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A practical algorithm for the computation of the genus

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Abstract

We describe a practical algorithm to compute the (orientable) genus of a graph, give results of the program implementing this algorithm, and compare the performance to existing algorithms. The aim of this algorithm is to be fast enough for many applications instead of focusing on the theoretical asymptotic complexity.
Apart from the specific problem and the results, the article can also be seen as an example how some design principles used to carefully develop and implement standard backtracking algorithms can still result in very competitive programs.

Keywords: Genus, NP-complete, backtracking.

Math. Subj. Class. (2020): 05C10, 05-04, 05C85

1 Introduction

Algorithms are studied in two different ways. First, as underlying methods of computer programs. These algorithms only come to their right when implemented as a program and used as a tool. Second, as objects of research themselves. In the latter case the emphasis lies on determining the asymptotic complexity of a problem – that is: of optimal algorithms solving the problem – often without the intention or realistic possibility to transform the described algorithms to a useful computer program. Even if such algorithms could be implemented and used, many would be extremely inefficient for real problem sizes and outperform standard algorithms only for problem sizes far beyond the limit where either of them can be used. Of course there are also some nice cases – e.g. the linear time planarity algorithm described in [3] – where both aspects meet and algorithms with the best asymptotic behaviour also perform well in practice.

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The general difference between these two approaches can very well be illustrated at the example of the problem of determining the genus of a graph, which is defined as the genus of the smallest orientable 2-manifold so that the graph can be embedded on its surface without crossing edges. The problem is NP-complete [16], but for any fixed g there is a linear time algorithm that can compute the genus $g' \leq g$ or decide that it has genus larger than g [14]. On the other hand, there is no algorithm implemented and available that is at least guaranteed to compute the genus of a single sparse graph with, say, 80 vertices, or determine that it has genus larger than, say, 20, in one year of CPU time.

Nevertheless, the interests in genus computations remain alive. Even lately papers have been published that theoretically determine the genus of specific relatively small graphs or describe algorithms fine tuned for these graphs [7, 8, 12]. In addition to that, researchers have also started to develop general purpose genus computation programs. In [1] such an algorithm based on an integer linear programming approach was published. Later, an improved approach described in [6] was – although also being a general purpose algorithm – able to achieve many of the results formerly obtained by individual theoretical research, automatically in a reasonable amount of time. Unfortunately, some of these programs are neither publicly available nor easy to use.

In this article we will describe an algorithm that – in spite of also being exponential already for small genus – outperforms these approaches for moderate genus is widely usable. The program *multi_genus* based on this algorithm has the options to compute the genus of a graph, one or all minimum genus embeddings, one or all embeddings on an orientable surface of given genus, or to filter large lists of graphs for those with genus at most or at least a given bound. When choosing for *all embeddings*, for graphs with a nontrivial symmetry group isomorphic embeddings can be generated, but no two embeddings that are (labelled) mirror images are generated. We use a carefully designed backtracking algorithm.

2 The algorithm and its implementation multi_genus

We assume all input graphs to be simple connected graphs. The embeddings are constructed by interpreting each undirected edge as a pair of oppositely directed edges. We build a *rotation system*, that is a cyclic ordering of all directed edges starting in a vertex and interpret this ordering as clockwise. Faces of an embedded graph are defined by the usual face tracing algorithm (see e.g. [10]) starting from a directed edge (v, w) and constructing the face containing (v, w) by going to the inverse edge (w, v), and then to the next edge in the orientation around w. This process is repeated until being back at (v, w). A face f is thus a set of oriented edges. We will use the following notation for a given embedded graph: with f^v we denote the set of all vertices contained in one of the directed edges of a face f and with $f(e)$ we denote the number of directed edges in the face containing the directed edge e. The set of all faces is denoted by F, the set of all vertices by V, and finally the set of all edges by E. Following the Euler formula, the genus g of the embedded graph is $g = 1 + (|E| - |V| - |F|)/2$.

Preprocessing

Vertices of degree 1 are irrelevant when computing the genus – they can simply be removed without any impact on the genus. Similarly vertices of degree 2 can be replaced by an edge connecting their two neighbours. If this operation produces a double edge, the new edge can be removed too without changing the genus of the graph. Except when all embeddings of a graph on a surface of a certain genus must be computed and there are at least three vertices, these operations are recursively applied before the real computation of the genus begins. This means that e.g. cycles, trees or complete bipartite graphs $K_{2,n}$ are all reduced to a single vertex. After having computed an embedding, reduced vertices are restored. For graphs with minimum degree at least 3 – which is almost always the case when mathematical research about the genus is done – this preprocessing step has of course no impact.

When computing the genus, the algorithm works by first searching for plane embeddings, then embeddings with genus 1, etc. until an embedding is found. The upper bound for the genus of the embedding that is to be constructed is used in the recursive routine embedding edges. When trying to embed the graph in genus $g > 0$, it has already been determined that there are no embeddings of genus at most $g - 1$ and the first embedding of genus g determines the genus of the graph. Sometimes – this also depends on chance – such an embedding can be found relatively fast and the real bottleneck is the complete search for embeddings of genus $g - 1$. By computing a lower bound on the genus, sometimes expensive complete searches can be avoided, but the lower bound must be fast to compute in order to have an advantage over the complete search. We will now first describe a method to compute a (cheap) lower bound:

Computing a lower bound for the genus

When embedding a graph $G = (V, E)$, the values of $|V|$ and $|E|$ are fixed, so a minimum genus embedding is in fact an embedding with a maximum number of faces and if f' is an upper bound on the number of faces in any embedding then $g' = \lceil 1 + \frac{|E|-|V|-f'}{2} \rceil$ is a lower bound on the genus.

A trivial upper bound on the number of faces is $\frac{2|E|}{3}$ as all faces have at least three edges. This lower bound comes for free and is always computed and used. Instead of the constant value 3, except for trees one could also use the girth of the graph, but that would also have to be computed. The following methods give a better bound if there are few cycles of minimum length.

For a given embedding, let $s[]$ denote the vector of size $2|E|$ indexed from 1 to $2|E|$ containing all values $f(e)$ of directed edges e in non-decreasing order. Then $|F| = F(s) = \sum_{i=1}^{2|E|}(1/s[i])$. A vector $s'[]$ of size $2|E|$ with $s'[i] \leq s[i]$ for $1 \leq i \leq 2|E|$, is said to be dominated by $s[]$. For a vector $s'[]$ dominated by $s[]$ we have $F(s') = \sum_{i=1}^{2|E|}(1/s'[i]) \geq f$.

We call a cyclic sequence $e_0, \ldots, e_{k-1}$ of k pairwise distinct directed edges a *facial-like walk* if and only if for $0 \leq i < k$ the starting vertex of $e_{i+1 \pmod k}$ is the end vertex of e_i and $e_{i+1 \pmod k}$ is the inverse $(e_i)^{-1}$ of e_i if and only if the degree of the end vertex of e_i is one. A first approximation $s_0[]$ of $s[]$ is obtained by taking for each directed edge e the length $f_w(e)$ of the shortest facial-like walk containing e. The value of $f_w(e)$ can be easily computed by a Breadth First Search.

As each facial walk in an embedded graph is also a facial like walk, we see immediately that the non-decreasing sequence $s_0[]$ is dominated by $s[]$, as $f_w(e) \leq f(e)$ for each directed edge e. Especially $s_0[2|E|] \leq s[2|E|]$ and as in $s[]$ at least $s[2|E|]$ edges – all directed edges in a longest facial walk – have value $s[2|E|]$, we can replace the last $s_0[2|E|]$ values of $s_0[]$ with $s_0[2|E|]$ and get another sequence $s_1[]$ dominated by $s[]$. We use $F(s_1)$ as a first nontrivial upper bound on the number of faces.

In fact the length of the shortest facial-like walk is the same for a directed edge and its reverse, but unless the graph is a cycle, one facial-like walk that does not also contain the reverse edge, can only form a face for at most one of them. This observation might lead to a better approximation, but in order to keep the computation of the approximation easy and fast, the length of the shortest facial-like walk is used for a directed edge and its inverse.

An angle α of a face is a pair of directed edges, following each other in the facial walk. The central vertex of the angle is the endpoint of the first edge – so except when this vertex has degree 1 it is the only common vertex of the two edges. In what follows we use that for an edge e and its inverse e^{-1} we have $f_w(e) = f_w(e^{-1})$.

Instead of summing over all edges, we can sum over all angles. With $f(\alpha)$ the size of the face that contains α and $A(v)$ the set of all angles with central vertex v, we have $|F| = \sum_{v\in V}(\sum_{\alpha\in A(v)} 1/f(\alpha))$. If for a vertex v the sequence $s'_v[1],\dots,s'_v[\deg(v)]$ is the non-decreasing sequence of all $f(\alpha)$ with $\alpha \in A(v)$, then $|F| = \sum_{v\in V}(\sum_{1\le i\le \deg(v)} 1/s'_v[i])$. Taking for each vertex v a vector dominated by $s'_v[]$ we again get an upper bound on $|F|$. If we take for a vertex v and each angle $\alpha \in A(v)$ instead of $f(\alpha)$ the value $\max\{f_w(e), f_w(e')\}$, with e, e' the edges in the angle, we get a non-decreasing sequence $s'_{0,v}[]$ dominated by $s'_v[]$. If $s'_{1,v}[]$ is the non-decreasing sequence of values of $f_w(e)$ with e starting at v, then we define $s'_{2,v}[] = s'_{1,v}[2], s'_{1,v}[3],\dots, s'_{1,v}[\deg(v)], s'_{1,v}[\deg(v)]$. So we remove the smallest value of $s'_{1,v}[]$ and add a copy of the largest value.

Remark 2.1. Let $G = (V, E)$ be an embedded graph. Then for each vertex $v \in V$ the sequence $s'_v[]$ dominates $s'_{2,v}[]$.

Proof. We know that $s'_v[]$ dominates $s'_{0,v}[]$. We will show that $s'_{0,v}[]$ dominates $s'_{2,v}[]$. As the maximum values of $s'_{0,v}[]$, $s'_{1,v}[]$ and $s'_{2,v}[]$ are the same, it is sufficient to prove $s'_{0,v}[i] \ge s'_{2,v}[i] = s'_{1,v}[i+1]$ for $i < \deg(v)$. Let $\alpha_1,\dots,\alpha_i$ be the angles (that is: pairs of edges) determining the values $s'_{0,v}[1],\dots,s'_{0,v}[i]$ and $S_{i,v}$ be the set of all directed edges starting at v, so that e or e^{-1} is in at least one of these angles. Then the value of $s'_{0,v}[i]$ is $\max\{f_w(e) | e \in S_{i,v}\}$ (here we use that $f_w(e) = f_w(e^{-1})$) and as $|S_{i,v}| \ge i+1$, we have that $s'_{0,v}[i] \ge s'_{1,v}[i+1]$. □

We use $\sum_{v\in V}(\sum_{1\le i\le \deg(v)} 1/s'_{2,v}[i])$ as a second nontrivial upper bound on the number of faces.

These upper bounds on the number of faces and the corresponding lower bounds on the genus are relatively fast to compute. Nevertheless they do not always speed up the program. Especially for small graphs or small genus they can even slow down the program, as the embedding algorithm can exclude low genus embeddings very fast. While for few small graphs this is no problem, for very large lists of small graphs it can have some impact. Of course the bounds can only increase the running time of the program by a small factor and never by a large factor, but they can sometimes speed the program up a lot:

In this article, all running times for the C-program multi_genus implementing the algorithm described here are on an *Intel Core i7-9700 CPU @ 3.00GHz* (running on one core at 4.4-4.7 Ghz). The prefix *multi* stands for the graph coding that is accepted as input: multi_code. Examples for the impact of the computation of a lower bound when computing the genera of graphs are:

All bipartite graphs on 14 vertices with degrees between 5 and 6 (73 graphs, genus 3 to 5): without lower bound 60.9 seconds, with lower bound 0.035 seconds.

All cubic graphs on 22 vertices ($7,319,447$ graphs, genus 0 to 3): without lower bound 300 seconds, with lower bound 364 seconds.
Checking $1,000,000$ random cubic graphs on 50 vertices, generated by *genrang* (which is part of the nauty-package [13]) for being planar: without lower bound 18.5 seconds, with lower bound 56.2 seconds.
Checking the same $1,000,000$ random cubic graphs on 50 vertices for having genus at most 1: without lower bound 144.8 seconds, with lower bound 85.5 seconds.

The default is that the nontrivial bounds are used, but the use can be switched off by an option to multi_genus.

Constructing an embedding

We begin by relabeling the graph in a BFS way. The time necessary to compute the genus can differ a lot for isomorphic graphs depending on the labelling. In some cases a BFS labelling results in a better performance, in others it slows down the program. We have chosen for the BFS labeling as the results for different, but isomorphic, input graphs often differ less when always using such a labeling. An example showing the large differences that can still occur can be seen when computing a genus 7 embedding (that is a minimum genus embedding [11, 4]) of $C_3 \square C_3 \square C_3$. Taking the first graph of the file `ucay27_05_k=06` provided by Gordon Royle in his list of Cayley graphs (and doing BFS), it takes 0.19 seconds to find an embedding, taking the same graph from a program constructing cartesian products and not doing BFS, it takes 6.2 seconds. Taking the graph from the second source and doing BFS, it takes 281 seconds. So even when relabeling the graph in a BFS manner the time still depends on the labeling of the input graph.

The algorithm works by first greedily embedding a subgraph so that for each embedding or its mirror image, the induced embedding of this subgraph is the one constructed. It has genus 0. If the maximum degree is smaller than 3, the graph is a path or a cycle, both of which can be uniquely embedded and only in the plane. Otherwise we construct the initial subgraph by taking a vertex with minimum degree among all vertices with degree at least 3, embedding this vertex and three of its edges in an arbitrary way (thereby fixing the orientation) and greedily extending the three edges – one after the other – to paths until they cannot be made longer. The result of this construction forms the root of the recursion tree of the Branch and Bound algorithm in which the remaining edges are added recursively .

In branch and bound algorithms, the performance is often improved if one manages to reduce the branching at every node of the recursion tree. This is not a mathematical theorem, but more a rule of thumb, as in some cases more branching might be beneficial if it allows earlier bounding. Nevertheless in our case we have chosen to take small branching as the base (but not only) criterion for the order in which the edges are inserted. As an expensive choice of the next edge to insert is sometimes more costly than more branching, we work in three parts:

(i) Before the recursion starts, the edges that are still to be embedded are sorted as $\{x_1, y_1\}$, $\{x_2, y_2\}, \dots, \{x_k, y_k\}$, so that with S_i the initial subgraph with edges $\{x_1, y_1\}, \dots, \{x_i, y_i\}$ added, for $1 \leq i < k$, we have for $\{x_i, y_i\}$ that at least one of x_i, y_i is in S_i and $\deg_{S_{i-1}}(x_i) \cdot \deg_{S_{i-1}}(y_i) \leq \deg_{S_{i-1}}(x_j) \cdot \deg_{S_{i-1}}(y_j)$ for all $i < j \leq k$ for which at least one of x_j, y_j is in S_i. If one of $\deg_{S_{i-1}}(x_i), \deg_{S_{i-1}}(y_i)$ is 0, then also $\deg_{S_{i-1}}(x_i) + \deg_{S_{i-1}}(y_i) \leq \deg_{S_{i-1}}(x_j) + \deg_{S_{i-1}}(y_j)$ for all j

for which one of $deg_{S_{i-1}}(x_j), \deg_{S_{i-1}}(y_j)$ is 0 and the other is in S_i. Informally speaking: first all edges leading from the already embedded part to not yet included vertices are added and then the edges with both endpoints in the embedded subgraph are added. In each case we choose an edge that has the smallest number of possibilities how it can be added if there are no other restrictions.

During the recursion we always have an upper bound `max_genus` on the genus. As long as this bound is not reached, edges can be inserted in all possible angles – sometimes increasing the genus and sometimes not. In order to reach this bound as fast as possible and therefore be able to reduce the branching, we check at each node of the recursion tree whether there is an edge that cannot be embedded in any existing face of the partial embedding – we call such an edge a *critical edge* – and therefore always increases the genus. If there is such an edge and the partial embedding has already genus `max_genus`, we can backtrack, otherwise the first such edge in the list gets priority above other edges that do not have this property and is inserted first.

We distinguish two cases:

(ii) If the recursion is still close to the root node and the genus of the partial embedding is still smaller than `max_genus`, we have relatively few nodes and the impact of a smaller branching is large. In this situation, also more expensive tests can pay and we do not only check for the existence of a critical edge, but do in fact look for an edge which has the smallest number of faces into which it can be embedded and take such an edge as the next one to be embedded. Among all edges with the same number of faces where they can be embedded, the first one in the sorted list is taken. Note that it is possible that an edge can be embedded into a face in more than one way, but this is not taken into account when counting the number of faces.

(iii) Close to the leaves of the recursion tree we have many nodes and the impact of a smaller branching is small. In this situation, or when the genus of the partial embedding is already `max_genus`, we only check for the existence of a critical edge.

The decision when we consider a node of the recursion to be close to the root or close to the leaves has an impact on the performance, but a simple rule for the optimal moment to switch is hard to determine. Tests on different kinds of graph showed that considering nodes where at most half of the edges (edges of the initial tree not counted) are embedded as close to the root and considering the others as close to a leave is often a good compromise.

The method to find critical edges fast is crucial for the performance. An edge can be embedded into a face if both endpoints of the edge are in the same face. In the C-implementation we use bit vectors – that is C data types: integers of type *unsigned long int* (64 bit) or *unsigned __int128* to represent sets of vertices. Especially for graphs with up to 64 vertices this allows to determine whether an edge can be embedded in a face in few CPU cycles – provided the fact that the set of end vertices of the edge and the set of vertices of the face are represented as bit vectors. Unless otherwise mentioned, up to 64 vertices the version using *unsigned long int* is used for the timings and the version using *unsigned __int128* for larger graphs on up to 128 vertices.

Lemma 2.2. *Assume that an algorithm to embed a graph $G = (V, E)$ starts with embedding a spanning tree and then inserts the remaining edges of a graph step by step, but in each step inserts non-critical edges only if there are no critical edges. Let $G' = (V', E')$*

be a subgraph with at least one cycle that was embedded by this algorithm and let e *be the last edge that was inserted in one face* f *and split it into two faces with vertex sets* f_1^v, f_2^v*. If there is a critical edge* e_c *for* G'*, then* $|e_c \cap (f_1^v \setminus f_2^v)| = |e_c \cap (f_2^v \setminus f_1^v)| = 1|$.

Proof. Note first that G' need not have faces with vertex sets f_1^v, f_2^v. The faces f_1, f_2 can not have been subdivided, as they are the result of the last subdivision, but after that subdivision they might have been united with other faces (or with each other) when an edge with endpoint in two different faces was inserted.

If there is no critical edge, the statement is trivially true, so assume that there is a critical edge e_c. Let G_0 be the embedded subgraph into which e was embedded to form G_1. As e was inserted into a face, there was no critical edge for G_0, so e_c could be embedded into a face f_0 of G_0 (so $e_c \subset f_0^v$). After e only critical edges were inserted, so the vertex sets of all faces of G' are unions of those of G_1. If $f \neq f_0$, $f_0^v \subseteq f_0'^v$ for some face f_0' of G', so e_c could be embedded into f_0' and would not be critical. So we have $f = f_0$. As $(f_1^v \setminus f_2^v) \cap (f_2^v \setminus f_1^v) = \emptyset$ it is sufficient to show that $e_c \cap (f_1^v \setminus f_2^v)$ and $e_c \cap (f_2^v \setminus f_1^v)$ are both not empty. Assume that (w.l.o.g.) $e_c \cap (f_1^v \setminus f_2^v) = \emptyset$. As $f_1^v \cup f_2^v = f^v$, we have $e_c \subset f_2^v$, but as f_2^v is a subset of the vertex set of a face of G', e_c would not be critical. So $e_c \cap (f_1^v \setminus f_2^v) \neq \emptyset$. □

Finding critical edges is a nontrivial task. The straightforward way is a loop over all edges that still need to be inserted and inside this loop a loop over all faces of the embedded graph. The previous lemma gives a very cheap criterion to decide for many edges that they are not critical – without using the inner loop. In fact one could even make a list of all candidates for critical edges whenever a face is subdivided, but in the implementation this is not done.

Performance

There are some programs available, where the exact algorithm is not published – e.g. *simple_connected_genus_backtracker* in the computer algebra package sage. As a backtracking program it seems to be related to the algorithm described here. The manual says that it is *an extremely slow but relatively optimized algorithm. This is "only" exponential for graphs of bounded degree, and feels pretty snappy for* 3*-regular graphs.* It also says that K_7 *may take a few days*, while multi_genus takes less than 0.001 seconds for K_7. So we tested it only for cubic graphs, but already for relatively small vertex numbers, it is very slow, e.g. more than 24 minutes for the unique cubic graph with girth 8 on 34 vertices (instead of less than 0.001 seconds of multi_genus), so tests on a larger scale were not possible.

The program used in the graph database *House of Graphs* – short HoG [5] at the moment (it will be replaced by multi_genus) is much faster. It is a Java program called *MinGenusEmbedder* written by Jasper Souffriau as a student project and it is also a backtracking algorithm using branch and bound. That program was also used for independent tests. For the generation of random graphs we use the program genrang [13] which allows to restrict the generation to regular graphs of a given degree or to graphs with a given number of edges. As genrang also generates graphs that are not connected, we filtered them for connected graphs. If we say that we tested n random graphs generated by genrang, this means that we generated random graphs by genrang and took the first n connected ones. In order to have the results completely reproducible, we always fixed the seed used by genrang to 0.

For 2000 random cubic graphs on 30 vertices, MinGenusEmbedder needed 16.3 seconds (compared to 0.6 seconds of multi_genus – a factor of 27) and for 2000 random cubic graphs on 40 vertices, MinGenusEmbedder needed 435 seconds (compared to 12.5 seconds, a factor of 34). For larger degrees the ratio grows. For 30 quartic graphs on 30 vertices MinGenusEmbedder already needs 792 seconds (compared to 6.6 seconds, a factor of 120) and for 30 5-regular graphs on 22 vertices 2, 226 seconds (compared to 13.7 seconds, a factor of 163). For 6-regular graphs, testing 30 graphs on 19 vertices already took quite some time: 10.56 hours for MinGenusEmbedder and 2 minutes for multi_genus (a factor of 315). Of course such small samples are not sufficient for reliable results and we should see these numbers just as a hint what the relation of the running times might be. Unfortunately the running times do not allow tests on large sets of data.

The fastest published general purpose program is the integer linear programming approach described in [6] and implemented in the program ILP^D_{Real} . The program $\mathrm{ILP}^{\mathrm{D}}_{\mathrm{Real}}$ is not publicly available, so we compare the running times for a data set they used in [6]: the Rome graphs, which can be downloaded from `http://graphdrawing.org/data.html`. This set of graphs contains 11, 534 graphs with (at least indicated by the file names) up to 100 vertices which are used e.g. for graph drawing and are said to come from practical applications. Among these graphs, 3 are disconnected and 3, 279 planar. In [6] only nonplanar graphs were tested. Note that the set of Rome graphs not only contains isomorphic graphs, but even identical copies. Some files also seem obscure: e.g. in `grafo6975.39.graphml` due to the otherwise used convention, there should be a graph with 39 vertices. Nevertheless it has 105 vertices. We filtered out the disconnected graphs, the planar graphs, and the *obscure* ones and – like [6] – received a list of 8, 249 nonplanar graphs for which the genus had to be computed. It should be mentioned that the sizes of the Rome graphs are a bit misleading when it comes to estimating the complexity of computing the genus: many of the graphs have vertices of degree 1 and 2, which do not increase the complexity of the computation of the genus. In [6] a Xeon Gold 6134 CPU was used to compute the genus of these graphs. For each graph a time limit of 10 minutes and a memory limit of 8 GB was given. With these restrictions $\mathrm{ILP}^{\mathrm{D}}_{\mathrm{Real}}$ was able to decide 82% of the instances. For multi_genus the memory consumption is negligible. On the Core i7 it could determine the genus of 98.57% of the graphs within the same time limit of 10 minutes and even with a time limit of only 0.25 seconds for each graph, it can decide 83.7% of the cases.

In Figure 1 the development of the running times for random cubic and quartic graphs are given. For all sizes the version for more than 64 vertices using *unsigned __int128* was used in order not to have some misleading behaviour around 64 vertices. As expected, for given fixed degree of the vertices, the measured times depend exponentially on the number of vertices.

If we fix the number of vertices, but vary the number of edges, we get – as expected – again an exponential growth, as shown in Figure 2.

Together with the number of vertices and edges, also the average genus increases, so it is also interesting to know how the running times develop, when it is only tested whether the graphs can be embedded in a surface of given genus – similar to planarity testing. In Figure 3 the running times for testing whether 1000 random cubic resp. quartic graphs can be embedded in a surface of genus at most 3 are given. In fact for practically all of the larger graphs tested, the answer was *no*. Nevertheless it is astonishing that from a certain point on the time necessary to test a graph decreases again.

If one wanted to apply the algorithm to perform very well when testing only planarity, one could apply the reasoning of Demoucron, Malgrange and Pertuiset [9] as soon as a

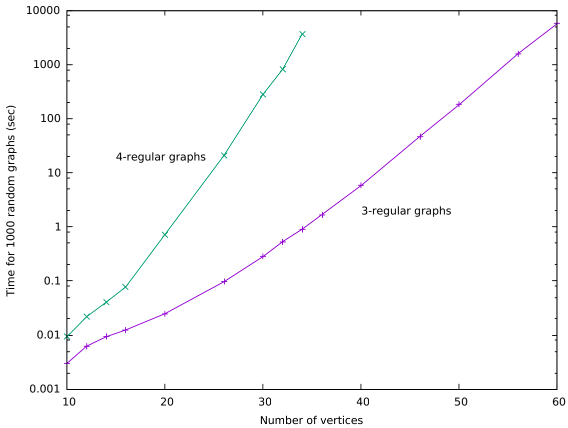

Figure 1: The running times for computing the genus of 1000 random cubic, resp. quartic graphs. Note that the time scale is logarithmic.

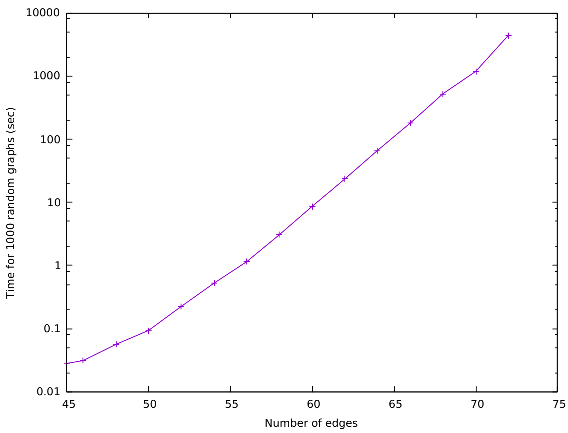

Figure 2: The running times for computing the genus of 1000 random graphs with 32 vertices and a given number of edges. Note that the time scale is logarithmic.

spanning tree and the first cycle is formed. Of course that would not be useful, as specialized and very efficient algorithms for planarity testing exist. Although no special adaptation for the planar case is implemented and this is not the task the algorithm was developed for, a comparison with specialized programs for planarity testing might be interesting. In [3] a practical linear time algorithm for planarity testing is presented. In the program *planarg* [13] an implementation of this algorithm by Paulette Lieby is available.

Testing $100,000$ random cubic graphs on 50 vertices with planarg and multi_genus with 0 as an upper bound for the genus, the times are 2.29 (planarg) resp. 5.5 (multi_genus) seconds. Testing $100,000$ random cubic graphs on 100 vertices, planarg is more than 4 times faster than multi_genus (5.0 seconds to 22.1 seconds). All these graphs were non-planar. Filtering the $11,529$ rome graphs for non-planar ones took 0.36 seconds with planarg and 0.29 seconds with multi_genus. In this case about 28% of the graphs were planar. In order

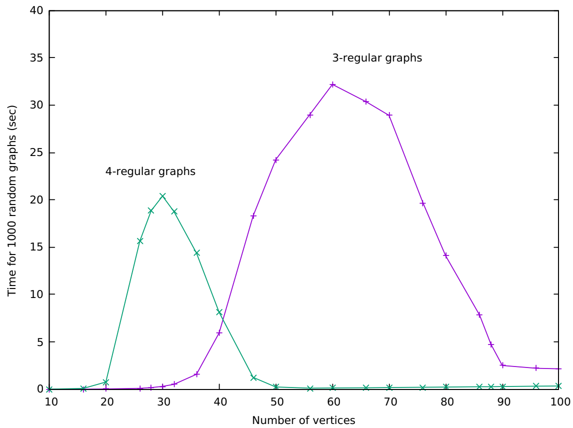

Figure 3: The running times for testing 1000 random cubic, resp. quartic graphs for having genus at most 3.

to test how fast embeddings are found if they exist, once sparse planar graphs – that is cubic graphs – were tested and once dense graphs – that is triangulations. For these tests, for multi_genus no lower bound for the genus was computed. We used the $285,914$ fullerenes on 100 vertices and their duals (triangulations on 52 vertices) randomly relabeled in order to avoid an impact of the embedding produced by the generation program. For the fullerenes, planarg needed 19.5 seconds (compared to 19.1 seconds for multi_genus) and for the triangulations planarg needed 11.85 seconds (compared to 11.1 seconds for multi_genus).

Testing

In order to test the implementation, independent programs were used and the results were compared to the results of multi_genus. For all cases tested the results agreed.

In order to test the option to compute the genus, the genus of each (connected) graph in the following set was computed by multi_genus and MinGenusEmbedder and the result was compared. The sets of graphs are: all graphs on up to 11 vertices, all 3-regular graphs on up to 24 vertices, all 4-regular graphs on up to 16 vertices, all 5-regular graphs on up to 14 vertices, all 3-regular graphs on up to 26 vertices with girth at least 5, all 3-regular graphs on up to 28 vertices with girth at least 6, all 3-regular graphs on up to 34 vertices with girth at least 7, all 3-regular graphs on up to 44 vertices with girth 8, all $(3, 9)$-cages – that is all 3-regular graphs on 58 vertices with girth 9, all 4-regular graphs on up to 24 vertices with girth 5, and finally all graphs with valence vector $(2, 2, 3, 3, 3)$.

In order to test the option that makes multi_genus generate all embeddings of a certain genus, a simple independent program was implemented that generated all combinations of all vertex orders around the vertices. Filtering the embeddings generated this way for those with a given genus we had a very slow but independent test. Then for each graph in one of the following sets, the range of possible genera was computed by the Euler formula and for each graph and each possible genus the embeddings of this genus were independently generated and the number of embeddings as well as the number of non-isomorphic embeddings (computed by an isomorphism checking program using lists) were compared. Due to the enormous number of embeddings already for small graphs, not too many graphs and no large graphs could be tested. The sets of graphs are: all 3-regular graphs on up to 18 vertices, all graphs on 7 vertices with 6 to 17 edges, all graphs on 8 vertices with 14 and with 15 edges, all graphs with valence vector $(1, 1, 1, 5)$, and finally all graphs with valence vector $(0, 2, 3, 2, 3)$ and girth at least 4.

Results obtained or confirmed by multi_genus

Times given for multi_genus in this section are again on an *Intel Core i7-9700 CPU @ 3.00GHz* running on one core at 4.4-4.7 Ghz.

In [15] Plummer and Zha prove a theorem describing the cases when K_{c+1} is the unique c-connected graph with smallest genus – except for the two cases $c = 9$ and $c = 13$ which are not decided and posed as an open question. This question is answered in [2] showing that in these cases the complete graphs are not unique, but that in these cases the graphs M_{c+2} on $c + 2$ vertices, obtained by deleting a maximum matching from K_{c+2}, have the same genus as K_{c+1}. The embeddings given in that article were computed by multi_genus. Computing the genus $g(M_{11}) = 4$ takes 0.005 seconds and computing the genus $g(M_{15}) = 10$ takes 7 hours and 6 minutes.

In [7] Conder and Grande determine all circulant graphs of genus 1 and 2. A large part of the proof discusses 12 specific circulant graphs and in order to prove that 11 of these graphs have genus larger than 2, next to several pages of theoretical argumentation also more than 80 CPU hours were needed. The program described in [1] confirms these results in 180 hours of CPU time (without additional theoretical arguments), and $\text{ILP}^{\text{D}}_{\text{Real}}$ computes the genera *in a matter of seconds* (the exact value isn't given). Multi_genus confirms the results of the paper in less than 0.03 seconds. Computing the exact genera (once genus 2, 7 times genus 3, 3 times genus 4, and once genus 5) and minimum genus embeddings takes 6.4 seconds.

In [12] the genus of the Gray graph is theoretically determined by a nontrivial construction. $\text{ILP}^{\text{D}}_{\text{Real}}$ confirms this result within 42 hours. Multi_genus confirms this in 28.3 seconds. In order to determine all 258,696 (labeled) minimum genus embeddings (219 non-isomorphic), multi_genus needed a bit less than 10 minutes. Isomorphism rejection is done by an independent program simply storing canonical embeddings in lists.

In [8] the genus (and also non-orientable genus) of several graphs was determined. They describe four specialized approaches they apply to some special graphs that have in general a large symmetry group. One of them – they call it the *subgroup orbit method* – is especially suited for as they write *graphs on surfaces with a certain degree of symmetry* and works well for graphs that allow an embedding with a face-transitive automorphism group. So the approach is not intended for general graphs and the program is also not available for everybody. Our general approach cannot reproduce their results for the Hoffman-Singleton graph, the Ljubljana graph or the Iofinova-Ivanov graph – at least not without an excessive amount of time and/or special adaptations. Some of the other examples they give can also be solved and sometimes extended by our general approach without any manual interference – just by piping the graph into multi_genus. The instances for which the results could be confirmed and sometimes extended are:

The graph $C_3 \square C_3 \square C_3$: In [8] it says that *with a natural vertex labeling* the subgroup orbit method *takes only a couple of minutes* to find a genus 7 embedding. The method described here takes – depending on the labeling – from 0.19 seconds to 281 seconds to find an embedding. Of course there may also be labelings that take even less or even more time. In total there are $188,211,024$ minimum genus embeddings, $145,468$ of them pairwise non-isomorphic, but computing these took almost 3 weeks of CPU time (on another, much older, machine used for the large memory available for isomorphism rejection).

Not only constructing a genus 7 embedding, but also proving its minimality by excluding the existence of an embedding of smaller genus takes between 1.5 and 4 hours depending on the labeling (both with BFS numbering first).

In [11] bounds for the genus of the cartesian product of four or five triangles are given. Determining the genus or useful bounds for the genus of these graphs or of $C_3 \square C_3 \square C_3 \square K_2$ is out of reach of the program described here.

The Tutte graph (or $(3,8)$-cage): In [8] no running times are given, but they construct a genus 4 embedding with cyclic automorphism group of order 3. The present approach takes 0.005 seconds to determine the genus as 4 and 0.13 seconds to construct all $13,440$ embeddings. Among these embeddings there are 15 non-isomorphic embeddings – 4 with a group of order 1, 10 with a group of order 2 (2 of them allowing a reflection) and one with a group of order 3.

The Gray graph: The running times for the Gray graph were already given. In [8] it is reported that there are minimum genus embeddings with an automorphism group of order 6. Checking all possible embeddings, the result is that there are 186 non-isomorphic embeddings with trivial symmetry, 23 with a group of order 2, 4 with a group of order 3, 4 with a group of order 6, and 2 with a group of order 18.

The Folkman graph: For the Folkman graph, in [8] minimum genus embeddings with a group of order 8 are constructed. The method described here takes less than 0.001 seconds to determine the genus as 3 and $0,037$ seconds to construct all $7,680$ minimum genus embeddings. Among these embeddings there are 7 pairwise non-isomorphic, 2 with a group of order 2, 3 with a group of order 4, and 2 with a group of order 8. All groups contain a reflection.

The Doyle-Holt graph: For the Doyle-Holt graph [8] describes a genus 5 embedding with an automorohism group of order 2. The present approach needs 0.23 seconds to determine the genus of the graph and 7.3 seconds to determine all $1,107$ minimum genus embeddings. There are 24 pairwise non-isomorphic embeddings – 17 with a trivial group and 7 with a group of order 2.

The dual Menger graph of the Gray configuration: For this 6-regular graph on 27 vertices, the present approach needs 20 seconds to determine the genus as 6. In a bit more than 26 minutes it constructed all 216 minimum genus embeddings – which turned out to be isomorphic. So the minimum genus embedding of the dual Menger graph of the Gray configuration is unique. It has an automorphism group of order 6.

Conclusion

The program described in this article can be a useful tool and has – among other applications – e.g. be used to solve an old question of Plummer and Zha [15] about the uniqueness of certain complete graphs K_{c+1} as the only c-connected graphs embeddable in a surface of minimal genus into which K_{c+1} can be embedded [2], determine the genus of the Georges-graph, etc. ... Nevertheless computing the genus of a graph is a very difficult problem and as can be deduced from the figures, the time grows fast with the genus and the size of the graph. Also in the future, theoretical approaches – maybe combined with specialized computer programs – will be necessary to determine the genus of some graphs of special interest.

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Multivariate polynomials for generalized permutohedra*

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Abstract

Using the notion of a Mahonian statistic on acyclic posets, we introduce a q-analogue of the h-polynomial of a simple generalized permutohedron. We focus primarily on the case of nestohedra and on explicit computations for many interesting examples, such as S_n-invariant nestohedra, graph associahedra, and Stanley-Pitman polytopes. For the usual (Stasheff) associahedron, our generalization yields an alternative q-analogue to the well-studied Narayana numbers.

Keywords: Generalized permutohedron, h-polynomial, q-analogues.

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1 Introduction

Given any combinatorially defined polynomial, a common theme in enumerative combinatorics is to consider multivariate analogues which further stratify and enrich the encoded data by an additional combinatorial statistic. A notable example of is the *Euler–Mahonian polynomial*

$$A_n(t,q) = \sum_{\pi \in S_n} t^{\mathrm{des}(\pi)} q^{\mathrm{maj}(\pi)}$$

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which is a bivariate generalization of the more foundational *Eulerian polynomial*

$$A_n(t) = \sum_{\pi \in S_n} t^{\mathrm{des}(\pi)},$$

both of which are specializations of the $n-1$ variable polynomial

$$A_n(t_1, t_2, \ldots, t_{n-1}) = \sum_{\pi \in S_n} \prod_{i \in \mathrm{Des}(\pi)} t_i.$$

In this case, we further stratify the descent statistic on permutations by the additional data of the major index. Such a generalization is commonly referred to as a q-analogue in reference to usual choice of added variable.

Given a convex polytope $P \subset \mathbb{R}^n$, the h-polynomial is an encoding of the face numbers of P obtained as a linear change of variables of the generating function for the face numbers. If P is simple or simplicial, then the Dehn–Sommerville equations for P are reflected in the palindromicity of the h-polynomial. For simple rational polytopes, the h-polynomial is the Poincaré polynomial of the cohomology groups of the toric variety attached to the polytope. Moreover, for simplicial polytopes, the h-polynomial is the generating function for facets of P according to the size of their restriction sets [25, Section 8.3].

Generalized permutohedra are a broad class of convex polytopes which exhibit many nice properties. First introduced by Postnikov [20], these polytopes have been the subject of much study and are of wide interest in many areas of algebraic and enumerative combinatorics, including the combinatorics of Coxeter groups, cluster algebras, combinatorial Hopf algebras and monoids, and polyhedral geometry (see, e.g., [2, 4, 15, 16]).

Of particular interest for our purposes, Postikov, Reiner, and Williams [21] give a combinatorial description of the h-polynomial for any simple generalized permutohedron using an Eulerian descent statistic on posets. Moreover, they provide a formula for well-behaved, special cases of generalized permutohedra. We give a bivariate generalization of their description for any simple generalized permutohedron: for P a simple generalized permutahedron and Q_σ the cone poset for a full dimensional cone σ in the normal fan $\mathcal{N}(P)$ (See Definition 2.1), we define

$$h_P(t,q) := \sum_{\sigma \in \mathcal{N}(P)} t^{\mathrm{des}(Q_\sigma)} q^{\mathrm{maj}(Q_\sigma)}$$

where des and maj are statistics defined below. Furthermore, we are able to be more explicit when restricting to particular classes of generalized permutahedra, specifically S_n-invariant nestohedra, graph associahedra, and Stanley–Pitman polytopes.

