1}, \ldots, y_{i,j,k_{i,j}}$ for all $i \in \{1, \ldots, p\}$, $j \in \{1, \ldots, j_i\}$. Now, we build an induced matching M, consisted of edges $x_{i,j}y_{i,j,k}$ in the following way. As long as this is possible, for each z_i choose a j from $\{1, \ldots, j_i\}$, and add an edge $x_{i,j}y_{i,j,k}$ to M, so that it does not destroy the property of M being an induced matching. Note that since girth is 7, the only possibility for destroying the property of M being an induced matching is that some vertex $y_{i,j,k}$ is adjacent to a vertex $y_{i',j',k'}$, which is already in M. More precisely, the procedure can end before an edge $x_{i,j}y_{i,j,k}$ has been added to M for all z_i, only if for some z_i and for all of its neighbors $x_{i,j}$ all of their neighbors $y_{i,j,k}$ cannot be chosen, because each of them is adjacent to some $y_{i',j',k'}$ that is an endvertex of an edge from M. In this case, by using Lemma 2.2, we infer that since M is an induced matching in G, and $2|M| < m$, we have $G - N[V(M)] \in \mathcal{D}_{m-2|M|}$. Now, this implies that all neighbors of $x_{i,1}, \ldots, x_{i,j_i}$ (except for z_i) are in $N[V(M)]$ and thus in $G - N[V(M)]$ all $x_{i,1}, \ldots, x_{i,j_i}$ are leaves. Hence z_i is a stem in $G - N[V(M)]$, which is at distance 2 from another stem v, a contradiction with Lemma 4.3.

Hence, the only possibility is that the procedure of building an induced matching M consisted from edges $x_{i,j}y_{i,j,k}$ ends, so that for each z_i we have chosen one edge $x_{i,j}y_{i,j,k}$ to belong to M. Since M is an induced matching and $2|M| < m$, we have $G - N[V(M)] \in \mathcal{D}_{m-2|M|}$ by Lemma 2.2. Note that in $G - N[V(M)]$, w is a leaf, adjacent only to v. Thus v is a stem, which is adjacent to two leaves, a contradiction with Lemma 4.1. □

Lemma 4.5. *If G is a uniformly dissociated graph with $g(G) \geq 7$ and has a leaf, then each vertex of G is either a leaf, or a stem or is adjacent to a stem.*

Proof. Let $G \in \mathcal{D}_m$ for some $m \geq 2$ with $g(G) \geq 7$ and with a leaf. We may assume that G is a connected graph. Suppose that there exists a vertex in G that is not a leaf, not a stem, and not adjacent to a stem. Since G is connected, there exists such a vertex w, which is, in addition, adjacent to u, which is in turn adjacent to a stem v.

Denote by $z_1, \ldots, z_p$ the neighbors of w, different from u, which are not leaves and not stems by our assumption. Hence each of them has a neighbor, and let us denote them by $x_{1,1}, \ldots, x_{1,j_1}, \ldots, x_{p,1} \ldots, x_{p,j_p}$, where $x_{i,j}$ are the neighbors of z_i for all $i \in \{1, \ldots, p\}, j \in \{1, \ldots, j_i\}$. Since $x_{i,j}$ are not leaves (because z_i are not stems), each of them has another neighbor, and let us denote the neighbors of $x_{i,j}$ by $y_{i,j,1}, \ldots, y_{i,j,k_{i,j}}$ for all $i \in \{1, \ldots, p\}$, $j \in \{1, \ldots, j_i\}$. Now, we build an induced matching M, consisted of edges $x_{i,j}y_{i,j,k}$ in the following way. As long as this is possible, for each z_i choose a j from $\{1, \ldots, j_i\}$, and add an edge $x_{i,j}y_{i,j,k}$ to M, so that it does not destroy the property of M being an induced matching. Suppose that the procedure of building an induced matching M consisted from edges $x_{i,j}y_{i,j,k}$ ends, so that for each z_i we have chosen one edge $x_{i,j}y_{i,j,k}$ to belong to M. Since M is an induced matching and $2|M| < m$, we have $G - N[V(M)] \in \mathcal{D}_{m-2|M|}$ by Lemma 2.2. Note that in $G - N[V(M)]$, w is a leaf, adjacent to u; thus u and v are two adjacent stems, a contradiction with Lemma 4.2. Thus the procedure of building an induced matching M such that all z_i would be in $N[V(M)]$

ends before each z_i has a neighbor $x_{i,j}$ added to $V(M)$. Let $z_{i'}$ be such a vertex that for all neighbors $x_{i',j'}$ all of their neighbors $y_{i',j',k'}$ cannot be chosen, because each of them is adjacent to some $y_{i,j,k}$ that is an endvertex of an edge from M.

Suppose $\deg(z_{i'}) > 2$. Note that for all neighbors $x_{i',j'}$ all of their neighbors $y_{i',j',k'}$ are adjacent to a vertex $y_{i,j,k} \in V(M)$. Since M an induced matching in G, and $2|M| < m$, we have $G - N[V(M)] \in \mathcal{D}_{m-2|M|}$. This implies that all neighbors of $x_{i',1}, \ldots, x_{i',j_{i'}}$ (except for $z_{i'}$) are in $N[V(M)]$ and thus in $G - N[V(M)]$ all $x_{i',1}, \ldots, x_{i',j_{i'}}$ are leaves. Hence $z_{i'}$ is a stem in $G - N[V(M)]$, which has at least two leaves, a contradiction with Lemma 4.1.

We may thus assume that $\deg(z_{i'}) = 2$, and let $x_{i'}$ be the neighbor of $z_{i'}$, different from w. Suppose that $\deg(x_{i'}) > 2$. By selecting the matching M', consisting only of the edge uv, we infer by Lemma 2.2 that $G - N[V(M')] \in \mathcal{D}_{m-2}$. Yet $z_{i'}$ is a leaf in $G - N[V(M')]$, and so $x_{i'}$ is a stem, whose degree is more than 2, a contradiction with Lemma 4.4. Hence, we infer that also $\deg(x_{i'}) = 2$, and let $y_{i'}$ be another neighbor of $x_{i'}$. By the property of M, established above, we know that $y_{i'}$ is adjacent to some $y_{i,j,k}$, which is at distance 3 from w. Now, let M'' be the matching consisting only of the edge $y_{i'}y_{i,j,k}$. Hence, $G - N[V(M'')] \in \mathcal{D}_{m-2}$, but in $G - N[V(M'')]$ the vertex $z_{i'}$ is a leaf, and so w is a stem. We derive that w and v are two stems in the uniformly dissociated graph $G - N[V(M'')]$, which are at distance 2, contradicting Lemma 4.3. □

We join the previous lemmas into the following fact.

Observation 4.6. *If G is a uniformly dissociated graph with $g(G) \geq 7$ and has a leaf, then every vertex that is not a stem nor a leaf, is adjacent to exactly one stem. Note that in that case G has the structure as presented in the construction from Theorem 3.2, where each of the extendable graphs, identified with a vertex from an arbitrary graph, is isomorphic to P_3.*

The above observation is correct, because if a vertex were adjacent to two stems, these two stems would be at distance 2, which is a contradiction with Lemma 4.3.

Lemma 4.7. *If G is a connected uniformly dissociated graph with $g(G) \geq 7$ and $\delta(G) \geq 2$, then G is isomorphic to C_7.*

Proof. Let $G \in \mathcal{D}_m$ for some $m \geq 2$, $g(G) \geq 7$, and $\delta(G) \geq 2$. Assume that there exists a vertex v, with $\deg(v) \geq 3$.

Suppose that there exists a neighbor w of v, with $\deg(w) = 2$. Let z be the neighbor of w, different from v; further let x be a neighbor of z, and y a neighbor of x, different from z. Note that y is not adjacent to v nor to any of its neighbors, due to the girth restriction. Let M be the matching consisting only of the edge xy. Hence, $G - N[V(M)] \in \mathcal{D}_{m-2}$, but in $G - N[V(M)]$ the vertex w is a leaf, and so v is a stem. Since $\deg_{G-N[V(M)]}(v) \geq 3$ we are in contradiction with Lemma 4.4.

The remaining possibility is that all neighbors of v have degree at least 3. Since G is connected, we derive that every vertex in G has degree at least 3. We conclude the proof by using the base technique from the proofs of previous lemmas.

Let $v \in V(G)$, w one of its neighbors, and denote by $z_1, \ldots, z_p$ the neighbors of w, different from v. Each of them has a neighbor, which we denote by

$$x_{1,1}, \ldots, x_{1,j_1}, \ldots, x_{p,1} \ \ldots, x_{p,j_p},$$

where $x_{i,j}$ are the neighbors of z_i for all $i \in \{1, \dots, p\}, j \in \{1, \dots, j_i\}$. Each of $x_{i,j}$ has another neighbor, and let us denote the neighbors of $x_{i,j}$ by $y_{i,j,1}, \dots, y_{i,j,k_{i,j}}$ for all $i \in \{1, \dots, p\}$, $j \in \{1, \dots, j_i\}$. Now, we build an induced matching M, consisted of edges $x_{i,j}y_{i,j,k}$ in the following way. As long as this is possible, for each z_i choose a j from $\{1, \dots, j_i\}$, and add an edge $x_{i,j}y_{i,j,k}$ to M, so that it does not destroy the property of M being an induced matching. Note that since girth is 7, the only possibility for destroying the property of M being an induced matching is that some vertex $y_{i,j,k}$ is adjacent to a vertex $y_{i',j',k'}$, which is already in M. More precisely, the procedure can end before an edge $x_{i,j}y_{i,j,k}$ has been added to M for all z_i, only if for some z_i and for all of its neighbors $x_{i,j}$ all of their neighbors $y_{i,j,k}$ cannot be chosen, because each of them is adjacent to some $y_{i',j',k'}$ that is an endvertex of an edge from M. In this case, by using Lemma 2.2, we infer that since M an induced matching in G, and $2|M| < m$, we have $G - N[V(M)] \in \mathcal{D}_{m-2|M|}$. Now, this implies that all neighbors of $x_{i,1}, \dots, x_{i,j_i}$ (except for z_i) are in $N[V(M)]$ and thus in $G - N[V(M)]$ all $x_{i,1}, \dots, x_{i,j_i}$ are leaves. Hence z_i is a stem in $G - N[V(M)]$, which has degree at least 3, a contradiction with Lemma 4.4.

Hence, the only possibility is that the procedure of building an induced matching M consisted from edges $x_{i,j}y_{i,j,k}$ ends, so that for each z_i we have chosen one edge $x_{i,j}y_{i,j,k}$ to belong to M. Since M is an induced matching and $2|M| < m$, we have $G - N[V(M)] \in \mathcal{D}_{m-2|M|}$ by Lemma 2.2. Note that in $G - N[V(M)]$, w is a leaf, adjacent only to v. Thus v is a stem with degree at least 3, again the contradiction with Lemma 4.4.

As a result of this we now conclude that G is a connected, uniformly dissociated, regular graph of degree 2 and girth at least 7. It is straightforward to check that C_7 is the only cycle of order seven or more that is uniformly dissociated. □

We are ready to state the main theorem.

Theorem 4.8. *If G is a uniformly dissociated graph with $g(G) \geq 7$, then each connected component of G is either isomorphic to C_7, or can be obtained from an arbitrary connected graph H with girth at least 7, by identifying each vertex of H with a leaf of a copy of P_3.*

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Balanced Abelian group-valued functions on directed graphs

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Abstract

We discuss functions from the edges and vertices of a directed graph to an Abelian group. A function is called balanced if the sum of its values along any cycle is zero. The set of all balanced functions forms an Abelian group under addition. We study this group in two cases: when we are allowed to walk against the direction of an edge taking the opposite value of the function and when we are not allowed to walk against the direction.

Keywords: Consistent graphs, balanced signed graphs, balanced labelings of graphs, gain graphs, weighted graphs.

Math. Subj. Class.: 05C22

1 Introduction

Let A be an Abelian group with the group operation denoted by $+$ and the identity element denoted by 0. Let G be a graph. Roughly speaking, an A-valued function f on vertices and/or edges of G is called *balanced* if the sum of its values along any cycle of G is 0. Our cycles are not permitted to have repeating edges.

The study of balanced functions can be conducted in three cases:

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1. The graph G is directed with the set of vertices V and the set of directed edges E. When traveling between the vertices, we are allowed to travel with or against the direction of the edges. The value of a function f on $\bar{e}$, which represents traveling the edge e against its direction, is equal to $-f(e)$. In this context, when the function is defined on edges only, the pair (G, f) is called a network or a directed network. In this paper we shall call this *the flexible case*, meaning that the direction of an edge does not forbid us to walk against it. The notion of balanced functions on edges for the flexible case, for functions taking values only on the edges, is introduced in the literature under different names. Thus, for example, in [1] the set of such functions is exactly $\mathrm{Im}(d)$, where d is the "exterior differential", which maps a function f defined on vertices to the function df on edges defined by the equality $(df)(e) = f(e_+) - f(e_-)$, where e_- and e_+ are the origin and the end of a directed edge e. In [11], in somewhat different language, that set is referred to as the set of consistent graphs. In [13] such functions have been introduced under the name "color-coboundaries". They also appear in literature under the name "tensions". They have been extensively studied, recent examples include [3, 4, 6, 10, 14]. In [5] balanced functions on edges appear in a certain connection with geometric representations of the Coxeter group associated to a graph. In a rather common terminology introduced by Zaslavsky, [15], a pair of a graph and such a function on the edges of a graph is called a "gain graph".

2. The graph G is directed with the set of vertices V and the set of directed edges E, but we are only allowed to travel with the direction of the edges. In this paper we shall call this *the rigid case*. When f takes values only on the edges then in some literature, following Serre, [12], the flexible case is described as a particular instance of the rigid case by introducing the set $\mathbb{E}$ as the new set of directed edges of G (the cardinality of $\mathbb{E}$ is twice that of E), denoting by $\bar{e} \in \mathbb{E}$ the inverse of the directed edge $e \in E$ and requiring $f(\bar{e}) = -f(e)$, [1, 12].

3. The graph G is undirected. The value of a function f on an edge e does not depend on the direction of the travel on e. The case of balanced functions $f : E \to \mathbb{R}$ is studied in [2], where these functions are called "cycle-vanishing edge valuations". The case of balanced functions $f : E \to A$ is studied in [7]. The case of balanced functions $f : V \bigcup E \to A$ is first introduced and studied in [9]. The group structure of the groups of balanced functions on the edges, balanceable functions on the vertices and balanced functions on the vertices and edges of an undirected graph with values in an Abelian group is studied in [7].

The subject of this paper is the group structure and the relations between groups of functions associated with the notion of balance on a directed graph. Namely, we study the group structures of the groups of balanced functions for the flexible and the rigid cases and the relations between these two cases.

In this article we calculate the groups of balanced functions on the edges, balanceable functions on the vertices and balanced functions on the vertices and edges of a directed graph with values in an Abelian group for both flexible and rigid cases.

In what follows, we say that a directed graph is *connected* if its underlying undirected graph is connected, and *strongly connected* whenever there exists a directed path between any ordered pair of vertices.

For the basics of Graph Theory we refer to [8].

2 The flexible case

Let $G = (V, E)$ be a connected directed graph, possibly with loops and multiple edges. Let v and w be two vertices connected by an edge e; v is the origin of e and w is the endpoint of e. For $e \in E$ denote by $\bar{e}$ the same edge as e but taken in the opposite direction. Thus $\bar{e}$ goes from w to v. Let $\mathbb{E} = \{e, \bar{e} \,|\, e \in E\}$.

Definition 2.1. A *path* from a vertex x to a vertex y is an alternating sequence v_1, e_1, v_2, e_2,...,v_n, e_n of vertices from V and different edges from $\mathbb{E}$ such that $v_1 = x$ and each e_j, for $j = 1, ..., n-1$, goes from v_j to v_{j+1} and e_n goes from v_n to y. We permit the same edge e to appear in a path twice - one time along and one time against its direction, since this is regarded as using two different edges from $\mathbb{E}$.

We require our graphs to be connected. Namely, any two different vertices of the graph G can be connected by a path.

Definition 2.2. A *cycle* is a path from a vertex to itself.

We permit the trivial cycle, which is the empty sequence containing no vertices and no edges.

Definition 2.3. The *length* of a cycle is the number of its edges.

Definition 2.4. A function $f : \mathbb{E} \to A$ such that $f(\bar{e}) = -f(e)$ is *balanced* if the sum $f(e_1) + ... + f(e_n)$ of the values of f over all the edges of any cycle of G is equal to 0.

Definition 2.5. The set of all the balanced functions $f : \mathbb{E} \to A$ is denoted by $\mathcal{HF}(\mathbb{E}, A)$. $\mathcal{HF}(\mathbb{E}, A)$ is a subgroup of the Abelian group $A^{\mathbb{E}}$ of all the functions from $\mathbb{E}$ to A.

Definition 2.6. A function $g : V \to A$ is *balanceable* if there exists some $f : \mathbb{E} \to A$ such that $f(\bar{e}) = -f(e)$ and the sum of all the values $g(v_1) + f(e_1) + g(v_2) + f(e_2) + ... + g(v_n) + f(e_n)$ along any cycle of G is zero. We say that this function $f : \mathbb{E} \to A$ balances the function $g : V \to A$.

Definition 2.7. The set of all the balanceable functions $g : V \to A$ is denoted by $\mathcal{BF}(V, A)$. The group $\mathcal{BF}(V, A)$ is a subgroup of the free Abelian group A^V of all the functions from V to A.

Definition 2.8. A function $h : V \bigcup \mathbb{E} \to A$, which takes both vertices and edges of G to some elements of A, is *balanced* if $h(\bar{e}) = -h(e)$ and the sum of its values $h(v_1) + h(e_1) + h(v_2) + h(e_2) + ... + h(v_n) + h(e_n)$ along any cycle of G is zero.

Definition 2.9. The set of all the balanced functions $h : V \bigcup \mathbb{E} \to A$ is denoted by $\mathcal{WF}(G, A)$. The group $\mathcal{WF}(G, A)$ is a subgroup of the Abelian group $A^{V \cup \mathbb{E}}$ of all the functions from $V \bigcup \mathbb{E}$ to A.

Clearly, any balanced function $f \in \mathcal{HF}(\mathbb{E}, A)$ can be viewed as a balanced function from $V \bigcup \mathbb{E}$ to A, which takes zero value on every vertex of G. Thus, we will regard $\mathcal{HF}(\mathbb{E}, A)$ as a subgroup of $\mathcal{WF}(V \bigcup \mathbb{E}, A)$.

Proposition 2.10. *The quotient* $\mathcal{WF}(V \bigcup \mathbb{E}, A)/\mathcal{HF}(\mathbb{E}, A)$ *is naturally isomorphic to* $\mathcal{BF}(V, A)$.

Proof. The natural isomorphism is defined by "forgetting" the values of $h \in \mathcal{WF}(V \bigcup \mathbb{E}, A)$ on the edges of G and regarding it just as a balanceable function from V to A. □

We review some basic definitions and facts regarding Abelian groups.

Definition 2.11. A natural number k is the *order* of an element $a \in A$ if it is the minimal positive integer such that $k \cdot a = 0$.

Definition 2.12. The set of all elements of A of order 2 is denoted by A_2. The set A_2 is a subgroup of A.

We provide a proof of a folklore result which describes the structure of the group $\mathcal{HF}(\mathbb{E}, A)$.

Proposition 2.13. *The group $\mathcal{HF}(\mathbb{E}, A)$ is isomorphic to $A^{|V|-1}$.*

Proof. Select a vertex v and consider the following bijection between the group of all A-valued functions g on V with $g(v) = 0$ and the group $\mathcal{HF}(\mathbb{E}, A)$. For any such g, since each edge $e \in \mathbb{E}$ goes from some vertex x to some vertex y, we define $f(e) = g(y) - g(x)$. A straightforward calculation shows that $f \in \mathcal{HF}(\mathbb{E}, A)$. In the other direction of the bijection, for $f \in \mathcal{HF}(\mathbb{E}, A)$ we inductively construct the function g as follows: we set $g(v) = 0$; if $g(u)$ has been defined for a vertex u then for every vertex w, for which there exists some edge e from u to w, we define $g(w) = g(u) + f(e)$. Since $f \in \mathcal{HF}(\mathbb{E}, A)$, any two calculations of the value of g on any vertex u will produce the same result. The connectivity implies that every vertex indeed receives a value. Thus, our g is well-defined. Obviously, the bijection, constructed above, is a group isomorphism. □

Now we can state and prove one of our main results.

Theorem 2.14. *Let $G = (V, E)$ be a connected directed graph and G' be its underlying undirected graph. Then:*

1. *If G' is bipartite, then the group $\mathcal{WF}(V \bigcup \mathbb{E}, A)$ is isomorphic to $A^{|V|}$.*
2. *If G' is not bipartite, then $\mathcal{WF}(V \bigcup \mathbb{E}, A)$ is isomorphic to $A_2 \times A^{|V|-1}$.*

Proof. If G consists only of one vertex then part (1) of our theorem is trivial. Otherwise, let us look at any one non-loop edge of G as it is depicted in Fig. 1:

$$a \xrightarrow{\quad p \quad} b$$

Figure 1: An edge with values on it and on its origin and end.

The letters on the edge and the vertices denote the values of a function $h : V \bigcup \mathbb{E} \to A$. Assume that h is balanced, that is, $h \in \mathcal{WF}(V \bigcup \mathbb{E}, A)$. Then for the cycle obtained by walking along this edge and returning back along it we have the following equation:

$$a + p + b - p = 0 \ ,$$

which immediately implies that

$$b = -a \, .$$

Thus h must have opposite values on any two vertices of G connected by an edge. Assume that G' is bipartite, which implies that G has no cycles of odd length. Then h, restricted to the edges, must be equal to some balanced function $f \in \mathcal{HF}(\mathbb{E}, A)$ on the edges, since values a and $-a$ on the vertices of an even-length cycle appear equally often. Now select any vertex $v \in V$. We can construct a balanced function h on vertices and edges by: for any element $a \in A$ define $h(v) = a$ and then define h for all the neighbors of v to be $-a$ and then for all the neighbors of the neighbors of v define h to be a and so on. Continuing this way we will assign values a or $-a$ to all the vertices of G. Since all the cycles are of even length, we will not get a contradiction in that process. Next we choose any function $f \in \mathcal{HF}(\mathbb{E}, A)$ and we set h on the edges to be equal to f. Hence, we have constructed a bijection between $\mathcal{WF}(V \bigcup \mathbb{E}, A)$ and the group of pairs $\{(a, f) \mid a \in A,\ f \in \mathcal{HF}(\mathbb{E}, A)\}$. This bijection is obviously also a group isomorphism. In addition, $\{(a, f) \mid a \in A,\ f \in \mathcal{HF}(\mathbb{E}, A)\}$ is isomorphic to $A^{|V|}$, since the group $\mathcal{HF}(\mathbb{E}, A)$ is isomorphic to $A^{|V|-1}$ by Proposition 2.13.

Now assume that G' is not bipartite, that is, G has a cycle of odd length. As we have already seen above, the values of a balanced function $h \in \mathcal{WF}(V \bigcup \mathbb{E}, A)$ on the vertices must be a and $-a$ for some $a \in A$. But walking along a cycle of odd length with vertices v_1, v_2,...,v_n, we get $h(v_1) = a$, $h(v_2) = -a$,..., $h(v_n) = a$, $h(v_1) = -a$, so $a = -a$, that is, $2a = 0$, which exactly means that $a \in A_2$. Thus we construct a bijection between $\mathcal{WF}(V \bigcup \mathbb{E}, A)$ and the group of pairs $\{(a, f) \mid a \in A_2,\ f \in \mathcal{HF}(\mathbb{E}, A)\}$ mapping $h \in \mathcal{WF}(V \bigcup \mathbb{E}, A)$ to the pair (a, f), where $a \in A_2$ is the value of h on any vertex, and f is a balanced function on edges defined as $f(e) = h(e) + a$ for every edge e; conversely, from a given $a \in A_2$ and a balanced function $f \in \mathcal{HF}(\mathbb{E}, A)$, we can construct a balanced function $h \in \mathcal{WF}(V \bigcup \mathbb{E}, A)$ assuming $h(v) = a$ for any vertex v and $h(e) = f(e) + a$ for any edge e. This bijection is a group isomorphism. In addition, $\{(a, f) \mid a \in A_2,\ f \in \mathcal{HF}(\mathbb{E}, A)\}$ is isomorphic to $A_2 \times A^{|V|-1}$, since the group $\mathcal{HF}(\mathbb{E}, A)$ is isomorphic to $A^{|V|-1}$ by Proposition 2.13. □

Remark 2.15. Let $G = (V, E)$ be a connected directed graph and G' be its underlying undirected graph. Notice that if the graph G' is bipartite, then the group of balanceable functions $\mathcal{BF}(V, A)$ is isomorphic to A and if G' is not bipartite, then the group of balanceable functions $\mathcal{BF}(V, A)$ is isomorphic to A_2 - the group of involutions of A.

3 The rigid case

Let $G = (V, E)$ be a connected directed graph. Recall that in the rigid case we are allowed to walk only in the direction of an edge but not against it. It naturally changes the notion of a path and of a cycle in comparison with the flexible case.

Definition 3.1. A *path* from a vertex x to a vertex y is an alternating sequence v_1, e_1, v_2, e_2,...,v_n, e_n of vertices from V and different edges from E (and not $\mathbb{E}$) such that $v_1 = x$ and each e_j, for $j = 1, ..., n-1$, goes from v_j to v_{j+1} and e_n goes from v_n to y.

For example, the triangle depicted in Fig. 2 is a cycle in the flexible case but is not a cycle in the rigid case.

Similarly to the flexible case denote by $\mathcal{BR}(V, A)$, $\mathcal{HR}(E, A)$ and $\mathcal{WR}(V \bigcup E, A)$ the groups of balanceable functions on vertices, balanced functions on edges and balanced functions of the entire graph G (vertices and edges), respectively.

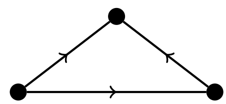

Figure 2: A flexible cycle, which is not a rigid cycle.

Proposition 3.2. *Every function on the set of vertices is balanceable. That is,*

$$\mathcal{BR}(V, A) = A^V.$$

Proof. Let us take a function $g : V \to A$. Define the function $h : V \bigcup E \to A$ in the following way: $h(v) = g(v)$ for each vertex $v \in V$, $h(e) = -g(v)$ for all the edges $e \in E$ which start at v. Obviously h is a balanced function. □

Definition 3.3. Two vertices x and y of G are *strongly connected* if there exists a path P_1 from x to y and a path P_2 from y to x. We also say that every vertex is strongly connected to itself.

Notice that we allow P_1 and P_2 to have common edges.

Definition 3.4. A *cycle* is a path P from a vertex x to itself.

Notice that, since the paths P_1 and P_2 mentioned above might have common edges, P_1 followed by P_2 may not be a cycle. It can even happen, that there exists no cycle, which contains both x and y. To illustrate it, consider the following.

Example 3.5. Consider the graph G depicted in Fig. 3 with $V(G) = \{x, v, w, y\}$ and $E(G) = \{e_1, e_2, e_3, e_4, e_5\}$. The path $P_1 = x, e_1, v, e_5, w, e_4$ is the only path which goes from x to y and the path $P_2 = y, e_2, v, e_5, w, e_3$ is the only path which goes from y to x. They have a common edge e_5. Thus, according to Definitions 3.1 and 3.4, there exists no cycle containing both x and y.

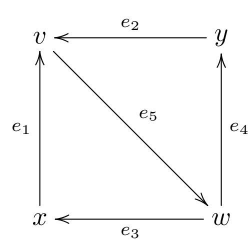

Figure 3: The vertices x, y are strongly connected but no cycle contains both of them.

Strong connectivity defines an equivalence relation on the vertices of G. The equivalence classes of strongly connected vertices, together with all the edges between the vertices of each class, are called the *strongly connected* components of G. We denote the number of strongly connected components of G by $\bar{k}(G)$. Obviously, G is strongly connected if and only if $\bar{k}(G) = 1$.

Lemma 3.6. *If G is strongly connected, then the group $\mathcal{HR}(E, A)$ is isomorphic to $A^{|V|-1}$, just like in the flexible case.*

Proof. Let $E = \{e_1, ..., e_n\}$. The edge e_1 goes from some x to some y. There is a path P which goes from y to x and does not contain e_1, since if P contains e_1 we can just delete this e_1 and all the vertices and edges that come after it from P. Thus, the sum of values of any $f \in \mathcal{HR}(E, A)$ along P must be equal to $-f(e_1)$. Hence, we can add a new edge $\bar{e}_1$ to G which goes from y to x and we can extend the function f to a balanced function on the edges of the new G if and only if we set $f(\bar{e}_1) = -f(e_1)$. So the group of the balanced functions on the edges of G after the addition of $\bar{e}_1$ is naturally isomorphic to the original group of the balanced functions on the edges of G before the addition. Repeating this process for all the edges of E we reduce G to the flexible case, while not changing the group of the balanced functions on the edges of G. □

Theorem 3.7. *The group $\mathcal{HR}(E, A)$ is isomorphic to $A^{|V|-\bar{k}(G)+r(G)}$, where $\bar{k}(G)$ is the number of strongly connected components of G, and $r(G)$ is the number of all the edges in G which go from a vertex in one strongly connected component of G to a vertex in a different strongly connected component of G.*

Proof. Let $V_1, ..., V_t$ be the equivalence classes of vertices of G with respect to strong connectivity. Denote the set of edges between the vertices of V_j by E_j. Obviously, $f \in \mathcal{HR}(E, A)$ if and only if $f|_{E_j} \in \mathcal{HR}(E_j, A)$ for each j, $1 \leqslant j \leqslant t$. Thus, $\mathcal{HR}(E, A) = \mathcal{HR}(E_1, A) \times \cdots \times \mathcal{HR}(E_t, A) \times A^U$, where U is the set of all the edges between the vertices in different strongly connected components of G. By Lemma 3.6 we conclude that $\mathcal{HR}(E, A)$ is isomorphic to $A^{|V_1|-1+|V_2|-1+\cdots+|V_t|-1+r(G)} = A^{|V|-\bar{k}(G)+r(G)}$. □

Theorem 3.8. *The group $\mathcal{WR}(V \bigcup E, A)$ is isomorphic to $A^{2|V|-\bar{k}(G)+r(G)}$, where $\bar{k}(G)$ is the number of strongly connected components of G, and $r(G)$ is the number of all the edges in G which go from a vertex in one strongly connected component of G to a vertex in a different strongly connected component of G.*

Proof. To each $h \in \mathcal{WR}(V \bigcup E, A)$ there corresponds the pair (g, f), where $g \in \mathcal{BR}(V, A)$ is just the restriction of h on the vertex set, and the value of $f \in \mathcal{HR}(E, A)$ on every edge e is equal to $h(e) + h(v)$, where the vertex v is the origin of the edge e. Such a function f is obviously a balanced function on the edge set since its value along any path is equal to the value of h along that path. This correspondence between the elements of $\mathcal{WR}(V \bigcup E, A)$ and the pairs from $\mathcal{BR}(V, A) \times \mathcal{HR}(E, A)$ is a bijection. Indeed, for a given pair (g, f), where g is any function on the vertex set and f is a balanced function on the edge set, we can construct $h : V \bigcup E \to A$ as follows: $h(v) = g(v)$ for all $v \in V$ and $h(e) = f(e) - g(v)$ for all $e \in E$, where the vertex v is the origin of the edge e. The constructed bijection is obviously a group isomorphism between the group $\mathcal{WR}(V \bigcup E, A)$ and the group $\mathcal{BR}(V, A) \times \mathcal{HR}(E, A)$, which is isomorphic to $A^{|V|} \times A^{|V|-\bar{k}(G)+r(G)}$. □

Thus, the flexible problem for a graph $G = (V, E)$ can be regarded as the rigid problem for the graph $G' = (V, \mathbb{E})$, where $\mathbb{E} = \{e, \bar{e} \,|\, e \in E\}$. Vice versa, the rigid problem for a graph G can be regarded as a free product of the rigid problems for the strongly connected components of G also multiplied by $A^{r(G)}$ where $r(G)$ is the number of edges between different strongly connected components of G.

The following simple claim connects this work to [7].

Proposition 3.9. *Let G be an undirected connected graph and let G_{dir} be a directed graph obtained from G by any assignment of directions to the edges of G. Denote by $H(E, A)$ the group of A-valued balanced functions on edges of G. Choose any order on edges of G and embed $H(E, A)$ and $\mathcal{HR}(E(G_{dir}), A)$ into $A^{|E|}$. For an undirected graph G the group of balanced functions on edges of G is equal to the intersection of all the groups $\mathcal{HR}(E(G_{dir}), A)$, where G_{dir} runs over all directed graphs for all $2^{|E|}$ possible direction assignments to the edges of G. The same is true for the groups of balanced functions on the entire graph (both vertices and edges). Namely, $W(V \bigcup E, A) = \bigcap \mathcal{WR}(V \bigcup E(G_{dir}), A)$.*

Proof. Let $Cyc = v_1, e_1, ..., v_k, e_k$ be a cycle in the undirected graph G. There exists a directed graph G_{dir} for which c is also a cycle. So any $f \in \bigcap \mathcal{HR}(E(G_{dir}), A)$ must satisfy the equation $\sum_{i=1}^{k} f(e_i) = 0$. Therefore $f \in H(E, A)$, since Cyc is an arbitrary cycle of G. Hence,

$$H(E, A) \supseteq \bigcap \mathcal{HR}(E(G_{dir}), A) .$$

The opposite inclusion is obvious, since any cycle of any G_{dir} is a cycle of G. The proof of the second statement of the proposition is similar. □

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A note on acyclic number of planar graphs*

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Abstract

The acyclic number $a(G)$ of a graph G is the maximum order of an induced forest in G. The purpose of this short paper is to propose a conjecture that $a(G) \geq \left(1 - \frac{3}{2g}\right) n$ holds for every planar graph G of girth g and order n, which captures three known conjectures on the topic. In support of this conjecture, we prove a weaker result that $a(G) \geq \left(1 - \frac{3}{g}\right) n$ holds. In addition, we give a construction showing that the constant $\frac{3}{2}$ from the conjecture cannot be decreased.

Keywords: Induced forest, acyclic number, planar graph, girth.

Math. Subj. Class.: 05C10, 05C15

1 Introduction

Throughout the paper n and g, respectively, stand for the order and girth of a (finite, simple, undirected) graph G. For other standard terminology and notation of graph theory we simply refer to [5]. The *acyclic number* of G, denoted $a(G)$, is the maximum order of an induced forest in G. This parameter has been well investigated (see e.g. [1, 4, 9, 10]), and its determination is NP-hard even in the case of planar graphs [7]. In [2], Albertson and Berman proposed the following lower bound for it.

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Conjecture 1.1. *If G is a planar graph, then*

$$a(G) \geq \frac{n}{2}.$$

This conjecture has drawn much attention since it implies that every planar graph has a stable set on at least a quarter of its vertices, a fact known to be true only as a consequence of the Four Color Theorem. It holds for planar graphs of girth at least 4 as Salavatipour [10] (see also [4]) proved that $a(G) \geq \frac{17n+24}{32}$ whenever G is such a graph. The best known lower bound for $a(G)$ over the class of all planar graphs G is the inequality $a(G) \geq \frac{2n}{5}$, which can be readily deduced from the acyclic 5-colorability of planar graphs (proven by Borodin in [6]). A similar problem to Conjecture 1.1 is Conjecture 1.2 below, raised by Akiyama and Watanabe [1].

Conjecture 1.2. *If G is a bipartite planar graph, then*

$$a(G) \geq \frac{5n}{8}.$$

Motivated by the last conjecture, the existence of large induced acyclic subgraphs in sparse bipartite graphs (resp. sparse graphs) was considered by Alon et al. in [3] (resp. [4]). Inspired by the fact that the dodecahedron attains the minimum possible ratio of order to size among all connected planar graphs of girth at least 5, Kowalik et al. [8] conjectured the following.

Conjecture 1.3. *If G is a planar graph of girth $g \geq 5$, then*

$$a(G) \geq \frac{7n}{10}.$$

The main purpose of this note is to generalize Conjectures 1.1, 1.2 and 1.3 through the following.

Conjecture 1.4. *If G is a planar graph of girth g, then*

$$a(G) \geq \left(1 - \frac{3}{2g}\right) n.$$

In particular, our conjecture reduces to Conjecture 1.1 (resp. Conjecture 1.3) for $g = 3$ (resp. $g = 5$), and for $g = 4$ strengthens Conjecture 1.2 by allowing odd 5^+-cycles. Moreover, it suggests a lower bound $a(G) \geq \frac{3n}{4}$ if $g \geq 6$, $a(G) \geq \frac{11n}{14}$ if $g \geq 7$, etc. Another way of stating Conjecture 1.4 is to claim that every non-acyclic planar graph G satisfies the inequality

$$\left(1 - \frac{a(G)}{n}\right) g \leq \frac{3}{2}. \tag{1.1}$$

Equivalently, we are looking for the smallest possible constant C, so that

$$\left(1 - \frac{a(G)}{n}\right) g \leq C, \tag{1.2}$$

holds for every planar graph of order n and finite girth g. If true, our conjecture is best possible in the sense that no excluding of a finite set of graphs could yield a better bound.

Indeed, take a tree T and let K be K_4, Q_3 or the dodecahedron. For any graph G obtained by blowing up every vertex of T to a copy of K, (1.1) becomes an equality.

In support to Conjecture 1.4, in the next section we prove that $C = 3$ is sufficient for (1.2).

Theorem 1.5. *If G is a planar graph of order n and girth $g = g(G) < \infty$, then*

$$a(G) > \left(1 - \frac{3}{g}\right) n. \tag{1.3}$$

Moreover, for every integer $g \geq 3$ there exists a planar graph G of girth g for which

$$a(G) = \left\lceil \left(1 - \frac{3}{2g}\right) n \right\rceil. \tag{1.4}$$

Notice that the first part of Theorem 1.5 implies Conjectures 1.1, 1.2, and 1.3, respectively, for girths $g \geq 6$, $g \geq 8$, and $g \geq 10$.

2 Proof of Theorem 1.5

The proof relies on an auxiliary result. Before stating it, let us recall some terminology. We use *k-vertex* and *k^+-vertex* to refer to a vertex of degree k and a vertex of degree at least k, respectively. Given a plane graph $G = (V, E)$, a *face* f is a region of $\mathbb{R}^2 \backslash (V \cup \bigcup E)$, and its *length* $\deg(f)$ is the degree of the corresponding vertex in the geometric dual G^* (thus every bridge incident to f is counted twice in the length); we speak of an *ℓ-face* f if $\deg(f) = \ell$, and an ℓ^+-face is a face of length at least ℓ. Recall that in case of a bridgeless plane graph, every cut-vertex is a 4^+-vertex and for every face f it holds that $\deg(f) = |E(f)|$ (since its topological boundary $\partial(f)$ is a union of simple curves). As usual, we say that a face f is *incident with* a vertex v if $v \in V(f)$. Here is our auxiliary result.

Lemma 2.1. *If G is a simple 2-edge-connected triangle-free plane graph with $\delta(G) \geq 3$, then there exists a face $f \in F(G)$ such that either:*

(i) f is a 4-face incident with at least one 3-vertex; or

(ii) f is a 5-face incident with at least four distinct 3-vertices.

Proof. We use the discharging method. By the Euler formula, it holds that

$$\sum_{v \in V(G)} (\deg(v) - 4) + \sum_{f \in F(G)} (\deg(f) - 4) = -8, \tag{2.1}$$

which leads to the following initial charge $w_0(x)$ for each $x \in V(G) \cup F(G)$:

$$w_0(x) = \deg(x) - 4. \tag{2.2}$$

By (2.1), the total charge is negative. On the other hand, (2.2) tells us that only the 3-vertices are with negative initial charge (equal to -1). Next, redistribute the initial charge according to the following simple rule:

(R) Every 5^+-face sends a charge of $\frac{1}{3}$ to each of its incident 3-vertices.

Let $w_1(x)$ denote the new charge of every $x \in V(G) \cup F(G)$ after applying (R). Assuming that a face satisfying (i) of Lemma 2.1 does not exist, for every $v \in V(G)$ it holds that $w_1(v) \geq 0$ (since G is bridgeless, any 3-vertex lies on the boundary of three faces, thus receives a combined charge of 1). The fact that the total charge remains negative implies the existence of a face f with $w_1(f) < 0$. Moreover, from

$$0 > w_1(f) \geq w_0(f) - \frac{\deg(f)}{3} = \frac{2}{3}(\deg(f) - 6),$$

it follows that every such f must be a 5-face incident with at least four 3-vertices. This completes the proof of the lemma. □

Proof of Theorem 1.5. We show (1.3) by contradiction. Suppose G is a minimal (under inclusion) counter-example to (1.3) among all non-acyclic planar graphs. Then G is clearly connected, of finite girth $g \geq 4$ and $\Delta(G) \geq 3$.

Claim 1: G is bridgeless. For otherwise, let e be a bridge and denote by G_1, G_2 the components of $G - e$. The choice of G combined with the fact that both subgraphs G_1, G_2 are of girth at least g, implies that $a(G_i) > \left(1 - \frac{3}{g}\right) n(G_i)$ for $i = 1, 2$. Summing up leads to the desired contradiction (1.3).

Let $\widetilde{G}$ be a plane embedding of the graph obtained by suppressing every 2-vertex in G. Then $\widetilde{G}$ is bridgeless and $\delta(\widetilde{G}) \geq 3$. Next we show that $\widetilde{G}$ meets all the requirements of Lemma 2.1.

Claim 2: $\widetilde{G}$ is simple and triangle-free. Supposing the opposite, there is a cycle C of G passing through at most three 3^+-vertices. Denote by S the set of 2-vertices in $V(C)$ and set $s = |S|$. In the graph $G' = G - V(C)$, let M be a maximum acyclic set. Then $M \cup S$ is an acyclic set of G, hence $a(G) \geq a(G') + s$. Combined with the choice of G, this would imply that

$$\left(1 - \frac{3}{g}\right)(n - s - 3) + s < \left(1 - \frac{3}{g}\right) n,$$

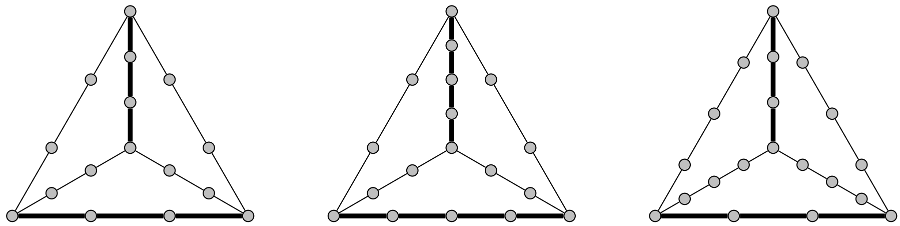
which is equivalent to $s + 3 < g$. However, the last inequality contradicts that the length of C is at least g, and thus settles the claim.

Our aim of contradicting the existence of G is now achievable. Select an $f \in F(\widetilde{G})$ as in Lemma 2.1, and denote $\ell = \deg(f)$. For this choice of f we can certainly find an independent (seen in $\widetilde{G}$) $(\ell - 3)$-subset $T \subseteq V(f)$ consisting entirely of 3-vertices. Indeed, in case $\ell = 4$ the last assertion is trivial; as for $\ell = 5$, it is enough to consider four consecutive 3-vertices v_1, v_2, v_3, v_4 on f and observe that, by planarity, v_1, v_3 or v_2, v_4 form an independent pair.

Returning back to G, every boundary edge of f becomes a path of G whose interior consists entirely of 2-vertices. Let $V_2(f)$ be the collection of all 2-vertices lying on f, and denote $r = |V_2(f)|$. Take from the graph $G' = G - (V(f) \cup V_2(f))$ a maximum acyclic set M. Then $M \cup V_2(F) \cup T$ is an acyclic set of G, giving that $a(G) \geq a(G') + r + \ell - 3$. Similarly to before, the last inequality would imply

$$\left(1 - \frac{3}{g}\right)(n - r - \ell) + r + \ell - 3 < \left(1 - \frac{3}{g}\right) n,$$

which is in turn equivalent to $r + \ell < g$. The last inequality is clearly impossible and thus validates (1.3).



(a) $k = 3, r = 0$ (b) $k = 3, r = 1$ (c) $k = 3, r = 2$

Figure 1: Three cases for G (edges coming from M bolded) when $k = 3$.

In regard to the second assertion of Theorem 1.5, we provide a constructive proof based on the fact that the removal of any two vertices decycles K_4: thus every subdivision of K_4 with order n has acyclic number $a = n - 2$. Given an integer $g \geq 3$, it is of the form $3k + r$ where r equals either 0, 1 or 2. Construct the graph G as follows. Consider a copy of K_4 and select a perfect matching M. If $r = 0$, then subdivide $k - 1$ times every $e \in E(K_4)$; else if $r = 1$, then subdivide k times each $e \in M$ and every other edge $k - 1$ times; finally, if $r = 2$, then subdivide $k - 1$ times each $e \in M$ and every other edge k times (see Fig. 1). In either case the constructed subdivision G has the desired girth g. Moreover, as can be readily checked, its order $n = 6k + 2(r - 1)$ and acyclic number $a = 6k + 2(r - 2)$ satisfy

$$\left(1 - \frac{3}{2g}\right) n = a - 1 + \frac{3}{g}, \tag{2.3}$$

since both sides of (2.3) are equal to $(6k + 2r - 3)(3k + r - 1)/(3k + r)$. Thus, it holds that

$$a = \left\lceil \left(1 - \frac{3}{2g}\right) n \right\rceil. \tag{2.4}$$

Additionally, observe that for $g = 3$, (2.3) becomes equal to a, which confirms that the left-hand side of (1.2) is at least $\frac{3}{2}$. This completes the proof of the theorem. □

3 Concluding remarks and further work

We are fully aware that a technically more involved argument could lower the bound $C \leq 3$ in (1.2), however that was not our main objective.

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Regular dessins d'enfants with field of moduli $\mathbb{Q}(\sqrt[p]{2})$

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Abstract

Herradon has recently provided an example of a regular dessin d'enfant whose field of moduli is the non-abelian extension $\mathbb{Q}(\sqrt[3]{2})$ answering in this way a question due to Conder, Jones, Streit and Wolfart. In this paper we observe that Herradon's example belongs naturally to an infinite series of such kind of examples; for each prime integer $p \geq 3$ we construct a regular dessin d'enfant whose field of moduli is the non-abelian extension $\mathbb{Q}(\sqrt[p]{2})$; for $p = 3$ it coincides with Herradon's example.

Keywords: Dessins d'enfants, Riemann surfaces, field of moduli and field of definition.

Math. Subj. Class.: 14H57, 30F10, 11G32

1 Introduction

A *dessin d'enfant* (or hypermap) of genus g, as defined by Grothendick in his *Esquisse d'un Programme* [8], is a bipartite map (vertices come in black and white colors and vertices of the same color are non-adjacent) on a closed orientable surface of genus g. The degree of the dessin d'enfant is the number of its edges. As a consequence of the classical uniformization theorem, a dessin d'enfant can also be seen as a pair (S, β), where S is a closed Riemann surface and $\beta : S \to \widehat{\mathbb{C}}$ is a non-constant meromorphic map whose branch values are contained in the set $\{\infty, 0, 1\}$; the degree of the dessin is the same as the degree of β. A dessin d'enfant (S, β) is called *regular* if β is a regular branched covering.

The *signature* of the dessin d'enfant is the tripe (a, b, c), where a (respectively, b and c) is the least common multiple of the local degrees of β at each preimage of 0 (respectively, 1 and ∞). In terms of the bipartite map, a is the least common multiple of the degrees of

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black vertices, b is the least common multiple of the degrees of white vertices and c is the least common multiple of the degrees of the faces (recall that a face of the dessin d'enfant must have an even number 2δ of boundary edges, where internal edges are counted twice; in this case δ is the degree of the face).

Two dessins d'enfant (S_1, β_1) and (S_2, β_2) are said to be *equivalent* (denoted this by the symbol $(S_1, \beta_1) \sim (S_2, \beta_2)$) if there is an isomorphisms $f : S_1 \to S_2$ so that $\beta_1 = \beta_2 \circ f$. Clearly, the signature is an invariant under this equivalence relation.

There is a natural bijection between dessins d'enfants (respectively, regular dessins d'enfants), of signature (a, b, c) and degree d, and conjugacy classes of subgroups (respectively, normal subgroups) of index d of the triangular group

$$\Delta(a, b, c) = \langle x, y : y^a = x^b = (xy)^c = 1\rangle.$$

By Belyi's theorem [2], each dessin d'enfant is equivalent to a dessin d'enfant (C, β) where C is an algebraic curve and β a rational map, both defined over the field $\overline{\mathbb{Q}}$ of algebraic numbers. This provides a natural action of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the set of (equivalence classes of) dessins d'enfants as follows. Start with a dessin d'enfant (C, β), defined algebraically over $\overline{\mathbb{Q}}$ and let $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Assume C is defined by the polynomials $P_1, \ldots, P_r$ and that $\beta = Q_1/Q_2$, where all polynomials have coefficients in $\overline{\mathbb{Q}}$. Let P_j^σ and Q_k^σ be the polynomials obtained by applying σ to the coefficients of P_j and Q_k, respectively. If C^σ is the algebraic curve defined by the polynomials P_j^σ and $\beta^\sigma = Q_1^\sigma/Q_2^\sigma$, then (C^σ, β^σ) still a dessin d'enfant. It is well known that the above action of the absolute Galois group is faithful [4, 5, 8, 12]. For many years, it was an open and difficult question if the absolute Galois group also acts faithfuly on the set of regular dessins d'enfants. Last year, this problem was solved by González-Diez and Jaikin-Zapirain in [6] and in a slightly weaker form by Bauer, Catanese and Grunewald in [1].

The *field of moduli* of a dessin d'enfant (C, β) is the fixed field of the subgroup of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ consisting of those σ for which $(C^\sigma, \beta^\sigma) \sim (C, \beta)$ (i.e., the field of definition of the equivalence class of the dessin d'enfant). The field of moduli is contained in any field of definition of the dessin (it is in fact the intersection of all of them by results due to Koizumi [10]), but there are examples for which the field of moduli is not a field of definition of it.

In [15], Wolfart observed that regular dessins d'enfants are definable over its field of moduli. The only explicit examples for such Galois Belyi actions were however known only for curves and dessins defined over abelian extensions of $\mathbb{Q}$. A question posed by Conder, Jones, Streit and Wolfart in [3] was if there were examples of regular dessins d'enfant with field of moduli being a non-abelian extension of $\mathbb{Q}$. In [9] Herradon answered the above positively by constructing a regular dessin d'enfant with field of moduli being $\mathbb{Q}\left(\sqrt[3]{2}\right)$. Herradon starts with the following genus one non-uniform dessin d'enfant of signature $(4, 6, 12)$

$$\left(C : y^2 = x\,(x-1)\left(x - \sqrt[3]{2}\right), \quad \beta(x, y) = x^3(2 - x^3)\right),$$

whose field of moduli is $\mathbb{Q}(\sqrt[3]{2})$, and then he observes that its normalizing regular dessin d'enfant has the same field of moduli (he also constructs another regular dessin d'enfant with the same property, this being a quotient of the previous one).

In this paper we observe that Herradon's example belongs to a infinite family with the same property which we proceed to describe in Section 3.

2 Preliminaries on triangle groups

If $l, m, n \geq 2$ are integers so that $l \leq m \leq n$ and $l^{-1}+m^{-1}+n^{-1} < 1$, then the triangular group

$$\Delta(l,m,n) = \langle x, y : y^l = x^m = (xy)^n = 1 \rangle$$

can be seen as a discrete group of isometries of the hyperbolic plane $\mathbb{H}$, that is a triangular Fuchsian group. In this case, $\mathbb{H}/\Delta(l,m,n)$ is an orbifold of genus zero having exactly three cone points of respective orders l, m and n. The triple (l,m,n) is called the signature of $\Delta(l,m,n)$.

A triangular Fuchsian group Δ is maximal if it is not a proper subgroup of finite index of another triangle group [7]. In [13], Singerman proved that $\Delta(l,m,n)$ is maximal if and only if

$$(l,m,n) \notin \{(l,l,l),(l,l,n),(l,m,m),(2,m,2m),(3,m,3m)\}.$$

A Fuchsian group Δ is called non-arithmetic if the commensurate group

$$\mathrm{Comm}(\Delta) = \{\gamma \in \mathrm{Aut}(\mathbb{H}) : [\Delta : \Delta \cap \gamma\Delta\gamma^{-1}] < \infty,\ [\gamma\Delta\gamma^{-1} : \Delta \cap \gamma\Delta\gamma^{-1}] < \infty\}$$

is discrete. This is not the original definition of a non-arithmetic group but it is equivalent due to a result of Margulis in [11]. The list of all the triples (l,m,n) for which Δ is arithmetic has been provided by Takeuchi in [14] (there are 76 such triples):

$$(2,3,7),(2,3,8),(2,3,9),(2,3,10),(2,3,11),(2,3,12),(2,3,14),(2,3,16),$$
$$(2,3,18),(2,3,24),(2,3,30),(2,4,5),(2,4,6),(2,4,7),(2,4,8),(2,4,10),$$
$$(2,4,12),(2,4,18),(2,5,5),(2,5,6),(2,5,8),(2,5,10),(2,5,20),(2,5,30),$$
$$(2,6,6),(2,6,8),(2,6,12),(2,7,7),(2,7,14),(2,8,8),(2,8,16),(2,9,18),$$
$$(2,10,10),(2,12,12),(2,12,24),(2,15,30),(2,18,18),(3,3,4),(3,3,5),(3,3,6),$$
$$(3,3,7),(3,3,8),(3,3,9),(3,3,12),(3,3,15),(3,4,4),(3,4,6),(3,4,12),$$
$$(3,5,5),(3,6,6),(3,6,18),(3,8,8),(3,8,24),(3,10,30),(3,12,12),(4,4,4),$$
$$(4,4,5),(4,4,6),(4,4,9),(4,5,5),(4,6,6),(4,8,8),(4,16,16),(5,5,5),$$
$$(5,5,10),(5,5,15),(5,10,10),(6,6,6),(6,12,12),(6,24,.24),(7,7,7),$$
$$(8,8,8),(9,9,9),(9,18,18),(12,12,12),(15,15,15).$$

All the above asserts the following simple fact.

Lemma 2.1. *If $p \geq 3$ is a prime integer, then $\Delta(4,2p,4p)$ is maximal and non-arithmetic. In particular, $\mathrm{Comm}(\Delta(4,2p,4p)) = \Delta(4,2p,4p)$ and, if there is a finite index subgroup Γ of $\Delta(4,2p,4p)$ and there is some $\gamma \in \mathrm{Aut}(\mathbb{H})$ so that $\gamma\Gamma\gamma^{-1} \in \Delta(4,2p,4p)$, then $\gamma \in \Delta(4,2p,4p)$.*

Proof. It follows from the above lists that $\Delta := \Delta(4,2p,4p)$ is maximal and non-arithmetic one. The non-arithmetic property asserts that $\mathrm{Comm}(\Delta)$ is a Fuchsian triangular group containing Δ; and by the maximal property, it must then follows the equality. Now, let Γ be a finite index subgroup of Δ and let $\gamma \in \mathrm{Aut}(\mathbb{H})$ so that $\gamma\Gamma\gamma^{-1} \in \Delta$. As $\gamma\Gamma\gamma^{-1}$ is a finite index subgroup of $\gamma\Delta\gamma^{-1}$ and also of Δ, and $\gamma\Gamma\gamma^{-1} < \Delta \cap \gamma\Delta\gamma^{-1}$, it follows that $\gamma \in \mathrm{Comm}(\Delta) = \Delta$. $\square$

3 Regular dessin d'enfants with field of moduli $\mathbb{Q}(\sqrt[p]{2})$

Let $p \geq 3$ be a prime integer and let us consider the elliptic curve

$$C_0 : y^2 = x(x-1)\left(x - \sqrt[p]{2}\right).$$

It is well known that the field of moduli of C_0 is $\mathbb{Q}\left(j\left(\sqrt[p]{2}\right)\right) = \mathbb{Q}\left(\sqrt[p]{2}\right)$, where j is the elliptic modular function

$$j(\lambda) = (1 - \lambda + \lambda^2)^3/\lambda^2(1-\lambda)^2.$$

On C_0 we consider the Belyi map

$$\beta(x, y) = x^p(2 - x^p).$$

The dessin d'enfant (C_0, β) has signature $(4, 2p, 4p)$, which is, by Lemma 2.1, maximal and non-arithmetic. This dessin d'enfant is non-uniform, in particular, it is non-regular (see Figure 1).

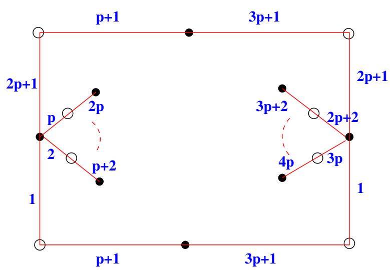

Figure 1: The dessin d'enfant (C_0, β)

The monodromy of the dessin (C_0, β) is

$$\omega_0 : \Delta = \langle x, y : y^4 = x^{2p} = (xy)^{4p} = 1 \rangle \to \mathfrak{S}_{4p},$$

$$\omega_0(x) = (1, 2, \ldots, p, 2p+1, 2p+2, \ldots, 3p)(p+1, 3p+1),$$

$$\omega_0(y) \\ || \\ (2, p+2)(3, p+3)\cdots(p, 2p)(2p+2, 3p+2)(2p+3, 3p+3)\cdots(3p, 4p) \\ (1, 3p+1, 2p+1, p+1),$$

and the dessin d'enfant corresponds to the Δ-conjugacy class of the subgroup

$$F_0 = \omega_0^{-1}\left(\mathrm{Stab}_{\omega_0(\Delta)}(1)\right).$$

Let us set $\Gamma_0 = \ker(\omega_0)$.

Next we list some properties of ω_0, the first three of them are immediate from its definition.

Lemma 3.1.

1. $\omega_0(x)^p = (1, 2p+1)(2, 2p+2)\cdots(p+1, 3p+1)$.
2. $\omega_0(y)^2 = (1, 2p+1)(p+1, 3p+1)$.
3. $\omega_0(xy)$ *has order* $4p$.
4. $\omega_0(\Delta)$ *is a group of order* $2^{2p}p^2$.
5. $$\prod_{j=0}^{p-1} x^{-j}\left(x^p y^2\right) x^j \in \Gamma_0.$$

Proof. Parts (1), (2) and (3) are direct to see from the definition of ω_0. Part (4) it is a little more difficult to see, but as we do not need it in the rest, we leave it to the reader. To check part (5) we only need to observe the following equalities:

$$\omega_0(x)^p \omega_0(y)^2 = (2, 2p+2)\cdots(p, 3p),$$

$$\omega_0(x)^{-1}\left(\omega_0(x)^p\omega_0(y)^2\right)\omega_0(x) = (1, 2p+1)(3, 2p+3)\cdots(p, 3p),$$

$$\omega_0(x)^{-2}\left(\omega_0(x)^p\omega_0(y)^2\right)\omega_0(x)^2 = (1, 2p+1)(2, 2p+2)(4, 2p+4)\cdots(p, 3p),$$

$$\vdots$$

$$\omega_0(x)^{-(p-1)}\left(\omega_0(x)^p\omega_0(y)^2\right)\omega_0(x)^{p-1} = (1, 2p+1)(2, 2p+2)\cdots(p-1, 3p-1).$$

□

The normal subgroup Γ_0 corresponds to a regular dessin d'enfant $(\widetilde{C}_0, \widetilde{\beta}_0)$ with signature $(4, 2p, 4p)$. As the previous signature is maximal (by Lemma 2.1), we have that

$$\mathrm{deck}(\widetilde{\beta}_0) = \mathrm{Aut}(\widetilde{C}_0) \cong \omega_0(\Delta).$$

Also, as a consequence of the Riemann-Hurwitz formula, the genus of $\widetilde{C}_0$ is

$$g_p = 1 + 3 \times 2^{2p-3}p(p-1).$$

The Galois orbit of (C_0, β) is given by the p dessins d'enfants (see Figure 2)

$$(C_k, \beta); \quad k = 0, 1, \ldots, p-1,$$

where

$$C_k : y^2 = x(x-1)\left(x - \rho_p^k \sqrt[p]{2}\right), \quad \rho_p = e^{2\pi i/p},$$

whose monodromy $\omega_k : \Delta \to \mathfrak{S}_{4p}$ is defined by

$$\omega_k(x)$$
$$||$$
$$(1, 2, \ldots, k+1, 2p+k+2, \ldots, 3p, 2p+1, 2p+2, \ldots, 2p+k+1, k+2, \ldots, p)$$
$$(p+k+1, 3p+k+1),$$

and

$$\begin{array}{c}\omega_k(y)\\ ||\\ (2,p+2)(3,p+3)\cdots(p,2p)(2p+2,3p+2)(2p+3,3p+3)\cdots(3p,4p)\\ \hfill (1,3p+1,2p+1,p+1),\end{array}$$

and the dessin d'enfant corresponds to the Δ-conjugacy class of the subgroup

$$F_k = \omega_k^{-1}\left(\mathrm{Stab}_{\omega_k(\Delta)}(1)\right).$$

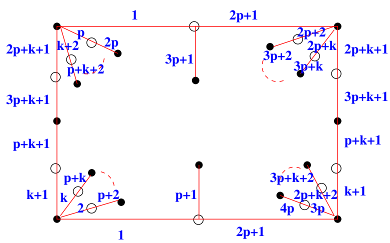

Figure 2: The dessin d'enfant (C_k, β)

The normal subgroup $\Gamma_k = \ker(\omega_k)$ corresponds to a regular dessin d'enfant $(\widetilde{C}_k, \widetilde{\beta}_k)$ with signature $(4, 2p, 4p)$. Again, by maximality of the signature,

$$\mathrm{deck}(\widetilde{\beta}_k) = \mathrm{Aut}(\widetilde{C}_k) \cong \omega_k(\Delta) \cong \omega_0(\Delta).$$

Theorem 3.2. *The field of moduli of* $(\widetilde{C}_0, \widetilde{\beta}_0)$ *is* $\mathbb{Q}\left(\sqrt[p]{2}\right)$.

Proof. As the regular dessin d'enfant $(\widetilde{C}_k, \widetilde{\beta}_k)$ is the normalization of the dessin d'enfant (C_k, β), we see that the Galois orbit of $(\widetilde{C}_0, \widetilde{\beta}_0)$ is given by the following p dessins d'enfants

$$(\widetilde{C}_k, \widetilde{\beta}_k); \quad k = 0, 1, \dots, p-1.$$

It follows that the field of moduli of $(\widetilde{C}_0, \widetilde{\beta}_0)$ is a subfield of $\mathbb{Q}\left(\sqrt[p]{2}\right)$. As $\mathbb{Q}\left(\sqrt[p]{2}\right)$ is an extension of degree p (a prime integer) of $\mathbb{Q}$, in order to see that the field of moduli is exactly $\mathbb{Q}(\sqrt[p]{2})$ we only need to check that Γ_0 and Γ_1 are not conjugated in $\mathrm{Aut}(\mathbb{H})$. As Γ_0 is a finite index subgroup of the maximal and non-arithmetic group $\Delta(4, 2p, 4p)$, it follows, from Lemma 2.1, that we only need to check that $\Gamma_0 \neq \Gamma_1$. This last can be noted by part (5) of Lemma 3.1 and the fact that $\prod_{j=0}^{p-1} x^{-j}\left(x^p y^2\right) x^j \notin \Gamma_1$; since

$$\prod_{j=0}^{p-1} \omega_1(x)^{-j}\left(\omega_1(x)^p \omega_1(y)^2\right) \omega_1(x)^j = (p+1, 3p+1)(p+2, 3p+2).$$

This last can be checked by observing that

$$\omega_0(x)^p = \omega_1(x)^p (p+1, 3p+1)(p+2, 3p+2),$$

$$\omega_1(y)^2 = \omega_0(y)^2,$$
$$\omega_1(x)^p\omega_1(y)^2 = \left(\omega_0(x)^p\omega_0(y)^2\right)(p+1, 3p+1)(p+2, 3p+2),$$

and, for $j = 1, \ldots, p-1$,

$$\omega_1(x)^{-j}\left(\omega_1(x)^p\omega_1(y)^2\right)\omega_1(x)^j$$
$$||$$
$$\omega_0(x)^{-j}\left(\omega_0(x)^p\omega_0(y)^2\right)\omega_0(x)^j(p+1, 3p+1)(p+2, 3p+2).$$

□

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Association schemes all of whose symmetric fusion schemes are integral*

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Abstract

In this paper we aim to characterize association schemes all of whose symmetric fusion schemes have only integral eigenvalues, and classify those obtained from a regular action of a finite group by taking its orbitals.

Keywords: Association schemes, groups.

Math. Subj. Class.: 05E15, 05E30

1 Introduction

In the history of algebraic combinatorics it has been one of the important topics to consider eigenvalues of the adjacency matrix of a graph. In [3] and [5] many criterions and conjectures on such problems are suggested and the eigenvalues of well-known distance-regular graphs are explicitly found. Together with web catalogue [7] this gives many association schemes with integral first eigenmatrices (see [3] and [5] for its definition). As mentioned in [3, Ex. 2.1] a transitive permutation group H on a finite set X induces an association scheme $(X, \mathcal{R}_H)$ where $\mathcal{R}_H$ is the set of orbitals of H. If G is a permutation group of X containing H, then each element in $\mathcal{R}_G$ is a union of elements of $\mathcal{R}_H$, and the first eigenmatrix of $(X, \mathcal{R}_G)$ is influenced by that of $(X, \mathcal{R}_H)$. In general, the fusion (and fission) of association relations gives rise to new association schemes from a given association scheme. In this paper we focus on association schemes whose adjacency matrices have only integral eigenvalues.