Our definition of the bivariate h-polynomial, which specializes to the usual h-polynomial is justified by analogy with the Euler–Mahonian polynomial. Other possible definitions exist. An inequivalent definition is the principal specialization of the Frobenius characteristic of the permutohedral toric variety. This definition does not extend to generalized permutohedra and is not discussed in the body of the paper. However, it does make use of the major index.

The structure of this note is as follows. In Section 2, we provide a review of necessary background and terminology on permutations, posets, polyhedral geometry, and generalized permutohedra. Section 3 defines and discusses the the q-analogue for the h-polynomial of any simple generalized permutohedron. In Section 4, we focus on general results for a large class of simple generalized permutohedra called *nestohedra*, including a palidromicity result for special cases. Section 5 is devoted to several explict examples, including S_n-invariant nestohedra, graph associahedra, the classical associahedron, the stellohedron, and the Stanley–Pitman polytope. These examples produce some *alternative* q-analogues of some well-known combinatorial sequences, including the Narayana numbers.

2 Background

In this section, we provide a brief review of basic properties of permutations statistics, posets, polytopes and normal fans, and generalized permutohedra.

2.1 Permutation statistics

Let $A = \{a_1 < a_2 < \ldots < a_n\}$ be a set of n elements. The *symmetric group on* A, denoted S_A, is the set of all permutations of the elements of A. In the case of $A = [n]$, we will simply write S_n. Given $\pi = \pi_1\pi_2 \cdots \pi_n \in S_A$, the *descent set* of π is

$$\mathrm{Des}(\pi) = \{i \in [n-1] \ : \ \pi_i > \pi_{i+1}\},$$

the *descent number* of π is $\mathrm{des}(\pi) = |\mathrm{Des}(\pi)|$, and the *major index* of π is

$$\mathrm{maj}(\pi) = \sum_{i \in \mathrm{Des}(\pi)} i.$$

The descent statistic is commonly referred to as an *Eulerian* statistic, due to the connection to polynomial first studied by Euler [14]. The *Eulerian polynomial* $A_n(t)$ is the unique polynomial which satisfies

$$\sum_{k \geq 0} (k+1)^n t^k = \frac{A_n(t)}{(1-t)^{n+1}}$$

However, this polynomial can be interpreted entirely combinatorially as

$$A_n(t) = \sum_{\pi \in S_n} t^{\mathrm{des}(\pi)}.$$

The major index, on the other hand, is commonly known as a *Mahonian statistic*, as it was introduced by MacMahon [18]. The descent statistic and major index statistic are naturally linked as they both encode information regarding the descent set of a permutation. Thus, it is fruitful to consider the joint distribution of these statistics, which motivates the *Euler–Mahonian polynomial*

$$A_n(t, q) = \sum_{\pi \in S_n} t^{\mathrm{des}(\pi)} q^{\mathrm{maj}(\pi)},$$

which specializes to the Eulerian polynomial under the substitution $q = 1$. This polynomial and various generalizations are widely of interest (see, e.g., [1, 5, 7, 9]).

2.2 Posets

Let Q be a partially ordered set (poset) on $[n]$ with relation $<_Q$. Given $x, y \in Q$, let $x \lessdot_Q y$ denote the covering relation. Two elements $x, y \in Q$ are *incomparable* if we have neither

$x <_Q y$ nor $y <_Q x$. A *chain* in Q, is a collection of elements $x_1, x_2, \ldots, x_k \in Q$ such that $x_1 <_Q \cdots <_Q x_k$. A chain $x_1, x_2, \ldots, x_k \in Q$ is called *saturated* if $x_1 \lessdot_Q \cdots \lessdot_Q x_k$. The *Hasse diagram* of Q is the graph with an oriented upwards direction such that there is an edge from x up to y if and only if $x \lessdot y$. We say that Q is *acyclic* if for all $x, y \in [n]$ with $x <_Q y$ there is a unique saturated chain from x to y.

Given two posets Q_1 and Q_2, the *ordinal sum* $Q_1 \oplus Q_2$ is the poset on the disjoint union of the ground sets of Q_1 and Q_2 such that $x < y$ if

(i) $x, y \in Q_1$ and $x <_{Q_1} y$,

(ii) $x, y \in Q_2$ and $x <_{Q_2} y$, or

(iii) $x \in Q_1$ and $y \in Q_2$.

The poset Q is called *graded* (or *ranked*) if there is a function $\rho\colon Q \to \mathbb{Z}_{\geq 0}$ such that if $x \lessdot_Q y$, then $\rho(y) = \rho(x) + 1$. While there are infinitely many rank functions for a graded Q, there is a unique minimal rank function ρ such that $\rho(x) - 1$ is not a valid rank function.

Given a poset Q on $[n]$, we can generalize the notion of the descent statistic for permutations. The *descent set* of Q is

$$\mathrm{Des}(Q) := \{(i, j) \,:\, i \lessdot_Q j \text{ and } i >_{\mathbb{Z}} j\}$$

and thus the *descent number* of Q is $\mathrm{des}(Q) := |\mathrm{Des}(Q)|$. If Q is a graded poset on $[n]$ with minimal rank function ρ, we further have a notion of *major index* of Q

$$\mathrm{maj}(Q) := \sum_{(i,j) \in \mathrm{Des}(Q)} \rho(j).$$

We note that if Q is a totally ordered set with labels $\pi_1 <_Q \pi_2 <_Q \cdots <_Q \pi_n$, these quantities are precisely $\mathrm{des}(\pi)$ and $\mathrm{maj}(\pi)$.

2.3 Polytopes, fans, and h-vectors

A *(convex) polytope* P is the convex hull of finitely many point $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k \in \mathbb{R}^d$. The *dimension* of P, denoted $\dim(P)$, is the dimension of the smallest affine subspace containing P. A *face* F of P is the collection of points where a linear functional $\ell \in (\mathbb{R}^n)^*$ is maximized on P. Faces of dimension 0 are called *vertices* and faces with $\dim(F) = \dim(P) - 1$ are called *facets*. A polytope P is called *simple* if every vertex is contained in exactly $\dim(P)$ many facets. The set of all faces of P forms a poset $L(P)$ under inclusion of faces, which we will the *face lattice* of P. We say that two polytopes P_1 and P_2 are *combinatorially equivalent* if $L(P_1) = L(P_2)$.

A *polyhedral cone* $\sigma \subset \mathbb{R}^n$ is solution set to the weak inequality $A\boldsymbol{x} \geq 0$ for some real matrix A. A cone σ is called *pointed* if σ contains no linear subspaces. The *dimension* of σ, denoted $\dim(\sigma)$, is the dimension of the smallest affine subspace containing σ. A cone is called *simplical* if it is defined by exactly $\dim(\sigma)$ many independent inequalities. A *face* of σ is the subset obtained by replacing some of the defining equalities with equality. Two cones σ_1 and σ_2 *intersect properly* if $\sigma_1 \cap \sigma_2$ is a face of both σ_1 and σ_2. A collection of cones $\mathcal{F}$ is called a *fan* if it is closed under taking faces, and any two cones $\sigma_1, \sigma_2 \in \mathcal{F}$ intersect properly. We say $\mathcal{F}$ is a *complete fan* if $\mathcal{F}$ covers $\mathbb{R}^n$.

Let P be a polytope with face F. The *normal cone of* F *in* P, denoted $\mathcal{N}_P(F)$ is the subset of linear functions $\ell \in (\mathbb{R}^n)^*$ whose maximum on P occurs at all points of F. That is,

$$\mathcal{N}_P(F) := \{\ell \in (\mathbb{R}^n)^* \ : \ \ell(x) = \max\{\ell(y) \ : \ y \in P\} \text{ for all } x \in F\}$$

The *normal fan* of P, denote $\mathcal{N}(P)$ is the complete fan formed by the normal cones of all faces. Note that $\mathcal{N}(P)$ is pointed if and only if $\dim(P) = n$. However, one can always reduce $\mathcal{N}(P)$ to a pointed fan in the space $(\mathbb{R}^n)^*/P^\perp$, where $P^\perp \subset (\mathbb{R}^n)^*$ is the subset of linear functionals constant on P.

Given a polytope P, the f*-vector* of P is the vector $(f_0(P), f_1(P), \ldots, f_{\dim(P)}(P))$ where $f_i(P)$ is the number of i-dimensional faces of P. The f*-polynomial* of P is the generating function $f_P(t) = \sum_{i=0}^{\dim(P)} f_i(P)t^i$. Moreover, one can define f-vector and f-polynomial of fan $\mathcal{F}$ in the obvious way. The f-vectors of a polytope P and its normal fan $\mathcal{N}(P)$ are related by $f_i(P) = f_{\dim(P)-i}(\mathcal{N}(P))$.

Given P a simple polytope, or equivalently if $\mathcal{F}$ is a simplical fan, one can instead consider a different vector. The h*-vector* of P is the vector $(h_0(P), \ldots, h_{\dim(P)}(P)) \in \mathbb{Z}_{\geq 0}^{\dim(P)+1}$ and the h*-polynomial* is $h_P(t) = \sum_{i=0}^{\dim(P)} h_i(P)t^i$ defined uniquely by the relation $f_P(t) = h_P(t+1)$. Likewise, the h-polynomial of $\mathcal{F}$, $h_{\mathcal{F}}(t) = \sum_{i=0}^{\dim(\mathcal{F})} h_i(\mathcal{F})t^i$ is given by the relation $t^{\dim(\mathcal{F})} f_{\mathcal{F}}(t^{-1}) = h_{\mathcal{F}}(t+1)$. Hence, the h-polynomial of a polytope P and the h-polynomial of its normal fan $\mathcal{N}(P)$ coincide. In this case, it happens that the h-polynomial satisfies the *Dehn-Sommerville relations* $h_i(P) = h_{\dim(P)-i}(P)$ for $i = 0, 1, \ldots, \dim(P)$ (see, e.g., [25, Section 8.3]).

2.4 Generalized permutohedra

Given $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{R}^n$ such that $0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_n$, the α-*permutohedron* or *usual permutohedron* $\Pi_n^\alpha \subset \mathbb{R}^n$ is the convex hull of the S_n-orbit of α. Note that this is an $(n-1)$-dimensional polytope, as it lies in the hyperplane $\sum_{i=1}^n x_i = \sum_{j=0}^n \alpha_j$. Regardless of the choice of α, the normal fan of Π_n^α is the *braid fan* is

$$\mathrm{Br}_n := \{\sigma(\pi) \ : \ \pi \in S_n\} \subseteq \mathbb{R}^n/(1, 1, \ldots, 1)$$

where the full dimensional cones $\sigma(\pi)$ are

$$\sigma(\pi) = \{\boldsymbol{x} \in \mathbb{R}^n/(1, 1, \ldots, 1) \ : \ x_{\pi_1} \leq x_{\pi_2} \leq \cdots \leq x_{\pi_n}\}.$$

See Figure 1 for the example of Br_3. Given that any choice of α produces the normal fan of Br_n, we will usually consider usual permutohedron for $\alpha = (0, 1, 2, \ldots, n-1)$, which we will simply denote Π_n. It is a well-known result that the h-polynomial for Π_n is given by the Eulerian polynomial

$$h_{\Pi_n}(t) = \sum_{\pi \in S_n} t^{\mathrm{des}(\pi)}.$$

Introduced by Postnikov in [20], a *generalized permutohedron* $P \subset \mathbb{R}^n$ is a convex polytope whose normal cone $\mathcal{N}(P) \subset \mathbb{R}^n/(1, 1, \ldots, 1)$ can be refined to Br_n. We say that $\mathcal{N}(P)$ is a *coarsening* of Br_n if there is a polytopal realization for $\mathcal{N}(P)$ which can be refined by Br_n.

Definition 2.1. Suppose that $\mathcal{F}$ is a coarsening of Br_n. Given a full-dimensional cone $\sigma \in \mathcal{F}$, the *cone poset* Q_σ is a poset on $[n]$ given by the relations $i <_{Q_\sigma} j$ if $x_i \leq x_j$ for all $\boldsymbol{x} \in \sigma$.

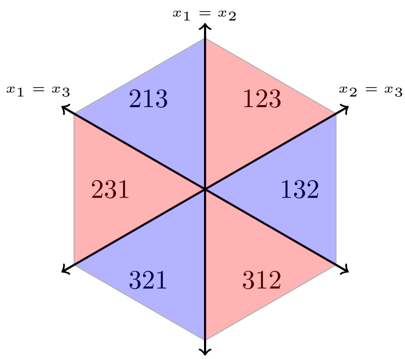

Figure 1: Br_3 in $\mathbb{R}^3/(1,1,1)$.

It follows immediately from the definition that the cone poset Q_σ is connected and acyclic if and only if σ is a full-dimensional simplical cone. For additional exposition and details on this correspondence, the reader should consult [21, Section 3]. Moreover, we make the following observation.

Proposition 2.2. *Given $\sigma \in \mathcal{N}(P)$ be a full dimensional cone for a simple generalized permutahedron P. Then the poset Q_σ is an connected, acyclic, graded poset with a unique minimal rank function $\rho\colon Q_\sigma \to \mathbb{Z}_{\geq 0}$.*

Proof. Since P is simple, this implies that σ is simplicial. As noted above, it follows directly from Definition 2.1 that the poset Q_σ is connecteded and acyclic. This implies that if $x <_{Q_\sigma} y$, there is a unique saturated chain from x to y. Hence, we can define a rank function ρ such that $\rho(x) \geq 0$ for all $x \in Q_\sigma$ and if $x \lessdot_{Q_\sigma} y$ then $\rho(y) = \rho(x) + 1$. To obtain the unique minimal rank function, let ρ be any valid function above and define $\tilde{\rho}(x) = \rho(x) - \alpha$, where $\alpha = \min_{y \in Q_\sigma} \rho(y)$. □

Remark 2.3. In [21], the authors use the alternative language of a *tree-poset*, which is poset whose Hasse diagram in a spanning tree on $[n]$. This is equivalent to a poset which is acyclic and connected.

In the case of a simple generalized permutohedron, or rather a simplicial coarsening of Br_n, one can give a combinatorial formula for the h-polynomial in terms of descents on acyclic posets, which is a natural generalization of the result for the usual permutohedron.

Theorem 2.4 ([21, Theorem 4.2])**.** *Let P be a simple generalized permutahedron and let $\{Q_\sigma\}_{\sigma \in \mathcal{N}(P)}$ be the cone posets for full dimensional cones in the normal fan $\mathcal{N}(P)$ as in* Definition 2.1. *Then*

$$h_P(t) = \sum_{\sigma \in \mathcal{N}(P)} t^{\mathrm{des}(Q_\sigma)}.$$

One should note that it is straightforward to verify that

$$h_{\Pi_n}(t) = \sum_{\pi \in S_n} t^{\mathrm{des}(\pi)}.$$

using Theorem 2.4, as the posets for the full dimensional cones are simply linear orderings of $[n]$.

3 Simple generalized permutohedra

In this section, we will introduce a bivariate generalization for the h-polynomial of any simple generalized permutohedron. Particularly, we will give a formula for a q-analogue of Theorem 2.4, which gives us the expected bivariate polynomial in the case of Π_n. Unfortunately, our generalization does not produce a polynomial invariant for the combinatorial type of $\mathcal{N}(P)$. Rather, the polynomials will vary based upon the particular choice of coarsening of Br_n, and thus one may have combinatorially equivalent generalized permutohedra with different polynomials.

Based on the obserations of Proposition 2.2, we can now give a q-analogue of the h-polynomial for a simple generalized permutohedron.

Definition 3.1. Let P be a simple generalized permutahedron and let $\{Q_\sigma\}_{\sigma\in\mathcal{N}(P)}$ be the posets for full dimensional cones in the normal fan $\mathcal{N}(P)$. Then, the *q-h-polynomial* is given by

$$h_P(t,q) := \sum_{\sigma\in\mathcal{N}(P)} t^{\mathrm{des}(Q_\sigma)} q^{\mathrm{maj}(Q_\sigma)}.$$

In the case of the usual permutohedron Π_n, this q-analogue gives us the expected generalization. The full dimensional cones of the braid fan correspond to permutations $\pi \in S_n$ giving the total order Q_π which is $\pi_1 < \pi_2 < \cdots < \pi_n$. By definition, $\mathrm{des}(Q_\pi) = \mathrm{des}(\pi)$ and $\mathrm{maj}(Q_\pi) = \mathrm{maj}(\pi)$. Thus we have

$$h_{\Pi_n}(t,q) = \sum_{\pi\in S_n} t^{\mathrm{des}(\pi)} q^{\mathrm{maj}(\pi)}$$

which is the Euler–Mahonian polynomial, an expected q-analogue of the Eulerian polynomial.

Unfortunately, this construction is not invariant under reordering of the ground set. That is, the q-analogue depends on the choice of embedding or (equivalently) the choice of coarsening of the braid fan, as demonstrated by the following example.

Example 3.2. Consider the associahedron $\mathsf{A}(3) \subset \mathbb{R}^3$ which is the polytope whose normal fan is obtained by merging exactly 2 full-dimensional cones that intersect in an edge in Br_3 (see Section 5.3 for an in depth discussion of $\mathsf{A}(n)$). Two different choices of coarsening will produce combinatorially equivalent fans (resp. polytopes), but different multivariate polynomials. If one coarsens the braid fan by merging the cones corresponding to the permutations 132 and 312, the obtained q-analogue is $h_{\mathcal{F}_1}(t,q) = 1 + tq + 2tq^2 + t^2q^3$. Alternatively, if one instead coarsens the braid fan by merging the cones corresponding to 231 and 321, the obtained q-analogue is $h_{\mathcal{F}_2}(t,q) = 1 + 2tq + tq^2 + t^2q^2$. Of course when $q = 1$ in either case we have $h_{\mathcal{F}}(t) = 1 + 3t + t^2$ as expected. These two choices of coarsening are depicted in Figure 2.

4 Nestohedra

In this section, we focus on a broad class of simple generalized permutohedra known as nestohedra, for which one can be more explicit in producing combinatorial definitions for these q-analogues. The nestohedra were first introduced by Postnikov [20]. To construct a nestohedron, we need the notion of a building set.

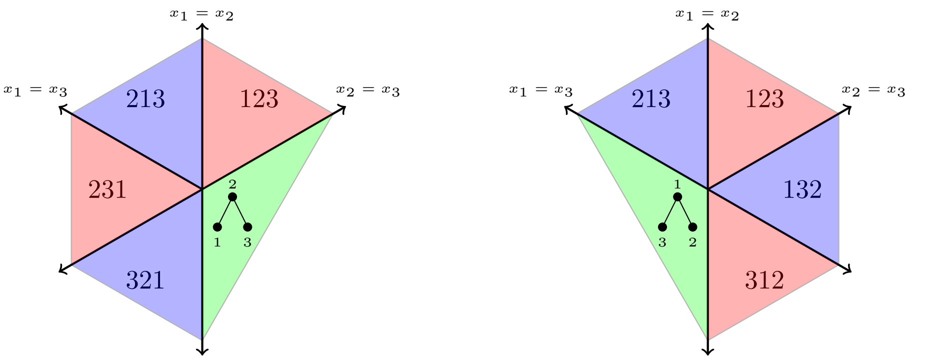

Figure 2: Two coarsenings of Br_3 which are combinatorially equivalent but produce different q-h-polynomials in Example 3.2.

Definition 4.1 ([20, Definition 7.1])**.** A collection $\mathcal{B}$ of nonempty subsets of $[n]$ is called a *building set* if it satisfies the following conditions:

1. If $I, J \in \mathcal{B}$ and $I \cap J \neq \varnothing$, then $I \cup J \in \mathcal{B}$.
2. $\mathcal{B}$ contains all singletons $\{i\}$, such that $i \in [n]$.

A building set $\mathcal{B}$ is *connected* if $[n] \in \mathcal{B}$. For any building set, one can define a nestohedron. For any subset $I \subseteq [n]$, let $\Delta_I := \mathrm{conv}\{e_i : i \in I\}$. The following definition appears implicitly in the results of [20, Section 7], but stated explicitly in this form in [21, Definition 6.3].

Definition 4.2 (see [21, Definition 6.3], [20, Section 7])**.** Given a building set $\mathcal{B}$ on $[n]$. The *nestohedron* $P_{\mathcal{B}}$ on the building set $\mathcal{B}$ is the polytope obtained from the Minkowski sum

$$P_{\mathcal{B}} := \sum_{I \in \mathcal{B}} y_I \Delta_I$$

for some strictly positive parameters y_I.

One can see that $P_{\mathcal{B}}$ is a generalized permutohedron because $\mathcal{N}(\Delta_I)$ is refined by Br_n and thus $\mathcal{N}(\sum_{I\in\mathcal{B}} y_I \Delta_I)$ must also be refined by Br_n [25, Proposition 7.12]. For our purposes, we are primarily interested in the explicit cones and associated posets in $\mathcal{N}(P_{\mathcal{B}})$. These can be described through combinatorial means. Given a rooted tree T on $[n]$ which is directed such that all edges are oriented away from the root and a vertex i in T, let $T_{\leq i}$ be the tree of descendants of i. That is, $j \in T_{\leq i}$ if there is a directed path from i to j in T. We define $\mathcal{B}$-trees for a connected building set $\mathcal{B}$.

Definition 4.3 ([20, Definition 7.7])**.** For a connected building set $\mathcal{B}$ on $[n]$, a $\mathcal{B}$*-tree* is a rooted tree T on the set $[n]$ such that

1. For any $i \in [n]$, one has $T_{\leq i} \in \mathcal{B}$
2. For any $k \geq 2$ incomparable nodes $i_1, \dots, i_k \in [n]$, one has $\bigcup_{j=1}^{k} T_{\leq i_j} \notin \mathcal{B}$.

One can algorithmically construct all of these $\mathcal{B}$-trees using the following proposition.

Proposition 4.4 ([21, Proposition 8.5], [20, Proposition 7.8])**.** *Let $\mathcal{B}$ be a connected building set on $[n]$ and let $i \in [n]$. Let $\mathcal{B}_1, \dots, \mathcal{B}_r$ be the connected components of of the restriction $\mathcal{B}|_{[n]\setminus\{i\}}$. Then all $\mathcal{B}$-trees with root at i are obtained by picking a $\mathcal{B}_j$-tree T_j, for each component $\mathcal{B}_j$, $j = 1, \dots, r$, and connecting the roots of $T_1, \dots, T_r$ to the node i by edges.*

For a building set $\mathcal{B}$, a $\mathcal{B}$-tree T has the structure of a poset by $x \lessdot y$ provided that the there is an edge (x, y) and y is closer to the root. For ease of notation, we will write $x \lessdot_T y$ to denote an edge (x, y) in T and to indicate which element is closer to the root. So,

$$\operatorname{Des}(T) = \{(i, j) \,:\, i \lessdot_T j \text{ and } i >_{\mathbb{N}} j\}$$

and $\operatorname{des}(T) = |\operatorname{Des}(T)|$. Given $x \in T$, we say that the *depth* of x, denoted $\operatorname{dp}(x)$, is the length of the unique path from x to the root. The *depth* of T is $\operatorname{depth}(T) := \max_{x\in T} \operatorname{dp}(x)$. The *major index* of T is

$$\operatorname{maj}(T) := \sum_{(i,j)\in\operatorname{Des}(T)} (\operatorname{depth}(T) - \operatorname{dp}(j))$$

Remark 4.5. Note that for any $x \in T$, the quantity $\operatorname{depth}(T) - \operatorname{dp}(x)$ is precisely $\rho(x)$ where ρ is the minimal rank function on the poset representation of T.

Proposition 4.6 ([21, Corollary 8.4])**.** *For any connected building set $\mathcal{B}$ on $[n]$, the h-polynomial of the generalized permutohedron $P_{\mathcal{B}}$ is*

$$h_{\mathcal{B}}(t) = \sum_T t^{\operatorname{des}(T)}$$

where the sum is over $\mathcal{B}$-trees T.

Given connected building sets $\mathcal{B}_1, \dots, \mathcal{B}_r$ on pairwise disjoint sets $S_1, \dots, S_r$, we can form the *combined connected building set* $\mathcal{B}$ on $S = \bigcup_{i=1}^r S_i$ by $\mathcal{B} = (\bigsqcup_{i=1}^r \mathcal{B}_i) \sqcup \{S\}$. We will now give a formula for the h-polynomial of such a building set.

Proposition 4.7. *Let $\mathcal{B}_1, \dots, \mathcal{B}_r$ be connected building sets on the pairwise disjoint sets $S_1, \dots, S_r$, and let $\mathcal{B}$ be the combined connected building set on $S = \bigcup_{i=1}^r S_i$. Then*

$$h_{\mathcal{B}}(t) = (1 + t + \cdots + t^{r-1}) \prod_{i=1}^r h_{\mathcal{B}_i}(t).$$

Proof. Without loss of generality, let $S = [n]$ and let the sets $S_1, \dots, S_r$ partition $[n]$ such that if $x \in S_i$ and $y \in S_j$, $x < y$ if and only if $i < j$ for every $1 \le i, j \le r$. Let T be a $\mathcal{B}$-tree with vertex i as the root. Suppose that $i \in S_j$ for some j. By Proposition 4.4, T is formed by connecting the root i to the roots of trees on the connected components of $\mathcal{B}|_{[n]\setminus\{i\}}$. Note that the connected components are precisely $\mathcal{B}_k$ where $k \neq j$ and the connected components of $\mathcal{B}_j|_{S_j\setminus\{i\}}$. Therefore, T is formed by $\mathcal{B}_k$-trees $T_1, T_2, \dots, T_r$ such that for all $k \neq j$, the root of T_k is connected to the root of T_j for some $j = 1, 2, \dots, r$. Additionally, given any collection of $\mathcal{B}_k$-trees, we can form a $\mathcal{B}$-tree by simply choosing one of the trees T_j to contain the root. Therefore, we will consider T as being partitioned into $\mathcal{B}_k$-trees $T_1, T_2, \dots, T_r$ with root in T_j in this way. Now, it is a straightforward computations to note that $\mathrm{des}(T) = r - j + \sum_{k=1}^{r} \mathrm{des}(T_k)$ as the construction preserves all existing descents in each tree T_k and introduces exactly one new descent between T_j and T_k where $k > j$. Since we the choices of trees for each k are independent, the contribution of all trees where T_j has the root to the h-polynomial is $t^{r-j} \prod_{k=1}^{r} h_{\mathcal{B}_k}(t)$. Thus, summing over all choices of j gives us the desired expression. □

Now we give a different characterization of the q-h-polynomial of the generalized permutohedron. This description comes from specializing Definition 3.1 to the case of nestohedra, making use of alternative descriptions of the descent set and major index.

Proposition 4.8. *For any connected building set $\mathcal{B}$ on $[n]$, the q-h-polynomial of the generalized permutohedron $P_{\mathcal{B}}$ is*

$$h_{\mathcal{B}}(t, q) = \sum_{T} t^{\mathrm{des}(T)} q^{\mathrm{maj}(T)}$$

where the sum is over $\mathcal{B}$-trees T.

Define the statistic $\mu(T) := \sum_{(i,j) \in T} (\mathrm{depth}(T) - \mathrm{dp}(j))$. Note that this statistic depends only on the isomorphism type of the rooted tree T not on the labeling. With this, we introduce a trivariate analogue of the h-polynomial of a nestohedron on connected building set

$$h_{\mathcal{B}}(t, q, u) := \sum_{T} t^{\mathrm{des}(T)} q^{\mathrm{maj}(T)} u^{\mu(T)}$$

By the Dehn-Sommerville relations, we have that the h-polynomial is palindromic. In certain cases, we can provide a multivariate analogue of palindromicity.

Theorem 4.9. *Let $\mathcal{B}$ be a connected building set on $[n]$ which is invariant under the involution $\omega\colon [n] \to [n]$ such that $\omega(i) = n - i + 1$. Then the h-polynomial for the nestohedron $P_{\mathcal{B}}$ is*

$$h_{\mathcal{B}}(t, q, u) = t^{n-1} h_{\mathcal{B}}(t^{-1}, q^{-1}, qu)$$

Proof. Let $\mathcal{B}$ be a building set such that $\omega(\mathcal{B}) = \mathcal{B}$. Suppose that T is a $\mathcal{B}$-tree. By Proposition 4.4, there exists a $\mathcal{B}$-tree $\tilde{T}$ such that T and $\tilde{T}$ such that $\tilde{T} = \omega(T)$. That is, the trees are isomorphic as unlabeled rooted trees, and one can obtain the appropriate labels of one tree by applying the involution. It is clear that $\mathrm{Des}(\tilde{T}) = \{(i, j) : (i, j) \notin \mathrm{Des}(T)\}$. Hence $\mathrm{des}(\tilde{T}) = n - 1 - \mathrm{des}(T)$ and $\mathrm{maj}(\tilde{T}) = \mu(T) - \mathrm{maj}(T)$. This gives the equality above. □

5 Examples

We conclude with a section computing explicit examples of q-h-polynomials for nestohedra of interest. Included in the list are S_n-invariant nestohedra, graph associahedra, the associahedron, the stellahedron, and the Stanley–Pitman polytope.

5.1 S_n-invariant nestohedra

We will now specialize to the case of building sets which are invariant under the action of S_n on the ground set $[n]$. Note that a connected building set $\mathcal{B}$ on $[n]$ is S_n-invariant if and only if

$$\mathcal{B} = \left\{ \{1\}, \ldots, \{n\}, \binom{[n]}{j}, \ j = k, \ldots, n \right\}$$

for some $2 \leq k \leq n$. Therefore, for a fixed n and fixed $2 \leq k \leq n$, we will denote this building set $\mathcal{B}_n^k$.

Proposition 5.1. *Let $\mathcal{B}_n^k$ be the S_n-invariant connected building set of $[n]$ with minimal nonsingleton set of cardinality k. Suppose that T_1 and T_2 are any two $\mathcal{B}$-trees. Then T_1 and T_2 are isomorphic as unlabeled rooted trees. Moreover, for any $\mathcal{B}$-tree T, $T \cong A_{k-1} \oplus C_{n-k+1}$ as a poset, where A_i is an antichain on i elements, C_j is a totally ordered chain on j elements, and $\oplus$ is ordinal sum.*

Proof. This follows from Proposition 4.4 with the observation that $\mathcal{B}_n^k|_{[n]\setminus\{i\}} \cong \mathcal{B}_{n-1}^{k-1}$ which is a connected building set. Continuing in this fashion, repeated restictions will result in connected building sets until we arrive at $\mathcal{B}_n^k|_{[n]\setminus W}$ where $W \subset [n]$ with $|W| = n-k+1$, which consists only of singleton elements. □

Theorem 5.2. *Let $\mathcal{B}_n^k$ be the S_n-invariant connected building set on $[n]$ with minimal nonsingleton set of cardinality k. The q-h-polynomial for the nestohedron $P_{\mathcal{B}_n^k}$ is*

$$h_{\mathcal{B}_n^k}(t,q) = \sum_{A \in \binom{[n]}{n-k+1}} \sum_{\pi \in S_A} t^{\mathrm{des}(\pi)+|\{j\in[n]\setminus A\,:\,j>\pi_1\}|} q^{\mathrm{maj}(\pi)+\mathrm{des}(\pi)+|\{j\in[n]\setminus A\,:\,j>\pi_1\}|}$$

Moreover, this polynomial satisfies

$$h_{\mathcal{B}_n^k}(t,q) = t^{n-1} q^{\frac{k^2-2kn-k+n^2+3n-2}{2}} h_{\mathcal{B}_n^k}(t^{-1},q^{-1}).$$

Prior to giving the proof of this formula, it is instructive to give concrete example of enumerating the descents in $\mathcal{B}_n^k$-trees.

Example 5.3. Consider the $\mathcal{B}_8^5$-tree T given in Figure 3. The descents which occur along the chain are precisely the descents of the permutation $\pi = 5481 \in S_{\{1,4,5,8\}}$ which has $\mathrm{Des}(5418) = \{1,3\}$ and $\mathrm{des}(5418) = 2$. Moreover, there are descents which occur between the antichain and the chain itself. The number of such descents is precisely the number of elements of $[8] \setminus \{1,4,5,8\}$ which are larger than 5. There are precisely 2, and hence yielding $\mathrm{des}(T) = \mathrm{des}(5481) + |\{j \in [8] \setminus \{1,4,5,8\} : j > 5\}| = 4$. When computing the major index, we note that the contributions of $\pi = 5418$ is $\sum_{i\in\mathrm{Des}(5418)}(i+1) = \mathrm{maj}(5418) + \mathrm{des}(5418) = 4+2 = 6$, to account for the correct rank. Moreover, every descent between the antichain and the chain has rank 1, so this contributes a total of 2. Thus, $\mathrm{maj}(T) = \mathrm{maj}(5418)+\mathrm{des}(5418)+|\{j \in [8]\setminus\{1,4,5,8\} : j > 5\}| = 8$.

Proof. By Proposition 5.1, we know that any T has the poset structure of $A_{k-1} \oplus C_{n-k+1}$. So any labeled tree is described by an $n-k+1$-element subset A of $[n]$ and a permutation $\pi \in S_A$. The permutation labels C_{n-k+1}, and the remaining elements of $[n] \setminus A$ label the antichain A_{k-1}. There are two types of descents in the labeling: descents in C_{n-k+1} which are enumerated by $\mathrm{des}(\pi)$, and descents where a label on the antichain A_{k-1} is greater than π_1 which is enumerated by $|\{j \in [n] \setminus A : j > \pi_1\}|$. To compute $\mathrm{maj}(T)$, note that if $i \in \mathrm{Des}(\pi)$ this corresponds to $(j,\ell) \in \mathrm{Des}(T)$ such that $\rho(\ell) = i+1$. So the contribution from descents of this form is $q^{\mathrm{maj}(\pi)+\mathrm{des}(\pi)}$. The other descents are of the form $(i,\pi_1) \in \mathrm{Des}(T)$ and since $\rho(\pi_1) = 1$, this contributes $q^{|\{j\in[n]\setminus A\,:\,j>\pi_1\}|}$.

To see the palidromicity statement, note that since $\mathcal{B}_n^k$ is S_n-invariant, then it is invariant under the involution $\omega(i) = n-i+1$. It is clear that $\mu(T) = k-2+\sum_{i=1}^{n-k+1} i =$

$\frac{k^2-2kn-k+n^2+3n-2}{2}$ for any $\mathcal{B}_n^k$-tree T. Subsequently, applying the result of Theorem 4.9 and setting $u = 1$ yields the desired statement. □

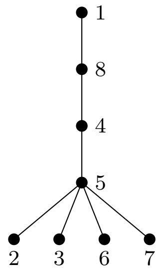

Figure 3: An example of a $\mathcal{B}_8^5$-tree T as appears in Example 5.3. By directly applying the definitions of descent and major index statistics, we can see that $\operatorname{des}(T) = 4$ and $\operatorname{maj}(T) = 8$.

5.2 Graph associahedra

We now consider a large family of examples of nestohedra arising from graphs. Given a graph $G = ([n], E)$, a *tube* of G is a proper, nonempty subset $I \subset [n]$ such that the induced subgraph $G|_I$ is connected. A *k-tubing* of G, χ, is a a collection of k distinct tubes subject to:

1. For all incomparable $A_1, A_2 \in \chi$, $A_1 \cup A_2 \notin \chi$ (*non-adjacency*);

2. For all incomparable $A_1, A_2 \in \chi$, $A_1 \cap A_2 = \varnothing$ (*non-intersecting*).

We do, however, allow for $A_1 \subset A_2$, which is called a *nesting*. We say that a tubing χ is *maximal* if it cannot add any additional tubes to χ, or equivalently, if $|\chi| = n - 1$. Given a graph G, the *graph associahedron of* G is the polytope P_G whose face lattice is given by the set of all tubings of G where $\chi < \chi'$ if χ is obtained from χ' by adding tubes. Subsequently, the vertices of P_G correspond to maximal tubings. This notion of graph associahedra originates with Carr and Devadoss [12, 13] and has been a well-studied family of examples of simple generalized permutohedra (see, e.g., [3, 6, 10, 11, 19]).

Remark 5.4. Given a simple graph $G = ([n], E)$, the graph associahedron P_G is an example of nestohedron on a connected building set, even when G is not a connected graph. The *graphical building set* of G, $\mathcal{B}(G)$ is the collection of nonempty $J \subseteq [n]$ such that the induced subgraph $G|_J$ is connected. While the building set $B(G)$ is connected if and only if G is connected (c.f. [21, Example 6.2]), the graph associahedra P_G using the notions of Carr and Devadoss [12, 13] is the nestahedron with building set $\widehat{\mathcal{B}(G)} = \mathcal{B}(G) \cup [n]$ which is always connected and $\widehat{\mathcal{B}(G)} = \mathcal{B}(G)$ if G connected.

In light of Remark 5.4, we can specialize Proposition 4.7 to determine the h-polynomial of a disconnected graph.

Corollary 5.5. *Let G be a simple graph on $[n]$ with connected components $G_1, G_2, \ldots, G_k$. Then*

$$h_G(t) = (1 + t + \cdots + t^{k-1}) \prod_{i=1}^{k} h_{G_i}(t).$$

Let $G = ([n], E)$ be a simple graph and let χ be a maximal tubing of G. Given $i \in [n]$, the *nesting index* of i, denoted $\nu_\chi(i)$, is the number of tubes containing i. The *nesting number* of χ is $\operatorname{nest}(\chi) := \max_{i\in[n]} \nu_\chi(i)$. Given any maximal χ, observe that for any tube $A_j \in \chi$, there exists a unique element $\alpha_j \in A_j$ such that for any tube $A_k \subset A_j$, we have $\alpha_j \notin A_k$. For convenience, we will write $A_k \lessdot A_j$ if $A_k \subset A_j$ and there is no tube A_ℓ such that $A_k \subset A_\ell \subset A_\ell$. Let α_n denote the unique element which is not contained in any tube of χ.

The *nesting descent set* is

$$\begin{aligned}\operatorname{NestDes}(\chi) :=& \{(\alpha_k, \alpha_j) \,:\, \alpha_k > \alpha_j \text{ and } A_k \lessdot A_j\} \\ & \cup \{(\alpha_\ell, \alpha_n) \,:\, \alpha_\ell > \alpha_n \text{ and } A_\ell \not\subset A_p \text{ for any } A_p\}.\end{aligned}$$

The *nesting descent number* is

$$\operatorname{nestDes}(\chi) := |\operatorname{NestDes}(\chi)|$$

and the *nesting major index* is

$$\operatorname{nestMaj}(\chi) := \sum_{(\alpha_k,\alpha_j)\in \operatorname{NestDes}(\chi)} (\operatorname{nest}(\chi) - \nu_\chi(\alpha_j))$$

We now state a formula for the q-h-polynomial of graph associahedra in terms of graph tubings.

Proposition 5.6. *Let G be a simple graph. The q-h-polynomial is*

$$h_G(t, q) = \sum_{\chi} t^{\operatorname{nestDes}(\chi)} q^{\operatorname{nestMaj}(\chi)}$$

where the sum is taken over all maximal tubings χ.

Proof. This follows by unpacking the definitions of $\mathcal{B}$-trees in terms of graph tubings and applying Proposition 4.8. □

Remark 5.7. As was the case with nestohedra in general, we should note that this polynomial is invariant only under labeled graph automorphisms. Under most circumstance, a different choice of labeling of the vertices G will produce a different bivariate polynomial. However, the specialization under $q = 1$ is invariant under permutation of the ground set.

Remark 5.8. As with nestohedra, we can similarly define a trivariant polynomial for graph associahedra, namely

$$h_G(t, q, u) = \sum_{\chi} t^{\operatorname{nestDes}(\chi)} q^{\operatorname{nestMaj}(\chi)} u^{\mu(\chi)}$$

where the sum ranges over all maximal and $\mu(\chi) = \sum_{(\alpha_k,\alpha_j)} (\operatorname{nest}(T) - \nu_T(\alpha_j))$ where this sum is over all pairs (α_k, α_j) such that $A_k \lessdot A_j$, which is a direct translation of the μ

statistic for nestohedra. If the involution $\omega\colon [n] \to [n]$ such that $\omega(i) = n - i + 1$ produces a labeled graph automorphism, then Theorem 4.9 gives us that palindromicity statement

$$h_G(t, q, u) = t^{n-1} h_G(t^{-1}, q^{-1}, qu).$$

There are only two S_n-invariant graphs, namely the complete graph K_n and the null graph $N_n = \overline{K_n}$ (i.e. the edgeless graph), which produce only the simplest examples of generalized permutohedra. P_{K_n} is the usual permutohedron Π_n, and hence $h_{K_n}(t, q)$ is the usual Euler–Mahonian polynomial. P_{N_n} is simply an $n-1$ dimensional simplex and thus $h_{N_n}(t, q) = \sum_{i=0}^{n-1} (tq)^i$.

5.3 The associahedron and a new q-analogue of Narayana numbers

The associahedron $\mathsf{A}(n)$, which first appeared in the work of Stasheff [24], as well as the notable work of Lee [17], is the graph associahedron for $G = \mathsf{Path}(n)$, where the vertices are labeled linearly. It is well-known that

$$h_{\mathsf{Path}(n)}(t) = \sum_{k=1}^{n} N(n, k) t^{k-1}$$

where $N(n, k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$ is the *Narayana number*, which refine the Catalan numbers. That is, $h_{\mathsf{Path}(n)}(1) = C_n$. To verify this formula, one should note that $\mathcal{B}$-trees, or graph tubings on $\mathsf{Path}(n)$, are in bijection with binary trees on n vertices (See [20, Section 8.2]). The bijection sends descents in a $\mathcal{B}$-tree to right edges in an unlabeled binary tree and $N(n, k)$ is known to enumerate the number of unlabeled binary trees on n vertices with $k - 1$ right edges. Subsequently, we will phrase all formulae in terms of binary trees.

Let T be a binary tree. Given an edge $e \in T$, let $\mathrm{dp}(e)$ be the length of the path from the root vertex to the closest vertex incident with e. Let $\mathrm{depth}(T) = \max_{e \in T} \mathrm{dp}(e)$. The *right multiset* of T is the multiset

$$\mathcal{R}(T) := \{\mathrm{dp}(e) \ : \ e \text{ is a right edge of } T\}.$$

The *right number* of T is $r(T) = |\mathcal{R}(T)|$ and the *right index* of T is

$$\mathrm{rindex}(T) := \mathrm{depth}(T) r(T) - \sum_{j \in \mathcal{R}(T)} j.$$

By translating the general results for nestohedra into the above language for binary trees, we have the following:

Corollary 5.9. *The q-h-polynomial for the associahedron is*

$$h_{\mathsf{Path}(n)}(t, q) = \sum_{T} t^{r(T)} q^{\mathrm{rindex}(T)}$$

where the sum ranges over all rooted unlabelled binary tree T on n vertices.

Remark 5.10. This theorem gives rise to *a* q-analogue of the Narayana numbers. We say the *(alternative) q-Narayana number* is

$$N(n, k, q) = \sum_{\substack{T \\ r(T) = k-1}} q^{\mathrm{rindex}(T)}.$$

It is clear that the substitution $q = 1$ yields $N(n, k)$ as desired. We call these the *alternative* q-Narayana numbers because, while this is the natural q-analogue in the context of generalized permutohedra as it arises from the major index, this does not agree with the usual q-Narayana number in the literature (see, e.g., [8, 22]).

5.4 The stellahedron

The *star graph* on $n + 1$ vertices is the complete bipartite graph $K_{1,n}$. The *stellohedron* is the graph associahedron associated to $K_{1,n}$. Let $K_{1,n}$ be labeled such that the center vertex is labeled $n + 1$. Recall that a *partial permutation* of $[n]$ is a linear ordering of a k-subset $L \subseteq [n]$ for some $k = 1, 2, \dots n$. The $\mathcal{B}$-trees for $K_{1,n}$ are in bijection with partial permutations of $[n]$. In particular, the structure of a $\mathcal{B}$-tree is given by the ordinal sum of an antichain with a totally ordered chain $A_{n-k-1} \oplus C_{k+1}$ for some $k = 0, \dots, n$ such that the minimal element of C_{k+1} has label $n + 1$.

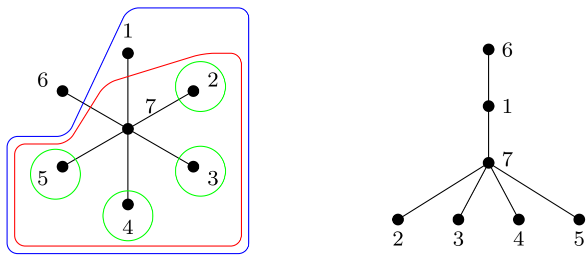

Figure 4: A tubing of $K_{1,6}$ and its corresponding $\mathcal{B}$-tree.

To see this, note that we can identify the $\mathcal{B}$-trees with graph tubings. Any tubing of $K_{1,n}$ is either

(i) the tubing where each vertex $i = 1, 2, \dots, n$ is in a singleton tube and $n + 1$ is the root, or

(ii) some vertex i is the root and we have a tube containing all other vertices.

In the case of (ii), once i is chosen, then the tubing directly arises from a tubing of $K_{1,n-1}$ on the labels $[n + 1] \setminus \{i\}$. Thus, by induction, we will have $\mathcal{B}$-trees of the proposed form. For example, consider the tubing and $\mathcal{B}$-tree given in Figure 4, which corresponds to the partial permutation $\pi = 61$ on $[6]$.

Subsequently, the elements of the C_{k+1} above the $n+1$ are the partial permutation (see [21, Section 10.4]) With this in mind, we can state the q-analogue of the h-polynomial for the stellohedron.

Proposition 5.11. *The q-h-polynomial for the stellohedron is*

$$h_{K_{1,n}}(t, q) = 1 + \sum_{w} t^{\mathrm{des}(w)+1} q^{\mathrm{maj}(w)+2\mathrm{des}(w)+2}$$

where the sum is over all nonempty partial permutations of $[n]$.

Proof. The labels on C_{k+1} correspond to a partial permutation of $\tilde{w}$ of $[n+1]$ where $\tilde{w}_1 = n+1$. Thus, we consider w to be the partial permutation of $[n]$ with this first element omitted. If $w = \varnothing$, the corresponding $\mathcal{B}$-tree has no descents. If $w \neq \varnothing$, then the corresponding $\mathcal{B}$-tree T has precisely $\mathrm{des}(w)+1$ descents, due to the guaranteed descent between $n+1$ and w_1. When computing the major index, note that if $i \in \mathrm{Des}(w)$, this means that we have an element of rank $i+2$ where a descent occurs in T. Hence, the contribution to the major index is $\sum_{i \in \mathrm{Des}(w)}(i+2) = \mathrm{maj}(w) + 2\mathrm{des}(w)$. Additionally, the descent between $n+1$ and w_1 contributes 2, as $\rho(w_1) = 2$. Thus, we have the desired formula. □

5.5 The Stanley-Pitman polytope

Introduced by Stanley and Pitman in [23], the *Stanley-Pittman polytope* is a integral polytope defined by the equations

$$\mathsf{PS}(n) := \left\{ \boldsymbol{x} \in \mathbb{R}^n \,:\, x_i \geq 0 \text{ and } \sum_{i=1}^{j} x_i \leq j \text{ for each } 1 \leq j \leq n \right\}.$$

This polytope is combinatorially equivalent to an n-cube, as illustrated in Figure 5. However, this polytope is of particular interest as it appears naturally when studying empirical distributions in statistics and has connections to many combinatorial objects, such as parking functions and plane trees. Postnikov [20, Section 8.5] observed that this polytope can be realized as the nestohedron from the building set

$$\mathcal{B}_{\mathsf{PS}} = \{[i,n],\ \{i\} \,:\, i \in [n]\},$$

where $[i,n] = \{i, i+1, \ldots, n\}$. Notably, this is not a graph associahedron. Given that this polytope is combinatorially equivalent to an n-cube, we have $h_{\mathcal{B}_{\mathsf{PS}}}(t) = (1+t)^{n-1}$ [23, Theorem 20]. We now give the q-analogue.

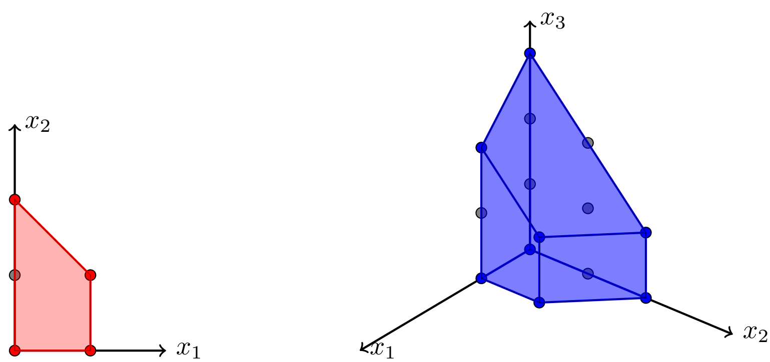

Figure 5: $\mathsf{PS}(2)$ and $\mathsf{PS}(3)$.

Proposition 5.12. *The q-h-polynomial for the Stanley-Pitman polytope is*

$$h_{\mathcal{B}_{\mathsf{PS}}}(t,q) = \sum_{\ell=0}^{n-2} \binom{n-2}{\ell} t^{\ell} q^{\frac{\ell^2+3\ell+2}{2}} \left(t + q^{\ell}\right).$$

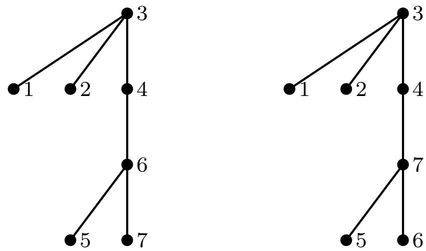

Figure 6: Two $\mathcal{B}_{\mathsf{PS}}$-trees for $n = 7$ from the increases sequences $I_1 = \{3 < 4 < 6 < 7\}$ and $I_2 = \{3 < 4 < 7\}$. Alternatively, these are the two trees from the set $\{3, 4\} \subset [5]$.

Proof. First note that $h_{\mathcal{B}_{\mathsf{PS}}}(t, 1) = (t + 1)^{n-1}$, so this agrees with the known results. To compute this, we will need $\mathcal{B}_{\mathsf{PS}}$-trees, which as determined by Postnikov, Reiner, and Williams [21, Section 10.5], are formed in the following way. Given any increasing sequence of positive integers $I = \{i_1 < i_2 < \cdots < i_k = n\}$ where we let i_1 be the root and form the chain of edges $(i_1, i_2), (i_2, i_3), \ldots, (i_{k-1}, i_k)$ and for all $j \in [n] \setminus I$ we have the edge (i_s, j) where i_s is the minimal element of I such that $i_s > j$. An example can be seen in Figure 6.

It is clear that all descents will be occur along the chain of edges. So, we must consider two cases:

(i) $i_{k-1} = n - 1$ and

(ii) $i_{k-1} \leq n - 2$.

In case (i), for convenience let $\ell = k - 2$. We form a tree T by choosing a subset $J \in \binom{[n-2]}{\ell}$ and arranging it increasing order to form a chain of edges which ends in $(i_\ell, n-1)$, $(n - 1, n)$. By definition, $\mathrm{depth}(T) = \ell + 1$, $\mathrm{des}(T) = \ell + 1$, and $\mathrm{maj}(T) = (\ell + 1)^2 - \sum_{i=0}^{\ell} i = \frac{\ell^2+3\ell+2}{2}$. So, the contribution of trees of this form to the q-h-polynomials is

$$\sum_{\ell=0}^{n-2} \binom{n-2}{\ell} t^{\ell+1} q^{\frac{\ell^2+3\ell+2}{2}}. \tag{5.1}$$

In case (ii) where $i_{k-1} \neq n - 1$, for ease of notation, let $\ell = k - 1$. Similarly, we form such a tree T by choosing $J \in \binom{[n-2]}{\ell}$ and arranging it increasing order to form a chain of edges which ends in (i_ℓ, n). Note that, when including the elements not in the chain, we gain edges from the vertex n going away from the root, in particular, the edge $(n, n-1)$. So, we again have $\mathrm{depth}(T) = \ell + 1$. However, we now have $\mathrm{des}(T) = \ell$, and $\mathrm{maj}(T) = (\ell + 1)^2 - \sum_{i=0}^{\ell-1} i = \frac{\ell^2+5\ell+2}{2}$. So the contribution of trees of this type to the q-h-polynomial is

$$\sum_{\ell=0}^{n-2} \binom{n-2}{\ell} t^{\ell} q^{\frac{\ell^2+5\ell+2}{2}}. \tag{5.2}$$

Summing (5.1) and (5.2) and simplifying gives the desired expression. □

Remark 5.13. We conclude our discussion by noting that our computation produces an *alternative q-analogue* of $\binom{n-1}{\ell}$, namely

$$\binom{n-2}{\ell-1}q^{\frac{\ell^2+\ell}{2}} + \binom{n-2}{\ell}q^{\frac{\ell^2+5\ell+2}{2}}.$$

This, of course, reduces to $\binom{n-1}{\ell}$ when $q = 1$ and arises quite naturally from generalizing the major index statistic. However, this is not the usual q-analogue of a binomial coefficient which arises in many natural ways, such as bit string inversions and lattice path areas.

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A note on the k-tuple domination number of graphs*

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Abstract

In a graph G, a vertex dominates itself and its neighbours. A set $D \subseteq V(G)$ is said to be a k-tuple dominating set of G if D dominates every vertex of G at least k times. The minimum cardinality among all k-tuple dominating sets is the k-tuple domination number of G. In this note, we provide new bounds on this parameter. Some of these bounds generalize other ones that have been given for the case $k = 2$.

Keywords: k-domination, k-tuple domination.

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1 Introduction

Throughout this note we consider simple graphs G with vertex set $V(G)$. Given a vertex $v \in V(G)$, $N(v)$ denotes the *open neighbourhood* of v in G. In addition, for any set $D \subseteq V(G)$, the *degree* of v in D, denoted by $\deg_D(v)$, is the number of vertices in D adjacent to v, i.e., $\deg_D(v) = |N(v) \cap D|$. The *minimum* and *maximum degrees* of G will be denoted by $\delta(G)$ and $\Delta(G)$, respectively. Other definitions not given here can be found in standard graph theory books such as [12].

Domination theory in graphs have been extensively studied in the literature. For instance, see the books [9, 10, 11]. A set $D \subseteq V(G)$ is said to be a *dominating set* of G if $\deg_D(v) \geq 1$ for every $v \in V(G) \setminus D$. The *domination number* of G is the minimum cardinality among all dominating sets of G and it is denoted by $\gamma(G)$. We define a $\gamma(G)$-set as a dominating set of cardinality $\gamma(G)$. The same agreement will be assumed for optimal parameters associated to other characteristic sets defined in the paper.

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In 1985, Fink and Jacobson [4, 5] extended the idea of domination in graphs to the more general notion of k-domination. A set $D \subseteq V(G)$ is said to be a *k-dominating set* of G if $\deg_D(v) \geq k$ for every $v \in V(G) \setminus D$. The *k-domination number* of G, denoted by $\gamma_k(G)$, is the minimum cardinality among all k-dominating sets of G. Subsequently, and as expected, several variants for k-domination were introduced and studied by the scientific community. In two different papers published in 1996 and 2000, Harary and Haynes [7, 8] introduced the concept of double domination and, more generally, the concept of k-tuple domination. Given a graph G and a positive integer $k \leq \delta(G) + 1$, a k-dominating set D is said to be a *k-tuple dominating set* of G if $\deg_D(v) \geq k - 1$ for every $v \in D$. The *k-tuple domination number* of G, denoted by $\gamma_{\times k}(G)$, is the minimum cardinality among all k-tuple dominating sets of G. The case $k = 2$ corresponds to double domination, in such a case, $\gamma_{\times 2}(G)$ denotes the double domination number of graph G.

In this note, we provide new bounds on the k-tuple domination number. Some of these bounds generalize other ones that have been given for the double domination number.

2 New bounds on the k-tuple domination number

Recently, Hansberg and Volkmann [6] put into context all relevant research results on multiple domination that have been found up to 2020. In that chapter, they posed the following open problem.

Problem 2.1 ([6, Problem 5.8, p. 194])**.** Give an upper bound for $\gamma_{\times k}(G)$ in terms of $\gamma_k(G)$ for any graph G of minimum degree $\delta(G) \geq k - 1$.

A fairly simple solution for the problem above is given by the straightforward relationship $\gamma_{\times k}(G) \leq k\gamma_k(G)$, which can be derived directly by constructing a set of vertices $D' \subseteq V(G)$ of minimum cardinality from a $\gamma_k(G)$-set D such that $D \subseteq D'$ and $\deg_{D'}(x) \geq k - 1$ for every vertex $x \in D$. From this construction above, it is easy to check that D' is a k-tuple dominating set of G and so,

$$\gamma_{\times k}(G) \leq |D'| = |D| + |D' \setminus D| \leq |D| + (k-1)|D| = k\gamma_k(G).$$

This previous inequality was surely considered by Hansberg and Volkmann and, in that sense, they have established the previous problem assuming that $\gamma_{\times k}(G) < k\gamma_k(G)$ for every graph G with $\delta(G) \geq k - 1$.

We next confirm their suspicions and provide a solution to Problem 2.1.

Theorem 2.2. *Let $k \geq 2$ be an integer. For any graph G with $\delta(G) \geq k - 1$,*

$$\gamma_{\times k}(G) \leq k\gamma_k(G) - (k-1)^2.$$

Proof. Let D be a $\gamma_k(G)$-set. As $\gamma_{\times k}(G) \leq |V(G)|$ we assume, without loss of generality, that $k|D| - (k-1)^2 \leq |V(G)|$. Now, let $U = \{u_1, \ldots, u_{k-1}\} \subseteq V(G) \setminus D$, $D' = D \cup U$ and $D_0 = \{v \in D : \deg_{D'}(v) < k - 1\}$. The following inequalities arise from counting arguments on the number of edges joining U with D_0 and U with $D \setminus D_0$, respectively.

$$\sum_{v \in D_0} \deg_{D'}(v) \geq \sum_{i=1}^{k-1} \deg_{D_0}(u_i) \quad \text{and} \quad |D \setminus D_0|(k-1) \geq \sum_{i=1}^{k-1} \deg_{D \setminus D_0}(u_i).$$

By the previous inequalities and the fact that D is a k-dominating set of G, we deduce that

$$\begin{aligned}\sum_{v\in D_0}\deg_{D'}(v)+|D\setminus D_0|(k-1) &\geq \sum_{i=1}^{k-1}\deg_{D_0}(u_i)+\sum_{i=1}^{k-1}\deg_{D\setminus D_0}(u_i)\\ &= \sum_{i=1}^{k-1}\deg_{D}(u_i)\\ &\geq k(k-1).\end{aligned}$$

Now, we define $D''\subseteq V(G)$ as a set of minimum cardinality among all supersets W of D' such that $\deg_W(x)\geq k-1$ for every vertex $x\in D$. Since $\deg_{D'}(x)\geq k-1$ for every $x\in D\setminus D_0$, the condition on W is equivalent to that every vertex $v\in D_0$ has at least $k-1-\deg_{D'}(v)$ neighbours in $W\setminus D$. Hence, by the minimality of D'' and the inequality chain above, we deduce that

$$\begin{aligned}|D''\setminus D'| &\leq |D_0|(k-1)-\sum_{v\in D_0}\deg_{D'}(v)\\ &= |D|(k-1)-\left(\sum_{v\in D_0}\deg_{D'}(v)+|D\setminus D_0|(k-1)\right)\\ &\leq |D|(k-1)-k(k-1).\end{aligned}$$

Moreover, it is easy to check that D'' is a k-tuple dominating set of G because each vertex in $V(G)\setminus D$ is dominated k times by vertices of $D\subseteq D''$ (recall that D is a k-dominating set of G) and the construction of D'' ensures that each vertex in D is dominated k times by vertices of D''. Hence,

$$\begin{aligned}\gamma_{\times k}(G)\leq |D''| &= |D'|+|D''\setminus D'|\\ &\leq |D|+k-1+|D|(k-1)-k(k-1)\\ &= k\gamma_k(G)-(k-1)^2,\end{aligned}$$

which completes the proof. □

The bound above is tight. For instance, it is achieved by any complete bipartite graph $K_{k,k'}$ with $k'\geq k$, as $\gamma_{\times k}(K_{k,k'})=2k-1$ and $\gamma_k(K_{k,k'})=k$. When $k=2$, Theorem 2.2 leads to the relationship $\gamma_{\times 2}(G)\leq 2\gamma_2(G)-1$ given in 2018 by Bonomo et al. [1].

A set $D\subseteq V(G)$ is a 2-packing of a graph G if $N[u]\cap N[v]=\emptyset$ for every pair of different vertices $u,v\in D$. The 2-packing number of G, denoted by $\rho(G)$, is the maximum cardinality among all 2-packings of G.

The next theorem relates the k-tuple domination number with the 2-packing number of a graph. Note that the bounds given in this result are generalizations of the bounds $\gamma_{\times 2}(G)\geq 2\rho(G)$ due to Chellali et al. [3], and $\gamma_{\times 2}(G)\leq |V(G)|-\rho(G)$ due to Chellali and Haynes [2].

Theorem 2.3. *Let $k\geq 2$ be an integer. For any graph G of order n and $\delta(G)\geq k$,*

$$k\rho(G)\leq\gamma_{\times k}(G)\leq n-\rho(G).$$

Proof. Let D be a $\rho(G)$-set and S a $\gamma_{\times k}(G)$-set. Since $\deg_S(v) \geq k$ for every $v \in D \setminus S$, and $\deg_S(v) \geq k-1$ for every $v \in D \cap S$, we deduce that

$$\gamma_{\times k}(G) = |S| \geq \sum_{v \in D \setminus S} \deg_S(v) + \sum_{v \in D \cap S} (\deg_S(v) + 1) \geq k|D| = k\rho(G),$$

and the lower bound follows.

Next, let us proceed to prove that $V(G) \setminus D$ is a k-tuple dominating set of G. Since $\delta(G) \geq k$, $N(D) \cap D = \emptyset$ and $\deg_D(x) \leq 1$ for every $x \in V(G) \setminus D$, we deduce that $\deg_{V(G)\setminus D}(v) \geq k$ for every $v \in D$ and $\deg_{V(G)\setminus D}(v) \geq k-1$ for every $v \in V(G) \setminus D$. Hence, $V(G) \setminus D$ is a k-tuple dominating set of G, as desired.

Therefore, $\gamma_{\times k}(G) \leq |V(G) \setminus D| = n - \rho(G)$, which completes the proof. □

Let $\mathcal{H}$ be the family of graphs $H_{k,r}$ defined as follows. For any pair of integers $k, r \in \mathbb{Z}$, with $k \geq 2$ and $r \geq 1$, the graph $H_{k,r}$ is obtained from a complete graph K_{kr} and an empty graph rK_1 such that $V(H_{k,r}) = V(K_{kr}) \cup V(rK_1)$, $V(K_{kr}) = \{v_1, \dots, v_{kr}\}$ and $V(rK_1) = \{u_1, \dots, u_r\}$ and $E(H_{k,r}) = E(K_{kr}) \cup (\bigcup_{i=0}^{r-1} \{u_{i+1}v_{ki+1}, \dots, u_{i+1}v_{ki+k}\})$. Figure 1 shows a graph of this family. Observe that $|V(H_{k,r})| = r(k+1)$, $\gamma_{\times k}(H_{k,r}) = kr$ and $\rho(H_{k,r}) = r$ for every $H_{k,r} \in \mathcal{H}$. Therefore, for these graphs the bounds given in Theorem 2.3 are tight, i.e., $\gamma_{\times k}(H_{k,r}) = k\rho(H_{k,r}) = |V(H_{k,r})| - \rho(H_{k,r})$.

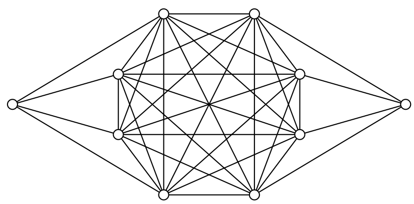

Figure 1: The graph $H_{4,2} \in \mathcal{H}$.

In [8], Harary and Haynes showed that $\gamma_{\times k}(G) \geq \frac{2kn-2m}{k+1}$ for any graph G of order n and size m with $\delta(G) \geq k-1$. The next result is a partial refinement of the bound above because it only considers graphs with minimum degree at least k.

Proposition 2.4. *Let $k \geq 2$ be an integer. For any graph G of order n and size m with $\delta(G) \geq k$,*

$$\gamma_{\times k}(G) \geq \frac{(\delta(G)+k)n - 2m}{\delta(G)+1}.$$

Proof. Let S be a $\gamma_{\times k}(G)$-set and $\overline{S} = V(G) \setminus S$. Hence,

$$\begin{aligned} 2m &= \sum_{v \in S} \deg_S(v) + 2\sum_{v \in \overline{S}} \deg_S(v) + \sum_{v \in \overline{S}} \deg_{\overline{S}}(v) \\ &= \sum_{v \in S} \deg_S(v) + \sum_{v \in \overline{S}} \deg_S(v) + \sum_{v \in \overline{S}} \deg_{V(G)}(v) \\ &\geq (k-1)|S| + k(n-|S|) + \delta(G)(n-|S|) \\ &= (k-1)|S| + (\delta(G)+k)(n-|S|) \\ &= (\delta(G)+k)n - (\delta(G)+1)|S|, \end{aligned}$$

which implies that $|S| \geq \frac{(\delta(G)+k)n-2m}{\delta(G)+1}$. Therefore, the proof is complete. □

The bound above is tight. For instance, it is achieved for the join graph $G = K_k + C_k$ obtained from the complete graph K_k and the cycle graph C_k, with $k \geq 3$. For this case, we have that $\gamma_{\times k}(G) = k$, $|V(G)| = 2k$, $\delta(G) = k + 2$ and $2|E(G)| = 3k^2 + k$. Also, it is achieved for the complete graph K_n ($n \geq 3$) and any $k \in \{2, \ldots, n-1\}$.

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Growable realizations: a powerful approach to the Buratti-Horak-Rosa Conjecture*

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Abstract

Label the vertices of the complete graph K_v with the integers $\{0, 1, \ldots, v-1\}$ and define the *length* of the edge between the vertices x and y to be $\min(|x-y|, v-|x-y|)$. Let L be a multiset of size $v-1$ with underlying set contained in $\{1, \ldots, \lfloor v/2 \rfloor\}$. The Buratti-Horak-Rosa Conjecture is that there is a Hamiltonian path in K_v whose edge lengths are exactly L if and only if for any divisor d of v the number of multiples of d appearing in L is at most $v-d$.

We introduce "growable realizations," which enable us to prove many new instances of the conjecture and to reprove known results in a simpler way. As examples of the new method, we give a complete solution when the underlying set is contained in $\{1, 4, 5\}$ or in $\{1, 2, 3, 4\}$ and a partial result when the underlying set has the form $\{1, x, 2x\}$. We believe that for any set U of positive integers there is a finite set of growable realizations that implies the truth of the Buratti-Horak-Rosa Conjecture for all but finitely many multisets with underlying set U.

Keywords: Hamiltonian path, complete graph, edge-length, growable realization.

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1 Introduction

Let K_v be the complete graph on v vertices, labeled with the integers $\{0, 1, \ldots, v-1\}$. For two vertices x and y, define the *length* of the edge between them to be

$$\ell(x, y) = \min(|x - y|,\ v - |x - y|),$$

which is an integer in the range $1 \leq \ell(x, y) \leq \lfloor v/2 \rfloor$.

A Hamiltonian path $\boldsymbol{h} = [h_1, h_2, \ldots, h_v]$ in K_v uses $v-1$ edges and gives a multiset $L = \{\ell(h_i, h_{i+1}) : 1 \leq i \leq v-1\}$ of edge-lengths. Call $\boldsymbol{h}$ a *realization* of L or say that $\boldsymbol{h}$ *realizes* L. For example, with $v = 7$ the Hamiltonian path $[0, 5, 1, 2, 6, 3, 4]$ has edge-length sequence $[2, 3, 1, 3, 3, 1]$ and hence realizes the multiset $\{1^2, 2, 3^3\}$ (where exponents indicate multiplicity).

Given a multiset L, its *underlying set* is given by $U = \{x : x \in L\}$.

The focus of our inquiry is the *Buratti-Horak-Rosa Conjecture*, or *BHR Conjecture*:

Conjecture 1.1. *Let L be a multiset of size $v-1$ with underlying set U contained in $\{1, \ldots, \lfloor v/2 \rfloor\}$. Then there is a realization of L in K_v if and only if for any divisor d of v the number of multiples of d in L is at most $v - d$.*

When v is prime, in which case the condition on divisors is always satisfied, we have the original *Buratti Conjecture*, see [1, 11]. Horak and Rosa [5] generalize this to composite v and show that the condition on divisors is necessary; Pasotti and Pellegrini [9] reformulate Horak and Rosa's statement into the one in Conjecture 1.1.

Call a multiset L of size $v-1$ *admissible* if it has underlying set $U \subseteq \{1, \ldots, \lfloor v/2 \rfloor\}$ and it satisfies the divisor condition of the BHR conjecture. Denote the BHR Conjecture for L by $\mathrm{BHR}(L)$.

Much work has been done on the BHR Conjecture. Theorem 1.2 captures the main progress that has been made to date.

Theorem 1.2. *Let L be a multiset of size $v-1$ with underlying set U. In each of the following cases, if L is admissible, then it is realizable.*

(1) $|U| \leq 2$ [3, 5],

(2) $U = \{1, 2, 4\}, \{1, 2, 6\}, \{1, 2, 8\}$ [10],

(3) $U \subseteq \{1, 2, 3, 5\}$ [2, 9],

(4) $L = \{1^a, 2^b, 3^c, 4^d\}$ *with either* $a \geq 3$ *and* $c, d \geq 1$ *or* $a = 2$ *and* $b, c, d \geq 1$ [8],

(5) $L = \{1^a, 2^b, x^c\}$ *when x is even and* $a + b \geq x - 1$ [10],

(6) $L = \{1^a, x^b, (x+1)^c\}$ *when x is odd and either* $a \geq \min(3x-3, b+2x-3)$ *or* $a \geq 2x-2$ *and* $c \geq 4b/3$ [8],

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(7) $L = \{1^a, x^b, (x+1)^c\}$ *when* x *is even and either* $a \geq \min(3x-1, c+2x-1)$ *or* $a \geq 2x-1$ *and* $b \geq c$ [8],

(8) $U \subseteq \{1, 2, 4, \ldots, 2x\}$ *and* $\{1^{2x-1}, 2x\} \subseteq L$ [8],

(9) $U \subseteq \{1, 2, 4, \ldots, 2x, 2x+1\}$ *and* $\{1^{6x-1}, 2x+1\} \subseteq L$ [8],

(10) $L = \{1^{a_1}, 2^{a_2}, \ldots, x^{a_x}\}$ *with* $a_1 \geq a_2 \geq \cdots \geq a_x$ [7, 8],

(11) $L = M \cup \{1^a\}$ *for any multiset* M *and* $a > a_M$*, where* a_M *is a constant that depends on* M [5],

(12) $v \leq 19$ *or* $v = 23$ [6].

After proving Theorem 1.2(11), Horak and Rosa observe that "to get an explicit bound... one only needs refer to lemmas used in the proof" [5]. It turns out that their methodology can be used to give a bound that is linear in the elements of the underlying set and independent of their multiplicities, neither of which is clear from the statement of the result. We believe that this is of interest and so give an explicit bound with these properties in Theorem 1.3.

Theorem 1.3. *Let* M *be a multiset with underlying set* $U = \{x_1, \ldots, x_k\}$*, where* $1 < x_1 < \cdots < x_k$*. Then* $L = M \cup \{1^s\}$ *is realizable for any* $s \geq 3x_k - 5 + \sum_{i=1}^{k} x_i$*.*

Proof outline. We give the steps required to establish the bound, referring to [5] for the specific details.

In the notation of [5, Theorem 3.4], we partition M as $L_1 \cup L_2 \cup L_3 \cup L_4$ in a certain way and then $M \cup \{1^s\}$ is realizable for all $s \geq s_1 + s_2 + s_3 + s_4 - 1$, where each s_i is dependent on L_i for $1 \leq i \leq 4$. Let U_i be the underlying set of L_i for $1 \leq i \leq 4$.

By [5, Lemma 3.12], we may take $s_1 = 1 - 2|U_1| + \sum_{x \in U_1} x$; hence $s_1 \leq \left(\sum_{i=1}^{k} x_i\right) - 1$. By [5, Lemma 3.9], we may take $s_2 = \max(U_2) - 1$; hence $s_2 \leq x_k - 1$. By [5, Lemma 3.7], we may take $s_3 = \max(U_3) - 1$; hence $s_3 \leq x_k - 1$. By [5, Lemma 3.13], we may take $s_4 = \max(U_4) - |U_4|$; hence $s_4 \leq x_k - 1$.

Combining these bounds we find that L is realizable for all $s \geq 3x_k - 5 + \sum_{i=1}^{k} x_i$. □

The BHR Conjecture has close connections to many other problems and conjectures concerning sequences with distinct partial sums or subgraphs of K_v other than paths; see [8] for more discussion of this. A recent paper also makes a connection between the BHR Conjecture and the Traveling Salesman Problem [4].

We are frequently concerned with the congruence classes of multiple elements with respect to multiple integers. The following notation is useful for these situations: if $x_i \equiv y_i \pmod{z_i}$ for $1 \leq i \leq k$, then write

$$(x_1, \ldots, x_k) \equiv (y_1, \ldots, y_k) \pmod{(z_1, \ldots, z_k)}.$$

We also need the notion of a *translation* of a sequence $\boldsymbol{h} = [h_1, \ldots, h_v]$ by an integer m:

$$\boldsymbol{h} + m = [h_1 + m, \ldots, h_v + m].$$

The translation of a sequence produces the same multiset of absolute differences as the original sequence.

We require a lot of small examples of realizations for particular multisets. These were mostly found using a heuristic algorithm implemented in GAP. Given a target multiset L of size $v-1$, the algorithm starts from a random Hamiltonian path in K_v and keeps trying to move to a Hamiltonian path that is "closer" to realizing L—in the sense of trying to increase $|L \cap L'|$, where L' is the multiset realized by the path under consideration—by removing an edge from the path and reconnecting the two resulting paths in a different way. If it gets stuck before finding a realization of L, then it tries again from a different starting path. For the fairly small values of v in which we are interested, this simple algorithm is sufficient to find the desired realizations quickly. The programs are available on the ArXiv page for this paper.

The most far-reaching components of Theorem 1.2 were proved using "linear" realizations. A Hamiltonian path $\boldsymbol{h} = [h_1, h_2, \ldots, h_v]$ of K_v defines a multiset of absolute differences $L = \{|h_i - h_{i+1}| : 1 \leq i \leq v-1\}$ with underlying set contained in $\{1, \ldots, v-1\}$. In this situation, $\boldsymbol{h}$ is a *linear realization* of L. If $h_1 = 0$, then the linear realization is *standard*; if $h_1 = 0$ and $h_v = v-1$, then the linear realization is *perfect*. To emphasize the distinction between linear realizations and realizations, realizations as defined above are sometimes referred to as *cyclic realizations*.

Linear realizations are closely related to cyclic realizations. For example, if each element in a multiset L of size $v-1$ is at most $\lfloor v/2 \rfloor$, then a linear realization of L is also a cyclic realization of L. See [5] for further discussion.

What makes linear realizations so useful in addressing the BHR Conjecture, standard and perfect ones especially, is their ability to be combined and hence used in inductive arguments. This is the approach taken by Horak and Rosa in [5] and the multisets L given in Theorem 1.3 in fact have linear realizations that are also cyclic realizations for the same L.

The main contribution of this work is to introduce an alternative object: the "growable" realization, which we define in the next section. These are cyclic realizations that can be used in inductive arguments in somewhat similar ways to linear ones.

In Section 3 we reprove, with a much shorter proof, the result from [2] that $\mathrm{BHR}(L)$ holds when L has underlying set $U = \{1, 2, 3\}$ to illustrate that this new tool is, in some ways, more powerful than existing ones. We go on to prove instances of the BHR Conjecture that seem beyond the reach of current techniques. In particular, we are able to add the following items to Theorem 1.2:

- $U = \{1, 4, 5\}$ (Section 3),
- $U \subseteq \{1, 2, 3, 4\}$ (Section 4),
- $L = \{1^a, x^b, (2x)^c\}$ when $a \geq x-2$, c is even and $b \geq 5x - 2 + c/2$ (Section 5),
- $L = \{1^a, 3^b, 6^c\}$ when c is odd and $b \geq 18 + (c-1)/2$ (Section 5).

2 Growable realizations

Growable realizations will let us move from solving $\mathrm{BHR}(L)$ to $\mathrm{BHR}(L \cup \{x^x\})$ under certain circumstances. When this can be done for multiple choices of x, this is a powerful tool.

Take x with $0 < x \leq v/2$. For a given m, with $0 \leq m < v$, we shall embed K_v into

K_{v+x} as follows:

$$y \mapsto \begin{cases} y & \text{when } y \le m, \\ y + x & \text{otherwise.} \end{cases}$$

This embedding preserves some edge lengths and increases others. Call it the *m-embedding of K_v into K_{v+x}*.

Remark 2.1. Let y, z be two vertices of K_v and assume, without loss of generality, that $y < z$. Let y' and z' be the images of y and z, respectively, by the m-embedding of K_v into K_{v+x}. One can check that $\ell(y,z) = \ell(y',z')$ if and only if one of the following holds:

- $y < z \le m$ and $z - y \le \frac{v}{2}$;
- $m < y < z$ and $z - y \le \frac{v}{2}$;
- $y \le m < z$ and $z - y > \frac{v}{2}$.

Definition 2.2. Let $\boldsymbol{h} = [h_1, \dots, h_v]$ be a cyclic realization of a multiset L. Take x and m with $0 < x \le v/2$ and $0 \le m < v$. If each y with $m - x < y \le m$ is incident with exactly one edge whose length is increased by the m-embedding of K_v into K_{v+x} and there is no other edge whose length is increased, then say that $\boldsymbol{h}$ is *x-growable at m*.

A realization is said to be x-growable, if it is x-growable at some m. If a realization is x-growable for each $x \in X$ for some set X, then say that it is *X-growable*.

Example 2.3. It is easy to see that the sequence $[6,4,3,0,7,1,5,2,8]$ is a cyclic realization of $\{1, 2^2, 3^4, 4\}$ and that it is 3-growable at 2. In fact, we can represent this Hamiltonian path of K_9 writing in bold the vertices not increased by the 2-embedding of K_9 into K_{12}, and using the symbol $-$ for each edge whose length does not change and the symbol $\cdots$ for each edge whose length increases by 3:

$$6 - 4 - 3 \cdots \mathbf{0} - 7 - \mathbf{1} \cdots 5 \cdots \mathbf{2} - 8.$$

Note that every vertex in bold is incident with exactly one edge $\cdots$. Also, note that the edges $0 - 7$, $7 - 1$ and $2 - 8$ do not change length, since their absolute differences are greater than $\left\lfloor \frac{9}{2} \right\rfloor$.

Theorem 2.4 and its immediate consequence Theorem 2.6 are the core results for using growable realizations.

Theorem 2.4. *Suppose a multiset L has an X-growable realization. Then for each $x \in X$, the multiset $L \cup \{x^x\}$ has an X-growable realization.*

Proof. Let $\boldsymbol{g} = [g_1, \dots, g_v]$ be an X-growable realization of a multiset L. Take $x \in X$ and m such that $\boldsymbol{g}$ is x-growable at m. Each element y with $m - x < y \le m$ is adjacent to exactly one element z such that the edge between them is lengthened by the m-embedding of K_v into K_{v+x}.