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The authors of [9] introduced fusion association schemes and presented some diagrams to enumerate all association schemes of given small orders according to the partial order defined by fusing. As shown in the enumeration, the association scheme induced by the icosahedron can be obtained as a fusion of the alternating group A_4 of degree 4 where we identify a finite group G with the association scheme obtained from a regular action of G on itself, but not all eigenvalues of the icosahedron are integral. On the other hand, every symmetric fusion association scheme of a non-cyclic abelian group of order 12 has the integral first eigenmatrix.

For more general cases, we introduce the following definition: an association scheme is said to be *desirable* if the first eigenmatrix of each of its symmetric fusion schemes is integral, otherwise it is said to be *undesirable*. Our main problem is the following:

Problem 1.1. Characterize desirable association schemes.

For the remainder of this article we shall write schemes instead of association schemes for short. It is obvious that every fusion scheme of a desirable scheme is desirable. Moreover given a desirable scheme (X, S), the subscheme induced by a closed subset and the quotient modulo a closed subset are desirable (see Lemma 2.2), while the direct product (or the other scheme products) of two desirable schemes are not necessarily desirable. The following are examples of desirable or undesirable association schemes:

Example 1.2.

(i) The scheme of a cyclic group G of order m is undesirable if $m \notin \{1, 2, 3, 4, 6\}$ by Corollary 2.4.

(ii) Every symmetric scheme with non-integral first eigenmatrix is undesirable since for any scheme it is also one of its fusion schemes.

(iii) Every symmetric scheme with integral first eigenmatrix is desirable by [2, Lemma 1 (2)].

(iv) Every association scheme of rank 2 is symmetric and integral. This implies that every non-symmetric scheme of rank 3 is desirable.

A group G is said to be *desirable* if it induces a desirable scheme by its regular action, otherwise it is said to be *undesirable*. Then the former example may lead readers to confront the following problem:

Problem 1.3. Find all *desirable* finite groups.

Remark 1.4. For a finite group G, there is a one-to-one correspondence between the set of fusions of the association scheme induced by G by its regular action and the set of Schur rings over G (see [8] for the definition of Schur ring). Thus Problem 1.3 can be stated in terms of Schur rings.

Remark 1.5. In connection with Problem 1.3 we mention that Bridge and Mena [4] give a criterion on Cayley graphs over abelian groups with integral eigenvalues which is obtained from a group action.

By Corollary 2.4, if a finite group G is desirable, then $|G| = 2^a 3^b$ for some nonnegative integers a, b and the order of each element of G belongs to the set $\{1, 2, 3, 4, 6\}$. But, the converse does not hold because of A_4. In [1] all *Cayley integral groups G* were classified;

the defining property of such a group is that the eigenvalues of any undirected Cayley graph over G are integral. Let C_n, S_n and Q_8 denote the cyclic group of order n, the symmetric group of degree n and the quaternion group, respectively. It is remarkable that any Cayley integral group is desirable. On the other hand, our main result show that the converse also holds:

Theorem 1.6. *Every desirable group is isomorphic to one of the following:*

(i) *an abelian group the exponent of which divides* 4 *or* 6*;*

(ii) $Q_8 \times C_2^m$ *for some nonnegative integer* m*;*

(iii) S_3*;*

(iv) $C_3 \rtimes C_4 = \langle x, y \mid x^3 = y^4 = 1, y^{-1}xy = x^{-1} \rangle$.

In Section 2, we prepare some terminologies on association schemes and groups. In Section 3, we show a number of desirable groups that will be used in the proof of Theorem 1.6. In Section 4, we give a proof of our main result, Theorem 1.6.

2 Preliminaries

Following [11] we prepare terminologies on association schemes. Let X be a finite set and S a partition of $X \times X$. We say that the pair (X, S) is an *association scheme* (or shortly *scheme*) if it satisfies the following:

(i) $1_X := \{(x, x) \mid x \in X\}$ is an element of S;

(ii) For any s in S, $s^* := \{(y, x) \mid (x, y) \in s\}$ is an element of S;

(iii) For all $s, t, u \in S$ the size of $\{z \in X \mid (x, z) \in s, (z, y) \in t\}$ is constant whenever $(x, y) \in u$. The constant is denoted by a_{stu}.

For the remainder of this section we assume that (X, S) is an association scheme. For $s \in S$ we define a matrix σ_s over $\mathbb{C}$, which is called the *adjacency matrix* of s, whose rows and columns are indexed by the elements of X as follows:

$$(\sigma_s)_{x,y} = \begin{cases} 1 & \text{if } (x, y) \in s \\ 0 & \text{if } (x, y) \notin s. \end{cases}$$

We shall write $\mathrm{ev}(A)$ as the set of all eigenvalues of a square matrix A over $\mathbb{C}$. We say that (X, S) is *integral* if $\bigcup_{s \in S} \mathrm{ev}(\sigma_s) \subseteq \mathbb{Z}$.

Remark 2.1. The first eigenmatrix of (X, S) is defined when (X, S) is commutative, i.e., $\sigma_s\sigma_t = \sigma_t\sigma_s$ for all $s, t \in S$. Then the first eigenmatrix of (X, S) is integral if and only if (X, S) is integral.

We say that (X, S) is *symmetric* if σ_s is symmetric for each $s \in S$, and *desirable* if (X, T) is integral for each symmetric fusion scheme (X, T) of (X, S).

For a finite group G we set

$$\widetilde{G} = \{\tilde{g} \mid g \in G\}$$

where $\tilde{g} = \{(a, b) \in G \times G \mid ag = b\}$. It is well-known that $(G, \widetilde{G})$ is an association scheme (see [10, Appendix]). We say that G is *desirable* if $(G, \widetilde{G})$ is desirable.

Following [11] we introduce a concept which corresponds to blocks in permutation groups. For $T \subseteq S$ we say that T is *closed* if

$$\{u \in S \mid a_{stu} > 0\} \subseteq T \text{ for all } s, t \in T,$$

equivalently, $\bigcup_{t \in T} t$ is an equivalence relation on X since each digraph (X, t) has a directed cycle because of $|X| < \infty$. We shall write the equivalence class containing x by $\bigcup_{t \in T} t$ as xT. It is well-known (see [10, 1.5]) that $(xT, \{t \cap (xT \times xT) \mid t \in T\})$ forms an association scheme, which is denoted by $(X, S)_{xT}$, and the quotient set X/T forms an association scheme, called the *factor scheme* of (X, S) over T, denoted by $(X/T, S/\!\!/T)$ where

$$S/\!\!/T = \{s^T \mid s \in S\} \quad \text{and} \quad s^T := \{(xT, yT) \mid (xT \times yT) \cap s \neq \emptyset\}.$$

Lemma 2.2. *Let (X, S) be a desirable scheme, $x \in X$ and T a closed subset of S. Then both of $(X, S)_{xT}$ and $(X/T, S/\!\!/T)$ are desirable.*

Proof. Let (xT, U) be a symmetric fusion scheme of $(X, S)_{xT}$ where U is a partition of $xT \times xT$. Since each $u \in U$ is a union of elements of the restrictions of T to $xT \times xT$, it allows us to fuse elements of S as follows:

$$\left\{ \bigcup_{s \in S \setminus T} s \right\} \cup \left\{ \bigcup_{s \in S; s \cap u \neq \emptyset} s \;\middle|\; u \in U \right\}$$

which forms a symmetric fusion scheme of (X, S).

Let $|X/T| = m$. Notice that $\bigcup_{s \in S; s \cap u \neq \emptyset} s$ is contained in $\bigcup_{i=1}(x_i T \times x_i T)$ where $x_i T$ with $i = 1, \dots, m$ are the equivalence classes induced by T. Thus, the characteristic polynomial of $\bigcup_{s \in S; s \cap u \neq \emptyset} s$ is the product of those of $(\bigcup_{s \in S; s \cap u \neq \emptyset} s) \cap (x_i T \times x_i T)$ with $i = 1, \dots, m$ which are mutually equal since $(X, S)_{x_i T}$ have the same structure constants. This implies that the characteristic polynomial of $\bigcup_{s \in S; s \cap u \neq \emptyset} s$ is equal to the m-th power of that of u. Since (X, S) is desirable, the eigenvalues of the adjacency matrix of $\bigcup_{s \in S; s \cap u \neq \emptyset} s$ are all integers. Therefore, $\mathrm{ev}(\sigma_u) \subseteq \mathbb{Z}$ for each $u \in U$.

Let $(X/T, U)$ be a symmetric fusion of $(X/T, S/\!\!/T)$, and for $u \in S$ let A_{TuT} denote the adjacency matrix of $\bigcup_{s \in S; s^T \subseteq u^T} s$. Notice that $\bigcup_{s \in TuT} s$ is a union of some of the $\{x_i T \times x_j T \mid i, j = 1, \dots, m\}$. Moreover, $(x_i T \times x_j T) \subseteq \bigcup_{s \in TuT} s$ if and only if $(x_i T, x_j T) \in u^T$. This implies that A_{TuT} is conjugate to the Kronecker's product $\sigma_{u^T} \otimes J$ under the group of permutation matrices where J is the all one matrix of degree m, so that it can be easily checked that

$$\left\{ 1_X, \bigcup_{t \in T; t \neq 1_X} t \right\} \cup \left\{ \bigcup_{s \in S; s^T \subseteq u^T} s \;\middle|\; u^T \in U \setminus \{1_{X/T}\} \right\}$$

is a symmetric fusion of (X, S).

Since each of the eigenvalues of A_{TuT} is an integral multiple of an eigenvalue of σ_{u^T}, it follows from the fact that each eigenvalue of σ_{u^T} is an algebraic integer that $\mathrm{ev}(\sigma_{u^T}) \subseteq \mathbb{Z}$ for each $u^T \in U$. Therefore, $(X/T, S/\!\!/T)$ is desirable. □

We frequently use the following without mentioning.

Corollary 2.3. *Any subgroup or any homomorphic image of a desirable group is desirable.*

Proof. Let G be a finite group and H a subgroup of G. Then $\widetilde{H}$ is a closed subset of $\widetilde{G}$, and if H is normal in G, then $(G/H, \widetilde{G/H})$ is isomorphic to the factor scheme of $(G, \widetilde{G})$ over $\widetilde{H}$. Applying Lemma 2.2 with the homomorphism theorem in group theory we obtain the result. □

Corollary 2.4. *The order of any element of a desirable group belongs to the set* $\{1, 2, 3, 4, 6\}$. *In particular, the order of a desirable group equals* $2^a 3^b$ *for some nonnegative integers* a, b.

Proof. Let G be a desirable group and $x \in G$ has order n. By Corollary 2.3, $H := \langle x \rangle$ is desirable. Since the symmetrization $\{\tilde{y} \cup \tilde{z} \mid y, z \in H; yz = 1\}$ forms a symmetric fusion of $(H, \widetilde{H})$ and $\mathrm{ev}(\sigma_{\tilde{y}\cup\tilde{z}}) = \{2\cos(2\pi k/n) \mid k \in \mathbb{Z}\}$ with $yz = 1$, it follows that $n \in \{1, 2, 3, 4, 6\}$. □

3 Undesirable groups of small orders

By Corollary 2.4, every desirable group has order $2^a 3^b$ for some nonnegative integers a, b. But, the converse does not necessarily hold. In this section we collect some undesirable groups of such orders.

Lemma 3.1. *The dihedral group of order* 8 *is undesirable.*

Proof. Let $G = \langle x, y \mid x^4 = y^2 = 1, yxy = x^{-1} \rangle$ be the dihedral group of order 8. Then the following partition of G induces a symmetric fusion of $(G, \widetilde{G})$:

$$\{\{1\}, \{x^2\}, \{x, x^3\}, \{y, yx\}, \{yx^2, yx^3\}\}.$$

Since $\tilde{y} \cup \widetilde{yx}$ forms the octagon whose eigenvalues are not all integral, the dihedral group of order 8 is undesirable. □

Lemma 3.2. *The direct product* $C_2 \times S_3$ *is undesirable.*

Proof. Let $G = \langle x \rangle \times \langle y, z \mid y^3 = z^2 = 1, zyz = y^{-1} \rangle$ denote the group $C_2 \times S_3$. Then the following partition of G induces a symmetric fusion of $(G, \widetilde{G})$:

$$\{\{1\}, \{y, y^2\}, \{x\}, \{xy, xy^2\}, \{z, xz\}, \{y^2 z, xyz\}, \{yz, xy^2 z\}\}.$$

Since $\widetilde{y^2 z} \cup \widetilde{xyz}$ forms a 12-gon whose eigenvalues are not all integral, the statement holds. □

Lemma 3.3. *The alternating group of degree* 4 *is undesirable.*

Proof. The following partition of A_4 induces a symmetric fusion of $(A_4, \widetilde{A}_4)$:

$$\{\{1\}, \{(12)(34)\}, \{(13)(24), (123), (132), (124), (142)\},$$
$$\{(14)(23), (134), (143), (234), (243)\}\}.$$

The third one induces the icosahedron, whose eigenvalues are not necessarily integral. □

Lemma 3.4. *The semidirect product $(C_3 \times C_3) \rtimes C_2$ by the action of the inverse map is undesirable.*

Proof. Let $\langle x, y, z \mid x^3 = y^3 = [x, y] = zxzx = zyzy = z^2 = 1\rangle$ denote the group given in the statement. Then the following partition of G induces a symmetric fusion of $(G, \widetilde{G})$:

$$\{\{1\}, \{x, x^2\}, \{y, xy, x^2y, y^2, xy^2, x^2y^2\}, \{z, yz, xy^2z\},\\ \{xz, x^2z, y^2z, xyz, x^2yz, x^2y^2z\}\}.$$

Since $\tilde{z} \cup \widetilde{yz} \cup \widetilde{xy^2z}$ forms a distance-regular graph with intersection array $\{3, 2, 2, 1; 1, 1, 2, 3\}$ whose eigenvalues are not all integral, the statement holds. □

Lemma 3.5. *The direct product $S_3 \times C_3$ is undesirable.*

Proof. Let $\langle x, y, z \mid x^3 = y^3 = [x, y] = [z, x] = zyzy = z^2 = 1\rangle$ denote the group given in the statement. Then the following partition of G induces a symmetric fusion of $(G, \widetilde{G})$:

$$\{\{1\}, \{y, y^2\}, \{x, xy, xy^2, x^2, x^2y, x^2y^2\}, \{z, xyz, x^2yz\},\\ \{yz, y^2z, xz, xy^2z, x^2z, x^2y^2z\}\}.$$

Since $\tilde{z} \cup \widetilde{xyz} \cup \widetilde{x^2yz}$ forms a distance-regular graph with intersection array $\{3, 2, 2, 1; 1, 1, 2, 3\}$ whose eigenvalues are not all integral, the statement holds. □

Lemma 3.6. *The semidirect product $(C_3 \rtimes C_4) \times C_2$ is undesirable.*

Proof. Let $G = H \cup Hy$ where $H = \langle x\rangle \times \langle y^2\rangle \times \langle z\rangle$ is the unique subgroup with index two with $|x| = 2$, $|y| = 4$ and $|z| = 3$. Then the following partition of G induces a symmetric fusion of $(G, \widetilde{G})$:

$$\big\{\{a\}_{a\in\langle x,y^2\rangle}, \{az, az^2\}_{a\in\langle x,y^2\rangle}, \langle x, y^2\rangle y, T, xT\big\}$$

where $T := \{zy, zy^3, xz^2y^3, z^2xy\}$. Since the minimal polynomial of the adjacency matrix of the graph induced by $\bigcup_{t\in T} \tilde{t}$ is $\lambda(\lambda^2 - 4)(\lambda^2 - 16)(\lambda^2 - 12)$ where λ is an indeterminate, the statement holds. □

Lemma 3.7. *There are no non-abelian desirable groups of order 27.*

Proof. Notice that there are two non-abelian groups of order 27 those exponents are 9 and 3. Every group with exponent 9 is undesirable by Corollary 2.4. Let $G = (\langle x\rangle \times \langle y\rangle) \rtimes \langle z\rangle$ where $|x| = |y| = |z| = 3$, $zx = xz$ and $z^{-1}yz = xy$. Then G is a unique non-abelian group of order 27 with exponent 3, and the following partition of G induces a symmetric fusion of $(G, \widetilde{G})$:

$$\big\{\{1\}, \{x, x^2\}, \{y, y^2, xy, xy^2, x^2y, x^2y^2\}, H_1, H_2, H_3\big\}$$

where $H_1 = \{z, z^2, yz, x^2y^2z^2, x^2y^2z, x^2yz^2\}$, $H_2 = \{xz, x^2z^2, xyz, xy^2z^2, y^2z, xyz^2\}$ and $H_3 = \{x^2z, xz^2, x^2yz, y^2z^2z, xy^2z, yz\}$. Since the minimal polynomial of the adjacency matrix of the graph induced by $\bigcup_{h\in H_1} \tilde{h}$ is $\lambda(\lambda + 3)(\lambda - 6)(\lambda^3 - 9\lambda - 9)$ where λ is an indeterminate, the statement holds. □

Proposition 3.8. *There are no non-abelian desirable groups of order* 18, 24 *or* 27.

Proof. Since the groups as in Lemma 3.4 and 3.5 are the non-abelian groups of order 18 without any element of order 9, there are no such groups of order 18.

The following are the non-abelian groups of order 24 without any element of order 8 or order 12:

$$SL(2,3),\ (C_3 \rtimes C_4) \times C_2,\ D_{12} \times C_2,\ S_4,\ C_2 \times A_4 \text{ and } C_2 \times C_2 \times S_3.$$

Among them the first, third, fourth, fifth and sixth ones are undesirable by Lemma 3.3 and Corollary 2.3 and the second is undesirable by Lemma 3.6. So there are no such groups of order 24.

By Lemma 3.7, there are no such groups of order 27, the statement holds. □

4 Proof of our main result

Lemma 4.1. *Let G be a desirable group and $a, b \in G$ non-commuting involutions. Then* $|\langle a, b\rangle| \in \{6, 12\}$.

Proof. Since $\langle a, b\rangle$ is isomorphic to a dihedral group, it follows from Corollary 2.4 and Lemma 3.1. □

Lemma 4.2. *Let G be a desirable group. If $a \in G$ normalizes an elementary abelian* 2*-subgroup H of G, then $ab = ba$ for each $b \in H$.*

Proof. Suppose the contrary, i.e., $ab \neq ba$ for $a \in N_G(H)$ and $b \in H$ where $N_G(H) = \{x \in G \mid x^{-1}Hx = H\}$. Since $\langle a^{-i}ba^i \mid i = 0, 1, \dots, 3\rangle$ is a subgroup of H, it is an elementary abelian 2-group of rank at least two. By Corollary 2.4, we divide our proof according to $|a| \in \{2, 3, 4, 6\}$.

If $|a| = 2$, then $\langle a, b\rangle$ is a non-abelian group of order 8, which is isomorphic to D_8, a contradiction to Lemma 3.1.

If $|a| = 4$, then $\langle a^{-i}ba^i \mid i = 0, 1, \dots, |a| - 1\rangle = \langle b, a^{-1}ba\rangle$ since a^2 commutes with b by what we proved in the last paragraph. If $a^2 \in \langle b, a^{-1}ba\rangle$, then $\langle a, b\rangle$ is isomorphic to D_8 since it has more than one involutions, a contradiction to Lemma 3.1. If $a^2 \notin \langle b, a^{-1}ba\rangle$, then $\langle a^2\rangle$ is central in $\langle a, b\rangle$ and $|\langle a, b\rangle/\langle a^2\rangle| = 8$ by $|a^2| = 2$. If $\langle a, b\rangle/\langle a^2\rangle$ is non-abelian, then it is isomorphic to D_8, a contradiction to Lemma 3.1. If $\langle a, b\rangle/\langle a^2\rangle$ is abelian, then $a^{-1}ba = ba^2$, which implies that $bab = a^{-1}$, and hence, $\langle a, b\rangle$ is isomorphic to D_8, a contradiction to Lemma 3.1.

If $|a| = 3$ and $\langle b, a^{-1}ba, aba^{-1}\rangle \simeq C_2^2$, then $\langle a, b\rangle = \langle a, b, a^{-1}ba, aba^{-1}\rangle$ is isomorphic to A_4, which contradicts Lemma 3.3.

If $|a| = 3$ and $\langle b, a^{-1}ba, aba^{-1}\rangle \simeq C_2^3$, then $\langle a, b\rangle$ is a non-abelian group of order 24, which contradicts Proposition 3.8.

If $|a| = 6$, then $a = cd = dc$ for some $d, c \in \langle a\rangle$ with $|d| = 2$ and $|c| = 3$. Since both d and c centralize H, a also centralizes H. □

Lemma 4.3. *If G is a desirable non-abelian* 2*-group, then G is isomorphic to $Q_8 \times C_2^m$ for some nonnegative integer m.*

Proof. By Lemma 4.1, all involutions of a desirable 2-group commute for each other. This implies that the subgroup, say K, generated by all involutions is a normal subgroup, which is isomorphic to an elementary abelian 2-group contained in the center of G by Lemma 4.2.

In order to prove the statement it suffices to show that each cyclic subgroup of G is normal by a well-known theorem by Baer and Dedekind (We mimic the same argument as in the proof of [1, Thm. 2.13]). Suppose the contrary, i.e., $a^{-1}ba \notin \langle b\rangle$ for $a, b \in G \setminus K$, namely, $|a| = |b| = 4$. Let L denote the subgroup of $\langle a, b\rangle$ generated by the involutions of $\langle a, b\rangle$. Since L is central in G and $b^2 \in L$, $b\langle b^2\rangle \in \langle a, b\rangle/\langle b^2\rangle$ is an element of order two. Since $\langle a, b\rangle/\langle b^2\rangle$ is a desirable 2-group by Corollary 2.3, it follows from the same argument as in the last paragraph that $b\langle b^2\rangle$ is contained in the center of $\langle a, b\rangle/\langle b^2\rangle$. Therefore, $a^{-1}ba \in b\langle b^2\rangle \subseteq \langle b\rangle$, a contradiction. □

Lemma 4.4. *If G is a desirable 3-group, then G is isomorphic to C_3^m for some nonnegative integer m.*

Proof. Suppose that G is a desirable non-abelian 3-group of the least order. Let $x, y \in G$ with $xy \neq yx$. By Corollary 2.3, $\langle x, y\rangle$ is a desirable non-abelian 3-group, which is of exponent three by Corollary 2.4. By the minimality of $|G|$ we have $G = \langle x, y\rangle$. Since G has a non-trivial center, there exists a non-identity element $z \in Z(G)$. By the minimality of $|G|$ and Corollary 2.3, $G/\langle z\rangle$ is an elementary abelian 3-group of rank two. This implies that $|G| = |\langle x, y\rangle| = |G/\langle z\rangle||\langle z\rangle| = 27$, which contradicts Proposition 3.8.

Next we suppose that G is a desirable abelian 3-group. Since there are no element of order lager than 3 in G by Corollary 2.4, G is isomorphic to C_3^m for some nonnegative integer m. □

Lemma 4.5. *Let G be a desirable group and $a \in G$. If a normalizes an elementary abelian 3-group H of G, then $a^{-1}ba \in \{b, b^{-1}\}$ for each $b \in H$.*

Proof. Suppose the contrary, i.e., $a^{-1}ba \notin \{b, b^{-1}\}$ for $a \in N_G(H)$ and $b \in H$. Since $\langle a^{-i}ba^i \mid i = 0, 1, \ldots, |a| - 1\rangle$ is a subgroup of H, it is an elementary abelian 3-group of rank at least two. By Corollary 2.4, we divide our proof according to $|a| \in \{2, 3, 4, 6\}$.

If $|a| = 2$, then $\langle a, b\rangle$ has order 18, which contradicts Corollary 2.3 and Proposition 3.8.

If $|a| = 4$, then

$$\langle b, a^{-1}ba, a^2ba^2, aba^{-1}\rangle$$

is an elementary abelian 3-group contained in H. Note that $\langle b, a^2ba^2\rangle$ is an elementary abelian 3-group of rank at most two. If $a^2ba^2 \notin \{b, b^{-1}\}$, then $\langle a^2, b\rangle$ is a non-abelian group of order 18, which contradicts Proposition 3.8. If $a^2ba^2 = b$, then a^2 is central in $\langle a, b\rangle$, and $a^{-1}ba \notin b\langle a^2\rangle$ by the assumption that $a^{-1}ba \in H$ and $a^2 \notin H$. This implies that $\langle a, b\rangle/\langle a^2\rangle$ is a non-abelian group of order 18, which contradicts Proposition 3.8. If $a^2ba^2 = b^{-1}$, then $\langle a^2, b, a^{-1}ba\rangle$ is a non-abelian group of order 18, which contradicts Proposition 3.8.

If $|a| = 3$, then $\langle a, H\rangle$ is abelian by Lemma 4.4, and hence a centralizes H.

If $|a| = 6$, then $a = cd = dc$ for some $d, c \in \langle a\rangle$ with $|d| = 2$ and $|c| = 3$. Since both d and c normalize $\langle b\rangle$, a also normalizes $\langle b\rangle$. □

It is well-known that a minimal normal subgroup of a finite group is the direct product of isomorphic simple groups (see [6]). Applying this fact with Corollary 2.4 we obtain that any minimal normal subgroup of a desirable group is isomorphic to an elementary abelian p-group for some $p \in \{2, 3\}$.

Lemma 4.6. *Let G be a desirable group and N a minimal normal subgroup of G. If $N \simeq C_2^m$ and there exist $a, b \in G$ with $|a| = |b| = 2$ and $|ab| = 3$, then $m = 1$.*

Proof. We claim that $N \cap \langle a, b\rangle = 1$, otherwise $\langle a, b\rangle$ contains an involution in N. Since all involutions of $\langle a, b\rangle$ are conjugate in $\langle a, b\rangle$, it follows that $\langle a, b\rangle \subseteq N$, a contradiction to $|ab| = 3$ and $N \simeq C_2^m$. By Lemma 4.2, N is central, and hence, if N has a subgroup H of order 4, then, by Lemma 4.2, $|H\langle a, b\rangle| = 24$, which contradicts Proposition 3.8. Therefore, $m = 1$. □

Lemma 4.7. *Let G be a desirable group and N a minimal normal subgroup of G. If $N \simeq C_3^m$ and $a, b \in G$ with $|a| = |b| = 2$ and $|ab| = 3$, then $N = \langle ab\rangle$.*

Proof. Suppose $c \in N\setminus\langle ab\rangle$. By Lemma 4.5, $\langle a, b\rangle$ normalizes $\langle c\rangle$, and hence, $|\langle a, b, c\rangle| = 18$, which contradicts Proposition 3.8. □

Lemma 4.8. *Let G be a desirable group. If G has two non-commuting involutions, then $|G| \in \{6, 12\}$.*

Proof. Use induction on $|G|$. Suppose that G is a desirable group with two non-commuting involutions and $|G|$ is minimal such that $|G| \notin \{6, 12\}$. Note that any two non-commuting involutions generate the dihedral group of degree 3 or 6 by Lemma 4.1, and each of the cases contains two non-commuting involutions whose product has order three. Let N be a minimal normal subgroup of G and $a, b \in G$ such that $|a| = |b| = 2$ and $|ab| = 3$. Recall that N is an elementary abelian p-group for some $p \in \{2, 3\}$. Applying Lemma 4.6 and 4.7 we obtain that $N \simeq C_2$ and $N \cap \langle a, b\rangle = 1$, or $N = \langle ab\rangle$.

If $N \simeq C_2$ and $N \cap \langle a, b\rangle = 1$, then G/N is a desirable group with two non-commuting involutions. By the minimality of $|G|$, $|G/N| \in \{6, 12\}$. Since $|G| \neq 12$ and $|N| = 2$, it follows that $|G/N| = 12$, and hence G is a non-abelian group of order 24, a contradiction to Proposition 3.8.

Suppose $N = \langle ab\rangle$.

We claim that $8 \nmid |G/N|$. Otherwise there exists a subgroup L/N of G/N such that $|L/N| = 8$ and $a, b \in L$ by Sylow's theorem, implying that L is a non-abelian group of order 24, a contradiction to Proposition 3.8.

If G/N has two non-commuting involutions, then $|G/N| \in \{6, 12\}$ by the minimality of $|G|$. By Proposition 3.8, $|G/N| = 12$. Since G/N is not isomorphic to A_4 by Lemma 3.3, G/N has a minimal normal subgroup N_1/N of order 3 by the classification of groups of order 12. Since $|N_1| = 9$, $\langle a, b, N_1\rangle$ is a non-abelian group of order 18 by Lemma 4.5, a contradiction to Proposition 3.8. Therefore, we conclude that all involutions of G/N commute for each other, and the subgroup of G/N generated by all involutions is a normal subgroup of G/N which is an elementary abelian 2-group.

By the last claim, it suffices to show that $3 \nmid |G/N|$. Otherwise, there exists $L/N \leq G/N$ such that $N \leq L$ and $|L/N| = 3$. Since aN is an involution of G/N, it is central by Lemma 4.2. Thus, $\langle aN, L/N\rangle$ is a subgroup of order 6. This implies that $\langle L, a, b\rangle$ is a non-abelian group of order 18, a contradiction to Proposition 3.8. □

Lemma 4.9. *Let G be a non-abelian desirable group any two involutions of which are commute. Then the subgroup of G generated by all involutions is normal in G, and G is isomorphic to $C_3 \rtimes C_4$ unless G is a 2-group.*

Proof. The first statement is obvious. Let K be the subgroup of G generated by all involutions of G. Since G is non-abelian, G is not a 3-group by Lemma 4.4. This implies that K has a subgroup L of order two. By Lemma 4.2, L is central in G.

We use the induction on $|G|$ to prove the second statement. Let G be a non-abelian desirable group of the least order such that all involutions of G commute for each other and G is neither 2-group nor $G \simeq C_3 \rtimes C_4$.

If G/L has two non-commuting involutions, then $|G/L| \in \{6, 12\}$ by Lemma 4.8. By Proposition 3.8, $|G| = 12$, and hence $G \simeq C_3 \rtimes C_4$ by the classification of groups of order 12, a contradiction.

If G/L has no two non-commuting involutions and non-abelian then, by the minimality of $|G|$, $G/L \simeq C_3 \rtimes C_4$ or a 2-group. But, the former case does not occur by Proposition 3.8, and the latter case implies that G is a 2-group, a contradiction.

Suppose that G/L is an abelian group any two involutions of which are commute. We claim that $a \in Z(G)$ for each element $a \in G$ with $|a| = 3$. Otherwise, $ab \neq ba$ for some $b \in G$. Since G/L is abelian, $b^{-1}ab = al$ for a non-identity $l \in L$. Since $l \in Z(G)$ and $|l| = 2$, it follows that $|al| = 6$, which contradicts $|a| = 3$. Applying the claim for an element $c \in G$ of order 6 we obtain from $c = c^4c^3$, $|c^4| = 3$ and $|c^3| = 2$ that each element of order 2, 3 or 6 is in the center of G. This implies that there exist $a, b \in G$ such that $|a| = |b| = 4$ and $ab \neq ba$ since G is non-abelian. Applying Lemma 4.3 we conclude that G has a unique Sylow 2-subgroup, which has a subgroup isomorphic to Q_8. Since G is not a 2-group by the assumption, it follows from the claim that there exists a subgroup of G isomorphic to $C_3 \times Q_8$, a contradiction to Proposition 3.8. □

4.1 Proof of Theorem 1.6

Proof. Suppose that G is a non-abelian desirable group. If G has two non-commuting involutions, then $|G| \in \{6, 12\}$ by Lemma 4.8. If G has no two non-commuting involutions, then $G \simeq C_3 \rtimes C_4$ or a 2-group by Lemma 4.9, which is eliminated by Lemma 4.3. Since all non-abelian desirable groups of order 6 or 12 are known to be S_3 or $C_3 \rtimes C_4$, this completes the proof. □

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Half-arc-transitive graphs of prime-cube order of small valencies*

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Abstract

A graph is called *half-arc-transitive* if its full automorphism group acts transitively on vertices and edges, but not on arcs. It is well known that for any prime p there is no half-arc-transitive graph of order p or p^2. In 1992, Xu classified half-arc-transitive graphs of order p^3 and valency 4. In this paper we classify half-arc-transitive graphs of order p^3 and valency 6 or 8. In particular, the first known infinite family of half-arc-transitive Cayley graphs on non-metacyclic p-groups is constructed.

Keywords: Cayley graph, half-arc-transitive graph, automorphism group.

Math. Subj. Class.: 05C10, 05C25, 20B25

1 Introduction

A (di)graph Γ consists of a pair of sets $(V(\Gamma), E(\Gamma))$, where $V(\Gamma)$ is its *vertex set*, and $E(\Gamma)$ is its *edge set*. For a graph, $E(\Gamma)$ is also called *undirected edge set* and is a subset of the set $\{\{u,v\} \mid u,v \in V(\Gamma)\}$, and for a digraph, $E(\Gamma)$ is also called *directed edge set* and is a subset of the set $\{(u,v) \mid u,v \in V(\Gamma)\}$. For an edge $\{u,v\}$ of a graph Γ, we call (u,v) an *arc* of Γ. An automorphism of a (di)graph Γ is a permutation on $V(\Gamma)$ preserving the adjacency of Γ, and all automorphisms of Γ form a group under the composition of permutations, called the *full automorphism group* of Γ and denoted by $\mathrm{Aut}(\Gamma)$. A (di)graph Γ is *vertex-transitive* or *edge-transitive* if $\mathrm{Aut}(\Gamma)$ acts transitively on $V(\Gamma)$ or $E(\Gamma)$, respectively. A graph Γ is *arc-transitive* or *symmetric* if $\mathrm{Aut}(\Gamma)$ is transitive on the arc set of Γ, and *half-arc-transitive* provided that it is vertex-transitive, edge-transitive, but not arc-transitive. Throughout this paper, all (di)graphs Γ are finite and simple, that is, $|V(\Gamma)|$ is finite and there are no loops or multiple edges.

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Let G be a permutation group on a set Ω and $\alpha \in \Omega$. Denote by G_α the stabilizer of α in G, that is, the subgroup of G fixing the point α. We say that G is semiregular on Ω if $G_\alpha = 1$ for every $\alpha \in \Omega$ and regular if G is transitive and semiregular.

Let G be a finite group and S a subset of G such that $1 \notin S$. The *Cayley digraph* $\Gamma = \mathrm{Cay}(G, S)$ on G with respect to S is defined as the digraph with vertex set $V(\Gamma) = G$ and directed edge set $\{(g, sg) \mid g \in G, s \in S\}$. The Cayley digraph $\mathrm{Cay}(G, S)$ is connected if and only if $G = \langle S \rangle$, and if S is symmetric, that is, $S^{-1} = \{s^{-1} \mid s \in S\} = S$, then $\mathrm{Cay}(G, S)$ can be viewed as a graph by identifying the two oppositely directed edges (g, sg) and (sg, g) as an undirected edge $\{g, sg\}$. Thus a Cayley graph can be viewed as a special case of a Cayley digraph. It is easy to see that $\mathrm{Aut}(\mathrm{Cay}(G, S))$ contains the right regular representation $\hat{G} = \{\hat{g} \mid g \in G\}$ of G, where $\hat{g}$ is the map on G defined by $x \mapsto xg$, $x \in G$, and $\hat{G}$ is regular on the vertex set $V(\Gamma)$. This implies that a Cayley digraph is vertex-transitive. Also, it is easy to check that $\mathrm{Aut}(G, S) = \{\alpha \in \mathrm{Aut}(G) \mid S^\alpha = S\}$ is a subgroup of $\mathrm{Aut}(\mathrm{Cay}(G, S))_1$, the stabilizer of the vertex 1 in $\mathrm{Aut}(\mathrm{Cay}(G, S))$. A Cayley digraph $\mathrm{Cay}(G, S)$ is said to be *normal* if $\hat{G}$ is normal in $\mathrm{Aut}(\mathrm{Cay}(G, S))$.

In 1966, Tutte [26] initiated an investigation of half-arc-transitive graphs by showing that a vertex- and edge-transitive graph with odd valency must be arc-transitive. A few years later, in order to answer Tutte's question of the existence of half-arc-transitive graphs of even valency, Bouwer [5] gave a construction of a $2k$-valent half-arc-transitive graph for every $k \geq 2$. Following these two classical articles, half-arc-transitive graphs have been extensively studied from different perspectives over decades by many authors. See, for example, [3, 13, 15, 18, 20, 23, 25, 33]. One of the standard problems in the study of half-arc-transitive graphs is to classify such graphs of certain orders. Let p be a prime. It is well known that there are no half-arc-transitive graphs of order p or p^2, and by Cheng and Oxley [6], there are no half-arc-transitive graphs of order $2p$. Alspach and Xu [2] classified half-arc-transitive graphs of order $3p$ and Dobson [9] classified half-arc-transitive graphs of order a product of two distinct primes. Classification of half-arc-transitive graphs of order $4p$ had been considered for more than 10 years by many authors, and recently was solved by Kutnar et al. [16]. Despite all of these efforts, however, further classifications of half-arc-transitive graphs with general valencies seem to be very difficult.

In view of the fact that 4 is the smallest admissible valency for a half-arc-transitive graph, special attention has rightly been given to the study of tetravalent half-arc-transitive graphs. In particular, constructing and classifying tetravalent half-arc-transitive graphs is currently an active topic in algebraic graph theory (for example, see [10, 11, 22, 28]). Marušič [20] and Šparl [27] classified tightly attached tetravalent half-arc-transitive graphs with odd and even radius, respectively. For quite some time, all known examples of tetravalent half-arc-transitive graphs had vertex-stabilizers that are either abelian or dihedral: For instance, Marušič [21] constructed an infinite family of tetravalent half-arc-transitive graphs having vertex stabilizers isomorphic to $\mathbb{Z}_2^m$ for each positive integer $m \geq 1$, and Conder and Marušič [7] constructed a tetravalent half-arc-transitive graph with vertex-stabilizer isomorphic to D_4 of order 8. Recently, a tetravalent half-arc-transitive graph with vertex-stabilizers that are neither abelian nor dihedral was constructed by Conder et al. [8].

Xu [31] classified tetravalent half-arc-transitive graphs of order p^3 for each prime p, and later this was extended to the case of p^4 by Feng et al. [10]. In this paper, we classify half-arc-transitive graphs of order p^3 and valency 6 or 8. In these new constructions, there is an infinite family of half-arc-transitive Cayley graphs on non-metacyclic p-groups, and to our best knowledge, this is the first known construction of such graphs.

Denote by $\mathbb{Z}_n$ the cyclic group of order n as well as the ring of order n. From elementary group theory we know that up to isomorphism there are only five groups of order p^3, that is, three abelian groups $\mathbb{Z}_{p^3}$, $\mathbb{Z}_{p^2} \times \mathbb{Z}_p$ and $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$, and two non-abelian groups $G_1(p)$ and $G_2(p)$ defined as

$$G_1(p) = \langle a, b \mid a^{p^2} = 1, b^p = 1, b^{-1}ab = a^{1+p} \rangle$$

and

$$G_2(p) = \langle a, b, c \mid a^p = b^p = c^p = 1, [a, b] = c, [a, c] = [b, c] = 1 \rangle.$$

It is easy to check that the center of $G_1(p)$ is $\langle a^p \rangle$ and the center of $G_2(p)$ is $\langle c \rangle$.

Denote by $\mathbb{Z}_n^*$ the multiplicative group of the ring $\mathbb{Z}_n$ consisting of numbers coprime to n. Let e be an element of order $j < p$ in $\mathbb{Z}_{p^2}^*$. Since $\mathbb{Z}_{p^2}^* \cong \mathbb{Z}_{p(p-1)}$, we have $j \mid (p-1)$. For each $k \in \mathbb{Z}_p^*$, let $T^{j,k} = \{b^k a, b^k a^e, \ldots, b^k a^{e^{j-1}}, (b^k a)^{-1}, (b^k a^e)^{-1}, \ldots, (b^k a^{e^{j-1}})^{-1}\}$ be a subset of $G_1(p)$ and define

$$\Gamma^{j,k}(p) = \mathrm{Cay}(G_1(p), T^{j,k}).$$

By Proposition 2.2, $\Gamma^{j,k}(p)$ does not depend on the choice of the element e of order j.

Suppose $4 \mid (p-1)$ and let λ be an element of order 4 in $\mathbb{Z}_p^*$. For each $k \in \mathbb{Z}_p$ with $k \neq 2^{-1}(1+\lambda)$, let $S_{4,k} = \{a, b, a^\lambda b^{\lambda-1} c^k, a^{-\lambda-1} b^{-\lambda} c^{1-k}, a^{-1}, b^{-1}, (a^\lambda b^{\lambda-1} c^k)^{-1}, (a^{-\lambda-1} b^{-\lambda} c^{1-k})^{-1}\}$ be a subset of $G_2(p)$ and define

$$\Gamma_{4,k}(p) = \mathrm{Cay}(G_2(p), S_{4,k}).$$

There are exactly two elements of order 4 in $\mathbb{Z}_p^*$, that is, λ and $\lambda^{-1} = -\lambda$. Let

$$\overline{S}_{4,s} = \{a, b, a^{-\lambda} b^{-\lambda-1} c^s, a^{\lambda-1} b^\lambda c^{1-s}, a^{-1}, b^{-1}, (a^{-\lambda} b^{-\lambda-1} c^s)^{-1}, (a^{\lambda-1} b^\lambda c^{1-s})^{-1}\},$$

where $s \in \mathbb{Z}_p$ and $s \neq 2^{-1}(1-\lambda)$. For each $k \in \mathbb{Z}_p$ and $k \neq 2^{-1}(1+\lambda)$, the automorphism of $G_2(p)$ induced by $a \mapsto a$, $b \mapsto a^{\lambda-1} b^\lambda c^{1-k+\lambda}$, $c \mapsto c^\lambda$, maps $S_{4,k}$ to $\overline{S}_{4,k-\lambda}$, and so $\mathrm{Cay}(G_2(p), S_{4,k}) \cong \mathrm{Cay}(G_2(p), \overline{S}_{4,k-\lambda})$. Since $k \neq 2^{-1}(1+\lambda)$, we have $k - \lambda \neq 2^{-1}(1-\lambda)$. Thus, $\Gamma_{4,k}(p)$ does not depend on the choice of λ. The following is the main result of the paper.

Theorem 1.1. *Let Γ be a graph of order p^3 for a prime p. Then we have:*

(1) *If Γ has valency 6 then Γ is half-arc-transitive if and only if $3 \mid (p-1)$ and $\Gamma \cong \Gamma^{3,k}(p)$. There are exactly $(p-1)/2$ nonisomorphic half-arc-transitive graphs of the form $\Gamma^{3,k}(p)$;*

(2) *If Γ has valency 8 then Γ is half-arc-transitive if and only if $4 \mid (p-1)$ and $\Gamma \cong \Gamma^{4,k}(p)$ or $\Gamma_{4,k}(p)$. There are exactly $p-1$ nonisomorphic half-arc-transitive graphs of the forms $\Gamma^{4,k}(p)$ and $\Gamma_{4,k}(p)$, with $(p-1)/2$ such graphs in each form.*

2 Preliminaries

We start by stating some group-theoretical results. For a group G and $x, y \in G$, denote by $[x, y]$ the commutator $x^{-1}y^{-1}xy$ and by x^y the conjugation $y^{-1}xy$. The following proposition is a basic property of commutators and its proof is straightforward (also see [24, Subsection 5.1.5]):

Proposition 2.1 ([14, Kapitel III, Hilfssätze 1.2 and 1.3]). *Let G be a group. Then, for any $x, y, z \in G$, we have $[x, y] = [y, x]^{-1}$, $[xy, z] = [x, z]^y [y, z]$ and $[x, yz] = [x, z][x, y]^z$. Furthermore, if $[x, y]$ commutes with x and y then for any integers i and j, $[x^i, y^j] = [x, y]^{ij}$, and for any positive integer n, $(xy)^n = x^n y^n [y, x]^{\binom{n}{2}}$.*

We remark that it is easy to see that the equality $(xy)^n = x^n y^n [y, x]^{\binom{n}{2}}$ holds also for negative integers n if we define $\binom{n}{2} = \frac{n(n-1)}{2}$. By Li and Sim [18, Theorem 1.1 and Lemma 2.6], we have the following proposition.

Proposition 2.2. *Let Γ be a Cayley graph on $G_1(p)$ of valency $2j$ with $1 < j < p$. Then Γ is half-arc-transitive if and only if $j \big| (p-1)$ and $\Gamma \cong \Gamma^{j,k}(p)$ for $1 \leq k \leq p-1$, and $\Gamma^{j,k}(p) \cong \Gamma^{j',k'}(p)$ if and only if $j = j'$ and $k = k'$ (mod p). Furthermore, for each $j \big| (p-1)$ there exist exactly $(p-1)/2$ nonisomorphic such graphs of the form $\Gamma^{j,k}(p)$.*

Since a transitive permutation group of prime degree p has a regular Sylow p-subgroup, every vertex-transitive digraph of order a prime must be a Cayley digraph. Together with the results given by Marušič [19], we have the following proposition.

Proposition 2.3. *Any vertex-transitive digraph of order p^k with $1 \leq k \leq 3$ is a Cayley digraph on a group of order p^k.*

For any abelian group H, the map $h \mapsto h^{-1}, h \in H$ is an automorphism of H. By [10, Proposition 2.10], we have the following proposition.

Proposition 2.4. *Let G be a finite group and $\mathrm{Cay}(G, S)$ a connected half-arc-transitive Cayley graph. Then, S does not contain an involution and for any $s \in S$, there is no $\alpha \in \mathrm{Aut}(G, S)$ satisfying $s^\alpha = s^{-1}$. Furthermore, every edge-transitive Cayley graph on an abelian group is also arc-transitive.*

The following proposition is about isomorphisms between Cayley graphs on p-groups.

Proposition 2.5 ([17, Theorem 1.1 (3)]). *Let $\mathrm{Cay}(G, S)$ and $\mathrm{Cay}(G, T)$ be two connected Cayley graphs on a p-group G with respect to subsets S and T, and let $|S| = |T| < 2p$. Then $\mathrm{Cay}(G, S)$ and $\mathrm{Cay}(G, T)$ are isomorphic if and only if there is an automorphism α of G such that $S^\alpha = T$.*

Let $\Gamma = \mathrm{Cay}(G, S)$ be a Cayley digraph on a finite group G. By Godsil [12, Lemma 2.2] (also see [32, Proposition 1.5]), we have $\mathrm{N}_{\mathrm{Aut}\Gamma}(\hat{G}) = \hat{G} \rtimes \mathrm{Aut}(G, S)$.

Proposition 2.6. *A Cayley digraph $\Gamma = \mathrm{Cay}(G, S)$ is normal if and only if $\mathrm{Aut}(\Gamma)_1 = \mathrm{Aut}(G, S)$.*

A finite group G is called 2-genetic if each normal subgroup of G can be generated by two elements. For a prime p, denote by $\mathrm{O}_p(G)$ the largest normal p-subgroup of G, and by $\Phi(G)$ the *Frattini subgroup* of G, that is, the intersection of all maximal subgroups of G. We call G a *p'-group* if the order of G is not divisible by p. The following proposition is about automorphism groups of Cayley digraphs on 2-genetic groups.

Proposition 2.7 ([30, Theorem 1.1]). *Let G be a nonabelian 2-genetic group of order p^n for an odd prime p and a positive integer n, and let $\Gamma = \mathrm{Cay}(G, S)$ be a connected Cayley digraph. Assume that $\mathrm{Aut}(G, S)$ is a p'-group and Γ is non-normal. Then $p \in \{3, 5, 7, 11\}$*

and $\mathrm{ASL}(2,p) \leq \mathrm{Aut}(\Gamma)/\Phi(\mathrm{O}_p(A)) \leq \mathrm{AGL}(2,p)$. *Furthermore, the kernel of* $A := \mathrm{Aut}(\Gamma)$ *acting on the quotient digraph* $\Gamma_{\Phi(\mathrm{O}_p(A))}$ *is* $\Phi(\mathrm{O}_p(A))$, *and one of the following happens:*

(1) $p = 3$, $n \geq 5$, *and* $\Gamma_{\Phi(\mathrm{O}_p(A))}$ *has out-valency at least* 8*;*

(2) $p = 5$, $n \geq 3$ *and* $\Gamma_{\Phi(\mathrm{O}_p(A))}$ *has out-valency at least* 24*;*

(3) $p = 7$, $n \geq 3$ *and* $\Gamma_{\Phi(\mathrm{O}_p(A))}$ *has out-valency at least* 48*;*

(4) $p = 11$, $n \geq 3$ *and* $\Gamma_{\Phi(\mathrm{O}_p(A))}$ *has out-valency at least* 120.

In Proposition 2.7, the quotient digraph $\Gamma_{\Phi(\mathrm{O}_p(A))}$ has the orbits of $\Phi(\mathrm{O}_p(A))$ on $V(\Gamma)$ as vertices, and for two orbits O_1 and O_2, (O_1, O_2) is a directed edge in $\Gamma_{\Phi(\mathrm{O}_p(A))}$ if and only if (u, v) is a directed edge in Γ for some $u \in O_1$ and $v \in O_2$.

3 Proof of Theorem 1.1

Let Γ be a half-arc-transitive graph and $A = \mathrm{Aut}(\Gamma)$. Let (u, v) be an arc of Γ and set $(u, v)^A = \{(u^a, v^a) \mid a \in A\}$. Define digraphs Γ_1 and Γ_2 having vertex set $V(\Gamma)$ and directed edge sets $(u, v)^A$ and $(v, u)^A$, respectively. Since Γ is half-arc-transitive, for every edge $\{x, y\} \in E(\Gamma)$, each of Γ_1 and Γ_2 contains exactly one of the directed edges (x, y) and (y, x), and Γ is connected if and only if Γ_i is connected for each $i = 1, 2$. Furthermore, $A = \mathrm{Aut}(\Gamma_i)$ and Γ_i is A-edge-transitive. In what follows we denote by $\overrightarrow{\Gamma}$ one of the digraphs Γ_1 and Γ_2.

Let Γ be a half-arc-transitive graph of order p^3 for a prime p. Since there exists no half-arc-transitive graph of order less than 27 (see [1]), we have $p \geq 3$. By Proposition 2.3, $\Gamma = \mathrm{Cay}(G, S)$ and $\overrightarrow{\Gamma} = \mathrm{Cay}(G, R)$. Since a group of order p or p^2 is abelian and there is no half-arc-transitive Cayley graph on an abelian group by Proposition 2.4, Γ is connected, and so $G = \langle R\rangle$ and $S = R \cup R^{-1}$. Furthermore, $G = G_1(p)$ or $G_2(p)$, where

$$G_1(p) = \langle a, b \mid a^{p^2} = 1, b^p = 1, b^{-1}ab = a^{1+p}\rangle,$$
$$G_2(p) = \langle a, b, c \mid a^p = b^p = c^p = 1, [a, b] = c, [a, c] = [b, c] = 1\rangle.$$

Since $G = \langle R\rangle$ is non-abelian, R contains two elements x and y such that $xy \neq yx$, and since $|G| = p^3$, we have $\langle x, y\rangle = G$. For $G = G_2(p)$, x and y have the same relations as do a and b, which implies that we may assume that $a, b \in R$. Similarly, for $G = G_1(p)$ we may assume that $a \in R$. Thus we have the following observation:

Observation 3.1. *Let* Γ *be a half-arc-transitive graph of order* p^3 *for a prime* p*. Then* $\Gamma = \mathrm{Cay}(G, S)$ *and* $\overrightarrow{\Gamma} = \mathrm{Cay}(G, R)$, *where* $G = G_1(p)$ *or* $G_2(p)$ *with* $p \geq 3$, $G = \langle R\rangle$ *and* $S = R \cup R^{-1}$. *Furthermore,*

(1) *if* $G = G_1(p)$ *then* $a \in R$*;*

(2) *if* $G = G_2(p)$ *then* $a, b \in R$*.*

Let us begin by considering normal half-arc-transitive Cayley graphs on $G_2(p)$ of valency 8. Since $G_2(p)$ has center $\langle c\rangle$, Proposition 2.1 implies $b^j a^i = a^i b^j c^{-ij}$ and $(a^i b^j)^k = a^{ki} b^{kj} c^{-2^{-1}k(k-1)ij}$ for $i, j, k \in \mathbb{Z}_p$. Our proofs will constantly be relying on these facts.

Lemma 3.2. *Let $\Gamma = \mathrm{Cay}(G_2(p), S)$ be a Cayley graph of valency 8. Then Γ is normal and half-arc-transitive if and only if $4 \big| (p-1)$ and $\Gamma \cong \Gamma_{4,k}(p)$ for some k.*

Proof. Let $\Gamma = \mathrm{Cay}(G_2(p), S)$ be normal and half-arc-transitive. Set $A = \mathrm{Aut}(\Gamma)$. By Observation 3.1, we have $\overrightarrow{\Gamma} = \mathrm{Cay}(G_2(p), R)$ with $p \geq 3$, $G_2(p) = \langle R \rangle$ and $S = R \cup R^{-1}$. We may further assume $a, b, a^i b^j c^k \in R$. Since Γ has valency 8, we have $|S| = 8$ and $|R| = 4$. Since Γ is normal, Proposition 2.6 implies that $A_1 = \mathrm{Aut}(G_2(p), S) = \mathrm{Aut}(G_2(p), R)$, which is transitive on R. Since $|R| = 4$, $\mathrm{Aut}(G_2(p), R) \leq S_4$. Thus, $\mathrm{Aut}(G_2(p), R)$ has a regular subgroup M on R such that $M \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_4$.

Case 1: $M \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Let $\alpha_1, \alpha_2 \in \mathrm{Aut}(G_2(p), R)$ and $M = \langle \alpha_1 \rangle \times \langle \alpha_2 \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Without loss of generality, we may assume that $a^{\alpha_1} = b$ and $b^{\alpha_1} = a$, and so $c^{\alpha_1} = c^{-1}$. This yields that $R = \{a, b, a^i b^j c^k, (a^i b^j c^k)^{\alpha_1}\} = \{a, b, a^i b^j c^k, a^j b^i c^{-ij-k}\}$. Since $\langle \alpha_1, \alpha_2 \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, we may assume that $a^{\alpha_2} = a^i b^j c^k$ and $(a^i b^j c^k)^{\alpha_2} = a$. Then $b^{\alpha_2} = a^j b^i c^{-ij-k}$ and so $c^{\alpha_2} = c^{i^2-j^2}$. By Proposition 2.1,

$$\begin{aligned} a = (a^i b^j c^k)^{\alpha_2} &= (a^i b^j c^k)^i (a^j b^i c^{-ij-k})^j (c^{i^2-j^2})^k = (a^i b^j)^i (a^j b^i)^j c^{k(i^2-j^2+i-j)-ij^2} \\ &= a^{i^2} b^{ij} c^{-2^{-1} i^2 j(i-1)} a^{j^2} b^{ij} c^{-2^{-1} ij^2 (j-1)} c^{k(i^2-j^2+i-j)-ij^2} \\ &= a^{i^2+j^2} b^{2ij} c^{-ij^3+k(i^2-j^2+i-j)-ij^2-2^{-1}i^2 j(i-1)-2^{-1}ij^2(j-1)}, \end{aligned}$$

implying the following equations:

$$i^2 + j^2 = 1; \tag{3.1}$$

$$2ij = 0; \tag{3.2}$$

$$-ij^3 + k(i^2 - j^2 + i - j) - ij^2 - 2^{-1} i^2 j(i-1) - 2^{-1} ij^2 (j-1) = 0. \tag{3.3}$$

As above, in what follows all equations are considered in $\mathbb{Z}_p$, unless otherwise stated. Since α_1 interchanges a and b, we can assume $i = 0$ by Eq. (3.2), and so $j = \pm 1$ by Eq. (3.1). If $j = -1$ then $S = \{a, b, a^{-1} c^{-k}, b^{-1} c^k\} \cup \{a^{-1}, b^{-1}, ac^k, bc^{-k}\}$, and the automorphism of $G_2(p)$ induced by $a \mapsto a^{-1}$, $b \mapsto bc^{-k}$, $c \mapsto c^{-1}$, fixes S setwise, contrary to Proposition 2.4. If $j = 1$ then $k = 0$ by Eq. (3.3), implying that $a^i b^j c^k = b$, a contradiction.

Case 2: $M \cong \mathbb{Z}_4$. Let $\alpha \in \mathrm{Aut}(G_2(p), R)$ and $M = \langle \alpha \rangle \cong \mathbb{Z}_4$. Then $R = \{a, a^\alpha, a^{\alpha^2}, a^{\alpha^3}\}$, and since $G_2(p) = \langle R \rangle$, we have $\langle a, a^\alpha \rangle = G_2(p)$ and so $a \mapsto a$, $b \mapsto a^\alpha$ induces an automorphism of $G_2(p)$. We thus assume that $a^\alpha = b$, and α is induced by $a \mapsto b$, $b \mapsto a^i b^j c^k$, $c \mapsto c^{-i}$. It follows that

$$R = \{a, b, a^i b^j c^k, a^{ij} b^{i+j^2} c^{-i^2 j + k(j-i) - 2^{-1} ij^2 (j-1)}\}.$$

Since

$$\begin{aligned} a = a^{\alpha^4} &= a^{i(i+j^2)} b^{j(2i+j^2)} c^{i^3 j + (k - i^2 j)(i+j^2) - ik(j-i) + 2^{-1} i^2 j^2 (j-1) - 2^{-1} ij(i+j^2)(i+j^2-1)} \\ &= a^{i(i+j^2)} b^{j(2i+j^2)} c^{k(i^2+j^2+i-ij) - i^2 j^3 + 2^{-1} ij[ij^2 - ij - (i+j^2)(i+j^2-1)]}, \end{aligned}$$

we have the following equations:

$$i(i + j^2) = 1; \tag{3.4}$$

$$j(2i + j^2) = 0; \tag{3.5}$$

$$k(i^2 + j^2 + i - ij) - i^2 j^3 + 2^{-1} ij[ij^2 - ij - (i+j^2)(i+j^2-1)] = 0. \tag{3.6}$$

By Eq. (3.5), either $j = 0$ or $2i + j^2 = 0$.

Case 2.1: $j = 0$. By Eq. (3.4), $i = \pm 1$. If $i = 1$ then $k = 0$ by Eq. (3.6), and hence $a^i b^j c^k = a$, a contradiction. If $i = -1$ then $S = \{a, b, a^{-1}c^k, b^{-1}c^k\} \cup \{a^{-1}, b^{-1}, ac^{-k}, bc^{-k}\}$, and the automorphism of $G_2(p)$ induced by $a \mapsto a^{-1}$, $b \mapsto bc^{-k}$, $c \mapsto c^{-1}$, fixes S setwise, contrary to Proposition 2.4.

Case 2.2: $2i + j^2 = 0$. Clearly, $i + j^2 = -i$. By Eq. (3.4), $i^2 = -1$ and so $ij^2 = 1 - i^2 = 2$. Since $j^2 = -2i$, Eq. (3.6) implies $2k(1+i+ij) = -ij(1+i+ij)$, and hence $2ki(1 + i + ij) = j(1 + i + ij)$, implying $2ki - j = 0$ or $1 + i + ij = 0$.

Suppose $2ki - j = 0$. Then $ij = -2k$ and $k = -2^{-1}ij$. Since $ij^2 = 2$, we have $k = -j^{-1}$ and $k(j-i)+1 = -k(j+i)-1$. It follows that $S = \{a, b, a^i b^j c^k, a^{-2k} b^{-i} c^{k(j-i)+1}\} \cup \{a^{-1}, b^{-1}, a^{-i} b^{-j} c^k, a^{2k} b^i c^{-k(j+i)-1}\}$. The automorphism of $G_2(p)$ induced by $a \mapsto a^{-1}$, $b \mapsto b^{-1}$, $c \mapsto c$, fixes S setwise, contrary to Proposition 2.4.

Thus, $1 + i + ij = 0$ and so $j = i - 1$. Then $S = \{a, b, a^i b^{i-1} c^k, a^{-i-1} b^{-i} c^{1-k}\} \cup \{a^{-1}, b^{-1}, a^{-i} b^{1-i} c^{-k+1+i}, a^{i+1} b^i c^{k-i}\}$. If $k = 2^{-1}(1 + i)$, then $k = -k + 1 + i$ and $1 - k = k - i$, and hence the automorphism of $G_2(p)$ induced by $a \mapsto a^{-1}$, $b \mapsto b^{-1}$, $c \mapsto c$, fixes S setwise, contrary to Proposition 2.4. Hence $k \neq 2^{-1}(1 + i)$. Note that $i^2 = -1$ implies that $4 \mid (p - 1)$ and $i = \lambda$ is an element of order 4 in $\mathbb{Z}_p^*$. Then $j = \lambda - 1$ and $k \neq 2^{-1}(1 + \lambda)$. By the definition of $\Gamma_{4,k}(p)$ before Theorem 1.1, $\Gamma \cong \Gamma_{4,k}(p)$.

To finish the proof, we only need to show that $\Gamma_{4,k}(p) = \mathrm{Cay}(G_2(p), S_{4,k})$ is normal and half-arc-transitive. Note that $S_{4,k} = \{a,\ a^{-1},\ b,\ b^{-1},\ a^\lambda b^{\lambda-1} c^k,\ a^{-\lambda} b^{1-\lambda} c^{-k+\lambda+1}, a^{-\lambda-1} b^{-\lambda} c^{1-k},\ a^{\lambda+1} b^\lambda c^{k-\lambda}\}$, λ is an element of order 4 in $\mathbb{Z}_p^*$, and $k \neq 2^{-1}(1 + \lambda)$. Let $A = \mathrm{Aut}(\Gamma_{4,k}(p))$ and set $R_{4,k} = \{a, b, a^\lambda b^{\lambda-1} c^k, a^{-\lambda-1} b^{-\lambda} c^{1-k}\}$. Then $S_{4,k} = R_{4,k} \cup R_{4,k}^{-1}$. Let α be the automorphism of $G_2(p)$ induced by $a \mapsto b$, $b \mapsto a^\lambda b^{\lambda-1} c^k$, $c \mapsto c^{-\lambda}$. By Proposition 2.1, $(a^\lambda b^{\lambda-1} c^k)^\alpha = a^{-\lambda-1} b^{-\lambda} c^{1-k}$ and $(a^{-\lambda-1} b^{-\lambda} c^{1-k})^\alpha = a$. Thus, $\alpha \in \mathrm{Aut}(G_2(p), S_{4,k})$ has order 4 and permutes the elements of $R_{4,k}$ cyclically, implying that $\hat{G}_2(p) \rtimes \langle\alpha\rangle$ is half-arc-transitive on $\Gamma_{4,k}(p)$. To prove the normality and the half-arc-transitivity of $\Gamma_{4,k}(p)$, it suffices to show that $A = \hat{G}_2(p) \rtimes \langle\alpha\rangle$.

Write $L = \mathrm{Aut}(G_2(p), S_{4,k})$. Then L acts on $S_{4,k}$ faithfully. Set $\Omega_1 = \{a, a^{-1}\}$, $\Omega_2 = \{b, b^{-1}\}$, $\Omega_3 = \{a^\lambda b^{\lambda-1} c^k, a^{-\lambda} b^{1-\lambda} c^{-k+\lambda+1}\}$, $\Omega_4 = \{a^{-\lambda-1} b^{-\lambda} c^{1-k}, a^{\lambda+1} b^\lambda c^{k-\lambda}\}$. Since $L \leq \mathrm{Aut}(G_2(p))$, $\{\Omega_1, \Omega_2, \Omega_3, \Omega_4\}$ is a complete imprimitive block system of L on $S_{4,k}$. Let $\Omega = \{\Omega_1, \Omega_2, \Omega_3, \Omega_4\}$. Since $\alpha \in L$, L is transitive on Ω.

Claim: $L_a = 1$ and $L_x = 1$ for any $x \in S_{4,k}$. Let $\beta \in L_a$. Then $a^\beta = a$ and $\Omega_1^\beta = \Omega_1$. Thus, $(\Omega_2 \cup \Omega_3 \cup \Omega_4)^\beta = \Omega_2 \cup \Omega_3 \cup \Omega_4$, and so $b^\beta \in \Omega_2 \cup \Omega_3 \cup \Omega_4$, that is, $b^\beta = b$, b^{-1}, $a^\lambda b^{\lambda-1} c^k$, $a^{-\lambda} b^{1-\lambda} c^{-k+\lambda+1}$, $a^{-\lambda-1} b^{-\lambda} c^{1-k}$ or $a^{\lambda+1} b^\lambda c^{k-\lambda}$. As λ is an element of order 4 in $\mathbb{Z}_p^*$, we have $\lambda \neq 0, \pm 1$. If $b^\beta = b^{-1} \in \Omega_2$ then $c^\beta = c^{-1}$ and $\Omega_2^\beta = \Omega_2$. It follows that $(\Omega_3 \cup \Omega_4)^\beta = \Omega_3 \cup \Omega_4$, implying that $(a^\lambda b^{\lambda-1} c^k)^\beta = a^\lambda b^{1-\lambda} c^{-k} \in \Omega_3 \cup \Omega_4$, which is impossible. If $b^\beta = a^\lambda b^{\lambda-1} c^k \in \Omega_3$ then $c^\beta = c^{\lambda-1}$ and $\Omega_2^\beta = \Omega_3$. Thus, $\Omega_3^\beta \subseteq \Omega_2 \cup \Omega_4$, but $(a^\lambda b^{\lambda-1} c^k)^\beta = a^{-1} b^{-2\lambda} c^{-\lambda+2+2k(\lambda-1)} \notin \Omega_2 \cup \Omega_4$, a contradiction. If $b^\beta = a^{-\lambda} b^{1-\lambda} c^{-k+\lambda+1} \in \Omega_3$, then $c^\beta = c^{1-\lambda}$ and $\Omega_2^\beta = \Omega_3$. Thus, $\Omega_3^\beta \subseteq \Omega_2 \cup \Omega_4$ and $(a^\lambda b^{\lambda-1} c^k)^\beta = a^{2\lambda+1} b^{2\lambda} c^{-\lambda-2k(\lambda-1)}$ implies that $(a^\lambda b^{\lambda-1} c^k)^\beta = b^{-1}$. It follows that $\Omega_3^\beta = \Omega_2$ and so $\Omega_4^\beta = \Omega_4$, which is impossible because $(a^{\lambda+1} b^\lambda c^{k-\lambda})^\beta = a^{\lambda+2} b^{\lambda+1} c^{-2k\lambda+k-\lambda-2} \notin \Omega_4$. If $b^\beta = a^{-\lambda-1} b^{-\lambda} c^{1-k} \in \Omega_4$, then $c^\beta = c^{-\lambda}$ and $\Omega_2^\beta = \Omega_4$. Thus, $\Omega_3^\beta \subseteq \Omega_2 \cup \Omega_3$ and $(a^\lambda b^{\lambda-1} c^k)^\beta = a^{\lambda+2} b^{\lambda+1} c^{-2k\lambda+k-\lambda-2}$ implies that $(a^\lambda b^{\lambda-1} c^k)^\beta = b^{-1}$ and $\lambda = -2$. It follows that $\Omega_3^\beta = \Omega_2$ and so $\Omega_4^\beta = \Omega_3$. This forces $\lambda = 2$ as $(a^{\lambda+1} b^\lambda c^{k-\lambda})^\beta = a^2 b c^{-2k\lambda+\lambda-2} \in \Omega_3$, and hence $\lambda = 2 = -2$, a contradiction. If $b^\beta = a^{\lambda+1} b^\lambda c^{k-\lambda} \in \Omega_4$ then $c^\beta = c^\lambda$ and $\Omega_2^\beta = \Omega_4$. Thus, $\Omega_3^\beta \subseteq \Omega_2 \cup \Omega_3$

and so $(a^{\lambda}b^{\lambda-1}c^k)^\beta = a^{\lambda-2}b^{-\lambda-1}c^{2k\lambda-k-\lambda} \in \Omega_2 \cup \Omega_3$, which is impossible. The above arguments mean that $b^\beta = b$, implying $\beta = 1$. Thus, $L_a = 1$, and since Ω_1 is a block of L, we have $L_{a^{-1}} \leq L_a = 1$. The transitivity of $\langle\alpha\rangle$ on Ω implies $L_x = 1$ for any $x \in S_{4,k}$, as claimed.