Applying the embedding we obtain a sequence $\boldsymbol{h}' = [h_1, \dots, h_v]$ in K_{v+x}. Each adjacent pair y, z in $\boldsymbol{g}$ as above becomes a subsequence $(y, z+x)$ or $(z+x, y)$ in $\boldsymbol{h}'$. Obtain a new sequence $\boldsymbol{h}$ in K_{v+x} by replacing each subsequence $(y, z+x)$ with $(y, y+x, z+x)$ and each subsequence $(z+x, y)$ with $(z+x, y+x, y)$. As there is one pair for each y in the range $m - x < y \le m$, this adds the elements $m+1, \dots, m+x$ to the sequence and hence $\boldsymbol{h}$ is a Hamiltonian path in K_{v+x}.

Now, $\boldsymbol{h}$ has the desired lengths because each pair of adjacent elements in $\boldsymbol{g}$ whose length was fixed by the embedding are still adjacent in $\boldsymbol{h}$ and each adjacent pair y, z whose length was not fixed is replaced by a triple whose lengths are the original length and x. There are x such pairs.

We now show that $\boldsymbol{h}$ is x-growable at m. Let (a, b) be an edge of $\boldsymbol{g}$ and let (a', b') be the corresponding edge of $\boldsymbol{h}$, obtained by the m-embedding of K_v into K_{v+x}. Clearly, we may assume $a < b$. Now, let (a'', b'') be the edge obtained from (a', b') applying the m-embedding of K_{v+x} into K_{v+2x}.

First, suppose $\ell(a', b') = \ell(a, b)$: we show that $\ell(a'', b'') = \ell(a', b')$. We have to distinguish three cases and apply Remark 2.1:

1. If $a', b' \leq m$ then $a'' = a' = a$ and $b'' = b' = b$: hence, $b'' - a'' = b - a \leq \frac{v}{2} \leq \frac{v+2x}{2}$.
2. If $a', b' > m$ then $a' = a + x$, $b' = b + x$, $a'' = a' + x$ and $b'' = b' + x$: hence $b'' - a'' = b - a \leq \frac{v}{2} < \frac{v+2x}{2}$.
3. If $a' \leq m < b'$ then $a \leq m$ and $b > m$, so $a' = a$, $b' = b + x$ and $b - a > \frac{v}{2}$: hence, $b'' - a'' = b + x - a > \frac{v+2x}{2}$.

In each case, by Remark 2.1, we get that $\ell(a'', b'') = \ell(a', b') = \ell(a, b)$.

Now, suppose $\ell(a', b') \neq \ell(a, b)$. Then $m - x < a \leq m$ and $b > m$. Take the edge $(a + x, b + x)$ of $\boldsymbol{h}$: the corresponding edge (a'', b'') by the m-embedding of K_{v+x} into K_{v+2x} is such that $a'' = a + 2x$, $b'' = b + 2x$, whence $b'' - a'' \leq \frac{v}{2}$. This implies that $\ell(a'', b'') = \ell(a', b')$. Finally, consider the edge $(a, a + x)$. Note that $(a + x) - a = x \leq \frac{v}{2} < \frac{v+x}{2}$, so the corresponding edge in K_{v+2x} has length which is increased by the m-embedding, and this is the unique edge whose length changes. We conclude that $\boldsymbol{h}$ is x-growable at m.

With similar reasoning, but with many more tedious calculations, one can prove that if $\boldsymbol{g}$ is x'-growable at m', then $\boldsymbol{h}$ is x'-growable at m' if $m' \leq m$ and x'-growable at $m' + x$ if $m' > m$. □

Example 2.5. Applying the 2-embedding of K_9 into K_{12} to the 3-growable realization of Example 2.3 we obtain the sequence

$$9 - 7 - 6 \cdots 0 - 10 - 1 \cdots 8 \cdots 2 - 11.$$

Now, following the proof of Theorem 2.4, we insert the vertices $3, 4, 5$, replacing the edges $6 \cdots 0$, $1 \cdots 8$ and $8 \cdots 2$ with $6 - 3 \cdots 0$, $1 \cdots 4 - 8$ and $8 - 5 \cdots 2$, respectively. In this way, the sequence

$$9 - 7 - 6 - 3 \cdots 0 - 10 - 1 \cdots 4 - 8 - 5 \cdots 2 - 11$$

is a cyclic realization of $\{1, 2^2, 3^7, 4\}$, which is still 3-growable at 2.

Theorem 2.6. *Suppose a multiset L has a realization that is $\{x_1, \ldots, x_k\}$-growable. Then the multiset $L \cup \{x_1^{x_1\ell_1}, x_2^{x_2\ell_2}, \ldots, x_k^{x_k\ell_k}\}$ has a $\{x_1, \ldots, x_k\}$-growable realization for any $\ell_1, \ell_2, \ldots, \ell_k \geq 0$.*

Proof. Repeatedly apply Theorem 2.4. □

Example 2.7. The sequence

$$[0, 3, 6, 2, 1, 13, 10, 11, 14, 12, 9, 8, 5, 4, 7]$$

is a cyclic realization of $L = \{1^4, 2, 3^8, 4\}$. It is 1-growable at 8 and 9; it is 2-growable at 3; it is 3-growable at 11; and it is 4-growable at 5.

If we apply Theorem 2.4 four times with $x = 2$ and then three times with $x = 3$ we get the sequence

$$[0, 3, 5, 7, 9, 11, 14, 10, 8, 6, 4, 2, 1, 30, 27, 24, 21, 18, 19, \\ 22, 25, 28, 31, 29, 26, 23, 20, 17, 16, 13, 12, 15],$$

which is a $\{1, 2, 3, 4\}$-growable realization of $\{1^4, 2, 3^8, 4\} \cup \{2^8, 3^9\} = \{1^4, 2^9, 3^{17}, 4\}$.

Any standard linear realization (and hence any perfect realization) is 1-growable at 0.

Suppose we are investigating multisets that have underlying set $U = \{x_1, \ldots, x_k\}$. Using Theorem 2.6, a U-growable realization for a multiset $L = \{x_1^{a_1}, \ldots, x_k^{a_k}\}$ is sufficient to cover all multisets $M = \{x_1^{b_1}, \ldots, x_k^{b_k}\}$ with $b_i \geq a_i$ for each i and

$$(b_1, \ldots, b_k) \equiv (a_1, \ldots, a_k) \pmod{(x_1, \ldots, x_k)}.$$

This means that the task frequently breaks naturally into considering $\prod_{i=1}^{k} x_i$ cases according to congruence modulo $(x_1, \ldots, x_k)$.

We conclude this section with two lemmas that allow the expansion of the range of values for which realizations are growable.

Lemma 2.8. *Suppose L has an X-growable realization with $1 \in X$ and K has a Y-growable perfect linear realization. Then $L \cup K$ has a $(X \cup Y)$-growable realization.*

Proof. Suppose $|K| = k$ and let $\boldsymbol{g} = [g_1, \ldots, g_{k+1}]$ be a Y-growable perfect linear realization of K.

Apply Theorem 2.4 k times with $x = 1$ to the X-growable realization of L to obtain an X-growable realization of $L \cup \{1^k\}$ with subsequence $m, m+1, \ldots, m+k$. Replace this subsequence with $\boldsymbol{g} + m$ to obtain the desired $(X \cup Y)$-growable realization of $L \cup K$. □

It is possible to take Y to be the empty set in Lemma 2.8 to construct an X-growable realization for $L \cup K$.

Lemma 2.9. *Suppose L has an X-growable realization with $2 \in X$. Let y and z be even (possibly with $y = z$). Then $L \cup \{1^{y+z-4}, y^{y+1}, z^{z+1}\}$ has an $(X \cup \{y, z\})$-growable realization.*

Proof. Apply Theorem 2.4 $y+z-1$ times with $x = 2$ to the X-growable realization of L to obtain an X-growable realization of $L \cup \{2^{2(y+z-1)}\}$ with the following two subsequences:

$$[m, m+2, \ldots, m+2y+2z-2], [m-1, m+1, \ldots, m+2y+2z-3].$$

The sequence

$$\boldsymbol{g} = [1, y+1, y+2, 2, 3, y+3, \ldots, y-1, 2y-1, 2y+z-1, 2y+2z-1]$$

uses the elements

$$\{1, 2, \dots, y-1, y+1, y+2, \dots, 2y-1, 2y+z-1, 2y+2z-1\}$$

and has edge-lengths $\{1^{y-2}, y^{y-1}, z^2\}$. The sequence

$$\boldsymbol{h} = [0, y, 2y, 2y+z, 2y+z+1, 2y+1, 2y+2, 2y+z+2, \dots, 2y+2z-2]$$

uses the elements

$$\{0, y, 2y, 2y+1, \dots, 2y+z-2, 2y+z, 2y+z+1, \dots, 2y+2z-2\},$$

and has edge-lengths $\{1^{z-2}, y^2, z^{z-1}\}$. The elements used by $\boldsymbol{g}$ and $\boldsymbol{h}$ together are exactly those used in the two subsequences from the realization of $L \cup \{2^{2(y+z-1)}\}$. Replace the two subsequences with $\boldsymbol{g} + m - 1$ and $\boldsymbol{h} + m - 1$ respectively to obtain a realization of $L \cup \{1^{y+z-4}, y^{y+1}, z^{z+1}\}$.

It is y-growable at $m + y - 1$ because each t in the range $m - 1 < t < m + y - 1$ is adjacent to $t + y > m + y - 1$ and to $t \pm 1 \leq m + y - 1$, and $m + y - 1$ is adjacent to $m - 1$ and $m + 2y - 1$. It is z-growable at $m + 2y + z - 2$ because each t in the range $m + 2y - 2 < t < m + 2y + z - 2$ is adjacent to $t + z > m + 2y + z - 2$ and to $t \pm 1 \leq m+2y+z-2$, and $m+2y+z-2$ is adjacent to $m+2y-2$ and $m+2y+2z-2$. □

3 Complete solutions for $U = \{1, 2, 3\}$ and $U = \{1, 4, 5\}$

Given any fixed set U, we may use growable realizations to try to prove $\mathrm{BHR}(L)$ for all but finitely many multisets L with underlying set U. To do this, divide the problem into $\prod_{x \in U} x$ cases, corresponding to the possible congruence classes of the number of occurrences of each element $x \pmod{x}$. For each case, a finite number—possibly one—of growable realizations can show that all but finitely many—possibly zero—admissible L matching these congruence classes has a realization. The finitely many exceptions can then be dealt with directly. In this section we illustrate this process for $U = \{1, 2, 3\}$ and $U = \{1, 4, 5\}$.

When $U = \{1, 2, 3\}$, the BHR Conjecture is already known to hold [2]. However, the self-contained proof given here in Theorem 3.1 is significantly shorter, which gives an indication of the power of the method of growable realizations compared to existing tools.

When $U = \{1, 4, 5\}$, from previous work we know that $\{1^a, 4^b, 5^c\}$ is realizable when $a \geq 11$ or when both $a \geq 7$ and $b \geq c$ [8]. However, the proof of Theorem 3.3 does not rely on this result.

Theorem 3.1. *Let $L = \{1^a, 2^b, 3^c\}$ be an admissible multiset with $a, b, c \geq 1$. Then $\mathrm{BHR}(L)$ holds.*

Proof. We start with the $\{1, 2, 3\}$-growable cyclic realizations of $\{1, 2^b, 3^c\}$ described in the first part of Table 1, which allow to cover all the 6 possibilities of the congruence class combinations of $(b, c) \pmod{(2, 3)}$. Using Theorem 2.6 this proves $\mathrm{BHR}(L)$ for all $a, b \geq 1$ and $c \geq 4$. To complete the case $b + c \geq 4$, we use the $\{1, 2\}$-growable realizations for $(b, c) \in \{(2, 2), (3, 1), (3, 2), (3, 3), (4, 1)\}$ from the second part of Table 1 and the 1-growable realization of $\{1, 2, 3^3\}$, described in Table 2.

Now, the cases when $b + c < 4$ can be solved using the 1-growable realizations of $\{1^a, 2^b, 3^c\}$, described in Table 2. □

Table 1: $\{1, 2, 3\}$-growable cyclic realizations for $\{1, 2^b, 3^c\}$: they are x-growable at m_x. The congruence classes of (b, c) are taken modulo $(2, 3)$.

Classes	Realizations	(b, c)	(m_1, m_2, m_3)	Missing cases
$(0, 0)$	$[2, 4, 1, 5, 3, 0, 6]$	$(2, 3)$	$(5, 1, 3)$	
$(0, 1)$	$[3, 6, 0, 5, 2, 1, 7, 4]$	$(2, 4)$	$(2, 3, 4)$	$c = 1$
$(0, 2)$	$[6, 5, 2, 8, 1, 4, 7, 0, 3]$	$(2, 5)$	$(7, 5, 2)$	$c = 2$
$(1, 0)$	$[8, 5, 2, 3, 6, 0, 7, 1, 4]$	$(1, 6)$	$(1, 6, 3)$	$c = 3$
$(1, 1)$	$[2, 5, 1, 3, 6, 0, 4]$	$(1, 4)$	$(1, 3, 6)$	$c = 1$
$(1, 2)$	$[6, 1, 4, 7, 5, 0, 3, 2]$	$(1, 5)$	$(4, 1, 2)$	$c = 2$
$(0, 1)$	$[0, 2, 4, 1, 6, 5, 3]$	$(4, 1)$	$(5, 2, 3)$	
$(0, 2)$	$[3, 1, 4, 5, 2, 0]$	$(2, 2)$	$(4, 1, -)$	
$(1, 0)$	$[7, 4, 2, 0, 3, 1, 6, 5]$	$(3, 3)$	$(4, 5, 2)$	$(b, c) = (1, 3)$
$(1, 1)$	$[4, 2, 5, 3, 1, 0]$	$(3, 1)$	$(1, 3, -)$	
$(1, 2)$	$[2, 4, 6, 5, 1, 3, 0]$	$(3, 2)$	$(4, 1, 2)$	

Table 2: 1-growable cyclic realizations for $\{1^a, 2^b, 3^c\}$: they are 1-growable at m_1.

(a, b, c)	Realizations	m_1	(a, b, c)	Realizations	m_1
$(1, 1, 3)$	$[2, 5, 4, 1, 3, 0]$	4	$(2, 1, 2)$	$[0, 3, 5, 4, 1, 2]$	3
$(2, 2, 1)$	$[3, 1, 0, 5, 2, 4]$	1	$(3, 1, 1)$	$[0, 5, 4, 1, 3, 2]$	4

We now move on to $U = \{1, 4, 5\}$.

Lemma 3.2. *Let* $L = \{1^a, 4^b, 5^c\}$ *be an admissible multiset with* $a \geq 2$*. Then* $\mathrm{BHR}(L)$ *holds.*

Proof. In view of Theorem 1.2(1), we may assume $b, c \geq 1$. We start with the $\{1, 4, 5\}$-growable cyclic realizations of $\{1^2, 4^b, 5^c\}$ described in the first part of Table 4 (note that in this case $b + c \geq 7$). These realizations allow to cover all the 20 possibilities of the congruence class combinations of $(b, c) \pmod{(4, 5)}$. Using Theorem 2.6, this proves $\mathrm{BHR}(L)$ for all $a \geq 2$, $b \geq 7$ and $c \geq 1$. The case $2 \leq b \leq 6$ with $b + c \geq 8$ can be solved using the $\{1, 5\}$-growable cyclic realizations of $\{1^2, 4^b, 5^c\}$ provided by Table 4, with the exception of $(b, c) \equiv (2, 4) \pmod{(4, 5)}$. Furthermore, the same table gives 5-growable cyclic realizations of $\{1^2, 4, 5^c\}$ for $c \geq 7$ with $c \not\equiv 1 \pmod 5$. Note that the multisets $\{1^2, 4, 5^{5k+6}\}$ are not admissible.

To complete the case $b + c \geq 7$ we consider the 5-growable cyclic realization of $\{1^2, 4^2, 5^9\}$ and the 1-growable cyclic realizations of $\{1^2, 4^b, 5^{7-b}\}$, $2 \leq b \leq 6$, given in Table 4, as well as the $\{1, 5\}$-growable cyclic realization of $\{1^3, 4^2, 5^9\}$ given in Table 3.

To conclude our proof, we use the 1-growable cyclic realizations of $\{1^a, 4^b, 5^c\}$ with $a + b + c = 9$, described in Table 3. □

Theorem 3.3. *Let* $L = \{1^a, 4^b, 5^c\}$ *be an admissible multiset with* $a, b, c \geq 0$*. Then* $\mathrm{BHR}(L)$ *holds.*

Table 3: $\{1,5\}$-growable cyclic realizations for $\{1^a, 4^b, 5^c\}$, $a \geq 3$: they are x-growable at m_x.

(a,b,c)	Realizations	(m_1, m_5)
$(3,1,6)$	$[6,7,2,1,5,0,10,4,9,3,8]$	$(9,4)$
$(3,2,9)$	$[9,14,0,10,5,4,8,13,3,7,12,2,1,11,6]$	$(3,9)$
$(3,1,5)$	$[8,3,2,7,1,6,5,0,9,4]$	$(2,-)$
$(3,2,4)$	$[7,2,6,1,5,0,9,8,3,4]$	$(1,-)$
$(3,3,3)$	$[3,2,8,4,9,0,5,1,6,7]$	$(8,-)$
$(3,4,2)$	$[6,2,8,7,3,4,9,0,5,1]$	$(7,-)$
$(3,5,1)$	$[5,6,2,8,9,4,0,1,7,3]$	$(7,-)$
$(4,1,4)$	$[2,1,6,7,3,8,9,4,5,0]$	$(1,-)$
$(4,2,3)$	$[7,8,4,9,3,2,1,6,5,0]$	$(4,-)$
$(4,3,2)$	$[9,5,0,6,1,2,3,4,8,7]$	$(4,-)$
$(4,4,1)$	$[0,9,4,3,7,8,2,6,5,1]$	$(8,-)$
$(5,1,3)$	$[8,3,2,7,6,5,1,0,9,4]$	$(6,-)$
$(5,2,2)$	$[5,4,9,0,1,2,8,3,7,6]$	$(8,-)$
$(5,3,1)$	$[4,5,9,8,3,7,6,2,1,0]$	$(2,-)$
$(6,1,2)$	$[3,4,8,9,0,5,6,7,2,1]$	$(8,-)$
$(6,2,1)$	$[8,4,3,2,1,7,6,5,0,9]$	$(4,-)$
$(7,1,1)$	$[3,2,1,0,4,9,8,7,6,5]$	$(8,-)$

Proof. By Lemma 3.2 we are left with the case $L = \{1, 4^b, 5^c\}$ with $b, c \geq 1$. The multiset L is admissible only if $b + c \geq 8$. Also, the following multisets are not admissible: $\{1, 4, 5^{5k+7}\}$, $\{1, 4^2, 5^{5k+6}\}$ and $\{1, 4^{4k+1}, 5\}$. The $\{4,5\}$-growable cyclic realizations of $\{1, 4^b, 5^c\}$ described in the first part of Table 5 allow to cover all the 20 possibilities of the congruence class combinations of $(b, c) \pmod{(4,5)}$. Using Theorem 2.6, this proves $\mathrm{BHR}(L)$ for all $b \geq 2$ and $c \geq 6$. To complete the case $b = 1$ we use the 5-growable cyclic realization of $\{1, 4, 5^{11}\}$ given in Table 5. Finally, the case $c \leq 5$ can be solved using the 4-growable cyclic realizations of Table 6, as well as the cyclic realization $[0,5,9,4,8,3,7,2,1,6]$ of $\{1, 4^3, 5^5\}$. □

4 A complete solution for $U \subseteq \{1,2,3,4\}$

In this section we prove $\mathrm{BHR}\left(\{1^a, 2^b, 3^c, 4^d\}\right)$. In view of Theorem 1.2(2) and 1.2(3), we may assume $c, d \geq 1$. Also, by Theorem 1.2(4) we have as a starting point that $\mathrm{BHR}(L)$ holds for $a \geq 3$ and also for $a = 2$ when $b \geq 1$. We begin by closing the case $a = 2$.

Lemma 4.1. *Let $L = \{1^2, 3^c, 4^d\}$ be an admissible multiset with $c, d \geq 1$. Then* $\mathrm{BHR}(L)$ *holds.*

Proof. First, note that L is admissible only if $c + d \geq 5$. The first part of Table 7 collects $\{3,4\}$-growable cyclic realizations for L in each of the 12 possibilities of congruence class combinations of $(c, d) \pmod{(3,4)}$. Using Theorem 2.6, this proves $\mathrm{BHR}(L)$ except in the following cases: $d = 1, 2$; $d = 3$ and $c \not\equiv 0 \pmod 3$; $d = 4$ and $c \equiv 1 \pmod 3$. So, we prove the validity of $\mathrm{BHR}(L)$ for these exceptional cases using the 3-growable

Table 4: $\{1, 4, 5\}$-growable cyclic realizations for $\{1^2, 4^b, 5^c\}$: they are x-growable at m_x. The congruence classes of (b, c) are taken modulo $(4, 5)$.

Classes	Realizations	(b, c)	(m_1, m_4, m_5)	Missing cases
$(0, 0)$	$[5, 9, 1, 6, 7, 2, 10, 3, 8, 4, 11, 0]$	$(4, 5)$	$(9, 4, 5)$	
$(0, 1)$	$[5, 9, 1, 6, 2, 10, 11, 3, 7, 8, 4, 0]$	$(8, 1)$	$(9, 3, 5)$	$b = 4$
$(0, 2)$	$[5, 6, 1, 10, 9, 0, 4, 8, 12, 3, 7, 2, 11]$	$(8, 2)$	$(8, 3, 5)$	$b = 4$
$(0, 3)$	$[1, 11, 12, 2, 7, 3, 13, 4, 8, 9, 5, 0, 10, 6]$	$(8, 3)$	$(10, 5, 6)$	$b = 4$
$(0, 4)$	$[1, 6, 7, 2, 9, 5, 0, 10, 3, 8, 4]$	$(4, 4)$	$(9, 3, 4)$	
$(1, 0)$	$[7, 8, 3, 11, 2, 10, 6, 1, 5, 9, 4, 0, 12]$	$(5, 5)$	$(10, 6, 7)$	$b = 1$
$(1, 1)$	$[10, 1, 6, 2, 11, 12, 3, 7, 8, 4, 0, 9, 5]$	$(9, 1)$	$(9, 3, 5)$	$b = 1, 5$
$(1, 2)$	$[5, 9, 13, 3, 7, 8, 4, 0, 10, 1, 6, 2, 12, 11]$	$(9, 2)$	$(9, 3, 5)$	$b = 1, 5$
$(1, 3)$	$[5, 9, 2, 7, 6, 1, 0, 4, 8, 3, 10]$	$(5, 3)$	$(9, 3, 5)$	$b = 1$
$(1, 4)$	$[0, 1, 5, 9, 4, 11, 3, 8, 7, 2, 10, 6]$	$(5, 4)$	$(10, 4, 6)$	$b = 1$
$(2, 0)$	$[4, 9, 13, 0, 10, 5, 1, 11, 6, 2, 3, 8, 12, 7]$	$(6, 5)$	$(1, 4, 7)$	$b = 2$
$(2, 1)$	$[7, 11, 1, 5, 9, 10, 6, 2, 12, 13, 3, 8, 4, 0]$	$(10, 1)$	$(11, 3, 7)$	$b = 2, 6$
$(2, 2)$	$[1, 6, 2, 9, 5, 0, 10, 3, 7, 8, 4]$	$(6, 2)$	$(9, 3, 5)$	$b = 2$
$(2, 3)$	$[5, 9, 1, 6, 2, 10, 3, 7, 8, 4, 11, 0]$	$(6, 3)$	$(9, 4, 5)$	$b = 2$
$(2, 4)$	$[5, 6, 1, 10, 9, 0, 8, 4, 12, 3, 7, 2, 11]$	$(6, 4)$	$(8, 4, 5)$	$b = 2$
$(3, 0)$	$[1, 6, 2, 8, 9, 3, 7, 0, 5, 4, 10]$	$(3, 5)$	$(7, 10, 4)$	
$(3, 1)$	$[10, 3, 7, 8, 4, 0, 1, 6, 2, 9, 5]$	$(7, 1)$	$(9, 3, 5)$	$b = 3$
$(3, 2)$	$[11, 3, 7, 2, 10, 6, 1, 0, 4, 8, 9, 5]$	$(7, 2)$	$(10, 3, 5)$	$b = 3$
$(3, 3)$	$[11, 12, 3, 8, 4, 0, 9, 5, 1, 10, 2, 7, 6]$	$(7, 3)$	$(9, 3, 6)$	$b = 3$
$(3, 4)$	$[11, 12, 2, 7, 6, 1, 10, 0, 4, 8, 3, 13, 9, 5]$	$(7, 4)$	$(9, 3, 5)$	$b = 3$
$(0, 1)$	$[12, 11, 3, 8, 4, 0, 5, 9, 1, 10, 2, 7, 6]$	$(4, 6)$	$(9, 5, 6)$	
$(0, 2)$	$[3, 13, 4, 9, 10, 5, 0, 1, 6, 11, 7, 2, 12, 8]$	$(4, 7)$	$(12, 7, 8)$	
$(0, 3)$	$[12, 2, 6, 1, 11, 7, 3, 13, 8, 9, 14, 10, 0, 5, 4]$	$(4, 8)$	$(7, 3, 4)$	
$(1, 1)$	$[0, 5, 9, 8, 4, 13, 12, 3, 7, 2, 11, 1, 10, 6]$	$(5, 6)$	$(10, 5, 6)$	
$(1, 2)$	$[4, 9, 5, 1, 12, 7, 2, 3, 8, 13, 14, 10, 0, 11, 6]$	$(5, 7)$	$(1, 5, 8)$	
$(2, 0)$	$[10, 0, 5, 4, 14, 9, 8, 13, 3, 7, 12, 2, 6, 1, 11]$	$(2, 10)$	$(7, 10, 4)$	
$(2, 1)$	$[2, 7, 0, 6, 1, 8, 3, 4, 9, 10, 5]$	$(2, 6)$	$(1, 4, 5)$	
$(2, 2)$	$[5, 10, 11, 6, 1, 9, 4, 3, 8, 0, 7, 2]$	$(2, 7)$	$(1, 4, 5)$	
$(2, 3)$	$[5, 10, 1, 6, 11, 12, 7, 2, 3, 8, 0, 9, 4]$	$(2, 8)$	$(1, 4, 5)$	
$(3, 1)$	$[2, 7, 0, 8, 3, 4, 9, 1, 5, 10, 11, 6]$	$(3, 6)$	$(1, 4, 6)$	
$(3, 2)$	$[10, 2, 7, 3, 11, 12, 4, 8, 0, 9, 1, 6, 5]$	$(3, 7)$	$(8, 4, 5)$	
$(3, 3)$	$[4, 9, 0, 1, 10, 5, 6, 11, 2, 12, 7, 3, 13, 8]$	$(3, 8)$	$(3, 6, 8)$	
$(3, 4)$	$[0, 5, 10, 6, 1, 11, 7, 2, 12, 13, 3, 14, 4, 9, 8]$	$(3, 9)$	$(11, 7, 8)$	
$(1, 0)$	$[11, 2, 7, 12, 3, 8, 13, 4, 9, 10, 6, 1, 0, 5]$	$(1, 10)$	$(-, 9, 4)$	
$(1, 2)$	$[8, 3, 2, 7, 1, 6, 0, 10, 4, 9, 5]$	$(1, 7)$	$(-, 4, 5)$	
$(1, 3)$	$[6, 11, 4, 9, 8, 1, 2, 7, 3, 10, 5, 0]$	$(1, 8)$	$(-, 5, 6)$	
$(1, 4)$	$[5, 10, 2, 7, 8, 3, 11, 6, 1, 0, 9, 4, 12]$	$(1, 9)$	$(-, 4, 5)$	
$(2, 4)$	$[6, 11, 2, 7, 12, 3, 8, 4, 5, 10, 1, 0, 9, 13]$	$(2, 9)$	$(-, 5, 6)$	
$(0, 3)$	$[5, 0, 6, 1, 7, 2, 3, 9, 8, 4]$	$(4, 3)$	$(4, -, -)$	
$(1, 2)$	$[4, 8, 3, 9, 5, 0, 6, 7, 1, 2]$	$(5, 2)$	$(4, -, -)$	
$(2, 0)$	$[9, 4, 0, 5, 6, 1, 7, 2, 3, 8]$	$(2, 5)$	$(7, -, -)$	
$(2, 1)$	$[9, 3, 4, 0, 6, 5, 1, 7, 2, 8]$	$(6, 1)$	$(6, -, -)$	
$(3, 4)$	$[9, 4, 5, 0, 1, 6, 2, 8, 3, 7]$	$(3, 4)$	$(8, -, -)$	

Table 5: $\{4,5\}$-growable cyclic realizations for $\{1,4^b,5^c\}$: they are x-growable at m_x. The congruence classes of (b,c) are taken modulo $(4,5)$.

Classes	Realizations	(b,c)	(m_4,m_5)	Missing cases
$(0,0)$	$[4,9,5,0,1,6,10,3,7,2,8]$	$(4,5)$	$(3,4)$	
$(0,1)$	$[9,2,7,3,10,5,1,0,8,4,11,6]$	$(4,6)$	$(4,6)$	$c=1$
$(0,2)$	$[4,9,1,5,0,8,12,11,6,10,2,7,3]$	$(4,7)$	$(3,4)$	$c=2$
$(0,3)$	$[6,10,5,0,9,13,4,8,3,12,7,2,1,11]$	$(4,8)$	$(5,6)$	$c=3$
$(0,4)$	$[3,14,4,9,5,0,10,6,1,11,12,7,2,13,8]$	$(4,9)$	$(7,8)$	$c=4$
$(1,0)$	$[6,11,3,8,0,12,7,2,10,5,1,9,4]$	$(1,10)$	$(4,6)$	$c=5$
$(1,1)$	$[4,9,5,0,8,3,12,7,11,10,1,6,2]$	$(5,6)$	$(3,4)$	$b=1$ or $c=1$
$(1,2)$	$[12,3,7,11,2,1,6,10,0,5,9,4,13,8]$	$(5,7)$	$(6,8)$	$b=1$ or $c=2$
$(1,3)$	$[4,9,10,3,8,2,7,1,6,0,5]$	$(1,8)$	$(3,4)$	$c=3$
$(1,4)$	$[10,3,8,1,2,7,0,5,9,4,11,6]$	$(1,9)$	$(5,6)$	$c=4$
$(2,0)$	$[7,2,11,6,1,10,5,0,9,13,12,3,8,4]$	$(2,10)$	$(3,4)$	$c=5$
$(2,1)$	$[7,11,2,12,3,8,4,13,9,5,0,1,10,6]$	$(6,6)$	$(6,7)$	$b=2$ or $c=1$
$(2,2)$	$[1,6,0,5,10,3,7,2,8,9,4]$	$(2,7)$	$(3,4)$	$c=2$
$(2,3)$	$[9,1,2,7,0,5,10,3,8,4,11,6]$	$(2,8)$	$(5,6)$	$c=3$
$(2,4)$	$[9,4,12,3,8,0,1,6,11,7,2,10,5]$	$(2,9)$	$(4,5)$	$c=4$
$(3,0)$	$[4,9,14,3,8,13,2,7,12,1,6,11,10,0,5]$	$(3,10)$	$(3,4)$	$c=5$
$(3,1)$	$[4,9,3,8,2,6,10,0,5,1,7]$	$(3,6)$	$(3,4)$	$c=1$
$(3,2)$	$[1,6,10,3,8,4,11,0,7,2,9,5]$	$(3,7)$	$(4,5)$	$c=2$
$(3,3)$	$[10,5,0,4,9,1,6,11,2,3,8,12,7]$	$(3,8)$	$(6,7)$	$c=3$
$(3,4)$	$[11,6,1,2,7,12,3,8,4,13,9,0,10,5]$	$(3,9)$	$(4,5)$	$c=4$
$(1,1)$	$[1,6,10,11,2,7,12,3,8,13,4,9,0,5]$	$(1,11)$	$(9,4)$	

cyclic realizations for the cases $(c,d) \in \{(4,1),(5,1),(6,1),(3,2),(4,2),(5,2),(2,3),(4,3),(4,4)\}$ given in Table 7. The last case left open is $L=\{1^2,3,4^4\}$, for which we take the following cyclic realization: $[0,4,5,1,2,6,3,7]$. □

Lemma 4.2. *Let $L=\{1,2^b,3^c,4^d\}$ be an admissible multiset, where $b \geq 0$ is even and $c,d \geq 1$. Then $\mathrm{BHR}(L)$ holds.*

Proof. Suppose first $d \geq 5$. We start with the $\{2,3,4\}$-growable cyclic realizations of $\{1,3^c,4^d\}$ described in the first part of Table 8 (note that in this case $c+d \geq 6$). These realizations allow to cover all the 12 possibilities of the congruence class combinations of $(c,d) \pmod{(3,4)}$. Using Theorem 2.6 this proves $\mathrm{BHR}(L)$ except when $c=1$ and $d \not\equiv 2 \pmod 4$. So, suppose $c=1$. Table 8 also gives $\{2,3,4\}$-growable cyclic realizations for $d=7,8$, proving the validity of $\mathrm{BHR}(L)$ for $b \geq 0$ even, $c=1$ and $d \equiv 0,3 \pmod 4$. Hence, we may assume $d \equiv 1 \pmod 4$. Note that the multiset $L=\{1,3,4^{4k+5}\}$ does not satisfy the necessary condition, as $4k+5 > v-4 = 4k+4$. On the other hand, a $\{2,3,4\}$-growable cyclic realization of $\{1,2^2,3,4^5\}$ is given in Table 9.

Next, we consider the cases $1 \leq d \leq 4$. Table 8 provides $\{2,3\}$-growable cyclic realizations for $\{1,3^c,4^d\}$, in the following cases: $d=1$ and $5 \leq c \leq 7$; $d=2$ and $4 \leq c \leq 6$; $d=3,4$ and $3 \leq c \leq 5$. It also provides a 2-growable cyclic realization for the multiset $\{1,3^2,4^4\}$. This completes the analysis for the admissible multisets with

Table 6: 4-growable cyclic realizations for $\{1, 4^b, 5^c\}$: they are 4-growable at m_4.

(b, c)	Realizations	m_4
$(4, 4)$	$[7, 3, 8, 4, 9, 0, 5, 1, 6, 2]$	4
$(5, 3)$	$[9, 4, 8, 3, 7, 2, 6, 0, 1, 5]$	4
$(5, 4)$	$[9, 10, 3, 7, 1, 5, 0, 6, 2, 8, 4]$	3
$(5, 5)$	$[4, 8, 9, 1, 6, 11, 3, 7, 2, 10, 5, 0]$	3
$(6, 2)$	$[7, 3, 8, 4, 0, 9, 5, 1, 6, 2]$	5
$(6, 3)$	$[2, 6, 1, 7, 0, 4, 8, 3, 10, 9, 5]$	3
$(6, 4)$	$[5, 6, 1, 9, 4, 0, 8, 3, 11, 7, 2, 10]$	3
$(6, 5)$	$[5, 10, 6, 1, 9, 0, 4, 8, 12, 11, 3, 7, 2]$	4
$(7, 1)$	$[7, 3, 9, 4, 8, 2, 6, 0, 1, 5]$	4
$(7, 2)$	$[8, 4, 0, 10, 3, 7, 1, 6, 2, 9, 5]$	4
$(7, 3)$	$[4, 8, 0, 1, 5, 9, 2, 6, 11, 7, 3, 10]$	7
$(7, 4)$	$[8, 3, 12, 0, 4, 9, 5, 1, 10, 2, 6, 11, 7]$	6
$(7, 5)$	$[5, 10, 0, 9, 13, 4, 8, 7, 3, 12, 2, 6, 1, 11]$	4
$(8, 1)$	$[7, 3, 10, 0, 4, 8, 1, 6, 2, 9, 5]$	3
$(8, 2)$	$[4, 8, 0, 5, 9, 1, 6, 2, 10, 11, 3, 7]$	3
$(8, 3)$	$[6, 10, 11, 2, 7, 3, 12, 8, 4, 0, 5, 9, 1]$	5
$(9, 2)$	$[4, 9, 8, 12, 3, 7, 11, 2, 6, 10, 1, 5, 0]$	3
$(10, 1)$	$[6, 10, 1, 5, 9, 0, 4, 8, 12, 11, 2, 7, 3]$	3

$b = 0$. Now, Table 9 gives 2-growable cyclic realizations for the multiset $\{1, 2^2, 3^c, 4^d\}$ when $(c, d) \in \{(1, 3), (1, 4), (2, 2), (2, 3), (3, 1), (3, 2), (4, 1)\}$, that is with $4 \le c + d \le 5$. This completes the analysis for the admissible multisets with $b = 2$. Finally, Table 9 also gives 2-growable cyclic realizations for the multiset $\{1, 2^4, 3^c, 4^d\}$ for each (c, d) in the set $\{(1, 1), (1, 2), (2, 1)\}$, concluding our proof. □

Lemma 4.3. *Let* $L = \{1, 2^b, 3^c, 4^d\}$ *be an admissible multiset, where* $b \ge 1$ *is odd and* $c, d \ge 1$. *Then* $\mathrm{BHR}(L)$ *holds.*

Proof. The first part of Table 10 gives $\{2, 3, 4\}$-growable cyclic realizations for $\{1, 2, 3^c, 4^d\}$ for each of the 12 possibilities of congruence class combinations of (c, d) $\pmod{(3, 4)}$. Note that $c + d \ge 5$. Using Theorem 2.6, this proves $\mathrm{BHR}(L)$ except for the following cases: $d = 1, 2$; $d = 3$ and $c \not\equiv 0 \pmod 3$; $d = 4$ and $c \equiv 1 \pmod 3$. So, we prove the validity of $\mathrm{BHR}(L)$ for these exceptional cases using $\{2, 3\}$-growable cyclic realizations for $\{1, 2, 3^c, 4^d\}$ and a 2-growable cyclic realization for $\{1, 2, 3, 4^4\}$, which can be found in Table 10.

To conclude the proof we have to consider the cases when $c + d \le 4$. Table 10 also provides 2-growable cyclic realizations for $\{1, 2^3, 3^c, 4^d\}$ when (c, d) is in the set $\{(1, 2), (1, 3), (2, 1), (2, 2), (3, 1)\}$, and for $\{1, 2^5, 3, 4\}$. □

Lemma 4.4. *Let* $L = \{2^b, 3^c, 4^d\}$ *be an admissible multiset, where* $b \ge 1$ *is odd and* $c, d \ge 1$. *Then* $\mathrm{BHR}(L)$ *holds.*

Table 7: $\{3,4\}$-growable cyclic realizations for $\{1^2, 3^c, 4^d\}$: they are x-growable at m_x. The congruence classes of (c,d) are taken modulo $(3,4)$.

Classes	Realizations	(c,d)	(m_3,m_4)	Missing cases
$(0,0)$	$[3,7,1,4,0,9,2,6,5,8]$	$(3,4)$	$(2,3)$	
$(0,1)$	$[3,7,10,2,6,5,1,9,8,4,0]$	$(3,5)$	$(2,5)$	$d=1$
$(0,2)$	$[6,9,10,2,1,5,8,0,4,7,3,11]$	$(3,6)$	$(5,6)$	$d=2$
$(0,3)$	$[1,5,6,2,8,0,3,7,4]$	$(3,3)$	$(3,4)$	
$(1,0)$	$[3,7,11,10,6,2,1,5,9,0,4,8]$	$(1,8)$	$(2,7)$	$d=4$
$(1,1)$	$[3,6,2,7,8,4,0,1,5]$	$(1,5)$	$(2,4)$	$d=1$
$(1,2)$	$[5,9,8,4,1,7,3,2,6,0]$	$(1,6)$	$(4,5)$	$d=2$
$(1,3)$	$[4,8,7,3,0,1,5,9,2,6,10]$	$(1,7)$	$(3,6)$	$d=3$
$(2,0)$	$[4,8,0,3,7,6,1,5,2]$	$(2,4)$	$(3,4)$	
$(2,1)$	$[5,9,3,6,2,8,7,4,0,1]$	$(2,5)$	$(4,5)$	$d=1$
$(2,2)$	$[4,0,10,6,9,2,5,1,8,7,3]$	$(2,6)$	$(2,3)$	$d=2$
$(2,3)$	$[8,0,4,3,11,7,6,9,1,5,2,10]$	$(2,7)$	$(8,3)$	$d=3$
$(0,1)$	$[5,8,9,2,6,3,0,1,4,7]$	$(6,1)$	$(4,5)$	
$(0,2)$	$[6,3,2,5,1,4,0,7]$	$(3,2)$	$(3,-)$	
$(1,0)$	$[8,5,6,2,10,9,1,4,0,7,3]$	$(4,4)$	$(2,3)$	$(c,d)=(1,4)$
$(1,1)$	$[2,5,6,3,7,4,1,0]$	$(4,1)$	$(4,-)$	
$(1,2)$	$[3,7,8,5,2,1,4,0,6]$	$(4,2)$	$(2,3)$	
$(1,3)$	$[2,6,3,0,9,5,1,8,7,4]$	$(4,3)$	$(3,5)$	
$(2,1)$	$[0,4,1,7,8,2,5,6,3]$	$(5,1)$	$(2,3)$	
$(2,2)$	$[5,8,1,4,7,3,2,6,9,0]$	$(5,2)$	$(4,5)$	
$(2,3)$	$[7,6,2,3,0,4,1,5]$	$(2,3)$	$(2,-)$	

Table 8: $\{2,3,4\}$-growable cyclic realizations for $\{1,3^c,4^d\}$: they are x-growable at m_x. The congruence classes of (c,d) are taken modulo $(3,4)$.