Let K be the kernel of L on Ω. Then K fixes each Ω_i setwise, and by Claim, $|K| = |K_a||a^K| \leq 2$. Suppose $|K| = 2$. Then the unique involution, say γ, in K interchanges the two elements in each Ω_i because $L_x = 1$. In particular, γ is induced by $a^\gamma = a^{-1}$, $b^\gamma = b^{-1}$ and $c^\gamma = c$. It follows that $(a^\lambda b^{\lambda-1}c^k)^\gamma = a^{-\lambda}b^{1-\lambda}c^k$, and since $(a^\lambda b^{\lambda-1}c^k)^\gamma \in \Omega_3$, we have $a^{-\lambda}b^{1-\lambda}c^k = a^{-\lambda}b^{1-\lambda}c^{-k+\lambda+1}$, forcing that $k = 2^{-1}(1+\lambda)$, a contradiction. Thus, $K = 1$ and $L \leq S_4$, the symmetric group of degree 4.

Since $L_x = 1$ for any $x \in S_{4,k}$, L is semiregular on $S_{4,k}$, and so $|L|$ is a divisor of 8. Since $\alpha \in L$, we have $|L| = 4$ or 8. Suppose $|L| = 8$. Since $L \leq S_4$, L is the dihedral group of order 8, and so $\alpha^2 \in Z(L)$. Note that α^2 interchanges Ω_1 and Ω_3, and Ω_2 and Ω_4. Then $L_{\Omega_1} = L_{\Omega_1}^{\alpha^2} = L_{\Omega_3}$. Since L is transitive on Ω, $|L_{\Omega_1}| = 2$. Let δ be the unique involution in L_{Ω_1}. Then $\Omega_1^\delta = \Omega_1$ and $\Omega_3^\delta = \Omega_3$. Since $L_a = 1$, we have $a^\delta = a^{-1}$, and since $K = 1$, we have $\Omega_2^\delta = \Omega_4$. On the other hand, $\langle\alpha\rangle \trianglelefteq L$ and so $R_{4,k}$ is an imprimitive block of L, yielding $R_{4,k}^\delta = R_{4,k}^{-1}$. It follows that $b^\delta \in \Omega_2^\delta \cap R_{4,k}^\delta = \Omega_4 \cap R_{4,k}^{-1}$, that is, $b^\delta = a^{\lambda+1}b^\lambda c^{k-\lambda}$. Thus, $c^\delta = c^{-\lambda}$, and since δ is an involution, $b = (a^{\lambda+1}b^\lambda c^{k-\lambda})^\delta = a^{-2}b^{-1}c^{-1}$, which is impossible. Thus, $|L| = 4$ and $L = \langle\alpha\rangle$. Clearly, $p \nmid |L| = |\mathrm{Aut}(G_2(p), S_{4,k})|$. By Proposition 2.7, $\Gamma_{4,k}(p)$ is a normal Cayley graph, and by Proposition 2.6, $A = \hat{G}_2(p) \rtimes \langle\alpha\rangle$. □

Remark 3.3. The above proof implies $\mathrm{Aut}(\Gamma_{4,k}(p)) = \hat{G}_2(p) \rtimes \langle\alpha\rangle$ and $\mathrm{Aut}(\Gamma_{4,k}(p))_1 = \mathrm{Aut}(G_2(p), S_{4,k}) = \langle\alpha\rangle$, where α is the automorphism of $G_2(p)$ of order 4 induced by $a \mapsto b$ and $b \mapsto a^\lambda b^{\lambda-1}c^k$. Moreover, the automorphism α cyclically permutes the elements in $\{a, b, a^\lambda b^{\lambda-1}c^k, a^{-\lambda-1}b^{-\lambda}c^{1-k}\}$.

Lemma 3.4. *There are exactly $(p-1)/2$ nonisomorphic graphs of the form $\Gamma_{4,k}(p)$.*

Proof. By definition, $S_{4,k} = \{a, a^{-1}, b, b^{-1}, a^\lambda b^{\lambda-1}c^k, a^{-\lambda}b^{1-\lambda}c^{\lambda-k+1}, a^{-\lambda-1}b^{-\lambda}c^{1-k}, a^{\lambda+1}b^\lambda c^{k-\lambda}\}$ and $\Gamma_{4,k}(p) = \mathrm{Cay}(G_2(p), S_{4,k})$, where λ is an element of order 4 in $\mathbb{Z}_p^*$, $k \in \mathbb{Z}_p$ with $k \neq 2^{-1}(1+\lambda)$. Thus, $4 \mid (p-1)$. Since $\Gamma_{4,k}(p)$ does not depend on the choice of λ (see the paragraph before Theorem 1.1), there are at most $p-1$ nonisomorphic half-arc-transitive graphs of the form $\Gamma_{4,k}(p)$ ($k \neq 2^{-1}(1+\lambda)$). To finish the proof, it suffices to show that $\Gamma_{4,k}(p) \cong \Gamma_{4,l}(p)$ ($k, l \neq 2^{-1}(1+\lambda)$) if and only if $l = k$ or $l = 1+\lambda-k$.

Let $l = 1+\lambda-k$. One may easily show that the automorphism of $G_2(p)$ induced by $a \mapsto a^{-1}$, $b \mapsto b^{-1}$, $c \mapsto c$, maps $S_{4,k}$ to $S_{4,1+\lambda-k} = S_{4,l}$, and so $\Gamma_{4,k}(p) \cong \Gamma_{4,l}(p)$.

Let $\Gamma_{4,k}(p) \cong \Gamma_{4,l}(p)$ ($k, l \neq 2^{-1}(1+\lambda)$). Set

$$R_{4,i} = \{a, b, a^\lambda b^{\lambda-1}c^i, a^{-\lambda-1}b^{-\lambda}c^{1-i}\}.$$

Then $S_{4,k} = R_{4,k} \cup R_{4,k}^{-1}$ and $S_{4,l} = R_{4,l} \cup R_{4,l}^{-1}$. Since $4 \mid (p-1)$, we have $p \geq 5$, and by Proposition 2.5, there exists $\sigma \in \mathrm{Aut}(G_2(p))$ such that $S_{4,k}^\sigma = S_{4,l}$. This implies that σ maps the stabilizer $\mathrm{Aut}(\Gamma_{4,k}(p))_1$ to the stabilizer $\mathrm{Aut}(\Gamma_{4,l}(p))_1$. It follows $\mathrm{Aut}(G_2(p), S_{4,k})^\sigma = \mathrm{Aut}(G_2(p), S_{4,l})$ because $\mathrm{Aut}(\Gamma_{4,k}(p))_1 = \mathrm{Aut}(G_2(p), S_{4,k})$ and $\mathrm{Aut}(\Gamma_{4,l}(p))_1 = \mathrm{Aut}(G_2(p), S_{4,l})$ by Remark 3.3. Moreover, $\mathrm{Aut}(G_2(p), S_{4,k})$ is regular on both $R_{4,k}$ and $R_{4,k}^{-1}$, and $\mathrm{Aut}(G_2(p), S_{4,l})$ is regular on both $R_{4,l}$ and $R_{4,l}^{-1}$.

Thus, $R_{4,k}^{\sigma} = R_{4,l}$ or $R_{4,l}^{-1}$, and replacing σ by a multiplication of σ and an element in $\mathrm{Aut}(G_2(p), S_{4,l})$, we may assume that $a^{\sigma} = a$ if $R_{4,k}^{\sigma} = R_{4,l}$, and $a^{\sigma} = a^{-1}$ if $R_{4,k}^{\sigma} = R_{4,l}^{-1}$.

Assume $R_{4,k}^{\sigma} = R_{4,l}$ with $a^{\sigma} = a$. Then $b^{\sigma} \in R_{4,l}$ and $b^{\sigma} = b$, $a^{\lambda}b^{\lambda-1}c^{l}$ or $a^{-\lambda-1}b^{-\lambda}c^{1-l}$. If $b^{\sigma} = a^{\lambda}b^{\lambda-1}c^{l}$ then $c^{\sigma} = c^{\lambda-1}$. By Proposition 2.1, $(a^{\lambda}b^{\lambda-1}c^{k})^{\sigma} = a^{-1}b^{-2\lambda}c^{-\lambda+2+(k+l)(\lambda-1)} \in R_{4,l}$, which is impossible. If $b^{\sigma} = a^{-\lambda-1}b^{-\lambda}c^{1-l}$ then $c^{\sigma} = c^{-\lambda}$, and hence $(a^{\lambda}b^{\lambda-1}c^{k})^{\sigma} = a^{\lambda+2}b^{\lambda+1}c^{-\lambda-2-l(\lambda-1)-k\lambda} \in R_{4,l}$, which is impossible. If $b^{\sigma} = b$ then $c^{\sigma} = c$, and hence $(a^{\lambda}b^{\lambda-1}c^{k})^{\sigma} = a^{\lambda}b^{\lambda-1}c^{k} \in R_{4,l}$, implying that $l = k$.

Assume $R_{4,k}^{\sigma} = R_{4,l}^{-1}$ with $a^{\sigma} = a^{-1}$. Then $b^{\sigma} \in R_{4,l}^{-1}$, and $b^{\sigma} = b^{-1}$, $a^{-\lambda}b^{1-\lambda}c^{-l+\lambda+1}$ or $a^{\lambda+1}b^{\lambda}c^{l-\lambda}$. If $b^{\sigma} = a^{-\lambda}b^{1-\lambda}c^{-l+\lambda+1}$ then $c^{\sigma} = c^{\lambda-1}$. By Proposition 2.1, we have $(a^{\lambda}b^{\lambda-1}c^{k})^{\sigma} = ab^{2\lambda}c^{-\lambda+(k-l)(\lambda-1)} \in R_{4,l}^{-1}$, which is impossible. If $b^{\sigma} = a^{\lambda+1}b^{\lambda}c^{l-\lambda}$ then $c^{\sigma} = c^{-\lambda}$ and hence $(a^{\lambda}b^{\lambda-1}c^{k})^{\sigma} = a^{-\lambda-2}b^{-\lambda-1}c^{-\lambda-k\lambda+l(\lambda-1)} \in R_{4,l}^{-1}$, which is impossible. If $b^{\sigma} = b^{-1}$ then we have $c^{\sigma} = c$ and $(a^{\lambda}b^{\lambda-1}c^{k})^{\sigma} = a^{-\lambda}b^{1-\lambda}c^{k} \in R_{4,l}^{-1}$, implying that $l = 1 + \lambda - k$. □

By Magma [4], a brute force computer search can be performed to verify the following lemma, but we have also verified the correctness of the lemma theoretically. Since the proof is rather long, we will not present it in the paper but are willing to provide it upon request (also see [29]).

Lemma 3.5. *There is no half-arc-transitive graph of order* 27 *and valency* 6 *or* 8.

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let Γ be a half-arc-transitive graph of order p^3 and valency 6 or 8, and let $A = \mathrm{Aut}(\Gamma)$. By Observation 3.1, $\Gamma = \mathrm{Cay}(G, S)$ and $\overrightarrow{\Gamma} = \mathrm{Cay}(G, R)$ for some group $G = G_1(p)$ or $G_2(p)$ with $p \geq 3$, where $G = \langle R \rangle$ and $S = R \cup R^{-1}$. By Lemma 3.5, $p \geq 5$, and by the half-arc-transitivity of Γ, $A = \mathrm{Aut}(\overrightarrow{\Gamma})$ and A_1 is transitive on R. Since $G = \langle R \rangle$, $\mathrm{Aut}(G, R)$ acts faithfully on R, and since $|R| < 5$, $\mathrm{Aut}(G, R)$ is a p'-group. Since G is a non-abelian group of order p^3, G is 2-genetic, that is, each normal subgroup of G can be generated by two elements. By Proposition 2.7, $\overrightarrow{\Gamma}$ is normal, and by Proposition 2.6, $A_1 = \mathrm{Aut}(G, R)$. Since $A = \mathrm{Aut}(\Gamma) = \mathrm{Aut}(\overrightarrow{\Gamma})$, we have $A_1 = \mathrm{Aut}(G, S)$, and so Γ is normal.

The theorem is true for $G = G_1(p)$ by Proposition 2.2. Now assume $G = G_2(p)$. If Γ has valency 8, the theorem is also true by Lemma 3.2. We may thus assume that Γ has valency 6, that is, $|R| = 3$. We prove that this is not possible.

By Observation 3.1, $R = \{a, b, a^{i}b^{j}c^{k}\}$, where $i, j, k \in \mathbb{Z}_p$. Since $\mathrm{Aut}(G, R)$ is transitive on R, there exists $\alpha \in \mathrm{Aut}(G_2(p))$ of order 3 permuting the elements in R cyclically. If necessary, replace α by α^2, and then we may assume that α is induced by $a \mapsto b$, $b \mapsto a^{i}b^{j}c^{k}$, and then $c \mapsto c^{-i}$ by Proposition 2.1. Thus, $a = (a^{i}b^{j}c^{k})^{\alpha} = a^{ij}b^{i+j^2}c^{-i^2j-2^{-1}ij^2(j-1)+k(j-i)}$, and so we have:

$$ij = 1; \tag{3.7}$$

$$i + j^2 = 0; \tag{3.8}$$

$$-i^2j + k(j-i) - 2^{-1}ij^2(j-1) = 0. \tag{3.9}$$

By Eqs. (3.7) and (3.8), $j^3 + 1 = 0$, implying $(j+1)(j^2 - j + 1) = 0$. Thus, either $j + 1 = 0$ or $j^2 - j + 1 = 0$. If $j + 1 = 0$ then $j = -1$. By Eq. (3.8), $i = -1$ and so $S = \{a, b, a^{-1}b^{-1}c^k\} \cup \{a^{-1}, b^{-1}, abc^{-1-k}\}$, but the automorphism of G induced by $a \mapsto a^{-1}$, $b \mapsto abc^{-1-k}$, $c \mapsto c^{-1}$, fixes S setwise, contrary to Proposition 2.4. If $j^2 - j + 1 = 0$ then by Eq. (3.8), $i = 1 - j$, and since $ij = 1$, Eq. (3.9) implies that $-i + k(j - i) - 2^{-1}j(j-1) = j - 1 + k(j + j - 1) + 2^{-1} = (2j-1)(k + 2^{-1}) = 0$. It follows that either $2j - 1 = 0$ or $k + 2^{-1} = 0$. For $2j - 1 = 0$, we have $j = 2^{-1}$ and $i = 1 - j = 1 - 2^{-1} = 2^{-1}$, but then Eq. (3.7) implies $4 = 1$ in $\mathbb{Z}_p$, contradicting $p \neq 3$. For $k + 2^{-1} = 0$, we have $S = \{a, b, a^{1-j}b^j c^{-2^{-1}}\} \cup \{a^{-1}, b^{-1}, a^{j-1}b^{-j}c^{-2^{-1}}\}$, and the automorphism of G induced by $a \mapsto a^{-1}$, $b \mapsto b^{-1}$, $c \mapsto c$, fixes S setwise, a contradiction. □

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More on the structure of plane graphs with prescribed degrees of vertices, faces, edges and dual edges

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Abstract

We study the families of plane graphs determined by lower bounds δ, ρ, w, w^* on their vertex degrees, face sizes, edge weights and dual edge weights, respectively. Continuing the previous research of such families comprised of polyhedral graphs, we determine the quadruples $(2, \rho, w, w^*)$ for which the associated family is non-empty. In addition, we determine all quadruples which yield extremal families (in the sense that the increase of any value of a quadruple results in an empty family).

Keywords: Plane graph, girth, edge weight, dual edge weight.

Math. Subj. Class.: 05C10

1 Introduction

Throughout this paper, we consider connected graphs without loops or multiple edges. Given a graph $G = (V, E)$, the degree $d(v)$ of a vertex $v \in V$ is the number of edges incident with v. By k^+ or k^- we denote any integer not smaller or not greater than k, respectively. Hence, a k-vertex (k^+-vertex, k^--vertex) is a vertex v with $d(v) = k$ ($d(v) \geq k$, $d(v) \leq k$, respectively). An edge uv is an (i, j)-edge, if $d(u) = i$ and $d(v) = j$. For an edge $e = uv \in E$, the *weight* $w(e)$ of e is the sum $d(u) + d(v)$. The *minimum vertex degree* of G is the number $\delta(G) = \min\{d(v) : v \in V\}$, and the *minimum edge weight of* G is $w(G) = \min\{w(e) : e \in E\}$. The *girth* $g(G)$ of G is the length of a shortest cycle of G

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and the *double girth* of G is defined as the minimum sum of lengths of two distinct cycles of G which share a common edge; it will be denoted as $\mathrm{dg}(G)$ (note that $g(G) = \infty$ if G is a tree, and $\mathrm{dg}(G) = \infty$ if no two cycles of G share an edge). A graph is called *planar* if it can be drawn in the plane in such a way that, in this drawing, no two edges cross (such a drawing is called a *plane graph* and it is determined by the triple (V, E, F), where F is the set of faces). The *face size* $d(\alpha)$ of a face $\alpha \in F$ is the number of edges incident with α (incident cut-edges being counted twice). A k-face (k^+-face, k^--face) is a face α with $d(\alpha) = k$ ($d(\alpha) \geq k$, $d(\alpha) \leq k$, respectively). The *minimum face size* of G, denoted $\rho(G)$, is defined as $\min\{d(\alpha) : \alpha \in F\}$ and the *minimum dual edge weight* of G is the number $w^*(G) = \min\{d(\alpha) + d(\beta) : \alpha, \beta \in F, \alpha \neq \beta, \alpha, \beta \text{ have a common edge}\}$. Note that $g(G) \leq \rho(G)$ and $\mathrm{dg}(G) \leq w^*(G)$.

For a general graph G, there are no special dependencies of the above mentioned graph invariants apart from the trivial ones: $w(G) \geq 2\delta(G)$ and $\mathrm{dg}(G) \geq 2g(G)$. On the other hand, these invariants are strongly dependent when additional graph constraints are considered. Particularly, if G is a plane graph, then $\min\{\delta(G), \rho(G)\} \leq 5$; additionally $\delta(G) \geq 4$ implies $\rho(G) = 3$ and $\rho(G) \geq 4$ implies $\delta(G) \leq 3$. These facts follow easily from Euler's formula for the numbers of vertices, edges and faces of a plane graph. A more subtle analysis of consequences of Euler's formula yields further dependencies: if $\delta(G) \geq 3$ then $w(G) \leq 13$, whereas $\delta(G) \geq 4$ gives $w(G) \leq 11$, see [1]. By considering dual versions of these results, we obtain a dependence between the minimum face size $\rho(G)$ and the minimum dual edge weight $w^*(G)$: if $\rho(G) \geq 3$ then $w^*(G) \leq 13$ and, for $\rho(G) \geq 4$, $w^*(G) \leq 11$. Furthermore, the results of the classical paper [9] give that if $\delta(G) \geq 3$ and $\rho(G) \geq 4$, then $w(G) \leq 8$, and $\delta(G) \geq 3$ together with $\rho(G) = 5$ yield $w(G) = 6$. The mutual dependence of all four values $\delta(G), \rho(G), w(G)$ and $w^*(G)$ for polyhedral (that is, 3-connected plane) graphs was studied in [4] giving the characterization of all quadruples (δ, ρ, w, w^*) for which the corresponding families of polyhedral graphs of minimum vertex degree at least δ, minimum face size at least ρ, minimum edge weight at least w and minimum dual edge weight at least w^* are non-empty.

The aim of this paper is to extend the results of [4] for wider families of plane graphs with $\delta = 2$. The graph $K_{2,r}$ shows that $w(G)$ is unbounded for $\rho(G) = 4$. On the other hand, recent results by Jendrol' and Maceková [7] and results from [2] show that if $g(G) \in \{5, 6\}$ then $w(G) \leq 7$ and, further, if $g(G) \in \{7, 8, 9, 10\}$, then $w(G) \leq 5$ as well as $g(G) \geq 11$ implies $w(G) = 4$. Denoting the set of all plane graphs of minimum degree at least δ, girth at least ρ, minimum edge weight at least w and minimum double girth at least w^* as $\mathcal{G}(\delta, \rho, w, w^*)$, the equivalent formulation of these results is that the families $\mathcal{G}(2, 5, 8, 10), \mathcal{G}(2, 7, 6, 14)$ and $\mathcal{G}(2, 11, 5, 22)$ are empty.

In this paper, we prove the following additional results:

Theorem 1.1. *The family* $\mathcal{G}(2, 3, 7, 15)$ *is empty.*

Theorem 1.2. *The family* $\mathcal{G}(2, 3, 9, 11)$ *is empty.*

Theorem 1.3. *The family* $\mathcal{G}(2, 3, 13, 9)$ *is empty.*

Theorem 1.4. *The families* $\mathcal{G}(2, 5, 5, 27)$ *and* $\mathcal{G}(2, 7, 5, 23)$ *are empty.*

Theorem 1.5. *The family* $\mathcal{G}(2, 5, 6, 17)$ *is empty.*

Theorem 1.6. *The family* $\mathcal{G}(2, 5, 7, 13)$ *is empty.*

For the non-empty families arising from admissible quadruples, we are interested in determining the extremal ones, that is, the families $\mathcal{G}(\delta, \rho, w, w^*)$ such that the increase of any of the values δ, ρ, w and w^* results in an empty family. We prove:

Theorem 1.7. *The families* $\mathcal{G}(2,4,8,14)$, $\mathcal{G}(2,4,12,10)$, $\mathcal{G}(2,6,5,26)$, $\mathcal{G}(2,6,6,16)$, $\mathcal{G}(2,6,7,12)$, $\mathcal{G}(2,10,5,22)$ *are non-empty and extremal.*

2 The proofs

For the needs of the proofs we will use the following consequence of Euler's formula with specified parameters a and b (without giving a proof):

Lemma 2.1. *Let G be a connected plane graph, a be a positive and b be a non-negative integer. Then*

$$\sum_{v \in V(G)} (a \cdot d(v) - 2(a+b)) + \sum_{\alpha \in F(G)} (b \cdot d(\alpha) - 2(a+b)) = -4(a+b).$$

The common approach used in the majority of proofs in this paper is the *discharging method*. Assuming the existence of a hypothetical plane counterexample $G = (V, E, F)$ for a particular statement of Theorems 1.1 – 1.7, we define the initial charges of vertices and faces by the function $\omega : V \cup F \to \mathbb{Z}$ assigning $\omega(v) = a \cdot d(v) - 2(a+b)$ for each $v \in V$, and $\omega(\alpha) = b \cdot d(\alpha) - 2(a+b)$ for each $\alpha \in F$. By Lemma 2.1, $\sum_{x \in V \cup F} \omega(x) = -4(a+b) < 0$. Next, we redistribute the initial charges of vertices and faces of G using certain rules which specify, in particular situations, the amount of charge transferred from one element to another; all transfers preserve the total sum of the initial charges. Finally, by case analysis, we show that the final charge $\varphi : V \cup F \to \mathbb{Q}$ is a non-negative function; this is, however, a contradiction since $0 > \sum_{x \in V \cup F} \omega(x) = \sum_{x \in V \cup F} \varphi(x) \geq 0$.

We note that, while checking the non-negativity of φ, we will usually mention just a minimal set of discharging rules that give $\varphi(x) \geq 0$ for an $x \in V \cup F$, although there may be additional transfers of a positive charge to x.

2.1 Proof of Theorem 1.1

Let the family $\mathcal{G}(2,3,7,15)$ be non-empty and let $G = (V, E, F)$ be its representative.

Without loss of generality, we can assume that 5^+-vertices are not adjacent in G (otherwise we subdivide each $(5^+, 5^+)$-edge with a new 2-vertex which yields a new graph G' being again from $\mathcal{G}(2,3,7,15)$). Therefore each k-face α of G, for k odd, is incident with at most $\frac{k-3}{2}$ 2-vertices (note that k-face α, for k even, is incident with at most $\frac{k}{2}$ 2-vertices).

The discharging procedure is based on Lemma 2.1 with $a = 1$ and $b = 0$ and the following discharging rules:

- R1 Each k-face α, $k \leq 7$, distributes its initial charge uniformly to all incident 3^+-vertices.
- R2 Each k-face α, $k \geq 8$, distributes its initial charge uniformly to all incident 4^+-vertices.

It follows from the discharging rules that $\varphi(\alpha) = 0$ for all $\alpha \in F$.

In Table 1 we give the lower bounds for charges received by vertices of graph G from k-faces of G ($k \geq 3$):

k	3	4	5	6	7	8	9	10	11	12^+
$d(v)=3$	$-\frac{2}{3}$	$-\frac{2}{3}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{2}{5}$	–	–	–	–	–
$d(v)=4$	$-\frac{2}{3}$	$-\frac{2}{3}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{2}{5}$	$-\frac{1}{2}$	$-\frac{2}{5}$	$-\frac{2}{5}$	$-\frac{1}{3}$	$-\frac{1}{3}$
$d(v)\geq 5$	$-\frac{2}{3}$	-1	$-\frac{1}{2}$	$-\frac{2}{3}$	$-\frac{2}{5}$	$-\frac{1}{2}$	$-\frac{2}{5}$	$-\frac{2}{5}$	$-\frac{1}{3}$	$-\frac{1}{3}$

Table 1: Lower bounds for charges sent to vertices of G from a k-face α.

Now, let $v \in V$ be a k-vertex, $k \geq 2$. We consider the following cases regarding k:

$k = 2$: Discharging rules do not involve 2-vertices, therefore $\varphi(v) = \omega(v) = 0$.

$k = 3$: The 3-vertices receive negative charge only from incident 7^--faces by R1. Since $w^*(G) \geq 15$, v is incident with at most one 7^--face. If a 3-vertex v is incident with an l-face α, $3 \leq l \leq 7$, then, using Table 1, $\varphi(v) \geq 1 + (-\frac{2}{3}) = \frac{1}{3}$. If v is incident with no such face, then $\varphi(v) = \omega(v) = 1$.

$k = 4$: Each 4-vertex v is incident with at most two 7^--faces (as $w^*(G) \geq 15$). If v is incident with a 4^--face, then it is incident with at least two 11^+-faces, and hence $\varphi(v) \geq 2 + (-\frac{2}{3}) + 2 \cdot (-\frac{1}{3}) + (-\frac{2}{3}) = 0$ due to Table 1. Otherwise, v is incident with four 5^+-faces and $\varphi(v) \geq 2 + 4 \cdot (-\frac{1}{2}) = 0$.

$k = 5$: Each 5-vertex v is incident with at most two 7^--faces (as $w^*(G) \geq 15$). If v is incident with a 3-face, then it is incident with at least two 12^+-faces, and hence, using Table 1, $\varphi(v) \geq 3 + (-\frac{2}{3}) + 2 \cdot (-\frac{1}{3}) + (-1) + (-\frac{1}{2}) = \frac{1}{6}$. If v is incident with one k-face, $4 \leq k \leq 7$, and four 8^+-faces then $\varphi(v) \geq 3 - 1 + 4 \cdot (-\frac{1}{2}) = 0$. If v is incident with two 4-faces, then it is incident with three 11^+-faces, and $\varphi(v) \geq 3 + 2 \cdot (-1) + 3 \cdot (-\frac{1}{3}) = 0$. If v is incident with a 4-face and a 5-face, then it is incident with two 11^+-faces and a 10^+-face, and hence $\varphi(v) \geq 3 + (-1) + (-\frac{1}{2}) + 2 \cdot (-\frac{1}{3}) + (-\frac{2}{5}) = \frac{13}{30}$. If it is incident with a 4-face and a 6-face, then it is incident with two 11^+-faces and a 9^+-face, and hence $\varphi(v) \geq 3 + (-1) + (-\frac{2}{3}) + 2 \cdot (-\frac{1}{3}) + (-\frac{2}{5}) = \frac{4}{15}$. If v is incident with a 4-face and a 7-face, then it is incident with two 11^+-faces and an 8^+-face, and hence $\varphi(v) \geq 3 + (-1) + (-\frac{2}{5}) + 2 \cdot (-\frac{1}{3}) + (-\frac{1}{2}) = \frac{13}{30}$. Finally, if v is incident with faces α and β, where $5 \leq d(\alpha), d(\beta) \leq 7$, then $\varphi(v) \geq 3 + 2 \cdot (-\frac{2}{3}) + 3 \cdot (-\frac{1}{2}) = \frac{1}{6}$.

$k \geq 6$: Each k-vertex v, $k \geq 6$, is incident with at most $\lfloor \frac{k}{2} \rfloor$ 7^--faces. To estimate the total reception of the vertex v we argue as follows. If v is incident with a 3-face, then it is incident with a 12^+-face and they send together a charge $-\frac{2}{3} + (-\frac{1}{3}) = -1$ to v (according to Table 1). If v is incident with a 4-face, then it is incident with an 11^+-face and they send together a charge $-1 + (-\frac{1}{3}) = -\frac{4}{3}$ to v. If v is incident with a 5-face, then it is incident with a 10^+-face and they send together a charge $-\frac{1}{2} + (-\frac{2}{5}) = -\frac{9}{10}$ to v. If v is incident with a 6-face, then it is incident with a 9^+-face and they send together a charge $-\frac{2}{3} + (-\frac{2}{5}) = -\frac{16}{15}$ to v. And finally, if v is incident with a 7-face, then it is incident with an 8^+-face and they send together a charge $-\frac{2}{5} + (-\frac{1}{2}) = -\frac{9}{10}$ to v. Thence it follows, that each face sends in average a charge at least $-\frac{2}{3}$ to v and therefore $\varphi(v) \geq k - 2 + k \cdot (-\frac{2}{3}) = \frac{k}{3} - 2 \geq 0$ for $k \geq 6$.

Hence, all elements of G have non-negative final charge, giving the desired contradiction.

2.2 Proof of Theorem 1.2

Let the family $\mathcal{G}(2,3,9,11)$ be non-empty and $G=(V,E,F)$ be its representative.

The discharging procedure is based on Lemma 2.1 with $a=1$ and $b=0$ and the following discharging rules:

R1 Each k-face, $k\leq 5$, divides its initial charge uniformly among all incident 3^+-vertices.

R2 Each k-face, $k\geq 6$, sends a charge of size $-\frac{1}{3}$ to each incident 4-vertex.

R3 Each k-face, $k\geq 6$, distributes its residual charge (after application of R2) uniformly to all incident 5^+-vertices.

It follows from the discharging rules that $\varphi(\alpha)=0$ for all $\alpha\in F$.

In Table 2 we give the lower bounds for charges received by vertices of graph G from k-faces of G, $k\geq 3$:

k	3	4	5	6	7	8	9^+
$d(v)=3$	$-\frac{2}{3}$	$-\frac{2}{3}$	$-\frac{1}{2}$	–	–	–	–
$d(v)=4$	$-\frac{2}{3}$	$-\frac{2}{3}$	$-\frac{1}{2}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$
$d(v)=5$	$-\frac{2}{3}$	$-\frac{2}{3}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{5}{12}$	$-\frac{2}{5}$	$-\frac{1}{3}$
$d(v)=6$	$-\frac{2}{3}$	$-\frac{2}{3}$	$-\frac{1}{2}$	$-\frac{2}{3}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{2}{5}$
$d(v)\geq 7$	-1	-1	$-\frac{2}{3}$	$-\frac{2}{3}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{2}{5}$

Table 2: Lower bounds for charges sent to vertices of G from a k-face α.

Now, let $v\in V$ be a k-vertex, $k\geq 2$. We consider the following cases regarding k:

$k=2$: Discharging rules do not involve 2-vertices, therefore $\varphi(v)=\omega(v)=0$.

$k=3$: Each 3-vertex v is incident with at most one 5^--face (as $w^*(G)\geq 11$). Hence, using Table 2, $\varphi(v)\geq 1+(-\frac{2}{3})=\frac{1}{3}$.

$k=4$: Each 4-vertex v is incident with at most two 5^--faces (as $w^*(G)\geq 11$). Hence, according to Table 2, $\varphi(v)\geq 2+2\cdot(-\frac{2}{3})+2\cdot(-\frac{1}{3})=0$.

$k=5$: Each 5-vertex v is incident with at most two 5^--faces (as $w^*(G)\geq 11$). Hence, $\varphi(v)\geq 3+2\cdot(-\frac{2}{3})+3\cdot(-\frac{1}{2})=\frac{1}{6}$.

$k=6$: Each 6-vertex v receives from each face charge at least $-\frac{2}{3}$ and therefore, $\varphi(v)\geq 4+6\cdot(-\frac{2}{3})=0$.

$k=7$: If v is incident with three 3- or 4-faces, then it is incident with four 8^+-faces and, using Table 2, $\varphi(v)\geq 5+3\cdot(-1)+4\cdot(-\frac{1}{2})=0$. If v is incident with two 3- or 4-faces, then it is incident with at least three 8^+-faces and $\varphi(v)\geq 5+2\cdot(-1)+3\cdot(-\frac{1}{2})+2\cdot(-\frac{2}{3})=\frac{1}{6}$. If v is incident with one 3- or 4-face, then it is incident with at least two 8^+-faces and $\varphi(v)\geq 5+(-1)+2\cdot(-\frac{1}{2})+4\cdot(-\frac{2}{3})=\frac{1}{3}$. Otherwise, it is incident only with 5^+-faces and $\varphi(v)\geq 5+7\cdot(-\frac{2}{3})=\frac{1}{3}$.

$k \geq 8$: Let s be the number of 3- and 4-faces incident with k-vertex v and t be the number of 8^+-faces incident with v. As $w^*(G) \geq 11$, we have $t \geq s$ and $s \leq \lfloor \frac{k}{2} \rfloor$. Then $\varphi(v) \geq k-2+s\cdot(-1)+t\cdot(-\frac{1}{2})+(k-s-t)\cdot(-\frac{2}{3}) = k-2-s\cdot\frac{1}{3}+t\cdot\frac{1}{6}-k\cdot\frac{2}{3} \geq \frac{k}{3} - s\cdot\frac{1}{6} - 2 \geq \frac{k}{3} - \frac{k}{12} - 2 = \frac{k}{4} - 2 \geq 0$ for $k \geq 8$.

Hence, all elements of G have non-negative final charge, a contradiction.

2.3 Proof of Theorem 1.3

Let the family $\mathcal{G}(2,3,13,9)$ be non-empty and $G = (V,E,F)$ be its representative.

The discharging procedure is based on Lemma 2.1 with $a = 1$ and $b = 0$ and the following discharging rules:

R1 a) Each face α sends a charge of size $-\frac{1}{3}$ to each incident 3-vertex.
b) Each face α sends a charge of size $-\frac{1}{2}$ to each incident 4-vertex.

R2 Each face α distributes its residual charge (after the application of R1a and R1b) uniformly among all incident 5^+-vertices.

It follows from the discharging rules that $\varphi(\alpha) = 0$ for all $\alpha \in F$ (the faces are able to distribute the charge, since there are always 5^+-vertices in the graph).

In Table 3 we give the lower bounds for charges received by vertices of graph G from k-faces of G ($k \geq 3$) after the application of the rule R1:

k	3	4	5	6^+
$5 \leq d(v) \leq 8$	$-\frac{2}{3}$	$-\frac{2}{3}$	$-\frac{1}{2}$	$-\frac{1}{2}$
$d(v) = 9$	$-\frac{3}{4}$	$-\frac{2}{3}$	$-\frac{1}{2}$	$-\frac{1}{2}$
$d(v) = 10$	$-\frac{5}{6}$	$-\frac{2}{3}$	$-\frac{5}{9}$	$-\frac{1}{2}$
$d(v) \geq 11$	-1	-1	$-\frac{2}{3}$	$-\frac{2}{3}$

Table 3: Lower bounds for charges sent to vertices of G from a k-face α.

Now, let $v \in V$ be a k-vertex, $k \geq 2$. We consider the following cases regarding k:

$k = 2$: Discharging rules do not involve 2-vertices, therefore $\varphi(v) = \omega(v) = 0$.

$k = 3$: Each 3-vertex v receives $-\frac{1}{3}$ from all incident faces, hence $\varphi(v) = 1+3\cdot(-\frac{1}{3}) = 0$.

$k = 4$: Each 4-vertex v receives $-\frac{1}{2}$ from all incident faces, hence $\varphi(v) = 2+4\cdot(-\frac{1}{2}) = 0$.

$5 \leq k \leq 8$: Let s and t be the numbers of 4^-- and 5^+-faces incident with a k-vertex v, respectively. As $w^*(G) \geq 9$, we have $t \geq s$ and $s \leq \lfloor \frac{k}{2} \rfloor$. Then, using Table 3, $\varphi(v) \geq k-2+s\cdot(-\frac{2}{3})+t\cdot(-\frac{1}{2}) \geq k-2+\lfloor \frac{k}{2} \rfloor\cdot(-\frac{2}{3})+\lceil \frac{k}{2} \rceil\cdot(-\frac{1}{2}) \geq \frac{5k}{12}-2 > 0$ for $k \geq 5$.

$k = 9$: As each 9-vertex receives, according to Table 3, a charge of at least $-\frac{3}{4}$ from each incident face, we have that $\varphi(v) \geq 7 + 9\cdot(-\frac{3}{4}) = \frac{1}{4} > 0$.

$k = 10$: Each 10-vertex is incident with at least five 5^+-faces (as $w^*(G) \geq 9$). Therefore $\varphi(v) \geq 8 + 5\cdot(-\frac{5}{9}) + 5\cdot(-\frac{5}{6}) = \frac{19}{18} > 0$.

$k = 11$: Each 11-vertex is incident with at least six 5^+-faces (as $w^*(G) \geq 9$). Therefore $\varphi(v) \geq 9 + 6 \cdot (-\frac{2}{3}) + 5 \cdot (-1) = 0$.

$k \geq 12$: Let s and t be the numbers of 4^-- and 5^+-faces incident with a k-vertex v, respectively. As $w^*(G) \geq 9$, we have $t \geq s$ and $s \leq \lfloor \frac{k}{2} \rfloor$. Then $\varphi(v) \geq k - 2 + s \cdot (-1) + t \cdot (-\frac{2}{3}) \geq k - 2 + \lfloor \frac{k}{2} \rfloor \cdot (-1) + \lceil \frac{k}{2} \rceil \cdot (-\frac{2}{3}) \geq \frac{k}{6} - 2 \geq 0$ for $k \geq 12$.

Hence, all elements of G have non-negative final charge, a contradiction.

2.4 Proof of Theorem 1.4

Let the families $\mathcal{G}(2, 5, 5, 27)$ and $\mathcal{G}(2, 7, 5, 23)$ be non-empty and G_1^* and G_2^* be their respective representatives. Each face α of G_1^* and G_2^* is incident with at most $\left\lfloor \frac{d(\alpha)}{2} \right\rfloor$ 2-vertices (as $w(G_i^*) \geq 5$ for $i \in \{1, 2\}$). Then, after suppressing all 2-vertices of G_1^* and G_2^*, respectively, we obtain graphs G_1, G_2 with $\delta(G_i) \geq 3$, $i \in \{1, 2\}$. Moreover, G_1 belongs to the family $\mathcal{G}(3, 3, 6, 14)$ and G_2 is from $\mathcal{G}(3, 4, 6, 12)$, which contradicts the fact that these families were proven to be empty (see [4]).

2.5 Proof of Theorem 1.5

Let $G = (V, E, F) \in \mathcal{G}(2, 5, 6, 17)$ be a counterexample to the theorem. Without loss of generality, we assume that 4^+-vertices are not adjacent in G (otherwise we subdivide each $(4^+, 4^+)$-edge in G with a new 2-vertex, which yields a new counterexample $G' \in \mathcal{G}(2, 5, 6, 17)$). Therefore, each k-face α of G, for k odd, is incident with at most $\frac{k-3}{2}$ 2-vertices (note that k-face α, for k even, is incident with at most $\frac{k}{2}$ 2-vertices).

The discharging procedure is based on Lemma 2.1 with $a = 2$, $b = 1$ and the following discharging rules:

- R1 Each vertex v distributes its initial charge uniformly to all incident faces.
- R2 Each 11^+-face α sends a charge of size $\frac{1}{4}$ to each adjacent face (through every common edge).

By R1, $\varphi(v) = 0$ for all $v \in V$. Since $w(G) \geq 6$, every 2-vertex of G is adjacent only to 4^+-vertices and every its 3-vertex is adjacent only to 3^+-vertices.

Let $\alpha \in F$ be a k-face, $k \geq 5$. We consider the following cases regarding k:

$k = 5$: All faces adjacent to α are 12^+-faces (as $w^*(G) \geq 17$) and α is incident with at most one 2-vertex. If α is incident with exactly one 2-vertex, then it is incident with at least two 4^+-vertices and hence $\varphi(\alpha) \geq -1 + (-1) + 2 \cdot \frac{1}{2} + 5 \cdot \frac{1}{4} = \frac{1}{4}$. Finally, if α is not incident with any 2-vertex, then $\varphi(\alpha) \geq -1 + 5 \cdot \frac{1}{4} = \frac{1}{4}$.

$k = 6$: All faces adjacent to α are 11^+-faces (as $w^*(G) \geq 17$). If α is incident with three 2-vertices, then it is incident with three 4^+-vertices. Hence $\varphi(\alpha) \geq 0 + 3 \cdot (-1) + 3 \cdot \frac{1}{2} + 6 \cdot \frac{1}{4} = 0$. If α is incident with two 2-vertices, then it is incident with at least three 4^+-vertices, giving $\varphi(\alpha) \geq 0 + 2 \cdot (-1) + 3 \cdot \frac{1}{2} + 6 \cdot \frac{1}{4} = 1$. If α is incident with at most one 2-vertex, then $\varphi(\alpha) \geq 0 + (-1) + 6 \cdot \frac{1}{4} = \frac{1}{2}$.

$k = 7$: α is incident with at most two 2-vertices. If α is incident with two 2-vertices, then it is incident with at least three 4^+-vertices. Hence $\varphi(\alpha) \geq 1 + 2 \cdot (-1) + 3 \cdot \frac{1}{2} = \frac{1}{2}$ by R1. Otherwise, if α is incident with at most one 2-vertex, then $\varphi(\alpha) \geq 1 + (-1) = 0$.

$k = 8$: If α is incident with four 2-vertices, then it is incident with four 4^+-vertices. Hence $\varphi(\alpha) \geq 2 + 4 \cdot (-1) + 4 \cdot \frac{1}{2} = 0$ by R1. If α is incident with three 2-vertices, then it is incident with at least four 4^+-vertices, giving $\varphi(\alpha) \geq 2 + 3 \cdot (-1) + 4 \cdot \frac{1}{2} = 1$. Finally, if α is incident with at most two 2-vertices, then $\varphi(\alpha) \geq 2 + 2 \cdot (-1) = 0$.

$k = 9$: α is incident with at most three 2-vertices, thus we have that $\varphi(\alpha) \geq 3 + 3 \cdot (-1) = 0$.

$k = 10$: If α is incident with five 2-vertices, then it is incident with five 4^+-vertices. Hence $\varphi(\alpha) \geq 4 + 5 \cdot (-1) + 5 \cdot \frac{1}{2} = \frac{3}{2}$ by R1. Otherwise, if α is incident with at most four 2-vertices, then $\varphi(\alpha) \geq 4 + 4 \cdot (-1) = 0$.

$k = 11$: α is incident with at most four 2-vertices. If α is incident with four 2-vertices, then it is incident with at least five 4^+-vertices. Hence $\varphi(\alpha) \geq 5 + 4 \cdot (-1) + 5 \cdot \frac{1}{2} - 11 \cdot \frac{1}{4} = \frac{3}{4}$ by R1 and R2. If α is incident with three 2-vertices, then it is incident with at least four 4^+-vertices, giving $\varphi(\alpha) \geq 5 + 3 \cdot (-1) + 4 \cdot \frac{1}{2} - 11 \cdot \frac{1}{4} = \frac{5}{4}$. Finally, if α is incident with at most two 2-vertices, then $\varphi(\alpha) \geq 5 + 2 \cdot (-1) - 11 \cdot \frac{1}{4} = \frac{1}{4}$.

$k \geq 12$: Let s and t be numbers of $2-$ and 4^+-vertices incident with α, respectively. As $w(G) \geq 6$, we have $t \geq s$ and $s \leq \lfloor \frac{k}{2} \rfloor$. Then $\varphi(\alpha) \geq k - 6 + s \cdot (-1) + t \cdot \frac{1}{2} - \frac{1}{4} \cdot k \geq \frac{3}{4} \cdot k - 6 - \frac{1}{2} \cdot s \geq \frac{3}{4} \cdot k - 6 - \frac{1}{2} \cdot \lfloor \frac{k}{2} \rfloor \geq 0$ for $k \geq 12$.

Hence, all elements of G have non-negative final charge, a contradiction.

2.6 Proof of Theorem 1.6

Let the family $\mathcal{G}(2, 5, 7, 13)$ be non-empty and $G = (V, E, F)$ be its representative. Without loss of generality, we can assume that 5^+-vertices are not adjacent in G (otherwise we subdivide each $(5^+, 5^+)$-edge in G with a new 2-vertex, which yields a new counterexample $G' \in \mathcal{G}(2, 5, 7, 13)$). Therefore, each k-face α of G, for k odd, is incident with at most $\frac{k-3}{2}$ 2-vertices (note that k-face α, for k even, is incident with at most $\frac{k}{2}$ 2-vertices).

The discharging procedure is based on Lemma 2.1 with $a = 2$, $b = 1$ and the following discharging rules:

- R1 Each vertex v divides its initial charge uniformly among all incident faces.
- R2 Each 7^+-face α sends a charge of size $\frac{3}{25}$ to each adjacent face (through every common edge).

By R1, $\varphi(v) = 0$ for all $v \in V$. Since $w(G) \geq 7$, every 2-vertex of G is adjacent only to 5^+-vertices and every its 3-vertex is adjacent only to 4^+-vertices.

Let $\alpha \in F$ be a k-face, $k \geq 5$. We consider the following cases regarding k:

$k = 5$: All faces adjacent to α are 8^+-faces (as $w^*(G) \geq 13$). If α is incident with a 2-vertex, then it is incident with at least two 5^+-vertices and hence $\varphi(\alpha) \geq -1 + (-1) + 2 \cdot \frac{4}{5} + 5 \cdot \frac{3}{25} = \frac{1}{5}$. Otherwise, if α is not incident with any 2-vertex, then it is incident with at least three 4^+-vertices, and therefore $\varphi(\alpha) \geq -1 + 3 \cdot \frac{1}{2} + 5 \cdot \frac{3}{25} = \frac{11}{10}$.

$k = 6$: All faces adjacent to α are 7^+-faces (as $w^*(G) \geq 13$). If α is incident with three 2-vertices, then it is incident with three 5^+-vertices. Hence $\varphi(\alpha) \geq 0 + 3 \cdot (-1) + 3 \cdot \frac{4}{5} + 6 \cdot \frac{3}{25} = \frac{3}{25}$. If α is incident with two 2-vertices, then it is incident with at least three 5^+-vertices, giving $\varphi(\alpha) \geq 0 + 2 \cdot (-1) + 3 \cdot \frac{4}{5} + 6 \cdot \frac{3}{25} = \frac{28}{25}$. If α is incident with exactly one 2-vertex, then it is incident with at least two 5^+-vertices and thus $\varphi(\alpha) \geq 0 + (-1) + 2 \cdot \frac{4}{5} + 6 \cdot \frac{3}{25} = \frac{33}{25}$. Finally, if α is not incident with a 2-vertex, then it receives non-negative charge from each incident vertex, therefore $\varphi(\alpha) \geq \omega(\alpha) = 0$.

$k = 7$: If α is incident with two 2-vertices, then it is incident with at least three 5^+-vertices, so $\varphi(\alpha) \geq 1 + 2 \cdot (-1) + 3 \cdot \frac{4}{5} + 7 \cdot (-\frac{3}{25}) = \frac{14}{25}$. If α is incident with exactly one 2-vertex, then it is incident with at least two 5^+-vertices and hence $\varphi(\alpha) \geq 1 + (-1) + 2 \cdot \frac{4}{5} + 7 \cdot (-\frac{3}{25}) = \frac{19}{25}$. Finally, if α is not incident with a 2-vertex, then it receives non-negative charge from each incident vertex and, by R2, $\varphi(\alpha) \geq 1 + 7 \cdot (-\frac{3}{25}) = \frac{4}{25}$.

$k \geq 8$: Let s and t be numbers of $2-$ and 5^+-vertices incident with α, respectively. As $w(G) \geq 7$, we have $t \geq s$ and $s \leq \lfloor \frac{k}{2} \rfloor$. Then $\varphi(\alpha) \geq k - 6 + s \cdot (-1) + t \cdot \frac{4}{5} - \frac{3}{25} \cdot k \geq \frac{22}{25}k - \frac{s}{5} - 6 \geq \frac{22}{25}k - \lfloor \frac{k}{2} \rfloor \cdot \frac{1}{5} - 6 \geq \frac{39 \cdot k}{50} - 6 > 0$ for $k \geq 8$.

Hence, all elements of G have non-negative final charge, a contradiction.

2.7 Proof of Theorem 1.7

For each of the mentioned six families, we describe a representative and show that the increase in any of the four parameters results in an empty family.

The family $\mathcal{G}(2,4,8,14)$ contains, as a representative, the graph obtained from the dodecahedron by replacing each edge uv by a 4-cycle $uxvy$ with x, y being 2-vertices. Furthermore, $\mathcal{G}(3,4,8,14) \subset \mathcal{G}(3,4,7,9) = \emptyset$ by [4], $\mathcal{G}(2,5,8,14) \subset \mathcal{G}(2,5,8,10) = \emptyset$ by [2] and [7], $\mathcal{G}(2,4,9,14) \subset \mathcal{G}(2,3,9,11) = \emptyset$ by Theorem 1.2, and $\mathcal{G}(2,4,8,15) \subset \mathcal{G}(2,3,7,15) = \emptyset$ by Theorem 1.1.

A representative of $\mathcal{G}(2,4,12,10)$ is obtained from the icosahedron by replacing each edge uv by a 4-cycle $uxvy$ with x, y being 2-vertices. Furthermore, $\mathcal{G}(3,4,12,10) \subset \mathcal{G}(3,4,7,9) = \emptyset$ by [4], $\mathcal{G}(2,5,12,10) \subset \mathcal{G}(2,5,8,10) = \emptyset$ by [2, 7], $\mathcal{G}(2,4,13,10) \subset \mathcal{G}(2,3,13,9) = \emptyset$ by Theorem 1.3, and finally $\mathcal{G}(2,4,12,11) \subset \mathcal{G}(2,3,9,11) = \emptyset$ by Theorem 1.2.

For $\mathcal{G}(2,6,5,26)$, a suitable representative can be obtained, for example, by subdividing each edge of the graph of the truncated dodecahedron. Note that $\mathcal{G}(3,6,5,26) = \emptyset$ (if $\delta(G) \geq 3$, then $\rho(G) \leq 5$). Furthermore, by Theorem 1.4, $\mathcal{G}(2,7,5,26) \subset \mathcal{G}(2,7,5,23) = \emptyset$, $\mathcal{G}(2,6,5,27) \subset \mathcal{G}(2,5,5,27) = \emptyset$ and, by Theorem 1.5, $\mathcal{G}(2,6,6,26) \subset \mathcal{G}(2,5,6,17) = \emptyset$.

By subdividing each edge of the graph of icosidodecahedron, we obtain a representative of $\mathcal{G}(2,6,6,16)$. Again, $\mathcal{G}(3,6,6,16) = \emptyset$ (if $\delta(G) \geq 3$, then $\rho(G) \leq 5$) and $\mathcal{G}(2,7,6,16) \subset \mathcal{G}(2,7,6,14) = \emptyset$ by [2, 7], $\mathcal{G}(2,6,7,16) \subset \mathcal{G}(2,5,7,13) = \emptyset$ by Theorem 1.6, $\mathcal{G}(2,6,6,17) \subset \mathcal{G}(2,5,6,17) = \emptyset$ by Theorem 1.5.

A representative of $\mathcal{G}(2,6,7,12)$ is obtained by subdividing each edge of the icosahedron graph. Further, $\mathcal{G}(3,6,7,12) = \emptyset$ (if $\delta(G) \geq 3$, then $\rho(G) \leq 5$), $\mathcal{G}(2,7,7,12) = \emptyset$ (if $\rho(G) \geq 7$, then $w^*(G) \geq 2\rho(G) = 14$), $\mathcal{G}(2,6,8,12) \subset \mathcal{G}(2,5,8,10) = \emptyset$ by [2, 7], and $\mathcal{G}(2,6,7,13) \subset \mathcal{G}(2,5,7,13) = \emptyset$ by Theorem 1.6.

A representative of $\mathcal{G}(2,10,5,22)$ is obtained by subdividing each edge of the truncated icosahedron. Further, $\mathcal{G}(3,10,5,22) = \emptyset$ (if $\delta(G) \geq 3$, then $\rho(G) \leq 5$), $\mathcal{G}(2,11,5,22) = \emptyset$ by [2] and [7], $\mathcal{G}(2,10,6,22) \subset \mathcal{G}(2,5,6,17) = \emptyset$ by Theorem 1.5, and finally $\mathcal{G}(2,10,5,23) \subset \mathcal{G}(2,7,5,23) = \emptyset$ by Theorem 1.4.

3 Concluding remarks

A possible common way how to visualize the dependence of δ, ρ, w, w^* for families of plane graphs is to construct a diagram of a partially ordered set depicting the hierarchy of all non-empty families (generated by quadruples (δ, ρ, w, w^*)) under the set inclusion partial ordering. For $\delta \geq 3$, a partially ordered set of generated families of polyhedral graphs is shown in Figure 1 (this also corrects the error in the original diagram in [4]):

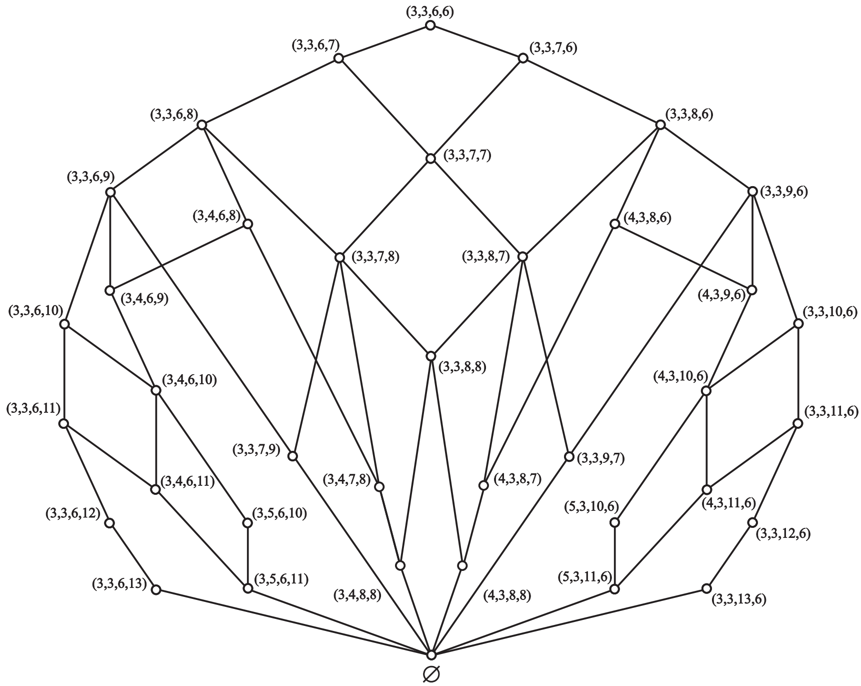

Figure 1: The hierarchy of families of polyhedral graphs generated by (δ, rho, w, w^*).

The results for $\delta = 2$ are presented in Table 4 indexed by values of girth (rows) and edge weight (columns) such that, the corresponding table entry shows the maximal admissible value of dual edge weight. The value ∞ in the first column is due to the fact that, in the graph obtained from C_n (n arbitrarily large) by replacing every edge with two disjoint paths of length 2, the dual edge weight is unbounded. The value 8 in the last column results from the graph $K_{2,r}$ for large r. The bold values correspond to extremal families.

The verification that we found all extremal classes can be done manually, or, as we did, using a simple computer program. Iterating over all possible classes $(2, \rho, w, w^*)$ check for every non-extremal class that it is either covered by an extremal class (all parameters are less or equal than for some extremal class) or by an empty class (all parameters are greater or equal than for some empty class).

Let us note that all extremal classes must have all parameters less or equal to 26, because every class that has at least one parameter greater than 26 is empty (it is a subset of $\mathcal{G}(2, 3, 13, 9)$, $\mathcal{G}(2, 5, 5, 27)$ or $\mathcal{G}(2, 3, 7, 15)$, which are all proven to be empty) or it is a part of one of two infinite chains.

ρ \ w	4	5	6	7	8	9	10	11	12	13^+
3	∞	∞	∞	14	14	10	10	10	10	8
4	∞	∞	$\boldsymbol{\infty}$	14	**14**	10	10	10	**10**	**8**
5	∞	26	16	12	–	–	–	–	–	–
6	∞	**26**	**16**	**12**	–	–	–	–	–	–
7	∞	22	–	–	–	–	–	–	–	–
8	∞	22	–	–	–	–	–	–	–	–
9	∞	22	–	–	–	–	–	–	–	–
10	∞	**22**	–	–	–	–	–	–	–	–
11^+	$\boldsymbol{\infty}$	–	–	–	–	–	–	–	–	–

Table 4: The table of admissible values for quadruples $(2, \rho, w, w^*)$.

The mutual dependence of the invariants δ, ρ, w, w^* can also be studied for graphs embedded into higher surfaces. Partial results were obtained for embedded graphs with $\delta(G) = 3$ and orientable genus $\gamma(G)$ in [6], it was proved that $w(G) \leq 2\gamma(G) + 13$ if $0 \leq \gamma(G) \leq 3$ and $w(G) \leq 4\gamma(G) + 7$ if $\gamma(G) \geq 3$, whereas $w(G) \leq 4\gamma(G) + 5$ if $\gamma(G) \geq 1$ and $g(G) \geq 4$. For graphs with non-orientable genus $\overline{\gamma}(G)$, it was proved in [8] that $w(G) \leq 2\overline{\gamma}(G) + 11$ if $1 \leq \overline{\gamma}(G) \leq 2$, and $w(G) \leq 2\overline{\gamma}(G) + 9$ in $3 \leq \overline{\gamma}(G) \leq 5$ with $w(G) \leq 2\overline{\gamma}(G) + 7$ for $\overline{\gamma}(G) \geq 6$; furthermore, if $g(G) \geq 4$, then $w(G) \leq 2\overline{\gamma}(G) + 5$ for $\overline{\gamma}(G) \geq 2$ and $w(G) \leq 8$ for $\overline{\gamma}(G) = 1$. Note, however, that for embedded graphs with fixed genus, the invariant $w^*(G)$ need not be well-defined, as G might have a single face. This could be overcome by considering polyhedral embeddings (whose facial walks are cycles and each two of them have at most a vertex or an edge in common).

There exist many graph families whose members do not involve a fixed genus embedding, but they possess structural properties which are analogous to ones for plane or embedded graphs nonetheless. A particularly interesting family in this direction is the family of *1-planar graphs*, that is, the family of graphs which can be drawn in the plane in such a way that each edge is crossed at most once. It is known that if G is a 1-planar graph, then $\delta(G) \leq 7$ and, in addition, $w(G) \leq 40$ if G is 3-connected, see [3]. For a 1-planar graph G with $\delta(G) \in \{5, 6, 7\}$ it was proved in [5] that $w(G) \leq 14$. A partial dependence between $\delta(G)$ and $g(G)$ is also known: if $\delta(G) \geq 5$, then $g(G) \leq 4$ and $g(G) = 3$ for $\delta(G) \in \{6, 7\}$, see [3]; however, for $\delta(G) \in \{3, 4\}$, an upper bound for $g(G)$ is still not known. Also, not much is known on the dependence of $\mathrm{dg(G)}$ (which is a vague analogue of $w^*(G)$ for non-embedded graphs) on $w(G), g(G)$ and $\delta(G)$: so far, the only result is the one from [10] that if $\delta(G) \geq 6$ and $w(G) \geq 13$, then $\mathrm{dg(G)} = 6$.

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The binary locating-dominating number of some convex polytopes

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Abstract

In this paper the binary locating-dominating number of convex polytopes is considered. The exact value is determined and proved for convex polytopes D_n and R''_n, while for the convex polytopes R_n, Q_n and U_n a tight upper bound of the locating-dominating number is presented.

Keywords: Locating-dominating number, convex polytopes.

Math. Subj. Class.: 05C69, 05C90

1 Introduction

Let G be a simple connected undirected graph $G = (V, E)$, where V is a set of vertices, and E is a set of edges. The *open neighborhood* of a vertex $v \in V$ is $N_G(v) = \{u \in V \mid (u, v) \in E\}$ and the *closed neighborhood* is $N_G[v] = \{u \in V \mid (u, v) \in E\} \cup \{v\}$. We write $N(v)$ or $N[v]$ if the graph G is clear from the context [4]. For a graph $G = (V, E)$ a *dominating set* is a vertex set $D \subseteq V$ such that the union of the closed neighborhoods of the vertices in D is all of V; that is, $\bigcup_{v \in D} N[D] = V$. Equivalently, each vertex not in D is adjacent to at least one vertex in D, e.g. for every vertex $v \in V \setminus D$, $N(v) \cap D \neq \emptyset$. The

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domination number of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G.

The concept of a dominating set can also be studied through assigning a weight of 1 will be assigned to the vertices in D and a weight of 0 to the vertices of $V \setminus D$. In this case, D is a dominating set of G if for every vertex in G the sum of weights for closed neighborhoods is at least 1, i.e. $|N[v] \cap S| \geq 1$ for each $v \in V$. A dominating set $S \subseteq V$ is a *binary locating-dominating set* if for every two different vertices $u, v \in V \setminus S$ holds $N(u) \cap S \neq N(v) \cap S$ ([12]). The *binary locating-dominating number of* G, denoted by $\gamma_{l-d}(G)$, is the minimum cardinality of a binary locating-dominating set. In the sequel all terms about the locating-dominating number or set is denoted by binary locating-dominating number or set.

The article [11] studies the smallest cardinalities of locating-dominating codes on chains and cycles and the extreme values of the cardinality of a minimum r-identifying or r-locating-dominating code in any connected undirected graph G having a given number, n, of vertices is studied in [8]. For more information about these issues, see [3, 9, 14, 15, 21]. The authors of the papers [23, 24, 27] study the single-fault-tolerant locating-dominating sets and an open neighborhood locating-dominating sets in trees. More information on locating-dominating sets can be found in [12, 13, 15, 25].

The identifying code problem and binary locating-dominating problem are NP-hard in a general case [6, 7]: I. Charon et al. proved in [7] that, given a graph G and an integer k, the decision problem of the existence of an r-identifying code, or of an r-locating-dominating code, of size at most k in G, is NP-complete for any r.

The comprehensive list of papers related to identifying code and binary locating-dominating problems were given in [19].

The following theorem gives a tight lower bound of binary locating-dominating number on regular graphs:

Theorem 1.1 (Slater [26])**.** *If G is a regular graph of degree r, then*

$$\gamma_{l-d}(G) \geq \left\lceil \frac{2 \cdot |V(G)|}{r+3} \right\rceil .$$

Graphs of convex polytopes were introduced by Bača [1]. The classes of convex polytopes Q_n and R_n were introduced in [2]. The metric dimension of convex polytopes D_n, Q_n and R_n are equal to 3, as was proved in [16]. In [17] it was proven that metric dimension of convex polytopes S_n, T_n and U_n is also equal to 3. Minimal doubly resolving sets and the strong metric dimension of convex polytopes D_n and T_n are studied in [18]. M. Salman et al. [22] were considering three similar optimization problems: the fault-tolerant metric dimension problem, the local metric dimension problem and the strong metric dimension problem of two convex polytopes S_n and U_n.

2 A modified integer linear programming formulation

An integer linear programming (ILP) formulation of minimum identifying code problem was given in [5]. If S is an identifying set, then decision variables x_i are defined as:

$$x_i = \begin{cases} 1, & i \in S \\ 0, & i \notin S \end{cases}$$

Then, the ILP formulation of minimum identifying code problem from [5] is presented as follows:

$$\min \sum_{i \in V} x_i \tag{2.1}$$

subject to

$$\sum_{j \in N[i]} x_j \geq 1, \qquad i \in V \tag{2.2}$$

$$\sum_{j \in N[i] \nabla N[k]} x_j \geq 1, \qquad i, k \in V, i \neq k \tag{2.3}$$

$$x_i \in \{0, 1\}, \qquad i \in V \tag{2.4}$$

The objective function (2.1) ensures that the identifying code set has a minimal cardinality, and constraints (2.2) defines S to be a dominating set. Identifying feature is represented by constraints (2.3) while the binary nature of decision variables x_i are given by constraints (2.4).

This formulation can not be directly used for the binary locating-dominating problem. Therefore, it needs to be adapted by changing constraints (2.3) into the constraints (2.5).

$$x_i + x_k + \sum_{j \in N(i) \nabla N(k)} x_j \geq 1, \qquad i, k \in V, i \neq k \tag{2.5}$$

Constraints (2.3) and (2.5) are the same when vertices i and k are not neighbors, e.g. $N[i] \nabla N[k] = \{i, j\} \cup (N(i) \nabla N(k))$. The change between (2.3) and (2.5) is reflected only when vertices i and k are neighbors, i.e. $i \in N(k)$. Then, by constraints (2.5), at least one of vertices i, k or some $j \in N(i) \nabla N(k)$ must be in S. When i and k are not neighbors, then $N[i] \nabla N[k] = \{i, j\} \cup (N(i) \nabla N(k))$, so constraints (2.3) and (2.5) are equal.

In [28] it was noted that if $d(u, v) \geq 3$ then u, v has no neighbors in common, therefore, $N(u) \cap S \neq N(v) \cap S$ need not be checked for equivalence. This becomes computationally important for large graphs as it allows us to minimize the number of constraints generated by the locating requirement. Using this idea, constraints (2.5) would be further improved:

$$x_i + x_k + \sum_{j \in N(i) \nabla N(k)} x_j \geq 1, \qquad i, k \in V, i \neq k, d(i, k) \leq 2 \tag{2.6}$$

The proposed formulation with a reduced number of constraints can be used to find the exact optimal values for problems of small dimensions. Moreover, as it can be seen from [10], ILP formulation can be tackled by efficient metaheuristic approaches for obtaining suboptimal solutions for large dimensions.

3 The exact values

3.1 Convex polytope D_n

The graph of convex polytope D_n, on Figure 1, was introduced in [16]. It consists of $2n$ 5-sided faces and a pair of n-sided faces. Mathematically, it has vertex set $V(D_n) = \{a_i, b_i, c_i, d_i \mid i = 0, 1, \ldots, n-1\}$ and edge set $E(D_n) = \{(a_i, a_{i+1}), (d_i, d_{i+1}), (a_i, b_i), (b_i, c_i), (c_i, d_i), (b_{i+1}, c_i) \mid i = 0, 1, \ldots, n-1\}$. Note that arithmetic in the subscripts is performed modulo n.

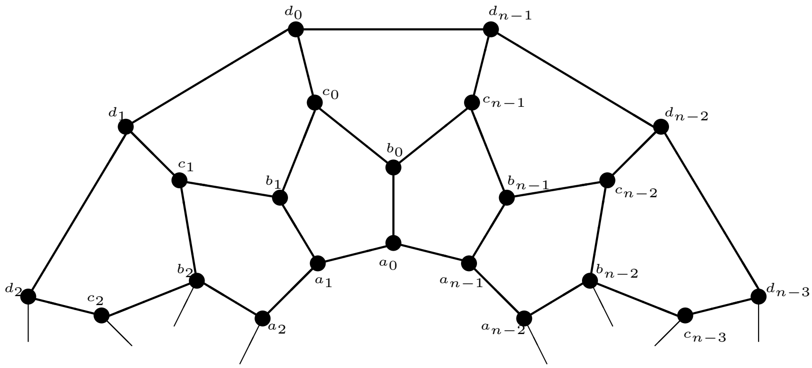

Figure 1: The graph of convex polytope D_n.

Table 1: Locating-dominating vertices in D_n.

n	$v \in V \setminus S$	$S \bigcap N[v]$	$v \in V \setminus S$	$S \bigcap N[v]$
$3k$	a_{3i}	$\{a_{3i+1}, b_{3i}\}$	a_{3i+2}	$\{a_{3i+1}\}$
	b_{3i+1}	$\{a_{3i+1}, c_{3i+1}\}$	b_{3i+2}	$\{c_{3i+1}\}$
	c_{3i}	$\{b_{3i}\}$	c_{3i+2}	$\{b_{3(i+1)}, d_{3i+2}\}$
	d_{3i}	$\{d_{3(i-1)+2}\}$	d_{3i+1}	$\{c_{3i+1}, d_{3i+2}\}$
$3k+1$	a_{3i}	$\{a_{3(i-1)+2}, b_{3i}\}$	a_{3i+1}	$\{a_{3i+2}\}$
	b_{3i+1}	$\{c_{3i+1}\}$	b_{3i+2}	$\{a_{3i+2}, c_{3i+1}\}$
	c_{3i}	$\{b_{3i}, d_{3i}\}$	c_{3i+2}	$\{b_{3(i+1)}\}$
	d_{3i+1}	$\{c_{3i+1}, d_{3i}\}$	d_{3i+2}	$\{d_{3(i+1)}\}$
	a_{3k}	$\{b_{3k}, a_{3(k-1)+2}\}$	c_{3k}	$\{b_{3k}, d_{3k}\}$
	a_0	$\{b_0\}$		
$3k+2$	a_{3i+1}	$\{a_{3i}, b_{3i+1}\}$	a_{3i+2}	$\{a_{3(i+1)}\}$
	b_{3i}	$\{a_{3i}, c_{3(i-1)+2}\}$	b_{3i+2}	$\{c_{3i+2}\}$
	c_{3i}	$\{b_{3i+1}, d_{3i}\}$	c_{3i+1}	$\{b_{3i+1}\}$
	d_{3i+1}	$\{d_{3i}\}$	d_{3i+2}	$\{c_{3i+2}, d_{3(i+1)}\}$
	b_{3k}	$\{a_{3k}, c_{3(k-1)+2}\}$	c_{3k}	$\{b_{3k+1}, d_{3k}\}$
	a_{3k+1}	$\{a_{3k}, b_{3k+1}\}$	c_{3k+1}	$\{b_{3k+1}\}$
	d_{3k+1}	$\{d_{3k}\}$	b_0	$\{a_0\}$

Theorem 3.1.

$$\gamma_{l-d}(D_n) = \left\lceil \frac{4 \cdot n}{3} \right\rceil.$$

Proof. Firstly, notice that D_n is a regular graph of degree 3, with $4n$ vertices. Then, by Theorem 1.1 it holds $\gamma_{l-d}(D_n) \geq \left\lceil \frac{2 \cdot 4 \cdot n}{3+3} \right\rceil = \left\lceil \frac{4 \cdot n}{3} \right\rceil$.

Let

$$S = \begin{cases} \{a_{3i+1}, b_{3i}, c_{3i+1}, d_{3i+2} | i = 0, \ldots, k-1\}, & n = 3k \\ \{b_{3k}, d_{3k}\} \bigcup \{a_{3i+2}, b_{3i}, c_{3i+1}, d_{3i} | i = 0, \ldots, k-1\}, & n = 3k+1 \\ \{a_{3k}, b_{3k+1}, d_{3k}\} \bigcup \{a_{3i}, b_{3i+1}, c_{3i+2}, d_{3i} | i = 0, \ldots, k-1\}, & n = 3k+2 \end{cases}$$

Now, let us prove that S is a locating-dominating set of D_n. In order to do that, we need to consider three possible cases:

Case 1: $n = 3k$. As can be seen from Table 1, neighborhoods of all vertices in $V \setminus S$ and their intersections with set S are non-empty and distinct. Although some formulas for some intersections can be somewhat similar, they are distinct. For example, $S \bigcap N[a_{3i+2}] = \{a_{3(i+1)}\} \neq \{a_{3i+1}\} = S \bigcap N[b_{3i+1}]$, since indices $3(i+1) = 3i+3 \neq 3i+1$. Similarly, $S \bigcap N[d_{3i+1}] = \{c_{3i+2}, d_{3i}\} \neq \{c_{3i+2}, d_{3(i+1)}\} = S \bigcap N[d_{3i+2}]$;

Case 2: $n = 3k+1$. As in the previous case, once again, all intersections of neighborhoods $N[v]$ with set S, i.e. $S \bigcap N[v]$, are non-empty and distinct. This also can be seen from Table 1;

Case 3: $n = 3k+2$. As in both previous cases, once again, all intersections of neighborhoods $N[v]$ with set S, i.e. $S \bigcap N[v]$, are non-empty and distinct, which also can be seen from Table 1. □

3.2 Convex polytope R''_n

The graph of convex polytope R''_n on Figure 2 is introduced in [20]. It has vertex set $V = \{a_i, b_i, c_i, d_i, e_i, f_i \mid i = 0, \ldots, n-1\}$ and edge set $E = \{(a_i, a_{i+1}), (a_i, b_i), (b_i, c_i), (b_{i+1}, c_i), (c_i, d_i), (d_i, e_i), (d_{i+1}, e_i), (e_i, f_i), (f_i, f_{i+1}) \mid i = 0, \ldots, n-1\}$.

Theorem 3.2.

$$\gamma_{l-d}(R''_n) = 2 \cdot n.$$

Proof. It can be seen that R''_n is a regular graph of degree 3, with $6n$ vertices. Then, by Theorem 1.1 it holds $\gamma_{l-d}(R''_n) \geq \left\lceil \frac{2 \cdot 6 \cdot n}{3+3} \right\rceil = 2 \cdot n$. Now, let us prove that a set $S = \{b_i, e_i \mid i = 0, \ldots, n-1\}$ is a binary locating-dominating set of R''_n. Indeed, it is easy to see that all intersections $S \bigcap N[a_i] = \{b_i\}$; $S \bigcap N[c_i] = \{b_i, b_{i+1}\}$; $S \bigcap N[d_i] = \{e_{i-1}, e_i\}$ and $S \bigcap N[f_i] = \{e_i\}$ are non-empty and distinct. Since S is a binary locating-dominating set of R''_n and $|S| = 2 \cdot n$ therefore, $\gamma_{l-d}(R''_n) \leq 2 \cdot n$. Due to the previously proved fact that $\gamma_{l-d}(R''_n) \geq 2 \cdot n$, it is proven that $\gamma_{l-d}(R''_n)$ is equal to $2 \cdot n$. □

4 The upper bounds

4.1 Convex polytope Q_n

The graph of convex polytope Q_n in Figure 3, is introduced in [2]. It has vertex set $V(Q_n) = \{a_i, b_i, c_i, d_i \,|\, i = 0, 1, \ldots, n-1\}$ and edge set $E(Q_n) = \{(a_i, a_{i+1}), (b_i, b_{i+1}),$

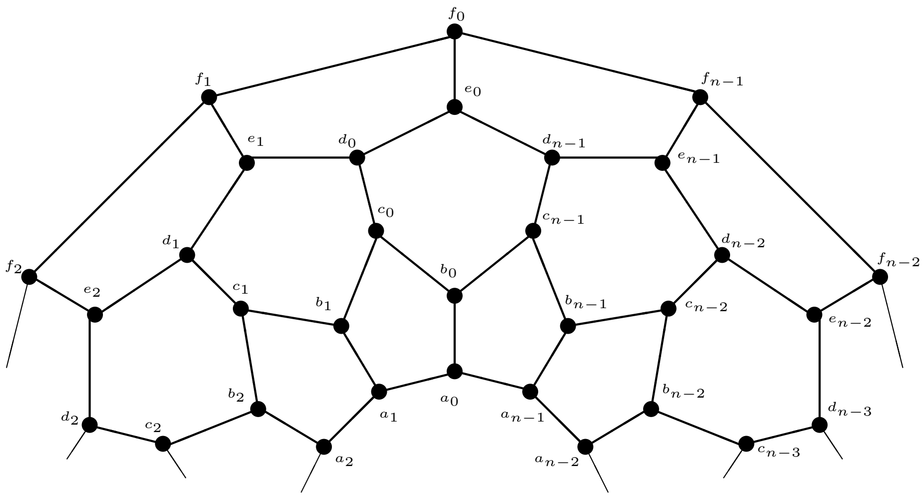

Figure 2: The graph of convex polytope R''_n.

$(d_i, d_{i+1}), (a_i, b_i), (b_i, c_i), (c_i, d_i), (b_{i+1}, c_i) \mid i = 0, 1, \ldots, n-1\}$. We call the cycle induced by set of vertices $\{a_0, a_1, \ldots, a_{n-1}\}$ the inner cycle, the cycle induced by $\{d_0, d_1, \ldots, d_{n-1}\}$ the outer cycle, and the middle cycle are induced by set of vertices $\{b_0, b_1, \ldots, b_{n-1}\}$. This polytope consists of n 5-sided faces, n 4-sided faces and n triangles.

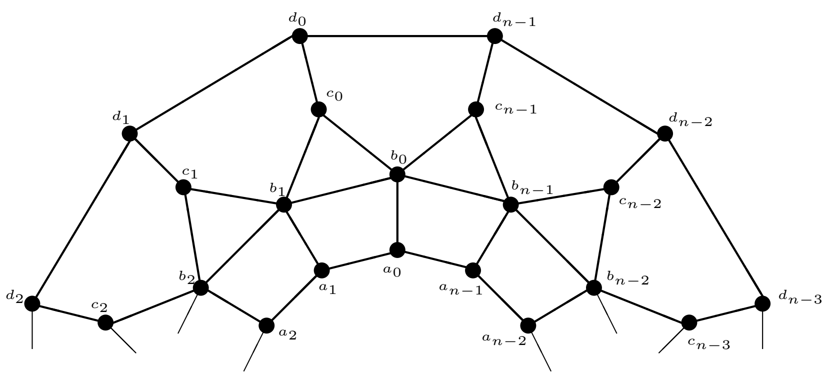

Figure 3: The graph of convex polytope Q_n.

Theorem 4.1.

$$\gamma_{l-d}(Q_n) \leq \left\lceil \frac{4 \cdot n}{3} \right\rceil,$$

and this bound is tight.

Table 2: Additional data for Q_n compared to D_n.

n	$v \in V \setminus S$	$S \bigcap N[v]$	$v \in V \setminus S$	$S \bigcap N[v]$
$3k$	b_{3i+1}	$\{a_{3i+1}, b_{3i}, c_{3i+1}\}$	b_{3i+2}	$\{b_{3(i+1)}, c_{3i+1}\}$
$3k+1$	b_{3i+1}	$\{b_{3i}, c_{3i+1}\}$	b_{3i+2}	$\{a_{3i+2}, b_{3(i+1)}, c_{3i+1}\}$
$3k+2$	b_{3i}	$\{a_{3i}, b_{3i+1}, c_{3(i-1)+2}\}$	b_{3i+2}	$\{b_{3i+1}, c_{3i+2}\}$
	b_{3k}	$\{a_{3k}, b_{3k+1}, c_{3(k-1)+2}\}$	b_0	$\{a_0, b_1, b_{3k+1}\}$

Proof. Let

$$S = \begin{cases} \{a_{3i+1}, b_{3i}, c_{3i+1}, d_{3i+2} | i = 0, \dots, k-1\}, & n = 3k \\ \{b_{3k}, d_{3k}\} \bigcup \{a_{3i+2}, b_{3i}, c_{3i+1}, d_{3i} | i = 0, \dots, k-1\}, & n = 3k+1 \\ \{a_{3k}, b_{3k+1}, d_{3k}\} \bigcup \{a_{3i}, b_{3i+1}, c_{3i+2}, d_{3i} | i = 0, \dots, k-1\}, & n = 3k+2 \end{cases}$$

Note that this set is the same as for convex polytopes D_n. This is not a surprise, since convex polytopes Q_n have only n additional edges $(b_i, b_{i+1}), i = 0, \dots, n-1$ compared to D_n. Therefore, except vertices $b_i, i = 0, \dots, n-1$, all neighborhoods of vertices in $V \setminus S$ and their intersection with set S are the same as in Table 1. Additional data is presented in Table 2.

As can be seen from Table 3 and additional data from Table 2, in all three cases, neighborhoods of all vertices in $V \setminus S$ and their intersection with set S are non-empty and distinct. Therefore set S is a locating-dominating set for Q_n. Since $|S| = \lceil \frac{4 \cdot n}{3} \rceil$ therefore, $\gamma_{l-d}(Q_n) \leq \lceil \frac{4 \cdot n}{3} \rceil$.

Using the CPLEX solver on the integer linear programming formulation (2.1), (2.2), (2.4), and (2.6) we have obtained optimal solutions: $\gamma_{l-d}(Q_5) = 7$, $\gamma_{l-d}(Q_6) = 8$, $\gamma_{l-d}(Q_7) = 10$, $\dots$, $\gamma_{l-d}(Q_{28}) = 38$, $\gamma_{l-d}(Q_{29}) = 39$ and $\gamma_{l-d}(Q_{30}) = 40$ which all match the proposed upper bound in this theorem. Therefore, the proposed upper bound is tight. □

4.2 Convex polytope R_n

The graph of convex polytope R_n, on Figure 4, has been introduced in [2]. It has vertex set $V = \{a_i, b_i, c_i \mid i = 0, \dots, n-1\}$ and edge set $E = \{(a_i, a_{i+1}), (a_i, b_i), (a_{i+1}, b_i), (b_i, b_{i+1}), (b_i, c_i), (c_i, c_{i+1}) \mid i = 0, \dots, n-1\}$. This graph consists of n 4-sided faces and $2n$ triangles.

Theorem 4.2.

$$\gamma_{l-d}(R_n) \leq n,$$

and this bound is tight.

Proof. Let $S = \{b_i \mid i = 0, \dots, n-1\}$. It is easy to see that all intersections $S \bigcap N[a_i] = \{b_{i-1}, b_i\}$ and $S \bigcap N[c_i] = \{b_i\}$ are non-empty and distinct. Since S is a binary locating-dominating set of R_n and $|S| = n$ therefore, $\gamma_{l-d}(R_n) \leq n$.

Using the CPLEX solver on integer linear programming formulation (2.1), (2.2), (2.4), and (2.6), we have obtained optimal solutions. For $5 \leq n \leq 31$, $\gamma_{l-d}(R_n) = n$, which match the proposed upper bound in this theorem. Therefore, the proposed upper bound is tight. □

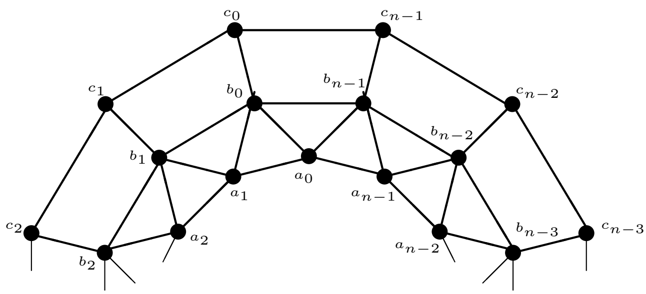

Figure 4: The graph of convex polytope R_n.

4.3 Convex polytope U_n

Mathematically, the graph of convex polytope U_n, on Figure 5, introduced in [17], has vertex set $V = \{a_i, b_i, c_i, d_i, e_i \mid i = 0, \ldots, n-1\}$ and edge set $E = \{(a_i, a_{i+1}), (a_i, b_i), (b_i, b_{i+1}), (b_i, c_i), (c_i, d_i), (c_{i+1}, d_i), (d_i, e_i), (e_i, e_{i+1}) \mid i = 0, \ldots, n-1\}$. This graph in Figure 5 has $2n$ 5-sided faces and n 4-sided faces.

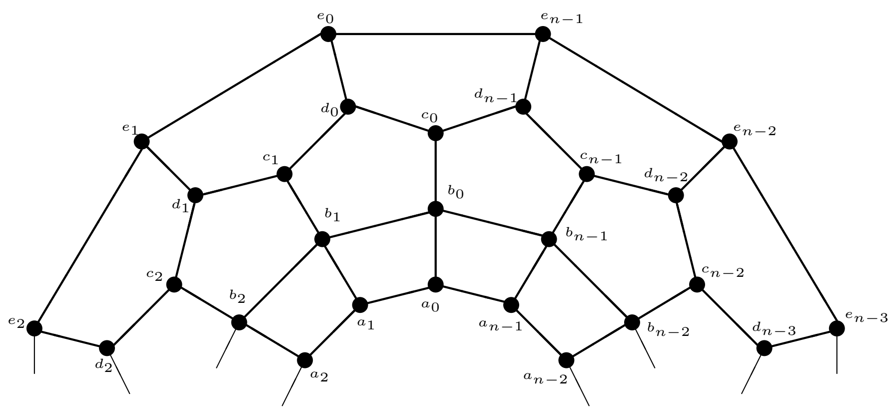

Figure 5: The graph of convex polytope U_n.

Theorem 4.3.

$$\gamma_{l-d}(U_n) \leq \left\lceil \frac{5 \cdot n}{3} \right\rceil,$$

and this bound is tight.

Proof. If $n = 3k$, let $S = \{a_{3i+1}, b_{3i}, c_{3i+1}, d_{3i+2}, e_{3i} \mid i = 0, \ldots, k-1\}$, if $n = 3k+1$ let $S = \{a_{3k}, c_{3k}, e_{3k}\} \bigcup \{a_{3i}, b_{3i+2}, c_{3i}, d_{3i+1}, e_{3i} \mid i = 0, \ldots, k-1\}$ and if $n = 3k+2$ let $S = \{a_{3k+1}, b_{3k}, c_{3k+1}, e_{3k+1}\} \bigcup \{a_{3i+1}, b_{3i}, c_{3i+1}, d_{3i+2}, e_{3i+1} \mid i = 0, \ldots, k-1\}$.

Now, let us prove that S is a locating-dominating set of U_n. In order to do that, as we did in proofs of previous Theorems, we need to consider three possible cases. As it can be seen from Table 3, in all three cases, neighborhoods of all vertices in $V \setminus S$ and their intersection with set S are non-empty and distinct.

Using the CPLEX solver on the integer linear programming formulation (2.1), (2.2), (2.4), and (2.6) we have obtained optimal solutions: $\gamma_{l-d}(U_5) = 9$, $\gamma_{l-d}(U_6) = 10$, $\gamma_{l-d}(U_7) = 12, \ldots, \gamma_{l-d}(U_{22}) = 38$, $\gamma_{l-d}(U_{23}) = 39$ and $\gamma_{l-d}(U_{24}) = 40$ which all match the proposed upper bound in this theorem. Therefore, the proposed upper bound is tight. □

Table 3: Locating-dominating vertices in U_n.

n	$v \in V \setminus S$	$S \bigcap N[v]$	$v \in V \setminus S$	$S \bigcap N[v]$
$3k$	a_{3i}	$\{a_{3i+1}, b_{3i}\}$	a_{3i+2}	$\{a_{3i+1)}\}$
	b_{3i+1}	$\{a_{3i+1}, b_{3i}, c_{3i+1}\}$	b_{3i+2}	$\{b_{3(i+1)}\}$
	c_{3i}	$\{b_{3i}, d_{3(i-1)+2}\}$	c_{3i+2}	$\{d_{3i+2}\}$
	d_{3i}	$\{c_{3i+1}, e_{3i}\}$	d_{3i+1}	$\{c_{3i+1}\}$
	e_{3i+1}	$\{e_{3i}\}$	e_{3i+2}	$\{d_{3i+2}, e_{3(i+1)}\}$
$3k+1$	a_{3i+1}	$\{a_{3i}\}$	a_{3i+2}	$\{a_{3(i+1)}, b_{3i+2}\}$
	b_{3i}	$\{a_{3i}, b_{3(i-1)+2}, c_{3i}\}$	b_{3i+1}	$\{b_{3i+2}\}$
	c_{3i+1}	$\{d_{3i+1}\}$	c_{3i+2}	$\{b_{3i+2}, d_{3i+1}\}$
	d_{3i}	$\{c_{3i}, e_{3i}\}$	d_{3i+2}	$\{c_{3(i+1)}\}$
	e_{3i+1}	$\{d_{3i+1}, e_{3i}\}$	e_{3i+2}	$\{e_{3(i+1)}\}$
	b_{3k}	$\{a_{3k}, b_{3(k-1)+2}, c_{3k}\}$	d_{3k}	$\{c_{3k}, e_{3k}\}$
	b_0	$\{a_0, c_0\}$		
$3k+2$	a_{3i}	$\{a_{3i+1}, b_{3i}\}$	a_{3i+2}	$\{a_{3i+1}\}$
	b_{3i+1}	$\{a_{3i+1}, b_{3i}, c_{3i+1}\}$	b_{3i+2}	$\{b_{3(i+1)}\}$
	c_{3i}	$\{b_{3i}, d_{3(i-1)+2}\}$	c_{3i+2}	$\{d_{3i+2}\}$
	d_{3i}	$\{c_{3i+1}\}$	d_{3i+1}	$\{c_{3i+1}, e_{3i+1}\}$
	e_{3i}	$\{e_{3i+1}\}$	e_{3i+2}	$\{d_{3i+2}, e_{3i+1}\}$
	a_{3k}	$\{a_{3k+1}, b_{3k}\}$	c_{3k}	$\{b_{3k}, d_{3(k-1)+2}\}$
	d_{3k}	$\{c_{3k+1}\}$	e_{3k}	$\{e_{3k+1}\}$
	b_{3k+1}	$\{a_{3k+1}, b_0, b_{3k}, c_{3k+1}\}$	d_{3k+1}	$\{c_{3k+1}, e_{3k+1}\}$
	a_0	$\{a_1, a_{3k+1}, b_0\}$	c_0	$\{b_0\}$
	e_0	$\{e_1, e_{3k+1}\}$		

5 Conclusions

In this paper, we are studying the locating-dominating sets and the binary locating-dominating number of some convex polytopes. We are dealing with some classes of convex polytopes by considering classes: D_n, R''_n, R_n, Q_n and U_n. For D_n and R''_n exact values are obtained and proved, while for R_n, Q_n and U_n tight upper bounds are given.

Future work can be directed towards determining a binary locating-dominating set of some other challenging classes of graphs. The other promising direction for future work is solving of some other similar graph problem on convex polytopes.

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Spectra and structural polynomials of graphs of relevance to the theory of molecular conduction

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Abstract

In chemistry and physics, distortivity of π-systems (stabilisation of bond-alternated structures) is an important factor in the calculation of geometric, energetic, and electronic properties of molecules *via* graph theoretical methods. We use the spectra of paths and cycles with alternating vertex and edge weights to obtain the eigenvalues and eigenvectors for a class of linear and cyclic ladders with alternating rung and backbone edge weights. We derive characteristic polynomials and other structural polynomials formed from the cofactors of the characteristic matrix for these graphs. We also obtain spectra and structural polynomials for ladders with flipped weights and/or Möbius topology. In all cases, the structural polynomials for the composite graphs are expressed in terms of products of polynomials for graphs of half order. This form of the expressions allows global deductions about the transmission spectra of molecular devices in the graph-theoretical theory of ballistic molecular conduction.

Keywords: Adjacency matrix, characteristic polynomial, molecular conduction, eigenvalues, weighted graphs.

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1 Introduction

Our aim is to use weighting of graphs as a tool for the study of ballistic molecular conduction in undistorted and distorted molecular and extended systems. In this article we derive the spectra and characteristic polynomials of a series of graphs that possess three common features. The first is that they are bipartite. The second is that they possess an involution that allows the graph to be expressed as a product of simpler graphs with known spectra. The third feature is termed 'distortivity' by physical scientists. This refers to the way that the spectrum changes with edge weights, and is of prime importance in theories of electronic structure, where molecular structures are modelled by graphs. It is well known to physicists and chemists that extended overlapping π-electron systems may achieve greater stability by distorting in such a way that bond lengths alternate, and the sharing of electron density across the π-system is reduced. This is known in the physics literature as Peierls distortion [13], and in the chemical literature as Jahn-Teller distortion [12]. It typically affects π-electron systems in such a way as to reduce their conductivity. In order to assess the importance of distortivity for the specific phenomenon of ballistic molecular conduction, we need explicit characteristic polynomials and spectra for families of weighted graphs representing molecules of chemical interest.

1.1 Graph theoretical background

The graphs in which we are interested are linear *ladders*, their cyclic analogues the *treadmills*, and graphs derivable from them by using (signed or zero) weights, such as linear polyacenes and (Möbius) cyclacenes, shown in Fig. 1. In graph theory terms, we can mimic

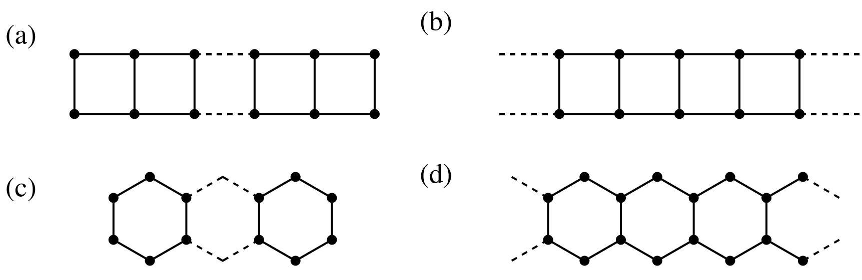


Figure 1: Families of graphs treated in this paper: (a) ladders; (b) treadmills; (c) linear polyacenes; (d) cyclic polyacenes.

geometric distortion of a molecular framework by studying weighted graphs in which edge weights alternate [11]. Adjacency matrices of such graphs have been studied by Gover [9] in the form of 2-Toeplitz matrices. Gover gave an explicit solution for the spectra of 2-Toeplitz matrices of odd dimension, and an implicit solution for even dimensions. These solutions form the basis for our treatment of ladders and treadmills. Ladders (treadmills) comprise two backbone chains (rings), that are linked by 'rungs'. We shall alternate the weights on the rungs, and separately on the edges comprising the two backbones, in such a way that an involution symmetry is preserved. This symmetry element swaps vertices in upper and lower backbone chains of the graphs and is crucial for the solution of the secular problem for distorted and undistorted systems. The use of symmetry splits the secular matrices of the graphs into two non-interacting blocks, each of which represents a single path (or cycle) with alternating weighted vertices and edges. It is these backbone graphs that possess the analytical solutions derived previously by Gover [9] and Shin [18].

An active avenue of research is exploration of the influence of molecular topology in conduction behaviour. We therefore include certain graphs with edge weights having flipped signs, and/or with a pair of crossed backbone edges. These flipped and crossed graphs are sometimes called Möbius graphs [6]. The cases that we consider here have closed-form spectra and structural polynomials that can be derived using the methodology used for unflipped, uncrossed graphs.

1.2 Physical motivation

The physical context for the present mathematical exploration is that electronic structure of unsaturated carbon networks is qualitatively modelled using spectral graph theory. In particular, the basic reason for our interest in the graphs described in this paper is our research into molecular conductivity in small molecules [14, 15] using the source-sink-potential (SSP) method of Ernzerhof *et al.* [3, 4, 5, 20]. This approach uses graph theory as a vehicle for showing important qualitative features in electron transmission for individual molecules. Central to the SSP method is the idea of a molecular device based on a molecular graph, in which the effects of infinite attached wires are represented by two special extra vertices, which behave respectively as a source and sink (of electrons). We have shown [14] that electronic transmission in this model can be expressed using a basic set of polynomials related to the molecular graph, $\mathcal{G}$. These are the characteristic polynomial,

$$s(E) = \det(E\mathbf{1} - \mathbf{A}), \tag{1.1}$$

and the cofactors of the characteristic matrix,

$$\jmath_{\mathrm{pq}}(E) = (-1)^{\mathrm{p+q}} \det(E\mathbf{1} - \mathbf{A})^{[\mathrm{p,q}]} = (E\mathbf{1} - \mathbf{A})^{-1}_{\mathrm{pq}} s(E), \tag{1.2}$$

where E is the energy of the transmitted electron, $\mathbf{1}$ is the $n \times n$ unit matrix and $\mathbf{A}$ is the $n \times n$ adjacency matrix of the graph $\mathcal{G}$ of order n. The indices in square-brackets refer to the sets of rows and columns deleted from the determinant of the characteristic matrix.

The eigenvalue problem,

$$\mathbf{A}\mathbf{c}_{\mathrm{k}} = \mathbf{c}_{\mathrm{k}}\epsilon_{\mathrm{k}}, \tag{1.3}$$

allows us to define the n eigenvalues $\{\epsilon_{\mathrm{k}}\}$, and the corresponding eigenvectors $\mathbf{c}_{\mathrm{k}}$. Spectral decomposition allows us to write

$$s(E) = \prod_{\mathrm{k}=1}^{n} (E - \epsilon_{\mathrm{k}}), \tag{1.4}$$

and spectral resolution of the inverse gives a general expression for all $\jmath_{\mathrm{pq}}(E)$ polynomials in terms of eigenvectors and eigenvalues of $\mathbf{A}$:

$$\jmath_{\mathrm{pq}}(E) = \sum_{\mathrm{k}=1}^{n} \frac{c_{\mathrm{pk}} c_{\mathrm{qk}}}{E - \epsilon_{\mathrm{k}}}, \tag{1.5}$$

where c_{pk} is the pth entry in the kth eigenvector, $\mathbf{c}_{\mathrm{k}}$. In what follows we will find it useful to switch between the two approaches, *viz.* calculating structural polynomials from determinants of characteristic matrices, or from explicit solutions of the eigenvalue problem Eq. (1.3).

For a specific device in which vertices p and q of the molecular graph are attached to infinite conducting wires, one needs just four polynomials, namely, s, $\jmath_{\mathrm{pp}}$, $\jmath_{\mathrm{qq}}$, and

$$v_{\mathrm{pq,pq}}(E) = \det(E\mathbf{1} - \mathbf{A})^{[\mathrm{pq,pq}]} \tag{1.6}$$

to deduce an expression for the transmission, $T(E)$, of an incoming stream of electrons [14]. These are all characteristic polynomials derived from vertex-deleted graphs:

$$\begin{aligned} s &\equiv \varphi(\mathcal{G}, E), \\ t &\equiv \jmath_{\mathrm{pp}} = \varphi(\mathcal{G} - \mathrm{p}, E), \\ u &\equiv \jmath_{\mathrm{qq}} = \varphi(\mathcal{G} - \mathrm{q}, E), \\ v &\equiv v_{\mathrm{pq,pq}}(E) = \varphi(\mathcal{G} - \mathrm{p} - \mathrm{q}, E), \end{aligned} \tag{1.7}$$

where $\varphi(\mathcal{G}, E)$ is the characteristic polynomial of graph $\mathcal{G}$, and the letters s, t, u, and v refer to literature notation [14]. The formula for $v_{\mathrm{pq,pq}}$ can be deduced using Jacobi's relation [19]:

$$s v_{\mathrm{pq,pq}} = \jmath_{\mathrm{pp}} \jmath_{\mathrm{qq}} - \jmath_{\mathrm{pq}}^2. \tag{1.8}$$

For convenience, we refer to $s(E)$, the $\jmath_{\mathrm{pq}}(E)$ and $v(E)$ as *structural polynomials.*

We have shown [15] that molecular conduction can be thought of in two different ways, *i.e.* either as occurring through molecular bonds (graph edges), or through individual molecular orbitals (eigenvectors of the graph adjacency matrix). We find that there are 11 basic categories of conduction [7, 15, 17] for molecules and that these are determined by the eigenvector coefficients. Conduction behaviour at eigenvalues of the adjacency matrix is particularly important [15]. Hence arises our interest in closed-form expressions for spectra and structural polynomials. Spectral representations of the structural polynomials are also informative, in that they allow elaboration of the SSP model to treat the physically important effects of Pauli exclusion, an effect that prevents current passing through filled orbitals. This extension of the theory is worked out in a recent paper [16].

We can summarise the key features of our approach and the main results as follows. Explicit expressions for structural polynomials, spectra and eigenvectors of weighted paths and cycles are obtained. These are useful in themselves for the discussion of distortivity and conduction. We then exploit the graph-product structure of the families of ladders, treadmills and Möbius forms to build analytical expressions for the structural polynomials and spectral properties of these graphs in terms of those of the simpler graphs. This gives compact formulas that are ultimately related to Chebyshev and similar orthogonal polynomials. It is this 'factorised' form of the final expressions that gives a powerful tool for interpretation of spectra and conduction properties of ladders, treadmills. This interpretation will be used to analyse the effects of flips and twists on conduction in physically realisable systems.

1.3 Plan

The plan of the paper is as follows. First we derive eigenvectors and eigenvalues for weighted alternating paths (Section 2), and then derive expressions for the important structural polynomials in Section 3. These results are used to derive spectra for ladders and their structural polynomials in Sections 4 and 5. The spectra for alternating cycles are derived in Section 6, and their structural polynomials in Section 7. Derivations of spectra and structural polynomials of treadmills then follow in Sections 8 and 9. Section 10 introduces important chemical graphs that can be derived from ladders and treadmills. We end with a brief conclusion.

Our explicit treatment of cases necessarily leads to a large number of equations, but the central results are Eqs. (5.6) and (9.3), which show the generic relationships between the structural polynomials of ladders and chains, and treadmills and cycles, respectively. The structural polynomials for weighted chains are given in Eqs. (3.12) and (3.13), and weighted cycles in Eqs. (7.13) and (7.14) and for flipped cycles in Eqs. (7.15) and (7.16). The blocks of equations giving the results are: Eqs. (5.7) to (5.10) for ladders; Eqs. (9.4) and (9.5) for treadmills; Eqs. (9.9) and (9.10) for flipped treadmills; Eqs. (9.14) and (9.15) for Möbius treadmills; Eqs. (9.19) and (9.20) for flipped Möbius treadmills.

2 The spectra of alternating weighted paths $P_M(a, b \mid c, d)$

We consider paths, $P_M(a, b \mid c, d)$, with alternating vertex weights a, b, and edge weights c, d. Eigenvalues and eigenvectors for such weighted paths have been deduced by Gover [9] and Shin [18]. Gover used recursion to show that the spectrum of the odd-vertex chain, P_{2N+1}, could be expressed in terms of two sets of polynomials. One is the Chebyshev polynomials of the second kind, U_N. The other set of polynomials satisfy the Chebyshev recursion relation, but with different initial values. The eigenvectors for the odd paths are evaluated at the zeroes of the polynomial U_N. The even paths have an analogous form for eigenvectors and eigenvalues, but one of the quantities cannot be evaluated analytically. We discuss odd and even paths separately.

2.1 The odd path, $P_{2N+1}(a, b \mid c, d)$

A path, $P_{2N+1}(a, b \mid c, d)$, with $2N+1$ vertices is shown in Fig. 2. It is convenient to write

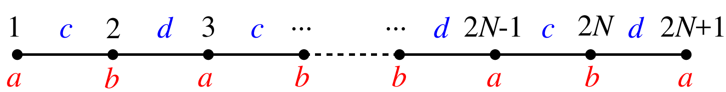


Figure 2: A chain, $P_{2N+1}(a, b \mid c, d)$, with $2N + 1$ vertices and alternating vertex weights (a, b), and edge weights (c, d).

the adjacency matrix, $\mathbf{A}^{\mathrm{P}}$, for this bipartite graph in the form

$$\mathbf{A}^{\mathrm{P}} = \begin{pmatrix} a\mathbf{1}_{N+1} & \mathbf{B}^{\mathrm{P}} \\ \left(\mathbf{B}^{\mathrm{P}}\right)^{\mathrm{T}} & b\mathbf{1}_N \end{pmatrix}, \tag{2.1}$$

where $\mathbf{1}_h$ symbolises a unit matrix of dimension h, and superscript T indicates a transpose. We place the $(N + 1)$ odd-numbered vertices shown in Fig. 2 in the first block, and the N

even-numbered vertices in the second. The $(N+1)\times N$-dimensional matrix $\mathbf{B}^{\mathrm{P}}$ is then

$$\mathbf{B}^{\mathrm{P}} = \begin{pmatrix} c & 0 & \cdots & 0 & 0 \\ d & c & \ddots & & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & & \ddots & d & c \\ 0 & 0 & \cdots & 0 & d \end{pmatrix}. \tag{2.2}$$

In order to find the eigenvalues of the matrix $\mathbf{A}^{\mathrm{P}}$ we can use the fact that the blocks on the diagonal are invariant to any unitary transformation. Therefore, a singular value decomposition of the off diagonal block will render the whole matrix in a form in which each block is diagonal or pseudo-diagonal. This technique has been used [15], for example, to provide a compact derivation of the Coulson-Rushbrooke theorem for bipartite graphs [1]. The singular value decomposition [8, Sections 2.5.3 and 2.5.6] of $\mathbf{B}^{\mathrm{P}}$ can be written as

$$\mathbf{B}^{\mathrm{P}}\mathbf{X}^{\mathrm{P}} = \mathbf{Y}^{\mathrm{P}}\boldsymbol{\sigma}^{\mathrm{P}}, \tag{2.3}$$

where $\mathbf{X}^{\mathrm{P}}$, and $\mathbf{Y}^{\mathrm{P}}$ are N- and $(N+1)$-dimensional orthogonal matrices, respectively. The $(N+1)\times N$-dimensional rectangular matrix, $\boldsymbol{\sigma}^{\mathrm{P}}$, is "diagonal", *i.e.*

$$\begin{pmatrix} \sigma_1^{\mathrm{P}} & 0 & \cdots & 0 \\ 0 & \sigma_2^{\mathrm{P}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & & \ddots & \sigma_N^{\mathrm{P}} \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \tag{2.4}$$

and the singular values $\sigma_{\mathrm{k}}^{\mathrm{P}} \geq 0$, have labels k. The theory of singular value decomposition tells us further that

$$\begin{aligned} \left(\mathbf{B}^{\mathrm{P}}\right)^{\mathrm{T}}\mathbf{B}^{\mathrm{P}}\mathbf{X}_k^{\mathrm{P}} &= \mathbf{X}_{\mathrm{k}}^{\mathrm{P}}(\sigma_{\mathrm{k}}^{\mathrm{P}})^2 \qquad \text{for } \mathrm{k}=1,\dots,N, \\ \mathbf{B}^{\mathrm{P}}\left(\mathbf{B}^{\mathrm{P}}\right)^{\mathrm{T}}\mathbf{Y}_{\mathrm{k}}^{\mathrm{P}} &= \mathbf{Y}_{\mathrm{k}}^{\mathrm{P}}(\sigma_{\mathrm{k}}^{\mathrm{P}})^2 \qquad \text{for } \mathrm{k}=1,\dots,N+1, \end{aligned} \tag{2.5}$$

with $\sigma_{\mathrm{k}}^{\mathrm{P}} > 0$ for $\mathrm{k}=1,\dots,N$, and $\sigma_{N+1}^{\mathrm{P}} = 0$. We note that the $N\times N$-dimensional positive definite tridiagonal matrix

$$\left(\mathbf{B}^{\mathrm{P}}\right)^{\mathrm{T}}\mathbf{B}^{\mathrm{P}} = \begin{pmatrix} c^2+d^2 & cd & 0 & \cdots & 0 & 0 \\ cd & c^2+d^2 & cd & \ddots & & 0 \\ 0 & cd & c^2+d^2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & & \ddots & \ddots & c^2+d^2 & cd \\ 0 & 0 & \cdots & 0 & cd & c^2+d^2 \end{pmatrix} \tag{2.6}$$

represents the adjacency matrix of a path of length N with equal vertex weights $(c^2 + d^2)$, and equal edge weights cd, so the eigenvalues are

$$(\sigma_{\mathrm{k}}^{\mathrm{P}})^2 = c^2 + d^2 + 2cd \cos \theta_{\mathrm{k}}^{\mathrm{P}} \qquad \text{for } \mathrm{k} = 1, 2, \ldots, N, \tag{2.7}$$

where the angle

$$\theta_{\mathrm{k}}^{\mathrm{P}} = \frac{\pi \mathrm{k}}{N+1} \tag{2.8}$$

also describes the orthonormal eigenvectors

$$X_{\mathrm{pk}}^{\mathrm{P}} = N_{\mathrm{k}}^{\mathrm{P}} \sin \mathrm{p}\theta_{\mathrm{k}}^{\mathrm{P}}, \tag{2.9}$$

and the normalisation factor is

$$N_{\mathrm{k}}^{\mathrm{P}} = \sqrt{\frac{2}{N+1}}. \tag{2.10}$$

The $(N+1) \times (N+1)$-dimensional semi-definite tridiagonal matrix, on the other hand,

$$\mathbf{B}^{\mathrm{P}} \left(\mathbf{B}^{\mathrm{P}}\right)^{\mathrm{T}} = \begin{pmatrix} c^2 & cd & 0 & \cdots & 0 & 0 \\ cd & c^2+d^2 & cd & \ddots & & 0 \\ 0 & cd & c^2+d^2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & & \ddots & \ddots & c^2+d^2 & cd \\ 0 & 0 & \cdots & 0 & cd & d^2 \end{pmatrix}, \tag{2.11}$$

has no such simple expressions for its eigenvectors, but they can be derived directly from the singular value decomposition. We can use Eq. (2.3) to deduce that

$$Y_{\mathrm{pk}}^{\mathrm{P}} = \frac{1}{\sigma_{\mathrm{k}}^{\mathrm{P}}} \left(cX_{\mathrm{pk}}^{\mathrm{P}} + dX_{\mathrm{p-1,k}}^{\mathrm{P}}\right) \qquad \text{for } \mathrm{k}, \mathrm{p} = 1, 2, \ldots, N. \tag{2.12}$$

We note from Eq. (2.9) that $\mathrm{p} = 0$ implies $X_{0\mathrm{k}}^{\mathrm{P}} = 0$, and $\mathrm{p} = N+1$, implies $X_{N+1,\mathrm{k}}^{\mathrm{P}} = 0$. The nullspace vector is

$$Y_{\mathrm{p},N+1}^{\mathrm{P}} = N_{N+1}^{\mathrm{P}} \left(-1\right)^{\mathrm{p}-1} c^{\mathrm{p}-1} d^{N-\mathrm{p}+1}, \tag{2.13}$$

where the normalisation factor is

$$N_{N+1}^{\mathrm{P}} = \sqrt{\frac{d^2 - c^2}{d^{2N+2} - c^{2N+2}}}. \tag{2.14}$$

We define the $(2N+1)$-dimensional orthogonal matrix

$$\mathbf{W}^{\mathrm{P}} = \begin{pmatrix} \mathbf{Y}^{\mathrm{P}} & \mathbf{0} \\ \mathbf{0} & \mathbf{X}^{\mathrm{P}} \end{pmatrix}, \tag{2.15}$$

which gives

$$\left(\mathbf{W}^{\mathrm{P}}\right)^{\mathrm{T}} \mathbf{A}^{\mathrm{P}} \mathbf{W}^{\mathrm{P}} = \begin{pmatrix} a\mathbf{1}_{N+1} & \left(\mathbf{Y}^{\mathrm{P}}\right)^{\mathrm{T}} \mathbf{B}^{\mathrm{P}} \mathbf{X}^{\mathrm{P}} \\ \left(\mathbf{X}^{\mathrm{P}}\right)^{\mathrm{T}} \left(\mathbf{B}^{\mathrm{P}}\right)^{\mathrm{T}} \mathbf{Y}^{\mathrm{P}} & b\mathbf{1}_N \end{pmatrix}. \tag{2.16}$$

The off-diagonal blocks in Eq. (2.16) simplify because

$$\left(\mathbf{Y}^{\mathrm{P}}\right)^{\mathrm{T}}\mathbf{B}^{\mathrm{P}}\mathbf{X}^{\mathrm{P}} = \boldsymbol{\sigma}^{\mathrm{P}}, \tag{2.17}$$

where $\boldsymbol{\sigma}^{\mathrm{P}}$ is given in Eq. (2.4). It is evident that each block of $\left(\mathbf{W}^{\mathrm{P}}\right)^{\mathrm{T}}\mathbf{A}^{\mathrm{P}}\mathbf{W}^{\mathrm{P}}$ is diagonal, so that it comprises N two-dimensional interacting blocks of the form

$$\begin{pmatrix} a & \sigma_k^{\mathrm{P}} \\ \sigma_k^{\mathrm{P}} & b \end{pmatrix} \tag{2.18}$$

and a single one-dimensional block with eigenvalue a. The two-dimensional blocks give $2N$ eigenvalues

$$E_{\mathrm{k}\pm}^{\mathrm{P}} = \frac{1}{2}(a+b) \pm \frac{1}{2}D_{\mathrm{k}}^{\mathrm{P}} \qquad \text{for } \mathrm{k} = 1, 2, \ldots, N \tag{2.19}$$

with discriminant

$$D_{\mathrm{k}}^{\mathrm{P}} = \sqrt{(a-b)^2 + 4(\sigma_{\mathrm{k}}^{\mathrm{P}})^2} = \sqrt{(a-b)^2 + 4(c^2 + d^2 + 2cd\cos\theta_{\mathrm{k}}^{\mathrm{P}})}. \tag{2.20}$$

The eigenvectors arising from these two-dimensional blocks can be written as

$$N_{\mathrm{k}\pm}^{\mathrm{P}}\begin{pmatrix} \sigma_{\mathrm{k}}^{\mathrm{P}} \\ E_{\mathrm{k}\pm}^{\mathrm{P}} - a \end{pmatrix}, \tag{2.21}$$

where the normalisation constants are

$$N_{\mathrm{k}+}^{\mathrm{P}} = \sqrt{\frac{1}{D_{\mathrm{k}}^{\mathrm{P}}\left(E_{\mathrm{k}+}^{\mathrm{P}} - a\right)}} \qquad \text{and} \qquad N_{\mathrm{k}-}^{\mathrm{P}} = \sqrt{\frac{1}{D_{\mathrm{k}}^{\mathrm{P}}\left(a - E_{\mathrm{k}-}^{\mathrm{P}}\right)}}. \tag{2.22}$$

We can write the $2N+1$ eigenvectors of $\mathbf{A}^{\mathrm{P}}$ in the form $\mathbf{c}_{\mathrm{k}\pm}^{\mathrm{P}}$ for $\mathrm{k} = 1, 2, \ldots, N$, and $\mathbf{c}_{N+1}^{\mathrm{P}}$, the latter arising from the extra null space eigenvector in the singular value decomposition. Using expression (2.12) for $\mathbf{Y}^{\mathrm{P}}$, we obtain expressions for the eigenvector coefficients

$$\begin{aligned} c_{2\mathrm{p},\mathrm{k}\pm}^{\mathrm{P}} &= N_{\mathrm{k}\pm}^{\mathrm{P}}\left(E_{\mathrm{k}\pm}^{\mathrm{P}} - a\right)X_{\mathrm{pk}}^{\mathrm{P}}, \\ c_{2\mathrm{p}-1,\mathrm{k}\pm}^{\mathrm{P}} &= N_{\mathrm{k}\pm}^{\mathrm{P}}\left(cX_{\mathrm{pk}}^{\mathrm{P}} + dX_{\mathrm{p}-1,\mathrm{k}}^{\mathrm{P}}\right). \end{aligned} \tag{2.23}$$

There is, in addition, a single eigenvalue arising from the one-dimensional block, and corresponding to the null-space eigenvector in the $\mathbf{Y}^{\mathrm{P}}$ subspace, that is of the form

$$E_{N+1}^{\mathrm{P}} = a. \tag{2.24}$$

The corresponding eigenvector has coefficients

$$\begin{aligned} c_{2\mathrm{p},N+1}^{\mathrm{P}} &= 0, \\ c_{2\mathrm{p}-1,N+1}^{\mathrm{P}} &= N_{N+1}^{\mathrm{P}}\,(-1)^{\mathrm{p}-1}\,c^{\mathrm{p}-1}d^{N-\mathrm{p}+1}. \end{aligned} \tag{2.25}$$

2.2 The even path, $P_{2N}(a, b \mid c, d)$

The adjacency matrix for the bipartite graph, $P_{2N}(a, b \mid c, d)$, can be written as

$$\mathbf{A}^{\mathrm{P}} = \begin{pmatrix} a\mathbf{1}_N & \mathbf{B}^{\mathrm{P}} \\ \left(\mathbf{B}^{\mathrm{P}}\right)^{\mathrm{T}} & b\mathbf{1}_N \end{pmatrix}, \tag{2.26}$$

using the same numbering scheme for vertices as that shown in Fig. 2. The adjacency matrix is identical to Eq. (2.1), but with one row missing, so that $\mathbf{B}^{\mathrm{P}}$ is a *square* $N \times N$ matrix.

We again use singular decomposition [8, Sections 2.5.3 and 2.5.6] of $\mathbf{B}^{\mathrm{P}}$ as shown in Eq. (2.3), where $\mathbf{X}^{\mathrm{P}}$ and $\mathbf{Y}^{\mathrm{P}}$ are both N-dimensional orthogonal matrices. We note that the $N \times N$-dimensional positive definite tridiagonal matrix has the form

$$\left(\mathbf{B}^{\mathrm{P}}\right)^{\mathrm{T}} \mathbf{B}^{\mathrm{P}} = \begin{pmatrix} c^2+d^2 & cd & 0 & \cdots & 0 & 0 \\ cd & c^2+d^2 & cd & \ddots & & 0 \\ 0 & cd & c^2+d^2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & & \ddots & \ddots & c^2+d^2 & cd \\ 0 & 0 & \cdots & 0 & cd & c^2 \end{pmatrix}. \tag{2.27}$$

We introduce an *Ansatz* for the eigenvectors of the matrix in Eq. (2.27) as

$$X^{\mathrm{P}}_{\mathrm{pk}} = N^{\mathrm{P}}_{\mathrm{k}} \sin \mathrm{p}\theta^{\mathrm{P}}_{\mathrm{k}}, \tag{2.28}$$

where $N^{\mathrm{P}}_{\mathrm{k}}$ is a normalization factor, and $\theta^{\mathrm{P}}_{\mathrm{k}}$ is an angle yet to be determined. Examining the first row of $\left(\mathbf{B}^{\mathrm{P}}\right)^{\mathrm{T}} \mathbf{B}^{\mathrm{P}}\mathbf{X}^{\mathrm{P}}$, leaving out the normalisation factor, we find

$$(c^2+d^2)\sin\theta^{\mathrm{P}}_{\mathrm{k}} + cd\sin 2\theta^{\mathrm{P}}_{\mathrm{k}} = {\sigma^{\mathrm{P}}_{\mathrm{k}}}^2 \sin\theta^{\mathrm{P}}_{\mathrm{k}}, \tag{2.29}$$

and expanding $\sin 2\theta^{\mathrm{P}}_{\mathrm{k}}$, we obtain an expression for the eigenvalue as

$$(\sigma^{\mathrm{P}}_{\mathrm{k}})^2 = c^2 + d^2 + 2cd\cos\theta^{\mathrm{P}}_{\mathrm{k}} \tag{2.30}$$

which should be compared with Eq. (2.7). Likewise, for row p,

$$cd\sin{(\mathrm{p}-1)\theta^{\mathrm{P}}_{\mathrm{k}}} + (c^2+d^2)\sin\mathrm{p}\theta^{\mathrm{P}}_{\mathrm{k}} + cd\sin{(\mathrm{p}+1)\theta^{\mathrm{P}}_{\mathrm{k}}} = {\sigma^{\mathrm{P}}_{\mathrm{k}}}^2 \sin\mathrm{p}\theta^{\mathrm{P}}_{\mathrm{k}}. \tag{2.31}$$

Noting that

$$\sin{(\mathrm{p}-1)\theta^{\mathrm{P}}_{\mathrm{k}}} + \sin{(\mathrm{p}+1)\theta^{\mathrm{P}}_{\mathrm{k}}} = 2\sin\mathrm{p}\theta^{\mathrm{P}}_{\mathrm{k}}\cos\theta^{\mathrm{P}}_{\mathrm{k}}, \tag{2.32}$$

it is easy to see that the eigenvalue equations are satisfied for rows $\mathrm{p} = 2$ up to row $N-1$ using the expression Eq. (2.30) for the eigenvalue. However, the Nth equation is

$$cd\sin{(N-1)\theta^{\mathrm{P}}_{\mathrm{k}}} + c^2\sin N\theta^{\mathrm{P}}_{\mathrm{k}} = \sigma^{\mathrm{P}2}_{\mathrm{k}}\sin N\theta^{\mathrm{P}}_{\mathrm{k}}. \tag{2.33}$$

In order to ensure that this equation be satisfied, we would need to add a factor

$$d^2\sin N\theta^{\mathrm{P}}_{\mathrm{k}} + cd\sin{(N+1)\theta^{\mathrm{P}}_{\mathrm{k}}} \tag{2.34}$$

to the left-hand side. Dividing by $\sin\theta_{\mathrm{k}}^{\mathrm{P}}$, it follows that we require a condition on $\theta_{\mathrm{k}}^{\mathrm{P}}$, such that

$$cU_N(\cos\theta_{\mathrm{k}}^{\mathrm{P}}) + dU_{N-1}(\cos\theta_{\mathrm{k}}^{\mathrm{P}}) = 0 \qquad \text{for } -\pi \le \theta_{\mathrm{k}}^{\mathrm{P}} \le \pi. \tag{2.35}$$

Eq. (2.35) is a polynomial of order N in the variable $\cos\theta_{\mathrm{k}}^{\mathrm{P}}$, and has N solutions that fully determine the spectrum.

The norm of the eigenvectors is obtained from

$$\begin{aligned}\sum_{\mathrm{p}=1}^{N}\left(X_{\mathrm{pk}}^{\mathrm{P}}\right)^2 &= \left(N_{\mathrm{k}}^{\mathrm{P}}\right)^2\sum_{\mathrm{p}=1}^{N}\sin^2 \mathrm{p}\theta_{\mathrm{k}}^{\mathrm{P}}\\ &= \left(N_{X\mathrm{k}}^{\mathrm{P}}\right)^2\frac{1}{4}\left\{2N+1-\frac{\sin\left(2N+1\right)\theta_{\mathrm{k}}^{\mathrm{P}}}{\sin\theta_{\mathrm{k}}^{\mathrm{P}}}\right\}\\ &= 1,\end{aligned} \tag{2.36}$$

where we use a well-known trigonometrical summation [10]. This simplifies to

$$N_{\mathrm{k}}^{\mathrm{P}} = \frac{2}{\sqrt{2N+1-U_{2N}(\cos\theta_{\mathrm{k}}^{\mathrm{P}})}}, \tag{2.37}$$

which has been expressed in terms of the Chebyshev polynomial of the second kind,

$$U_{2N}(\cos\theta) = \frac{\sin(2N+1)\theta}{\sin\theta}.$$

The $N\times N$-dimensional semi-definite tridiagonal matrix $\mathbf{B}^{\mathrm{P}}\left(\mathbf{B}^{\mathrm{P}}\right)^{\mathrm{T}}$, has eigenvectors $\mathbf{Y}^{\mathrm{P}}$ that can be derived directly from the singular value decomposition as

$$Y_{\mathrm{pk}}^{\mathrm{P}} = \frac{N_{\mathrm{k}}^{\mathrm{P}}}{\sigma_{\mathrm{k}}^{\mathrm{P}}}\left[c\sin \mathrm{p}\theta_{\mathrm{k}}^{\mathrm{P}} + d\sin\left(\mathrm{p}-1\right)\theta_{\mathrm{k}}^{\mathrm{P}}\right]. \tag{2.38}$$

The expression for the eigenvalues,

$$E_{\mathrm{k}\pm}^{\mathrm{P}} = \frac{1}{2}(a+b) \pm \frac{1}{2}D_{\mathrm{k}}^{\mathrm{P}} \qquad \text{for } \mathrm{k} = 1,2,\ldots,N, \tag{2.39}$$

where

$$D_{\mathrm{k}}^{\mathrm{P}} = \sqrt{(a-b)^2 + 4(c^2+d^2+2cd\cos\theta_{\mathrm{k}}^{\mathrm{P}})}, \tag{2.40}$$

is identical to that for the odd path (Eq. (2.19)), apart from the difference in angle $\theta_{\mathrm{k}}^{\mathrm{P}}$. The eigenvector entries are

$$\begin{aligned} c_{2\mathrm{p},\mathrm{k}\pm}^{\mathrm{P}} &= N_{\mathrm{k}\pm}^{\mathrm{P}}\left(E_{\mathrm{k}\pm}^{\mathrm{P}} - a\right)X_{\mathrm{pk}}^{\mathrm{P}},\\ c_{2\mathrm{p}-1,\mathrm{k}\pm}^{\mathrm{P}} &= N_{\mathrm{k}\pm}^{\mathrm{P}}\left(cX_{\mathrm{pk}}^{\mathrm{P}} + dX_{\mathrm{p}-1,\mathrm{k}}^{\mathrm{P}}\right).\end{aligned} \tag{2.41}$$

We note that the expressions for eigenvalues and eigenvectors of odd and even paths are substantially the same for the pairs $E_{\mathrm{k}\pm}^{\mathrm{P}}$. The odd path has an extra eigenvector arising from the null space of $\mathbf{B}^{\mathrm{P}}\left(\mathbf{B}^{\mathrm{P}}\right)^{\mathrm{T}}$. The expression for the angle $\theta_{\mathrm{k}}^{\mathrm{P}}$, however, is different in the two cases (*c.f.* Eqs. (2.8) and (2.35)), as are the normalisation factors (*c.f.* Eqs. (2.10) and (2.37)). The angle $\theta_{\mathrm{k}}^{\mathrm{P}}$ is the sole quantity that cannot be determined in closed form for the even chain. For some values of the edge weights c, d, the angle $\theta_{\mathrm{k}}^{\mathrm{P}}$ may be equal to $\pm\pi$. Such cases must be treated separately as the expressions for the norm (2.37) and for the eigenvector entries in Eq. (2.41) vanish, but this is not difficult.

3 Structural polynomials of alternating paths

The characteristic polynomials for alternating paths can be written

$$s\left(P_{2N+1}(a,b \mid c,d),E\right) = (E-a)\prod_{\mathrm{k}=1}^{N}(E-E^{\mathrm{P}}_{\mathrm{k}+})(E-E^{\mathrm{P}}_{\mathrm{k}-}),$$
$$s\left(P_{2N}(a,b \mid c,d),E\right) = \prod_{\mathrm{k}=1}^{N}(E-E^{\mathrm{P}}_{\mathrm{k}+})(E-E^{\mathrm{P}}_{\mathrm{k}-}). \tag{3.1}$$

We can combine these factors in pairs as

$$\left(E-E^{\mathrm{P}}_{\mathrm{k}+}\right)\left(E-E^{\mathrm{P}}_{\mathrm{k}-}\right) = (E-a)(E-b) - c^2 - d^2 - 2cd\cos\theta^{\mathrm{P}}_{\mathrm{k}}, \tag{3.2}$$

and using the product expression for the Chebyshev function of the second kind,

$$U_N(z) = \prod_{\mathrm{k}=1}^{N}\left(2z - 2\cos\frac{\mathrm{k}\pi}{N+1}\right), \tag{3.3}$$

it can be shown that the '*half-chain*' expression for the *odd* path is

$$s\left(P_{2N+1}(a,b \mid c,d),E\right) = (E-a)(cd)^N U_N(x), \tag{3.4}$$

with

$$x = \frac{(E-a)(E-b) - \left(c^2+d^2\right)}{2cd}. \tag{3.5}$$

The expression in Eq. (3.3) cannot be used for even paths because of the more complicated formula for the angle, $\theta^{\mathrm{P}}_{\mathrm{k}}$. It has been shown by Gover [9], however, that the even chain has a related 'half-chain' form

$$s\left(P_{2N}(a,b \mid c,d),E\right) = d(cd)^{N-1}\widetilde{U}_N(x;c,d), \tag{3.6}$$

where

$$\widetilde{U}_N(x;c,d) = cU_N(x) + dU_{N-1}(x). \tag{3.7}$$

We can derive an expression in terms of Chebyshev polynomials for the full chain by using the standard formula

$$U_{2N+1}(z) = 2zU_N(2z^2-1) \tag{3.8}$$

which, when applied to Eqs. (3.4) and (3.6) gives

$$s\left(P_{2N+1}(a,b \mid c,d),E\right) = \frac{(cd)^N(E-a)}{2y}U_{2N+1}(y),$$
$$s\left(P_{2N}(a,b \mid c,d),E\right) = \frac{d(cd)^{N-1}}{2y}\left(cU_{2N+1}(y) + dU_{2N-1}(y)\right), \tag{3.9}$$

where, comparing Eqs. (3.4), (3.6) and (3.8) gives

$$y = \sqrt{\frac{(E-a)(E-b)-(c-d)^2}{4cd}}. \tag{3.10}$$

In the case of a *non-alternating* linear chain, with $a = b$ and $c = d$, Eqs. (3.9) simplify to

$$\begin{aligned} s\left(P_{2N+1}(a,a\mid c,c),E\right) &= c^{2N+1}U_{2N+1}\left(\frac{E-a}{2c}\right),\\ s\left(P_{2N}(a,a\mid c,c),E\right) &= \frac{c^{2N+1}}{E-a}\left(U_{2N+1}\left(\frac{E-a}{2c}\right)+U_{2N-1}\left(\frac{E-a}{2c}\right)\right)\\ &= c^{2N}U_{2N}\left(\frac{E-a}{2c}\right), \end{aligned} \tag{3.11}$$

where we have used the Chebyshev recursion relations in the last step.