Classes	Realizations	(c,d)	(m_2,m_3,m_4)	Missing cases
$(0,0)$	$[3,6,1,4,0,5,2,7,8]$	$(3,4)$	$(6,2,3)$	
$(0,1)$	$[3,6,2,8,4,1,7,0,9,5]$	$(3,5)$	$(2,4,5)$	$d=1$
$(0,2)$	$[4,5,1,8,0,3,7,10,6,2,9]$	$(3,6)$	$(7,3,4)$	$d=2$
$(0,3)$	$[3,7,6,2,10,1,5,9,0,4,8,11]$	$(3,7)$	$(9,2,6)$	$d=3$
$(1,0)$	$[4,7,0,6,3,9,8,2,5,1]$	$(4,4)$	$(7,3,4)$	$c=1$
$(1,1)$	$[6,9,2,10,3,7,4,0,1,8,5]$	$(4,5)$	$(4,5,6)$	$c=1$ or $d=1$
$(1,2)$	$[2,6,1,5,0,3,7,8,4]$	$(1,6)$	$(1,3,4)$	$d=2$
$(1,3)$	$[4,7,1,5,0,8,2,6,3]$	$(4,3)$	$(2,3,4)$	$c=1$
$(2,0)$	$[7,11,3,4,0,8,5,1,9,6,2,10]$	$(2,8)$	$(6,8,3)$	$d=4$
$(2,1)$	$[3,7,2,6,0,1,5,8,4]$	$(2,5)$	$(2,3,4)$	$d=1$
$(2,2)$	$[4,5,1,8,2,6,0,7,3,9]$	$(2,6)$	$(7,3,4)$	$d=2$
$(2,3)$	$[4,8,1,0,7,10,3,6,2,9,5]$	$(2,7)$	$(3,4,5)$	$d=3$
$(1,0)$	$[4,5,1,8,0,7,3,10,6,2,9]$	$(1,8)$	$(7,3,4)$	
$(1,3)$	$[3,7,1,4,8,2,6,0,9,5]$	$(1,7)$	$(2,4,5)$	
$(0,1)$	$[7,1,4,5,8,2,6,0,3]$	$(6,1)$	$(6,2,-)$	
$(0,2)$	$[5,6,3,9,2,8,1,4,7,0]$	$(6,2)$	$(4,5,-)$	
$(0,3)$	$[0,3,7,6,2,5,1,4]$	$(3,3)$	$(1,2,-)$	
$(1,1)$	$[5,2,9,8,1,4,7,0,6,3]$	$(7,1)$	$(7,4,9)$	
$(1,2)$	$[0,5,2,6,1,4,3,7]$	$(4,2)$	$(2,3,-)$	
$(2,0)$	$[10,9,2,6,3,0,7,4,1,8,5]$	$(5,4)$	$(8,4,5)$	$(c,d)=(2,4)$
$(2,1)$	$[2,5,0,1,6,3,7,4]$	$(5,1)$	$(3,4,-)$	
$(2,2)$	$[2,5,8,7,3,0,6,1,4]$	$(5,2)$	$(1,3,-)$	
$(2,3)$	$[3,0,6,7,1,4,8,5,2,9]$	$(5,3)$	$(5,2,7)$	
$(2,0)$	$[2,6,7,3,0,4,1,5]$	$(2,4)$	$(1,-,-)$	

Table 9: $\{2,3,4\}$-growable cyclic realizations for $\{1,2^b,3^c,4^d\}$, with $b\geq 2$ even: they are x-growable at m_x. The congruence classes of (c,d) are taken modulo $(3,4)$.

Classes	Realizations	(b,c,d)	(m_2,m_3,m_4)
$(1,1)$	$[7,8,2,6,0,4,1,9,5,3]$	$(2,1,5)$	$(6,2,3)$
$(0,1)$	$[6,4,1,2,5,7,3,0]$	$(2,3,1)$	$(5,2,-)$
$(0,2)$	$[4,7,5,1,8,0,3,6,2]$	$(2,3,2)$	$(1,3,4)$
$(1,0)$	$[1,5,8,6,2,7,0,4,3]$	$(2,1,4)$	$(6,2,3)$
$(1,1)$	$[3,6,0,4,1,8,7,5,2]$	$(2,4,1)$	$(6,2,3)$
$(1,3)$	$[1,0,4,6,2,5,3,7]$	$(2,1,3)$	$(3,4,-)$
$(2,2)$	$[7,6,2,4,1,5,3,0]$	$(2,2,2)$	$(1,3,-)$
$(2,3)$	$[3,5,7,8,2,6,1,4,0]$	$(2,2,3)$	$(6,2,3)$
$(1,1)$	$[0,3,1,7,5,4,2,6]$	$(4,1,1)$	$(1,2,-)$
$(1,2)$	$[3,5,7,8,6,2,0,4,1]$	$(4,1,2)$	$(6,2,3)$
$(2,1)$	$[1,3,5,8,2,4,0,7,6]$	$(4,2,1)$	$(6,2,-)$

Table 10: $\{2,3,4\}$-growable cyclic realizations for $\{1,2^b,3^c,4^d\}$, with $b \geq 1$ odd: they are x-growable at m_x. The congruence classes of (c,d) are taken modulo $(3,4)$.

Classes	Realizations	(b,c,d)	(m_2,m_3,m_4)	Missing cases
$(0,0)$	$[9,2,6,0,4,1,7,8,5,3]$	$(1,3,4)$	$(6,2,3)$	
$(0,1)$	$[5,8,1,9,10,2,6,4,0,7,3]$	$(1,3,5)$	$(8,4,5)$	$d=1$
$(0,2)$	$[3,5,9,1,4,0,8,7,10,6,2,11]$	$(1,3,6)$	$(6,2,3)$	$d=2$
$(0,3)$	$[4,7,3,0,1,5,8,6,2]$	$(1,3,3)$	$(1,3,4)$	
$(1,0)$	$[10,6,2,11,3,7,9,1,5,4,0,8]$	$(1,1,8)$	$(7,8,4)$	$d=4$
$(1,1)$	$[2,6,7,3,0,5,1,8,4]$	$(1,1,5)$	$(1,3,4)$	$d=1$
$(1,2)$	$[3,4,0,6,2,8,1,5,9,7]$	$(1,1,6)$	$(6,2,3)$	$d=2$
$(1,3)$	$[9,2,6,5,1,10,3,7,0,8,4]$	$(1,1,7)$	$(8,3,5)$	$d=3$
$(2,0)$	$[2,6,1,4,0,8,5,7,3]$	$(1,2,4)$	$(1,2,3)$	
$(2,1)$	$[8,1,5,4,0,6,2,9,7,3]$	$(1,2,5)$	$(7,2,4)$	$d=1$
$(2,2)$	$[3,7,0,4,6,10,2,5,1,8,9]$	$(1,2,6)$	$(7,2,4)$	$d=2$
$(2,3)$	$[7,11,3,4,0,9,1,5,8,10,6,2]$	$(1,2,7)$	$(6,8,3)$	$d=3$
$(0,1)$	$[4,7,9,2,6,3,0,1,8,5]$	$(1,6,1)$	$(3,4,5)$	
$(0,2)$	$[2,5,1,0,6,3,7,4]$	$(1,3,2)$	$(3,4,-)$	
$(1,0)$	$[3,7,10,8,0,4,5,1,9,6,2]$	$(1,4,4)$	$(7,2,4)$	$(c,d)=(1,4)$
$(1,1)$	$[0,6,3,7,4,1,2,5]$	$(1,4,1)$	$(2,4,-)$	
$(1,2)$	$[3,7,6,0,4,1,8,5,2]$	$(1,4,2)$	$(1,2,3)$	
$(1,3)$	$[3,6,9,7,0,1,5,2,8,4]$	$(1,4,3)$	$(2,3,4)$	
$(2,1)$	$[3,6,7,1,4,0,2,5,8]$	$(1,5,1)$	$(6,2,3)$	
$(2,2)$	$[7,1,4,0,2,5,8,9,6,3]$	$(1,5,2)$	$(6,2,3)$	
$(2,3)$	$[4,0,3,1,5,2,6,7]$	$(1,2,3)$	$(1,2,-)$	
$(1,0)$	$[3,7,6,2,0,4,1,5]$	$(1,1,4)$	$(1,-,-)$	
$(0,1)$	$[3,6,8,7,5,2,0,4,1]$	$(3,3,1)$	$(6,2,3)$	
$(1,0)$	$[4,6,0,8,7,3,9,1,5,2]$	$(3,1,4)$	$(7,3,4)$	
$(1,1)$	$[4,2,0,8,6,3,1,5,7]$	$(5,1,1)$	$(6,3,-)$	
$(1,2)$	$[4,6,2,5,3,7,1,0]$	$(3,1,2)$	$(3,4,-)$	
$(1,3)$	$[2,6,7,0,3,5,1,8,4]$	$(3,1,3)$	$(1,3,4)$	
$(2,1)$	$[5,2,0,1,7,3,6,4]$	$(3,2,1)$	$(3,4,-)$	
$(2,2)$	$[4,6,0,2,5,1,8,7,3]$	$(3,2,2)$	$(2,3,4)$	

Proof. The first part of Table 11 gives $\{2,3,4\}$-growable cyclic realizations for $\{2,3^c,4^d\}$ in each of the 12 possibilities of congruence class combinations of $(c,d) \pmod{(3,4)}$. Note that $c+d \geq 6$. Using Theorem 2.6, this proves $\mathrm{BHR}(L)$ except for the following cases: $d=1,2,3$; $d=4,5$ and $c \equiv 2 \pmod 3$; $c=1$ and $d \equiv 0,1 \pmod 4$. Next, we consider the case $c \geq 2$ and $1 \leq d \leq 5$, using $\{2,3\}$-growable cyclic realizations for $\{2,3^c,4^d\}$. For the exceptional case $\{2,3^2,4^4\}$ we use a 2-growable cyclic realization. Now we complete the case $c \geq 2$: Table 11 also provides 2-growable cyclic realization for the multisets $\{2^3,3^c,4^d\}$ when $(c,d) \in \{(2,2),(2,3),(3,1),(3,2),(4,1)\}$, and for the multiset $\{2^5,3^2,4\}$.

Finally, we assume $c=1$. Note that the multiset $\{2,3,4^{4k+5}\}$ does not satisfy the necessary condition. For the multisets $\{2,3,4^{4k+8}\}$ we use the 4-growable realization $[2,6,10,3,7,4,0,9,5,1,8]$ of $\{2,3,4^8\}$. Table 11 gives $\{2,4\}$-growable cyclic realizations for the multisets $\{2^3,3,4^4\}$ and $\{2^3,3,4^5\}$; it also gives 2-growable cyclic realizations for the multisets $\{2^3,3,4^3\}$, $\{2^5,3,4\}$ and $\{2^5,3,4^2\}$. □

Theorem 4.5. *Let $L=\{1^a,2^b,3^c,4^d\}$ be an admissible multiset with $a,b,c,d \geq 0$. Then* $\mathrm{BHR}(L)$ *holds.*

Proof. By Theorem 1.2(2)–(4) we may assume $0 \leq a \leq 2$ and $c,d \geq 1$. If $a=2$ the result follows from Theorem 1.2(4) and Lemma 4.1. Suppose now $a=1$. If b is even we apply Lemma 4.2, otherwise we apply Lemma 4.3. Finally, assume $a=0$. By [5] we may assume $b \geq 1$. If b is odd we apply Lemma 4.4 and so, we may also assume $b \geq 2$ is even.

We start with the $\{2,3,4\}$-growable cyclic realizations of $L=\{2^2,3^c,4^d\}$ for each of the 12 possibilities of congruence class combinations of $(c,d) \pmod{(3,4)}$ described in the first part of Table 12. Note that $c+d \geq 5$. Using Theorem 2.6, this proves $\mathrm{BHR}(L)$ except for the following cases: $d=1,2$; $d=3$ and $c \not\equiv 0 \pmod 3$; $d=4$ and $c \equiv 1 \pmod 3$. For these exceptions, Table 12 also gives $\{2,3\}$-growable cyclic realizations when $(c,d) \in \{(4,1),(5,1),(6,1),(3,2),(4,2),(5,2),(2,3),(4,3),(4,4)\}$ and a 2-growable cyclic realization for $\{2^2,3,4^4\}$.

We are left to the cases $c+d \leq 4$. This table also provides a 2-growable cyclic realization for the multiset $\{2^4,3^c,4^d\}$ when $(c,d) \in \{(1,2),(1,3),(2,1),(2,2),(3,1)\}$, and for the multiset $\{2^6,3,4\}$. □

The success in proving these small cases leads us to make the following conjecture, which says that the method of the previous section and this one is always successful.

Conjecture 4.6. *For any fixed set U, there is a finite set of growable realizations with underlying set U that implies the existence of realizations for all but finitely many admissible multisets L that have underlying set U.*

If Conjecture 4.6 is true, then the BHR Conjecture for any given underlying set can be proved with a finite set of realizations.

5 A partial solution for $U=\{1,x,2x\}$

In previous sections we have seen how it is possible to completely prove the BHR Conjecture for a fixed U by the construction of one or more base case realizations for each of $\prod_{x \in U} x$ cases. In this section we develop ways to produce general results with fewer base cases.

Table 11: $\{2,3,4\}$-growable cyclic realizations for $\{2^b, 3^c, 4^d\}$, with $b \geq 1$ odd: they are x-growable at m_x. The congruence classes of (c,d) are taken modulo $(3,4)$.

Classes	Realizations	(b,c,d)	(m_2,m_3,m_4)	Missing cases
$(0,0)$	$[2,6,8,5,0,4,1,7,3]$	$(1,3,4)$	$(1,2,3)$	
$(0,1)$	$[2,5,1,8,4,0,6,9,7,3]$	$(1,3,5)$	$(1,2,4)$	$d=1$
$(0,2)$	$[3,6,10,7,0,9,2,5,1,8,4]$	$(1,3,6)$	$(2,3,4)$	$d=2$
$(0,3)$	$[6,10,2,5,9,0,8,4,1,3,11,7]$	$(1,3,7)$	$(5,6,7)$	$d=3$
$(1,0)$	$[3,6,9,5,2,8,0,4,1,7]$	$(1,4,4)$	$(6,2,3)$	$c=1$
$(1,1)$	$[2,5,9,6,3,10,8,1,4,0,7]$	$(1,4,5)$	$(6,7,3)$	$c=1$ or $d=1$
$(1,2)$	$[3,7,2,6,0,5,1,8,4]$	$(1,1,6)$	$(2,3,4)$	$d=2$
$(1,3)$	$[2,6,0,4,8,1,5,9,7,3]$	$(1,1,7)$	$(1,2,5)$	$d=3$
$(2,0)$	$[10,2,6,3,11,7,5,1,9,0,8,4]$	$(1,2,8)$	$(8,3,5)$	$d=4$
$(2,1)$	$[7,11,8,12,3,5,9,0,4,1,10,6,2]$	$(1,2,9)$	$(6,7,3)$	$d=1,5$
$(2,2)$	$[2,6,0,3,7,9,5,1,8,4]$	$(1,2,6)$	$(1,3,5)$	$d=2$
$(2,3)$	$[9,2,5,1,8,0,7,3,10,6,4]$	$(1,2,7)$	$(7,3,4)$	$d=3$
$(0,1)$	$[3,6,0,4,1,7,5,2,8]$	$(1,6,1)$	$(6,2,3)$	
$(0,2)$	$[4,7,1,5,2,9,6,3,0,8]$	$(1,6,2)$	$(7,3,4)$	
$(0,3)$	$[0,4,1,7,3,6,2,5]$	$(1,3,3)$	$(2,3,-)$	
$(1,1)$	$[3,6,9,2,5,1,8,0,7,4]$	$(1,7,1)$	$(7,3,4)$	
$(1,2)$	$[7,1,4,0,5,2,6,3]$	$(1,4,2)$	$(2,3,-)$	
$(1,3)$	$[3,6,0,4,2,7,1,5,8]$	$(1,4,3)$	$(6,2,3)$	
$(2,0)$	$[6,10,2,9,1,4,8,0,3,7,5]$	$(1,5,4)$	$(4,5,6)$	$(c,d)=(2,4)$
$(2,1)$	$[2,6,1,3,7,4,0,5,8]$	$(1,2,5)$	$(1,4,-)$	
	$[1,4,7,3,6,0,5,2]$	$(1,5,1)$	$(2,4,-)$	
$(2,2)$	$[3,6,0,4,1,8,5,2,7]$	$(1,5,2)$	$(6,2,3)$	
$(2,3)$	$[9,6,2,8,1,5,3,0,7,4]$	$(1,5,3)$	$(7,3,4)$	
$(2,0)$	$[7,3,6,2,4,0,5,1]$	$(1,2,4)$	$(3,-,-)$	
$(0,1)$	$[1,6,0,2,5,3,7,4]$	$(3,3,1)$	$(3,4,-)$	
$(0,2)$	$[7,1,4,0,2,6,8,5,3]$	$(3,3,2)$	$(6,2,3)$	
$(1,1)$	$[3,6,8,2,5,7,0,4,1]$	$(3,4,1)$	$(6,2,3)$	
$(2,1)$	$[5,7,3,1,8,2,0,6,4]$	$(5,2,1)$	$(4,5,-)$	
$(2,2)$	$[2,4,1,5,7,3,0,6]$	$(3,2,2)$	$(1,2,-)$	
$(2,3)$	$[3,5,7,0,4,1,6,2,8]$	$(3,2,3)$	$(6,2,3)$	
$(1,0)$	$[3,7,5,0,4,1,8,6,2]$	$(3,1,4)$	$(1,2,3)$	
$(1,1)$	$[5,7,1,3,0,6,2,4]$	$(5,1,1)$	$(1,2,-)$	
$(1,2)$	$[4,6,8,1,5,2,0,7,3]$	$(5,1,2)$	$(2,3,4)$	
$(1,3)$	$[1,7,3,5,2,6,4,0]$	$(3,1,3)$	$(3,4,-)$	
$(1,4)$	$[5,9,1,7,3,0,8,2,6,4]$	$(3,1,5)$	$(3,4,5)$	

Our main goal is Theorem 5.10, which says that $\mathrm{BHR}(L)$ holds for $L = \{1^a, x^b, (2x)^c\}$ when $a \geq x-2$, c is even, and $b \geq 5x-2+c/2$. When x is even, this covers many instances not covered by Theorem 1.2(8). When x is odd, the instances covered are all new.

Lemma 5.1. *Suppose L has an X-growable realization and take $x \in X$. Take i with $1 \leq i \leq x$. Then $L \cup \{x^{3x-2i}, (2x)^{2i}\}$ has an X-growable realization.*

Table 12: $\{2,3,4\}$-growable cyclic realizations for $\{2^b,3^c,4^d\}$, with $b \geq 2$ even: they are x-growable at m_x. The congruence classes of (c,d) are taken modulo $(3,4)$.

Classes	Realizations	(b,c,d)	(m_2,m_3,m_4)	Missing cases
$(0,0)$	$[2,5,9,6,8,0,4,1,7,3]$	$(2,3,4)$	$(1,2,3)$	
$(0,1)$	$[8,0,7,3,10,1,5,2,9,6,4]$	$(2,3,5)$	$(7,3,4)$	$d=1$
$(0,2)$	$[8,0,4,2,10,1,5,7,11,3,6,9]$	$(2,3,6)$	$(7,8,3)$	$d=2$
$(0,3)$	$[3,6,1,5,8,2,4,0,7]$	$(2,3,3)$	$(6,2,3)$	
$(1,0)$	$[4,8,0,2,6,10,1,9,5,3,11,7]$	$(2,1,8)$	$(3,6,7)$	$d=4$
$(1,1)$	$[4,8,5,1,6,2,0,7,3]$	$(2,1,5)$	$(2,3,4)$	$d=1$
$(1,2)$	$[4,8,0,2,6,9,5,1,7,3]$	$(2,1,6)$	$(2,3,5)$	$d=2$
$(1,3)$	$[4,8,1,5,7,0,9,2,6,3,10]$	$(2,1,7)$	$(8,3,5)$	$d=3$
$(2,0)$	$[2,6,0,3,7,5,1,8,4]$	$(2,2,4)$	$(1,3,4)$	
$(2,1)$	$[4,7,1,5,3,9,6,2,8,0]$	$(2,2,5)$	$(7,3,4)$	$d=1$
$(2,2)$	$[5,8,10,3,7,0,4,6,2,9,1]$	$(2,2,6)$	$(8,4,5)$	$d=2$
$(2,3)$	$[10,2,6,8,0,9,5,1,11,3,7,4]$	$(2,2,7)$	$(9,3,6)$	$d=3$
$(0,1)$	$[4,7,9,6,3,0,8,1,5,2]$	$(2,6,1)$	$(7,3,4)$	
$(0,2)$	$[2,5,7,3,6,0,4,1]$	$(2,3,2)$	$(2,3,-)$	
$(1,0)$	$[0,3,7,10,8,1,5,2,9,6,4]$	$(2,4,4)$	$(7,3,4)$	$(c,d)=(1,4)$
$(1,1)$	$[0,3,1,6,2,5,7,4]$	$(2,4,1)$	$(5,2,-)$	
$(1,2)$	$[3,6,1,4,0,7,5,2,8]$	$(2,4,2)$	$(6,2,3)$	
$(1,3)$	$[8,1,5,2,0,7,3,9,6,4]$	$(2,4,3)$	$(7,3,4)$	
$(2,1)$	$[7,5,2,8,1,4,0,6,3]$	$(2,5,1)$	$(6,2,3)$	
$(2,2)$	$[9,1,5,2,8,6,3,0,7,4]$	$(2,5,2)$	$(7,3,4)$	
$(2,3)$	$[7,3,6,4,0,2,5,1]$	$(2,2,3)$	$(3,4,-)$	
$(1,0)$	$[0,4,2,6,1,5,3,7]$	$(2,1,4)$	$(3,-,-)$	
$(0,1)$	$[2,5,7,0,4,1,8,6,3]$	$(4,3,1)$	$(6,2,3)$	
$(1,1)$	$[7,5,2,0,4,6,8,1,3]$	$(6,1,1)$	$(6,2,-)$	
$(1,2)$	$[7,1,3,5,2,6,4,0]$	$(4,1,2)$	$(3,4,-)$	
$(1,3)$	$[3,7,0,4,6,8,1,5,2]$	$(4,1,3)$	$(1,2,4)$	
$(2,1)$	$[1,3,0,6,2,4,7,5]$	$(4,2,1)$	$(5,2,-)$	
$(2,2)$	$[8,6,2,0,4,1,7,5,3]$	$(4,2,2)$	$(6,2,3)$	

Proof. Apply Theorem 2.4 three times to the x-growable realization of L to obtain an X-growable realization of $L \cup \{x^{3x}\}$ with each of the subsequences

$$[m,m+x,m+2x,m+3x],[m-1,m-1+x,m-1+2x,m-1+3x],\ldots,$$
$$[m-x+1,m+1,m+1+x,m+1+2x]$$

appearing, possibly reversed. Each subsequence has differences $\{x^3\}$. Take i of the subsequences and in each switch the middle two elements (so, for example, the first would become $[m,m+2x,m+x,m+3x]$). Each time we perform this operation we obtain a subsequence with differences $\{x,(2x)^2\}$ instead of $\{x^3\}$. After performing it i times the new differences are $\{x^{3x-2i},(2x)^{2i}\}$.

These operations do not interfere with growability: if the original realization is y-growable at m', then the new realization is y-growable at m' if $m' \leq m$ and at $m'+3x$

otherwise. □

Let $L = \{1^a, x^b, (2x)^c\}$. When $x = 1$ or 2 $\mathrm{BHR}(L)$ follows from Theorem 1.2(1) or 1.2(2) respectively, so $x = 3$ is the first open case. We treat the $x = 3$ case first both as an illustration of the general method and because some of the later constructions require $x > 3$.

Lemma 5.2. *Let $L = \{1^a, 3^b, 6^c\}$. If c is even and $b \geq 13 + c/2$, then $\mathrm{BHR}(L)$ holds; if c is odd and $b \geq 18 + (c-1)/2$, then $\mathrm{BHR}(L)$ holds.*

Proof. By Theorem 1.2(1) we may assume that $a, b, c \geq 1$. The multiset L is not admissible when $a = 1$ and $b + c \equiv 1 \pmod 3$.

Table 13 gives $\{1, 3\}$-growable realizations for

$$(a, b, c) \in \{(2, 4, 0), (1, 5, 0), (1, 6, 0), (1, 10, 1), (1, 11, 1), (2, 9, 1)\}.$$

First we use Lemma 5.1 along with the realizations of Table 13 to obtain a $\{1, 3\}$-growable realization of $L' = \{1^{a'}, 3^{b'}, 6^{c'}\}$ with $a' \in \{1, 2\}$, $b' \equiv b \pmod 3$ and $c' \leq 5$ such that $c' \equiv c \pmod 6$. We have to distinguish cases according to the congruence class of c modulo 6.

Table 13: $\{1, 3\}$-growable realizations of $\{1^a, 3^b, 6^c\}$. Where they are 1- or 3-growable is indicated by (m_1, m_3).

name	realization	(a, b, c)	(m_1, m_3)
$\boldsymbol{g}_1$	$[6, 5, 1, 4, 0, 3, 2]$	$(2, 4, 0)$	$(4, 2)$
$\boldsymbol{g}_2$	$[3, 0, 6, 2, 5, 1, 4]$	$(1, 5, 0)$	$(5, 2)$
$\boldsymbol{g}_3$	$[5, 2, 7, 0, 3, 6, 1, 4]$	$(1, 6, 0)$	$(6, 2)$
$\boldsymbol{g}_4$	$[2, 12, 9, 6, 3, 0, 10, 7, 4, 5, 8, 1, 11]$	$(1, 10, 1)$	$(3, 5)$
$\boldsymbol{g}_5$	$[5, 2, 13, 10, 7, 8, 11, 0, 3, 6, 9, 12, 4, 1]$	$(1, 11, 1)$	$(6, 8)$
$\boldsymbol{g}_6$	$[9, 6, 3, 0, 10, 7, 8, 11, 1, 4, 5, 12, 2]$	$(2, 9, 1)$	$(6, 8)$

When c is even, we start with $\boldsymbol{g}_1$, $\boldsymbol{g}_2$ or $\boldsymbol{g}_3$. If $c \equiv 0 \pmod 6$, then start by taking $\boldsymbol{g}_1$, $\boldsymbol{g}_2$ or $\boldsymbol{g}_3$ according to whether b is congruent to $1, 2$ or $0 \pmod 3$ respectively. If $c \equiv 2 \pmod 6$, then start by taking $\boldsymbol{g}_1$, $\boldsymbol{g}_2$ or $\boldsymbol{g}_3$ according to whether b is congruent to $2, 0$ or $1 \pmod 3$ respectively and apply Lemma 5.1 with $i = 1$. If $c \equiv 4 \pmod 6$, then start by taking $\boldsymbol{g}_1$, $\boldsymbol{g}_2$ or $\boldsymbol{g}_3$ according to whether b is congruent to $0, 1$ or $2 \pmod 3$ respectively and apply Lemma 5.1 with $i = 2$.

When c is odd, we start with $\boldsymbol{g}_4$, $\boldsymbol{g}_5$ or $\boldsymbol{g}_6$. If $c \equiv 1 \pmod 6$, then start by taking $\boldsymbol{g}_4$, $\boldsymbol{g}_5$ or $\boldsymbol{g}_6$ according to whether b is congruent to $1, 2$ or $0 \pmod 3$ respectively. If $c \equiv 3 \pmod 6$, then start by taking $\boldsymbol{g}_4$, $\boldsymbol{g}_5$ or $\boldsymbol{g}_6$ according to whether b is congruent to $2, 0$ or $1 \pmod 3$ respectively and apply Lemma 5.1 with $i = 1$. If $c \equiv 5 \pmod 6$, then start by taking $\boldsymbol{g}_4$, $\boldsymbol{g}_5$ or $\boldsymbol{g}_6$ according to whether b is congruent to $0, 1$ or $2 \pmod 3$ respectively and apply Lemma 5.1 with $i = 2$.

In each case we obtain the required realization of L'. Next, apply Lemma 5.1 $(c-c')/6$ times with $x = i = 3$ to obtain a $\{1, 3\}$-growable realization of $\{1^{a'}, 3^{b'+(c-c')/2}, 6^c\}$. Finally, complete to the required realization using $a - a'$ applications of Theorem 2.4 with $x = 1$ and $\frac{b-b'}{3} - \frac{c-c'}{6}$ applications with $x = 3$.

When c is even, the method requires up to six 3's in the $\boldsymbol{g}_i$, up to seven 3's to adjust the congruency class of the number of 6's, and then $c/2$ 3's to obtain the correct number of 6's. Hence it always works for $b \geq 6+7+c/2 = 13+c/2$. When c is odd, the method requires up to eleven 3's in the $\boldsymbol{g}_i$, up to seven 3's to adjust the congruency class of the number of 6's, and then up to $(c-1)/2$ 3's to obtain the correct number of 6's. Hence it always works for $b \geq 11+7+(c-1)/2 = 18+(c-1)/2$. $\square$

Example 5.3. Let $L = \{1^3, 3^{18}, 6^{10}\}$. Since $b \equiv 0 \pmod 3$ and $c \equiv 4 \pmod 6$, we start applying Lemma 5.1 with $i = 2$ to $\boldsymbol{g}_1$. In this way we obtain the realization

$$[15, 14, 1, 7, 4, 10, 13, 0, 6, 3, 9, 12, 11, 8, 5, 2]$$

of the multiset $\{1^2, 3^9, 6^4\}$. Now we apply Lemma 5.1 once with $i = x = 3$ to this new multiset and we get a realization of $\{1^2, 3^{12}, 6^{10}\}$:

$$[24, 23, 1, 7, 4, 10, 16, 13, 19, 22, 0, 6, 3, 9, 15, 12, 18, 21, 20, 17, 14, 11, 5, 8, 2].$$

We now apply Theorem 2.4 twice with $x = 3$ to get a realization of $\{1^2, 3^{18}, 6^{10}\}$ and then once with $x = 1$ to get a realization of L:

$$[31, 30, 1, 4, 7, 13, 10, 16, 22, 19, 25, 28, 29, 0, 3, 6, 12, \\ 9, 15, 21, 18, 24, 27, 26, 23, 20, 17, 11, 14, 8, 5, 2].$$

To use the method of proof of Lemma 5.2 for $x > 3$ we require $\{1, x\}$-growable realizations for $\{1^a, x^b\}$ for a as small as possible and for a b in each congruence class modulo x. Lemmas 5.4, 5.6 and 5.8 provide these. Each of the constructions has at least one subsequence consisting of multiple instances of pairs $[t, t+x]$, triples $[t, t+x, t+2x]$ or their reverses; we indicate these pairs and triples with underbraces to help illuminate the overall structure.

Lemma 5.4. *Let $x \geq 4$. The multisets $\{1^{x-1}, x^{x+1}\}$, $\{1^{x-2}, x^{x+2}\}$ and $\{1^{x-2}, x^{2x}\}$ have $\{1, x\}$-growable realizations.*

Proof. First, we cover $\{1^{x-1}, x^{x+1}\}$, in which case $v = 2x+1$. When x is even, the sequence

$$[1, x+1, 0, 2x, x, \underbrace{x-1, 2x-1}, \underbrace{2x-2, x-2}, \underbrace{x-3, 2x-3}, \ldots, \underbrace{x+2, 2}]$$

has edge-lengths $[x, x, 1, x, 1, x, \ldots, 1, x]$ and so realizes $\{1^{x-1}, x^{x+1}\}$. It is 1-growable at 1 and x-growable at x. When x is odd, the sequence

$$[x, x+1, 1, 0, 2x, x-1, 2x-1, \underbrace{x-2, 2x-2}, \underbrace{2x-3, x-3}, \underbrace{x-4, 2x-4}, \ldots, \underbrace{x+2, 2}]$$

has edge-lengths $[1, x, 1, 1, x, x, x, x, 1, x, \ldots, 1, x]$ and so realizes $\{1^{x-1}, x^{x+1}\}$. It is 1-growable at $2x-1$ and x-growable at x.

Next, consider $\{1^{x-2}, x^{x+2}\}$ and so $v = 2x+1$. When x is even, the sequence

$$[x, 2x, 0, x+1, 1, x+2, 2, \underbrace{3, x+3}, \underbrace{x+4, 4}, \underbrace{5, x+5}, \ldots, \underbrace{x-1, 2x-1}]$$

has edge-lengths $[x, 1, x, x, x, x, 1, x, 1, x, \dots, 1, x]$ and so realizes $\{1^{x-2}, x^{x+2}\}$. It is 1-growable at 1 and x-growable at x. When x is odd, the sequence

$$[0, x, x-1, 2x, 2x-1, x-2, 2x-2, \underbrace{x-3, 2x-3}, \underbrace{2x-4, x-4}, \underbrace{x-5, 2x-5}, \dots, \underbrace{x+1, 1}]$$

has edge-lengths $[x, 1, x, 1, x, x, x, x, 1, x, 1, x, \dots, 1, x]$ so this realizes $\{1^{x-2}, x^{x+2}\}$. It is 1-growable at $2x-2$ and x-growable at $x-1$.

Finally, consider $\{1^{x-2}, x^{2x}\}$ and so $v = 3x-1$. When x is even, the sequence

$$[\underbrace{0, x, 2x}, \underbrace{2x+1, x+1, 1}, \underbrace{2, x+2, 2x+2}, \dots, \underbrace{x-4, 2x-4, 3x-4}, x-3, 2x-3, 2x-2, x-2, 3x-3, 3x-2, x-1, 2x-1]$$

has edge-lengths $[x, x, 1, x, x, 1, \dots, 1, x, x, x, x, 1, x, x, 1, x, x]$ and so realizes $\{1^{x-2}, x^{2x}\}$. It is 1-growable at $3x-4$ and x-growable at $x-1$. When x is odd the sequence

$$[3x-2, x-1, 2x-1, \underbrace{2x, x, 0}, \underbrace{1, x+1, 2x+1}, \underbrace{2x+2, x+2, 2}, \dots, \underbrace{x-4, 2x-4, 3x-4}, x-3, 2x-3, 2x-2, x-2, 3x-3]$$

has edge-lengths $[x, x, 1, x, x, 1, x, x, \dots, 1, x, x, x, x, 1, x, x]$ and so realizes $\{1^{x-2}, x^{2x}\}$. It is 1-growable at $3x-4$ and x-growable at x. □

Example 5.5. Let $x = 8$. Lemma 5.4 gives the $\{1, 8\}$-growable realizations

$$[1, 9, 0, 16, 8, 7, 15, 14, 6, 5, 13, 12, 4, 3, 11, 10, 2],$$
$$[8, 16, 0, 9, 1, 10, 2, 3, 11, 12, 4, 5, 13, 14, 6, 7, 15],$$
$$[0, 8, 16, 17, 9, 1, 2, 10, 18, 19, 11, 3, 4, 12, 20, 5, 13, 14, 6, 21, 22, 7, 15]$$

of $\{1^7, 8^9\}$, $\{1^6, 8^{10}\}$ and $\{1^6, 8^{16}\}$ respectively.

Let $x = 9$. Lemma 5.4 gives the $\{1, 9\}$-growable realizations

$$[9, 10, 1, 0, 18, 8, 17, 7, 16, 15, 6, 5, 14, 13, 4, 3, 12, 11, 2],$$
$$[0, 9, 8, 18, 17, 7, 16, 6, 15, 14, 5, 4, 13, 12, 3, 2, 11, 10, 1],$$
$$[25, 8, 17, 18, 9, 0, 1, 10, 19, 20, 11, 2, 3, 12, 21, 22, 13, 4, 5, 14, 23, 6, 15, 16, 7, 24]$$

of $\{1^8, 9^{10}\}$, $\{1^7, 9^{11}\}$ and $\{1^7, 9^{18}\}$ respectively.

Lemma 5.6. *Let $x \geq 4$ be even. There is a $\{1, x\}$-growable realization for $\{1^{x-2}, x^b\}$ for b in range $x+3 \leq b \leq 2x-1$.*

Proof. We consider odd b and even b separately, starting with odd b.

Take r in the range $0 \leq r \leq (x-4)/2$. Write $x = 2r + 2s + 4$ for some $s \geq 0$. We construct a realization for

$$L = \{1^{2r+2s+2}, x^{4r+2s+7}\} = \{1^{x-2}, x^{x+2r+3}\}.$$

We have $v = (2r + 2s + 2) + (4r + 2s + 7) + 1 = 6r + 4s + 10$.

We build the required realization by concatenating three sequences. First:

$$[\underbrace{2r+1, 4r+2s+5}, \underbrace{4r+2s+6, 2r+2}, \underbrace{2r+3, 4r+2s+7}, \ldots, \underbrace{4r+4s+6, 2r+2s+2}],$$

which has $2s+2$ pairs and produces edge-lengths $\{1^{2s+1}, x^{2s+2}\}$. Second:

$$[\underbrace{6r+4s+8, 4r+2s+4, 2r}, \underbrace{2r-1, 4r+2s+3, 6r+4s+7}, \underbrace{6r+4s+6, 4r+2s+2, 2r-2}, \ldots, \underbrace{4r+4s+8, 2r+2s+4, 0}],$$

which has $2r+1$ triples and produces $\{1^{2r}, x^{4r+2}\}$. Third:

$$[6r+4s+9, 2r+2s+3, 4r+4s+7]$$

which produces $\{x^2\}$.

Upon concatenation we have a difference of x where the first and second sequences join and a difference of 1 where the second and third join. Hence we have a realization of

$$L = \{1^{2s+1}, x^{2s+2}\} \cup \{1^{2r}, x^{4r+2}\} \cup \{x^2\} \cup \{1, x\} = \{1^{x-2}, x^{x+2r+3}\}.$$

It is 1-growable at $v-2 = 6r+4s+8$: when embedding with $m = 6r+4s+8$, the only lengthened edge is $(2r+2s+2, 6r+4s+8)$. It is x-growable at $x-1 = 2r+2s+3$: when embedding with $m = 2r+2s+3$ the lengthened edges are $(i, i+x)$ for $0 \leq i \leq x-1$.

For the case with b even, first note that when $x = 4$ there are no values of b to be considered. Let $x \geq 6$ be even and take r in the range $0 \leq r \leq (x-6)/2$. Write $x = 2r+2s+6$ for some $s \geq 0$. We construct a realization for

$$L = \{1^{2r+2s+4}, x^{4r+2s+10}\} = \{1^{x-2}, x^{x+2r+4}\}.$$

We have $v = (2r+2s+4) + (4r+2s+10) + 1 = 6r+4s+15$.