Closed-form expressions for the other structural polynomials can be derived using the explicit inverse for 2-Toeplitz matrices derived by da Fonseca and Petronilho [2], and using Eqs. (1.1) and (1.2). Making the necessary translation of notation, we find that for paths of order $2N+1$ and assuming that $\mathrm{p} \le \mathrm{q}$,

$$\begin{aligned} \jmath_{2\mathrm{p},2\mathrm{q}}\left(P_{2N+1}(a,b\mid c,d),E\right) &= (cd)^{N-1}(E-a)^2U_{\mathrm{p}-1}(x)U_{N-\mathrm{q}}(x),\\ \jmath_{2\mathrm{p}-1,2\mathrm{q}-1}\left(P_{2N+1}(a,b\mid c,d),E\right) &= (cd)^{N-1}\widetilde{U}_{\mathrm{p}-1}(x;c,d)\widetilde{U}_{N+1-\mathrm{q}}(x;d,c),\\ \jmath_{2\mathrm{p},2\mathrm{q}+1}\left(P_{2N+1}(a,b\mid c,d),E\right) &= (cd)^{N-1}(E-a)U_{\mathrm{p}-1}(x)\widetilde{U}_{N-\mathrm{q}}(x;d,c),\\ \jmath_{2\mathrm{p}-1,2\mathrm{q}}\left(P_{2N+1}(a,b\mid c,d),E\right) &= (cd)^{N-1}(E-a)\widetilde{U}_{\mathrm{p}-1}(x;c,d)U_{N-\mathrm{q}}(x). \end{aligned} \tag{3.12}$$

The expressions for $2N$-vertex paths, again assuming that $\mathrm{p} \le \mathrm{q}$, are

$$\begin{aligned} \jmath_{2\mathrm{p},2\mathrm{q}}\left(P_{2N}(a,b\mid c,d),E\right) &= d(cd)^{N-2}(E-a)U_{\mathrm{p}-1}(x)\widetilde{U}_{N-\mathrm{q}}(x;c,d),\\ \jmath_{2\mathrm{p}-1,2\mathrm{q}-1}\left(P_{2N}(a,b\mid c,d),E\right) &= d(cd)^{N-2}(E-b)\widetilde{U}_{\mathrm{p}-1}(x;c,d)U_{N-\mathrm{q}}(x),\\ \jmath_{2\mathrm{p},2\mathrm{q}+1}\left(P_{2N}(a,b\mid c,d),E\right) &= d(cd)^{N-2}(E-a)(E-b)U_{\mathrm{p}-1}(x)U_{N-\mathrm{q}-1}(x),\\ \jmath_{2\mathrm{p}-1,2\mathrm{q}}\left(P_{2N}(a,b\mid c,d),E\right) &= d(cd)^{N-2}\widetilde{U}_{\mathrm{p}-1}(x;c,d)\widetilde{U}_{N-\mathrm{q}}(x;c,d). \end{aligned} \tag{3.13}$$

For cases where $\mathrm{q} < \mathrm{p}$, one needs to swap indices p and q in Eqs. (3.12) and (3.13).

4 Alternating ladders $L_{2M}(a, b \mid c, d)$

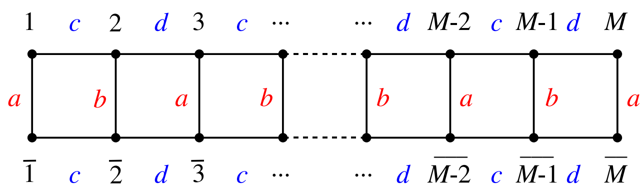


Figure 3: A linear ladder, $L_{2M}(a, b \mid c, d)$, with $2M$ vertices and alternating rung (a, b) and riser weights (c, d).

The $2M$-vertex linear ladder, $L_{2M}(a, b \mid c, d)$, has two alternating M-vertex paths $(1, 2, \dots, M)$ and $(\overline{1}, \overline{2}, \dots, \overline{M})$ joined by rungs between like-numbered vertices, $(\mathrm{p}, \overline{\mathrm{p}})$, with alternating weights as displayed in Fig. 3. The quantities (a, b) are, in this case, *rung* weights, and (c, d) are edge weights for the two *riser* chains.

The ladder has an involution involving simultaneous exchange of all vertices attached to the ends of the rungs. This involution can be exhibited by arranging the adjacency matrix so that the vertices are first put into 2×2 blocks with vertices 1 to M of the first path followed by vertices $\bar{1}$ to $\bar{M}$ of the lower path. Next, odd vertices $1, 3, \ldots$ are placed in a block together, and then the even vertices, $2, 4, \ldots$. The same procedure is adopted for the lower path. We now have a 4×4 blocked adjacency matrix in the form:

$$\begin{pmatrix} \mathbf{0} & \mathbf{B}^{\mathrm{P}} & a\mathbf{1} & \mathbf{0} \\ \left(\mathbf{B}^{\mathrm{P}}\right)^{\mathrm{T}} & \mathbf{0} & \mathbf{0} & b\mathbf{1} \\ a\mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{B}^{\mathrm{P}} \\ \mathbf{0} & b\mathbf{1} & \left(\mathbf{B}^{\mathrm{P}}\right)^{\mathrm{T}} & \mathbf{0} \end{pmatrix}, \tag{4.1}$$

where $\mathbf{B}^{\mathrm{P}}$ is the same matrix as in Eq. (2.2). This matrix can be block-diagonalised by an orthogonal transformation of the form

$$\begin{pmatrix} \frac{1}{\sqrt{2}}\mathbf{1}_M & \frac{1}{\sqrt{2}}\mathbf{1}_M \\ \frac{1}{\sqrt{2}}\mathbf{1}_M & -\frac{1}{\sqrt{2}}\mathbf{1}_M \end{pmatrix} \tag{4.2}$$

where the blocks are over all M vertices of top and bottom paths. This transformation leads to the adjacency matrix

$$\begin{pmatrix} a\mathbf{1} & \mathbf{B}^{\mathrm{P}} & \mathbf{0} & \mathbf{0} \\ \left(\mathbf{B}^{\mathrm{P}}\right)^{\mathrm{T}} & b\mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -a\mathbf{1} & \mathbf{B}^{\mathrm{P}} \\ \mathbf{0} & \mathbf{0} & \left(\mathbf{B}^{\mathrm{P}}\right)^{\mathrm{T}} & -b\mathbf{1} \end{pmatrix} = \begin{pmatrix} \mathbf{A}^{\mathrm{P}} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}^{\overline{\mathrm{P}}} \end{pmatrix}, \tag{4.3}$$

in which $\mathbf{A}^{\mathrm{P}}$ is the adjacency matrix of $P_M(a, b \mid c, d)$ as in Eq. (2.1), and $\mathbf{A}^{\overline{\mathrm{P}}}$ is the adjacency matrix of $P_M(-a, -b \mid c, d)$, the path with vertex weights of opposite sign. It follows that we can use the results of Section 2 to derive expressions for all relevant quantities. The eigenvalues for L_{2M} are thus

$$\begin{aligned} E_\mu^{\mathrm{L}} &= E_\mu^{\mathrm{P}}, \\ E_{\bar{\mu}}^{(\mathrm{L})} &= E_\mu^{\overline{\mathrm{P}}} \qquad \text{for } \mu = 1, 2, \ldots, M, \end{aligned} \tag{4.4}$$

where E_μ^{P} is an index ranging over the eigenvalues of the path $P_M(a, b|c, d)$ as given by Eq. (2.39), or (2.19) and (2.24), depending on whether M is even or odd. The eigenfunctions can be written using the vector $\mathbf{c}^{\mathrm{L}}$ using Eq. (2.23) as

$$\begin{aligned} c_{\mathrm{p}\mu}^{\mathrm{L}} &= \frac{c_{\mathrm{p}\mu}^{\mathrm{P}}}{\sqrt{2}}, \qquad & c_{\bar{\mathrm{p}}\mu}^{\mathrm{L}} &= \frac{c_{\bar{\mathrm{p}}\mu}^{\mathrm{P}}}{\sqrt{2}}, \\ c_{\mathrm{p}\bar{\mu}}^{\mathrm{L}} &= \frac{c_{\mathrm{p}\mu}^{\bar{\mathrm{P}}}}{\sqrt{2}}, \qquad & c_{\bar{\mathrm{p}}\bar{\mu}}^{\mathrm{L}} &= -\frac{c_{\bar{\mathrm{p}}\mu}^{\bar{\mathrm{P}}}}{\sqrt{2}} \end{aligned} \tag{4.5}$$

where the index, $\mu = 1, 2, \ldots, M$, labels the eigenvectors in each of the symmetric and antisymmetric blocks, labelled μ and $\bar{\mu}$, respectively.

5 Structural polynomials of alternating ladders

The characteristic polynomial, $s^{L_M}(E)$, can be derived directly from the expressions in Section 3 to give

$$\begin{aligned} s^{L_{4N}} &= d^2(cd)^{2N-2}\widetilde{U}_N(x;c,d)\,\widetilde{U}_N(\bar{x};c,d)\,, \\ s^{L_{4N+2}} &= (E^2-a^2)(cd)^{2N}U_N(x)\,U_N(\bar{x})\,, \end{aligned} \tag{5.1}$$

where we use x defined in Eq. (3.5) along with the antisymmetric analogue

$$\bar{x} = \frac{(E+a)(E+b)-c^2-d^2}{2cd}. \tag{5.2}$$

We can also use the results of Section 3 to write

$$\begin{aligned} s^{L_{4N}} &= \frac{d^2}{4y\bar{y}}(cd)^{2N-2}\left(cU_{2N+1}(y)+dU_{2N-1}(y)\right)\left(cU_{2N+1}(\bar{y})+dU_{2N-1}(\bar{y})\right), \\ s^{L_{4N+2}} &= \frac{(E^2-a^2)}{4y\bar{y}}(cd)^{2N}U_{2N+1}(y)U_{2N+1}(\bar{y}), \end{aligned} \tag{5.3}$$

in which we use

$$\bar{y} = \sqrt{\frac{(E+a)(E+b)-(c-d)^2}{4cd}}, \tag{5.4}$$

along with the definition of y in Eq. (3.10). The characteristic polynomials for the ladder, therefore, are written in Eqs. (5.1) and (5.3) as *products* of characteristic polynomials for the *part* systems $P_M(a,b\mid c,d)$ and $P_M(-a,-b\mid c,d)$.

We can easily deduce the forms of the $\jmath_{\mathrm{pq}}$ structural polynomials, since, using the spectral expansion of the structural polynomials in Eq. (1.5),

$$\jmath_{\mathrm{pq}}^{L_M} = (E\mathbf{1}-\mathbf{A}^{\mathrm{L}})^{-1}_{\mathrm{pq}}s^{\mathrm{L}}(E) = \frac{1}{2}\sum_{\mu=1}^{M}\left\{\frac{c^{\mathrm{P}}_{\mathrm{p}\mu}c^{\mathrm{P}}_{\mathrm{q}\mu}}{E-E^{\mathrm{P}}_{\mu}} + \frac{c^{\overline{\mathrm{P}}}_{\mathrm{p}\mu}c^{\overline{\mathrm{P}}}_{\mathrm{q}\mu}}{E-E^{\overline{\mathrm{P}}}_{\mu}}\right\}s^{\mathrm{L}}(E). \tag{5.5}$$

It follows immediately that,

$$\jmath_{\mathrm{pq}}^{L_M} = \frac{1}{2}\left\{\jmath^{\mathrm{P}}_{\mathrm{pq}}(E)s^{\bar{\mathrm{P}}}(E) + s^{\mathrm{P}}(E)\jmath^{\bar{\mathrm{P}}}_{\mathrm{pq}}(E)\right\}, \tag{5.6}$$

which also exhibits a simple structure in terms of $P_M(a,b\mid c,d)$ and $P_M(-a,-b\mid c,d)$.

The structural polynomials for $L_{4N+2}(a,b\mid c,d)$, expressed in terms of the half-ladder, where both vertices are on the same backbone path, and assuming $\mathrm{p} \leq \mathrm{q}$, are

$$\begin{aligned}
\jmath^{L_{4N+2}}_{2\mathrm{p},2\mathrm{q}} &= \frac{E^2-a^2}{2}(cd)^{2N-1}\Big\{(E-a)U_{\mathrm{p}-1}(x)U_{N-\mathrm{q}}(x)U_N(\bar{x}) \\
&\qquad +(E+a)U_N(x)U_{\mathrm{p}-1}(\bar{x})U_{N-\mathrm{q}}(\bar{x})\Big\}, \\
\jmath^{L_{4N+2}}_{2\mathrm{p}-1,2\mathrm{q}-1} &= \frac{1}{2}(cd)^{2N-1}\left\{(E+a)\widetilde{U}_{\mathrm{p}-1}(x;c,d)\widetilde{U}_{N+1-\mathrm{q}}(x;d,c)U_N(\bar{x})\right. \\
&\qquad \left. +\,(E-a)U_N(x)\widetilde{U}_{\mathrm{p}-1}(\bar{x};c,d)\widetilde{U}_{N+1-\mathrm{q}}(\bar{x};d,c)\right\}, \\
\jmath^{L_{4N+2}}_{2\mathrm{p},2\mathrm{q}+1} &= \frac{E^2-a^2}{2}(cd)^{2N-1}\left\{U_{\mathrm{p}-1}(x)\widetilde{U}_{N-\mathrm{q}}(x;d,c)U_N(\bar{x})\right. \\
&\qquad \left. +\,U_N(x)U_{\mathrm{p}-1}(\bar{x})\widetilde{U}_{N-\mathrm{q}}(\bar{x};d,c)\right\}, \\
\jmath^{L_{4N+2}}_{2\mathrm{p}-1,2\mathrm{q}} &= \frac{E^2-a^2}{2}(cd)^{2N-1}\left\{\widetilde{U}_{\mathrm{p}-1}(x;c,d)U_{N-\mathrm{q}}(x)U_N(\bar{x})\right. \\
&\qquad \left. +\,U_N(x)\widetilde{U}_{\mathrm{p}-1}(\bar{x};c,d)U_{N-\mathrm{q}}(\bar{x})\right\}. \qquad (5.7)
\end{aligned}$$

If $\mathrm{p} > \mathrm{q}$, then the indices p and q are swapped on the right-hand side of Eq. (5.7). For vertices on different backbone chains, and assuming again $\mathrm{p} \leq \mathrm{q}$,

$$\begin{aligned}
\jmath^{L_{4N+2}}_{2\mathrm{p},2\overline{\mathrm{q}}} &= \frac{E^2-a^2}{2}(cd)^{2N-1}\Big\{(E-a)U_{\mathrm{p}-1}(x)U_{n-\mathrm{q}}(x)U_N(\bar{x}) \\
&\qquad -(E+a)U_N(x)U_{\mathrm{p}-1}(\bar{x})U_{N-\mathrm{q}}(\bar{x})\Big\}, \\
\jmath^{L_{4N+2}}_{2\mathrm{p}-1,2\overline{\mathrm{q}}-1} &= \frac{1}{2}(cd)^{2N-1}\left\{(E+a)\widetilde{U}_{\mathrm{p}-1}(x;c,d)\widetilde{U}_{N+1-\mathrm{q}}(x;d,c)U_N(\bar{x})\right. \\
&\qquad \left. -\,(E-a)U_N(x)\widetilde{U}_{\mathrm{p}-1}(\bar{x};c,d)\widetilde{U}_{N+1-\mathrm{q}}(\bar{x};d,c)\right\}, \\
\jmath^{L_{4N+2}}_{2\mathrm{p},2\overline{\mathrm{q}}+1} &= \frac{E^2-a^2}{2}(cd)^{2N-1}\left\{U_{\mathrm{p}-1}(x)\widetilde{U}_{N-\mathrm{q}}(x;d,c)U_N(\bar{x})\right. \\
&\qquad \left. -\,U_N(x)U_{\mathrm{p}-1}(\bar{x})\widetilde{U}_{N-\mathrm{q}}(\bar{x};d,c)\right\}, \\
\jmath^{L_{4N+2}}_{2\mathrm{p}-1,2\overline{\mathrm{q}}} &= \frac{E^2-a^2}{2}(cd)^{2N-1}\left\{\widetilde{U}_{\mathrm{p}-1}(x;c,d)U_{N-\mathrm{q}}(x)U_N(\bar{x})\right. \\
&\qquad \left. -\,U_N(x)\widetilde{U}_{\mathrm{p}-1}(\bar{x};c,d)U_{N-\mathrm{q}}(\bar{x})\right\}. \qquad (5.8)
\end{aligned}$$

If $\mathrm{p} > \mathrm{q}$, then the indices p and q are swapped on the right-hand side of Eq. (5.8). Comparing Eqs. (5.7) and (5.8), we observe a sign change in the expressions that arises from the sign patterns of the antisymmetric functions in Eq. (4.5).

The structural polynomials, expressed in terms of the half-ladder for $L_{4N}(a, b \mid c, d)$, where both vertices are on the same backbone path and $\mathrm{p} \leq \mathrm{q}$, are

$$\begin{aligned}
\jmath^{L_{4N}}_{2\mathrm{p},2\mathrm{q}} &= \frac{1}{2} d^2 (cd)^{2N-3} \Big\{ (E-a) U_{\mathrm{p}-1}(x) \widetilde{U}_{N-\mathrm{q}}(x;c,d) \widetilde{U}_N(\bar{x};c,d) \\
&\qquad + (E+a) \widetilde{U}_N(x;c,d) U_{\mathrm{p}-1}(\bar{x}) \widetilde{U}_{N-\mathrm{q}}(\bar{x};c,d) \Big\}, \\
\jmath^{L_{4N}}_{2\mathrm{p}-1,2\mathrm{q}-1} &= \frac{1}{2} d^2 (cd)^{2N-3} \Big\{ (E-b) \widetilde{U}_{\mathrm{p}-1}(x;c,d) U_{N-\mathrm{q}}(x) \widetilde{U}_N(\bar{x};c,d) \\
&\qquad + (E+b) \widetilde{U}_N(x;c,d) \widetilde{U}_{\mathrm{p}-1}(\bar{x};c,d) U_{N-\mathrm{q}}(\bar{x}) \Big\}, \\
\jmath^{L_{4N}}_{2\mathrm{p},2\mathrm{q}+1} &= \frac{1}{2} d^2 (cd)^{2N-3} \Big\{ (E-a)(E-b) U_{\mathrm{p}-1}(x) U_{N-\mathrm{q}-1}(x) \widetilde{U}_N(\bar{x};c,d) \\
&\qquad + (E+a)(E+b) \widetilde{U}_N(x;c,d) U_{\mathrm{p}-1}(\bar{x}) U_{N-\mathrm{q}-1}(\bar{x}) \Big\}, \\
\jmath^{L_{4N}}_{2\mathrm{p}s-1,2\mathrm{q}} &= \frac{1}{2} d^2 (cd)^{2N-3} \Big\{ \widetilde{U}_{\mathrm{p}-1}(x;c,d) \widetilde{U}_{N-\mathrm{q}}(x;c,d) \widetilde{U}_N(\bar{x};c,d) \\
&\qquad + \widetilde{U}_N(x;c,d) \widetilde{U}_{\mathrm{p}-1}(\bar{x};c,d) \widetilde{U}_{N-\mathrm{q}}(\bar{x};c,d) \Big\}. \qquad (5.9)
\end{aligned}$$

If $\mathrm{p} > \mathrm{q}$, p and q are swapped on the RHS of Eq. (5.9). For vertices on different backbone paths and $\mathrm{p} \leq \mathrm{q}$,

$$\begin{aligned}
\jmath^{L_{4N}}_{2\mathrm{p},2\mathrm{q}} &= \frac{1}{2} d^2 (cd)^{2N-3} \Big\{ (E-a) U_{\mathrm{p}-1}(x) \widetilde{U}_{N-\mathrm{q}}(x;c,d) \widetilde{U}_N(\bar{x};c,d) \\
&\qquad - (E+a) \widetilde{U}_N(x;c,d) U_{\mathrm{p}-1}(\bar{x}) \widetilde{U}_{N-\mathrm{q}}(\bar{x};c,d) \Big\}, \\
\jmath^{L_{4N}}_{2\mathrm{p}-1,2\mathrm{q}-1} &= \frac{1}{2} d^2 (cd)^{2N-3} \Big\{ (E-b) \widetilde{U}_{\mathrm{p}-1}(x;c,d) U_{N-\mathrm{q}}(x) \widetilde{U}_N(\bar{x};c,d) \\
&\qquad - (E+b) \widetilde{U}_N(x;c,d) \widetilde{U}_{\mathrm{p}-1}(\bar{x};c,d) U_{N-\mathrm{q}}(\bar{x}) \Big\}, \\
\jmath^{L_{4N}}_{2\mathrm{p},2\mathrm{q}+1} &= \frac{1}{2} d^2 (cd)^{2N-3} \Big\{ (E-a)(E-b) U_{\mathrm{p}-1}(x) U_{N-\mathrm{q}-1}(x) \widetilde{U}_N(\bar{x};c,d) \\
&\qquad - (E+a)(E+b) \widetilde{U}_N(x;c,d) U_{\mathrm{p}-1}(\bar{x}) U_{N-\mathrm{q}-1}(\bar{x}) \Big\}, \\
\jmath^{L_{4N}}_{2\mathrm{p}s-1,2\mathrm{q}} &= \frac{1}{2} d^2 (cd)^{2N-3} \Big\{ \widetilde{U}_{\mathrm{p}-1}(x;c,d) \widetilde{U}_{N-\mathrm{q}}(x;c,d) \widetilde{U}_N(\bar{x};c,d) \\
&\qquad - \widetilde{U}_N(x;c,d) \widetilde{U}_{\mathrm{p}-1}(\bar{x};c,d) \widetilde{U}_{N-\mathrm{q}}(\bar{x};c,d) \Big\}. \qquad (5.10)
\end{aligned}$$

which exhibit the same sign change as in Eq. (5.8) for L_{4N+2} ladder. If $\mathrm{p} > \mathrm{q}$, then p and q are swapped on the RHS of Eq. (5.10).

6 Alternating cycles

We restrict our attention to even cycles with alternating weights. We consider separately the standard cycle, $C_{2N}(a, b \mid c, d)$, and the flipped cycle, $C^{f}_{2N}(a, b \mid c, d)$.

6.1 Alternating cycles, $C_{2N}(a, b \mid c, d)$

The $2N$-vertex cycle, $C_{2N}(a, b \mid c, d)$, has alternating weights as displayed in Fig. 4. The quantities (a, b) are in this case vertex weights, and (c, d) are edge weights. It is convenient

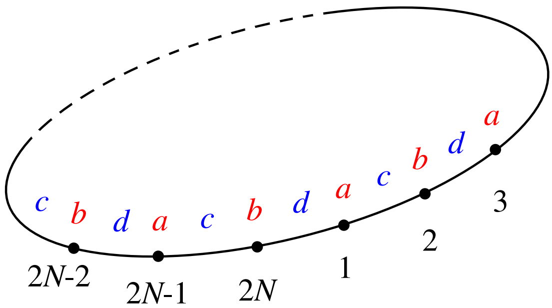


Figure 4: A ring, $C_{2N}(a, b \mid c, d)$, with $2N$ vertices and alternating vertex weights (a, b) and edge weights (c, d).

to write the adjacency matrix for this bipartite graph as

$$\mathbf{A}^{\mathrm{C}} = \begin{pmatrix} a\mathbf{1}_N & \mathbf{B}^{\mathrm{C}} \\ \left(\mathbf{B}^{\mathrm{C}}\right)^{\mathrm{T}} & b\mathbf{1}_N \end{pmatrix}, \tag{6.1}$$

where we have placed the N odd-numbered vertices in the first block, and the N even-numbered vertices in the second. The $N \times N$ matrix $\mathbf{B}^{\mathrm{C}}$ is

$$\mathbf{B}^{\mathrm{C}} = \begin{pmatrix} c & 0 & \cdots & 0 & d \\ d & c & \ddots & & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & d & c \end{pmatrix}. \tag{6.2}$$

$\mathbf{B}^{\mathrm{C}}$ is the adjacency matrix of a directed, weighted N-cycle. It is easy to show that

$$\mathbf{B}^{\mathrm{C}}\mathbf{X}^{\mathrm{C}} = \mathbf{X}^{\mathrm{C}}\mathbf{\Omega}^{\mathrm{C}}, \tag{6.3}$$

where

$$\mathbf{\Omega}^{\mathrm{C}}_{\mathrm{kk}'} = \delta_{\mathrm{kk}'}(c + d(\omega^{\mathrm{C}})^{-\mathrm{k}}), \tag{6.4}$$

with

$$\omega^{\mathrm{C}} = \exp\left(\frac{2\pi\imath}{N}\right). \tag{6.5}$$

The kth eigenvector has entries

$$X^{\mathrm{C}}_{\mathrm{pk}} = \frac{1}{\sqrt{N}} \exp\left(\frac{2\pi\imath\mathrm{kp}}{N}\right) \qquad \text{for } \mathrm{p}, \mathrm{k} = 1, 2, \ldots, N. \tag{6.6}$$

We use Eq. (6.6) to define the $2N$-dimensional unitary matrix

$$\mathbf{W} = \begin{pmatrix} \mathbf{X}^{\mathrm{C}} & \mathbf{0} \\ \mathbf{0} & \mathbf{X}^{\mathrm{C}} \end{pmatrix} \tag{6.7}$$

so that

$$\mathbf{W}^{\dagger}\mathbf{A}^{\mathrm{C}}\mathbf{W} = \begin{pmatrix} a\mathbf{1}_N & \mathbf{X}^{\mathrm{C}\dagger}\mathbf{B}^{\mathrm{C}}\mathbf{X}^{\mathrm{C}} \\ \mathbf{X}^{\mathrm{C}\dagger}\left(\mathbf{B}^{\mathrm{C}}\right)^{\mathrm{T}}\mathbf{X}^{\mathrm{C}} & b\mathbf{1}_N \end{pmatrix}, \tag{6.8}$$

where the † sign denotes the Hermitian conjugate. This transformation achieves a block diagonalization comprising N two-dimensional blocks of the form

$$\begin{pmatrix} a & c + d(\omega^{\mathrm{C}})^{-\mathrm{k}} \\ c + d(\omega^{\mathrm{C}})^{\mathrm{k}} & b \end{pmatrix}, \tag{6.9}$$

each with eigenvalues

$$E^{\mathrm{C}}_{\mathrm{k}\pm} = \frac{1}{2}(a+b) \pm \frac{1}{2}D^{\mathrm{C}}_{\mathrm{k}} \qquad \text{for } \mathrm{k} = 1, \dots, N, \tag{6.10}$$

where the discriminant is

$$D^{\mathrm{C}}_{\mathrm{k}} = \sqrt{(a-b)^2 + 4\left(c^2 + d^2 + 2cd\cos\left(\frac{2\pi\mathrm{k}}{N}\right)\right)}. \tag{6.11}$$

The eigenvectors can be written as entries in the vector $\mathbf{c}^{\mathrm{C}}$:

$$\begin{aligned} c^{\mathrm{C}}_{2\mathrm{p}-1,\mathrm{k}\pm} &= N^{\mathrm{C}}_{\mathrm{k}\pm} X^{\mathrm{C}}_{\mathrm{pk}}\left(E^{\mathrm{C}}_{\mathrm{k}\pm} - b\right), \\ c^{\mathrm{C}}_{2\mathrm{p},\mathrm{k}\pm} &= N^{\mathrm{C}}_{\mathrm{k}\pm}(cX^{\mathrm{C}}_{\mathrm{pk}} + dX^{\mathrm{C}}_{\mathrm{p}+1,\mathrm{k}}). \end{aligned} \tag{6.12}$$

The normalisation constants are

$$N^{\mathrm{C}}_{\mathrm{k}+} = \sqrt{\frac{1}{D^{\mathrm{C}}_{\mathrm{k}}\left(E^{\mathrm{C}}_{\mathrm{k}+} - b\right)}} \qquad \text{and} \qquad N^{\mathrm{C}}_{\mathrm{k}-} = \sqrt{\frac{1}{D^{\mathrm{C}}_{\mathrm{k}}\left(b - E^{\mathrm{C}}_{\mathrm{k}-}\right)}}. \tag{6.13}$$

6.2 Flipped alternating cycles, $C^f_{2N}(a, b \mid c, d)$

The $2N$-vertex cycle, $C^f_{2N}(a, b \mid c, d)$, has alternating weights as displayed in Fig. 4, except that a single weight has a changed sign; without loss of generality, we shall flip the $(1, 2N)$ edge. It is convenient to write the adjacency matrix as

$$\mathbf{A}^{\mathrm{C}^f} = \begin{pmatrix} a\mathbf{1}_N & \mathbf{B}^{\mathrm{C}^f} \\ \left(\mathbf{B}^{\mathrm{C}^f}\right)^{\mathrm{T}} & b\mathbf{1}_N \end{pmatrix}, \tag{6.14}$$

in the same manner as in Section 6. The N-dimensional matrix $\mathbf{B}^{\mathrm{C}^f}$ is hence defined by

$$\mathbf{B}^{\mathrm{C}^f} = \begin{pmatrix} c & 0 & \cdots & 0 & -d \\ d & c & \ddots & & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & & \ddots & d & c \\ 0 & 0 & \cdots & 0 & d \end{pmatrix}. \tag{6.15}$$

Proceeding as before,

$$\mathbf{B}^{\mathrm{C}^f}\mathbf{X}^{\mathrm{C}^f} = \mathbf{X}^{\mathrm{C}^f}\mathbf{\Omega}^{\mathrm{C}^f}, \tag{6.16}$$

where

$$\Omega^{\mathrm{C}^f}_{\mathrm{kk}'} = \delta\mathrm{kk}'(c + d(\omega^{\mathrm{C}^f})^{-(2\mathrm{k}-1)}), \tag{6.17}$$

with

$$\omega^{\mathrm{C}^f} = \exp\left(\frac{\imath\pi}{N}\right), \tag{6.18}$$

and the eigenfunctions are

$$X^{\mathrm{C}^f}_{\mathrm{pk}} = \frac{1}{\sqrt{N}}\exp\left\{\frac{\imath\pi(2\mathrm{k}-1)\mathrm{p}}{N}\right\} \qquad \text{for } \mathrm{p}, \mathrm{k} = 1, 2, \ldots, N. \tag{6.19}$$

The derivation proceeds exactly as for the simple cycle. The eigenvalues are

$$E^{\mathrm{C}^f}_{\mathrm{k}\pm} = \frac{1}{2}(a+b) \pm \frac{1}{2}D^{\mathrm{C}^f}_{\mathrm{k}} \qquad \text{for } \mathrm{k} = 1, \ldots, N \tag{6.20}$$

and the discriminant is

$$D^{\mathrm{C}^f}_{\mathrm{k}} = \sqrt{(a-b)^2 + 4\left(c^2 + d^2 + 2cd\cos\left(\frac{\pi(2\mathrm{k}-1)}{N}\right)\right)}. \tag{6.21}$$

The eigenvectors are written as entries in the vector $\mathbf{c}^{\mathrm{C}f}$ as

$$\begin{aligned} c^{\mathrm{C}^f}_{2\mathrm{p}-1,\mathrm{k}\pm} &= N^{\mathrm{C}^f}_{\mathrm{k}\pm}X^{\mathrm{C}^f}_{\mathrm{pk}}\left(E^{\mathrm{C}^f}_{\mathrm{k}\pm} - b\right), \\ c^{\mathrm{C}^f}_{2\mathrm{p},\mathrm{k}\pm} &= N^{\mathrm{C}^f}_{\mathrm{k}\pm}(cX^{\mathrm{C}^f}_{\mathrm{pk}} + dX^{\mathrm{C}^f}_{\mathrm{p}+1,\mathrm{k}}), \end{aligned} \tag{6.22}$$

which is the analogue of Eq. (6.12), and the normalisation constants are

$$N^{\mathrm{C}^f}_{\mathrm{k}+} = \sqrt{\frac{1}{D^{\mathrm{C}^f}_{\mathrm{k}}\left(E^{\mathrm{C}^f}_{\mathrm{k}+} - a\right)}} \qquad \text{and} \qquad N^{\mathrm{C}^f}_{\mathrm{k}-} = \sqrt{\frac{1}{D^{\mathrm{C}^f}_{\mathrm{k}}\left(a - E^{\mathrm{C}^f}_{\mathrm{k}-}\right)}}. \tag{6.23}$$

7 Structural polynomials of alternating cycles

We derive expressions for the structural polynomials of even cycles and flipped cycles in this section. The characteristic polynomial, $s^{C_{2N}}(E)$, for the graph $C_{2N}(a, b \mid c, d)$ is

$$\begin{aligned} s^{C_{2N}} &= \prod_{\mathrm{k}=1}^{N}(E - E^{\mathrm{C}}_{\mathrm{k}+})(E - E^{\mathrm{C}}_{\mathrm{k}-}) \\ &= \prod_{\mathrm{k}=1}^{N}\left((E-a)(E-b) - c^2 - d^2 - 2cd\cos\frac{2\pi\mathrm{k}}{N}\right). \end{aligned} \tag{7.1}$$

Expressing the cosine in terms of the half-angle, we find that

$$\begin{aligned} s^{C_{2N}} &= (4cd)^N\prod_{\mathrm{k}=1}^{N}\left(\frac{(E-a)(E-b) - (c-d)^2}{4cd} - \cos^2\frac{2\pi\mathrm{k}}{2N}\right) \\ &= (4cd)^N\prod_{\mathrm{k}=1}^{2N}\left\{y - \cos\frac{2\pi\mathrm{k}}{2N}\right\} \end{aligned} \tag{7.2}$$

where y is defined in Eq. (3.10). We use the well-known relation

$$2T_N(z) - 2 = \prod_{\mathrm{k}=1}^{N} \left(2z - 2\cos\frac{2\pi \mathrm{k}}{N} \right), \tag{7.3}$$

where the Chebyshev polynomial of the first kind is

$$T_N(\cos\theta) = \cos(n\theta).$$

We conclude that

$$s^{C_{2N}} = 2(cd)^N \left\{ T_{2N}(y) - 1 \right\}. \tag{7.4}$$

We can also derive a formula for the half ring by using Eq. (7.4) in conjunction with the standard formula

$$T_{2N}(z) = T_N\left(2z^2 - 1\right), \tag{7.5}$$

to give

$$s^{C_{2N}} = 2(cd)^N \left(T_N(x) - 1 \right), \tag{7.6}$$

where we have used the definition of x in Eq. (3.5).

The characteristic polynomial, $s^{C^f_{2N}}(E)$, for the graph $C^f_{2N}(a, b \mid c, d)$ is

$$s^{C^f_{2N}} = \prod_{\mathrm{k}=1}^{N} (E - E^{\mathrm{C}^f}_{\mathrm{k}+})(E - E^{\mathrm{C}^f}_{\mathrm{k}-}) = \prod_{\mathrm{k}=1}^{N} \left(2cdx^2 - 2cd\cos\frac{\pi(2\mathrm{k}-1)}{N} \right). \tag{7.7}$$

Expressing the cosine in terms of the half-angle, we find that

$$s^{C^f_{2N}} = (4cd)^N \prod_{\mathrm{k}=1}^{N} \left(y^2 - \cos^2\frac{\pi(2\mathrm{k}-1)}{2N} \right) = 4(-cd)^N T_N(y) T_N(-y), \tag{7.8}$$

where we have used y as in Eq. (3.10), and the well-known relation

$$T_N(z) = 2^{N-1} \prod_{\mathrm{k}=1}^{N} \left\{ z - \cos\left(\frac{\pi(2\mathrm{k}-1)}{2N} \right) \right\}. \tag{7.9}$$

The product formula

$$2T_N^2(z) = T_{2N}(z) + 1 \tag{7.10}$$

and the parity of the Chebychev polynomials gives the final 'full-ring' expression

$$s^{C^f_{2N}} = 2(cd)^N \left(T_{2N}(y) + 1 \right). \tag{7.11}$$

We can also derive a formula for the half ring using the transformation in Eq. (7.5), to give

$$s^{C^f_{2N}} = 2(cd)^N \left(T_N(x) + 1 \right). \tag{7.12}$$

The remaining structural polynomials can be deduced using the fact that removal of a vertex from a cycle gives rise to a path, and we have already derived the characteristic polynomials of even and odd vertex paths in Section 3. There are two kinds of vertex in our alternating cycles, odd-numbered vertices with weight a, and even-numbered vertices with weight b. When one of these vertices is removed we create a path of length $2N-1$. The equations for the diagonal parts of $\jmath_{\mathrm{pq}}$ for $C_{2N}(a,b \mid c,d)$ are:

$$\begin{aligned} \jmath^{C_{2N}}_{\mathrm{2p,2p}} &= (cd)^{N-1}(E-a)U_{N-1}(x), \\ \jmath^{C_{2N}}_{\mathrm{2p+1,2p+1}} &= (cd)^{N-1}(E-b)U_{N-1}(x). \end{aligned} \tag{7.13}$$

The formulae for the off-diagonal $\jmath$ polynomials can be deduced using Eq. (1.6), and by noting that if we remove a second vertex, then we form two (or possibly one) paths. This requires some trivial but lengthy trigonometry. The results are

$$\begin{aligned} \jmath^{C_{2N}}_{\mathrm{2p,2p+2q}} &= (E-a)(cd)^{N-1}\left\{U_{N-\mathrm{q}-1}(x)+U_{\mathrm{q}-1}(x)\right\}, \\ \jmath^{C_{2N}}_{\mathrm{2p+1,2p+2q+1}} &= (E-b)(cd)^{N-1}\left\{U_{N-\mathrm{q}-1}(x)+U_{\mathrm{q}-1}(x)\right\}, \\ \jmath^{C_{2N}}_{\mathrm{2p+1,2p+2q+2}} &= (cd)^{N-1}\left\{\widetilde{U}_{N-\mathrm{q}-1}(x;c,d)+\widetilde{U}_{\mathrm{q}}(x;d,c)\right\}. \end{aligned} \tag{7.14}$$

Note that the formulae do not depend upon p, but only upon q, the offset along the ring. The results for the flipped cycle can be calculated in the same manner. The diagonal $\jmath$ quantities are identical to those for the cycle. This is because the deletion of a vertex creates a chain, and flipped edges can be removed from a tree using an orthogonal transformation. Hence,

$$\begin{aligned} \jmath^{C^f_{2N}}_{\mathrm{2p,2p}} &= (cd)^{N-1}(E-a)U_{N-1}(x), \\ \jmath^{C^f_{2N}}_{\mathrm{2p+1,2p+1}} &= (cd)^{N-1}(E-b)U_{N-1}(x), \end{aligned} \tag{7.15}$$

and further

$$\begin{aligned} \jmath^{C^f_{2N}}_{\mathrm{2p,2p+2q}} &= (E-a)(cd)^{N-1}\left\{U_{N-\mathrm{q}-1}(x)-U_{\mathrm{q}-1}(x)\right\}, \\ \jmath^{C^f_{2N}}_{\mathrm{2p+1,2p+2q+1}} &= (E-b)(cd)^{N-1}\left\{U_{N-\mathrm{q}-1}(x)-U_{\mathrm{q}-1}(x)\right\}, \\ \jmath^{C^f_{2N}}_{\mathrm{2p+1,2p+2q+2}} &= (cd)^{N-1}\left\{\widetilde{U}_{N-\mathrm{q}-1}(x;c,d)-\widetilde{U}_{\mathrm{q}}(x;d,c)\right\}. \end{aligned} \tag{7.16}$$

The changes in sign between Eqs. (7.14) and (7.16) arise because of the sign change between Eqs. (7.6) and (7.12).

8 Alternating treadmills

Treadmills are cyclic ladders. We consider a $4N$-vertex treadmill, $T_{4N}(a,b \mid c,d)$, with rung edge weights (a,b), and backbone weights (c,d) displayed in Fig. 5. We shall also consider a related treadmill, namely the 'flipped' treadmill, $T^f_{4N}(a,b \mid c,d)$, which has a pair of symmetrically related backbone edges with weights having a changed sign. We also include in our discussion the Möbius treadmill, $T^M_{4N}(a,b \mid c,d)$, which has a pair of crossed edges connecting top and bottom rings. There is also the flipped Möbius treadmill, $T^{M^f}_{4N}(a,b \mid c,d)$, which has a pair of crossed bottom and top edges with weights having a changed sign. All of these treadmills possess the same involution symmetry, and hence can be treated in the same way.

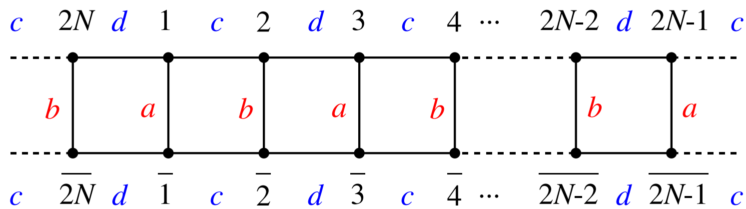


Figure 5: A treadmill, $T_{4N}(a, b \mid c, d)$, with $4N$ vertices and alternating rung (a, b) and ring edge weights (c, d).

8.1 The treadmill $T_{4N}(a, b \mid c, d)$

This system has an involution symmetry based upon exchange of vertices attached to the ends of rungs. The methodology proceeds in exactly the same way as for the ladder example in Section 4. Hence, the 4×4 blocks of vertices (top ring odd, top ring even, bottom ring odd, and bottom ring even) produce the adjacency matrix

$$\begin{pmatrix} \mathbf{0} & \mathbf{B}^{\mathrm{C}} & a\mathbf{1} & \mathbf{0} \\ (\mathbf{B}^{\mathrm{C}})^{\mathrm{T}} & \mathbf{0} & \mathbf{0} & b\mathbf{1} \\ a\mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{B}^{\mathrm{C}} \\ \mathbf{0} & b\mathbf{1} & (\mathbf{B}^{\mathrm{C}})^{\mathrm{T}} & \mathbf{0} \end{pmatrix}, \tag{8.1}$$

where $\mathbf{B}^{\mathrm{C}}$ is the same matrix as in Eq. (6.2). This matrix can be block-diagonalised by an orthogonal transformation of the form

$$\begin{pmatrix} \frac{1}{\sqrt{2}}\mathbf{1}_{2N} & \frac{1}{\sqrt{2}}\mathbf{1}_{2N} \\ \frac{1}{\sqrt{2}}\mathbf{1}_{2N} & -\frac{1}{\sqrt{2}}\mathbf{1}_{2N} \end{pmatrix}. \tag{8.2}$$

The adjacency matrix after the transformation is:

$$\begin{pmatrix} a\mathbf{1} & \mathbf{B}^{\mathrm{C}} & \mathbf{0} & \mathbf{0} \\ (\mathbf{B}^{\mathrm{C}})^{\mathrm{T}} & b\mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -a\mathbf{1} & \mathbf{B}^{\mathrm{C}} \\ \mathbf{0} & \mathbf{0} & (\mathbf{B}^{\mathrm{C}})^{\mathrm{T}} & -b\mathbf{1} \end{pmatrix} = \begin{pmatrix} \mathbf{A}^{\mathrm{C}} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}^{\overline{\mathrm{C}}} \end{pmatrix}, \tag{8.3}$$

in which $\mathbf{A}^{\mathrm{C}}$ is the adjacency matrix of $C_{2N}(a, b \mid c, d)$ as shown in Eq. (6.1), and $\mathbf{A}^{\overline{\mathrm{C}}}$ is the adjacency matrix of $C_{2N}(-a, -b \mid c, d)$, the ring with vertex weights of opposite sign. It follows that we can use the results of Section 6 to derive expressions for all relevant quantities. The eigenvalues for $T_{4N}(a, b \mid c, d)$ are therefore

$$\begin{aligned} E_{\mathrm{k}\pm}^{\mathrm{T}(s)} &= E_{\mathrm{k}\pm}^{\mathrm{C}}, \\ E_{\mathrm{k}\pm}^{\mathrm{T}(a)} &= E_{\mathrm{k}\pm}^{\overline{\mathrm{C}}} \qquad \text{for } \mathrm{k} = 1, 2, \ldots, N, \end{aligned} \tag{8.4}$$

where E^{C} is the expression given in Eq. (6.10) for the eigenvalues of the cycle $C_{2N}(a, b \mid c, d)$, and $E^{\overline{\mathrm{C}}}$ refers to the eigenvalues of the cycle $C_{2N}(-a, -b \mid c, d)$ with reversed vertex weights. The eigenvectors for the treadmill can be placed in a vector $\mathbf{c}^{\mathrm{T}}$, with entries

$$\begin{aligned} c_{\mathrm{pk}\pm}^{\mathrm{T}(s)} &= \;\; c_{\overline{\mathrm{pk}}\pm}^{\mathrm{T}(s)} = \frac{1}{\sqrt{2}} c_{\mathrm{pk}\pm}^{C}, \\ c_{\mathrm{pk}\pm}^{\mathrm{T}(a)} &= -c_{\overline{\mathrm{pk}}\pm}^{\mathrm{T}(a)} = \frac{1}{\sqrt{2}} c_{\mathrm{pk}\pm}^{\overline{C}}, \end{aligned} \tag{8.5}$$

where the superscripts (s), and (a) indicate the symmetric and antisymmetric eigenvectors, respectively.

8.2 Flipped treadmills, $T_{4N}^{f}(a, b \mid c, d)$

Flipped treadmills can be obtained by simply changing the sign of a pair of symmetrically positioned edges on the top and bottom rings. We shall take edges $(1, 2N)$ and $(\overline{1}, \overline{2N})$ to have weights $-d$, without loss of generality, since a series of orthogonal transformations can move the flipped pair of edges to any position around the rings. We use the same diagonalization methodology as for the standard treadmill, whence the adjacency matrix is

$$\begin{pmatrix} a\mathbf{1} & \mathbf{B}^{\mathrm{C}^f} & \mathbf{0} & \mathbf{0} \\ \left(\mathbf{B}^{\mathrm{C}^f}\right)^{\mathrm{T}} & b\mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -a\mathbf{1} & \mathbf{B}^{\mathrm{C}^f} \\ \mathbf{0} & \mathbf{0} & \left(\mathbf{B}^{\mathrm{C}^f}\right)^{\mathrm{T}} & -b\mathbf{1} \end{pmatrix} = \begin{pmatrix} \mathbf{A}^{\mathrm{C}^f} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}^{\overline{\mathrm{C}}^f} \end{pmatrix}, \tag{8.6}$$

where the matrix $\mathbf{B}^{\mathrm{C}^f}$ is shown in Eq. (6.15), $\mathbf{A}^{\mathrm{C}^f}$ is the adjacency matrix of $C_{2N}^{f}(a, b \mid c, d)$ as shown in Eq. (6.14), and $\mathbf{A}^{\overline{\mathrm{C}}^f}$ is the adjacency matrix of $C_{2N}^{f}(-a, -b \mid c, d)$, the ring with vertex weights of opposite sign. The eigenvalues for $T_{4N}^{f}(a, b \mid c, d)$ are thus

$$\begin{aligned} E_{\mathrm{k}\pm}^{\mathrm{T}^f(s)} &= E_{\mathrm{k}\pm}^{\mathrm{C}^f}, \\ E_{\mathrm{k}\pm}^{\mathrm{T}^f(a)} &= E_{\mathrm{k}\pm}^{\overline{\mathrm{C}}^f} \qquad \text{for } \mathrm{k} = 1, 2, \ldots, N, \end{aligned} \tag{8.7}$$

where $E_{\mathrm{k}\pm}^{\mathrm{C}^f}$, given in Eq. (6.20), is an eigenvalue of $C_{2N}^{f}(a, b \mid c, d)$, and $E_{\mathrm{k}\pm}^{\overline{\mathrm{C}}^f}$ is an eigenvalue of $C_{2N}^{f}(-a, -b \mid c, d)$.

The eigenfunction entries can be written as

$$\begin{aligned} c_{\mathrm{pk}\pm}^{\mathrm{T}^f(s)} &= \;\; c_{\overline{\mathrm{pk}}\pm}^{\mathrm{T}^f(s)} = \frac{1}{\sqrt{2}} c_{\mathrm{pk}\pm}^{C^f}, \\ c_{\mathrm{pk}\pm}^{\mathrm{T}^f(a)} &= -c_{\overline{\mathrm{pk}}\pm}^{\mathrm{T}^f(a)} = \frac{1}{\sqrt{2}} c_{\mathrm{pk}\pm}^{\overline{C}^f}, \end{aligned} \tag{8.8}$$

with $2N$ eigenvectors in the symmetric and the antisymmetric blocks (*i.e.* $\mathrm{k} = 1, 2, \ldots, N$).

8.3 Möbius treadmills, $T_{4N}^{M}(a, b \mid c, d)$

The Möbius treadmill has the same involution symmetry as the other treadmills defined in Section 8. Using the treadmill block diagonalisation procedure, the adjacency matrix

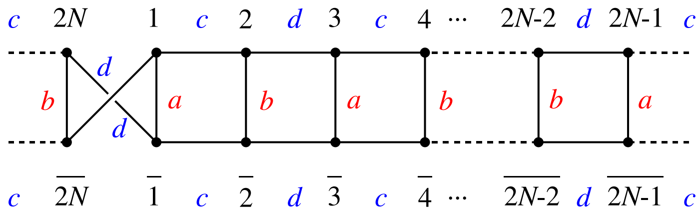


Figure 6: A Möbius treadmill, $T^M_{4N}(a, b \mid c, d)$, with $4N$ vertices and alternating rung (a, b) and ring edge weights (c, d).

becomes

$$\begin{pmatrix} a\mathbf{1} & \mathbf{B}^{\mathrm{C}} & \mathbf{0} & \mathbf{0} \\ \left(\mathbf{B}^{\mathrm{C}}\right)^{\mathrm{T}} & b\mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -a\mathbf{1} & \mathbf{B}^{\mathrm{C}^f} \\ \mathbf{0} & \mathbf{0} & \left(\mathbf{B}^{\mathrm{C}^f}\right)^{\mathrm{T}} & -b\mathbf{1} \end{pmatrix} = \begin{pmatrix} \mathbf{A}^{\mathrm{C}} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}^{\overline{\mathrm{C}}^f} \end{pmatrix}, \tag{8.9}$$

which has the structure of $C_{2N}(a, b \mid c, d)$ in the top block, and $C^f_{2N}(-a, -b \mid c, d)$ in the lower one. The eigenvalues for $T^M_{4N}(a, b \mid c, d)$ are thus

$$\begin{aligned} E^{\mathrm{T}^M(s)}_{\mathrm{k}\pm} &= E^{\mathrm{C}}_{\mathrm{k}\pm}, \\ E^{\mathrm{T}^M(a)}_{\mathrm{k}\pm} &= E^{\overline{\mathrm{C}}^f}_{\mathrm{k}\pm} \qquad \text{for } \mathrm{k} = 1, 2, \ldots, N. \end{aligned} \tag{8.10}$$

The eigenfunction entries can be written as

$$\begin{aligned} c^{\mathrm{T}^M(s)}_{\mathrm{pk}\pm} &= \quad c^{\mathrm{T}^M(s)}_{\overline{\mathrm{p}}\mathrm{k}\pm} = \frac{1}{\sqrt{2}} c^{C}_{\mathrm{pk}\pm}, \\ c^{\mathrm{T}^M(a)}_{\mathrm{pk}\pm} &= -c^{\mathrm{T}^M(a)}_{\overline{\mathrm{p}}\mathrm{k}\pm} = \frac{1}{\sqrt{2}} c^{\overline{C}^f}_{\mathrm{pk}\pm}, \end{aligned} \tag{8.11}$$

with $2N$ eigenvectors in the symmetric and the antisymmetric blocks.

8.4 Flipped Möbius treadmills, $T^{M^f}_{4N}(a, b \mid c, d)$

The block diagonalisation procedure gives the adjacency matrix

$$\begin{pmatrix} a\mathbf{1} & \left(\mathbf{B}^{\mathrm{C}^f}\right)^{\mathrm{T}} & \mathbf{0} & \mathbf{0} \\ \mathbf{B}^{\mathrm{C}^f T} & b\mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -a\mathbf{1} & \mathbf{B}^{\mathrm{C}} \\ \mathbf{0} & \mathbf{0} & \left(\mathbf{B}^{\mathrm{C}}\right)^{\mathrm{T}} & -b\mathbf{1} \end{pmatrix} = \begin{pmatrix} \mathbf{A}^{\mathrm{C}^f} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}^{\overline{\mathrm{C}}} \end{pmatrix}, \tag{8.12}$$

which has the structure of $C^f_{2N}(a, b \mid c, d)$ in the top block, and $C_{2N}(-a, -b \mid c, d)$ in the lower. The eigenfunction entries, in this case, are

$$\begin{aligned} c^{\mathrm{T}^{Mf}(s)}_{\mathrm{pk}\pm} &= \quad c^{\mathrm{T}^{Mf}(s)}_{\overline{\mathrm{p}}\mathrm{k}\pm} = \frac{1}{\sqrt{2}} c^{C^f}_{\mathrm{pk}\pm}, \\ c^{\mathrm{T}^{Mf}(a)}_{\mathrm{pk}\pm} &= -c^{\mathrm{T}^{Mf}(a)}_{\overline{\mathrm{p}}\mathrm{k}\pm} = \frac{1}{\sqrt{2}} c^{\overline{C}}_{\mathrm{pk}\pm}. \end{aligned} \tag{8.13}$$

9 Structural polynomials of alternating treadmills

The characteristic polynomial, $s^{T_{4N}}(E)$, can be written immediately because we have been able to split the secular equations into two annulene terms, $C_{2N}(a,b \mid c,d)$ and $C_{2N}(-a,-b \mid c,d)$, to give

$$\begin{aligned} s^{T_{4N}} &= s(C_{2N}(a,b \mid c,d),E)s(C_{2N}(-a,-b \mid c,d),E) \\ &= 4(cd)^{2N}\left(T_N(x)-1\right)\left(T_N(\bar{x})-1\right), \end{aligned} \tag{9.1}$$

with x and $\bar{x}$ defined in Eqs. (3.5) and (5.2). We can also write

$$s^{T_{4N}} = 4(cd)^{2N}\left(T_{2N}(y)-1\right)\left(T_{2N}(\bar{y})-1\right) \tag{9.2}$$

with y and $\bar{y}$ defined in Eqs. (3.10) and (5.4). The factoring of the secular problem for the treadmill allows the characteristic polynomials to be written as a *product* of the characteristic polynomials of $C_{2N}(a,b \mid c,d)$ and $C_{2N}(-a,-b \mid c,d)$.

The other structural polynomials can be obtained using the same logic as in Section 5,

$$\begin{aligned} \jmath^{T_{4N}}_{\mathrm{pq}} &= \frac{1}{2}\left(\jmath(C_{2N}(a,b \mid c,d),E)s(C_{2N}(-a,-b \mid c,d),E)\right. \\ &\quad \left.+\, s(C_{2N}(a,b \mid c,d),E)\jmath(C_{2N}(-a,-b \mid c,d),E)\right). \end{aligned} \tag{9.3}$$

It follows that, for pairs of indices on the same ring,

$$\begin{aligned} \jmath^{T_{4N}}_{2\mathrm{p},2\mathrm{p}+2\mathrm{q}} &= (cd)^{2N-1}\left\{(E-a)\left[U_{N-\mathrm{q}-1}(x)+U_{\mathrm{q}-1}(x)\right]\left[T_N(\bar{x})-1\right]\right. \\ &\qquad \left.+\,(E+a)\left[U_{N-\mathrm{q}-1}(\bar{x})+U_{\mathrm{q}-1}(\bar{x})\right]\left[T_N(x)-1\right]\right\}, \\ \jmath^{T_{4N}}_{2\mathrm{p}+1,2\mathrm{p}+2\mathrm{q}+1} &= (cd)^{2N-1}\left\{(E-b)\left[U_{N-\mathrm{q}-1}(x)+U_{\mathrm{q}-1}(x)\right]\left[T_N(\bar{x})-1\right]\right. \\ &\qquad \left.+\,(E+a)\left[U_{N-\mathrm{q}-1}(\bar{x})+U_{\mathrm{q}-1}(\bar{x})\right]\left[T_N(x)-1\right]\right\}, \\ \jmath^{T_{4N}}_{2\mathrm{p}+1,2\mathrm{p}+2\mathrm{q}+2} &= (cd)^{2N-1}\left\{\left[U_{N-\mathrm{q}-1}(x)+U_{\mathrm{q}}(x)\right]\left[T_N(\bar{x})-1\right]\right. \\ &\qquad \left.+\left[U_{N-\mathrm{q}-1}(\bar{x})+U_{\mathrm{q}}(\bar{x})\right]\left[T_N(x)-1\right]\right\}, \end{aligned} \tag{9.4}$$

and for pairs of indices on different rings,

$$\begin{aligned} \jmath^{T_{4N}}_{2\mathrm{p},2\overline{\mathrm{p}}+2\overline{\mathrm{q}}} &= (cd)^{2N-1}\left\{(E-a)\left[U_{N-\mathrm{q}-1}(x)+U_{\mathrm{q}-1}(x)\right]\left[T_N(\bar{x})-1\right]\right. \\ &\qquad \left.-\,(E+b)\left[U_{N-\mathrm{q}-1}(\bar{x})+U_{\mathrm{q}-1}(\bar{x})\right]\left[T_N(x)-1\right]\right\}, \\ \jmath^{T_{4N}}_{2\mathrm{p}+1,2\overline{\mathrm{p}}+2\overline{\mathrm{q}}+1} &= (cd)^{2N-1}\left\{(E-b)\left[U_{N-\mathrm{q}-1}(x)+U_{\mathrm{q}-1}(x)\right]\left[T_N(\bar{x})-1\right]\right. \\ &\qquad \left.-\,(E+b)\left[U_{N-\mathrm{q}-1}(\bar{x})+U_{\mathrm{q}-1}(\bar{x})\right]\left[T_N(x)-1\right]\right\}, \\ \jmath^{T_{4N}}_{2\mathrm{p}+1,2\overline{\mathrm{p}}+2\overline{\mathrm{q}}+2} &= (cd)^{2N-1}\left\{\left[U_{N-\mathrm{q}-1}(x)+U_{\mathrm{q}}(x)\right]\left[T_N(\bar{x})-1\right]\right. \\ &\qquad \left.-\left[U_{N-\mathrm{q}-1}(\bar{x})+U_{\mathrm{q}}(\bar{x})\right]\left[T_N(x)-1\right]\right\}, \end{aligned} \tag{9.5}$$

where we observe the expected sign change arising from the antisymmetry.

9.1 Structural polynomials of $T^f_{4N}(a,b \mid c,d)$

The characteristic polynomial, $s^{T^f_{4N}}(E)$, can be written immediately as we have been able to split the secular equations into contributions from two flipped annulenes, $C^f_{2N}(a,b \mid c,d)$

and $C^f_{2N}(-a,-b \mid c,d)$, to give

$$\begin{aligned} s^{T^f_{4N}} &= s(C^f_{2N}(a,b \mid c,d),E)s(C^f_{2N}(-a,-b \mid c,d),E) \\ &= 4(cd)^{2N}\left(T_N(x)+1\right)\left(T_N(\bar{x})+1\right), \end{aligned} \tag{9.6}$$

with x and $\bar{x}$ defined in Eqs. (3.5) and (5.2). We can also write a product form

$$s^{T^f_{4N}} = 4(cd)^{2N}\left(T_{2N}(y)+1\right)\left(T_{2N}(\bar{y})+1\right) \tag{9.7}$$

with y and $\bar{y}$ defined in Eqs. (3.10) and (5.4).

The $\jmath$ structural polynomials can be obtained using the same logic as in Section 5:

$$\begin{aligned} \jmath^{T^f_{4N}}_{\mathrm{pq}} = \frac{1}{2}\Big(&\jmath(C^f_{2N}(a,b \mid c,d),E)s(C^f_{2N}(-a,-b \mid c,d),E) \\ &+ s(C^f_{2N}(a,b \mid c,d),E)\jmath(C^f_{2N}(-a,-b \mid c,d),E)\Big), \end{aligned} \tag{9.8}$$

so that, for indices on the same ring,

$$\begin{aligned} \jmath^{T^f_{4N}}_{2\mathrm{p},2\mathrm{p}+2\mathrm{q}} &= (cd)^{2N-1}\{(E-a)\left[U_{N-\mathrm{q}-1}(x)-U_{\mathrm{q}-1}(x)\right]\left[T_N(\bar{x})+1\right] \\ &\qquad + (E+a)\left[U_{N-\mathrm{q}-1}(\bar{x})-U_{\mathrm{q}-1}(\bar{x})\right]\left[T_N(x)+1\right]\}, \\ \jmath^{T^f_{4N}}_{2\mathrm{p}+1,2\mathrm{p}+2\mathrm{q}+1} &= (cd)^{2N-1}\{(E-b)\left[U_{N-\mathrm{q}-1}(x)-U_{\mathrm{q}-1}(x)\right]\left[T_N(\bar{x})+1\right] \\ &\qquad + (E+a)\left[U_{N-\mathrm{q}-1}(\bar{x})-U_{\mathrm{q}-1}(\bar{x})\right]\left[T_N(x)+1\right]\}, \\ \jmath^{T^f_{4N}}_{2\mathrm{p}+1,2\mathrm{p}+2\mathrm{q}+2} &= (cd)^{2N-1}\{\left[U_{N-\mathrm{q}-1}(x)-U_{\mathrm{q}}(x)\right]\left[T_N(\bar{x})+1\right] \\ &\qquad + \left[U_{N-\mathrm{q}-1}(\bar{x})-U_{\mathrm{q}}(\bar{x})\right]\left[T_N(x)+1\right]\}, \end{aligned} \tag{9.9}$$

and, for indices on different rings,

$$\begin{aligned} \jmath^{T^f_{4N}}_{2\mathrm{p},2\overline{\mathrm{p}}+2\overline{\mathrm{q}}} &= (cd)^{2N-1}\{(E-a)\left[U_{N-\mathrm{q}-1}(x)-U_{\mathrm{q}-1}(x)\right]\left[T_N(\bar{x})+1\right] \\ &\qquad - (E+b)\left[U_{N-\mathrm{q}-1}(\bar{x})-U_{\mathrm{q}-1}(\bar{x})\right]\left[T_N(x)+1\right]\}, \\ \jmath^{T^f_{4N}}_{2\mathrm{p}+1,2\overline{\mathrm{p}}+2\overline{\mathrm{q}}+1} &= (cd)^{2N-1}\{(E-b)\left[U_{N-\mathrm{q}-1}(x)-U_{\mathrm{q}-1}(x)\right]\left[T_N(\bar{x})+1\right] \\ &\qquad - (E+b)\left[U_{N-\mathrm{q}-1}(\bar{x})-U_{\mathrm{q}-1}(\bar{x})\right]\left[T_N(x)+1\right]\}, \\ \jmath^{T^f_{4N}}_{2\mathrm{p}+1,2\overline{\mathrm{p}}+2\overline{\mathrm{q}}+2} &= (cd)^{2N-1}\{\left[U_{N-\mathrm{q}-1}(x)-U_{\mathrm{q}}(x)\right]\left[T_N(\bar{x})+1\right] \\ &\qquad - \left[U_{N-\mathrm{q}-1}(\bar{x})-U_{\mathrm{q}}(\bar{x})\right]\left[T_N(x)+1\right]\}. \end{aligned} \tag{9.10}$$

The differences between equations for plain (*c.f.* Eqs. (9.4) and (9.5)) and flipped treadmills occur because of the sign changes arising from edge flips. These affect all terms inside the square brackets because symmetry divides the adjacency matrix into *two* flipped rings C^f_{2N}.

9.2 Structural polynomials of $T^M_{4N}(a,b \mid c,d)$

The characteristic polynomial, $s^{T^M_{4N}}(E)$, can be written immediately as we have been able to split the secular equations into contributions from two annulenes, $C_{2N}(a,b \mid c,d)$ and

$C^f_{2N}(-a,-b \mid c,d)$, to give

$$s^{T^M_{4N}} = s(C_{2N}(a,b \mid c,d),E)s(C^f_{2N}(-a,-b \mid c,d),E)$$
$$= 4(cd)^{2N}\left(T_N(x)-1\right)\left(T_N(\bar{x})+1\right), \tag{9.11}$$

with x and $\bar{x}$ defined in Eqs. (3.5) and (5.2). We can also write

$$s^{T^M_{4N}} = 4(cd)^{2N}\left(T_{2N}(y)-1\right)\left(T_{2N}(\bar{y})+1\right), \tag{9.12}$$

with y and $\bar{y}$ defined in Eqs. (3.10) and (5.4).

The $\jmath$ structural polynomials can be obtained using the same logic as in Section 5,

$$\jmath^{T^M_{4N}}_{\mathrm{pq}} = \frac{1}{2}\Big(\jmath(C_{2N}(a,b \mid c,d),E)s(C^f_{2N}(-a,-b \mid c,d),E)$$
$$+ s(C_{2N}(a,b \mid c,d),E)\jmath(C^f_{2N}(-a,-b \mid c,d),E)\Big), \tag{9.13}$$

so that, for indices on the same ring,

$$\jmath^{T^M_{4N}}_{2\mathrm{p},2\mathrm{p}+2\mathrm{q}} = (cd)^{2N-1}\left\{(E-a)\left[U_{N-\mathrm{q}-1}(x)+U_{\mathrm{q}-1}(x)\right]\left[T_N(\bar{x})+1\right]\right.$$
$$\left.+\,(E+a)\left[U_{N-\mathrm{q}-1}(\bar{x})-U_{\mathrm{q}-1}(\bar{x})\right]\left[T_N(x)-1\right]\right\},$$
$$\jmath^{T^M_{4N}}_{2\mathrm{p}+1,2\mathrm{p}+2\mathrm{q}+1} = (cd)^{2N-1}\left\{(E-b)\left[U_{N+\mathrm{q}-1}(x)+U_{\mathrm{q}-1}(x)\right]\left[T_N(\bar{x})+1\right]\right.$$
$$\left.+\,(E+a)\left[U_{N-\mathrm{q}-1}(\bar{x})-U_{\mathrm{q}-1}(\bar{x})\right]\left[T_N(x)-1\right]\right\},$$
$$\jmath^{T^M_{4N}}_{2\mathrm{p}+1,2\mathrm{p}+2\mathrm{q}+2} = (cd)^{2N-1}\left\{\left[U_{N-\mathrm{q}-1}(x)+U_{\mathrm{q}}(x)\right]\left[T_N(\bar{x})+1\right]\right.$$
$$\left.+\left[U_{N-\mathrm{q}-1}(\bar{x})-U_{\mathrm{q}}(\bar{x})\right]\left[T_N(x)-1\right]\right\}, \tag{9.14}$$

whilst for indices on different rings,

$$\jmath^{T^M_{4N}}_{2\mathrm{p},2\overline{\mathrm{p}}+2\overline{\mathrm{q}}} = (cd)^{2N-1}\left\{(E-a)\left[U_{N-\mathrm{q}-1}(x)+U_{\mathrm{q}-1}(x)\right]\left[T_N(\bar{x})+1\right]\right.$$
$$\left.-\,(E+b)\left[U_{N-\mathrm{q}-1}(\bar{x})-U_{\mathrm{q}-1}(\bar{x})\right]\left[T_N(x)-1\right]\right\},$$
$$\jmath^{T^M_{4N}})_{2\mathrm{p}+1,2\overline{\mathrm{p}}+2\overline{\mathrm{q}}+1} = (cd)^{2N-1}\left\{(E-b)\left[U_{N-\mathrm{q}-1}(x)+U_{\mathrm{q}-1}(x)\right]\left[T_N(\bar{x})+1\right]\right.$$
$$\left.-\,(E+b)\left[U_{N-\mathrm{q}-1}(\bar{x})-U_{\mathrm{q}-1}(\bar{x})\right]\left[T_N(x)-1\right]\right\},$$
$$\jmath^{T^M_{4N}}_{2\mathrm{p}+1,2\overline{\mathrm{p}}+2\overline{\mathrm{q}}+2} = (cd)^{2N-1}\left\{\left[U_{N-\mathrm{q}-1}(x)+U_{\mathrm{q}}(x)\right]\left[T_N(\bar{x})+1\right]\right.$$
$$\left.-\left[U_{N-\mathrm{q}-1}(\bar{x})-U_{\mathrm{q}}(\bar{x})\right]\left[T_N(x)-1\right]\right\}. \tag{9.15}$$

The sign changes in this case affect only one of the sets of terms inside square brackets, because symmetry divides the adjacency matrix into contributions from one ring C_{2N} and one flipped ring C^f_{2N}.

9.3 Structural polynomials of $T^{M^f}_{4N}(a,b \mid c,d)$

The characteristic polynomial, $s^{T^{M^f}}_{4N}(E)$, can be written immediately as we have been able to split the secular equations into contributions from annulenes, $C^f_{2N}(a,b \mid c,d)$ and $C_{2N}(-a,-b \mid c,d)$, to give

$$s^{T^{M^f}}_{4N} = s(C^f_{2N}(a,b \mid c,d),E)s(C_{2N}(-a,-b \mid c,d),E)$$
$$= 4(cd)^{2N}\left(T_N(x)+1\right)\left(T_N(\bar{x})-1\right), \tag{9.16}$$

with x and $\bar{x}$ defined in Eqs. (3.5) and (5.2). We can also write

$$s_{4N}^{T^{Mf}} = 4(cd)^{2N}\left(T_{2N}(y)+1\right)\left(T_{2N}(\bar{y})-1\right), \tag{9.17}$$

with y and $\bar{y}$ defined in Eqs. (3.10) and (5.4).

The $\jmath$ structural polynomials can be obtained using the same logic as in Section 5,

$$\begin{aligned} \jmath_{\mathrm{pq}}^{T_{4N}^{Mf}} &= \frac{1}{2}\Big(\jmath(C_{2N}^{f}(a,b \mid c,d),E)s(C_{2N}(-a,-b \mid c,d),E) \\ &\quad + s(C_{2N}^{f}(a,b \mid c,d),E)\jmath(C_{2N}(-a,-b \mid c,d),E)\Big), \end{aligned} \tag{9.18}$$

so that, for indices on the same ring,

$$\begin{aligned} \jmath_{2\mathrm{p},2\mathrm{p}+2\mathrm{q}}^{T_{4N}^{Mf}} &= (cd)^{2N-1}\left\{(E-a)\left[U_{N-\mathrm{q}-1}(x)-U_{\mathrm{q}-1}(x)\right]\left[T_N(\bar{x})-1\right]\right. \\ &\qquad \left. + (E+a)\left[U_{N-\mathrm{q}-1}(\bar{x})+U_{\mathrm{q}-1}(\bar{x})\right]\left[T_N(x)+1\right]\right\}, \\ \jmath_{2\mathrm{p}+1,2\mathrm{p}+2\mathrm{q}+1}^{T_{4N}^{Mf}} &= (cd)^{2N-1}\left\{(E-b)\left[U_{N-\mathrm{q}-1}(x)-U_{\mathrm{q}-1}(x)\right]\left[T_N(\bar{x})-1\right)\right. \\ &\qquad \left. + (E+a)\left[U_{N-\mathrm{q}-1}(\bar{x})+U_{\mathrm{q}-1}(\bar{x})\right]\left(T_N(x)+1\right]\right\}, \\ \jmath_{2\mathrm{p}+1,2\mathrm{p}+2\mathrm{q}+2}^{T_{4N}^{Mf}} &= (cd)^{2N-1}\left\{\left[U_{N-\mathrm{q}-1}(x)-U_{\mathrm{q}}(x)\right]\left[T_N(\bar{x})-1\right]\right. \\ &\qquad \left. + \left[U_{N-\mathrm{q}-1}(\bar{x})+U_{\mathrm{q}}(\bar{x})\right]\left[T_N(x)+1\right]\right\}, \end{aligned} \tag{9.19}$$

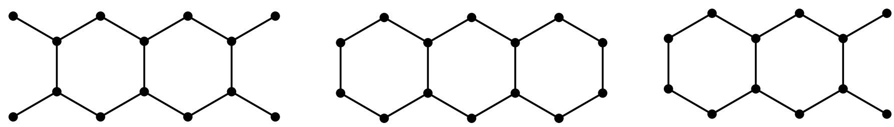
and for indices on different rings,

$$\begin{aligned} \jmath_{2\mathrm{p},2\overline{\mathrm{p}}+2\overline{\mathrm{q}}}^{T_{4N}^{Mf}} &= (cd)^{2N-1}\left\{(E-a)\left[U_{N-\mathrm{q}-1}(x)-U_{\mathrm{q}-1}(x)\right]\left[T_N(\bar{x})-1\right]\right. \\ &\qquad \left. - (E+b)\left[U_{N-\mathrm{q}-1}(\bar{x})+U_{\mathrm{q}-1}(\bar{x})\right]\left[T_N(x)+1\right]\right\}, \\ \jmath_{2\mathrm{p}+1,2\overline{\mathrm{p}}+2\overline{\mathrm{q}}+1}^{T_{4N}^{Mf}} &= (cd)^{2N-1}\left\{(E-b)\left[U_{N-\mathrm{q}-1}(x)-U_{\mathrm{q}-1}(x)\right]\left[T_N(\bar{x})-1\right)\right. \\ &\qquad \left. - (E+b)\left[U_{N-\mathrm{q}-1}(\bar{x})+U_{\mathrm{q}-1}(\bar{x})\right]\left[T_N(x)+1\right]\right\}, \\ \jmath_{2\mathrm{p}+1,2\overline{\mathrm{p}}+2\overline{\mathrm{q}}+2}^{T_{4N}^{Mf}} &= (cd)^{2N-1}\left\{\left[U_{N-\mathrm{q}-1}(x)-U_{\mathrm{q}}(x)\right]\left[T_N(\bar{x})-1\right]\right. \\ &\qquad \left. - \left[U_{N-\mathrm{q}-1}(\bar{x})+U_{\mathrm{q}}(\bar{x})\right]\left[T_N(x)+1\right]\right\}. \end{aligned} \tag{9.20}$$

The sign changes in this case affect only one of the sets of terms inside square brackets, because symmetry divides the adjacency matrix into a flipped ring C_{2N}^{f} and one plain ring C_{2N}.

10 Graphs derived from alternating ladders and treadmills

A series of interesting graphs can be derived from our alternating ladders and treadmills by putting either $a = 0$, or $b = 0$. Some of the graphs that can be derived from ladders are shown in Fig. 7. Ladders with backbone chains with odd numbers of vertices lead to polyacenes with arms and legs, or to polyacenes themselves (Fig. 7(a) and 7(b)), by putting the first rung edge parameter or the second to zero. Even-vertex backbones give polyacenes with a single arm and leg, as shown in Fig. 7(c), whichever rung weight is set to zero. In the case of treadmills, it does not matter which rung weight is set to zero. In either case one obtains cyclic polyacenes. The appropriate formulae in Sections 4 and 8 for eigenvalues, eigenvectors and structural polynomials can be used in these cases.



(a) (b) (c)

Figure 7: Graphs derived from ladders by zeroing rung parameters to zero: (a) $L_{14}(0,1 \mid 1,1)$, with two 7-vertex backbone chains, $a = 0$; (b) $L_{14}(0,1 \mid 1,1)$, with two 7-vertex backbone chains, $b = 0$; (c) $L_{12}(1,0 \mid 1,1)$, with two 6-vertex backbone chains, $b = 0$.

11 Conclusions

The algebraic development of structural polynomials reported here has been carried out in order to have exact results on which to base an elaboration of the theory of molecular conduction. The new formulae will allow us to treat π distortivity and its effect on ballistic conduction through conjugated molecular frameworks, as predicted within the source-sink potential (SSP) approach, a model that has a very direct connection to graph theory. In the simplest picture, a *molecular device* consists of a molecule attached to two semi-infinite wires. Such a device can be modelled qualitatively by replacing the system by a graph in which the electronic interactions are replaced by a series of edge weights. Furthermore, the infinite wires can be replaced by source and sink vertices, with complex vertex weights modified to reproduce the physics of a current of electrons down the wires. The formula for the transmission, $T(E)$, of current of electrons through a device with energy E, can then be expressed in terms of four polynomials derived from the graph of the molecule. These polynomials can be chosen as the characteristic polynomial $s(E)$ of the graph itself, $\jmath_{\mathrm{pp}}(E)$, $\jmath_{\mathrm{qq}}(E)$, and $\jmath_{\mathrm{pq}}(E)$ (*c.f.* Eqs. (1.1) and (1.2)), where vertices p and q of the molecule are attached to the source and sink vertices in the device.

The formulae we have obtained for ladders and the various forms of treadmills show that the structural polynomials can be written in a simple manner. We have shown that the existence of an involution allows the characteristic polynomials to be written neatly as a product of the characteristic polynomials of certain 'half' graphs comprising vertex-weighted backbones. The remaining structural polynomials are also expressed in terms of half graphs, albeit in a slightly more complicated form.

Representation of the various structural polynomials in this 'factorised' form has advantages for understanding the structure of the spectrum and has implications for the physics of the transmission as a function of energy. The different sign patterns of the structural polynomials exhibited in, for example, the varieties of treadmill (flipped, Möbius, *etc.*) will have a profound effect on transmission, $T(E)$, as a function of the energy of the incoming electrons. In certain cases, for example, conduction is switched off for the whole range of accessible energies, E. A detailed account of the SSP modelling of conduction in systems represented by graphs with these exotic topologies will be published elsewhere.

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On line and pseudoline configurations and ball-quotients

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Abstract

In this note we show that there are no real configurations of $d \geq 4$ lines in the projective plane such that the associated Kummer covers of order 3^{d-1} are ball-quotients and there are no configurations of $d \geq 4$ lines such that the Kummer covers of order 4^{d-1} are ball-quotients. Moreover, we show that there exists only one configuration of real lines such that the associated Kummer cover of order 5^{d-1} is a ball-quotient. In the second part we consider the so-called topological (n_k)-configurations and we show, using Shnurnikov's inequality, that for $n < 27$ there do not exist (n_5)-configurations and and for $n < 41$ there do not exist (n_6)-configurations.

Keywords: Line configurations, Hirzebruch inequality, Melchior inequality, Shnurnikov inequality, ball-quotients.

Math. Subj. Class.: 14C20, 52C35, 32S22

1 Preliminaries

In his pioneering paper Hirzebruch [5] constructed some new examples of algebraic surfaces which are ball-quotients, i.e., surfaces of general type satisfying equality in the Bogomolov-Miyaoka-Yau inequality [8]

$$K_X^2 \leq 3e(X),$$

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where K_X denotes the canonical divisor and $e(X)$ is the topological Euler characteristic. The key idea of Hirzebruch, which enabled constructing these new ball-quotients, is that one can consider abelian covers of the complex projective plane branched along line configurations. Let us recall briefly how the celebrated construction of Hirzebruch works (for more details please consult for instance [1]).

Let $\mathcal{L} = \{l_1, ..., l_d\} \subset \mathbb{P}^2$ be a configuration of $d \geq 4$ lines such that there is no point p where all d-lines meet and pick $n \in \mathbb{Z}_{\geq 2}$. Now we can consider the Kummer extension having degree n^{d-1} and Galois group $(\mathbb{Z}/n\mathbb{Z})^{d-1}$ defined as the function field

$$K := \mathbb{C}\left(z_1/z_0, z_2/z_0\right)\left((l_2/l_1)^{1/n}, ..., (l_d/l_1)^{1/n}\right).$$

This Kummer extension is an abelian extension of the function field of the complex projective plane. It can be shown that K determines an algebraic surface X_n with normal singularities which ramifies over the plane with the arrangement as the locus of the ramification. Hirzebruch showed that X_n is singular exactly over a point p iff p is a point of multiplicity ≥ 3 in $\mathcal{L}$. After blowing up these singular points we obtain a smooth surface $Y_n^{\mathcal{L}}$. It turns out that the Chern numbers of $Y_n^{\mathcal{L}}$ can be read off directly from combinatorics of line arrangements, i.e.,

$$\frac{c_2(Y_n^{\mathcal{L}})}{n^{d-3}} = n^2(3 - 2d + f_1 - f_0) + 2n(d - f_1 + f_0) + f_1 - t_2,$$

$$\frac{c_1^2(Y_n^{\mathcal{L}})}{n^{d-3}} = n^2(-5d + 9 + 3f_1 - 4f_0) + 4n(d - f_1 + f_0) + f_1 - f_0 + d + t_2,$$

where t_r denotes the number of r-fold points (i.e. points where exactly r lines meet), $f_0 = \sum_{r\geq 2} t_r$ and $f_1 = \sum_{r\geq 2} rt_r$. Moreover, it can be shown that $Y_n^{\mathcal{L}}$ has non-negative Kodaira dimension if $t_d = t_{d-1} = t_{d-2} = 0$ and $n \geq 2$, or $t_d = t_{d-1} = 0$ and $n \geq 3$ (we assume additionally that $d \geq 6$), and in these cases we have $K^2_{Y_n^{\mathcal{L}}} \leq 3e(Y_n^{\mathcal{L}})$. Now we can define the following Hirzebruch polynomial (for more details, please consult the original paper due to Hirzebruch [5, Section 3.1]):

$$P_{\mathcal{L}}(n) = \frac{3e(Y_n^{\mathcal{L}}) - K^2_{Y_n^{\mathcal{L}}}}{n^{d-3}} = n^2(f_0 - d) + 2n(d - f_1 + f_0) + 2f_1 + f_0 - d - 4t_2 \quad (1.1)$$

and by the construction $P_{\mathcal{L}}(n) \geq 0$ provided that $n \geq 2$. If there exists a configuration of lines $\mathcal{A}$ such that there exists $m \in \mathbb{Z}_{\geq 2}$ with $P_{\mathcal{A}}(m) = 0$, then $Y_m^{\mathcal{A}}$ is a ball quotient. There are some examples of line configurations which allow us to construct ball quotients via Hirzebruch's construction.

Example 1.1. ([5, p. 133]) Let us consider the following configuration, which is denoted in the literature by $\mathcal{A}_1(6)$.

Simple computations give

$$P_{\mathcal{A}_1(6)}(n) = n^2 - 10n + 25,$$

which means that $Y_5^{\mathcal{A}_1(6)}$ is a ball-quotient.

Example 1.2. ([5, p. 133]) Let us now consider the Hesse configuration $\mathcal{H}$ of lines (which cannot be drawn over the real numbers) having the following combinatorics:

$$d = 12, t_2 = 12, t_4 = 9.$$

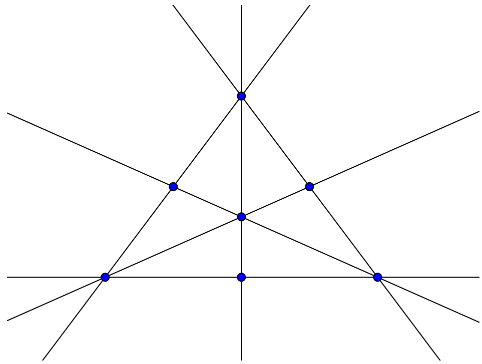

Figure 1: $\mathcal{A}_1(6)$ configuration.

Then

$$P_{\mathcal{H}}(n) = 9(n^2 - 6n + 9),$$

which means that $Y_3^{\mathcal{H}}$ is a ball-quotient.

It is known that there are only a few examples of ball-quotients provided by line arrangements and it seems to be extremely difficult to find other examples. In this note we study a natural question about the existence of new ball quotients constructed via Hirzebruch's method. Before we formulate our main results let us define the following object.

Definition 1.3. Let $Y_n^{\mathcal{L}}$ be the minimal desingularization of X_n constructed as the Kummer extension. Then $Y_n^{\mathcal{L}}$ is called the Kummer cover of order n^{d-1}.

Question 1.4. Does a real line configuration $\mathcal{L} \subset \mathbb{P}^2_{\mathbb{C}}$ exist such that $Y_3^{\mathcal{L}}$ is a ball quotient?

Remark 1.5. In this note by a real line configuration we mean a configuration of lines which is realizable over the real numbers. For instance, the Hesse line configuration is not realizable over the real numbers.

Our main results of this paper are the following strong classification results (our proofs are purely combinatorial).

Theorem A. There does not exist any real line configuration $\mathcal{L}$ with $d \geq 4$ lines and $t_d = t_{d-1} = 0$ such that $Y_3^{\mathcal{L}}$ is a ball quotient.

Theorem B. There does not exist any line configuration $\mathcal{L}$ with $d \geq 4$ lines and $t_d = t_{d-1} = 0$ such that $Y_4^{\mathcal{L}}$ is a ball-quotient.

As a simple application of our methods we show the following results.

Theorem C. The configuration $\mathcal{A}_1(6)$ is (up to projective equivalence) the only configuration for $d \geq 4$ real lines such that the Kummer cover of order 5^{d-1} is a ball quotient.

In our proof of Theorem A we use, in a very essential way, Shnurnikov's inequality (2.4) for pseudoline configurations. Using this inequality we can prove the following result about topological (n_k)-configurations.

Theorem D. For $n < 27$ there does not exist a topological (n_5)-configuration and for $n < 41$ there does not exist a topological (n_6)-configuration.

2 Real line configurations and ball-quotients

Firstly, we recall that the Hirzebruch polynomial, depending on $n \in \mathbb{Z}_{\geq 2}$, parameterizes the whole family of Hirzebruch's inequalities. Taking this into account, observe that if $n = 3$, then we have the following inequality (we assume here that $t_d = t_{d-1} = 0$):

$$t_2 + t_3 \geq d + \sum_{r \geq 5} (r-4)t_r. \tag{2.1}$$

It is worth pointing out that in a subsequent paper on the topic [6] Hirzebruch has improved his inequality (here we assume that $t_d = t_{d-1} = t_{d-2} = 0$):

$$t_2 + \frac{3}{4}t_3 \geq d + \sum_{r \geq 5} (2r-9)t_r, \tag{2.2}$$

and we should notice that this improvement comes from the Hirzebruch polynomial for $n = 2$ with some extra effort – please consult [6] for further details.

We will also need the following Melchior's inequality [7], which is true for real line configurations with $d \geq 3$ lines and $t_d = 0$:

$$t_2 \geq 3 + \sum_{r \geq 4} (r-3)t_r. \tag{2.3}$$

Finally, let us recall the notion of (real) pseudoline configurations.

Definition 2.1. We say that $\mathcal{C} \subset \mathbb{P}^2_{\mathbb{R}}$ is a configuration of pseudolines if it is a configuration of $n \geq 3$ smooth closed curves such that

- every pair of pseudolines meets exactly once at a single crossing (i.e., locally this intersection looks like $xy = 0$),
- curves do not intersect simultaneously at a single point.

In particular, every real line configuration is a pseudoline configuration. Recently I. N. Shnurnikov [9] has shown the following beautiful inequality.

Theorem 2.2. *Let $\mathcal{C}$ be a configuration of n pseudolines such that $t_n = t_{n-1} = t_{n-2} = t_{n-3} = 0$. Then*

$$t_2 + \frac{3}{2}t_3 \geq 8 + \sum_{r \geq 4} (2r-7.5)t_r. \tag{2.4}$$

Now we are ready to prove Theorem A.

Proof. Our problem boils down to show that there does not exist a real line configuration satisfying

$$t_2 + t_3 = d + \sum_{r \geq 5} (r-4)t_r. \tag{2.5}$$

We start with excluding the case of $t_{d-2} = 1$ for which two possibilities remain (we assume here that $d \geq 6$)

- $\mathcal{A}_1 : t_{d-2} = 1, t_2 = 2d-3,$

- $\mathcal{A}_2 : t_{d-2} = 1, t_3 = 1, t_2 = 2d-6$,

but it is easy to see that $\mathcal{A}_1$ and $\mathcal{A}_2$ do not satisfy (2.5).

From this point on we consider only real line configurations with d lines where $t_d = t_{d-1} = t_{d-2} = 0$. Assume there exists a real line configuration $\mathcal{L}$ such that $Y_3^{\mathcal{L}}$ is a ball-quotient. Using (2.2) and (2.5) we obtain

$$-\frac{1}{4}t_3 \geq \sum_{r\geq 5}(r-5)t_r,$$

which means that if $d \geq 4$ we have $t_2 \geq 3$, $t_3 = 0$ and $t_r = 0$ for $r \geq 6$. Moreover, it might happen that t_4 or t_5 are non-zero. This reduces (2.5) to

$$t_2 = d + t_5.$$

On the other hand, we have the following combinatorial equality

$$d(d-1) = \sum_{r\geq 2} r(r-1)t_r = 2t_2 + 12t_4 + 20t_5,$$

and combining this with $t_2 = d + t_5$ we obtain

$$d(d-3) = 12t_4 + 22t_5.$$

Using (2.3) we get

$$d - 3 \geq t_4 + t_5$$

and finally

$$12t_4 + 22t_5 = d(d-3) \geq d(t_4 + t_5),$$

which leads to

$$d \leq \frac{12t_4 + 22t_5}{t_4 + t_5} \leq 22.$$

Summing up, $\mathcal{L}$ satisfies the following conditions:

$$d \in \{4, ..., 22\}, \quad t_2 = d + t_5, \quad d(d-3) = 12t_4 + 22t_4, \quad d-3 \geq t_4 + t_5.$$

It can be checked (for instance using a computer program) that the above constraints result in the following combinatorics (using the following convention in our listing : $\mathcal{L} = [d, t_4, t_5]$):

$$\mathcal{L}_1 = [10,4,1], \quad \mathcal{L}_2 = [11,0,4], \quad \mathcal{L}_3 = [12,9,0], \quad \mathcal{L}_4 = [13,9,1], \quad \mathcal{L}_5 = [14,0,7],$$
$$\mathcal{L}_6 = [15,4,6], \quad \mathcal{L}_7 = [17,7,7], \quad \mathcal{L}_8 = [18,6,9], \quad \mathcal{L}_9 = [22,0,19].$$

Now we need to check whether the above combinatorics can be realized over the real numbers. To this end, first observe that $\mathcal{L}_1, ..., \mathcal{L}_9$ satisfy the assumptions of Theorem 2.2. Combining Shnurnikov‘s inequality with $t_2 = d + t_5$ we obtain

$$d - 8 \geq \frac{1}{2}t_4 + \frac{3}{2}t_5, \tag{2.6}$$

and it is easy to check that none of $\mathcal{L}_i$ satisfies (2.6). This contradiction finishes the proof. □

Next, we show Theorem B.

Proof. Suppose that there exists a line configuration $\mathcal{L}$ such that $Y_4^{\mathcal{L}}$ is a ball-quotient. This implies that $\mathcal{L}$ satisfies the following equality:

$$9t_2 + 7t_3 + t_4 = 9d + \sum_{r\geq 5}(6r-25)t_r. \tag{2.7}$$

Let us recall that Hirzebruch in [5, p. 140] pointed out that one can improve (2.1), namely

$$t_2 + \frac{3}{4}t_3 \geq d + \sum_{r\geq 5}(r-4)t_r. \tag{2.8}$$

Now let us rewrite (2.8) as follows

$$9t_2 + \frac{27}{4}t_3 \geq 9d + \sum_{r\geq 5}(9r-36)t_r. \tag{2.9}$$

On the other hand, we have

$$9t_2 + \frac{27}{4}t_3 = -t_4 - \frac{1}{4}t_3 + 9d + \sum_{r\geq 5}(6r-25)t_r. \tag{2.10}$$

Combining (2.9) with (2.10) we obtain

$$-t_4 - \frac{1}{4}t_3 + 9d + \sum_{r\geq 5}(6r-25)t_r \geq 9d + \sum_{r\geq 5}(9r-36)t_r, \tag{2.11}$$

which implies $t_r = 0$ for $r \geq 3$ and (2.7) has the following form

$$t_2 = d.$$

However, using the combinatorial equality one gets

$$d(d-1) = 2t_2 = 2d,$$

which implies that either $d = 3$ or $d = 0$, a contradiction. □

Remark 2.3. Using almost the same proof one can show that there does not exist any line configuration $\mathcal{L}$ of $d \geq 4$ lines with $t_d = t_{d-1} = 0$ such that $Y_7^{\mathcal{L}}$ is a ball-quotient.

Finally, we show Theorem C.

Proof. Again, our problem boils down to classifying all real line configurations that satisfy the following equality:

$$4t_2 + 3t_3 + t_4 = 4d + \sum_{r\geq 5}(2r-9)t_r. \tag{2.12}$$

It is easy to see that one can automatically exclude the case $t_{d-2} = 1$, thus from now on we assume that $t_d = t_{d-1} = t_{d-2} = 0$. Rewriting (2.12) in a slightly different way we get

$$t_2 + \frac{3}{4}t_3 = d - \frac{1}{4}t_4 + \sum_{r\geq 5}\left(\frac{1}{2}r - \frac{9}{4}\right)t_r.$$

Now combining this with (2.2), we obtain

$$d - \frac{1}{4}t_4 + \sum_{r \geq 5} \left(\frac{1}{2}r - \frac{9}{4}\right) t_r \geq d + \sum_{r \geq 5} (2r-9)t_r$$

and finally

$$-\frac{1}{4}t_4 \geq \sum_{r \geq 5} \left(\frac{3}{2}r - \frac{27}{4}\right) t_r.$$

This implies $t_r = 0$ for $r \geq 4$ and it leads to

$$t_2 + \frac{3}{4}t_3 = d. \tag{2.13}$$

Using the combinatorial equality with (2.13) one gets

$$\frac{2}{9}d(d-3) = t_3. \tag{2.14}$$

On the other hand, by Melchior's inequality

$$t_2 \geq 3$$

and

$$d(d-1) = 2t_2 + 6t_3 \geq 6(1 + t_3).$$

Now using (2.14) we obtain

$$d^2 - 9d + 18 \leq 0,$$

which means $d \in \{4, 5, 6\}$. It is easy to verify now that all these constraints lead to $d = 6, t_2 = 3$ and $t_3 = 4$, which completes the proof. □

3 Topological (n_k)-configurations

A topological (n_k) point-line configuration, or simply a topological (n_k)-configuration, is a set of n points and n pseudolines in the real projective plane, such that each point is incident with k pseudolines and each pseudoline is incident with k points. Much work has been done [4] to study the existence of (n_k)-configurations in which all pseudolines are straight lines. In these cases it is useful to know whether there exists at least a topological (n_k)-configuration. For $k = 4$ the existence of topological (n_4)-configurations is known for all $n \geq 17$, see [3].

Using the inequality of Shnurnikov (2.4), we obtain lower bounds for smallest topological (n_k)-configurations for $k > 4$. The corresponding bound for $k = 4$ is not sharp and leads to $n \geq 16$, however for $k = 5$ not much is known so far.

Now we prove Theorem D.

Proof. When we have a topological (n_k)-configuration, we can change the configuration locally (if neccessary) such that $t_s = 0$ for $2 < s < k$ and for $k < s$. This implies that the number of single crossings is

$$t_2 = \binom{n}{2} - n \cdot \binom{k}{2}$$

and the inequality of Shnurnikov becomes

$$n \cdot (n-1) - n \cdot k \cdot (k-1) > 16 + n \cdot (4 \cdot k - 15)$$

$$n \cdot (n - 1 - k \cdot (k-1) - 4 \cdot k + 15) > 16$$

$$n \cdot (n + 14 - k \cdot (k+3)) > 16$$

This implies especially that there are no topological (n_5)-configurations for $n < 27$ and there are no topological (n_6)-configurations for $n < 41$. □

The smallest known topological (n_5)-configuration with $n = 36$ is due to Leah Wrenn Berman, constructed from two (18_4)-configurations, [2]. It will be published elsewhere. An open problem remains to find topological (n_5)-configurations for $27 \leq n \leq 35$.

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On zero sum-partition of Abelian groups into three sets and group distance magic labeling

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Abstract

We say that a finite Abelian group Γ has the *constant-sum-partition property into t sets* (CSP(t)-property) if for every partition $n = r_1 + r_2 + \ldots + r_t$ of n, with $r_i \geq 2$ for $2 \leq i \leq t$, there is a partition of Γ into pairwise disjoint subsets $A_1, A_2, \ldots, A_t$, such that $|A_i| = r_i$ and for some $\nu \in \Gamma$, $\sum_{a \in A_i} a = \nu$ for $1 \leq i \leq t$. For $\nu = g_0$ (where g_0 is the identity element of Γ) we say that Γ has *zero-sum-partition property into t sets* (ZSP(t)-property).

A Γ-*distance magic labeling* of a graph $G = (V, E)$ with $|V| = n$ is a bijection ℓ from V to an Abelian group Γ of order n such that the weight $w(x) = \sum_{y \in N(x)} \ell(y)$ of every vertex $x \in V$ is equal to the same element $\mu \in \Gamma$, called the *magic constant*. A graph G is called a *group distance magic graph* if there exists a Γ-distance magic labeling for every Abelian group Γ of order $|V(G)|$.

In this paper we study the CSP(3)-property of Γ, and apply the results to the study of group distance magic complete tripartite graphs.

Keywords: Abelian group, constant sum partition, group distance magic labeling.

Math. Subj. Class.: 05C25, 05C78

1 Introduction

All graphs considered in this paper are simple finite graphs. Consider a simple graph G whose order we denote by $n = |G|$. We denote by $V(G)$ the vertex set and $E(G)$ the edge set of a graph G. The *open neighborhood* $N(x)$ of a vertex x is the set of vertices adjacent to x, and the degree $d(x)$ of x is $|N(x)|$, the size of the neighborhood of x.

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Let the identity element of Γ be denoted by g_0. Recall that any group element $\iota \in \Gamma$ of order 2 (i.e., $\iota \neq g_0$ such that $2\iota = g_0$) is called an *involution*.

In [8] Kaplan, Lev and Roditty introduced a notion of zero-sum partitions of subsets in Abelian groups. Let Γ be an Abelian group and let A be a finite subset of $\Gamma - \{g_0\}$, with $|A| = n - 1$. We shall say that A has the *zero-sum-partition property* (*ZSP-property*) if every partition $n - 1 = r_1 + r_2 + \ldots + r_t$ of $n - 1$, with $r_i \geq 2$ for $1 \leq i \leq t$ and for any possible positive integer t, there is a partition of A into pairwise disjoint subsets $A_1, A_2, \ldots, A_t$, such that $|A_i| = r_i$ and $\sum_{a \in A_i} a = g_0$ for $1 \leq i \leq t$. In the case that Γ is finite, we shall say that Γ has the ZSP-property if $A = \Gamma - \{g_0\}$ has the ZSP-property.

They proved the following theorem for cyclic groups of odd order.

Theorem 1.1 ([8])**.** *The group $\mathbb{Z}_n$ has the ZSP-property if and only if n is odd.*

Moreover, Kaplan, Lev and Roditty showed that if Γ is a finite Abelian group of even order n such that the number of involutions in Γ is different from 3, then Γ does not have the ZSP-property [8]. Their results along with results proved by Zeng [10] give necessary and sufficient conditions for the ZSP-property for a finite Abelian group.

Theorem 1.2 ([8, 10])**.** *Let Γ be a finite Abelian group. Then Γ has the ZSP-property if and only if either Γ is of odd order or Γ contains exactly three involutions.*

They apply those results to the study of anti-magic trees [8, 10].

We generalize the notion of ZSP-property. We say that a finite Abelian group Γ has the *constant-sum-partition property into t sets* (*CSP(t)-property*) if for every partition $n = r_1 + r_2 + \ldots + r_t$ of n, with $r_i \geq 2$ for $2 \leq i \leq t$, there is a partition of Γ into pairwise disjoint subsets $A_1, A_2, \ldots, A_t$, such that $|A_i| = r_i$ and for some $\nu \in \Gamma$, $\sum_{a \in A_i} a = \nu$ for $1 \leq i \leq t$. For $\nu = g_0$ we say that Γ has *zero-sum-partition property into t sets* (*ZSP(t)-property*).

In this paper we investigate also distance magic labelings, which belong to a large family of magic type labelings.

A *distance magic labeling* (also called *sigma labeling*) of a graph $G = (V, E)$ of order n is a bijection $\ell\colon V \to \{1, 2, \ldots, n\}$ with the property that there is a positive integer k (called the *magic constant*) such that

$$w(x) = \sum_{y \in N(x)} \ell(y) = k \text{ for every } x \in V(G),$$

where $w(x)$ is the *weight* of vertex x. If a graph G admits a distance magic labeling, then we say that G is a *distance magic graph*.

The concept of distance magic labeling has been motivated by the construction of magic rectangles, since we can construct a distance magic complete r-partite graph with each part size equal to n by labeling the vertices of each part by the columns of the magic rectangle. Although there does not exist a 2×2 magic rectangle, observe that the partite sets of $K_{2,2}$ can be labeled $\{1, 4\}$ and $\{2, 3\}$, respectively, to obtain a distance magic labeling. The following result was proved in [9].

Observation 1.3 ([9])**.** There is no distance magic r-regular graph with r odd.

Froncek in [7] defined the notion of group distance magic graphs, i.e., the graphs allowing a bijective labeling of vertices with elements of an Abelian group resulting in constant sums of neighbor labels.

A Γ-*distance magic labeling* of a graph $G = (V, E)$ with $|V| = n$ is a bijection ℓ from V to an Abelian group Γ of order n such that the weight $w(x) = \sum_{y \in N(x)} \ell(y)$ of every vertex $x \in V$ is equal to the same element $\mu \in \Gamma$, called the *magic constant*. A graph G is called a *group distance magic graph* if there exists a Γ-distance magic labeling for every Abelian group Γ of order $|V(G)|$.

The connection between distance magic graphs and Γ-distance magic graphs is as follows. Let G be a distance magic graph of order n with the magic constant μ'. If we replace the label n in a distance magic labeling for the graph G by the label 0, then we obtain a $\mathbb{Z}_n$-distance magic labeling for the graph G with the magic constant $\mu = \mu' \pmod{n}$. Hence every distance magic graph with n vertices admits a $\mathbb{Z}_n$-distance magic labeling. However a $\mathbb{Z}_n$-distance magic graph on n vertices is not necessarily a distance magic graph. Moreover, there are some graphs that are not distance magic while at the same time they are group distance magic (see [4]).

A general theorem for Γ-distance magic labeling similar to Observation 1.3 was proved recently.

Theorem 1.4 ([5]). *Let G be an r-regular graph on n vertices, where r is odd. There does not exist an Abelian group Γ of order n with exactly one involution ι such that G is Γ-distance magic.*

Notice that the constant sum partitions of a group Γ lead to complete multipartite Γ-distance magic labeled graphs. For instance, the partition $\{0\}$, $\{1, 2, 4\}$, $\{3, 5, 6\}$ of the group $\mathbb{Z}_7$ with constant sum 0 leads to a $\mathbb{Z}_7$-distance magic labeling of the complete tripartite graph $K_{1,3,3}$. More general, let G be a complete t-partite graph of order n with the partition sets $V_1, V_2, \ldots, V_t$. Note that G is Γ-distance magic if and only if $\sum_{i=1, i \neq j}^{t} \sum_{x \in V_i} \ell(x) = \mu$ for $j \in \{1, 2, \ldots, t\}$ which implies that $\sum_{x \in V_j} \ell(x) = \nu$ for $j \in \{1, 2, \ldots, t\}$ and some $\nu \in \Gamma$. Therefore we can see that G is Γ-distance magic if and only if Γ has the CSP(t)-property. The following theorems were proven in [3].

Theorem 1.5 ([3]). *Let $G = K_{n_1, n_2, \ldots, n_t}$ be a complete t-partite graph and $n = n_1 + n_2 + \ldots + n_t$. If $n \equiv 2 \pmod{4}$ and t is even, then there does not exist an Abelian group Γ of order n such that G is a Γ-distance magic graph.*

Theorem 1.6 ([3]). *The complete bipartite graph K_{n_1, n_2} is a group distance magic graph if and only if $n_1 + n_2 \not\equiv 2 \pmod{4}$.*

Therefore it follows that an Abelian group Γ of order n has the CSP(2)-property if and only if $n \not\equiv 2 \pmod{4}$.

In this paper we study the CSP(3)-property of Γ, and apply the results to an investigation of the necessary and sufficient conditions for complete tripartite graphs to be group distance magic. This work will also be potentially useful for group theorists working on Abelian groups.

2 Preliminaries

Assume Γ is an Abelian group of order n with the operation denoted by $+$. For convenience we will write ka to denote $a + a + \ldots + a$ (where the element a appears k times), $-a$ to denote the inverse of a and we will use $a - b$ instead of $a + (-b)$. Recall that a non-trivial finite group has elements of order 2 if and only if the order of the group is even. The fundamental theorem of finite Abelian groups states that a finite Abelian group Γ of order n can be expressed as the direct product of cyclic subgroups of prime-power order. This implies that

$$\Gamma \cong \mathbb{Z}_{p_1^{\alpha_1}} \times \mathbb{Z}_{p_2^{\alpha_2}} \times \ldots \times \mathbb{Z}_{p_k^{\alpha_k}} \quad \text{where} \quad n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \ldots \cdot p_k^{\alpha_k}$$

and p_i for $i \in \{1, 2, \ldots, k\}$ are not necessarily distinct primes. This product is unique up to the order of the direct product. When t is the number of these cyclic components whose order is a multiple of 2, then Γ has $2^t - 1$ involutions. In particular, if $n \equiv 2 \pmod 4$, then $\Gamma \cong \mathbb{Z}_2 \times A$ for some Abelian group A of odd order $n/2$. Moreover every cyclic group of even order has exactly one involution. The sum of all the group elements is equal to the sum of the involutions and the neutral element.

The following lemma was proved in [6] (see [6], Lemma 8).

Lemma 2.1 ([6]). *Let Γ be an Abelian group.*

1. *If Γ has exactly one involution ι, then $\sum_{g \in \Gamma} g = \iota$.*
2. *If Γ has no involutions, or more than one involution, then $\sum_{g \in \Gamma} g = g_0$.*

Anholcer and Cichacz proved the following (see [1], Lemma 2.4).

Lemma 2.2 ([1]). *Let Γ be an Abelian group with involutions set $I^* = \{\iota_1, \iota_2, \ldots, \iota_{2^k-1}\}$, $k > 1$ and let $I = I^* \cup \{g_0\}$. Given positive integers n_1, n_2 such that $n_1 + n_2 = 2^k$. There exists a partition $A = \{A_1, A_2\}$ of I such that*

1. *$n_1 = |A_1|$, $n_2 = |A_2|$,*
2. *$\sum_{a \in A_i} a = g_0$ for $i = 1, 2$,*

if and only if none of n_1, n_2 is 2.

3 Constant sum partition of Abelian groups

Note that if Γ has odd order, then it has the ZSP-property by Theorem 1.2, thus one can check that it has the ZSP(3)-property. We now generalize Lemma 2.2.

Lemma 3.1. *Let Γ be an Abelian group with involutions set $I^* = \{\iota_1, \iota_2, \ldots, \iota_{2^k-1}\}$, $k > 2$ and let $I = I^* \cup \{g_0\}$. Given positive integers n_1, n_2, n_3 such that $n_1 + n_2 + n_3 = 2^k$. There exists a partition $A = \{A_1, A_2, A_3\}$ of I such that*

1. *$n_1 = |A_1|$, $n_2 = |A_2|$, $n_3 = |A_3|$,*
2. *$\sum_{a \in A_i} a = g_0$ for $i \in \{1, 2, 3\}$,*

if and only if $n_1, n_2, n_3 \notin \{2, 2^k - 2\}$.

Proof. For $n_i = n_j = 1$ we have that $\sum_{a\in A_i} a \neq \sum_{a\in A_j} a$. For $n_i = 2$, it is easy to see $\sum_{a\in A_i} a \neq g_0$.

Let $\iota_0 = g_0$. Recall that since $I = \{\iota_0, \iota_1, \ldots, \iota_{2^k-1}\}$ is a subgroup of Γ, we have $I \cong (\mathbb{Z}_2)^k$. One can check that the lemma is true for $k \in \{3,4\}$. The sufficiency will be proved then by induction on k. Namely, suppose the assertion is true for some $m = k \geq 4$. We want to prove it is true for $m = k+1$. Let (n_1, n_2, n_3) be a triple such that $n_1, n_2, n_3 \notin \{2, 2^{k+1}-2\}$ and $n_1 + n_2 + n_3 = 2^{k+1}$. For $i \in \{1,2,3\}$ let $n_i = 4q_i + r_i$, where $r_i \in \{1,3,4,5,6\}$ and 1 appears at most once as a value of some r_i. Observe that $r_1 + r_2 + r_3 \leq 18$, but because $n_1 + n_2 + n_3 \equiv 0 \pmod 4$ and $n_1 + n_2 + n_3 = 4(q_1 + q_2 + q_3) + r_1 + r_2 + r_3$, we must have $r_1 + r_2 + r_3 \equiv 0 \pmod 4$, which implies that $r_1 + r_2 + r_3 \leq 16$. Thus $4(q_1 + q_2 + q_3) \geq 2^k$.

Now we select t_1, t_2, t_3 such that $t_i \leq q_i$ and $4(t_1+t_2+t_3) = 2^k$. Denote $n_i' = n_i - 4t_i$ and $n_i'' = 4t_i$. Obviously, $n_1' + n_2' + n_3' = n_1'' + n_2'' + n_3'' = 2^k$ and $n_i' \notin \{0, 2, 2^k - 2\}$ for any $i \in \{1,2,3\}$.

If also $n_i'' \neq 0$, then both triples (n_1', n_2', n_3') and (n_1'', n_2'', n_3'') satisfy the inductive hypothesis and there exist partitions of $(\mathbb{Z}_2)^k$ into sets S_1', S_2', S_3' and S_1'', S_2'', S_3'' of respective orders n_1', n_2', n_3' and n_1'', n_2'', n_3''. If we now replace each element $(x_1, x_2, \ldots, x_k)$ of $(\mathbb{Z}_2)^k$ in any S_i' by the $(x_1, x_2, \ldots, x_k, 0)$ of $(\mathbb{Z}_2)^{k+1}$, it should be clear that the sum of elements in each S_i' is the identity of $(\mathbb{Z}_2)^{k+1}$.

Similarly, we replace each element $(y_1, y_2, \ldots, y_k)$ of $(\mathbb{Z}_2)^k$ in any S_i'' by the element $(y_1, y_2, \ldots, y_k, 1)$ of $(\mathbb{Z}_2)^{k+1}$. Now because the order of each S_i'' is even, the ones in last entries add up to zero and the sum of elements in each S_i'' is again the identity of $(\mathbb{Z}_2)^{k+1}$. Now set $S_i = S_i' \cup S_i''$ to obtain the desired partition of $(\mathbb{Z}_2)^{k+1}$.

The case when $n_i'' = 0$ and $n_j'', n_l'' \neq 0$ can be treated using Lemma 2.2, and the case when $n_i'' = n_j'' = 0$ and $n_l'' = 2^k$ is obvious. □

Theorem 3.2. *Let Γ be an Abelian group of even order n. Γ has the CSP(3) -property if and only if $\Gamma \ncong (\mathbb{Z}_2)^t$ for some positive integer t. Moreover, $\Gamma \ncong (\mathbb{Z}_2)^t$ has the ZSP(3)-property if and only if Γ has more than one involution.*

Proof. For a given partition $n = n_1+n_2+n_3$ we will construct a partition $\Gamma = A_1 \cup A_2 \cup A_3$ such that $A_i = \{a_0^i, a_1^i, \ldots, a_{n_i-1}^i\}$ for $i \in \{1,2,3\}$. Let $\Gamma = \{g_0, g_1, \ldots, g_{n-1}\}$. Recall that by g_0 we denote the identity element of Γ.

Assume first that $\Gamma \cong (\mathbb{Z}_2)^t$ for $t > 1$ has the CSP(3)-property. Let A_1, A_2, A_3 be the desired partition of Γ for $n_2 = 2$. Hence $\sum_{a\in A_i} a = \iota \neq g_0$ for $i \in \{1,2,3\}$. Therefore $\sum_{g\in\Gamma} g = \sum_{i=1}^{3} \sum_{a\in A_i} a = 3\iota = \iota \neq g_0$, a contradiction with Lemma 2.1.

Suppose now that Γ has the ZSP(3)-property and there is the only one involution $\iota \in \Gamma$. Let A_1, A_2, A_3 be the desired partition of Γ, therefore $\sum_{a\in A_i} a = g_0$ for $i \in \{1,2,3\}$. Hence, $g_0 = \sum_{i=1}^{3} \sum_{a\in A_i} a = \sum_{g\in\Gamma} g$, on the other hand by Lemma 2.1 we have $\sum_{g\in\Gamma} g = \iota$, a contradiction.

We will prove sufficiency now. Let us consider two cases on the number of involutions in Γ.

Case 1. There is exactly one involution ι in Γ.
Notice that in that case $|\Gamma| \geq 6$. By fundamental theorem of finite Abelian groups

$$\Gamma \cong \mathbb{Z}_{2^{\alpha_1}} \times \mathbb{Z}_{p_2^{\alpha_2}} \times \ldots \times \mathbb{Z}_{p_k^{\alpha_k}} \quad \text{where} \quad n = 2^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \ldots \cdot p_k^{\alpha_k}, \quad \alpha_1 \geq 1$$

and $p_i \geq 3$ for $i \in \{2, 3, \ldots, k\}$ are not necessarily distinct primes. Since $|\Gamma| \geq 6$ we have $\Gamma \cong \mathbb{Z}_{2m} \times A$ for $m \geq 3$ and some Abelian group A of order $n/2m$. Let $g_1 = \iota$ and $g_{i+1} = -g_i$ for $i \in \{2, 4, 6, \ldots, n-2\}$. Using the isomorphism $\varphi\colon \Gamma \to \mathbb{Z}_{2m} \times A$, we can identify every $g \in \Gamma$ with its image $\varphi(g) = (j, a_i)$, where $j \in \mathbb{Z}_{2m}$ and $a_i \in A$ for $i \in \{0, 1, \ldots, \frac{n}{2m} - 1\}$ and a_0 is the identity element in A. Observe that $g_1 = \iota = (m, a_0)$. Because $m > 2$ we can set $g_2 = (1, a_0)$, $g_3 = (2m-1, a_0)$, $g_4 = (m-1, a_0)$, $g_5 = (m+1, a_0)$.
Without loss of generality we can assume that n_1 is even and $n_2 \geq n_3$. Let $a_0^1 = g_2$, $a_1^1 = g_4$ and $a_i^1 = g_{i+4}$ for $i \in \{2, 3, \ldots, n_1 - 1\}$.

Case 1.1. n_2, n_3 are both odd.
Let: $a_0^2 = g_0$, $a_1^2 = g_3$, $a_2^2 = g_5$ and $g_i^1 = a_{n_1+1+i}$ for $i \in \{3, 4, \ldots, n_2 - 1\}$.
$a_0^3 = g_1$ and $a_i^3 = g_{n_1+n_2+i}$ for $i \in \{1, 2, \ldots, n_3 - 1\}$.

Case 1.2. n_2, n_3 are both even.
Let: $a_0^2 = g_3$, $a_1^2 = g_5$ and $a_i^1 = g_{n_1+2+i}$ for $i \in \{2, 3, \ldots, n_2 - 1\}$.
$a_0^3 = g_0$, $a_1^3 = g_1$ and $a_i^3 = g_{n_1+n_2+i}$ for $i \in \{2, 3, \ldots, n_3 - 1\}$.

Note that in both Cases 1.1 and 1.2 we obtain that $\sum_{a \in A^i} a = (m, a_0) = \iota$ for $i \in \{1, 2, 3\}$.

Case 2. There is more that one involution ι in Γ.
By fundamental theorem of finite Abelian groups Γ has $2^t - 1$ involutions $\iota_1, \iota_2, \ldots, \iota_{2^t-1}$ for $t > 1$. Let $g_i = \iota_i$ for $i \in \{1, 2, \ldots, 2^t - 1\}$, and $g_{i+1} = -g_i$ for $i \in \{2^t, 2^t + 2, 2^t + 4, \ldots, n-2\}$. By the above arguments on necessity we obtain that $\Gamma \not\cong (\mathbb{Z}_2)^t$, therefore $2^t \leq n/2$. One can check, that we can choose integers t_1, t_2 and t_3 such that:

$$t_1 + t_2 + t_3 = 2^t,$$

with

$$n_i - t_i \equiv 0 \pmod 2, \quad t_i \geq 0, \quad t_i \notin \{2, 2^t - 2\} \quad \text{for} \quad i \in \{1, 2, 3\}.$$

By Lemmas 2.2 and 3.1 it follows that there exists a partition $B = \{B_1, B_2, B_3\}$ of $I = \{g_0, g_1, \ldots, g_{2^t-1}\}$ such that $t_1 = |B_1|$, $t_2 = |B_2|$, $t_3 = |B_3|$, and if $B_i \neq \emptyset$, then $\sum_{b \in B_i} b = g_0$ for $i \in \{1, 2, 3\}$. Let $B_i = \{b_0^i, b_1^i, \ldots, b_{t_i-1}^i\}$ for $i \in \{1, 2, 3\}$. Let us set now:
$a_i^1 = b_i^1$ for $i \in \{1, 2, \ldots, t_1 - 1\}$ and $a_i^1 = g_{i+t_2+t_3}$ for $i \in \{t_1, t_1 + 1 \ldots, n_1 - 1\}$,
$a_i^2 = b_i^2$ for $i \in \{1, 2, \ldots, t_2 - 1\}$ and $a_i^2 = g_{i+t_3+n_1}$ for $i \in \{t_2, t_2 + 1 \ldots, n_2 - 1\}$,
$a_i^3 = b_i^3$ for $i \in \{1, 2, \ldots, t_3 - 1\}$ and $a_i^3 = g_{i+n_1+n_2}$ for $i \in \{t_3, t_3 + 1 \ldots, n_3 - 1\}$.
In this case $\sum_{a \in A_i} a = g_0$ for $i \in \{1, 2, 3\}$. □

4 Group distance magic graphs

Observe that for G being an odd regular graph of order n, by hand shaking lemma n is even. Thus, the below theorem is a generalization of Theorem 1.4.

Theorem 4.1. *Let G have order $n \equiv 2 \pmod 4$ with all vertices having odd degree. There does not exist an Abelian group Γ of order n such that G is a Γ-distance magic graph.*

Proof. Assumption $n \equiv 2 \pmod 4$ implies that $\Gamma \cong \mathbb{Z}_2 \times A$ for some Abelian group A of odd order $n/2$ and there exists exactly one involution $\iota \in \Gamma$. Let $g_{n/2} = \iota$, $g_{n/2+i} = -a_i$ for $i \in \{1, 2, \dots, n/2-1\}$. Let $V(G) = \{x_0, x_1, \dots, x_{n-1}\}$.

Suppose that ℓ is a Γ-distance labeling for G and μ is the magic constant. Without loss of generality we can assume that $\ell(x_i) = a_i$ for $i \in \{0, 1, \dots, n-1\}$. Recall that $ng = 0$ for any $g \in \Gamma$ and $\deg(x_{n/2})g_{n/2} = g_{n/2} = \iota$ since $\deg(x_{n/2})$ is odd. Notice that $\deg(x_i) - \deg(x_{n-i}) = 2d_i$ for some integer d_i for $i \in \{1, 2, \dots, n/2-1\}$, because all vertices have odd degree. Let now

$$w(G) = \sum_{x \in V(G)} \sum_{y \in N(x)} w(y) = \sum_{i=0}^{n-1} \deg(x_i) g_i =$$

$$\sum_{i=1}^{n/2-1} \deg(x_i) g_i + \deg(x_{n/2}) g_{n/2} + \sum_{i=1}^{n/2-1} \deg(x_{n-i}) g_{n-i} =$$

$$\sum_{i=1}^{n/2-1} \deg(x_i) g_i - \sum_{i=1}^{n/2-1} \deg(x_{n-i}) g_i + g_{n/2} =$$

$$\sum_{i=1}^{n/2-1} (\deg(x_i) - \deg(x_{n-i})) g_i + g_{n/2} = 2 \sum_{i=1}^{n/2-1} d_i g_i + g_{n/2}$$

On the other hand, $w(G) = \sum_{x \in V(G)} w(x) = n \cdot \mu = g_0$. Therefore we obtain that $2v = g_{n/2}$ for some element $v \in \Gamma$. Since $n/2$ is odd and $\Gamma \cong \mathbb{Z}_2 \times A$, such an element v does not exist, a contradiction. □

From the above Theorem 4.1 we obtain the following.

Theorem 4.2. *If G have order $n \equiv 2 \pmod 4$ with all vertices having odd degree, then G is not distance magic.*

Proof. The graph G is not $\mathbb{Z}_n$-distance magic by Theorem 4.1, therefore it is not distance magic. □

We prove now the following useful lemma.

Lemma 4.3. *Let $G = K_{n_1, n_2, \dots, n_t}$ be a complete t-partite graph and $n = n_1 + n_2 + \dots + n_t$. If $n_1 \le n_2 \le \dots \le n_t$ and $n_2 = 1$, then there does not exist an Abelian group Γ of order n such that G is a Γ-distance magic graph.*

Proof. Let G have the partition vertex sets V_i such that $|V_i| = n_i$ for $i \in \{1, 2, \dots, t\}$. Let $x \in V_1$ and $y \in V_2$. Suppose that the graph G is Γ-distance magic for some Abelian group Γ of order n and that ℓ is a Γ-distance magic labeling of G, then $w(x) = \sum_{g \in \Gamma} g - \ell(x) = w(y) = \sum_{g \in \Gamma} g - \ell(y)$. Thus $\ell(y) = \ell(x)$, a contradiction. □

Theorem 4.4. *Let $G = K_{n_1,n_2,n_3}$ be a complete tripartite graph such that $1 \le n_1 \le n_2 \le n_3$ and $n = n_1 + n_2 + n_3$. The graph G is a group distance magic graph if and only if $n_2 > 1$ and $n_1 + n_2 + n_3 \ne 2^p$ for any positive integer p.*

Proof. Let G have the partition vertex sets V_i such that $|V_i| = n_i$ for $i \in \{1, 2, 3\}$. We can assume that $n_2 > 1$ by Lemma 4.3.

Suppose now that $\Gamma \cong (\mathbb{Z}_2)^p$ for some integer p. Let $n_1 = 2$ and ℓ be a Γ-distance magic labeling of G. Thus $\sum_{x\in V_1} \ell(x) = \iota \ne g_0$. Since G is Γ-distance magic we obtain that $\sum_{x\in V_i} \ell(x) = \iota$ for $i \in \{1, 2, 3\}$. Therefore $\sum_{g\in\Gamma} g = \sum_{i=1}^{3} \sum_{x\in V_i} \ell(x) = 3\iota = \iota \ne g_0$, a contradiction with Lemma 2.1.

If $\Gamma \not\cong (\mathbb{Z}_2)^p$ and $n_i \ge 2$ for $i \in \{2, 3\}$, then the group Γ can be partitioned into pairwise disjoint sets A_1, A_2, A_3 such that for every $i \in \{1, 2, 3\}$, $|A_i| = n_i$ with $\sum_{a\in A_i} a = \nu$ for some element $\nu \in \Gamma$ by Theorem 1.2 or 3.2. Label the vertices from a vertex set V_i using elements from the set A_i for $i \in \{1, 2, 3\}$. □

Theorem 4.5. *Let $G = K_{n_1,n_2,n_3}$ be a complete tripartite graph such that $1 \le n_1 \le n_2 \le n_3$ and $n_1 + n_2 + n_3 = 2^p$, then*

1. *G is Γ-distance magic for any Abelian group $\Gamma \not\cong (\mathbb{Z}_2)^p$ of order n if and only if $n_2 > 1$,*
2. *G is $(\mathbb{Z}_2)^p$-distance magic if and only if $n_1 \ne 2$ and $n_2 > 2$.*

Proof. Let G have the partition vertex sets V_i such that $|V_i| = n_i$ for $i \in \{1, 2, 3\}$.

We can assume that $n_2 > 1$ by Lemma 4.3. If ($n_1 = 2$ or $n_2 \ge 2$) and $\Gamma \cong (\mathbb{Z}_2)^p$ then Γ does not have a partition $A = \{A_1, A_2, A_3\}$ such that $\sum_{a\in A_i} a = \nu$ for $i \in \{1, 2, 3\}$ by Theorem 3.2. Thus one can check that then there does not exist a Γ-distance labeling of G.

If $\Gamma \not\cong (\mathbb{Z}_2)^p$ and $n_i \ge 2$ for $i \in \{2, 3\}$, or $\Gamma \cong (\mathbb{Z}_2)^p$ for some integer p and $n_1 \ne 2$, $n_2 > 2$, then the group Γ can be partitioned into pairwise disjoint sets A_1, A_2, A_3 such that for every $i \in \{1, 2, 3\}$, $|A_i| = n_i$ with $\sum_{a\in A_i} a = \nu$ for some element $\nu \in \Gamma$ by Theorem 3.2, or Lemma 3.1, resp. Label the vertices from a vertex set V_i using elements from the set A_i for $i \in \{1, 2, 3\}$. □

At the end of this section we put some observations that are implications of Theorem 1.2 for complete t-partite graphs. But first we need the following theorem proved in [2] (see Theorem 2.2, [2]).

Theorem 4.6 ([2]). *Let G be a graph for which there exists a distance magic labeling $\ell\colon V(G) \to \{1, 2, \ldots, |V(G)|\}$ such that for every $w \in V(G)$ the following holds: if $u \in N(w)$ with $\ell(u) = i$, then there exists $v \in N(w)$, $v \ne u$, with $\ell(v) = |V(G)| + 1 - i$. The graph G is a group distance magic graph.*

Observation 4.7. Let $G = K_{n_1,n_2,\ldots,n_t}$ be a complete t-partite graph such that $1 \le n_1 \le n_2 \le \ldots \le n_t$ and $n = n_1 + n_2 + \ldots + n_t$. Let Γ be an Abelian group of order n with exactly three involutions. The graph G is Γ-distance magic graph if and only if $n_2 > 1$.

Proof. Let G have the partition vertex sets $V_i = \{x_1^i, x_2^i, \ldots, x_{n_i}^i\}$ for $i \in \{1, 2, \ldots, t\}$. By Lemma 4.3 we can assume that $n_2 > 1$.
Suppose first that $n_1 = n_2 = \ldots = n_t = 2$. Note that a labeling $\ell\colon V(G) \to \{1, 2, \ldots, 2t\}$ defined as $\ell(x_1^i) = i$, $\ell(x_2^i) = 2t + 1 - i$ for $i \in \{1, 2, \ldots, t\}$ is distance magic, hence G is

a group distance magic graph by Theorem 4.6. This implies that there exists a Γ-distance magic labeling of G.
We can assume now that $n_t \geq 3$. If $n_1 > 1$, then $n_t \geq 4$ or $n_{t-1} = n_t = 3$. Therefore there exists a zero-sum partition $A'_1, A'_2, \ldots, A'_t$ of the set $\Gamma - \{g_0\}$ such that $|A'_t| = n_t - 1$ and $|A'_i| = n_i$ for every $1 \leq i \leq t-1$ by Theorem 1.2. Set $A_t = A'_t \cup \{g_0\}$ and $A_i = A'_i$ for every $1 \leq i \leq t-1$. If $n_1 = 1$ then there exists a zero-sum partition $A'_2, A'_3, \ldots, A'_t$ of the set $\Gamma - \{g_0\}$ such that $|A'_i| = n_i$ for every $2 \leq i \leq t$ by Theorem 1.2. In this case put $A_1 = \{g_0\}$ and $A_i = A'_i$ for every $2 \leq i \leq t$. Label now the vertices from a vertex set V_i using elements from the set A_i for $i \in \{1, 2, \ldots, t\}$. □

Observation 4.8. Let $G = K_{n_1,n_2,\ldots,n_t}$ be a complete t-partite graph such that $1 \leq n_1 \leq n_2 \leq \ldots \leq n_t$ and $n = n_1 + n_2 + \ldots + n_t$ is odd. The graph G is a group distance magic graph if and only if $n_2 > 1$.

Proof. Let G have the partition vertex sets V_i such that $|V_i| = n_i$ for $i \in \{1, 2, \ldots, t\}$. We can assume that $n_2 > 1$ by Lemma 4.3. If $n_1 > 1$, then $n_t \geq 3$. Therefore there exists a zero-sum partition $A'_1, A'_2, \ldots, A'_t$ of the set $\Gamma - \{g_0\}$ such that $|A'_t| = n_t - 1$ and $|A'_i| = n_i$ for every $1 \leq i \leq t-1$ by Theorem 1.2. Set $A_t = A'_t \cup \{g_0\}$ and $A_i = A'_i$ for every $1 \leq i \leq t-1$. If $n_1 = 1$ then there exists a zero-sum partition $A'_2, A'_3, \ldots, A'_t$ of the set $\Gamma - \{g_0\}$ such that $|A'_i| = n_i$ for every $2 \leq i \leq t$ by Theorem 1.2. In this case put $A_1 = \{g_0\}$ and $A_i = A'_i$ for every $2 \leq i \leq t$. Label now the vertices from a vertex set V_i using elements from the set A_i for $i \in \{1, 2, \ldots, t\}$. □

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Properties, proved and conjectured, of Keller, Mycielski, and queen graphs

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Abstract

We prove several results about three families of graphs. For queen graphs, defined from the usual moves of a chess queen, we find the edge-chromatic number in almost all cases. In the unproved case, we have a conjecture supported by a vast amount of computation, which involved the development of a new edge-coloring algorithm. The conjecture is that the edge-chromatic number is the maximum degree, except when simple arithmetic forces the edge-chromatic number to be one greater than the maximum degree. For Mycielski graphs, we strengthen an old result that the graphs are Hamiltonian by showing that they are Hamilton-connected (except M_3, which is a cycle). For Keller graphs G_d, we establish, in all cases, the exact value of the chromatic number, the edge-chromatic number, and the independence number; and we get the clique covering number in all cases except $5 \leq d \leq 7$. We also investigate Hamiltonian decompositions of Keller graphs, obtaining them up to G_6.

Keywords: Edge coloring, Keller graphs, Mycielski graphs, queen graphs, Hamiltonian, class one.

Math. Subj. Class.: 05C15, 05C45

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1 Introduction

Inspired by computational experiments, we prove several results about some families of graphs. We show in §5 that all Mycielski graphs (except the 5-cycle M_3) are Hamilton-connected. In §6, we establish the size of a maximum independent set for all Keller graphs and investigate some other parameters, determining the chromatic number of both the graphs and their complements, and also the edge-chromatic number. In particular, we prove that the edge-chromatic number of each Keller graph equals its degree. We also find the clique covering number for all cases except dimension 5, 6, and 7. And in §2–§4 we present a detailed study of queen graphs, resolving the edge-chromatic number in most cases.

Recall that the problem of coloring the edges of a graph is much simpler than the classic vertex-coloring problem. There are only two possibilities for the edge-chromatic number because of Vizing's classic theorem [2, §6.2] that the edge-chromatic number $\chi'(G)$ is either $\Delta(G)$ or $\Delta(G)+1$, where $\Delta(G)$ is the maximum vertex degree; the first case is called *class 1*; the second, *class 2*. Let $n_e(G)$ denote the number of edges, n_v the number of vertices, and $\rho(G)$ the number of edges in a maximum matching. Some graphs have too many edges to be class 1. An *overfull* graph G is one for which

$$n_e(G) > \Delta(G)\left\lfloor \frac{n_v(G)}{2} \right\rfloor.$$

For such a graph, $\Delta(G)\rho(G) < n_e(G)$, and this inequality implies that G must be class 2; so any overfull graph is class 2. The reason for this is that each color class is a matching and so has size at most $\rho(G)$; if class 1, the number of colored edges would be at most $\Delta(G)\rho(G)$ which is too small to capture all edges. In §2, we present results and an intriguing conjecture related to edge coloring of the standard queen graph $Q_{m,n}$: the conjecture is that $Q_{m,n}$ is class 1 whenever it is not overfull. Computation and proofs yield the truth of this conjecture for $m \leq 10$ and all values of $n \geq m$; the exact conjecture is that the queen is class 1 for $n \leq \frac{1}{3}(2m^2 - 11m + 12)$. In Theorem 4.1, we prove this for $n \leq \frac{1}{2}(m^2 - 3m + 2)$. For the extensive computations we developed a general edge-coloring algorithm that succeeded in finding class-1 colorings for some queen graphs having over two million edges.

Our notation is fairly standard: K_n is the complete graph on n vertices; C_n is an n-cycle; $\chi(G)$ is the chromatic number; $\chi_{\mathrm{frac}}(G)$ is the fractional chromatic number; $\alpha(G)$ is the size of a largest independent set; $\omega(G)$ is the size of a largest clique; $\theta(G)$ is the clique covering number (same as $\chi(G^c)$). Occasionally G will be omitted from these functions where the context is clear. A vertex of G is called *major* if its degree equals $\Delta(G)$. Graphs are always simple graphs, with the exception of some queen graph discussions, where multigraphs appear.

A *Hamiltonian path* (resp. *cycle*) is a path (resp. cycle) that passes through all vertices and does not intersect itself. A graph is *Hamiltonian* if there is a Hamiltonian cycle; a graph is *Hamilton-connected* (HC) if, for any pair u, v of vertices, there is a Hamiltonian path from the u to v. We will make use of Fournier's Theorem [9, 10] that a graph is class 1 if the subgraph induced by the vertices of maximum degree is a forest. This theorem is a straightforward consequence of Vizing's adjacency lemma [20, pp. 24, 54, 55].