We build the required realization by concatenating three sequences. First:

$$[\underbrace{4r+2s+11, 2r+5}, \underbrace{2r+6, 4r+2s+12}, \underbrace{4r+2s+13, 2r+7}, \ldots, \underbrace{4r+4s+11, 2r+2s+5}],$$

which has $2s+1$ pairs and produces the edge-lengths $\{1^{2s}, x^{2s+1}\}$. Second:

$$[2r+2s+6, 4r+4s+12, 4r+4s+13, 2r+2s+7, 1, 4r+2s+10, 2r+4, 2r+3, 4r+2s+9, 0]$$

which produces $\{1^2, x^7\}$. Third:

$$[\underbrace{6r+4s+14, 4r+2s+8, 2r+2}, \underbrace{2r+1, 4r+2s+7, 6r+4s+13}, \underbrace{6r+4s+12, 4r+2s+6, 2r}, \ldots, \underbrace{4r+4s+14, 2r+2s+8, 2}],$$

which has $2r+1$ triples and produces $\{1^{2r}, x^{4r+2}\}$.

Upon concatenation we have a difference of 1 at each of the joins. Hence we have a realization of

$$L = \{1^{2s}, x^{2s+1}\} \cup \{1^2, x^7\} \cup \{1^{2r}, x^{4r+2}\} \cup \{1^2\} = \{1^{x-2}, x^{x+2r+4}\}.$$

It is 1-growable at 1: when embedding with $m=1$, the only lengthened edge is $(1, 2r+2s+7)$. It is x-growable at $x = 2r+2s+6$: when embedding with $m = 2r+2s+6$ the lengthened edges are $(i, i+x)$ for $1 \le i \le x$. □

Example 5.7. To construct a $\{1,8\}$-growable realization of $\{1^6, 8^{13}\}$ using the proof of Lemma 5.6 we take $r = s = 1$ to obtain

$$[3, 11, 12, 4, 5, 13, 14, 6, 18, 10, 2, 1, 9, 17, 16, 8, 0, 19, 7, 15],$$

which is 1-growable at 18 and 8-growable at 7.

To construct a $\{1,10\}$-growable realization of $\{1^8, 10^{16}\}$ we take $r = s = 1$ to obtain

$$[17, 7, 8, 18, 19, 9, 10, 20, 21, 11, 1, 16, 6, 5, 15, 0, 24, 14, 4, 3, 13, 23, 22, 12, 2]$$

which is 1-growable at 1 and 10-growable at 10.

Lemma 5.8. *Let $x \ge 5$ be odd. There is a $\{1,x\}$-growable realization for $\{1^{x-2}, x^b\}$ for b in range $x+3 \le b \le 2x-1$.*

Proof. The constructions are similar to those of Lemma 5.6 in that they each are built from the concatenation of three sequences and we need to consider odd and even b separately. We start with odd b.

Take r in the range $0 \le r \le (x-5)/2$. Write $x = 2r+2s+5$ for some $s \ge 0$. We construct a realization for

$$L = \{1^{2r+2s+3}, (2r+2s+5)^{4r+2s+9}\} = \{1^{x-2}, x^{x+2r+4}\}.$$

We have $v = (2r+2s+3) + (4r+2s+9) + 1 = 6r+4s+13$.

The first sequence is

$$[\underbrace{6r+4s+10, 4r+2s+5, 2r}, \underbrace{2r-1, 4r+2s+4, 6r+4s+9},$$
$$\underbrace{6r+4s+8, 4r+2s+3, 2r-2}, \ldots, \underbrace{4r+4s+10, 2r+2r+5, 0}].$$

There are $2r+1$ triples, so this sequence produces edge-lengths $\{1^{2r}, x^{4r+2}\}$. The second sequence is

$$[6r+4s+12, 2r+2s+4, 4r+4s+9, 4r+4s+8, 2r+2s+3, 6r+4s+11].$$

This has internal differences $[2r+2s+5, 2r+2s+5, 1, 2r+2s+5, 2r+2s+5]$ and so produces $\{1, x^4\}$. The third sequence is

$$[\underbrace{4r+2s+6, 2r+1}, \underbrace{2r+2, 4r+2s+7}, \underbrace{4r+2s+8, 2r+3}, \ldots,$$
$$\underbrace{2r+2s+2, 4r+4s+7}].$$

There are $2s+2$ pairs, so this sequence produces $\{1^{2s+1}, x^{2s+2}\}$.

Upon concatenation, we have a difference of 1 generated where the first and second sequences join and a difference of $2r+2s+5 = x$ where the second and third join. Hence we have a realization of

$$L = \{1^{2r}, x^{4r+2}\} \cup \{1, x^4\} \cup \{1^{2s+1}, x^{2s+2}\} \cup \{1, x\} = \{1^{x-2}, x^{x+2r+4}\}.$$

The realization is 1-growable at $v-2 = 6r+4s+11$: when embedding with $m = 6r+4s+11$, the only lengthened edge is $(2r+2s+3, 6r+4s+11)$. It is x-growable at $x-1 = 2r+2s+4$: when embedding with $m = 2r+2s+4$ the lengthened edges are $(i, i+x)$ for $0 \leq i \leq x-1$.

Moving to even b, take r in the range $0 \leq r \leq (x-5)/2$ and write $x = 2r+2s+5$ for some $s \geq 0$. We construct a realization for

$$L = \{1^{2r+2s+3}, (2r+2s+5)^{4r+2s+8}\} = \{1^{x-2}, x^{x+2r+3}\}.$$

We have $v = (2r+2s+3) + (4r+2s+8) + 1 = 6r+4s+12$.

The first sequence is

$$[\underbrace{4r+2s+6, 2r+1}, \underbrace{2r+2, 4r+2s+7}, \underbrace{4r+2s+8, 2r+3}, \ldots, \underbrace{4r+4s+8, 2r+2s+3}].$$

There are $2s+3$ pairs, so this sequence produces edge-lengths $\{1^{2s+2}, x^{2s+3}\}$. The second sequence is the same as the first sequence of the previous construction:

$$[\underbrace{6r+4s+10, 4r+2s+5, 2r}, \underbrace{2r-1, 4r+2s+4, 6r+4s+9}, \underbrace{6r+4s+8, 4r+2s+3, 2r-2}, \ldots, \underbrace{4r+4s+10, 2r+2r+5, 0}].$$

As before, there are $2r+1$ triples, so this sequence produces $\{1^{2r}, x^{4r+2}\}$. The third sequence is

$$[6r+4s+11, 2r+2s+4, 4r+4s+9]$$

which has internal differences $[2r+2s+5, 2r+2s+5]$ and so produces $\{x^2\}$.

Upon concatenation, we have a difference of $2r+2s+5 = x$ generated where the first and second sequences join and a difference of 1 where the second and third join. Hence we have a realization of

$$L = \{1^{2s+2}, x^{2s+3}\} \cup \{1^{2r}, x^{4r+2}\} \cup \{x^2\} \cup \{1, x\} = \{1^{x-2}, x^{x+2r+3}\}.$$

The realization is 1-growable at $v-2 = 6r+4s+10$: when embedding with $m = 6r+4s+10$, the only lengthened edge is $(2r+2s+3, 6r+4s+10)$. It is x-growable at $x-1 = 2r+2s+4$: when embedding with $m = 2r+2s+4$ the lengthened edges are $(i, i+x)$ for $0 \leq i \leq x-1$. □

Example 5.9. To construct a $\{1, 13\}$-growable realization of $\{1^{11}, 13^{21}\}$ using the proof of Lemma 5.8 we take $r = s = 2$ to obtain

$$[30, 17, 4, 3, 16, 29, 28, 15, 2, 1, 14, 27, 26, 13, 0,$$
$$32, 12, 25, 24, 11, 31, 18, 5, 6, 19, 20, 7, 8, 21, 22, 9, 10, 23],$$

which is 1-growable at 31 and 13-growable at 12.

To construct a $\{1,13\}$-growable realization of $\{1^7, 9^{14}\}$ we take $r = s = 1$ to obtain

$$[12, 3, 4, 13, 14, 5, 6, 15, 16, 7, 20, 11, 2, 1, 10, 19, 18, 9, 0, 21, 8, 17]$$

which is 1-growable at 20 and 9-growable at 8.

We can now prove the main result of the section.

Theorem 5.10. *Let $L = \{1^a, x^b, (2x)^c\}$. If $a \geq x - 2$, c is even and $b \geq 5x - 2 + c/2$, then $\mathrm{BHR}(L)$ holds.*

Proof. If $x \leq 2$, then $\mathrm{BHR}(L)$ holds without restriction and the case $x = 3$ is covered in Lemma 5.2, so assume $x \geq 4$. We follow the method of proof of Lemma 5.2, with Lemmas 5.4, 5.6 and 5.8 providing the realizations to get started.

Take i in the range $0 \leq i < c/2$ such that $2i \equiv c \pmod{2x}$.

To construct the required realization for L, start with the realization of $\{1^{a'}, x^{b'}\}$ that has $b' \equiv b + 2i \pmod{x}$ given by Lemma 5.4, 5.6 or 5.8. So $a' = x - 2$, except when $b + 2i \equiv 1 \pmod{x}$ and admissibility forces us to use $a' = x - 1$.

If $c \not\equiv 0 \pmod{2x}$, then apply Lemma 5.1 using i to give a realization whose number of occurrences c' of $2x$ differs from c by a multiple of $2x$ and whose number of occurrences of x differs from b by a multiple of x. (If $c \equiv 0 \pmod{2x}$, then this is already the case.)

Apply Lemma 5.1 a further $(c - c')/2x$ times with $i = x$ to obtain a $\{1, x\}$-growable realization of $\{1^{a'}, x^{b''}, (2x)^c\}$ where $b'' \equiv b \pmod{x}$. Complete to the required realization using the appropriate number of applications of Theorem 2.4 with 1 and x.

The method requires up to $2x$ occurrences of x in the initial realization, up to $3x - 2$ occurrences of x to adjust the congruency class of the number of occurrences of $2x$, and $c/2$ occurrences of x to obtain the correct number of occurrences of $2x$. Hence it always works for $b \geq 2x + 3x - 2 + c/2 = 5x - 2 + c/2$. □

When c is odd, we are not aware of any reason why the same approach will not work. However, without new ideas, it will take more work to get weaker results than in the even case. This is because the starter realizations now need to be for $\{1^{x-2}, x^b, 2x\}$, which means that we must have $v \geq 4x$, compared to the constructions here which all have $v < 3x$. As well as being larger, using the same approach as Lemmas 5.6 and 5.8 would probably take more cases to cover all required values of b. Some of these issues are already apparent in Lemma 5.2.

Lemma 5.1 can be thought of as combining the notion of growability with that of a particular perfect realization. This can be generalized to other perfect realizations, which we now do.

For a multiset L, define $sL = \{sy : y \in L\}$. When we apply Theorem 2.4 k times to an x-growable realization we produce a realization with the x subsequences

$$[m + 1 - x, m + 1, m + 1 + x, \ldots, m + 1 + (k - 1)x] + t$$

for $0 \leq t \leq x - 1$. If we have a perfect linear realization of length k of a multiset L, then we can multiply each element by x to get a sequence that realizes xL and then take a translate of it to replace a subsequence of the above form.

Lemma 5.1 uses this process with the perfect linear realization $[0, 2, 1, 3]$ of $\{1, 2^2\}$. In general the approach gives the following lemma.

Lemma 5.11. *Let L have an X-growable realization with $x \in X$. Let $L_1, \ldots, L_x$ be multisets of size $k-1$ that have perfect linear realizations. Then*

$$L \cup xL_1 \cup \cdots \cup xL_x$$

has an X-growable realization.

If we use the perfect linear realization $[0, 1]$ of $\{1\}$ in Lemma 5.11, then we end up back at Theorem 2.4.

Example 5.12. Let $x \geq 3$. In this section we have constructed $\{1, x\}$-growable realizations of the multisets $\{1^{x-1}, x^{x+1}\}$ and $\{1^{x-2}, x^b\}$ for $x+2 \leq b \leq 2x$. Let $c, d, e, f, g \geq 0$ with $c + d + e + f + g \equiv 0 \pmod{x}$. Take c copies of the perfect linear realization $[0, 1, 2, 3, 4, 5]$ of $\{1^5\}$, d copies of the perfect realization $[0, 2, 1, 3, 4, 5]$ of $\{1^3, 2^2\}$, e copies of the perfect linear realization $[0, 3, 1, 2, 4, 5]$ of $\{1^2, 2^2, 3\}$, f copies of the perfect linear realization $[0, 3, 1, 4, 2, 5]$ of $\{2^2, 3^3\}$, and g copies of the perfect linear realization $[0, 2, 4, 1, 3, 5]$ of $\{2^4, 3\}$. Lemma 5.11 proves $\mathrm{BHR}(L)$ for

$$L = \{1^a, x^{b+5c+3d+2e}, (2x)^{2d+2e+2f+4g}, (3x)^{e+3f+g}\}$$

for $a \geq x-2$ and $b \geq x+1$.

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Ordering signed graphs with large index

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Abstract

The index of a signed graph is the largest eigenvalue of its adjacency matrix. We establish the first few signed graphs ordered decreasingly by the index in classes of connected signed graphs, connected unbalanced signed graphs and complete signed graphs with a fixed number of vertices.

Keywords: Adjacency matrix, largest eigenvalue, edge relocation, unbalanced signed graph, complete signed graph.

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1 Introduction

A *signed graph* $\dot{G}$ is a pair (G, σ), where $G = (V, E)$ is an unsigned graph, called the *underlying graph*, and $\sigma\colon E \longrightarrow \{1, -1\}$ is the *sign function* or the *signature*. The number of vertices of $\dot{G}$ is called the *order* and denoted by n. The edge set of $\dot{G}$ is composed of the subset E^+ of positive edges and the subset E^- of negative edges. We interpret an unsigned graph as a signed graph with the all positive signature, that is the signature which assigns 1 to every edge.

The *adjacency matrix* $A_{\dot{G}}$ of $\dot{G}$ is obtained from the standard adjacency matrix of its underlying graph by switching the sign of all 1's which correspond to negative edges. The *eigenvalues* of $\dot{G}$ are identified to be the eigenvalues of its adjacency matrix; they form

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the *spectrum* of $\dot{G}$. The largest eigenvalue of $\dot{G}$ is called the *index* and denoted by λ_1 (or $\lambda_1(\dot{G})$).

If S is a set of vertices of $\dot{G}$, the switched signed graph $\dot{G}^S$ is obtained from $\dot{G}$ by reversing the signs of the edges in the cut $[S, V(\dot{G}) \setminus S]$. The signed graphs $\dot{G}$ and $\dot{G}^S$ are said to be *switching equivalent*. The switching equivalence is an equivalence relation that preserves the eigenvalues, and the *switching class* of $\dot{G}$ is denoted by $[\dot{G}]$.

A signed graph is said to be *balanced* if it switches to the signed graph with all positive signature. Otherwise, it is said to be *unbalanced*. Equivalently, $\dot{G}$ is balanced if every cycle contained in $\dot{G}$ is balanced [15].

Ordering of unsigned graphs by the largest eigenvalue of some associated matrix has received a great deal of attention in literature. Many results can be found in [12]. More recently, there has been a growing interest for extremal problems in the framework of signed graphs. For instance, in [9] Koledin and the second author studied connected signed graphs of fixed order, size and number of negative edges that maximize the index. In the wake of that paper, signed graph maximizing the index in suitable subsets of complete signed graphs have been studied in [2]. Let $\mathfrak{U}_n$ (resp. $\mathfrak{B}_n$) denote the class of unbalanced unicyclic (resp. bicyclic) signed graphs of order n. Akbari et al. [1] determined the signed graphs attaining the extremal indices in $\mathfrak{U}_n$. Some of the same authors studied in [10] signed graphs achieving the maximum index among signed graphs in $\mathfrak{U}_n$ of fixed girth. The first five largest indices among signed graphs in $\mathfrak{B}_n$ with $n \geqslant 36$ are detected by He et al. [8]. Signed graphs in $\mathfrak{U}_n$ and $\mathfrak{B}_n$ with extremal spectral radius were identified in [4]. Finally, extremal graphs in $\mathfrak{U}_n$ and $\mathfrak{B}_n$ with respect to the least Laplacian eigenvalue were studied in [5] and [3], respectively.

In [6] we determined the unbalanced signed graph with largest index, for every order n. In this paper we continue this research by presenting a general method for ordering the signed graphs with a fixed number of vertices by their index. We demonstrate the method by determining the first few signed graphs ordered by the index in the class of connected signed graphs, or connected unbalanced signed graphs, or complete signed graphs with n vertices.

The paper is organized as follows. Section 2 contains a preliminary setting related to the graphical representations of signed graphs in this paper along with terminology, notation, a few known results and the proofs of two preliminary lemmas. The main result that provides the subsequent orderings is formulated in Theorem 3.2 of Section 3. Orderings in the mentioned classes are considered in Sections 3–5. Further computations, including orderings of signed graphs with a comparatively small number of vertices, are given in Section 6.

2 Preparatory

We introduce a way of depicting signed graphs that will be used in the subsequent sections. For a signed graph of order n, we draw only the negative edges and the non-edges, along with the assumption that all non-depicted edges are positive. By convention, a negative edge is represented by a full line and a non-edge is represented by a dotted line. Accordingly, the complete signed graph with the all positive signature (i.e. the complete unsigned graph) is represented by an empty figure, the complete signed graph with a single negative edge is represented by a negative edge, and so on.

The following lemmas are taken from [13, 14].

Lemma 2.1 ([14]). *For a connected signed graph $\dot{G} = (G, \sigma)$, we have $\lambda_1(\dot{G}) \leq \lambda_1(G)$ with equality if and only if $\dot{G}$ switches to G.*

Lemma 2.2 ([13]). *For an eigenvalue λ of a signed graph $\dot{G}$, there is a switching equivalent signed graph for which the λ-eigenspace contains an eigenvector whose non-zero coordinates are of the same sign.*

For the sake of completeness we say that a signed graph of the previous lemma can be constructed by taking $\dot{G}$ with $A_{\dot{G}}\mathbf{x} = \lambda\mathbf{x}$ and considering $D^{-1}A_{\dot{G}}D$ where D is the diagonal matrix of ± 1s whose negative entries correspond to negative coordinates of $\mathbf{x}$.

We proceed with some notation. For a signed graph $\dot{G}$ we denote by $\mathcal{R}(\dot{G})$ the set of signed graphs obtained by taking a positive edge e of some signed graph of the switching class $[\dot{G}]$, and then either removing e or reversing its sign.

Let $\mathcal{S} = (\dot{G}_1, \dot{G}_2, \ldots, \dot{G}_g)$ be a sequence which consists of the representatives of all switching equivalence classes of connected signed graphs with n vertices such that the representatives are ordered non-increasingly by the index and chosen in such a way that, for $1 \leq i \leq g$, the λ_1-eigenspace of $\dot{G}_i$ contains an eigenvector whose non-zero coordinates are positive. (The existence of $\dot{G}_i$ is provided by Lemma 2.2.)

We now prove the following lemmas. They generalize known results for unsigned graphs that can be found in [12, Lemma 1.28].

Lemma 2.3 (**Changing an edge**). *Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)^\intercal$ be an eigenvector associated with the index of a signed graph $\dot{G}$ and let r, s be fixed vertices of $\dot{G}$.*

(i) *If $x_r x_s \geq 0$ and rs is a non-edge (resp. rs is a negative edge), then for a signed graph $\dot{G}'$ obtained by inserting a positive edge between r and s (resp. deleting rs or reversing its sign) we have $\lambda_1(\dot{G}') \geq \lambda_1(\dot{G})$. If at least one of x_r, x_s is non-zero, the previous inequality is strict.*

(ii) *If $x_r x_s < 0$ and rs is a non-edge (resp. rs is a positive edge), then for a signed graph $\dot{G}'$ obtained by inserting a negative edge between r and s (resp. deleting rs or reversing its sign) we have $\lambda_1(\dot{G}') > \lambda_1(\dot{G})$.*

Proof. We only demonstrate the proof of (i), as (ii) is proved analogously. If $\mathbf{y}$ is an eigenvector associated with $\lambda_1(\dot{G}')$, using the Rayleigh principle we get

$$\begin{aligned}
&\lambda_1(\dot{G}') - \lambda_1(\dot{G}) \\
&\quad = \mathbf{y}^\intercal A_{\dot{G}'}\mathbf{y} - \mathbf{x}^\intercal A_{\dot{G}}\mathbf{x} \geq \mathbf{x}^\intercal A_{\dot{G}'}\mathbf{x} - \mathbf{x}^\intercal A_{\dot{G}}\mathbf{x} = \mathbf{x}^\intercal (A_{\dot{G}'} - A_{\dot{G}})\mathbf{x} \\
&\quad = \begin{cases} 2x_r x_s & \text{if } \big(rs \notin E(\dot{G}) \wedge rs \in E^+(\dot{G}')\big) \vee \big(rs \in E^-(\dot{G}) \wedge rs \notin E(\dot{G}')\big), \\ 4x_r x_s & \text{if } rs \in E^-(\dot{G}) \wedge rs \in E^+(\dot{G}'). \end{cases}
\end{aligned}$$

Hence, $\lambda_1(\dot{G}') \geq \lambda_1(\dot{G})$.

Assume that $x_r \neq 0$ and, by way of contradiction, that $\lambda_1(\dot{G}') = \lambda_1(\dot{G})$. In this case, the inequality in the previous chain reduces to equality, which means that $\mathbf{x}$ is an eigenvector afforded by $\lambda_1(\dot{G}')$. Using the eigenvalue equations at vertex s in $\dot{G}$ and $\dot{G}'$, we get

$$0 = \big(\lambda_1(\dot{G}') - \lambda_1(\dot{G})\big)x_s = \sum_{i\,:\,is \in E(\dot{G}')} \sigma_{\dot{G}'}(is)x_i - \sum_{i\,:\,is \in E(\dot{G})} \sigma_{\dot{G}}(is)x_i = \alpha x_r, \tag{2.1}$$

where α depends on (r,s)-entries in $A_{\dot{G}}$ and $A_{\dot{G}'}$, but it is always non-zero; for example, if $rs \notin E(\dot{G}) \wedge rs \in E^+(\dot{G}')$ then $\alpha = \sigma_{\dot{G}'}(rs) = 1$, and similarly for the remaining possibilities listed in the statement formulation. Together with (2.1), this leads to $x_r = 0$, which in turn contradicts the initial assumption and we are done. □

Let r, s, t, u be fixed vertices of a signed graph. A relocation $\mathrm{Rot}(r, s, t)$ (called a *rotation*) is realised in the adjacency matrix by replacing the entries a_{rs}, a_{sr} with the entries a_{rt}, a_{tr}, and vice versa. In simple words, this relocation is realised by taking the object (which can be a positive edge, or a negative edge, or a non-edge) located between r and s and the object located between r and t and then inserting the first object between r and t and the second object between r and s.

A relocation $\mathrm{Shift}(r, s, t, u)$ (called a *shifting*) is realised in the adjacency matrix by replacing the entries a_{rs}, a_{sr} with a_{tu}, a_{ut}, and vice versa.

Lemma 2.4 (Rotation and shifting). *Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)^\intercal$ be an eigenvector associated with the index of a signed graph $\dot{G}$ and let r, s, t, u be fixed vertices of $\dot{G}$.*

(i) *Let $\dot{G}'$ be obtained from $\dot{G}$ by the relocation $\mathrm{Rot}(r, s, t)$. If $\big(x_r\big(x_s - x_t\big) > 0 \vee (x_s = x_t \wedge x_r \neq 0)\big) \wedge \big((rs$ is a non-edge $\wedge$ rt is a positive edge$) \vee (rs$ is a negative edge and rt is a positive edge$) \vee (rs$ is a negative edge and rt is a non-edge$)\big)$ then $\lambda_1(\dot{G}') > \lambda_1(\dot{G})$.*

(ii) *Let $\dot{G}'$ be obtained from $\dot{G}$ by the relocation $\mathrm{Shift}(r, s, t, u)$. If $\big(x_t x_u > x_r x_s \vee (x_t x_u = x_r x_s \wedge$ at least one of x_r, x_s, x_t, x_u is non-zero$)\big) \wedge \big((rs$ is a positive edge $\wedge$ tu is a non-edge$) \vee (rs$ is a positive edge and tu is a negative edge$) \vee (rs$ is a non-edge and tu is a negative edge$)\big)$ then $\lambda_1(\dot{G}') > \lambda_1(\dot{G})$.*

Proof. This proof is similar to the proof of the previous lemma. If rs and rt are as in (i), then we compute

$$\lambda_1(\dot{G}') - \lambda_1(\dot{G}) \geq \mathbf{x}^\intercal(A_{\dot{G}'} - A_{\dot{G}})\mathbf{x} = 2\alpha x_r(x_s - x_t),$$

where $\alpha = 1$ for the first and the third assumption on rs and rt, and $\alpha = 2$ for the second assumption. Now, for $x_r(x_s - x_t) > 0$ we get $\lambda_1(\dot{G}') > \lambda_1(\dot{G})$. For $x_s = x_t$ we have $\lambda_1(\dot{G}') \geq \lambda_1(\dot{G})$. In case of equality, we have that $\mathbf{x}$ is afforded by $\lambda_1(\dot{G}')$. Considering the eigenvalue equation at the vertex s in $\dot{G}$ and $\dot{G}'$, we get $x_r = 0$, which completes (i).

If rs and rt are as in (ii), then we compute

$$\lambda_1(\dot{G}') - \lambda_1(\dot{G}) \geq \mathbf{x}^\intercal(A_{\dot{G}'} - A_{\dot{G}})\mathbf{x} = 2\alpha(x_t x_u - x_r x_s),$$

with $\alpha \in \{1, 2\}$, as before. We are done for $x_t x_u > x_r x_s$, while for $x_t x_u = x_r x_s$, using the previous reasoning we get that the equality between the indices necessarily leads to the conclusion that $\mathbf{x}$ takes zero at the corresponding four vertices. □

3 Ordering signed graphs by the index

We start our considerations with an example.

Example 3.1. Clearly, there are just 3 connected signed graphs of order 3, up to switching: the positive triangle (with index 2), the 3-vertex path (with index $\sqrt{2}$) and the negative triangle (with index 1). We know from [11] that there are exactly 12 connected signed graphs of order 4 (again, up to switching). Their indices are computed directly, and the corresponding ordering is given in Figure 1.

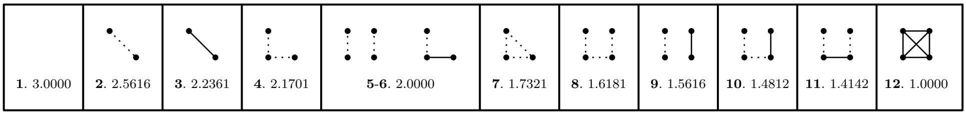


Figure 1: Connected signed graphs with 4 vertices ordered by the index. Here and in the subsequent graphical representations, signed graphs are depicted according to the convention explained in Section 2.

In what follows, we determine the first 5 connected signed graphs with n vertices ordered by the index, for every $n \geq 5$. In other words, we determine the signed graphs $\dot{G}_1 - \dot{G}_5$ of the sequence $\mathcal{S}$ defined in the previous section. First, $\dot{G}_1$ is the complete signed graph with the all positive signature, which follows from the well-known Perron-Frobenius Theorem and Lemma 2.1. We now prove the following theorem, crucial for our considerations.

Theorem 3.2. *Let $\mathcal{S}' = (\dot{G}_{k+1}, \dot{G}_{k+2}, \dots, \dot{G}_{k+\ell})$ be a subsequence of $\mathcal{S}$ such that*

$$\lambda_1(\dot{G}_k) > \lambda_1(\dot{G}_{k+1}) = \lambda_1(\dot{G}_{k+2}) = \cdots = \lambda_1(\dot{G}_{k+\ell}) > \lambda_1(\dot{G}_{k+\ell+1}). \tag{3.1}$$

Then, for every $\dot{G} \in \mathcal{S}'$, we have $\dot{G} \in \mathcal{R}(\dot{H})$ where $\dot{H} \in \{\dot{G}_1, \dot{G}_2, \dots, \dot{G}_{k+\ell}\} \setminus \dot{G}$.

In addition:

(a) *For at least one $\dot{G} \in \mathcal{S}'$ we have $\dot{H} \in \{\dot{G}_1, \dot{G}_2, \dots, \dot{G}_k\}$;*

(b) *If $\dot{H} \notin \{\dot{G}_1, \dot{G}_2, \dots, \dot{G}_k\}$ then a non-negative λ_1-eigenvector for $\dot{G}$ has at least two zero coordinates and the same eigenvector is afforded by $\lambda_1(\dot{H})$.*

Proof. Assume by way of contradiction that for some $\dot{G} \in \mathcal{S}'$, $\dot{G} \notin \mathcal{R}(\dot{H})$, for every $\dot{H}$ that belongs to the set given in the statement formulation. Let $\mathbf{x} = (x_1, x_2, \dots, x_n)^\intercal$ be an eigenvector with non-negative coordinates afforded by the index of $\dot{G}$.

Assume that at least one of x_r, x_s is non-zero for some vertices r, s of $\dot{G}$. If rs is not a positive edge, then Lemma 2.3(i) produces a signed graph $\dot{G}'$ that differs from $\dot{G}$ only in the positive edge rs, along with $\lambda_1(\dot{G}') > \lambda_1(\dot{G})$. Since $\dot{G} \notin \bigcup_{i=1}^{k} \mathcal{R}(\dot{G}_i)$, we have $\dot{G}' \notin \bigcup_{i=1}^{k} [\dot{G}_i]$, i.e. there are at least $k+1$ signed graphs whose index is larger than $\lambda_1(\dot{G})$, which contradicts (3.1). Hence, rs is a positive edge.

Further, if $x_r = x_s = 0$, then reversing the sign of rs, or removing rs, or adding rs do not affect the existence of the corresponding eigenvalue; indeed, it is afforded by the same eigenvector. Thus if for any such r, s there is no positive edge between them, as before we get $\dot{G}'$ with $\lambda_1(\dot{G}') \geq \lambda_1(\dot{G})$. Since $\dot{G} \notin \bigcup_{i=1}^{k} \mathcal{R}(\dot{G}_i)$, we have $\lambda_1(\dot{G}') = \lambda_1(\dot{G})$, but then $\dot{G} \in \bigcup_{i=1}^{\ell} \mathcal{R}(\dot{G}_{k+i})$ which together with the initial assumption leads to $\dot{G} \in \mathcal{R}(\dot{G})$, i.e. $\dot{G}$ is isomorphic to the signed graph obtained by inserting a positive edge rs. Replacing $\dot{G}$ with this signed graph we get that rs is positive.

Amalgamating the previous conclusions we get that $\dot{G}$ is the complete signed graph with the all positive signature, which together with Lemma 2.1 contradicts (3.1) (since we assumed in (3.1) that $\dot{G}$ is not $\dot{G}_1$).

Consider now (a). Take an arbitrary $\dot{G} \in \mathcal{S}'$. If $\dot{G} \in \bigcup_{i=1}^{k} \mathcal{R}(\dot{G}_i)$, we are done. Assume that $\dot{G} \notin \bigcup_{i=1}^{k} \mathcal{R}(\dot{G}_i)$. If $\mathbf{x}$ is the previously defined eigenvector, then there is a positive edge rs for every pair r, s such that at least one of x_r, x_s is non-zero, as otherwise by inserting a positive edge between such vertices we get $\dot{G} \in \bigcup_{i=1}^{k} \mathcal{R}(\dot{G}_i)$. If at most one coordinate of $\mathbf{x}$ is zero, then $\dot{G}$ switches to a complete unsigned graph, which contradicts (3.1). Therefore there exist at least 2 vertices at which $\mathbf{x}$ takes zero. In addition, there is a negative edge between at least one such a pair, since otherwise $\dot{G}$ has the all positive signature and then $\mathbf{x}$ has no zero coordinates by the Perron-Frobenius Theorem. If $\dot{G}'$ is obtained by switching the sign of such a negative edge, then $\lambda_1(\dot{G}') = \lambda_1(\dot{G})$, as otherwise we get $\dot{G} \in \bigcup_{i=1}^{k} \mathcal{R}(\dot{G}_i)$. Moreover, $\lambda_1(\dot{G}')$ is afforded by the same eigenvector, so we may repeat the previous consideration with $\dot{G}'$ in the role of $\dot{G}$. In this way we necessarily arrive at some $\dot{G} \in \mathcal{S}' \cap \big(\bigcup_{i=1}^{k} \mathcal{R}(\dot{G}_i)\big)$ since the number of negative edges strictly decreases in passing from $\dot{G}$ to $\dot{G}'$.

It remains to consider (b). Let $\dot{G} \in \mathcal{R}(\dot{H})$. If at most one coordinate of $\mathbf{x}$ is zero, then $\lambda_1(\dot{H}) > \lambda_1(\dot{G})$ (by Lemma 2.3(i)), which implies $\dot{H} \in \{\dot{G}_1, \dot{G}_2, \dots, \dot{G}_k\}$. Further, the assumption that $\dot{G} \in \mathcal{R}(\dot{H})$ together with $\lambda_1(\dot{H}) = \lambda_1(\dot{G})$ leads to the conclusion that $x_r = x_s = 0$ for a non-positive (resp. positive) edge rs in $\dot{G}$ (resp. $\dot{H}$), and thus $A_{\dot{H}}\mathbf{x} = A_{\dot{G}}\mathbf{x} = \lambda_1(\dot{H})\mathbf{x}$. □

Remark 3.3. Theorem 3.2 gives a method for the ordering of signed graphs by the index. Let $\dot{G}_1, \dot{G}_2, \dots, \dot{G}_k$ be the first k signed graphs ordered by the index such that all signed graphs (if any) sharing the index with $\dot{G}_k$ are listed before it (so, as in the theorem). Then the sequence is extended by the signed graph(s) belonging to $\bigcup_{i=1}^{k} \mathcal{R}(\dot{G}_i)$. The candidates must be connected and the λ_1-eigenspace for each of them must contain an eigenvector with non-negative coordinates. They are compared on the basis of an algebraic computation that relies on Lemmas 2.3 and 2.4. As long as we deal with signed graphs whose λ_1-eigenspaces do not contain an eigenvector with at least two zero coordinates, there are no other candidates. In case of such eigenvectors, signed graphs with equal indices sharing the same eigenvector may appear.

Remark 3.4. To determine the set $\mathcal{R}(\dot{G})$ we need to consider the entire switching class of $\dot{G}$. For example, if $\dot{G}$ is the complete signed graph with exactly one negative edge, say e, then the signed graphs obtained from $\dot{G}$ by removing a positive edge or reversing its sign are the following 4:

By making a switch at a vertex incident with e and either removing e or reversing its sign, we get 2 additional members of $\mathcal{R}(\dot{G})$ that are not switching equivalent to the previous ones. But in both cases the λ_1-eigenspace does not contain a non-negative eigenvector (the condition required in Remark 3.3). A method for computing the λ_1-eigenvectors is demonstrated in the proof of the forthcoming Lemma 3.5.

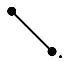
Now we proceed with the ordering.

Lemma 3.5. *$\dot{G}_2$ is* .

Proof. There are exactly two candidates for $\dot{G}_2$: $\dot{F}$ obtained by removing an edge of $\dot{G}_1$ and $\dot{H}$ obtained by reversing the sign of an edge of $\dot{G}_1$. If the vertices joined by the unique negative edge of $\dot{H}$ are labelled by 1 and 2, using the eigenvalue equation for $\lambda_1(\dot{H})$ we get

$$\begin{aligned} \lambda_1 a &= -a + (n-2)b \\ \lambda_1 b &= \ \ 2a + (n-3)b \end{aligned}$$

which leads to the $\lambda_1(\dot{H})$-eigenvector $b(\frac{\lambda_1+1}{\lambda_1+3}, \frac{\lambda_1+1}{\lambda_1+3}, 1, 1, \dots, 1)^\intercal$, $b \neq 0$. By virtue of Lemma 2.3(i) (applied to $\dot{H}$), we have $\lambda_1(\dot{F}) > \lambda_1(\dot{H})$. Hence $\dot{G}_2 \cong \dot{F}$. □

Lemma 3.6. *$\dot{G}_3$ is* .

Proof. The candidates for $\dot{G}_3$ are illustrated in Figure 2. They are obtained by considering $\mathcal{R}(\dot{G}_1) \cup \mathcal{R}(\dot{G}_2)$; we also include the transposes of the corresponding positive eigenvectors afforded by the index.

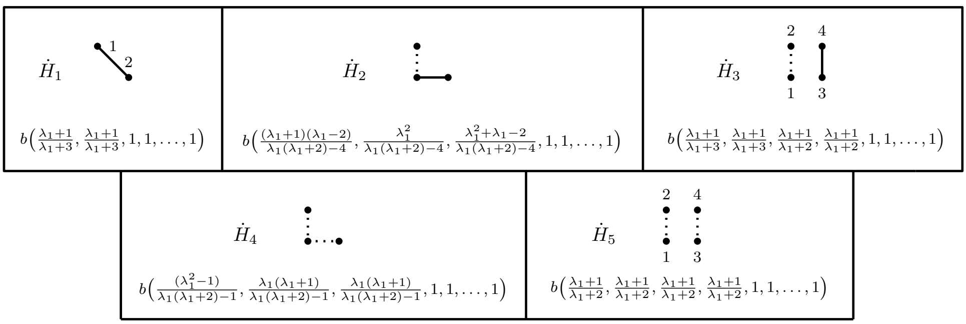


Figure 2: The candidates for $\dot{G}_3$.

From Lemma 2.3(i), we have $\lambda_1(\dot{H}_1) > \max\{\lambda_1(\dot{H}_2), \lambda_1(\dot{H}_3)\}$. We further apply Lemma 2.4(i) to $\dot{H}_5$ with $(r, s, t) = (1, 2, 3)$ to conclude that $\lambda_1(\dot{H}_4) > \lambda_1(\dot{H}_5)$. To show that $\dot{G}_3 \cong \dot{H}_1$ it remains prove that $\lambda_1(\dot{H}_1) > \lambda_1(\dot{H}_4)$. The adjacency matrix of $\dot{H}_1$ is

$$A_{\dot{H}_1} = \begin{pmatrix} \begin{matrix} 0 & -1 \\ -1 & 0 \end{matrix} & J^\intercal \\ J & A_{K_{n-2}} \end{pmatrix},$$

(where J is the all-1 matrix) which leads to the quotient matrix (i.e. the matrix of row sums in the corresponding blocks of $A_{\dot{H}_1}$):

$$Q_{\dot{H}_1} = \begin{pmatrix} -1 & n-2 \\ 2 & n-3 \end{pmatrix}.$$

We know from [7] that every eigenvalue whose eigenspace does not contain an eigenvector orthogonal to the all-1 vector $\mathbf{j}$ belongs to the spectrum of the quotient matrix. In our case, this means that $\lambda_1(\dot{H}_1)$ is an eigenvalue of $Q_{\dot{H}_1}$, i.e. $\lambda_1(\dot{H}_1)$ is the largest root of

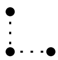
$$x^2 + (4-n)x - 3n + 7. \tag{3.2}$$

In the same way, we get that $\lambda_1(\dot{H}_4)$ is the largest root of

$$f(x) = x^3 - (n-3)x^2 - (2n-5)x + n - 3. \tag{3.3}$$

Now, computing the largest root of (3.2) we get $\lambda_1(\dot{H}_1) = \frac{n+\sqrt{(n-2)(n+6)}}{2} - 2$. Inserting it in (3.3), we get $f\big(\frac{n+\sqrt{(n-2)(n+6)}}{2} - 2\big) = \sqrt{(n-2)(n+6)} - n > 0$, which leads to the conclusion that either $\lambda_1(\dot{H}_1) > \lambda_1(\dot{H}_4)$ or the two roots of f are larger than $\lambda_1(\dot{H}_1)$. The latter is not true because $f(0) > 0, f(1) < 0$ which means that f has a negative root and a root in $(0, 1)$. □

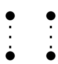
To avoid repetitive proofs, in the remainder of this section and the next two sections we omit the parts in which we compute the λ_1-eigenvectors of potential candidates since these are technical algebraic computations performed in exactly the same way as in the previous proof.

Lemma 3.7. *$\dot{G}_4$ is* .