We thank Joan Hutchinson for a careful reading and helpful suggestions, David Pike for the interesting comment about edge-coloring pioneer F. Walecki, and a referee for some valuable comments.

2 Rook and bishop graphs

The family of (not necessarily square) queen graphs presents a number of well-known combinatorial challenges. In this and the following two sections, we study the Vizing classification of queen graphs, a problem that turns out to have unexpected complexity. Queens on a chessboard can make all the moves of rooks and bishops, and thus queen graphs are the union of the rook graph and (white and black) bishop graphs. We therefore start, in this section, by looking at rook and bishop graphs separately. Then in §§3 and 4, we will show how these rook and bishop results lead to a variety of class-1 queen colorings.

It is well known that rook graphs behave similarly to their one-dimensional cousins, the complete graphs: they are class 2 if and only if both dimensions are odd. Perhaps more surprising, all bishop graphs are class 1. These two results already suffice to show that queen graphs are class 1 when at least one of the dimensions is even: just take the union of a class 1 rook coloring and a class 1 bishop coloring. When both dimensions are odd, however, the classification of queen graphs becomes much harder. A straightforward counting argument shows that such odd queen graphs are eventually class 2: for m and n odd, $Q_{m,n}$ is class 2 if $n \geq \frac{1}{3}(2m^3 - 11m + 18)$. On the other hand, we prove below (Theorem 4.1) that for m and n odd, $Q_{m,n}$ is class 1 if $m \leq n \leq \frac{1}{2}(m^2 - 3m + 2)$. As we will also show, the method we use cannot produce class-1 colorings all the way up to the cubic limit, and thus we leave essentially open the problem of determining whether there are any class-2 queen graphs when $\frac{1}{2}(m^2 - 3m + 4) \leq n \leq \frac{1}{3}(2m^3 - 11m + 12)$. We do, however, describe an algorithmic approach that gives lots of data to support the conjecture that there are no such graphs.

Recall that bishops move diagonally on a chessboard and rooks (Fig. 1) move horizontally or vertically. Because a queen can move diagonally, horizontally, or vertically, $Q_{m,n}$, the graph of queen moves, is the union of its two edge subgraphs $B_{m,n}$ and $R_{m,n}$, where $B_{m,n}$ denotes the graph of bishop moves on an $m \times n$ board, and $R_{m,n}$ denotes the rook graph; the latter is just the Cartesian product $K_m \square K_n$. The bishop graph is disconnected: it is the union of graphs corresponding to a white bishop and a black bishop (where we take the lower left square as being white). We will use $WB_{m,n}$ for the white bishop graph. It is natural to try to get edge-coloring results for the queen by combining such results for bishops and rooks, so we review the situation for those two pieces. The classic result on edge-coloring complete graphs is also essential, so we start there. Lucas [13, p. 177] attributes the first part of Proposition 2.1 to Felix Walecki.

Proposition 2.1. *$\chi'(K_n)$ is $n-1$ when n is even (and so the graph is class 1) and n when n is odd (the graph is class 2). Moreover, when n is odd every coloring has the property that no missing color at a vertex is repeated. Also, for all n, if M is a maximum matching of K_n, then $\chi'(K_n \setminus M) = n - 1$.*

Proof. For the even case, take the vertices to be v_i where $v_1, \ldots, v_{n-1}$ are the vertices of a regular $(n-1)$-gon, and v_n is the center. Use color i on $v_i \leftrightarrow v_n$ and on edges perpendicular to this edge. For n odd, one can use a regular n-gon to locate all the vertices and use n colors for the exterior n-cycle; then color any other edge with the same color used for the exterior edge that parallels it. (Alternatively, add a dummy vertex v_{n+1} and use the even-order result, discarding at the end any edges involving the dummy vertex.) Note that K_n, with n odd, is overfull, so the preceding coloring is optimal. Further, the coloring has the property that the missing colors at the vertices are $1, 2, \ldots, n$. This phenomenon, that no missing color is repeated, is easily seen to hold for any class-2 coloring of K_n, with n

odd. Because $\chi'(K_n) = n_e(K_n)/\rho(K_n)$, each color class in any optimal coloring of K_n is a maximum matching. These graphs are edge transitive, so all maximum matchings are the same, which yields the final assertion of the Proposition. □

Theorem 2.2. *The rook graph $R_{m,n}$ is class 1 except when both dimensions are odd, in which case it is overfull, and so is class 2.*

Proof. For the class-2 result, we have $\Delta = m + n - 2$, and $n_e = m\binom{n}{2} + n\binom{m}{2}$, which leads to $n_e - \frac{1}{2}(mn - 1)\Delta = \frac{\Delta}{2} > 0$. For class 1, the even case is trivial by Proposition 2.1, since we can use colors 1 through $n - 1$ on each row and n through $m + n - 2$ on each column. For the case of m even and n odd (which suffices by symmetry; see Fig. 1), use colors 1 through n on each complete row, and ensure that color 1 is missing at the vertices in the first column, color 2 is missing on the second column, and so on. The color set consisting of $i, n + 1, \ldots, n + m - 2$ can be used on the ith column. □

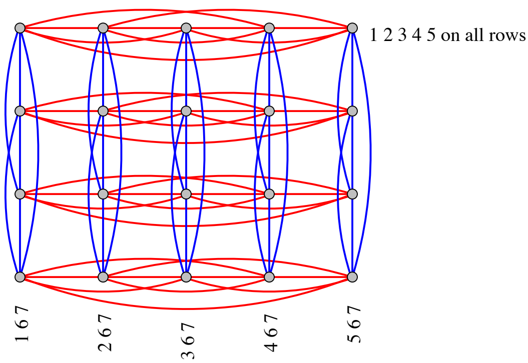

Figure 1: The rook graph $R_{4,5}$ is class 1. Use colors 1–5 on the rows, with color i missing on the vertices in the ith column. Then use i, 6, and 7 on the ith column.

For the class-2 case of the preceding result one can easily give an explicit $\Delta+1$ coloring, either by the method used in Theorem 3.1 or the ladder method of Proposition 4.3.

The bishop situation had not been investigated until the work of Saltzman and Wagon [16, 17, 18], who proved that all bishop graphs are class 1.

Theorem 2.3. *All bishop graphs $B_{m,n}$ are class 1.*

Proof. Assume $m \le n$. We have $\Delta(B_{m,n}) = 2m - 2$, except for one case: if m is even, then $\Delta(B_{m,m}) = 2m - 3$. The graph can be decomposed into paths as follows. Note that any bishop edge is a diagonal line with a natural "length": the Euclidean distance between the vertices divided by $\sqrt{2}$. Let G_1^+ consist of all edges of length 1 having negative slope and paths of length $m-1$, with edges having positive slope (in Fig. 2 this graph is the set of green and red edges). This subgraph consists of disjoint paths; the edges of each path can be 2-colored. Define G_1^- the same way, but with the slopes reversed. Get the full family by defining $G_1^+, G_1^-, G_2^+, G_2^-, \ldots, G_{\lfloor m/2 \rfloor}^+, G_{\lfloor m/2 \rfloor}^-$, where $G_i^\pm$ is defined similarly to $G_1^\pm$, but using edges of length i and $m - i$. The proof that these edge subgraphs partition the bishop edges is easy (see [16, 17]). Each of these subgraphs, being a collection of disjoint paths, can be 2-edge colored (for definiteness and because it plays a role in later work, we will always use the first of the two colors on the leftmost edge of each path; and the resulting coloring will be referred to as the *canonical bishop edge-coloring*). When m is odd the color count is $4\frac{m-1}{2} = 2m-2$. When m is even and $n > m$, the graphs $G^+_{m/2}$ and $G^-_{m/2}$ coincide and the color count is $4(\frac{m}{2}-1)+2 = 2m-2$. But when m is even and $n = m$, then $G^+_{m/2} = G^-_{m/2}$ and this subgraph consists of only disjoint edges; it is therefore 1-colorable and the color count is $4(\frac{m}{2}-1)+1 = 2m-3$. Note that for odd m, some of the edge subgraphs when restricted to the black bishop will be empty, but that is irrelevant. The black bishop will use fewer colors, but the colors are disjoint from the ones used for the white bishop and it is the latter that determines χ'. □

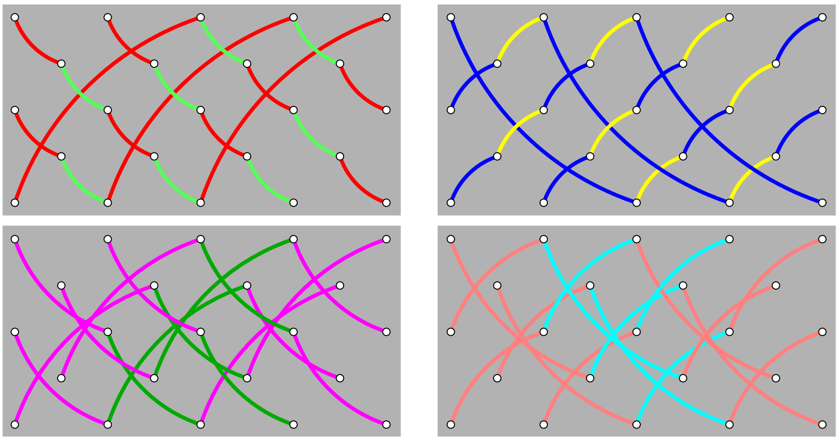

Figure 2: Top: The red and green edges form the subgraph G^+_1 of $WB_{5,9}$, the white bishop graph; blue and yellow form G^-_1. Bottom: The purple and green edges form G^+_2; cyan and pink are G^-_2. The total color count is 8, the maximum degree of the graph.

An interesting and useful type of coloring is one in which one color is as rare as can be. The next lemma shows that, for $B_{n,n}$ with n odd, the canonical coloring is such that the rarest color occurs the smallest possible number of times: once.

Lemma 2.4. *In the canonical coloring of $B_{n,n}$, n odd, the rarest color occurs on one edge only.*

Proof. Referring to the subgraphs of Theorem 2.3's proof, all the paths in the black bishop part of $G^-_{(m-1)/2}$ are isolated edges, and the same is true for the white bishop except for the single path $Z \leftrightarrow X \leftrightarrow Y$ where X is the central vertex and Y, Z are, respectively, the upper-right and lower-left corners (Fig. 3). Therefore the coloring used in the proof of Theorem 2.3 will use the last color only on $X \leftrightarrow Y$. □

A version of Lemma 2.4, with proof similar to the one given, but requiring some color switching, holds for even bishops; we do not need the result so just sketch the proof.

Lemma 2.5. *The bishop graph $B_{2k,2k}$ admits a class-1 coloring where one of the colors appears on only 2 edges; and the 2 cannot be replaced by 1.*

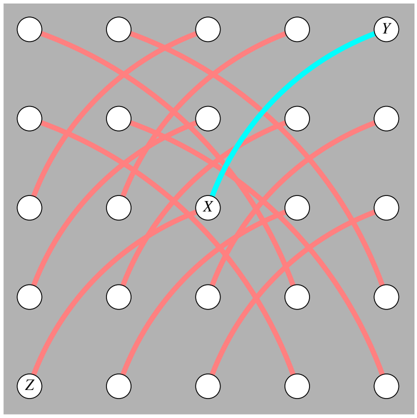

Figure 3: The edges shown form the subgraph G_3^- of $B_{5,5}$. This subgraph consists of many isolated edges and one 2-edge path: $Z \leftrightarrow X \leftrightarrow Y$, and so can be edge-colored so that one color occurs only once, at $X \leftrightarrow Y$.

Sketch of proof. The proof focuses on the white bishop and uses some color switching in the subgraphs G_1^- and G_{k-1}^+ to get the uniquely appearing color at the edge connecting the two major vertices. That 2 is best possible follows from the fact that the graph splits into two isomorphic components. □

In fact, this whole discussion can be generalized. Let $M(G)$ be the number of major vertices of G. Then in any coloring of G using $\Delta(G)$ colors, each color must occur at least $\lceil \frac{1}{2}M \rceil$ times (because all colors appear at every major vertex). Moreover, a coloring using the rarest color exactly $\lceil \frac{1}{2}M \rceil$ times cannot exist unless there is a perfect matching (or almost perfect matching if $M(G)$ is odd) of the major subgraph, since such a matching is needed to make each account for two major vertices. Call a class-1 coloring of G *extremal* if the rarest color occurs exactly $\lceil \frac{1}{2}M \rceil$ times. Then Lemma 2.4 states that $B_{n,n}$ admits an extremal class-1 coloring when n is odd, and Lemma 2.5 implies that $B_{n,n}$ does not admit such a coloring when n is even. Computations support the following conjecture, where *WB* denotes the white bishop graph.

Conjecture 2.6. *$WB_{m,n}$ always admits an extremal class-1 coloring.*

3 Queen graphs

The graph of queen moves on an $m \times n$ chessboard is the *queen graph* $Q_{m,n}$ (Fig. 4 shows $Q_{3,3}$). The vertices of $Q_{m,n}$ are arranged in an $m \times n$ grid and each vertex is adjacent to all vertices in the same row, in the same column, and on the same diagonal or back diagonal. We always assume $m \leq n$. Easy counting and summation leads to

$$\Delta(Q_{m,n}) = \begin{cases} 3m + n - 5 & \text{if } m = n \text{ and } n \text{ is even} \\ 3m + n - 4 & \text{otherwise} \end{cases}$$

$$n_e(Q_{m,n}) = \frac{1}{6} m(2 - 2m^2 - 12n + 9mn + 3n^2).$$

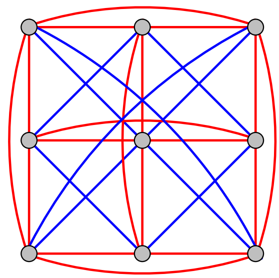

Figure 4: The queen graph $Q_{3,3}$ has 28 edges and maximum degree 8. It combines the bishop graph (blue) with the rook graph (red).

Queen graphs have served as a challenging benchmark for vertex coloring algorithms [1]. The values of $\chi(Q_{n,n})$ are known for $n \leq 26$. For $11 \leq n \leq 26$, $\chi(Q_{n,n}) = n$; the first open case is $Q_{27,27}$. So a case involving $27^2 = 729$ vertices is unresolved. As discussed later in this section, we developed an algorithm that succeeds in finding the edge-chromatic number for cases as large as $Q_{11,707}$, which has 7777 vertices and requires coloring almost three million edges. But many cases are resolved by relatively straightforward arguments and Theorems 3.1 and 4.1 find the edge-chromatic number in all cases except when m, n are odd and $\frac{1}{2}(m^2 - 3m + 4) \leq n \leq \frac{1}{3}(2m^3 - 11m + 12)$.

Any queen graph is the edge-union of a bishop subgraph and a rook subgraph (Fig. 4); the maximum degrees add: $\Delta(Q_{m,n}) = \Delta(B_{m,n}) + \Delta(R_{m,n})$. Theorem 2.3 shows that all bishop graphs are class 1 and Theorem 2.2 shows that the rooks are class 1 except when both m and n are odd. Thus one can often get a class-1 queen coloring by forming the union of optimal colorings of the bishop and rook subgraphs. When the rook is class 1, one can simply combine class-1 colorings for the rook and bishop to get a class-1 queen coloring. This yields the class-1 part of the next theorem in all cases except one: $Q_{n,n}$, n odd.

Theorem 3.1 (Joseph DeVincentis, Witold Jarnicki, and Stan Wagon)**.** *The queen graph $Q_{m,n}$ is class 1 if at least one of m and n is even, or if m and n are equal and odd. The graph is class 2 if m and n are odd and $n \geq \frac{1}{3}(2m^3 - 11m + 18)$.*

Proof. The last assertion follows from the fact that $Q_{m,n}$ in that case is overfull and therefore is class 2. This is because the overfull condition becomes

$$\frac{1}{2}(mn-1)(3m+n-4) \leq \frac{1}{6}m(2-2m^2-12n+9mn+3n^2)-1,$$

which simplifies to the stated inequality $n \geq \frac{1}{3}(2m^3 - 11m + 18)$.

The class-1 result is proved by combining class-1 colorings of the rook and bishop subgraphs, except in the one case that the rooks are class 2. Thus a different argument is needed for $Q_{n,n}$ where n is odd.

Consider $Q_{n,n}$ with n odd. The central vertex is the only vertex of maximum degree, so the result follows from Fournier's theorem (§1). It also follows from Theorem 4.1 below, but we can give a direct construction of a class-1 coloring, using a special property of the square bishop graph.

Start with a coloring of $B_{n,n}$ as in Lemma 2.4. Then we can color the corresponding rook graph $R_{n,n}$ using only new colors in such a way that a color is free to replace color $2n-2$ at its single use on the bishop edge $X \leftrightarrow Z$. The result will be a class-1 coloring of $Q_{n,n}$.

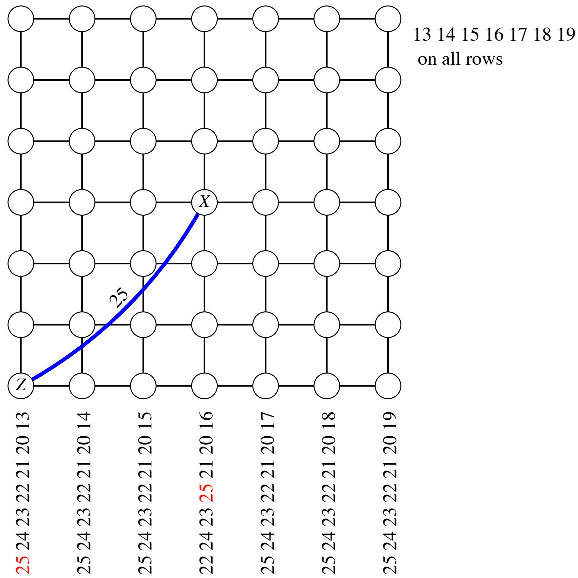

Figure 5: A class-2 coloring of $R_{7,7}$ with color 25 placed so it does not interfere with the bishop edge $X \leftrightarrow Z$; therefore 25 can replace 12 on that bishop edge, reducing the total color count to the desired 24.

We will build a class-2 coloring of $R_{n,n}$ using colors $2n-1, \ldots, 4n-3$, which are unused in the bishop coloring (recall $\Delta(R_{n,n}) = 2n-2$). Color each row with $2n-1, \ldots, 3n-2$ so that this order indicates the missing colors in each row (Fig. 5). Now color the columns by using the remaining colors together with the appropriate missing color; e.g., the first column gets colors $3n-1, \ldots, 4n-3$ together with $2n-1$. Arrange the column coloring so that the missing colors at the rows are in reverse numerical order (as in Fig. 5), except that, in the central column, color $4n-3$ is missing at the central vertex X. Then we can finish by replacing color $2n-2$ on $X \leftrightarrow Z$ in the bishop coloring by color $4n-3$ (25 in Fig. 5). So the total number of colors used is now $4n-2$, and combining the two colorings gives a class-1 coloring of $Q_{n,n}$. In fact, it is a also an extremal coloring (see end of §2), as the rarest color appears only once. □

Returning to the general edge-coloring questions left open by Theorem 3.1, the most natural conjecture is that $Q_{m,n}$ is class 1 whenever it is not overfull. The first cases are: $Q_{3,n}$, $3 \le n \le 11$; $Q_{5,n}$, $5 \le n \le 69$; $Q_{7,n}$, $7 \le n \le 207$; $Q_{9,n}$, $9 \le n \le 457$; and $Q_{11,n}$, $11 \le n \le 851$.

Conjecture 3.2. *The queen graph $Q_{m,n}$ is class 2 iff it is overfull.*

We have some positive steps toward Conjecture 3.2. A first step was a generalization of the $Q_{m,m}$ case that combined bishop and rook colorings and worked for $m \le n \le 2m-1$;

we omit the details because Theorem 4.1 in §4 uses more delicate arguments to get the much stronger result that $Q_{m,n}$ is class 1 when $m \le n \le \frac{1}{2}(m^2 - 3m + 2)$. For small values (e.g, $m = 3, 5$) the quadratic result is not as good as the $2m - 1$ result, but that is not a problem because various computations, which we describe in a moment, show that Conjecture 3.2 is true for $m \le 9$.

If $f(m) = \frac{1}{3}(2m^3 - 11m + 18)$, then $Q_{m,f(m)}$ is “just overfull” [20, p. 71], in that $n_e = \Delta \left\lfloor \frac{1}{2} n_v \right\rfloor + 1$. The *Just Overfull Conjecture* [20, p. 71] states that for any simple graph G such that $\Delta(G) \ge \frac{1}{2} n_v(G)$, G is just overfull iff G is “edge-critical” (meaning, χ' decreases upon the deletion of any edge). Computations show that $Q_{3,13}$ is edge-critical. Since deletion of a single queen edge cannot reduce the maximum degree, this is the same as saying that the deletion of any queen edge leads to a class-1 graph. So we have the following additional conjecture about the structure of queen edge colorings.

Conjecture 3.3. *The queen graph $Q_{m,n}$ is just overfull iff it is edge critical.*

An algorithm based on Kempe-style color switches yielded class-1 colorings for queen graphs that verify Conjecture 3.2 for $m = 3, 5, 7, 9$, and for $m = 11$ up to $n = 551$. A straightforward bootstrapping approach, where $Q_{m,n}$ was used to generate a precoloring for $Q_{m,n+2}$ and random Kempe color-switches were used to resolve impasses, worked for $m = 3$ and 5 (and 7 up to $Q_{7,199}$); an example of a class-1 coloring of $Q_{3,7}$ is in Figure 6; it was found by a method similar to the general Kempe method, but with an effort to find an extremal bishop coloring, which is shown at top with white the rarest color. But a more subtle method yielded a much faster algorithm, which resolved Conjecture 3.2 for $m = 7$ and 9 (and 11 up to $Q_{11,559}$, and also $Q_{11,707}$). The largest case required the coloring of 2,861,496 edges! This faster algorithm uses an explicit method to get a $\Delta + 1$ coloring (e.g., one can combine optimal colorings of the rook and bishop subgraphs) and then Kempe-type switches to eliminate the least popular color. This last step is based on a local search method that assigns a heuristic score to the possible switches and chooses the one with the highest score. This approach is quite general, using no information specific to the queen graph (except the speedy generation of the initial $\Delta + 1$ coloring, a task that can also be done via the algorithm inherent in Vizing’s proof that a coloring in $\Delta + 1$ colors always exists).

4 Quadratic class 1 queen graphs

In this section we show how a certain multigraph defined from the canonical bishop coloring can help prove that many queen graphs are class 1; the method can be called the *ring-and-ladder method.* Throughout this section m and n are odd, $m \le n$, and $k = \frac{1}{2}(m - 1)$. The main result, proved in §4.2, is the following.

Theorem 4.1. *$Q_{m,n}$ is class 1 for all $m \le n \le \frac{1}{2}(m^2 - 3m + 2)$.*

4.1 The derived ring of a bishop coloring

We will here use only the canonical path-based class-1 coloring of the bishop graph $B_{m,n}$, as described in Theorem 2.3. From such a coloring, we can define a derived multigraph; edge-coloring information about the multigraph can yield edge-coloring information about the corresponding queen graph $Q_{m,n}$. The multigraph is in fact a *ring*, by which we mean a multigraph on vertex-set V with edges being edges of the associated simple cycle on V.

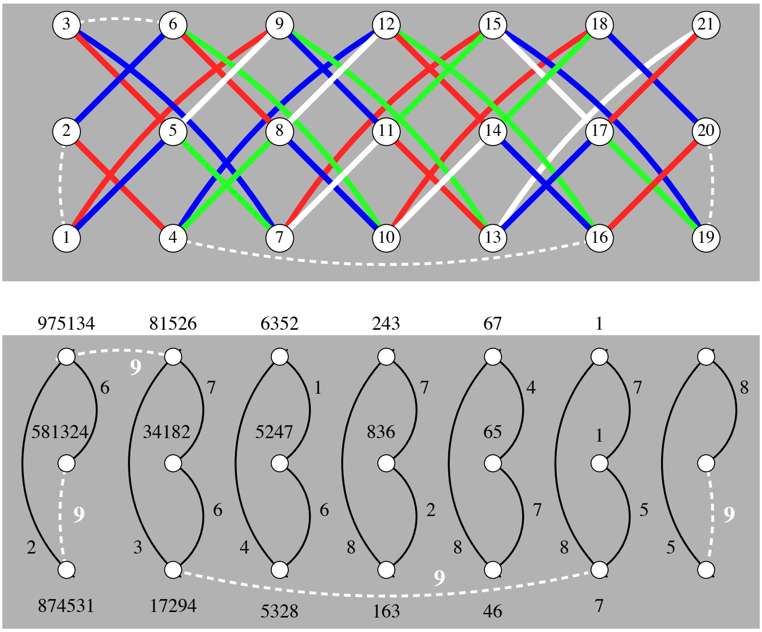

Figure 6: Top: An extremal edge coloring of $B_{3,7}$; there are four colors and white is avoided at vertices 1, 2, 3, 4, 6, 16, 18, 19, and 20. The dashed arcs indicated how white can then be used on four rook edges. Take the four colors, starting with white, to be 9, 10, 11, 12. Bottom: A class-1 coloring using 1 through 8 of the rook graph $R_{3,7}$ less the four edges from (a); only the vertical edges are shown as arcs in the edge-deleted graph. The four dashed white edges get the shared color, 9. The three sets of horizontal labels indicate edge colors on the horizontal edges, moving to the right. This rook coloring combines with the bishop coloring to yield a class-1 coloring of $Q_{3,7}$ using 1 through 12.

Note that a ring might be missing some edges from the underlying cycle; in that case it is called a *multipath* (it could have several connected components). In [20, p. 157], the term "ring" is used for a multigraph as we have defined it, but excluding multipaths; but allowing multipaths is a minor addition and causes no problems.

Recall from Theorem 2.3 that the last two colors ($2m-3$ and $2m-2$; in this section we use cyan for color $2m-2$) in the canonical bishop coloring are used on the subgraph G_k^-, which consists of paths that start with edges of positive slope and length k, then negative slope edges of length $k+1$, then positive slope edges of length k, and so on (Fig. 2, bottom right). We call such a path a *special path*.

Definition 4.2. For any bishop graph $B_{m,n}$, let $\hat{B}_{m,n}$, the *derived ring*, be the multigraph on vertices $\{1, 2, \ldots, m\}$ given in the order $\{1 + jk : j = 0, \ldots, m-1\}$ (where the numbers are reduced $\bmod m$ starting from 1); the edges arise from the $(2m-2)$-colored bishop edges: the bishop edge $(x_1, y_1) \leftrightarrow (x_2, y_2)$ induces the edge $y_1 \leftrightarrow y_2$ in $\hat{B}_{m,n}$. The case of $\hat{B}_{5,11}$ is shown in Figure 7.

The multiplicities of the edges in $\hat{B}_{m,n}$ ($(3, 5, 3, 4, 4)$ in Fig. 7) play a key role in

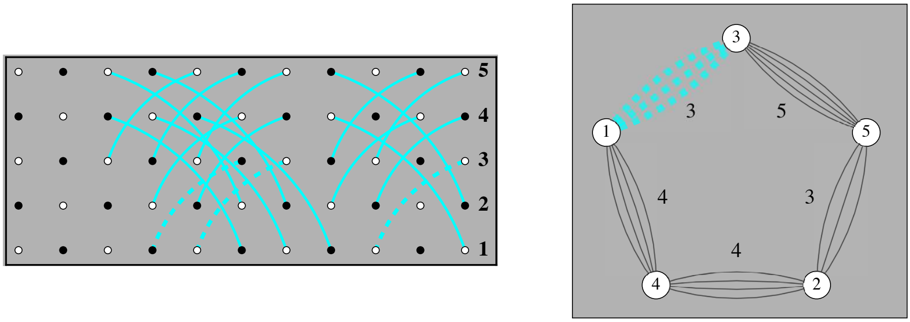

Figure 7: The last color (color $8 = 2 \cdot 5 - 2$) of the canonical coloring of $B_{5,11}$ (left) and the derived ring $\hat{B}_{5,11}$ with edge-multiplicities shown. The dashed edges correspond.

the proof that follows. The critical parameters of $\hat{B}_{m,n}$ are Δ and χ', and the minimum multiplicity μ^-. We also use the maximum multiplicity μ^+ and $\sigma_{m,n}$, the edge-count of $\hat{B}_{m,n}$ (i.e., the number of cyan edges in the canonical coloring). Now here is the key result that relates the chromatic index of $\hat{B}_{m,n}$ to that of $Q_{m,n}$.

Proposition 4.3 (The Ring-and-Ladder Method)**.** *Suppose $\hat{B}_{m,n}$ can be edge-colored using $n-1$ colors. then $Q_{m,n}$ is class 1. That is, $\chi'(\hat{B}_{m,n}) \leq n-1$ implies $\chi'(Q_{m,n}) = 3m+n-4$.*

Proof. We start with a special class-2 coloring (a "ladder coloring") of $R_{m,n}$ using colors $\{1, 2, \ldots, m-n-1\}$. Because the rook graph is regular, each vertex in any class-2 coloring misses exactly one color. We start by using 1 through m on the columns (and some row edges), and will then use the $n-1$ colors $A = \{m+1, \ldots, m+n-1\}$ on the uncolored row edges. Each vertex in the leftmost column will use all colors in A, but the sequence of $n-1$ missing colors in each row excluding its leftmost vertex is a permutation of A; moreover, the colors in A may be arranged so that, for each row, any preselected permutation is the missing-color permutation for that row. To define the ladder coloring, use colors 1 through m on each column, ensuring that color i is missing at vertices in the ith row. Now use 1 to color the horizontal edges in the bottom row that connect vertices in successive columns after the leftmost; i.e., the edges connecting the vertices in columns 2 and 3, columns 4 and 5, and so on. Do the same for row 2 but using color 2, and so on (Fig. 8). We have now used m colors to color all vertical edges and the edges of one maximum matching in each row. But each row is a K_n, and K_n minus any maximum matching can be colored with $n-1$ colors (Prop. 2.1). Thus the colors in A suffice to color all uncolored horizontal edges. The vertices in the leftmost column see all the colors in A, while the remaining vertices (which already have edges colored 1 through m) each miss exactly one color in A. Thus the missing colors in each left-deleted row form a permutation of A, and it is clear from the construction that the A-colors can be arranged independently in the rows, so that any set of m permutations can be assumed to be the missing-color permutations on the rows, excluding the leftmost vertices.

The proof of the theorem now proceeds as follows. We assume that the bishop graph is colored in the canonical way (Theorem 2.3) using colors from 1 to $2m-2$. Let ξ be

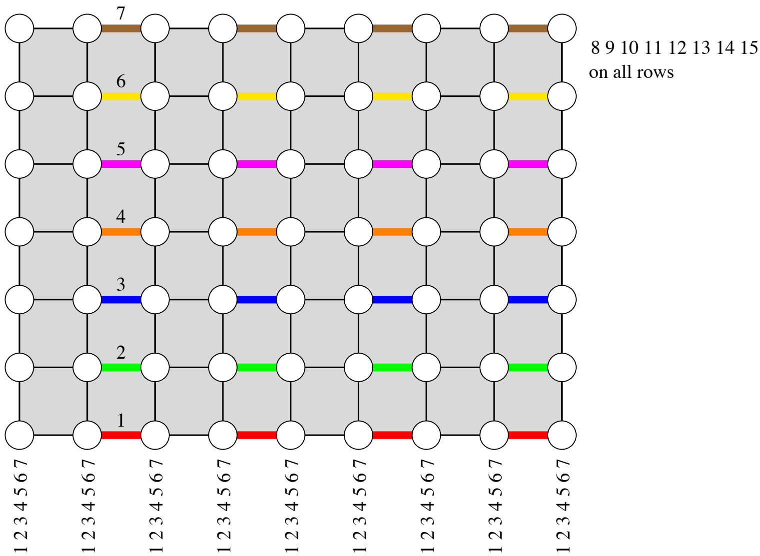

Figure 8: A class-2 ladder coloring of $R_{7,9}$ using 15 colors. The column colors are listed in missing order.

the hypothesized edge-coloring of $\hat{B}_{m,n}$ using colors $\{m+1,\ldots,m+r\}$, $r \leq n-1$; this is a subset of A. If ξ assigns k to edge $e = y \leftrightarrow y'$ of $\hat{B}_{m,n}$, assign k as a missing color to the endpoints of e viewed as a bishop edge; that is, if e arises from the $(2m-2)$-colored bishop edge $(x,y) \leftrightarrow (x',y')$, assign k to be used as a missing rook color at both vertices (x,y) and (x',y'). Because ξ is a proper multigraph coloring of $\hat{B}_{m,n}$, no k will be assigned to more than one vertex in any row; and because no bishop edge incident with the leftmost column gets color $2m-2$, no missing color will be assigned to vertices in the leftmost column. We therefore get, for each row, an injection from $\{m+1,\ldots,m+r\}$ to the vertices of that row excluding its leftmost vertex, and these maps can be extended to full permutations of A arbitrarily. Since, in the class-2 rook coloring, we can arrange the missing A-colors to match these missing-color permutations, we can now recolor each $(2m-2)$-colored bishop edge with the common missing rook color at its endpoints, thus eliminating $2m-2$ as a bishop color. So now the two colorings combine to give a coloring of $Q_{m,n}$ with color count equal to $\Delta(B_{m,n}) - 1 + \Delta(R_{m,n}) + 1$, which is $\Delta(Q_{m,n})$. □

We will use Proposition 4.3 to obtain an infinite family of queen class-1 colorings, but first we need careful analysis of the rings $\hat{B}_{m,n}$. For general rings there is a beautiful exact formula due to Rothschild and Stemple and, independently, Gallai (see [20, Thm. 6.3]).

Theorem 4.4 (Ring Chromatic Index). *Let G be a ring with n vertices; then* $\chi'(G) = \max\left(\Delta(G), \left\lceil \frac{n_e(G)}{\lfloor n/2 \rfloor} \right\rceil\right)$.

Proof. If G is a ring, but not a multipath, this is exactly the formula of Rothschild et al. For a multipath, $\chi'(G) = \Delta(G)$ (Lemma 4.6), and it is not hard to see that $\Delta(G) \geq \left\lceil \frac{n_e(G)}{\lfloor n/2 \rfloor} \right\rceil$. □

For the regular ring $C_{m,a}$ of length m with a edges in each group, this reduces to $\left\lceil \frac{2am}{m-1} \right\rceil$; we here present a direct proof of the regular case that is slightly different from the proof in [20] as it avoids induction. Recall also the two classic upper bounds for multigraphs: Vizing's bound is $\chi'(G) \leq \Delta(G) + \mu^+(G)$ and Shannon's bound is $\chi'(G) \leq \left\lfloor \frac{3}{2}\Delta(G) \right\rfloor$.

Proposition 4.5 (Regular Ring Chromatic Index)**.** $\chi'(C_{m,a}) = \left\lceil \frac{2am}{m-1} \right\rceil$.

Proof. View $C_{m,a}$ as a collection of a simple cycles and partition them into groups of size $k = (m-1)/2$, or less for the last group. Each group can be edge-colored using two colors for each cycle plus one extra color that is used on all the cycles in the group, spreading the extra color around a maximum matching in the m-cycle so that the edges with this extra color are disjoint (Fig. 9). The total color count is $2a + \left\lceil \frac{a}{k} \right\rceil$, which equals $\left\lceil \frac{am}{k} \right\rceil$, as in the Proposition. This upper bound is sharp because $\rho(C_{m,a}) = k$. If the color count was less than the upper bound, the edge count would be at most $k\left(\left\lceil \frac{a}{k} \right\rceil + 2a - 1\right)$; using the identity $k\left\lceil \frac{a}{k} \right\rceil \leq a+k-1$ simplifies this to $ma-1$, one less than the number of edges. □

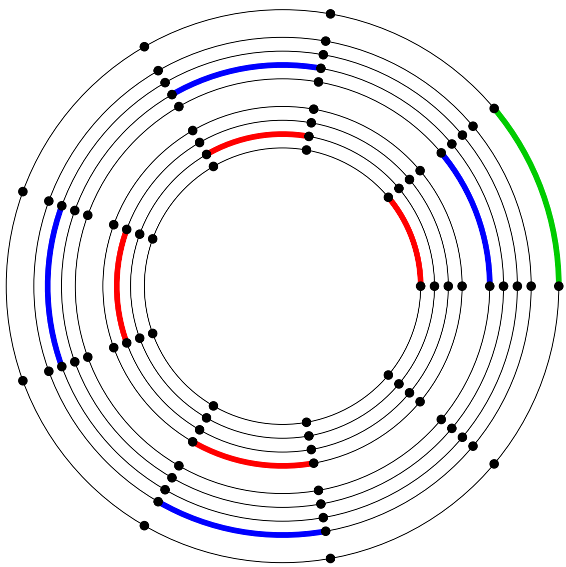

Figure 9: The regular ring $C_{9,9}$ (shown split apart into 9 cycles) can be edge-colored using 21 colors. Each cycle gets 2 colors (for 18), with three shared colors (one for each group of four or less; red, green, blue) each placed in a matching.

If m is even, then $\chi'(m, a) = 2a$, but this is irrelevant to our work. More important here is the simple case of a multipath.

Lemma 4.6. *If G is a multipath, then* $\chi'(G) = \Delta(G)$.

Proof. Enumerate the edges in the order they appear in the path as $\{e_i\}$ and assign color i (mod Δ) to e_i. □

The preceding cases lead to a simple upper bound for any ring (this also follows from Theorem 4.4 and the easily proved $\lceil \frac{n_e}{k} \rceil \leq \Delta + \left\lceil \frac{\mu^-}{k} \right\rceil$).

Corollary 4.7 (Ring Chromatic Index Bound)**.** *Let G be any ring with vertex count m. Then $\chi'(G) \leq \chi'(C_{m,\mu^-}) + \Delta(G) - 2\mu^-$. Using k for $\frac{1}{2}(m-1)$ as usual, this becomes $\chi'(G) \leq \left\lceil \frac{\mu^-}{k} \right\rceil + 2\mu^- + \Delta(G) - 2\mu^- = \Delta(G) + \left\lceil \frac{\mu^-}{k} \right\rceil$.*

Proof. Split G into the regular "kernel" — the ring C_{m,μ^-} — and the "residual", which is a multipath (because kernel removal leaves a 0 multiplicity) and so has chromatic index $\Delta(G) - 2\mu^-$ by Lemma 4.6. The kernel is colorable as in the Regular Ring Chromatic Index Theorem, and summing the two yields the claimed bound. □

Compare the preceding bound to the general Vizing bound: $\chi' \leq \Delta + \mu^+$. For rings, $\chi' \leq \Delta + \left\lfloor \frac{\mu^-}{k} \right\rfloor$. The values $\mu^\pm(\hat{B}_{m,n})$ do not differ by much (in general of course they can differ arbitrarily), but division by k is a big improvement. Recall the trivial lower bound $\left\lceil \frac{n_e(G)}{\rho(G)} \right\rceil \leq \chi'(G)$. For our bishop rings this becomes $\left\lceil \frac{\sigma(\hat{B}_{m,n}}{k} \right\rceil \leq \chi'(\hat{B}_{m,n}$. In fact, it appears that this lower bound is always equal to the chromatic index of the derived ring. We have checked this for $m \leq 19$ and $n \leq 199$. The computation is simplified by the use of Theorem 4.4 to compute χ'; by that theorem, the conjecture follows from $\Delta \leq \lceil \frac{\sigma_{m,n}}{k} \rceil$ for the bishop rings, and this is easy to check.

Conjecture 4.8. $\chi'(\hat{B}_{m,n}) = \lceil \frac{\sigma_{m,n}}{k} \rceil$.

There are many many patterns in the data one can compute for the derived ring of $B_{m,n}$; key parameters are the total edge count σ, the minimum multiplicity μ^-, and the maximum degree Δ. The next conjecture summarizes the results of many computations. Figure 10 presents some evidence for Conjecture 4.9, and also shows the periodicity in the edge counts of $\hat{B}_{m,n}$ that appears to arise in all cases.

Conjecture 4.9. *For odd m, n, with $m \leq n$, $\frac{1}{2}mn - (\frac{1}{2}m^2 - 1) \leq \sigma_{m,n} \leq \frac{1}{2}mn - \frac{1}{4}(m^2 + 1)$.*

4.2 The fine structure of the canonical bishop coloring

We can use the color-sharing theorem and a detailed study of the properties of the last color in the canonical bishop coloring to prove the queen coloring result of Theorem 4.1.

Proof of Theorem 4.1. Since we have already shown that $Q_{m,n}$ is class 1 if either m or n is even, we assume m and n are odd and write $m = 2k+1$. Recall (Thm. 2.3) that in $B_{m,n}$ only two colors (one of them being cyan) were needed to color all special paths. As we are at liberty to start each special path with either color, we assume that no special path starts with cyan.

Let C be the set of cyan edges in all special paths. Because k and m are relatively prime, the derived multigraph is a ring. The following result will yield Theorem 4.1 as a consequence of the ring-and-ladder method (Proposition 4.3) and the Ring Chromatic Index Bound (Cor. 4.7). It is not hard to see that $n - 2k \leq \Delta(\hat{B}_{m,n})$. Proposition 4.10, the final step in the proof, puts an upper bound on the maximum degree; so we see that, as m is fixed and n rises, the derived ring is close to being regular.

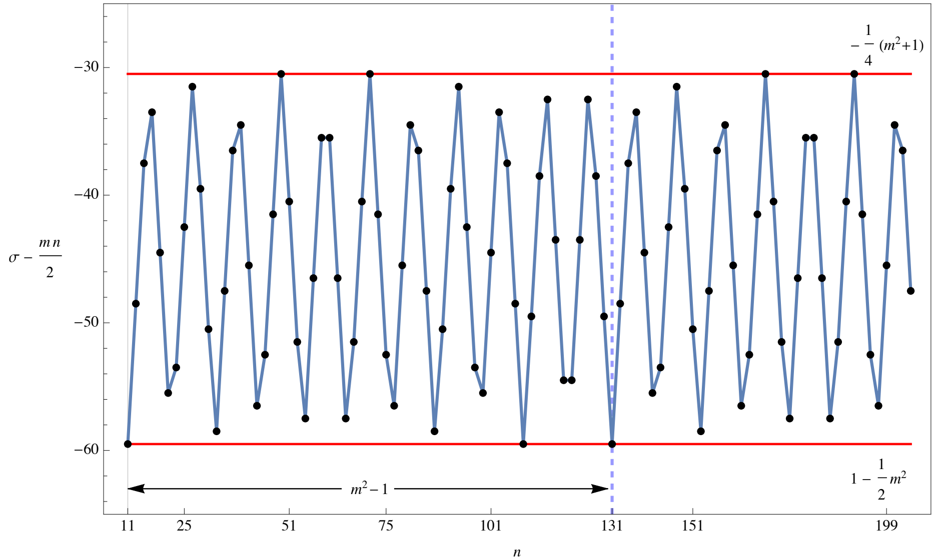

Figure 10: Computed values of $\sigma_{m,n} - \frac{mn}{2}$, where $m = 11$ and with the red lines being $-\frac{1}{4}(m^2+1)$ (upper) and $1 - \frac{1}{2}m^2$ (lower). This illustrates the bounds of Conjecture 4.9. Also note the periodicity of this reduced data set with period $m^2 - 1$ (dashed blue line).

Proposition 4.10. *When m and n are odd, $m \leq n$, $m = 2k+1$, we have $\Delta(\hat{B}_{m,n}) \leq n-k$.*

We can then complete the proof of Theorem 4.1 as follows. For any ring G, $\mu^-(G) \leq \frac{1}{2}\Delta(G)$, so by Corollary 4.7 we have $\chi'(\hat{B}_{m,n}) \leq \lceil n - k + \frac{n-k}{2k} \rceil$. By assumption, $n \leq \frac{1}{2}(m^2 - 3m + 2) = k(2k-1)$, and therefore $\frac{n-k}{2k} \leq k - 1$ so that $\lceil \frac{n-k}{2k} \rceil \leq k - 1$ and $\chi'(\hat{B}_{m,n}) \leq n - 1$. Proposition 4.3 now concludes the proof. □

We now prove Proposition 4.10, starting with some definitions: call the leftmost vertex of a special path in a bishop graph an *initial vertex* and let $\ell(i,j)$ be the length (i.e., edge count) of the special path that starts at the initial vertex (i,j). Furthermore, call an initial vertex (i,j) for which $\ell(i,j)$ is even an *even vertex* and one for which $\ell(i,j)$ is odd an *odd vertex*. For bishop vertices v viewed as points in the plane, we use $v \leq w$ to mean that v is first in the lexicographic ordering: $v_x \leq w_x - 1$, or $v_x = w_x$ and $v_y \leq w_y$. We break the proof into a series of claims.

Claim 4.11. *Let $I(j)$ be the number of initial vertices in row j, and let $O(j)$ be the number of odd vertices in row j. Then the degree of row j in $\hat{B}_{m,n}$ is $\deg(j) = n - I(j) - O(m + 1 - j)$.*

Proof. Note that the degree of a vertex in $\hat{B}_{m,n}$ is the number of bishop vertices in the row represented by the multigraph vertex that are incident with an edge in C. Every vertex in $B_{m,n}$ that is not an endpoint of a special path lies on a cyan edge, so is counted toward the degree of the row in which it sits. Because we have arranged for all paths to start on the left without cyan, we eliminate all initial vertices in the row from the count; we also eliminate all vertices in the row that terminate an odd-length path, as they will not be incident with a cyan edge. Note that $\pi(i,j) = (n+1-i, m+1-j)$, a vertical reflection followed by a horizontal reflection, is an automorphism of $B_{m,n}$ that takes special paths to special paths (and odd length special paths to odd length special paths). Therefore if v is an odd terminal point of path p, $\pi(v)$ is an odd vertex beginning path $\pi[p]$. Thus the number of vertices in row j that terminate an odd-length path is the same as the number of odd (initial) vertices in row $m+1-j$, and the claim follows. □

Claim 4.12. *$I(j) = k+1$ for $j \leq k$; $I(j) = k$ for $j > k$.*

Proof. This follows from the fact that the set of initial points of all special paths is $\{(i,j) : (i,j) \leq (k+1,k)\}$; see Figure 11. □

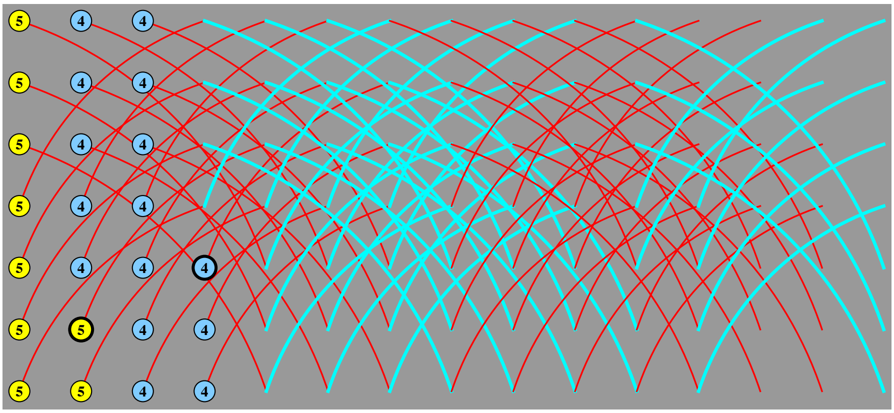

Figure 11: The values of $I(j)$ for $B_{7,15}$ are $4,4,4,3,3,3$ (Claim 4.12); these are all the vertices lexicographically below or equal to $(4,3)$. The yellow vertices are even, the blue ones odd.

Claim 4.13. *There is at least one even vertex and at least one odd vertex.*

Proof. Because C is a matching, $e = v/2$ where $e = |C|$ and v is the number of vertices in $B_{m,n}$ that are included in a C edge. But the vertices included in such an edge are those that don't begin a special path or terminate an odd-length special path. There are $k(m+1)$ vertices that begin a special path, so letting p be the number of odd vertices, we have $e = \frac{1}{2}(nm - k(m+1) - p)$. Because nm is odd and $k(m+1)$ is even, this implies that p must be odd, and thus there are an odd number of odd vertices, and hence also an odd number of even vertices. □

Claim 4.14. *$\ell(i,j)$ is the largest integer q such that $qk + i + \lfloor \frac{qk+j-1}{m} \rfloor \leq n$.*

Proof. A special path moves k units to the right (i.e., from i to $i+k$) when it moves upwards and $k+1$ units to the right when it moves downwards, and only moves downwards if $ck + j \leq bm < (c+1)k + j$, where c is the number of edges the path has moved along thus far, and bm is any multiple of m. Thus a path that starts at (i,j) and moves to the right q edges ends at (i',j'), where $i' = qk + i + \lfloor (qk+j-1)/m \rfloor$, and the claim follows. □

Claim 4.15. *There is an initial vertex* $(i^*, j^*) < (k+1, k)$ *such that* $\ell(i,j) = \ell(1,1)$ *for all* $(i,j) \le (i^*, j^*)$*, and* $\ell(i,j) = \ell(1,1) - 1$ *for all initial vertices* $(i,j) > (i^*, j^*)$*.*

Proof. See Figure 11 for an example where $(i^*, j^*) = (2,2)$. Let $L = \ell(k+1, k)$. It follows easily from Claim 4.14 that $\ell(i,j)$ is decreasing with respect to lexicographic order; that is, $(i,j) \le (i', j')$ implies $\ell(i,j) \ge \ell(i', j')$. Thus $L \le \ell(1,1)$, and the inequality must be strict by Claim 4.13. Also, by Claim 4.14, we have that $\ell(1,1)$ is the largest integer q such that $qk + \lfloor \frac{qk}{m} \rfloor \le n-1$, while L is the largest integer q such that $qk+k+\lfloor \frac{qk+k-1}{m} \rfloor = (q+1)k + \lfloor \frac{(q+1)k-1}{m} \rfloor \le n-1$. This implies that $\ell(1,1) \le L+1$. Thus $L \le \ell(1,1) \le L+1$, and the claim follows. More precisely, (i^*, j^*) is the smaller of the largest even initial point and largest odd initial point, where comparisons are lexicographic. □

Claim 4.16. *If* $\ell(1,1)$ *is odd,* $\Delta(\hat{B}_{m,n}) \le n - k$*.*

Proof. If $\ell(1,1)$ is odd, Claim 4.15 implies that the odd vertices are all $(i,j) \le (i^*, j^*)$. Thus

$$O(j) = \begin{cases} i^* & \text{if } j \le j^* \\ i^* - 1 & \text{if } j > j^* \end{cases}$$

Then

$$O(m+1-j) = \begin{cases} i^* - 1 & \text{if } j < m+1-j^* \\ i^* & \text{if } j \ge m+1-j^* \end{cases}$$

By Claim 4.12, we therefore have

$$I(j) + O(m+1-j) = \begin{cases} i^* + k & \text{if } j \le \min(k+1, m+1-j^*) \\ i^* + k - 1 & \text{if } k+1 \le j < m+1-j^* \\ i^* + k + 1 & \text{if } m+1-j^* \le j < k+1 \\ i^* + k & \text{if } j \ge \max(k+1, m+1-j^*) \end{cases}$$

Thus $\Delta(\hat{B}_{m,n}) \le n - k - i^* + 1$ by Claim 4.11, and because $i^* \ge 1$, $n - k - i^* + 1 \le n - k$. □

We complete the proof of Proposition 4.10, and thus Theorem 4.1, with:

Claim 4.17. *If* $\ell(1,1)$ *is even,* $\Delta(\hat{B}_{m,n}) \le n - k$*.*

Proof. If $\ell(1,1)$ is even, Claim 4.15 implies that the odd vertices are all (i,j) such that $(i^*, j^*) < (i,j) \le (k+1, k)$. Thus

$$O(j) = \begin{cases} k+1-i^* & \text{if } j \le \min(j^*, k) \\ k - i^* & \text{if } k < j \le j^* \\ k+2-i^* & \text{if } j^* < j \le k \\ k+1-i^* & \text{if } j > \max(j^*, k) \end{cases}$$

Then

$$O(m+1-j) = \begin{cases} k+1-i^* & \text{if } j < m+1-\max(j^*, k) \\ k+2-i^* & \text{if } m+1-k = k+2 \le j < m+1-j^* \\ k-i^* & \text{if } m+1-j^* \le j < k+2 \\ k+1-i^* & \text{if } j \ge m+1-\min(j^*, k) \end{cases}$$

By Claim 4.12, we therefore have

$$I(j)+O(m+1-j)=\begin{cases} m-i^*+1 & \text{if } j<m+1-\max(j^*,k) \text{ and } j\le k \\ m-i^* & \text{if } j<m+1-\max(j^*,k) \text{ and } j>k \\ m-i^*+1 & \text{if } k+2\le j<m+1-j^* \\ m-i^* & \text{if } m+1-j^*\le j\le k \\ m-i^*-1 & \text{if } m+1-j^*\le j=k+1 \\ m-i^* & \text{if } j\ge m+1-\min(j^*,k) \end{cases}$$

Note that if $i^*=k+1$, j^* must be less than k (by Claim 4.13), and thus the fifth of these cases ($m+1-j^*\le k+1$) cannot occur. In that case, $\Delta(\hat{B}_{m,n})\le n-m+i^*=n-m+k+1=n-k$ by Claim 4.11. If $i^*<k+1$, we similarly have $\Delta(\hat{B}_{m,n})\le n-m+i^*+1\le n-k$, and the claim follows. □

4.3 Limits to the ring-and-ladder method

The quadratic lower bound, $n\ge\frac{1}{2}(m^2-3m+4)$, for class-2 queen graphs of Theorem 4.1 provides substantial support for the queen chromatic index conjecture (Conjecture 3.2). Assuming the truth of Conjecture 4.8, we can improve the bound slightly, likely up to $\frac{1}{2}(m^2+2m-3)$, but the ring-and-ladder method will not reach beyond a quadratic bound and thus will not settle the full Conjecture 3.2. For suppose that $2m-1\le n$; then the number of major vertices in $B_{m,n}$ is $mn-2m(m-1)+2k+4\sum_{i=1}^{k-1} i=mn-(3m+1)k$; therefore, for such n, any bishop color class from a class-1 coloring must have size at least $a=\frac{1}{2}(mn-(3m+1)k-1)+1$. Now suppose that $\chi'(\hat{B}_{m,n})\le n-1$; then $\hat{B}_{m,n}$ is covered by at most $n-1$ matchings, each of size at most k, and thus $\hat{B}_{m,n}$ can have at most $b=k(n-1)$ edges. But this is the same as the size of the $(2m-2)$-color class, so we must have $a\le b$, which simplifies to $n\le\frac{3}{2}m^2-2m-\frac{1}{2}$. In fact, assuming the truth of Conjecture 4.9, the ring-and-ladder method can only work up to about m^2. For then $\sigma_{m,n}\ge\frac{1}{2}mn+1-\frac{1}{2}m^2$. Therefore $\chi'(\hat{B}_{m,n})\ge(\frac{1}{2}mn+1-\frac{1}{2}m^2)/k$. Now for this to be at most $n-1$ means $n\le m^2-m-1$, so this is a likely bound for the ring-and-ladder method.

Thus a quadratic bound is the best we can achieve with our methods, and a proof of Conjecture 3.2 would seem to require an altogether different approach of even greater intricacy. This highlights the surprising difficulty of the Vizing classification problem for queen graphs. It is well known that the general classification problem is $\mathcal{NP}-complete$ ([20, p. 18]), but one would expect that queen graphs — being a union of rook and bishop graphs whose classifications are relatively straightforward — would be amenable to a complete classification. So as we leave behind the quadratic bound obtained here, it is hard to resist the thought that we may be entering terrain of intractable complexity. If so, the computational evidence supporting Conjecture 3.2 seems all the more remarkable and may be the best we can hope for.

5 Mycielski graphs

The *Mycielskian* $\mu(G)$ of a graph G with vertex set X is an extension of G to the vertex set $X\cup Y\cup\{z\}$, where $|Y|=|X|$ and with new edges $z \leftrightarrow y_i$ for all i and $x_i \leftrightarrow y_j$ for each edge $x_i \leftrightarrow x_j$ in G (see Fig. 12). The *Mycielski graphs* M_n are formed by iterating

μ on the singleton graph M_1, but ignoring the isolated vertex that arises in $\mu(M_1)$. Thus $M_1 = K_1$, $M_2 = K_2$, $M_3 = C_5$, and M_4 is the Grötzsch graph (Fig. 13). They are of interest because M_n is a triangle-free graph of chromatic number n having the smallest possible vertex count. Fisher et al. [8] proved that if G is Hamiltonian, then so is $\mu(G)$. We extend that to a Hamilton-connected (HC) result, provided $n_v(G)$ is odd. This is sufficient to show that all Mycielski graphs M_n, except $M_3 = C_5$, are HC.

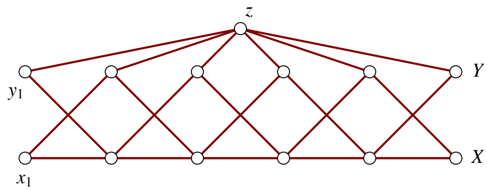

Figure 12: The Mycielskian of the 6-path formed by the lowest six vertices.

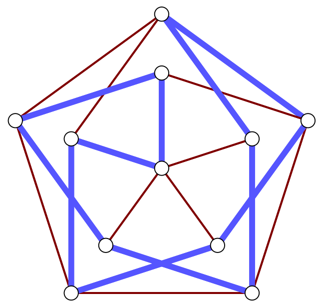

Figure 13: A Hamiltonian cycle in the Mycielski graph M_4, which is the Grötzsch graph.

The key result is the following.

Theorem 5.1. *If G is an odd cycle, then $\mu(G)$ is Hamilton-connected.*

We will prove this shortly; note that it yields the fact that μ preserves HC for graphs with an odd number of vertices.

Corollary 5.2. *If G is Hamilton-connected and $n_v(G)$ is odd, then $\mu(G)$ is Hamilton-connected.*

Proof. We may skip the trivial case that G has one vertex. Therefore G is Hamiltonian, with Hamiltonian cycle C. Since $\mu(G)$ contains $\mu(C)$ as an edge subgraph, and since $\mu(G)$ is HC by Theorem 5.1, so is $\mu(G)$. □

Theorem 5.1 does not extend to the even case.

Proposition 5.3. *If G is an even cycle, then $\mu(G)$ is not HC.*

Proof. It is easy to use parity to show that there is no Hamiltonian path from any vertex in X to z. This is because such a path can get to z only from Y, and hence must alternate from X to Y; but then if the path starts at x_1 it can never visit x_i, where i is even. □

Even cycles are not HC, so the negative result for even cycles does not mean that HC-preservation fails in general for even graphs (an exception being K_2: $\mu(K_2)$ is a 5-cycle, which is not HC, even though K_2 is HC). Computations support the following strengthening of Corollary 5.2, but some new ideas are needed.

Conjecture 5.4. *If G is Hamilton-connected and not K_2, then $\mu(G)$ is Hamilton-connected.*

Easy computation shows that M_4, the 11-vertex Grötzsch graph, is HC and so Corollary 5.2 means that all Mycielski graphs are HC, except the 5-cycle M_3.

Corollary 5.5. *The Mycielski graph M_n is Hamilton-connected iff $n \neq 3$.*

The stronger assertion that $\mu(G)$ is HC whenever G is Hamiltonian is false. Counterexamples include $C_4, K_{3,3}, K_{1,1,2}, \text{Grid}_{2,3}$. Indeed, the first two here are Hamilton-laceable, but their Mycielskians are not HC.

Proof of Theorem 5.1. Assume that the cycle G has n vertices, given in cyclic order as $X = \{x_i\}$, where n is odd. Then $\mu(G)$ has as vertices X, and also $Y = \{y_i\}$ and a single vertex z. The subgraph corresponding to $Y \cup \{z\}$ forms a $K_{1,n}$. Then, as in [8], we have the following Hamiltonian cycle in $\mu(G)$ (see Fig. 14):

$$C = y_1 \leftrightarrow x_2 \leftrightarrow y_3 \leftrightarrow x_4 \leftrightarrow \cdots$$
$$\cdots \leftrightarrow x_{n-1} \leftrightarrow y_n \leftrightarrow x_1 \leftrightarrow x_n \leftrightarrow y_{n-1} \leftrightarrow \cdots \leftrightarrow y_4 \leftrightarrow x_3 \leftrightarrow y_2 \leftrightarrow z \leftrightarrow y_1$$

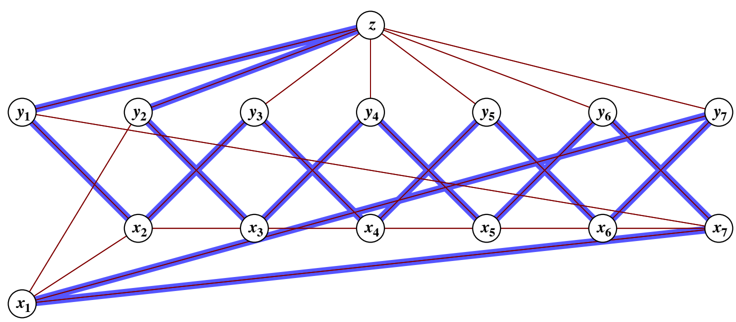

Figure 14: A Hamiltonian cycle in the Mycielskian of an odd cycle $\{x_i\}$.

Now consider any two distinct vertices A, B of $\mu(G)$.

Case 1. $A \leftrightarrow B$ is an edge in $\mu(G)$.

It suffices to show that there is a Hamiltonian cycle containing $A \leftrightarrow B$. By symmetry, we may assume $A \leftrightarrow B$ is one of $x_1 \leftrightarrow x_n$, $x_1 \leftrightarrow y_n$, or $y_1 \leftrightarrow z$. In all cases cycle C above contains the edge.

Case 2. $A \leftrightarrow B$ is not an edge of $\mu(G)$.

Case 2.1. $\{A, B\} \subset X$; say x_i, x_j.

Without loss of generality, assume $i = 1$. The same proof works for both even and odd j. Zigzag up from x_1 until y_{j-1} is reached (when reaching the end, carry on at the beginning in the obvious way). Then jump via z to y_n and zigzag left (past y_1 if necessary) until x_j is reached. Formally:

$$x_1 \leftrightarrow y_2 \leftrightarrow x_3 \leftrightarrow \cdots \leftrightarrow y_{j-1} \leftrightarrow z \leftrightarrow y_n \leftrightarrow x_{n-1} \leftrightarrow y_{n-2} \leftrightarrow \cdots \leftrightarrow x_j.$$

Figure 15 shows how this works when j is even, followed by the odd j case.

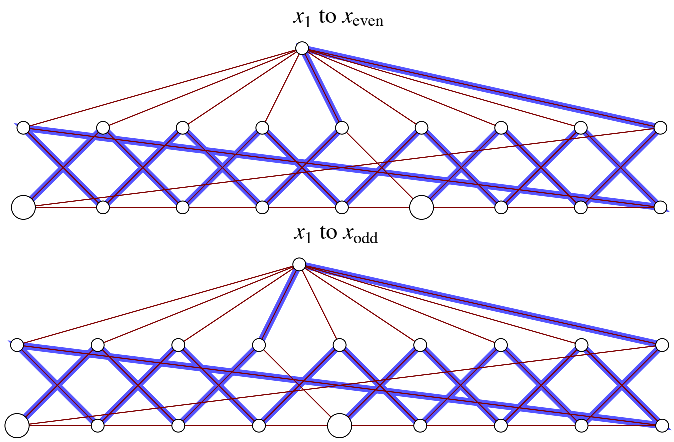

Figure 15: Typical Hamiltonian paths from X to X.

Case 2.2. $A \in X$ and $B \in Y$.

Assume $A = x_1$ and $B = y_j$. Then $x_1 \leftrightarrow x_j$ is a nonedge in G. If j is even: Zigzag up to $x_{j-1} \leftrightarrow x_j$ then zigzag to $y_n \leftrightarrow z \leftrightarrow y_{j-1}$ and zigzag left and through y_1 to the target x_j, as in Figure 16.

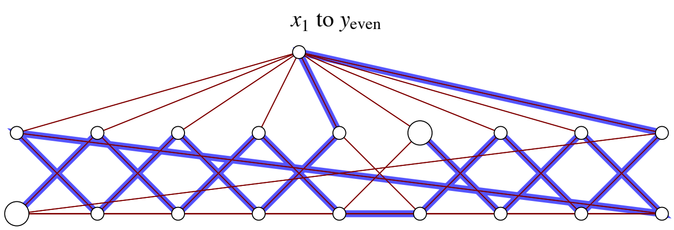

Figure 16: A typical Hamiltonian path from X to an even vertex in Y.

If j is odd, zigzag up to x_j then left to x_{j-1} and down through y_1 to y_{j+1}, then up to z and y_n and zigzag down to the finish at y_j, as in Figure 17. This works fine even if $j = n$.

Case 2.3. $A \in X$ and $B = z$.

Assume $A = x_1$. Zigzag through to y_n and then finish up at z:

$$x_1 \leftrightarrow y_2 \leftrightarrow x_3 \leftrightarrow y_4 \leftrightarrow x_5 \leftrightarrow y_6 \leftrightarrow \cdots$$
$$\cdots \leftrightarrow x_n \leftrightarrow y_1 \leftrightarrow x_2 \leftrightarrow y_3 \leftrightarrow x_4 \leftrightarrow y_5 \leftrightarrow \cdots \leftrightarrow y_n \leftrightarrow z;$$

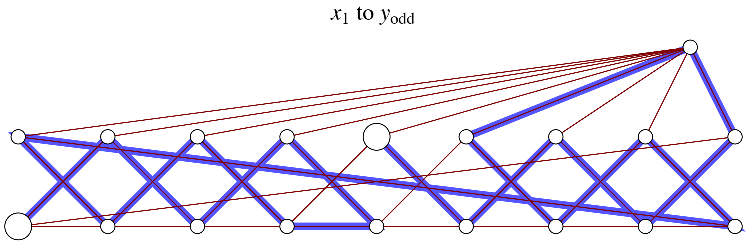

Figure 17: A typical Hamiltonian path from X to an odd vertex in Y.

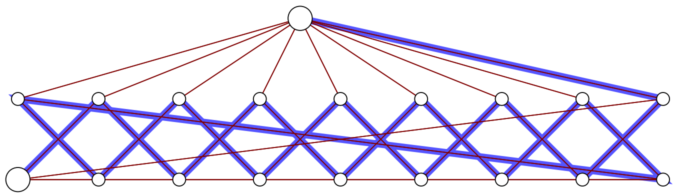

Figure 18: A typical Hamiltonian path from X to z.

see Figure 18.

Case 2.4. $\{A, B\} \subset Y$.

Assume first that $A = y_1$ and $B = y_j$, with j even.

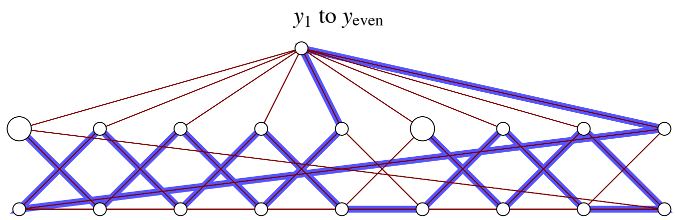

Figure 19: A typical Hamiltonian path from a vertex in Y to a vertex of different parity in Y.

Zigzag up from y_1 to y_{j-1}, then up to z and down to y_n, then back to x_1 and zigzag up to x_{j-1}, then to x_j and zigzag up to x_{n-1}, then x_n and zigzag down to y_j. See Figure 19. Formally:

$$y_1 \leftrightarrow x_2 \leftrightarrow y_3 \leftrightarrow \cdots \leftrightarrow y_{j-1} \leftrightarrow z \leftrightarrow y_n \leftrightarrow x_1 \leftrightarrow y_2 \leftrightarrow x_3 \leftrightarrow \cdots$$
$$\cdots \leftrightarrow x_{j-1} \leftrightarrow x_j \leftrightarrow y_{j+1} \leftrightarrow x_{j+2} \leftrightarrow \cdots \leftrightarrow x_{n-1} \leftrightarrow x_n \leftrightarrow y_{n-1} \leftrightarrow x_{n-2} \cdots \leftrightarrow y_j$$

Finally assume j is odd. Zigzag from y_1 to x_{j-1}, then back to x_{j-2} and zigzag down to x_1 and up to y_n, then up to z, down to y_{j-1}, and zigzag up to x_n, then back to x_{n-1} and zigzag to y_j; see Figure 20. This completes the proof. □

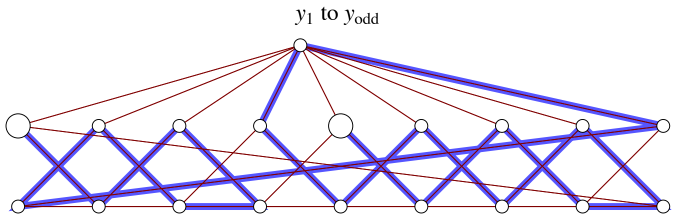

Figure 20: A typical Hamiltonian path from a vertex in Y to a vertex of the same parity in Y.

The proof technique does not work directly to settle the even case. But we have barely used the edges in G; because G is assumed HC, there might be a way to use more of those edges to extend Corollary 5.2 to all graphs, proving Conjecture 5.4. Computation also leads to a conjecture about edge coloring. All the Mycielski graphs (except M_3) are class 1 because of Fournier's theorem (§1); for M_4 and beyond, z is the unique vertex of maximum degree. But perhaps much more is true: computation supports the following conjecture.

Conjecture 5.6. *For any graph G other than K_2, $\mu(G)$ is class 1.*

6 Keller graphs

The *Keller graph G_d* of dimension d is defined as follows [6, 21]: the 4^d vertices are all d-tuples from $\{0, 1, 2, 3\}$. Two tuples form an edge if they differ in at least two coordinates and if in at least one coordinate the difference of the entries is 2 (mod 4). We ignore G_1, which simply consists of four isolated points. These graphs are vertex transitive and therefore regular; it is easy to work out the degree of G_d, which is $4^d - 3^d - d$. The graph G_2 is also known as the Clebsch graph. The Keller graphs play a critical role in the Keller conjecture [6], which, in its unrestricted form, states that any tiling of $\mathbb{R}^d$ by unit cubes contains two cubes that meet face-to-face. This conjecture is closely related to $\omega(G_d)$. The value of $\omega(G_d)$ is known for all d: when $d \geq 8$, $\omega(G_d) = 2^d$, while $\omega(G_d) < 2^d$ for $d \leq 7$ (Table 2; see [6]). These ω values imply that the Keller conjecture with the restriction that all cube centers involve only integers or half-integers is true for $d \leq 7$ and false for $d \geq 8$. The unrestricted Keller conjecture is known to be true for $d \leq 6$ and false for $d \geq 8$, but is unresolved in $\mathbb{R}^7$.

Note that G_d always admits a 2^d-vertex coloring, defined this way: There are 2^d vertices using only 0s and 2s; they each receive a distinct color. Give any other vertex (v_i) the same color as $\left(2\left\lfloor\frac{v_i}{2}\right\rfloor\right)$; the "differ by 2" condition is never satisfied by both vertices (v_i) and $\left(2\left\lfloor\frac{v_i}{2}\right\rfloor\right)$. Therefore $\chi(G_d) \leq 2^d$ (proved independently by Fung [11] and Debroni et al. [6]). This coloring is also implicit in the proof of Theorem 6.4 below: the 0-and-2 set is the diagonal of the array shown. We will show in Corollary 6.5 that this coloring is optimal for all d.

A classic theorem of Dirac [2, Thm. 4.3] states that a graph with minimum degree greater or equal to $n_v/2$ is Hamiltonian; this applies to G_d when $d \geq 3$. We can give an explicit Hamiltonian cycle for all Keller graphs.

Theorem 6.1. *All Keller graphs are Hamiltonian.*

Proof. For G_2, a Hamiltonian cycle is

$$(00, 23, 01, 20, 02, 21, 03, 22, 10, 33, 11, 30, 12, 31, 13, 32).$$

This ordering alternates 0 and 2 in the first coordinate for the first half, and then 1 and 3. And in the second coordinate, the leading 0s and 1s are matched, in order, with 0, 1, 2, 3, and the 2s and 3s with 3, 0, 1, 2. For larger d, just append vectors to the scheme for G_2, thus: $(00X, 23X, 01X, \ldots, 13X, 32X, 00Y, 23Y, \ldots)$, where $X, Y, \ldots$ exhaust all tuples in $\mathbb{Z}_4^{d-2}$. This repetition still yields a cycle and because all vertices are included, it is Hamiltonian. □

Another classic result [15] states that if the minimum degree of G is greater than or equal to $\frac{1}{2}(n_v + 1)$, then G is Hamilton-connected. The condition holds for G_d when $d \geq 3$ and a simple computation using an algorithm described in [7] verifies that G_2 is Hamilton-connected, so all Keller graphs are HC.

The Keller graphs are vertex-transitive and so provide an infinite family of examples for the conjecture in [7] that vertex-transitive, Hamiltonian graphs — except cycles and the dodecahedral graph — are HC. Computations also support the conjecture that Keller graphs have *Hamiltonian decompositions* (meaning that the edges can be partitioned into disjoint Hamiltonian cycles, plus a perfect matching if the degree is odd; see Fig. 21 for such a decomposition of G_2). We found Hamiltonian decompositions up through G_6 and conjecture that they exist for all Keller graphs. Table 1 shows such a decomposition for G_3: 17 Hamiltonian cycles.

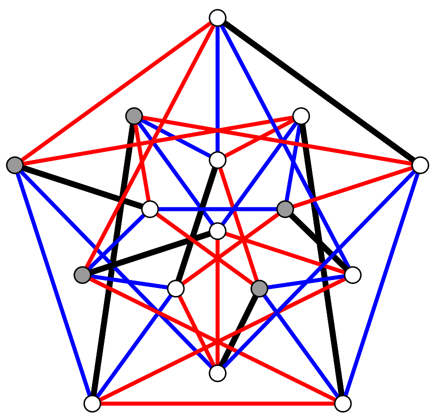

Figure 21: A Hamiltonian decomposition of G_2, also known as the Clebsch graph: two Hamiltonian cycles (red, blue) and one perfect matching (black). The gray vertices are a maximum independent set.

Conjecture 6.2. *All Keller graphs have a Hamiltonian decomposition.*

Conjecture 6.2 is related to deep work of Kühn et al. [12, 5]. Theorem 1.7 of [12] implies that for sufficiently large odd d, there is a Hamilton decomposition of G_d, while

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
41	27	54	26	46	30	45	9	22	58	43	25	24	44	59	37	62
2	9	32	62	60	36	19	31	31	19	53	49	19	1	50	35	22
45	50	27	55	38	54	8	17	52	48	44	58	33	11	25	1	61
21	42	60	20	20	8	49	62	46	59	52	12	23	16	63	51	7
46	48	14	49	53	2	30	29	15	32	62	18	28	40	16	43	21
16	17	56	15	14	32	21	50	41	11	6	44	51	19	41	50	55
53	41	42	52	16	56	15	56	5	29	12	14	21	26	3	24	16
7	34	1	19	51	22	39	27	3	22	40	39	57	63	17	46	58
28	59	40	25	29	44	33	49	34	40	17	46	43	49	15	32	27
62	49	33	51	39	9	12	7	40	30	56	22	9	21	45	10	17
8	23	26	9	12	7	7	38	49	24	29	51	2	52	31	12	35
39	25	57	39	2	42	40	32	56	2	36	31	59	27	36	21	59
17	16	20	13	33	3	50	8	30	26	5	21	22	5	9	3	41
47	62	29	42	25	58	22	18	4	54	42	59	16	39	46	8	19
1	7	52	33	52	23	4	52	18	27	51	61	6	60	55	47	54
27	43	12	47	58	29	43	45	42	3	18	35	56	31	28	2	18
48	13	34	3	17	5	61	54	19	13	24	4	34	4	18	16	53
25	46	10	61	42	46	52	47	57	51	35	42	13	46	60	59	12
32	37	45	16	35	6	34	22	16	61	9	16	54	7	22	45	50
42	14	22	43	41	52	14	13	26	31	38	56	46	57	53	5	28
24	4	58	45	18	63	42	63	28	53	61	1	3	29	47	34	49
10	63	18	14	9	17	9	5	61	46	25	10	53	58	6	9	43
51	53	48	41	59	60	17	35	27	10	3	43	17	32	30	19	37
40	26	24	31	26	21	54	15	29	4	57	17	10	22	39	56	31
13	36	63	7	56	39	25	1	59	38	15	24	37	55	11	23	48
37	18	36	37	28	28	58	25	33	28	20	34	4	61	42	30	26
19	32	50	59	6	59	36	11	58	34	54	15	60	17	2	58	60
29	15	16	23	13	20	3	53	2	1	16	33	25	51	40	48	42
43	5	9	63	31	50	26	59	60	43	30	6	7	33	58	22	6
3	40	47	6	50	57	44	19	23	18	48	48	32	43	8	52	20
33	3	5	36	21	24	2	60	13	57	46	57	12	48	44	42	63
4	29	30	10	27	61	35	20	32	35	23	63	38	8	62	49	33
34	6	55	60	63	15	56	61	50	49	37	37	5	38	56	26	9
26	8	19	58	54	25	51	23	11	41	28	47	31	3	20	17	32
50	22	49	34	15	34	53	57	47	12	4	52	39	59	51	11	5
52	28	3	27	22	43	27	55	38	62	58	28	49	25	27	38	51
23	57	44	50	36	12	57	33	24	21	31	36	47	18	35	15	23
9	51	7	4	11	26	1	24	51	56	59	8	55	62	7	53	45
49	58	41	32	49	51	23	3	37	25	1	41	18	53	29	25	1
31	24	23	30	55	45	47	37	8	39	8	27	27	24	49	55	36
56	60	17	53	45	4	20	44	45	63	26	45	62	6	24	27	2
18	54	59	29	3	41	46	42	53	7	55	7	40	15	32	13	25
59	61	13	38	10	55	28	28	39	14	13	13	48	9	43	36	47
30	47	21	1	48	37	63	58	6	5	19	20	41	37	19	7	29
54	12	28	46	23	27	18	21	44	33	63	26	1	2	5	60	4
14	55	53	56	44	16	10	43	10	8	21	32	58	28	23	29	15
38	1	35	2	5	49	41	2	20	16	47	3	44	10	12	63	40
63	39	25	57	43	18	13	48	48	52	7	9	50	56	37	41	46
22	10	31	17	8	11	11	14	9	17	33	40	8	54	1	39	11
57	19	38	8	34	33	48	36	63	50	27	54	42	23	26	44	34
11	21	62	35	7	13	29	46	1	23	2	29	36	14	61	4	44
44	11	39	12	30	53	37	12	7	15	39	55	15	35	21	62	13
15	35	61	22	37	19	62	30	17	42	45	5	29	13	14	20	52
55	45	4	24	61	1	32	51	55	20	11	62	35	47	33	18	24
60	38	2	54	19	31	6	41	14	44	41	23	26	41	10	61	14
6	44	46	44	62	62	31	6	43	55	32	53	52	50	52	6	8
35	30	8	21	24	48	55	34	25	9	14	60	14	30	38	40	30
58	52	51	48	1	38	24	16	62	45	22	30	20	20	13	14	57
20	2	11	28	32	40	59	10	36	6	49	38	45	34	4	28	39
12	20	37	5	57	35	5	40	12	37	10	2	63	36	54	54	3
5	31	15	11	47	10	60	26	54	60	50	11	30	45	48	31	56
61	33	43	40	4	47	16	4	21	36	60	19	61	12	34	57	38
36	56	6	18	40	14	38	39	35	47	34	50	11	42	57	33	10

Table 1: A decomposition of G_3 into 17 Hamiltonian cycles. The vertices are encoded, using base 4, by integers from 0 to 63. Because $\Delta = 34$, there are 17 Hamiltonian cycles.

the improvement in [5, Thm. 1.1.3] handles the even case too. So for sufficiently large d, G_d is known to have a Hamiltonian decomposition.

Our algorithm for finding these decompositions starts with the simple idea of trying random class-1 colorings (obtained by using the methods of Vizing and Kempe on a random permutation of the graph) and checking to see if the color-sets, which are matchings, can be paired up to form the desired cycles; the pairing is generally done using the classic blossom algorithm of Edmonds. A more sophisticated approach is needed for large cases such as G_5 and G_6. We again start with a class-1 coloring, but then apply Kempe switches in the hope of obtaining pairs of matchings that link to form cycles. The heuristic used to decide which Kempe switches to make is a scoring function that compares the number of Hamiltonian cycles obtainable by pairing up matchings (primary key), the minimum number of cycles a pair of remaining matchings produces (secondary key), and the total number of cycles that all pairs of remaining matchings produce (tertiary key). Additionally, a Hamiltonian decomposition of a significant part of the edges of G_6 can be effectively constructed from a decomposition of G_5. Therefore, to find a decomposition of G_6, we first find one for G_5. We then apply the local search method using the heuristic function described above, but only to the subgraph of G_6 consisting of the uncovered edges.

Although it seemed plausible that connected, vertex-transitive graphs always have Hamiltonian decompositions (excluding a few small examples), that was recently shown by Bryant and Dean [3] to be false. An even stronger property is that of having a perfect 1-factorization: a collection of matchings such that *any* two form a Hamiltonian cycle. That is a much more difficult subject — it is unresolved even for complete graphs — and all we can say is that an exhaustive search established that G_2 does not have a perfect 1-factorization. In the other direction, a weaker conjecture than the false one just mentioned is that all vertex-transitive graphs with even order are class 1 except the Petersen graph and the triangle-replaced Petersen graph; no counterexample is known.

Note that an even-order graph with a Hamiltonian decomposition is necessarily class 1. One can show that all Keller graphs are class 1 by explicit computation up to G_6 and then calling on a famous theorem of Chetwynd and Hilton for the rest ([4]; see also [20, Thm. 4.17]); their theorem applies to graphs for which $\Delta > \frac{1}{2}(\sqrt{7}-1)n_v$, which holds for G_7 and beyond. But in fact there is a uniform and constructive way to present class-1 colorings of all G_d, which we now describe. Note that this result also follows from the class-1 conjecture of the preceding paragraph.

Theorem 6.3. *All Keller graphs are class 1.*

Proof. All arithmetic here is mod 4. Call a vertex — a d-tuple — *even* if all entries are even; otherwise odd. The class-1 coloring can be constructed explicitly as follows. Define the color set S to consist of all vertices whose coordinates have at least one 2, but excluding the d vectors consisting of just $d-1$ 0s and one 2. This set is a type of kernel: the set of all differences $u-v$ for edges $v \leftrightarrow u$. This set satisfies (1) $|S| = \Delta$, and (2) for each vertex v, its neighbors are $v + S$. Partition S into its even vectors, S_0, and its odd ones, S_1.

For each $s \in S_1$, define an equivalence relation $\sim_s$ on the vertices: $u \sim_s v$ iff $u - v$ is a multiple of s. Each equivalence class has the form $\{v, v+s, v+2s, v+3s\}$; because s is odd, each such class has four distinct elements. Note that the collection of classes for s is identical to the collection of classes for $-s$. For each $s \in S_1$, define a choice set C_s for the equivalence classes; use the lexicographically first vector in each class. Then $C_s = C_{-s}$.

Define the edge coloring as follows (see Fig. 22). For each even color $s \in S_0$, use it for all edges $v \leftrightarrow v+s$. Each s colors $\frac{1}{2}n_v$ edges because $v \leftrightarrow v \pm s$ both get color s, but are the same edge (because $s=-s$). So in all, this colors $\frac{1}{2}n_v|S_0|$ edges. For each odd color $s \in S_1$, use it for the edges $v \leftrightarrow v+s$ and $v+2s \leftrightarrow v+3s$, but, in both cases, only for vertices $v \in C_s$. Because $|C_s| = \frac{1}{4}n_v$, each color applies to $\frac{1}{2}n_v$ edges, and so the odd colors taken together color $\frac{1}{2}n_v|S_1|$ edges. Thus the number of edges that are colored is $\frac{1}{2}n_v(|S_0|+|S_1|) = \frac{1}{2}n_v|S| = \frac{1}{2}n_v\Delta = n_e$, the total number of edges.

Claim. Every edge receives only one color.

Proof. Given edge $u \leftrightarrow w$, let $s = w-u$. If s is even, then $s=-s$ and this easily yields the claim. The odd case is more delicate. Suppose $u \leftrightarrow w$ is assigned color s; then $s = \pm(w-u)$. We may assume $s = w-u$. If the edge is assigned another color distinct from s, that color must therefore be $-s$. Now u and w are equivalent under both relations $\sim_s$ and $\sim_{-s}$. And the class representatives from C_s and C_{-s} agree. This means that $u \leftrightarrow w$ must be one of the edges $\{v \leftrightarrow v+s, v+2s \leftrightarrow v+3s\}$ and also one of the edges $\{v \leftrightarrow v-s, v-2s \leftrightarrow v-3s\}$. But the latter set equals $\{v+3s \leftrightarrow v, v+s \leftrightarrow v+2s\}$, which is disjoint from the first pair. ☐

The claim and the fact that n_e edges are colored means that every edge receives a color. So it remains only to show that the coloring is proper. Suppose not. Then we have edges $u \leftrightarrow w$ and $u \leftrightarrow y$ receiving the same color s. If s is even, this is not possible because the edges would have to be of the form $v \leftrightarrow v+s$ and $v \leftrightarrow v-s$, which are equal because $s=-s$. Suppose s is odd and color s is assigned to edge $u \leftrightarrow w$. If $u \in C_s$, then $w = u+s$; if $u = v+s$ where $v \in C_s$, then $w=v$; if $u = v+2s$ where $v \in C_s$, then $w = v+3s$, and if $u = v+3s$ where $v \in C_s$, then $w = v+2s$. In all cases there is only one choice for w. ☐

colors S	size-4 equivalence classes of vertices with edges colored by the odd element of S							
odd: 23	00 ⟷ 23	02 ⟷ 21	01 ⟷ 20	03 ⟷ 22	10 ⟷ 33	12 ⟷ 31	11 ⟷ 30	13 ⟷ 32
odd: 21	00 ⟷ 21	02 ⟷ 23	01 ⟷ 22	03 ⟷ 20	10 ⟷ 31	12 ⟷ 33	11 ⟷ 32	13 ⟷ 30
odd: 32	00 ⟷ 32	20 ⟷ 12	01 ⟷ 33	21 ⟷ 13	02 ⟷ 30	22 ⟷ 10	03 ⟷ 31	23 ⟷ 11
odd: 12	00 ⟷ 12	20 ⟷ 32	01 ⟷ 13	21 ⟷ 33	02 ⟷ 10	22 ⟷ 30	03 ⟷ 11	23 ⟷ 31
even: 22	00 ⟷ 22	01 ⟷ 23	02 ⟷ 20	03 ⟷ 21	10 ⟷ 32	11 ⟷ 33	12 ⟷ 30	13 ⟷ 31

Figure 22: The class-1 Keller coloring for the edges of G_2, using five colors. The even case has only one entry; $S_0 = \{22\}$. The odd case has four colors and the four equivalence classes of the full vertex set are shown, with the matchings within each class. Note that the classes for $\pm s$ are the same sets (e.g., $s = 12$ and 32).

We can also investigate some familiar parameters for Keller graphs. The standard parameters α, θ, χ, and χ_{frac} are defined in §1. Let θ_{frac} be the fractional clique covering number (same as χ_{frac} of the complementary graph). Table 2 shows the known results, including results proved here. It is clear that $\alpha(G_d) \geq 2^d$ since the tuples using only 0s

d	all d	2	3	4	5	6	7	$d \geq 8$
independence number, α		5	8	16	32	64	128	2^d
chromatic number, χ	2^d	4	8	16	32	64	128	2^d
fractional chromatic number, χ_{frac}		$\frac{16}{5}$	8	16	32	64	128	2^d
class 1 for edge coloring; $\chi' = \Delta$	Yes							
maximum clique size, ω		2	5	12	28	60	124	2^d
clique covering number, θ		8	13	22	$37 \leq \theta \leq 40$	$69 \leq \theta \leq 80$	$133 \leq \theta \leq 160$	2^d
fractional clique covering number, θ_{frac}		8	$\frac{64}{5}$	$\frac{64}{3}$	$\frac{256}{7}$	$\frac{1024}{15}$	$\frac{4096}{31}$	2^d
Hamiltonian, Hamilton-connected	Yes							
Hamiltonian decomposition	Conjectured yes	Yes	Yes	Yes	Yes	Yes	?	?
perfect 1-factorization		No	?	?	?	?	?	?
degree, Δ	$4^d - 3^d - d$	5	34	171	776	3361	14190	

Table 2: Properties of the Keller graphs G_d. The number of vertices of G_d is 4^d and the edge count is $\frac{1}{2}4^d(4^d - 3^d - d)$.

and 1s are independent. A larger independent set can exist, but only in G_2, as Theorem 6.4 shows.

Theorem 6.4. *The independence number of G_d is 2^d, except that $\alpha(G_2) = 5$.*

Proof. for $d \leq 5$, this was known; direct computational methods work. The anomalous case has maximum independent set $\{(0,3),(1,0),(1,2),(1,3),(2,3)\}$; see Figure 21. For $d = 6$ or 7, one can again use computation, but some efficiencies are needed since the graphs are large. The graphs are vertex-transitive, so we may assume the first vertex is in the largest independent set. Thus, if A consists of the first vertex together with its neighbors, we can look at H_d, the subgraph of G_d generated by the vertices not in A. This is substantially smaller, and we need only show that $\alpha(H_d) = 2^d - 1$. That can be done by standard algorithms for finding independent sets; in *Mathematica* it takes a fraction of a second to show that this is the case for H_6 and only a few seconds to do the same for H_7.

Now suppose $d \geq 8$. Recall that it is known that $\omega(G_d) = 2^d$ in this case (Mackey [14] for $d = 8$; see [6, Thm. 4.2] for larger d). As in [6], place the vertex labels in a $2^d \times 2^d$ grid, called the *independence square*. The row position of a tuple is computed by converting 0 or 1 to 0 and also converting 2 or 3 to 1 and then treating the result as a binary number. The column position of a tuple is computed by converting 0 or 3 to 0 and also converting 1 or 2 to 1 and then interpreting this in binary. The array for G_4 is shown in Table 3.

The tuples in the same row of the square form an independent set because, in each digit, the value is always either 0 or 1, or it is 2 or 3. Therefore there is no position where the difference is 2 $(\text{mod}\, 4)$. Similarly, the tuples in a column form an independent set. The independence square also proves that $\chi(G_d) \leq 2^d$ for any d.

0000	0001	0010	0011	0100	0101	0110	0111	1000	1001	1010	1011	1100	1101	1110	1111
0003	0002	0013	0012	0103	0102	0113	0112	1003	1002	1013	1012	1103	1102	1113	1112
0030	0031	0020	0021	0130	0131	0120	0121	1030	1031	1020	1021	1130	1131	1120	1121
0033	0032	0023	0022	0133	0132	0123	0122	1033	1032	1023	1022	1133	1132	1123	1122
0300	0301	0310	0311	0200	0201	0210	0211	1300	1301	1310	1311	1200	1201	1210	1211
0303	0302	0313	0312	0203	0202	0213	0212	1303	1302	1313	1312	1203	1202	1213	1212
0330	0331	0320	0321	0230	0231	0220	0221	1330	1331	1320	1321	1230	1231	1220	1221
0333	0332	0323	0322	0233	0232	0223	0222	1333	1332	1323	1322	1233	1232	1223	1222
3000	3001	3010	3011	3100	3101	3110	3111	2000	2001	2010	2011	2100	2101	2110	2111
3003	3002	3013	3012	3103	3102	3113	3112	2003	2002	2013	2012	2103	2102	2113	2112
3030	3031	3020	3021	3130	3131	3120	3121	2030	2031	2020	2021	2130	2131	2120	2121
3033	3032	3023	3022	3133	3132	3123	3122	2033	2032	2023	2022	2133	2132	2123	2122
3300	3301	3310	3311	3200	3201	3210	3211	2300	2301	2310	2311	2200	2201	2210	2211
3303	3302	3313	3312	3203	3202	3213	3212	2303	2302	2313	2312	2203	2202	2213	2212
3330	3331	3320	3321	3230	3231	3220	3221	2330	2331	2320	2321	2230	2231	2220	2221
3333	3332	3323	3322	3233	3232	3223	3222	2333	2332	2323	2322	2233	2232	2223	2222

Table 3: The independence square for G_4: each row and each column is an independent set.

000	001	010	011	100	101	110	111
003	002	013	012	103	102	113	112
030	031	020	021	130	131	120	121
033	032	023	022	133	132	123	122
300	301	310	311	200	201	210	211
303	302	313	312	203	202	213	212
330	331	320	321	230	231	220	221
333	332	323	322	233	232	223	222

(a)

001	000	011	010	101	100	111	110
002	003	012	013	102	103	112	113
031	030	021	020	131	130	121	120
032	033	022	023	132	133	122	123
301	300	311	310	201	200	211	210
302	303	312	313	202	203	212	213
331	330	321	320	231	230	221	220
332	333	322	323	232	233	222	223

(b)

Table 4: (a) The independence square for G_3. (b) After application of the automorphism defined by 001.

Let X be a clique of order 2^d in G_d. It has exactly one entry per row in the independence square. Given a d-digit bit-string b, use it to define an associated automorphism of the graph. In the positions where b has 0, leave the corresponding position of all the vertex entries alone. In the places where b has 1, do the following in those positions: switch 1 and 0, and switch 2 and 3. This preserves adjacency because positions that are different in value are still different and positions that differed by 2 (mod 4) still differ by 2 (mod 4).

If the bit string is 0011, then the first two columns stay the same and the last two columns have the swaps: for example, 0213 becomes 0202. The complete action of the automorphism on G_3 using the bit-string 001 is shown in Table 4.

Note that this automorphism maps each row of the square to itself. The collection of automorphisms that correspond to all 0-1 bit strings will map a 2^d-clique of the Keller graph to a partitioning of the vertices of the Keller graph into 2^d disjoint cliques each of size 2^d. So $\theta(G_d) = 2^d$ for such graphs.

Since any coloring of the graph can have at most one vertex per clique, for Keller graphs that have a 2^d clique (i.e., for $d \geq 8$, which we have assumed), it is not possible to find an independent set of size bigger than 2^d. □

Corollary 6.5. *For all Keller graphs, $\chi(G_d) = 2^d$.*

1	113	130	232	300	312
2	110	131	212	320	332
3	100	121	202	310	322
4	021	033	203	220	301
5	031	112	133	303	311
6	012	020	132	213	230
7	013	032	111	223	231
8	011	030	201	233	313
9	010	022	200	221	302
10	003	101	122	323	331
11	001	103	120	321	333
12	000	023	102	210	222
13	002	123	211	330	

Table 5: A covering of the 64 vertices of G_3 by 13 disjoint complete subgraphs.

Proof. The constructive coloring at the beginning of the section gives 2^d as an upper bound. Theorem 6.4 gives 2^d as a lower bound, because $4^d/\alpha(G_d) = 2^d$. □

Fung [11, Cor. 6.7] observed that $\chi(G_d) = 2^d$ for $d \geq 4$, $\chi(G_3) \geq 7$, and $\chi(G_2) \geq 3$. Corollary 6.5 establishes the validity of $\chi = 2^d$ for all Keller graphs.

The graph G_2 is an anomaly, with independence number 5. Because $\chi_{\text{frac}}(G) = \frac{n_v}{\alpha(G)}$ for vertex-transitive graphs (see [19]), we get the following result.

Corollary 6.6. *If $d \geq 3$, then $\chi_{\text{frac}}(G_d) = 2^d$; $\chi_{\text{frac}}(G_2) = 16/5$.*

So for $d \geq 8$, we have that each parameter $\alpha, \chi, \chi_{\text{frac}}$ and ω equals 2^d.

Computing $\theta(G_d)$ when $d \leq 7$ is difficult. A general lower bound is $\left\lceil \frac{n_V(H)}{\omega(H)} \right\rceil \leq \theta(H)$, which yields $8, 13, 22, 37, 69, 133$, the lower bounds of Table 2 (note also that $\alpha \leq \theta$); the first three are sharp. But for $5 \leq d \leq 7$, we do not know $\theta(G_d)$. For G_5, we have only that $37 \leq \theta \leq 40$. because G_2 has no triangles, it is clear that $\theta(G_2) = 8$. A 13-coloring of the complement of G_3 is shown in Table 5. Table 6 shows a covering of G_4 by 22 cliques, the method for which we will explain shortly. That same method found $\theta(G_5) \leq 40$ (see Table 7). The values of θ_{frac} in Table 2 arise from the vertex-transitive formula $\chi_{\text{frac}} = n_v/\alpha$ on the complement, which becomes n_v/ω.

The method of getting a minimal clique covering for G_4 uses backtracking and the structure of the independence square (Table 3). As discussed, any clique cover for G_4 has at least 22 cliques. One way to search for a cover using 22 cliques is to use twenty 12-cliques and two 8-cliques. So an initial goal was to search for 20 disjoint 12-cliques. The complete set of all 86,012 12-cliques was generated. We then tried backtracking on these to find a set of 20 pairwise disjoint cliques that could extend to a clique cover but the problem size proved unmanageable. Many search paths would get stuck after including only 16 of the 12-cliques. A second problem is that if after including the twenty 12-cliques, there is a row or column in the independence square that is not covered k times, then it is necessary to add at least k more cliques to complete the cover. So an auspicious selection of twenty 12-cliques should leave each row uncovered at most two times each.

1	0233	1003	1020	1213	1221	1301	2101	3033	3113	3121	3312	3331
2	0013	0030	0200	1212	1220	1332	2012	2020	2100	3032	3202	3221
3	0100	0132	0212	0310	0333	1331	2010	2033	2131	2211	2223	3012
4	0223	0303	0331	1102	1121	1311	2123	2313	2330	3011	3103	3131
5	0101	1022	1103	1120	1302	1330	2222	3013	3021	3203	3220	3301
6	0001	0033	0113	1021	1211	1232	2000	2023	2213	3201	3233	3321
7	0111	0123	0203	0301	0322	1320	2001	2022	2120	2200	2232	3003
8	0302	1111	1132	1230	1310	1322	2030	3010	3022	3102	3200	3223
9	0003	0020	0122	0202	0230	1001	2113	2121	2201	2303	2320	3322
10	0011	0103	0131	0313	0330	1123	2002	2021	2203	2231	2323	3211
11	0110	1033	1112	1131	1313	1321	2233	3002	3030	3212	3231	3310
12	0012	0031	0133	0213	0221	1010	2102	2130	2210	2312	2331	3333
13	0220	1011	1023	1201	1222	1303	2103	3020	3101	3122	3300	3332
14	0000	0032	0120	0201	0222	1012	2110	2133	2212	2300	2332	3320
15	0231	1000	1032	1210	1233	1312	2112	3031	3110	3133	3311	3323
16	0121	0311	0332	1013	1101	1133	2221	2301	2333	3100	3123	3313
17	0002	0021	0211	1203	1231	1323	2003	2031	2111	3023	3213	3230
18	0232	0312	0320	1113	1130	1300	2132	2302	2321	3000	3112	3120
19	0010	0022	0102	1030	1200	1223	2011	2032	2202	3210	3222	3330
20	0130	0300	0323	1002	1110	1122	2230	2310	2322	3111	3132	3302
21	0112	0321	1100	1333	2122	2311	3130	3303				
22	0023	0210	1031	1202	2013	2220	3001	3232				

Table 6: A covering of the 256 vertices of G_4 by 22 disjoint complete subgraphs.

To attempt to deal with both of these problems, the search was restricted to only cliques that had certain subsets of the rows missing. After inspecting the subsets of rows that could be missing from one of the cliques, the following selection was made (where the rows are indexed by $0, 1, \ldots, 15$.

Group 1 Rows 0, 3, 12, and 15 are missing.

Group 2 Rows 1, 2, 13, and 14 are missing.

Group 3 Rows 4, 7, 8, and 11 are missing.

Group 4 Rows 5, 6, 9, and 10 are missing.

Backtracking on just these cliques led to several sets of twenty 12-cliques. A backtracking program was used to try to complete the clique cover, and this quickly led to a solution (most of the sets of 20 do not extend to a clique cover of size 22 but it did not take long to find one that did); see Table 6. The set of 20 that completed had 5 tuples from each group meaning that each row was used exactly 15 times (and was missing once). Similar ideas yielded the 44-clique for G_5 (Table 7).

It is well-known (see [6, Thm. 4.2]) that a clique in G_d can be used to create a clique twice as large in G_{d+1} by making two copies of it, prefacing the first copy with the digit 0, adding 1 modulo 4 to each position of each tuple in the second copy, and then prefacing each tuple in the second copy with the symbol 2. The clique also can be doubled using digits 1 and 3 as the first digits instead of 0 and 2. To double a clique cover, start with each clique C. Double C using first digits 0 and 2. Then double the clique C again using 1 and 3 as the initial digits. The original clique cover used all the d-tuples exactly once. For first digit 0 and 1 it is easy to see that each tuple is used exactly once. Similarly for first digits 1 and 3, each tuple appears exactly once, because adding $11\ldots1$ to every tuple of dimension d gives back the complete set of tuples in dimension d. This construction gives a clique

1	0	18	328	346	44	382	401	147	457	203	176	242	424	490	529	595	777	843	624	882	552	810	925	719	753	739	953	939
2	65	343	25	31	33	51	56	314	393	399	421	439	172	430	533	791	524	542	564	562	620	890	641	899	904	922	932	930
3	276	22	332	78	117	55	365	303	149	135	477	463	497	163	784	578	628	614	828	814	660	726	908	974	1013	759	941	687
4	260	6	348	94	357	295	125	63	389	407	205	223	225	435	848	514	884	870	572	558	964	902	732	670	741	999	701	959
5	5	263	93	351	356	294	124	62	388	406	204	222	224	434	513	851	869	887	557	575	901	967	669	735	740	998	700	958
6	257	275	73	91	109	319	128	386	216	474	481	419	249	187	768	834	536	602	609	867	569	827	652	990	992	1010	680	698
7	1	19	329	347	45	383	400	146	456	202	177	243	425	491	528	594	776	842	625	883	553	811	924	718	752	738	952	938
8	4	262	92	350	293	359	61	127	405	391	221	207	433	227	512	850	868	886	556	574	900	966	668	734	997	743	957	703
9	69	339	269	267	289	307	40	298	157	155	165	183	188	446	517	775	780	798	800	806	616	894	657	915	648	666	688	694
10	320	326	97	355	376	362	208	214	408	142	245	503	252	238	837	855	844	590	873	879	705	723	984	730	673	951	761	767
11	80	86	113	371	104	122	448	454	136	414	229	487	492	510	581	599	604	862	637	635	961	979	712	970	677	947	1005	1003
12	341	67	29	27	49	35	312	58	397	395	437	423	428	174	789	535	540	526	560	566	888	622	897	643	920	906	928	934
13	325	323	352	98	361	379	213	211	141	411	500	246	237	255	852	838	589	847	876	874	720	706	729	987	948	674	764	762
14	277	23	333	79	52	118	300	366	132	150	460	478	160	498	785	579	629	615	829	815	661	727	909	975	756	1014	684	942
15	16	2	344	330	380	46	145	403	201	459	240	178	488	426	593	531	841	779	880	626	808	554	717	927	737	755	937	955
16	340	66	28	26	48	34	313	59	396	394	436	422	429	175	788	534	541	527	561	567	889	623	896	642	921	907	929	935
17	81	87	112	370	105	123	449	455	137	415	228	486	493	511	580	598	605	863	636	634	960	978	713	971	676	946	1004	1002
18	273	259	89	75	317	111	384	130	472	218	417	483	185	251	832	770	600	538	865	611	825	571	988	654	1008	994	696	682
19	84	82	369	115	120	106	452	450	412	138	485	231	508	494	597	583	860	606	633	639	977	963	968	714	945	679	1001	1007
20	272	258	88	74	316	110	129	387	217	475	416	482	184	250	769	835	537	603	864	610	824	570	653	991	993	1011	681	699
21	336	70	264	270	304	290	297	43	152	158	180	166	445	191	772	518	797	783	805	803	893	619	912	658	665	651	693	691
22	85	83	368	114	121	107	453	451	413	139	484	230	509	495	596	582	861	607	632	638	976	962	969	715	944	678	1000	1006
23	17	3	345	331	381	47	144	402	200	458	241	179	489	427	592	530	840	778	881	627	809	555	716	926	736	754	936	954
24	337	71	265	271	305	291	296	42	153	159	181	167	444	190	773	519	796	782	804	802	892	618	913	659	664	650	692	690
25	256	274	72	90	108	318	385	131	473	219	480	418	248	186	833	771	601	539	608	866	568	826	989	655	1009	995	697	683
26	21	279	77	335	116	54	364	302	148	134	476	462	496	162	577	787	613	631	813	831	725	663	973	911	1012	758	940	686
27	324	322	353	99	360	378	212	210	140	410	501	247	236	254	853	839	588	846	877	875	721	707	728	986	949	675	765	763
28	261	7	349	95	292	358	60	126	404	390	220	206	432	226	849	515	885	871	573	559	965	903	733	671	996	742	956	702
29	64	342	24	30	32	50	57	315	392	398	420	438	173	431	532	790	525	543	565	563	621	891	640	898	905	923	933	931
30	68	338	268	266	288	306	41	299	156	154	164	182	189	447	516	774	781	799	801	807	617	895	656	914	649	667	689	695
31	20	278	76	334	53	119	301	367	133	151	461	479	161	499	576	786	612	630	812	830	724	662	972	910	757	1015	685	943
32	321	327	96	354	377	363	209	215	409	143	244	502	253	239	836	854	845	591	872	878	704	722	985	731	672	950	760	766
33	12	10	36	102	196	194	168	234	520	586	544	550	644	710	744	750												
34	13	11	37	103	197	195	169	235	521	587	545	551	645	711	745	751												
35	8	14	100	38	192	198	232	170	584	522	548	546	708	646	748	746												
36	9	15	101	39	193	199	233	171	585	523	549	547	709	647	749	747												
37	284	282	308	374	468	466	440	506	792	858	816	822	916	982	1016	1022												
38	285	283	309	375	469	467	441	507	793	859	817	823	917	983	1017	1023												
39	280	286	372	310	464	470	504	442	856	794	820	818	980	918	1020	1018												
40	281	287	373	311	465	471	505	443	857	795	821	819	981	919	1021	1019												

Table 7: A covering of the 1024 vertices of G_5 by 40 disjoint complete subgraphs of sizes 28 and 16. The encoding is as in Table 1.

cover in G_{d+1} whose size is twice that of the cover of G_d. The 40-cover of G_4 therefore yields the upper bounds of 80 and 160 for the next two Keller graphs.

The success in finding clique covers whose size equals the lower bound suggests that this holds true in the remaining three cases, $d = 5$, 6, and 7.

Conjecture 6.7. *For all d, $\theta(G_d) = \left\lceil \frac{4^d}{\omega(G_d)} \right\rceil$.*

7 Conclusion

The first investigations for all our results involved computer experimentation. The patterns that one finds by such work often lead to new observations, which can sometimes be proved by classical methods. But one can be led astray. The assertion that all connected, vertex-transitive graphs (except five small examples) have Hamilton decompositions was conjectured to be true by Wagon based on extensive computations on over 100,000 graphs, including all graphs of 30 or fewer vertices. But the assertion is now known to be false [3]; however, the related conjecture [7] that all Hamiltonian vertex-transitive graphs are Hamilton-connected (except cycles and the dodecahedral graph) is still open. Very general conjectures such as these can be tricky; we believe that the more specific conjectures presented here about queen graphs, Keller graphs, and the Mycielskian operation are quite plausible and easier to contemplate.

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Semiregular automorphisms in vertex-transitive graphs with a solvable group of automorphisms

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Abstract

It has been conjectured that automorphism groups of vertex-transitive (di)graphs, and more generally 2-closures of transitive permutation groups, must necessarily possess a fixed-point-free element of prime order, and thus a non-identity element with all orbits of the same length, in other words, a *semiregular* element. The known affirmative answers for graphs with primitive and quasiprimitive groups of automorphisms suggest that solvable groups need to be considered if one is to hope for a complete solution of this conjecture. It is the purpose of this paper to present an overview of known results and suggest possible further lines of research towards a complete solution of the problem.

Keywords: Solvable group, semiregular automorphism, fixed-point-free automorphism, polycirculant conjecture.

Math. Subj. Class.: 20B25, 05C25

1 Introduction

It is known that each finite transitive permutation group contains a fixed-point-free element of prime power order (see [8, Theorem 1]), but not necessarily a fixed-point-free element of prime order (which is equivalent to existence of a semiregular element) [2, 8]. In 1981 it was asked if every vertex-transitive digraph admits a semiregular automorphism (see [18, Problem 2.4]). The existence of such automorphisms plays an important role in solutions to many important open problems in algebraic graph theory, such as, for example, in the classifications of graphs satisfying certain prescribed symmetry conditions (see [15, 16,

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20, 22, 25]). Semiregular automorphisms have also proved useful in the long standing hamiltonicity problem for connected vertex-transitive graphs and in a recently explored dichotomy of even/odd automorphisms (see [1, 13, 17]).

In 1997 Klin generalized the semiregularity problem conjecturing that every transitive 2-closed permutation group contains a semiregular element (see [4]) – the term *polycirculant conjecture* is sometimes used for the semiregularity problem in this wider context. (Recall that for a finite permutation group G on a set V the 2-*closure* $G^{(2)}$ of G is the largest subgroup of the symmetric group $\mathrm{Sym}(V)$ containing G and having the same orbits as G in the induced action on $V \times V$.) Both terms will be used throughout the paper, this should cause no confusion. The problem has spurred a lot of interest in the mathematical community producing several partial results – addressing graphs with valency and/or order restrictions – with varying degrees of difficulties involved in their proofs (see for instance [2, 3, 5, 6, 7, 8, 9, 10, 12, 14, 19, 21, 23, 24]). Recently, Giudici, Potočnik and Verret [11] considered the problem in the context of graphs whose automorphism group acts transitivity on edges and not necessarily on vertices. They proved that every regular edge-transitive graph of valency three or four has a semiregular automorphism. Also, in 2003 Giudici [9] proved the polycirculant conjecture for quasiprimitive groups, and in 2007 Giudici and Xu [12] proved it for biquasiprimitive groups.

Since the automorphism group of a vertex-transitive graph is a transitive 2-closed group these results imply that the only graphs for which the semiregularity problem has not yet been settled are graphs whose automorphism groups contain a non-identity normal subgroup with at least three orbits. Clearly, since disconnected vertex-transitive graphs must contain semiregular automorphisms, we can restrict ourselves to connected graphs.

It is usually the case that algebraic graph theory problems dealing with group actions on graphs are considerably harder to address for nonsolvable groups than for the solvable ones. Such is the case, for example, with various types of classification problems for arc-transitive graphs and vertex-transitive graphs in general. Counter-intuitively, this does not seem to be the case with the polycirculant conjecture. While, as already mentioned it has been proved that the polycirculant conjecture holds for quasiprimitive groups [9], nothing of that nature is known for groups at the opposite end of the spectrum. For example, solvable groups turned out to be the steepest hill to climb in the completion of the proof that groups of square-free degree satisfy the polycirculant conjecture, see [5].

Our aim is to discuss possible ways of approaching the semiregularity problem for solvable groups, trying to single out certain idiosyncrasies of this class of groups relevant to the problem.

Problem 1.1. Does the 2-closure $G^{(2)}$ of a solvable group G contain a semiregular element?

In Proposition 2.3 a partial solution to this problem is given for groups of degree mp^2, where $m < p$ is square-free. As a consequence a partial solution to existence of semiregular automorphism in vertex-transitive graphs of order p^2q, where p and q are primes, is shown (see Theorem 2.4 and Corollary 2.5). Since disconnected vertex-transitive graphs clearly contain semiregular automorphisms, the graphs considered in Section 2 are connected.

2 Searching for semiregular elements

First let us recall the definition of a pseudometric first defined in [5] where it was used as one of the tools in proving the existence of semiregular elements in transitive permutation groups of square-free degree.

Let G be a transitive permutation group with a complete block system $\mathcal{B}$ such that $\mathrm{fix}_G(\mathcal{B})$ contains a subgroup $K \cong U^s$, for some $s \geq 1$ and such that the restriction $K^B \cong U^r$, $1 \leq r \leq s$, acts transitively on B, for each $B \in \mathcal{B}$. Then in view of [5, Proposition 3.1] we can define a *pseudometric* on $\mathcal{B}$ by letting

$$\mathrm{Dist}_K(B, B') = \log_{|U|} |K_{(B)}^{B'}|.$$

(For the proof that Dist_K is symmetric and that it satisfies the triangle inequality see [5, Proposition 3.1].)

In Proposition 2.1 below the extremal case where the possible distances in this pseudometric are only 0 and 1 is considered.

Proposition 2.1. *Let p be a prime, let $s \geq 1$ be an integer, let U be a simple group and let G be a transitive permutation group on a set V admitting an imprimitivity block system $\mathcal{B}$ with blocks of length divisible by p. If $\mathrm{fix}_G(\mathcal{B})$ contains a subgroup $K \cong U^s$ such that for each block $B \in \mathcal{B}$, the restriction K^B is isomorphic to U, acts transitively on B and contains a semiregular element of order p, then $G^{(2)}$ contains a semiregular element of order p.*

Proof. Observe that the assumptions in the statement of the proposition imply that in the above pseudometric language the possible distances between any two blocks in $\mathcal{B}$ are either 0 or 1. Namely, since $K_{(B)}^{B'}$ is a normal subgroup of $K^{B'} \cong U$ it follows that $K_{(B)}^{B'}$ is either 1 or U. In particular, for $B, B' \in \mathcal{B}$, the following holds

$$\mathrm{Dist}_K(B, B') = 1 \Leftrightarrow K_{(B)} \text{ is transitive on } B' \text{ and } K_{(B')} \text{ is transitive on } B. \tag{2.1}$$

This will allow us to construct a semiregular element in $G^{(2)}$ by a succession of superpositions of permutations acting independently on collections of blocks at distance 0. First, if $s = 1$ then the distance between any two blocks in $\mathcal{B}$ is equal to 0, and thus the element of K whose restriction to a block $B \in B$ is semiregular on B is semiregular on V too. We may therefore assume that the maximal distance between blocks in $\mathcal{B}$ is precisely 1. One can easily see that each of the subsets of those blocks in $\mathcal{B}$ being at mutual distance 0 forms a block of G. More precisely,

$$\mathcal{C} = \{\{B_{i_0} \cup \ldots \cup B_{i_k} \mid \mathrm{Dist}_K(B_{i_j}, B_{i_t}) = 0 \text{ for all } i_j, i_t \in \{i_0, \ldots, i_k\}\} \mid i \in \{0, \ldots, e\}\},$$

where $e = |\mathcal{B}|$, is an imprimitivity block system of G. Moreover, in view of (2.1) for every block $C_i = B_{i_0} \cup \ldots \cup B_{i_k} \in \mathcal{C}$ there exists an element $\gamma_i \in K$ such that $\gamma_i^{C_i}$ is semiregular and $\gamma_i^{C_j} = 1$ for all blocks $C_j \in \mathcal{C}$, $i \neq j$. Consequently, $\gamma_0 \gamma_1 \cdots \gamma_k$ is semiregular on V, completing the proof of Proposition 2.1. □

Corollary 2.2. *Let G be a permutation group acting transitively on a set V and let M be a minimal normal subgroup of G having orbits of prime length p on V. Then $G^{(2)}$ contains a semiregular element of order p.*

Proof. Since M is a minimal normal subgroup of G it is isomorphic to a direct product of isomorphic simple groups, that is, $M \cong U^s$, $s \geq 1$, where U is a simple group. The orbits of M form an imprimitivity block system $\mathcal{B}$ consisting of blocks of length p. For $B \in \mathcal{B}$ the restriction M^B is therefore a transitive group of prime degree p, and thus $M^B \cong U$. Hence Proposition 2.1 applies. □

Proposition 2.3. *Let G be a transitive solvable group of degree mp^2, where $m < p$ is square-free. Then $G^{(2)}$ contains a semiregular element of prime order.*

Proof. Let M be a minimal normal subgroup of G. Since G is solvable we have $M \cong \mathbb{Z}_q^s$, where q is a prime and $s \geq 1$. The orbits of M give rise to an imprimitivity block system $\mathcal{B}$. If the blocks in $\mathcal{B}$ are of prime length then Corollary 2.2 applies. We may therefore assume that $\mathcal{B}$ consists of blocks of size p^2 and $M \cong \mathbb{Z}_p^s$, $s \geq 1$.

Let $\mathcal{B} = \{B_0, \ldots, B_{m-1}\}$. Assume that $G^{(2)}$ does not contain semiregular elements, and let $\alpha \in M$ be an element of order p with a minimal number of orbits of M on which the restriction of α is the identity. Without loss of generality we may assume that

$$\alpha^{B_i} = \begin{cases} 1; & 0 \leq i \leq t \\ \neq 1; & t < i \leq m-1 \end{cases}$$

where $t < m - 1$. Let $B \in \mathcal{B}$ be a block for which $\alpha^B = 1$. Because of transitivity of G there exists $\beta \in M$, a conjugate of α, such that $\beta^B \neq 1$. Since $m < p$ there exists $k \in \mathbb{Z}_p$ such that $\alpha\beta^k$ is semiregular and of order p on each of the blocks B_i, $t < i \leq m-1$, as well as on the block B. Therefore, the number of blocks on which the restriction of $\alpha\beta^k$ is the identity is at least one less than the number of blocks on which the restriction of α is the identity, contradicting the minimality condition. It follows that $G^{(2)}$ must contain semiregular elements as claimed. (Note that the assumption that $m < p$ was essential in this respect.) □

With the use of Corollary 2.2 and Proposition 2.3 we can now prove the following result about existence of semiregular automorphisms in vertex-transitive graphs of order qp^2, where p and q are primes.

Theorem 2.4. *Let X be a connected vertex-transitive graph of order p^2q, where p and q are primes, and either $q \leq p$ or $p^2 < q$. Then either*

(i) X admits a semiregular automorphism, or

(ii) $2 < q < p$ and Aut(X) is nonsolvable with an intransitive non-abelian minimal normal subgroup whose orbits are either of length p^2 or of length pq.

Proof. First, we may assume that $p > 3$ and $q > 2$ and that $q \neq p$. Namely, if $q = 2$ or $q = p$ then the order of X equals $2p^2$ or p^3, and the existence of semiregular automorphisms was proved in [19] and [18], respectively. If $q > p^2$ then the existence of semiregular automorphisms follows from results in [18]. Therefore, we may assume that $q < p^2$.

If Aut(X) is primitive or quasiprimitive then, by [9], X contains a semiregular automorphism. We may therefore assume that there exists a minimal normal subgroup M of Aut(X) whose orbits give rise to a non-trivial imprimitivity block system $\mathcal{B}$.

If Aut(X) is solvable then M is abelian and isomorphic to $\mathbb{Z}_r^k$, where $r \in \{q, p\}$ and $k \geq 1$. Hence $\mathcal{B}$ consists of blocks of length q, p or p^2, and Corollary 2.2 and Proposition 2.3 imply the existence of semiregular automorphisms of X. If, however, Aut(X)

is nonsolvable then M is non-abelian. If the orbits of M are of prime length then Corollary 2.2 applies. Otherwise the orbits of M are either of length p^2 or qp, completing the proof of Theorem 2.4. □

The following corollary is an immediate consequence of Theorem 2.4.

Corollary 2.5. *Let q and p be primes such that either $q \leq p$ or $p^2 < q$. A connected vertex-transitive graph of order p^2q admitting a transitive solvable group of automorphisms has a semiregular automorphism.*

In our search for semiregular group elements we now turn to vertex-transitive graphs admitting a solvable group of automorphisms and satisfying certain valency restrictions. For a graph X admitting a transitive action of a group G with an imprimitivity block system $\mathcal{B}$ arising from the orbits of a normal subgroup $M \leq G$, we let $X/\mathcal{B}$ denote the corresponding quotient graph having vertex set $\mathcal{B}$ with two blocks $B, B' \in \mathcal{B}$ being adjacent if there is an edge in X joining a vertex in B to a vertex in B'. Further, for $B, B' \in \mathcal{B}$ we let $[B, B']$ denote the bipartite graph induced by the edges of X joining blocks B and B', and we let $\mathrm{val}(B, B')$ denote the valency of $[B, B']$. Also let $\mathrm{val}(X)$ denote the valency of X.

Lemma 2.6. *Let X be a connected vertex-transitive graph admitting a transitive action of a solvable group G with a minimal normal subgroup $M = \mathbb{Z}_q^k$, and suppose further that $\mathrm{val}(X) < pq$, where $p > q$ is the largest prime dividing $|G|$. Then either*

(i) G contains a semiregular subgroup or

(ii) M is intransitive and there exist orbits B, B' of M such that $\mathrm{val}(B, B') \geq mq$, where mp is the smallest multiple of q exceeding p.

Proof. Assume that (i) does not hold. It follows that M is intransitive for otherwise X would be a Cayley graph of M and so M would act semiregularly on $V(X)$. Let $\mathcal{B}$ be the imprimitivity block system arising from the orbits of M, and let $\mathrm{Dist} = \mathrm{Dist}_M$ be the associated pseudometric on $\mathcal{B}$. If $\mathrm{Dist}(B, B') = 0$ for every two adjacent blocks B and B' in $X/\mathcal{B}$, then clearly M contains a semiregular element of order q. We may therefore assume that there are adjacent blocks B and B' such that $d = \mathrm{Dist}(B, B') \geq 1$. Furthermore, we may assume that G_v, for $v \in V(X)$, contains elements of order q as well as elements of order p.

Fix a vertex $v \in B$. Since $d \geq 1$, we have that $\mathrm{val}(B, B')$ is a positive multiple of q, and so is at least q. By assumption, there exists an element $\gamma \in G_v$ of order p which cyclically permutes at least p neighbors $w_0, w_1, \ldots, w_{p-1}$ of v, where $\gamma^i(w_0) = w_i$, and clearly fixes all other neighbors.

Without loss of generality let $w_0 \in B'$. We claim that w_i belongs to B' for every $i \in \{0, \ldots, p-1\}$. If that was not the case there would be p distinct blocks $\gamma^i(B')$, with $\mathrm{Dist}(B, \gamma^i(B')) \geq 1$, $i \in \{0, \ldots, p-1\}$. Consequently, v would have at least q neighbors in each of these p blocks and so the valency of X would be at least pq, which contradicts the assumption. We conclude that each $w_i \in B'$. It follows that $\mathrm{val}(B, B') \geq p$. But $\mathrm{val}(B, B')$ is a multiple of q, and hence at least mq. □

Proposition 2.7. *Let $p > q$ be primes and let X be a connected vertex-transitive graph admitting a transitive solvable $\{p, q\}$-group G, and let M be a minimal normal elementary abelian subgroup of G. Then one of the following possibilities occurs:*

(i) G contains a semiregular subgroup, or

(ii) $M \cong \mathbb{Z}_q^k$ and $\mathrm{val}(X) > mq$, where mq is the smallest multiple of q exceeding p, or

(iii) $M \cong \mathbb{Z}_p^k$ and $\mathrm{val}(X) > p$.

Proof. Assuming that G does not contain a semiregular subgroup and assuming that $M \cong \mathbb{Z}_q^k$ we have, by Lemma 2.6, that the valency $\mathrm{val}(X)$ of X is at least mq. Further if it is exactly mq then the imprimitivity block system $\mathcal{B}$ arising from the orbits of M consists of two blocks alone, that is, $\mathcal{B} = \{B, B'\}$ with valency $\mathrm{val}(B, B') = mq$. It follows that X is a bipartite graph (with bipartition $\{B, B'\}$). As the blocks of $\mathcal{B}$ have order q^j for some j it must be that either X has order a power of 2 or $p = 2$. But $q < p$ which is not possible. Therefore $\mathrm{val}(X) > mq$. Finally, suppose that $M \cong \mathbb{Z}_p^k$. Then the nonexistence of a semregular subgroup implies that there must exist a pair of adjacent blocks B and B' in $X/\mathcal{B}$ such that in the above defined pseudometrc Dist we have $\mathrm{Dist}(B, B') \geq 1$. This implies that $\mathrm{val}(X) \geq p$. But if $\mathrm{val}(X)$ was equal to p then a semiregular automorphism could be produced in an analogous way to the previous case. □

3 Conclusions

Special cases for valencies 3 and 4 of Lemma 2.6 and Proposition 2.7 played an important role in the proofs of these results, see [6, 19]. For example, in a cubic vertex-transitive graph an automorphism of prime order greater than 3 is clearly semiregular. In fact, in a vertex-transitive graph an automorphism of order greater than the valency of the graph is necessarily semiregular. Therefore, in order to complete the proof of the existence of semiregular automorphisms in such graphs it suffices to deal with graphs having a group of automorphisms which is a $\{2, 3\}$-group. Clearly, Proposition 2.7 applies. As for quartic vertex-transitive graphs again assuming that all automorphisms are of order 2 and 3, and hence a group in question is a $\{2, 3\}$-group, Proposition 2.7 implies that the minimal normal elementary abelian subgroup M has to be a 3-group, and furthermore that the quotient graph with respect to the imprimitivity block system arising from the orbits of M is a multicycle of even length obtained from a cycle with every second edge replaced by a triple of edges. A delicate analysis of this case is then needed in order to prove the existence of a semiregular automorphism (see [6]).

An obvious possible next goal would be to prove the existence of semiregular automorphisms in quintic vertex-transitive graphs. In 2007 Giudici and Xu [12] proved that all vertex-transitive locally-quasiprimitive graphs have a semiregular automorphism, implying that arc-transitive graphs of prime valency have semiregular automorphisms. This result combined together with the above remark about automorphisms of order greater than the valency of the graph allows us to assume that the group of automorphisms in question is a $\{2, 3\}$-group. Two cases may occur depending on whether the elementary abelian subgroup is a 2-group or a 3-group. If $M \cong \mathbb{Z}_2^k$ then, using Proposition 2.7, one can prove that the quotient graph with respect to the imprimitivity block system arising from the orbits of M is a multicycle of even length obtained from a cycle with every second edge replaced by a quadruple of edges. It is reasonable to expect that an approach similar to the one used in [5] for the quartic case would result in a construction of semiregular automorphisms. If on the other hand $M \cong \mathbb{Z}_3^k$, two possibilities needing further analysis may arise from Proposition 2.7. First, the quotient graph is a multicycle of even length obtained from a cycle with every other edge replaced, respectively, by a pair of edges and a triple of edges.

Second, the quotient graph is a vertex-transitive multigraph obtained from a cubic graph with every edge in a perfect matching replaced by a triple of edges. A complete solution for quintic vertex-transitive graphs depends heavily on a successful analysis of these three remaining cases.

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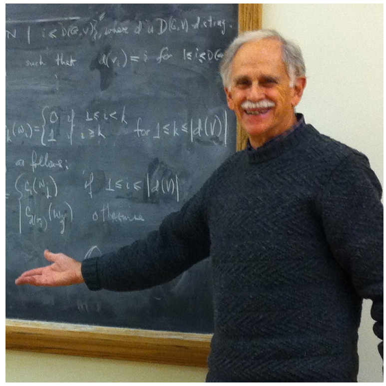
A Tribute to Mark E. Watkins on the Occasion of his 80th Birthday



A generation of undergraduate students passing through the halls of Carnegie Library at Syracuse University recall the name of Prof. Watkins as synonymous with rigorous Calculus classes taught by a passionate and engaging man with a moustache. Doctoral students likely have more personal memories of Mark, whether they be of his mentorship through coursework in graph theory and combinatorics, fall campouts in the Adirondacks, or simply conversations in the corridors.

Outside of the classroom, Mark was a constant presence at department colloquia and the much appreciated Coffee Time, where students and faculty take some time in the afternoon to relax and mingle. Mark could often be found on these afternoons regaling the graduate students with stories of some of the great names in graph theory, along with his close friend and collaborator Jack Graver. For many years, Mark and Jack were the conveners of a weekly combinatorics seminar at Syracuse, drawing together colleagues from neighboring universities as well as from neighboring departments within Syracuse University. Many students (this author included) were drawn to first appreciate and then to deeply love topics in graph theory and combinatorics through this seminar.

A native of the suburbs of Philadelphia, Pennsylvania in the United States, Mark first earned an AB at Amherst College in Massachusetts before proceeding to graduate study at Yale University, where he completed a PhD in 1964 under the direction of Oystein Øre [11]. He then spent several years at the University of North Carolina at Chapel Hill and one year at the University of Waterloo before arriving at Syracuse University as an Associate Professor in 1968. He has worked for the entirety of his career in the field of graph theory, beginning his work in areas of connectivity (e.g., [8]) before moving to more algebraic graph theory (e.g., [1, 5, 7]). Among the highlights of an excellent career, Mark is responsible for naming the generalized Petersen graphs ([1, 12]), for posing the problem of graphical regular representations ([9, 10, 15]), and for a long series of articles and investigations into the nature and structure of various families of infinite graphs (e.g., [3, 4, 6, 7, 13]). Mark has mentored six Ph.D. students through his career: James Uebelacker and Alwin Green 1972, John Kevin Doyle in 1976, Jennifer Ann Bruce in 2002, and finally Adam McCaffery and me in 2009. Mark has traveled extensively pursuing his love of mathematics, with academic terms in Vienna, Waterloo, and Paris, as well as scholarly visits to Oberwolfach, West Berlin, Montréal, Auckland, Marseille-Luminy, Canberra, Leoben, and Ljubljana. Mark has coauthored three books, *Combinatorics with Emphasis on the Theory of Graphs* in 1977 with Jack Graver [2], the AMS Memoir *Locally Finite, Planar Edge-transitive Graphs* in 1997 also with Jack Graver [3], and *Passage to Abstract Mathematics* in 2011 with Jeffrey Meyer [14].

Beyond academia, Mark spent many years playing oboe and English horn until 2006, when due to the effect of medical difficulty he was required to take up trombone. He is an avid outdoorsman, having been a hiker, camper, canoeist, and kayaker for many years, as well as a swimmer and cyclist. For many years, Mark and Jack Graver sponsored a weekend camping trip for graduate students to the Adirondack mountains of New York, and he still enjoys such adventures. In 2012 Mark retired from Syracuse University, and is now an Emeritus Professor of Mathematics. His mathematical contributions have not retired, however, and he continues to work and publish with several prior coauthors.

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Graphs, groups, and more: celebrating Brian Alspach's 80th and Dragan Marušič's 65th birthdays

Koper, Slovenia, May 28 – June 1, 2018

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It is our great pleasure to announce the conference "Graphs, groups, and more: celebrating Brian Alspach's 80th and Dragan Marušič's 65th birthdays" that will take place in Koper, Slovenia, from 28 May to 1 June, 2018. The conference is dedicated to our colleagues Brian Alspach, on the occasion of his 80th birthday, and Dragan Marušič, on the occasion of his 65th birthday.

Brian Alspach has had a distinguished career, primarily at Simon Fraser University in Canada. Since his retirement in 1999 he has held adjunct positions at the Universities of Regina (Canada) and Newcastle (Australia). He supervised 13 PhD students and has been a keen advocate and mentor for young mathematicians, and for establishing innovative programs of study. His research interests have included permutation groups and their actions on graphs; tournaments and digraphs; decompositions and factorizations of graphs; Hamilton cycles and other cycles in graphs, and more.

As one of the first mathematicians from the former Yugoslavia with a PhD obtained abroad, Dragan Marušič returned home to take a teaching position at the University of Ljubljana after having spent ten years at various universities in England and USA. He then moved to the University of Primorska where he now serves as the third rector. He has had a profound influence on mathematics in Slovenia, and is regarded as the founder of the Slovenian school of algebraic graph theory. He has been passionately involved in the promotion of mathematics and mathematicians in Slovenia, having supervised 7 PhD students and mentored many other young mathematicians. His research interests focus on the concept of symmetry in the broadest sense, with permutation groups and their actions on graphs as a primary "point of interest".

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