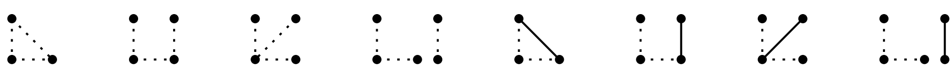
Proof. The 6 candidates for $\dot{G}_4$ are (those of Figure 2 that have not passed for $\dot{G}_3$ are included, of course):

We note that there are two additional members of $\mathcal{R}(\dot{G}_3)$ mentioned in Remark 3.3, but the non-existence of a required λ_1-eigenvector eliminates them.

The 3rd and the 5th candidate are eliminated since their indices are dominated by the index of the 1st one, while the 4th and the 6th are eliminated by the 2nd one in the same way – all this by virtue of Lemma 2.3(i). Finally, the fact that the index of the 1st signed graph is larger than that of the 2nd one is established in the proof of Lemma 3.6, and we are done. □

Lemma 3.8. *$\dot{G}_5$ is* .

Proof. The candidates for $\dot{G}_5$ are the 5 signed graphs that are eliminated in the proof of the previous lemma (when we considered $\dot{G}_4$) and the following 8 signed graphs:



Lemma 2.3(i) eliminates all except the 2nd and the 3rd of the previous proof; in Figure 2 they are denoted by $\dot{H}_5$ and $\dot{H}_2$) and the 1st and the 3rd of the additional candidates (we denote them by $\dot{F}_1$ and $\dot{F}_2$). By Lemma 2.4, $\lambda_1(\dot{F}_2)$ dominates $\lambda_1(\dot{F}_1)$; we already had this in the proof of Lemma 3.6.

Thus, it remains to prove that $\lambda_1(\dot{H}_5) > \max\{\lambda_1(\dot{H}_2), \lambda_1(\dot{F}_2)\}$. As in the proof of Lemma 3.6 we deduce that these indices are the largest roots of characteristic polynomials of the corresponding quotient matrices. These polynomials are:

$$h_5(x) = x^2 - (n-3)x - 2(n+3)$$
$$h_2(x) = x^4 - (n-4)x^3 - (3n-7)x^2 + 2(n-4)x + 4(n-3)$$
$$f_2(x) = x^3 - (n-3)x^2 - 2(n-3)x + 2(n-4)$$

The largest root of h_5 is $\frac{1}{2}(n-3+\sqrt{(n-3)(n+5)})$. Concerning h_2 we get $h_2(-4) = 12(n+11) > 0$, $h_2(-2) = -4(n-4) < 0$, $h_2(0) = 4(n-3) > 0$ and $h_2(n-2) = -(n-1)(n-4)^2 < 0$, which together with $n-2 < \lambda_1(\dot{H}_5)$ leads to the conclusion that at least 3 roots of h_2 are less than $\lambda_1(\dot{H}_5)$. Since $h_2(\lambda_1(\dot{H}_5)) = (n-4)(n-3+\sqrt{(n-3)(n+5)}) > 0$, we conclude that the fourth root of h_2 is also less than $\lambda_1(\dot{H}_5)$.

Similarly, we have $f_2(-3) = -n-26 < 0$, $f_2(0) = 2(n-4) > 0$ and $f_2(1) = 2-n < 0$, which means that that two roots of f_2 are less than $\lambda_1(\dot{H}_5)$, while $f_2(\lambda_1(\dot{H}_5)) = 2(n-4) > 0$ confirms the same for the third root, and we are done. □

Amalgamating the previous results we arrive at the following theorem.

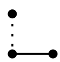
Theorem 3.9. *The first* 5 *connected signed graphs with* $n \geq 5$ *vertices ordered by their indices are:*

$\dot{G}_1$ $\dot{G}_2$ $\dot{G}_3$ $\dot{G}_4$ $\dot{G}_5$

4 Unbalanced signed graphs

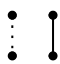
Let now $(\dot{U}_1, \dot{U}_2, \ldots, \dot{U}_u)$ be the subsequence of $\mathcal{S}$ (defined in Section 2) containing only unbalanced signed graphs. In other words, the previous sequence ignores the balanced ones. In what follows, we determine $\dot{U}_1 - \dot{U}_4$ for $n \geq 6$.

We know from [6] that $\dot{U}_1$ is obtained by reversing the sign of a single edge in the complete graph of order n.

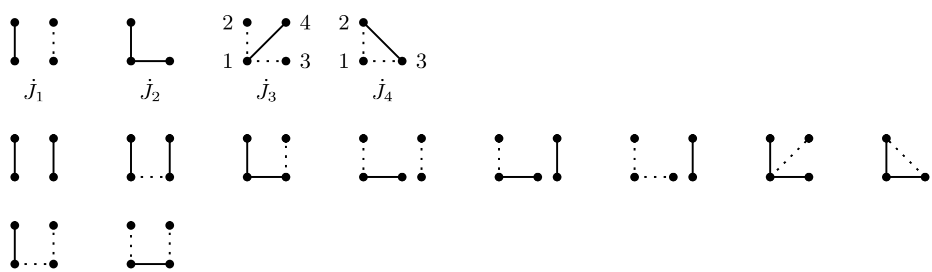
Lemma 4.1. $\dot{U}_2$ *is* .

Proof. The candidates for $\dot{U}_2$ are the last 4 signed graphs considered as the candidates in the proof of Lemma 3.7. (As before, it is not complicated to show that these are the only candidates with positive λ_1-eigenvectors).

The latter two candidates are eliminated by Lemma 2.3(i) – they are dominated by the 1st candidate. Observe that the λ_1-eigenvector for the 2nd candidate is given in Figure 2. Using the same vertex labelling and applying the relocation $\mathrm{Rot}(3,4,1)$ we arrive at the result formulated in this statement. □

Lemma 4.2. $\dot{U}_3$ *is* .

Proof. The candidates for $\dot{U}_3$ are the following 14 signed graphs:



All in the second row are easily eliminated on the basis of Lemma 2.3(i). The two in the third row are eliminated by Lemma 2.4(i). Namely, if we denote the vertices in the representing path by $1, 2, 3, 4$ (in the natural order) and if $x_2 \geq x_3$, then $\mathrm{Rot}(1, 2, 3)$ implies that the index of the signed graph under consideration is less than that of $\dot{J}_3$. Otherwise, we can apply $\mathrm{Rot}(4, 3, 2)$ with the same result.

It remains to consider the indices of $\dot{J}_1 - \dot{J}_4$. We first show that $\lambda_1(\dot{J}_2) > \max\{\lambda_1(\dot{J}_3), \lambda_1(\dot{J}_4)\}$. As in the proof of Lemma 3.5 we can show that $\dot{J}_3$ and $\dot{J}_4$ have a positive λ_1-eigenvector. (Namely, we compute $b\big(\frac{(\lambda_1+1)(\lambda_1-3)}{\lambda_1(\lambda_1+2)-5}, \frac{(\lambda_1^2+\lambda_1-2)}{\lambda_1(\lambda_1+2)-5}, \frac{(\lambda_1^2+\lambda_1-2)}{\lambda_1(\lambda_1+2)-5}, \frac{\lambda_1^2+1}{\lambda_1(\lambda_1+2)-5}, 1, 1, \ldots, 1\big)^{\intercal}$ for $\dot{J}_3$ which is positive for every $b > 0$, as $\lambda_1 > 3$ when $n \geq 6$. Similarly, we get $b\big(\frac{(\lambda_1+1)^2}{\lambda_1(\lambda_1+4)+1}, \frac{\lambda_1(\lambda_1+1)}{\lambda_1(\lambda_1+4)+1}, \frac{\lambda_1(\lambda_1+1)}{\lambda_1(\lambda_1+4)+1}, 1, 1, \ldots, 1\big)^{\intercal}$ for $\dot{J}_4$, which is positive for $b > 0$, as well.) Set $\dot{J}_* \in \{\dot{J}_3, \dot{J}_4\}$, and let $\mathbf{x}$ be a positive eigenvector afforded by $\lambda_1(\dot{J}_*)$. Observe that $\dot{J}_2$ is obtained by inserting a positive edge 12 and a negative edge 13 in $\dot{J}_*$. Therefore, we have

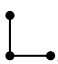
$$\lambda_1(\dot{J}_2) - \lambda_1(\dot{J}_*) \geq \mathbf{x}^{\intercal}(A_{\dot{J}_2} - A_{\dot{J}_*})\mathbf{x} = 2(x_1x_2 - x_1x_3) = 0,$$

where the last equality follows since $x_2 = x_3$ (by the symmetry in $\dot{J}_*$). Hence, $\lambda_1(\dot{J}_2) \geq \lambda_1(\dot{J}_*)$. If $\lambda_1(\dot{J}_2) = \lambda_1(\dot{J}_*)$, then $\mathbf{x}$ is afforded by $\lambda_1(\dot{J}_2)$, but this is impossible since the eigenvalue equation at the vertex 2 cannot hold in $\dot{J}_2$ and $\dot{J}_*$.

Characteristic polynomials of quotient matrices of $\dot{J}_1$ and $\dot{J}_2$ are:

$$\begin{aligned} j_1(x) &= x^3 - (n-6)x^2 - (5n-17)x - 6n + 20 \\ j_2(x) &= x^3 - (n-3)x^2 - (2n-3)x + 7n - 23 \end{aligned}$$

We compute $j(x) = j_1(x) - j_2(x) = 3x^2 - (3n - 14)x - 13n + 43$, with roots: $x_1, x_2 = \frac{1}{6}(3n - 14 \pm \sqrt{9n(8+n) - 320})$. It follows that $j_1(x) < j_2(x)$ for $x \in (x_1, x_2)$. For the larger root x_2 we have $j_1(x_2) = j_2(x_2) = \frac{1}{27}\big((3n-16)\sqrt{9n(8+n) - 320} + 9n^2 - 96n + 248\big) > 0$ where the inequality follows since $9n^2 - 96n + 248 > 0$ for $n \geq 7$, while for $n = 6$ it is confirmed directly. Taking into account that x_1 is negative (the easiest way to see this is to compute $j(0)$), we conclude that $\lambda_1(\dot{J}_1), \lambda_1(\dot{J}_2) \in (x_1, x_2)$. Together with $j_1(x) < j_2(x)$ on the same interval, this leads to $\lambda_1(\dot{J}_1) > \lambda_1(\dot{J}_2)$. □

Lemma 4.3. *$\dot{U}_4$ is* .

Proof. Besides the 13 signed graphs listed in the previous lemma, we have other 4 candidates for $\dot{U}_4$ (that arise from $\dot{U}_3$ but not from $\dot{U}_1$ or $\dot{U}_2$):

The former two are eliminated by Lemma 2.3(i), the latter two by Lemma 2.4(i). Therefore, it remains to consider $\dot{J}_2 - \dot{J}_4$, but they have been already considered in the proof of the previous lemma, when we proved that $\lambda_1(\dot{J}_2) > \max\{\lambda_1(\dot{J}_3), \lambda_1(\dot{J}_4)\}$, as desired. □

The previous results lead to the following theorem.

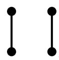
Theorem 4.4. *The first 4 connected unbalanced signed graphs with $n \geq 6$ vertices ordered by their indices are:*

$\dot{U}_1$ $\dot{U}_2$ $\dot{U}_3$ $\dot{U}_4$

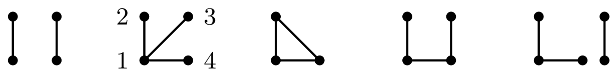
5 Complete signed graphs

As before, let $(\dot{C}_1, \dot{C}_1, \ldots, \dot{C}_c)$ be the subsequence of $\mathcal{S}$ containing complete signed graphs. Clearly, the complete signed graph with the largest index switches to the one with the all positive signature. The next one contains exactly one negative edge. There are 2 candidates for $\dot{C}_3$, both with 2 negative edges. By Lemma 2.4(i), $\dot{C}_3$ is the one in which negative edges are adjacent.

In what follows we set $n \geq 10$.

Lemma 5.1. *$\dot{C}_4$ is* .

Proof. The candidates are:



$\dot{L}_1$ $\dot{L}_2$ $\dot{L}_3$

The latter two are eliminated by Lemma 2.4(i). By inserting the largest eigenvalue of the quotient matrix $Q_{\dot{L}_3}$ into the characteristic polynomial ℓ_2, we get

$$\ell_2\big(\tfrac{1}{2}(n-6+\sqrt{n(n+8)-32})\big) = 2(-n-4+\sqrt{n(n+8)-32}) < 0$$

as $n(n+8)-32 < (n-4)^2$. The latter inequality implies that the largest root of ℓ_2 is larger than the largest eigenvalue of $Q_{\dot{L}_3}$, i.e. $\lambda_1(\dot{L}_2) > \lambda_1(\dot{L}_3)$.

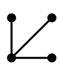
If the vertices of $\dot{L}_2$ are labelled as above then the λ_1-eigenvector has the form

$$\mathbf{b} = b\Big(\frac{(\lambda_1+1)(\lambda_1-5)}{\lambda_1(\lambda_1+2)-11}, \frac{\lambda_1^2+1}{\lambda_1(\lambda_1+2)-11}, \frac{\lambda_1^2+1}{\lambda_1(\lambda_1+2)-11}, 1, 1, \ldots, 1\Big)^{\intercal},$$

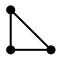
for $b > 0$. Now, $\dot{L}_1$ is obtained by reversing the sign of edges 12, 13 and 23, and thus we have

$$\begin{aligned}
\lambda_1(\dot{L}_1) - \lambda_1(\dot{L}_2) &\geq \mathbf{b}^{\intercal}(A_{\dot{L}_1} - A_{\dot{L}_2})\mathbf{b} \\
&= \frac{4(\lambda_1^2+1)b^2}{\lambda_1(\lambda_1+2)-11}\Big(\frac{2(\lambda_1+1)(\lambda_1-5)}{\lambda_1(\lambda_1+2)-11} - \frac{\lambda_1^2+1}{\lambda_1(\lambda_1+2)-11}\Big) \\
&= \frac{4(\lambda_1^2+1)b^2}{\lambda_1(\lambda_1+2)-11} \cdot \frac{\lambda_1^2-8\lambda_1-9}{\lambda_1(\lambda_1+2)-11} > 0 \ \text{ for } \ \lambda_1 > 9.
\end{aligned}$$

We compute $\lambda_1(\dot{L}_2) > 9$ for $n = 11$, and then by eigenvalue interlacing we have the same inequality for $n \geq 12$. For $n = 10$, the inequality $\lambda_1(\dot{L}_1) > \lambda_1(\dot{L}_2)$ is confirmed directly, and we are done. □

Lemma 5.2. *$\dot{C}_5$ is* .

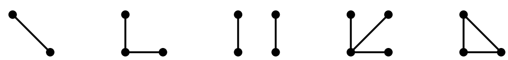
Proof. Apart from the signed graphs faced in the proof of the previous lemma, there is exactly one additional candidate: it contains exactly 3 non-adjacent negative edges. This candidate is eliminated on the basis of Lemma 2.4(i), while the remaining ones are already considered in the previous proof. In particular, we know that $\lambda_1(\dot{L}_2) > \lambda_1(\dot{L}_3)$, and the proof is completed. □

Lemma 5.3. *$\dot{C}_6$ is* .

Proof. The only critical case is the comparison of the indices of $\dot{L}_3$ and the signed graph, say $\dot{L}$, containing 4 negative edges that share the same vertex. Computing the λ_1-eigenvector for $\dot{L}$ and following the proof of Lemma 5.1, we get $\lambda_1(\dot{L}_3) > \lambda_1(\dot{L})$ for $\lambda_1^2 - 12\lambda_1 - 13 > 0$, i.e. for $\lambda_1 = \lambda_1(\dot{L}) > 13$. This proves this lemma for $n \geq 15$ (as there $\lambda_1(\dot{L}) > 13$). The case $10 \leq n \leq 14$ is considered directly, and we are done. □

We arrive at the following result.

Theorem 5.4. *The first 6 complete signed graphs with $n \geq 10$ vertices ordered by their indices are:*



$\dot{C}_1$ $\dot{C}_2$ $\dot{C}_3$ $\dot{C}_4$ $\dot{C}_5$ $\dot{C}_6$

Remark 5.5. With a slight modification in which a full line represents a positive edge and an unpictured line represents a non-edge, the result of Theorem 5.4 remains valid for the ordering of unsigned graphs by the index of the Seidel matrix. Indeed, the Seidel matrix of an unsigned graph G coincides with the adjacency matrix of the complete signed graph in which negative edges are induced by the edges of G.

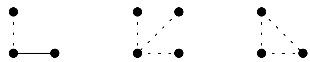
6 Further computations

We complete the results of Sections 4 and 5 by determining the 6 signed graphs with largest indices for every order that is not covered by Theorem 4.4 and the 7 signed graphs with largest indices for every order that is not covered by Theorem 5.4.

There is exactly one connected unbalanced signed graph with 3 vertices (the unbalanced triangle), while the ordering for $n \in \{4, 5\}$ is given in the first part of Figure 3. We note that there are exactly 6 connected unbalanced signed graphs for $n = 4$, so in this case the given list is complete.

There are exactly 3 complete signed graphs with 4 vertices and their ordering does not deviate from the general case considered in Theorem 5.4. For $5 \leq n \leq 9$ the one with the largest index switches to the signed graph with all positive signature, while the remaining 6 are given in the second part of Figure 3. Again, for $n = 5$ the list is complete.

In this paper our idea was to give a general method for the ordering by the index and to demonstrate its use by determining the lists of the first few signed graphs as reported in the previous sections. Of course, these results can be extended, but the theoretical approach is becoming more complicated as the number of candidates increases and comparison of their indices requires more sophisticated methods. However, it occurs that the list of Theorem 3.9 continues with:



$\dot{G}_6$ $\dot{G}_7$ $\dot{G}_8$

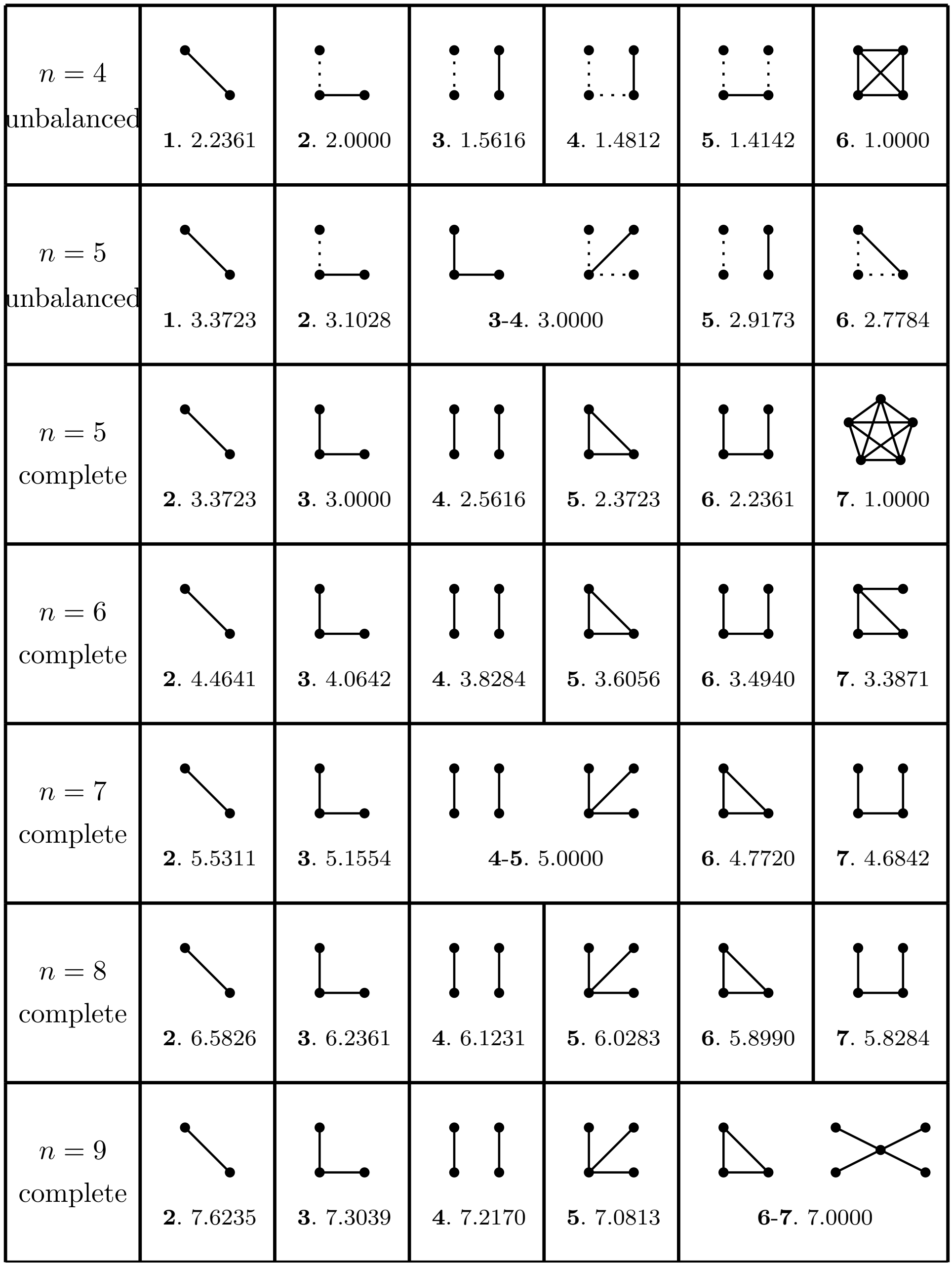


Figure 3: Orderings of small signed graphs that are uncovered by Theorem 4.4 or Theorem 5.4.

We skip the details and note that the proof relies on an intensive algebraic computation that basically does not deviate from those of the previous sections.

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The diameter of products of finite simple groups

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Abstract

Following partially a suggestion by Pyber, we prove that the diameter of a product of non-abelian finite simple groups is bounded linearly by the maximum diameter of its factors. For completeness, we include the case of abelian factors and give explicit constants in all bounds.

Keywords: Finite simple groups, diameter.

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1 Introduction

An important area of research in finite group theory in the last decades has been the production of upper bounds for the diameter of Cayley graphs of such groups. For any finite group G, the maximum diameter over all Cayley graphs defined by *symmetric* sets of generators of G (i.e. sets S with $S = S^{-1}$ and $e \in S$) is called the *diameter* of G. Arguably the best known conjecture in the area is Babai's conjecture [1]: every non-abelian finite simple group G has diameter $\leq \log^k |G|$, where k is an absolute constant; the conjecture is still open, despite great progress towards a solution both for alternating groups and for groups of Lie type.

A more modest question is that of producing bounds for the diameter of direct products of finite simple groups, depending on the diameter of their factors. This is not an idle question, for bounds of this sort have been used more than once as intermediate steps towards the proof of bounds for simple groups themselves: Babai and Seress have done so in [2, Lemma 5.4], as well as Helfgott more than two decades later in [5, Lemma 4.13]. We improve on both results in the following theorem, which also features explicit constants.

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Theorem 1.1. *Let $n \geq 2$. Let $G = \prod_{i=1}^{n} T_i$, where the T_i are finite simple groups.*

(a) *If the T_i are all abelian (say $G = \prod_{j=1}^{s} (\mathbb{Z}/p_j\mathbb{Z})^{e_j}$, where the p_j are distinct primes and $e_j \geq 1$), then:*

$$\operatorname{diam}(G) < \frac{2}{3} \max\{e_j | 1 \leq j \leq s\} \prod_{j=1}^{s} p_j.$$

(b) *If the T_i are all non-abelian, call $d = \max\{\operatorname{diam}(T_i) | 1 \leq i \leq n\}$; then:*

$$\operatorname{diam}(G) < \frac{196}{243} n^3 \max\{C_A, C_L, C_S\}(4d + 1) + d,$$

where:

$$C_A = \begin{cases} \max\left\{3, \left\lfloor \frac{m}{2} \right\rfloor\right\} & \text{if there are alternating groups among the } T_i \\ & \text{and where } m \text{ is their maximum degree,} \\ 0 & \text{if there are no alternating groups among the } T_i, \end{cases}$$

$$C_L = \begin{cases} 8(5r + 7) & \text{if there are groups of Lie type among the } T_i \\ & \text{and where } r \text{ is their maximum untwisted rank,} \\ 0 & \text{if there are no groups of Lie type among the } T_i, \end{cases}$$

$$C_S = \begin{cases} 6 & \text{if there are sporadic or Tits groups among the } T_i, \\ 0 & \text{if there are no sporadic or Tits groups among the } T_i. \end{cases}$$

(c) *If there are abelian and non-abelian T_i, write $G = G_A \times G_{NA}$, where G_A collects the abelian factors and G_{NA} collects the non-abelian ones; then:*

$$\operatorname{diam}(G) \leq d_A + 4d_{NA},$$

where $d_A = \operatorname{diam}(G_A), d_{NA} = \operatorname{diam}(G_{NA})$.

The result of part (a) is known and elementary: see [2, Lemma 5.2], where the constant is marginally worse only due to the fact that sets of generators are not required to be symmetric (cfr. also [5, Lemma 4.14], which treats the case of $G = (\mathbb{Z}/p\mathbb{Z})^e$ under this assumption). Part (c) is quite natural, given the different (in some sense, opposite) behaviour of abelian and non-abelian factors, as it can be readily observed in its short proof.

Part (b) is where the novelty of the result resides. Dependence on the maximum of the diameter of the components, instead of dependence on their product as Schreier's lemma (see Lemma 2.1) would naturally give us, was already established in [2, Lemma 5.4]: in that case, the diameter was bounded as $O(d^2)$, where the dependence of the constant on n was polynomial as in our statement. This result was improved in [5, Lemma 4.13] to $O(d)$, but only in the case of alternating groups: this was done in part to fix a mistake in the use of the previously available result in Babai-Seress, which is why only alternating groups were considered, as permutation subgroups were the sole concern in both papers; a suggestion by Pyber, reported in Helfgott's paper, points at the results by Liebeck and Shalev [8] as a way to prove a bound of $O(d)$ for a product of arbitrary non-abelian finite simple groups.

Indeed, the general approach that we follow in our proof owes its validity to [8, Theorem 1.6], although we do not explicitly use the statement of that theorem: rather, we closely follow the proof of [5, Lemma 4.13] and show that the same reasoning applies to groups of Lie type as well. The way that the lemma is related to Liebeck-Shalev is through the use of the fact that every element in $\mathrm{Alt}(m)$ is a commutator ([5, Lemma 4.12], first proved in [9, Theorem I]), which is essentially [8, Theorem 1.6] with $w = xyx^{-1}y^{-1}$ and a c that is just equal to 1 for $\mathrm{Alt}(m)$; the same can be said for all non-abelian finite simple groups (i.e., $c = 1$ in general) since Ore's conjecture [10] was established to be true in [7], a fact yet unproved at the time of [8].

2 Preliminaries

Before we turn to the proof of Theorem 1.1, we will need a certain number of group-theoretic results.

Lemma 2.1 (Schreier's Lemma). *Let G be a finite group, let $N \trianglelefteq G$, and let S be a set of generators of G with $e \in S = S^{-1}$. Then $S^{2d+1} \cap N$ generates N, where $d = \mathrm{diam}(G/N)$.*

Proof. This is a standard result dating back to Schreier [11], written in various fashions across the literature according to the needs of the user; let us prove here the present version.

Calling $\pi\colon G \to G/N$ the natural projection, by definition we have $\pi(S)^d = G/N$; this equality means that S^d contains at least one representative for each coset gN in G. For any coset gN, choose a representative $\tau(g) \in S^d$. Then, for any $h \in N$ and any way to write h as a product of elements $s_i \in S$, we have:

$$\begin{aligned} h &= s_1 s_2 \dots s_k \\ &= (s_1\tau(s_1)^{-1}) \cdot (\tau(s_1)s_2\tau(\tau(s_1)s_2)^{-1}) \cdots (\tau(\tau(\tau(\dots)s_{k-2})s_{k-1})s_k). \end{aligned}$$

Each element of the form $\tau(x)s_i\tau(\tau(x)s_i)^{-1}$ is contained in $S^{2d+1} \cap N$, so the same can be said about the last element of the form $\tau(x)s_k$ (since h itself is in N); therefore $S^{2d+1} \cap N$ is a generating set of N. □

Proposition 2.2 (Ore's Conjecture). *Let G be a finite non-abelian simple group. Then, for any $g \in G$, there exist $g_1, g_2 \in G$ such that $g = [g_1, g_2]$.*

Proof. See [7], for references to previously known results and for the proof of the final case. □

Notice that, for any finite non-abelian simple group G, any nontrivial conjugacy class C must generate the whole G (because $\langle C \rangle$ would be a normal subgroup). This observation justifies the following definition.

Definition 2.3. Let G be a finite non-abelian simple group. The conjugacy diameter $\mathrm{cd}(G)$ is the smallest m such that $(C \cup C^{-1} \cup \{e\})^m = G$ for all nontrivial conjugacy classes C.

We will need to have bounds for $\mathrm{cd}(G)$.

Proposition 2.4. *Let G be a finite non-abelian simple group.*

(a) If G is an alternating group of degree m, then $\mathrm{cd}(G) \leq \max\left\{3, \left\lfloor \frac{m}{2} \right\rfloor\right\}$.

(b) If G is a group of Lie type of untwisted rank r, then $\mathrm{cd}(G) \leq 8(5r + 7)$.

(c) If G is a sporadic group or the Tits group, then $\mathrm{cd}(G) \leq 6$.

Proof. First of all, $\mathrm{cd}(G)$ is trivially bounded by definition by the *covering number* of G, which is defined as $\mathrm{cn}(G) = \min\{m | \forall C \neq \{e\}(C^m = G)\}$; therefore it suffices to give bounds for $\mathrm{cn}(G)$.

For (a), see [4, Theorem 9.1] (our specific result is credited therein to a manuscript by J. Stavi). For (b), see [6, Theorem 1]. To prove (c), the sporadic groups all satisfy $\mathrm{cn}(G) \leq 6$: this inequality can be checked directly from [13, Table 1]; if $G = {}^2F_4(2)'$ is the Tits group, we can show the same inequality using [13, Lemma 3] and the character values reported in the ATLAS of Finite Groups [3]. □

Let us also perform a side computation separately from the proof of the main theorem, so as not to bog down the exposition there.

Lemma 2.5. *Let $n \geq 2$. Then:*

$$\sum_{i=1}^{n-1} 4^{\lceil \log_2 i \rceil} < \frac{196}{243} n^3.$$

Proof. Call $m = \lceil \log_2(n-1) \rceil$, and write $n - 1 = 2^{m-1} + l$, where $1 \leq l \leq 2^{m-1}$; $\lceil \log_2 i \rceil = j$ for all $i \in (2^{j-1}, 2^j]$, hence we can rewrite the sum in the statement as:

$$\begin{aligned}\sum_{i=1}^{n-1} 4^{\lceil \log_2 i \rceil} &= 1 + \sum_{j=1}^{m-1} 4^j 2^{j-1} + 4^m l = \frac{1}{2} + \frac{1}{2}\frac{8^m - 1}{7} + 4^m \left(2^{\log_2(n-1)} - 2^{m-1}\right) \\ &= \frac{3}{7} + 4^m 2^{\log_2(n-1)} - \frac{3}{7}8^m = \frac{3}{7} + 2^{2m'}\left(1 - \frac{3}{7}2^{m'}\right)(n-1)^3,\end{aligned}$$

where $m' = m - \log_2(n-1) \in [0, 1)$. We have $x^2\left(1 - \frac{3}{7}x\right) \leq \frac{196}{243}$ for $x \in [1, 2)$, and $\frac{3}{7} < \frac{196}{243}(3n^2 - 3n + 1)$ for all $n \geq 2$, so the result is proved. □

3 Proof of the main theorem

Proof of Theorem 1.1(a). Let $G = (\mathbb{Z}/p_1\mathbb{Z})^{e_1} \times (\mathbb{Z}/p_2\mathbb{Z})^{e_2} \times \ldots \times (\mathbb{Z}/p_s\mathbb{Z})^{e_s}$, with primes $p_1 < p_2 < \ldots < p_s$; we have:

$$G = A_1 A_2 \ldots A_s \tag{3.1}$$

(we are using multiplicative notation even if G is abelian) where the A_i are any sets such that:

$$A_{i,i} = (\mathbb{Z}/p_i\mathbb{Z})^{e_i} \qquad\qquad A_{i,j} = (0)^{e_j} \quad (\forall j < i) \tag{3.2}$$

where $A_{i,j}$ is the projection of A_i to the j-th component of G.

Let S be a set of generators of G with $e \in S = S^{-1}$: $\{t^{p_1 \cdots p_{i-1}} | t \in S\} \subseteq S^{p_1 \cdots p_{i-1}}$ has elements that are all 0 on the first $i-1$ components of G and that still generate the i-th one since $(p_1 \ldots p_{i-1}, p_i) = 1$; from now on, let us focus exclusively on the i-th component. $(\mathbb{Z}/p_i\mathbb{Z})^{e_i}$ is also a vector space over $\mathbb{Z}/p_i\mathbb{Z}$, so there must be e_i generators that also form a basis: any element of the space can be written as a linear combination of those generators with coefficients in $\left[-\left\lfloor \frac{p_i}{2} \right\rfloor, \left\lfloor \frac{p_i}{2} \right\rfloor\right]$, which corresponds to a word of length $\leq e_i \left\lfloor \frac{p_i}{2} \right\rfloor$; thus, each set A_i with the properties in (3.2) is covered in $e_i \left\lfloor \frac{p_i}{2} \right\rfloor p_1 \dots p_{i-1}$ steps. This fact and (3.1) imply that G has diameter bounded by:

$$\sum_{i=1}^{s}\left(e_i \left\lfloor \frac{p_i}{2} \right\rfloor \prod_{j=1}^{i-1} p_j\right) \leq \frac{1}{2}\max\{e_j|1 \leq j \leq s\} \prod_{j=1}^{s} p_j \cdot \sum_{i=1}^{s}\left(\prod_{j=i+1}^{s} \frac{1}{p_j}\right). \tag{3.3}$$

The sum in (3.3) is maximized when each p_j is the j-th prime number: for $s = 1$ the sum is 1 and for $s = 2$ it is bounded by $\frac{4}{3}$; for $s \geq 3$, we use $p_s \geq 5$ and $p_j \geq 3$ for all $1 < j < s$, so that the sum is bounded by $1 + \frac{1}{5}\frac{1}{1-\frac{1}{3}} = \frac{13}{10}$. The result follows. □

Proof of Theorem 1.1(b). Calling $G_j = \prod_{i=1}^{j} T_i$, we have natural projections $\pi_j : G = G_n \to G_j$ and $\rho_{j_1,j_2} : G_{j_1} \to T_{j_2}$ for any $j_1 \geq j_2$. As in (3.1), we write G as a product of subsets A_i with $\rho_{n,i}(A_i) = T_i$ and $\rho_{n,j}(A_i) = \{e\}$ for all $j < i$, and our aim is to cover each one of them.

Suppose that we have two subsets X_1, X_2 of G for which $\rho_{n,i}(X_1) = \rho_{n,i}(X_2) = T_i$ for some fixed $i \in \{1, \dots, n\}$ and that have $\rho_{n,j_1}(X_1) = \{e\} = \rho_{n,j_2}(X_2)$ for all $j_1 \in I_1, j_2 \in I_2$, where I_1, I_2 are two subsets of indices in $\{1, \dots, n\} \setminus \{i\}$: then, the set $X = \{[x_1, x_2]|x_1 \in X_1, x_2 \in X_2\}$ has $\rho_{n,i}(X) = T_i$ by Proposition 2.2 (Ore's conjecture) and $\rho_{n,j}(X) = \{e\}$ for all $j \in I_1 \cup I_2$. Now consider the set of indices $I = \{1, \dots, i-1\}$: if $|I| > 1$ we can partition I into two parts of size $\left\lfloor \frac{|I|}{2} \right\rfloor, \left\lceil \frac{|I|}{2} \right\rceil$, then partition each part I' with $|I'| > 1$ into two new parts again of size $\left\lfloor \frac{|I'|}{2} \right\rfloor, \left\lceil \frac{|I'|}{2} \right\rceil$, and continue until we reach a subdivision where all sets have size 1; the tree of partitions that we constructed to reach this subdivision will have exactly $\lceil \log_2 |I| \rceil$ layers. Notice that, given any two parts I_1, I_2 inside the tree, if we have two subsets X_1, X_2 (as described before) that are covered by a certain S^a, the resulting set X will be covered by S^{4a}: this observation, together with the information about the layers, tells us that if we can cover sets $X_{i,j}$ with $\rho_{n,i}(X_{i,j}) = T_i$ and $\rho_{n,j}(X_{i,j}) = \{e\}$ in a steps (for a fixed $i > 1$ and all $j < i$) then we are able to cover a set A_i defined as at the beginning of the proof in $4^{\lceil \log_2(i-1) \rceil} a$ steps as well.

Let us start now with a generating set S with $e \in S = S^{-1}$ and fix two indices $i \geq j$: $\pi_i(S)$ is a set of generators for G_i, and the set $\pi_i(S)^{2d+1}$ contains generators for the whole $T_1 \times \ldots \times T_{j-1} \times \{e\} \times T_{j+1} \times \ldots \times T_i = G_i \cap \ker(\rho_{i,j})$ by Lemma 2.1 (Schreier's lemma), where d is as in the statement. In particular, there is an element $x \in S^{2d+1}$ with $\rho_{n,i}(x) \neq e$ and $\rho_{n,j}(x) = e$; by hypothesis $\rho_{n,i}(S^d) = T_i$, which means that there is a set $S' = \{yxy^{-1}|y \in S^d\} \cup \{yx^{-1}y^{-1}|y \in S^d\} \cup \{e\} \subseteq S^{4d+1}$ with $\rho_{n,i}(S') = C \cup C^{-1} \cup \{e\}$ and $\rho_{n,j}(S') = \{e\}$, where C is the conjugacy class of $\rho_{n,i}(x)$. By Proposition 2.4, $\rho_{n,i}(S'^{\max\{3, \lfloor \frac{m_i}{2} \rfloor\}}) = T_i$ if $T_i = \mathrm{Alt}(m_i)$, $\rho_{n,i}(S'^{8(5r_i+7)}) = T_i$ if T_i is of Lie type of untwisted rank r_i, and $\rho_{n,i}(S'^6) = T_i$ otherwise; in all three cases, the projection to T_j is still $\{e\}$, therefore we managed to cover a set $X_{i,j}$ of the aforementioned form.

A set A_1 is reached in d steps, hence the final count for the whole G following the reasoning above is:

$$\mathrm{diam}(G) \leq d + \sum_{i=2}^{n} 4^{\lceil \log_2(i-1) \rceil} x_i (4d + 1),$$

where x_i is either $\max\left\{3, \left\lfloor \frac{m_i}{2} \right\rfloor\right\}$, $8(5r_i + 7)$ or 6, accordingly. The result follows by Lemma 2.5. □

A note on the connection between the proof given above and [8]. As mentioned before, Pyber pointed at [8] as a way to prove linear dependence on d for products of arbitrary non-abelian finite simple groups. In particular, [8, Theorem 1.6] seems to fit the bill: it states that for any word w that is not a law in a finite simple group T there is $c_w \in \mathbb{N}$, depending on w but not on T, such that any element of T can be written as a product of at most c_w values of w. We use this property, in disguise, when we want to pass from two subsets being indentically e at indices I_1, I_2 and filling an entire component T_i to a third subset that also fills the same component and is e for the whole $I_1 \cup I_2$: the creation of the new subset is made possible by taking c_w values of a word w, so that T_i remains filled, where w has two distinct letters x_1, x_2 and presents the same number of x_i and x_i^{-1} for $i \in \{1, 2\}$, so that when any one x_i is equal to e on a given factor of the product G the result is e on that factor; in our case, w was the shortest nontrivial word with these characteristics, namely the commutator $[x_1, x_2] = x_1x_2x_1^{-1}x_2^{-1}$ (not a law for any non-abelian group), and $c_w = 1$ by Ore's conjecture. In this sense $w = [x_1, x_2]$ is also computationally the best word we can expect, for it yields the lowest possible value of $|w|c_w$, the 4 that we find in Lemma 2.5.

Proof of Theorem 1.1(c). Define the two projections π_A, π_{NA} in the obvious way; for any generating set S of G, by definition there is a subset $X_A \subseteq S^{d_A}$ with $\pi_A(X_A) = G_A$ and there is a subset $X_{NA} \subseteq S^{d_{NA}}$ with $\pi_{NA}(X_{NA}) = G_{NA}$, and then:

$$G = X_A[X_{NA}, X_{NA}] \subseteq S^{d_A+4d_{NA}},$$

again by the fact that $[T, T] = T$ for non-abelian finite simple groups by Ore's conjecture and $[T, T] = \{e\}$ for abelian groups. □

4 Concluding remarks

One could wonder how tight the inequalities in Theorem 1.1 are. The results are essentially in line with what is generally expected from the behaviour of the diameter of finite groups. The abelian case is tight up to constant: for the group $G(x) = \prod_{p\leq x} \mathbb{Z}/p\mathbb{Z}$ (nontrivial for $x \geq 2$) one generator $s = (1, 1, \dots, 1)$ is enough, and then the diameter of $\mathrm{Cay}(G(x), \{s, s^{-1}, e\})$ is $\frac{1}{2}|G(x)|$; the fact that abelian groups behave in the worst possible way, i.e. linearly in the size of the group, should not be a surprise for anyone.

The non-abelian bound of case (b) also matches what is anticipated in general. Babai's conjecture posits a polylogarithmic bound on the diameter of finite simple groups: the natural extension to direct products of such groups would suggest a bound of the form $n^k d$, which is exactly what we have obtained. Case (c) also fits into the same idea, as a product $|G| = |G_A||G_{NA}|$ becomes a sum of the corresponding diameters.

The dependence on d in Theorem 1.1(b) is almost best possible by definition (we cannot drop the "almost", as m, r are not independent from d). It would be more interesting to understand which power of n is the correct one: here we have proved $O_{m,r,d}(n^3)$, and we can quickly show that the bound is $\Omega_{m,r,d}(n)$, as illustrated in the following example.

Example 4.1. If $G = (\mathrm{Alt}(m))^n$ then $\mathrm{diam}(G) = \Omega(m^2n)$. We prove it for $m \geq 5$ odd and n even, but the proof is analogous for the general case.

Consider the two permutations $\sigma = (1\ 2\ 3\ \dots\ m)$ and $\tau = (1\ 2\ 3\ \dots\ m-2)$; they generate $\mathrm{Alt}(m)$, and the elements:

$$\begin{aligned} s_0 &= (\sigma, \sigma, \dots, \sigma, \sigma), \\ s_1 &= (\tau, \sigma, \dots, \sigma, \sigma), \\ s_2 &= (\sigma, \tau, \dots, \sigma, \sigma), \\ &\dots \\ s_n &= (\sigma, \sigma, \dots, \sigma, \tau) \end{aligned}$$

generate G. Let $S = \{e\} \cup \{s_i, s_i^{-1}\}_{0 \leq i \leq n}$: to prove the lower bound on the diameter of G, we construct a function $f\colon G \to \mathbb{N}$ such that there are two elements $g_1, g_2 \in G$ with $|f(g_1) - f(g_2)|$ large and such that $|f(g) - f(gs)|$ is small for any $g \in G, s \in S$; this is a known technique to prove lower bounds for the diameter of $\mathrm{Sym}(m)$, as shown for instance in [12, Proposition 3.6].

Call $c(g, i, j) = (g(i))(j)$ the image of $j \in \{1, \dots, m\}$ under the i-th component of $g \in G$, for $1 \leq i \leq n$; define:

$$f(g) = \sum_{j=1}^{m} \sum_{i=1}^{n} ||c(g, i+1, j) - c(g, i, j)||_{\mathbb{Z}/m\mathbb{Z}},$$

where $||a||_{\mathbb{Z}/m\mathbb{Z}} = \min\{a, m-a\}$ (in the case $i = n$, $c(g, n+1, j)$ means $c(g, 1, j)$). First, $f(e) = 0$; also, if we call e_m the identity element in $\mathrm{Alt}(m)$ and $\eta = \left(1\ \frac{m+1}{2}\right)\left(2\ \frac{m+3}{2}\right) \cdots \left(\frac{m-1}{2}\ m-1\right)$, for $g \in G$ that has e_m at all odd components and η at all even ones we have $f(g) = \frac{1}{2}(m-1)^2 n$. Finally, notice that σ simply adds 1 modulo m to all the elements of $\{1, \dots, m\}$, so that $f(g) = f(gs_0^{\pm 1})$, while τ is defined so that it adds 1 for $m-3$ elements, adds 3 (modulo m) for one element and fixes two elements, which means that $|f(g) - f(gs_i^{\pm 1})| \leq 10$; these facts taken together imply that $\mathrm{diam}(G, S) \geq \frac{1}{20}(m-1)^2 n$.

The correct (or even expected) order of magnitude for a bound of the form $\mathrm{diam}(G) = O_{m,r}(n^k d)$ for a generic product G is not known to the author, besides knowing that $1 \leq k \leq 3$ by Theorem 1.1 and Example 4.1.

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S^2 coverings by isosceles and scalene triangles – adjacency case II*

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Abstract

The aim of this paper is to complete the study and classification of spherical f-tilings by scalene triangles T and isosceles triangles T' within a subclass defined by the adjacency of the lower side of T and the longest side of T'. It consists of eight families of f-tilings (two families with one continuous parameter, one family with one discrete parameter and one continuous parameter, and five families with one discrete parameter). We also analyze the combinatorial structure of all these families of f-tilings, as well as the group of symmetries of each tiling; the transitivity classes of isogonality are included.

Keywords: Dihedral f-tilings, combinatorial properties, spherical trigonometry.

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1 Introduction

A *folding tessellation* or *folding tiling* (f-tiling, for short) of the sphere S^2 is an edge-to-edge finite polygonal tiling τ of S^2 such that all vertices of τ satisfy the angle-folding relation, i.e., each vertex is of even valency and the sums of alternating angles around each vertex are equal to π.

F-tilings are intrinsically related to the theory of isometric foldings of Riemannian manifolds, introduced by Robertson [10] in 1977. In several situations (beyond the scope of this paper), the edge-complex associated to a spherical f-tiling is the set of singularities of some spherical isometric folding.

The classification of f-tilings was initiated by Breda [2], with a complete classification of all spherical monohedral (triangular) f-tilings. Afterwards, in 2002, Ueno and Agaoka [11] have established the complete classification of all triangular monohedral tilings of the sphere (without any restrictions on angles). Curiously, the triangular tilings of even valency at any vertex are necessarily f-tilings. Dawson has also been interested in special classes of spherical tilings, see [3, 4, 5], for instance. Spherical f-tilings by two noncongruent classes of isosceles triangles have recently been obtained [6, 7].

The study of dihedral triangular f-tilings involving scalene triangles is clearly more unwieldy and was initiated in [1]. In this paper we complete the classification of spherical f-tilings by scalene triangles T and isosceles triangles T' resulting from the adjacency of the lower side of T and the longest side of T'.

From now on,

(i) T denotes a spherical scalene triangle with internal angles $\alpha > \beta > \gamma$ and side lengths $a > b > c$;

(ii) T' denotes a spherical isosceles triangle with internal angles $(\delta, \delta, \varepsilon)$, $\delta \neq \varepsilon$, and side lengths (d, d, e),

as illustrated in Figure 1.

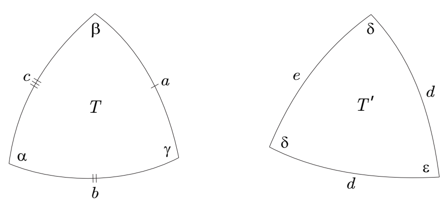


Figure 1: A spherical scalene triangle, T, and a spherical isosceles triangle, T'.

We shall denote by $\Omega(T, T')$ the set, up to isomorphism, of all dihedral folding tilings of S^2 whose prototiles are T and T' in which the lower side of T is equal to the longest side T'.

Taking into account the area of the prototiles T and T', we have

$$\alpha + \beta + \gamma > \pi \quad \text{and} \quad 2\delta + \varepsilon > \pi.$$

As $\alpha > \beta > \gamma$, we also have $\alpha > \frac{\pi}{3}$. In [8] it was established that any $\tau \in \Omega(T, T')$ has necessarily vertices of valency four.

We begin by pointing out that any element of $\Omega\,(T,T')$ has at least two cells congruent to T and T', respectively, such that they are in adjacent positions and in one and only one of the situations illustrated in Figure 2.

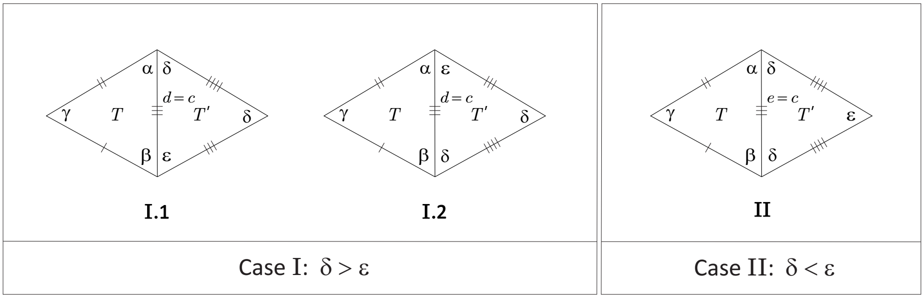


Figure 2: Distinct cases of adjacency.

In this paper we will consider the second case of adjacency. The next section contains the main result of this paper. In Section 2 we describe the eight families of spherical f-tilings that we may obtain from the second case of adjacency (Figure 2-II). The combinatorial structure of these tilings, the classification of the group of symmetries and also the transitivity classes of isogonality are presented. The proof of the main result consists of a long and exhaustive method and it is presented in Section 3.

2 Main result - Elements of $\Omega\,(T,T')$ in the case of Adjacency II

Theorem 2.1. *Let T and T' be a spherical scalene triangle and a spherical isosceles triangle, respectively, such that they are in adjacent positions as illustrated in* Figure 2-II. *Within this case, the f-tilings of $\Omega\,(T,T')$ are*

$$\mathcal{L}_\beta,\quad \mathcal{D}^k_\varepsilon\ (k\geq 4),\quad \mathcal{M}_\gamma,\quad \mathcal{N}^k\ (k\geq 6),\quad \mathcal{P}^k\ (k\geq 3),\quad \mathcal{Q}^k\ (k\geq 4),\quad \mathcal{R}^k\ (k\geq 6)\quad \text{and}\quad \mathcal{S}^k\ (k\geq 7),$$

that satisfy, respectively:

(i) $\alpha+\delta+\beta=\pi,\ \varepsilon=\frac{\pi}{2},\ \gamma=\frac{\pi}{3}$, *where α and β satisfy*

$$\sin^2(\alpha+\beta)\,(1+2\cos(\alpha-\beta))=2\sin\alpha\sin\beta \quad \text{and} \quad \beta\in\left(\frac{\pi}{3},\arccos\frac{\sqrt{6}}{6}\right);$$

(ii) $\alpha+\delta=\pi,\ \delta+\beta+\varepsilon=\pi,\ k\gamma=\pi,\ \delta=\delta^1_k(\varepsilon),\ \varepsilon\in\left(\varepsilon_{\min},\frac{(k-1)\pi}{k}\right), k\geq 4,$

$$\text{where}\quad \delta^1_k(\varepsilon)=\arctan\frac{2\sin\varepsilon\cos^2\frac{\varepsilon}{2}}{\cos\frac{\pi}{k}-\cos^2\varepsilon} \quad \text{and} \quad \varepsilon_{\min}=\arccos\frac{\sqrt{1+8\cos\frac{\pi}{k}}-1}{4};$$

(iii) $\alpha+\delta=\pi,\ \varepsilon=\frac{\pi}{2},\ \beta+\delta+\gamma=\pi,\ \delta=\gamma$ *and* $\gamma\in\left(\frac{\pi}{4},\frac{\pi}{3}\right)$;

(iv) $\alpha + \delta = \pi,\ \varepsilon = \frac{\pi}{2},\ \beta + 3\delta = \pi,\ k\gamma = \pi$ *and* $\delta = \delta_k^2 = \arccos\sqrt{\frac{1}{2}\cos\frac{\pi}{k}},\ k \geq 6$*;*

(v) $\alpha + \delta = \pi,\ \varepsilon = \frac{\pi}{2},\ 2\beta + 2\delta = \pi,\ \beta + \delta + k\gamma = \pi$ *and* $\delta = \delta_k^3 = \arctan\left(\sec\frac{\pi}{2k}\right)$, $k \geq 3$*;*

(vi) $\alpha + \delta = \pi,\ \varepsilon = \frac{\pi}{2},\ 2\beta + 2\delta + \gamma = \pi,\ \beta + \delta + k\gamma = \pi,\ \delta = \delta_k^4,\ k \geq 4$, *where*

$$\delta_k^4 = \arctan\left(\sin\frac{(k-1)\pi}{2k-1}\sec\frac{\pi}{2k-1}\right);$$

(vii) $\alpha + \delta = \pi,\ \varepsilon = \frac{\pi}{2},\ 2\beta + 2\delta = \pi,\ k\gamma = \pi$ *and* $\alpha = \alpha_k^2 = 2\arctan(\cos\frac{\pi}{k} + \sqrt{1+\cos^2\frac{\pi}{k}}),\ k \geq 6$*;*

(viii) $\alpha + \delta = \pi,\ \varepsilon = \frac{\pi}{2},\ 2\beta + 2\delta + \gamma = \pi,\ k\gamma = \pi$ *and* $\delta = \delta_k^5,\ k \geq 7$, *where*

$$\delta_k^5 = \arctan\left(\sin\frac{(k-1)\pi}{2k}\sec\frac{\pi}{k}\right).$$

For each family of f-tilings we present the distinct classes of congruent vertices in Figure 3 (including the respective number of vertices in each tiling).

Particularizing suitable values for the parameters involved in each case, the corresponding 3D representations of these families of f-tilings are given in Figure 4. In each case, we present two perspectives in order to provide a more effective visualization of each f-tiling's combinatorial structure. Regarding the f-tiling P^k, $k \geq 3$, it can be observed that, if we consider the great circle that contains the four vertices surrounded by $(\beta, \delta, \delta, \beta, \gamma, \gamma, ..., \gamma)$ (marked in red) as the equator line and rotating the southern hemisphere 90 degrees (around the "vertical" axis) we obtain the f-tiling R^{2k}. Also, it is interesting to relate the monohedral edge-to-edge tilings TI_{16n+8} and I_{8n} described by Ueno and Agaoka in [11] with the families of f-tilings Q^k, $k \geq 4$, and S^k, $k \geq 7$, obtained by subdividing the prototypes in the monohedral tilings into two triangles satisfying the conditions of Figure 2-II. Seeing from another perspective, we obtain TI_{16n+8} and I_{8n} eliminating the vertices surrounded by $(\alpha, \alpha, \delta, \delta)$ (marked in green) and two suitable edges emanating from those vertices of Q^k, $k \geq 4$, and S^k, $k \geq 7$, respectively.

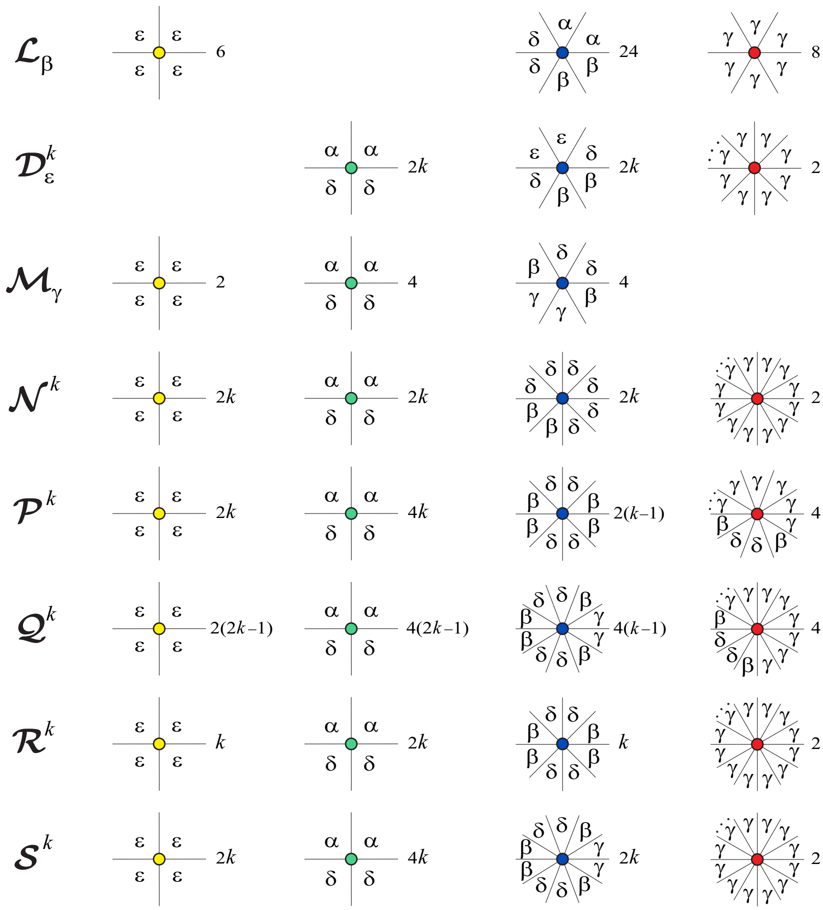
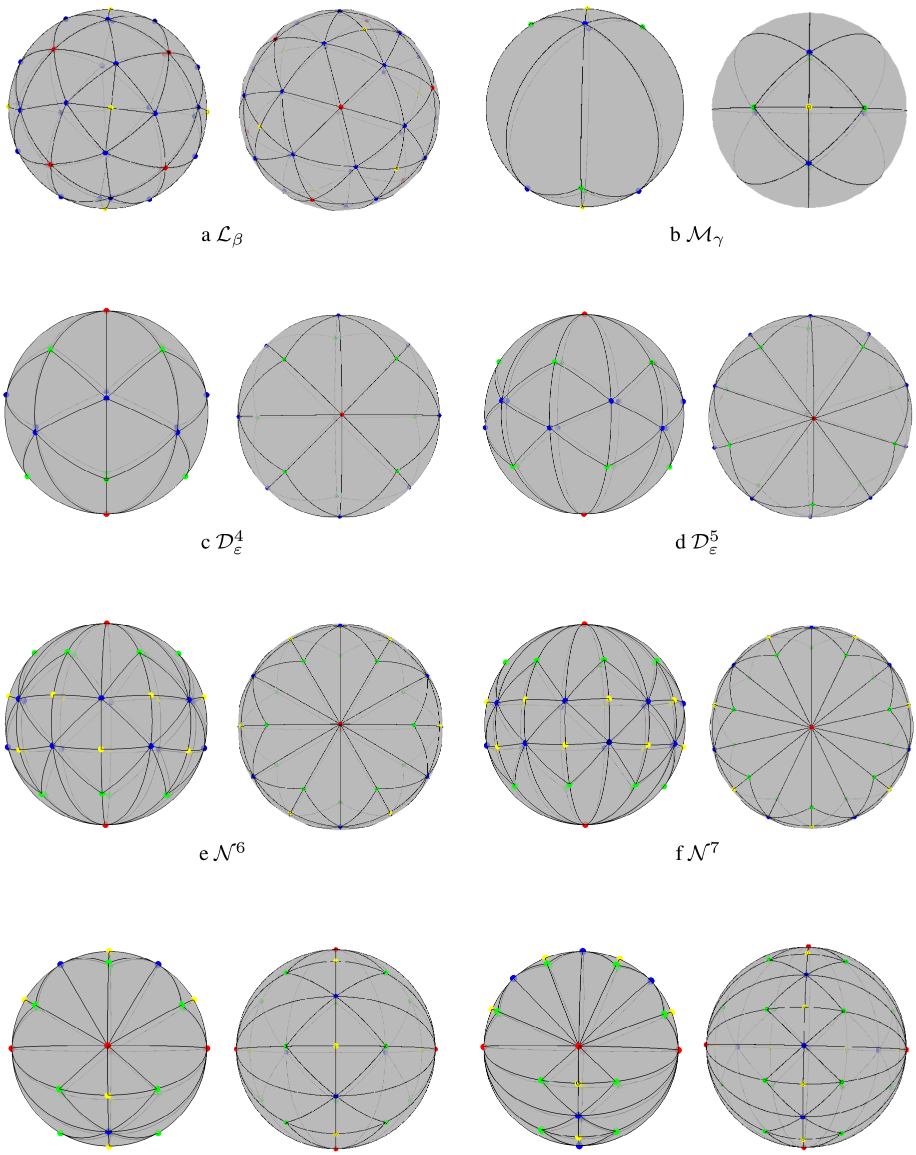


Figure 3: Distinct classes of congruent vertices.



a $\mathcal{L}_\beta$ b $\mathcal{M}_\gamma$

c $\mathcal{D}^4_\varepsilon$ d $\mathcal{D}^5_\varepsilon$

e $\mathcal{N}^6$ f $\mathcal{N}^7$

g $\mathcal{P}^3$ h $\mathcal{P}^4$

Figure 4: Elements of $\Omega(T, T')$ in the case of adjacency II.

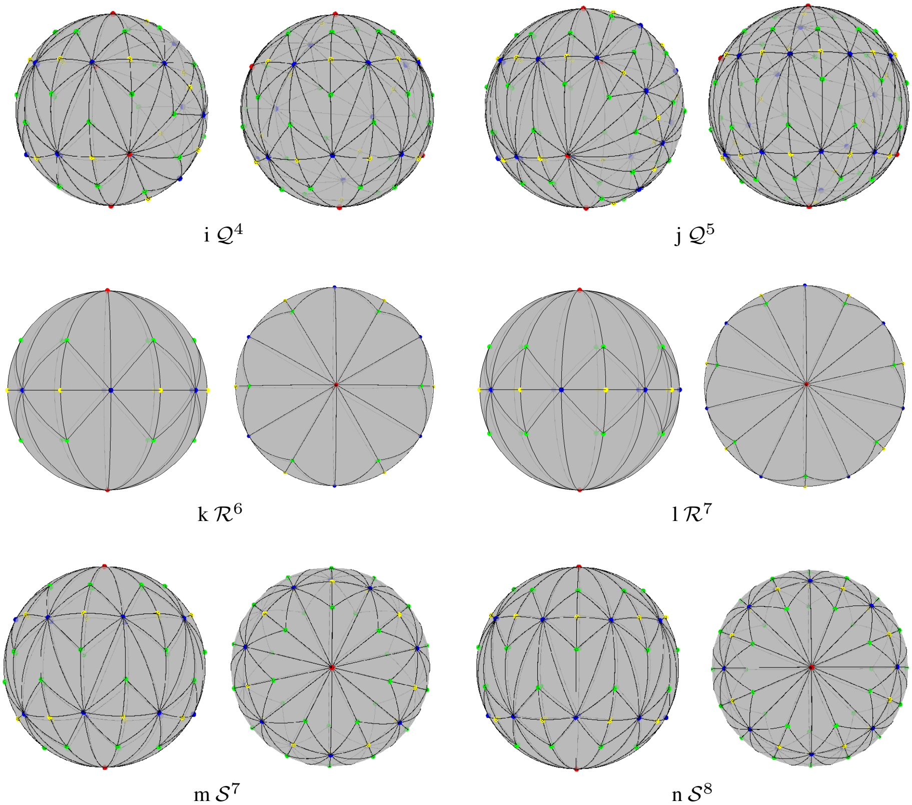


Figure 4: Elements of $\Omega(T, T')$ in the case of adjacency II.

The combinatorial structure of the classes of spherical f-tilings mentioned in Theorem 2.1, including the symmetry groups, is summarized in Table 1 (the analysis of the symmetry groups is similar to that applied in previous articles, *e.g.* [9]). Our notation is as follows:

- $|V|$ is the number of distinct classes of congruent vertices;
- N_1 and N_2 are, respectively, the number of triangles congruent to T and T', respectively;
- $G(\tau)$ is the symmetry group of each tiling $\tau \in \Omega\,(T, T')$ and the index of isogonality for the symmetry group is denoted by #isog.;
- C_n is the cyclic group of order n;
- $V \simeq C_2 \times C_2$ is the Klein group;
- D_n is the n^{th} dihedral group (it consists of n rotations and n reflections);
- O is the chiral group with 24 elements;

- $i(k) = \begin{cases} \frac{3k}{2} + 1 & \text{if } k \text{ even} \\ \frac{3k+1}{2} + 1 & \text{if } k \text{ odd.} \end{cases}$

f-tiling	α	β	γ	δ	ε	$\|V\|$	N_1	N_2	$G(\tau)$	#isog.
$\mathcal{L}_\beta$	$\alpha(\beta)$	$\left(\frac{\pi}{3}, \arccos\frac{\sqrt{6}}{6}\right)$	$\frac{\pi}{3}$	$\pi-\alpha-\beta$	$\frac{\pi}{2}$	3	48	24	O	3
$\mathcal{D}^k_\varepsilon$, $k \geq 4$	$\pi-\delta$	$\pi-\delta-\varepsilon$	$\frac{\pi}{k}$	$\delta^1_k(\varepsilon)$	$(\varepsilon_{\min}, \varepsilon_{\max})$	3	$4k$	$4k$	D_{2k}	3
$\mathcal{M}_\gamma$	$\pi-\gamma$	$\pi-2\gamma$	$\left(\frac{\pi}{4}, \frac{\pi}{3}\right)$	γ	$\frac{\pi}{2}$	3	8	8	V	3
$\mathcal{N}^k$, $k \geq 6$	$\pi-\delta$	$\pi-3\delta$	$\frac{\pi}{k}$	δ^2_k	$\frac{\pi}{2}$	4	$4k$	$8k$	D_{2k}	4
$\mathcal{P}^k$, $k \geq 3$	$\pi-\delta$	$\frac{\pi}{2}-\delta$	$\frac{\pi}{2k}$	δ^3_k	$\frac{\pi}{2}$	4	$8k$	$8k$	$C_2 \times C_2 \times C_2$	$i(k)$
$\mathcal{Q}^k$, $k \geq 4$	$\pi-\delta$	$\frac{(k-1)\pi}{2k-1}-\delta$	$\frac{\pi}{2k-1}$	δ^4_k	$\frac{\pi}{2}$	4	$14k$	$14k$	V	$4k-2$
$\mathcal{R}^k$, $k \geq 6$	α^2_k	$\frac{\pi}{2}-\delta$	$\frac{\pi}{k}$	$\pi-\alpha$	$\frac{\pi}{2}$	4	$4k$	$4k$	$C_2 \times D_k$	4
$\mathcal{S}^k$, $k \geq 7$	$\pi-\delta$	$\frac{(k-1)\pi}{2k}-\delta$	$\frac{\pi}{k}$	δ^5_k	$\frac{\pi}{2}$	4	$8k$	$8k$	D_{2k}	4

Table 1: Combinatorial structure of the dihedral f-tilings of S^2 by scalene triangles T and isosceles triangles T' performed by the lower side of T and the longest side of T' in the case of adjacency II.

3 Proof of Theorem 2.1

In order to better understand the structure of each tiling and due to the complexity of a global planar representation, in the following proof some f-tilings τ are illustrated only by a fundamental region F that generates τ by successive reflections and rotations of F. Comparing the fundamental region F with its associated f-tiling τ (in Figure 4), it becomes clear how it is generated. In two of the situations (tilings Q^k and S^k), instead of a fundamental region, we illustrate planar representations that correspond to a half of the f-tilings.

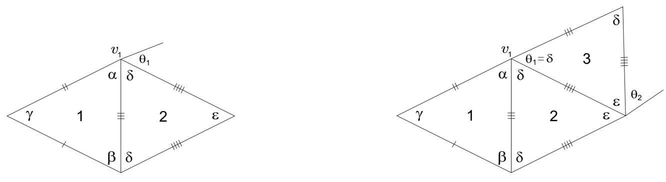
In the case of adjacency II, any element of $\Omega(T, T')$ has at least two cells congruent to T and T', respectively, such that they are in adjacent positions and in one and only one of the situations illustrated in Figure 2. After certain initial assumptions are made, it is usually possible to deduce sequentially the nature and orientation of most of the other tiles. Eventually, either a complete tiling or an impossible configuration proving that the hypothetical tiling fails to exist is reached. In the diagrams that follow, the order in which these deductions can be made is indicated by the numbering of the tiles. For $j \geq 2$, the location of tiling j can be deduced directly from the configurations of tiles $(1, 2, \ldots, j-1)$ and from the hypothesis that the configuration is part of a complete tiling, except where otherwise indicated.

Observe that we have $\varepsilon > \frac{\pi}{3}$ (since we are considering the case of adjacency II). Also, as $e = c$ and using spherical trigonometric formulas, we get

$$\frac{\cos\gamma + \cos\alpha\cos\beta}{\sin\alpha\sin\beta} = \frac{\cos\varepsilon + \cos^2\delta}{\sin^2\delta}. \tag{3.1}$$

Proof of Theorem 2.1. Suppose that any element of $\Omega(T, T')$ has at least two cells congruent, respectively, to T and T', such that they are in adjacent positions as illustrated in Figure 2-II.

With the labeling of Figure 5a, we have $\theta_1 \in \{\varepsilon, \delta\}$. It is easy to verify that θ_1 must be δ. In fact, if $\theta_1 = \varepsilon$, v_1 cannot have valency four (see side lengths), $\alpha + \varepsilon + \rho > \pi$,



a b

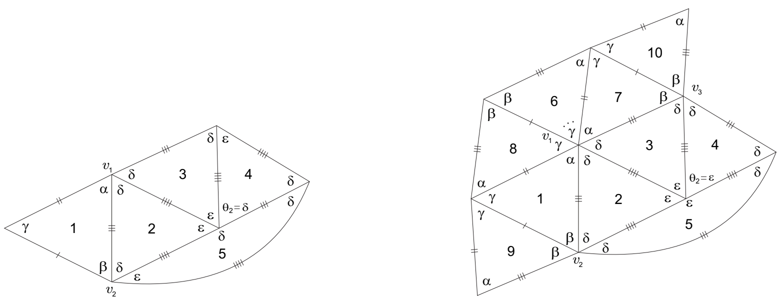
Figure 5: Local configurations.

$\forall \rho \in \{\alpha, \beta, \delta, \varepsilon\}$, and if $\alpha + \varepsilon + k\gamma = \pi$, $k \geq 1$, an incompatibility between sides cannot be avoided.

Now, at vertex v_1 (see Figure 5b) we must have

$$\alpha + \delta < \pi \quad \text{or} \quad \alpha + \delta = \pi.$$

1. Suppose firstly that $\alpha + \delta < \pi$. If $\theta_2 = \delta$ and $\varepsilon + \delta = \pi$ (Figure 6a), we reach a contradiction at vertex v_2, as $\varepsilon + \beta + \rho > \pi$, for all $\rho \in \{\alpha, \beta, \gamma, \delta, \varepsilon\}$. In fact, taking into account the side lengths, v_2 cannot have valency four and also observe that $\varepsilon + \beta + \rho_1 \geq \alpha + \beta + \gamma$, $\rho_1 \in \{\alpha, \beta, \gamma\}$, and $\varepsilon + \beta + \rho_2 > \varepsilon + \delta = \pi$, $\rho_2 \in \{\delta, \varepsilon\}$.



a b

Figure 6: Local configurations.

On the other hand, if $\theta_2 = \delta$ and $\varepsilon + \delta < \pi$, we must have $\varepsilon + \delta + \rho \leq \pi$, for some $\rho \in \{\alpha, \beta, \gamma\}$. If $\rho = \alpha$, we get $\varepsilon > \delta > \alpha > \beta > \gamma$; but then $\varepsilon + \delta + \alpha > \alpha + \beta + \gamma > \pi$, which is not possible. If $\rho = \beta$, we obtain $\delta > \beta$ and $\alpha > \varepsilon$, which implies $\alpha + \delta + \bar{\rho} > \pi$, $\forall \bar{\rho}$, which is a contradiction. Finally, due to an incompatibility between sides, it is not possible to have $\varepsilon + \delta + k\gamma = \pi$, $k \geq 1$.

Therefore, $\theta_2 = \varepsilon$ and, due to the side lengths, we must have $\varepsilon + \varepsilon = \pi$, and obviously $\alpha > \delta$, with $\delta \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$.

1.1 If $\alpha \geq \varepsilon$, at vertex v_1 (Figure 5b) we must have $\alpha + \delta + k\gamma = \pi$, with $k \geq 1$, and $\alpha > \beta > \delta > \gamma$. The last configuration extends to the one illustrated in Figure 6b.

If, at vertices v_2 and v_3, we have

(i) $\beta + \delta + \delta = \pi$, we reach a vertex surrounded by six angles δ, implying $\delta = \frac{\pi}{3} = \beta$, which is not possible as $\beta > \delta$;

(ii) $\beta + \delta + \beta = \pi$, we obtain the configuration illustrated in Figure 7a. Taking into account the edge lengths and the fact that $\beta > \delta > \frac{\pi}{4}$, at vertex v_4 we reach a contradiction.

Note that it is easy to conclude that is not possible to include angles γ in the previous sums.

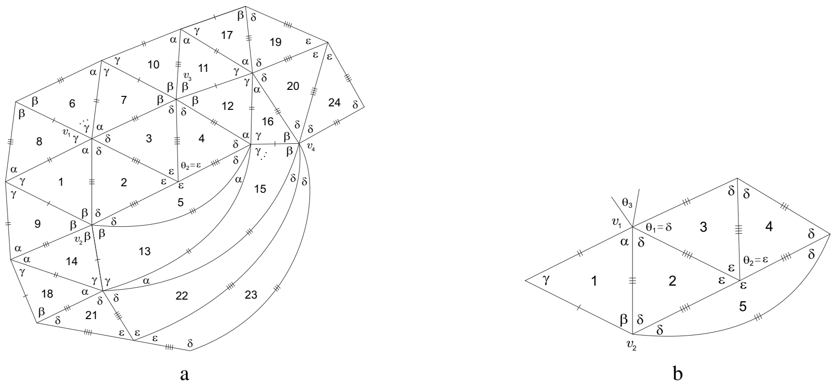


Figure 7: Local configurations.

1.2 Suppose now that $\alpha < \varepsilon$.

1.2.1 If $\delta \geq \gamma$, with the labeling of Figure 7b, we have $\theta_3 \in \{\delta, \gamma\}$. Additionally is important to note that $\varepsilon = \frac{\pi}{2} > \alpha > \beta > \delta \geq \gamma$, $\delta > \frac{\pi}{4}$ and $\beta + \gamma > \frac{\pi}{2}$.

If $\theta_3 = \delta$, we obtain the configuration of Figure 8a. Observe that θ_4 cannot be δ, as $\delta + \delta + \delta < \delta + \delta + \alpha = \pi$ and $\delta + \delta + \delta + \rho > \pi$, with $\rho \in \{\alpha, \beta, \delta, \varepsilon\}$; ρ_2 cannot be γ due to an incompatibility between sides. Moreover, θ_4 cannot be β, as $\delta + \delta + \beta < \pi$ and $\delta + \delta + \beta + \gamma > \pi$.

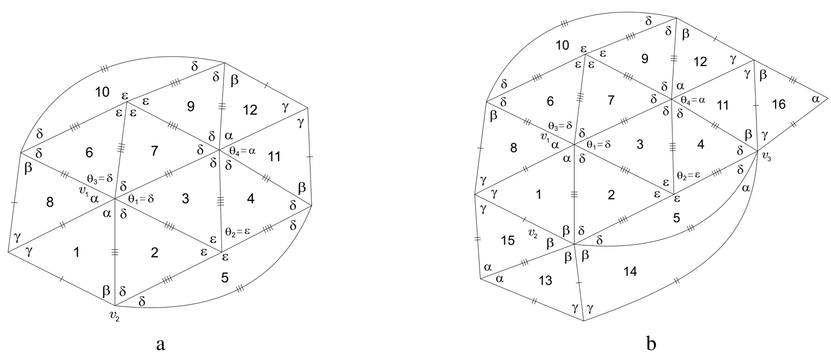
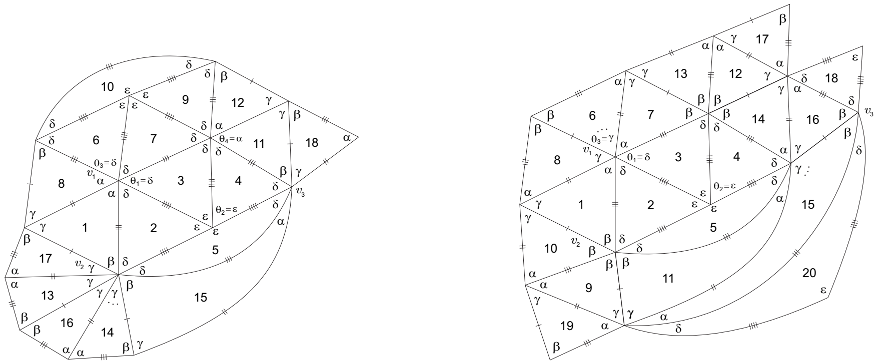


Figure 8: Local configurations.

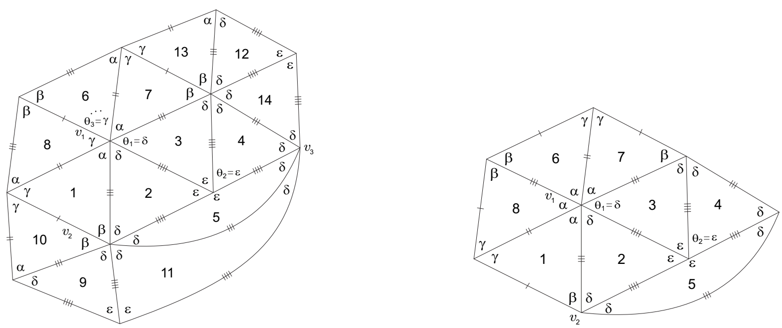
Now, at vertex v_2 we have necessarily $\beta + \delta + \beta = \pi$ or $\beta + \delta + k\gamma = \pi$, with $k \geq 2$. These cases lead to the configurations illustrated in Figure 8b and Figure 9a, respectively. In both cases, at vertex v_3 we reach a contradiction. In fact, due to the edge and angles lengths there is no way to satisfy the angle-folding relation around this vertex.



a b

Figure 9: Local configurations.

If $\theta_3 = \gamma$ (Figure 7b), at vertex v_2 we have necessarily $\beta+\delta+\beta = \pi$ or $\beta+\delta+\delta = \pi$. These cases lead to the configurations illustrated in Figure 9b and Figure 10a, respectively. In the first case, at vertex v_3 we have $\beta + \delta + \delta < \pi$ and $\beta + \delta + \delta + \rho > \pi$, for all $\rho \in \{\alpha, \beta, \gamma, \delta, \varepsilon\}$. In the last case, at vertex v_3 we also reach a contradiction, as $\delta = \frac{\pi}{3}$ implies $\beta = \frac{\pi}{3}$ and, due to the edge and angles lengths, it is not possible that this vertex has valency greater than three.



a b

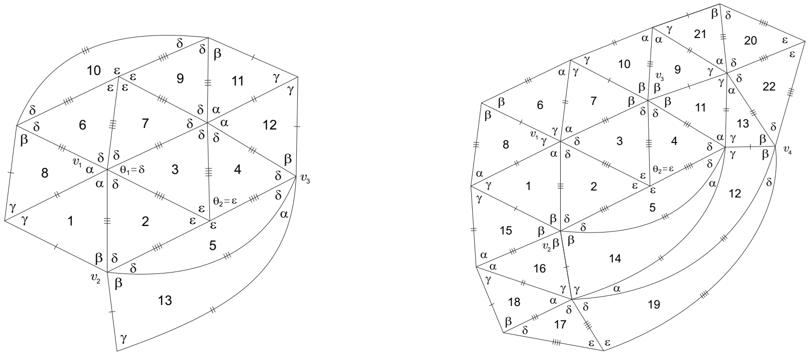
Figure 10: Local configurations.

1.2.2 If $\delta < \gamma$, then $\varepsilon = \frac{\pi}{2} > \alpha > \beta > \gamma > \delta > \frac{\pi}{4}$. At vertex v_1 (Figure 7b) we must have one of the following situations:

(i) $\alpha + \delta + \alpha = \pi$; in this case (Figure 10b), there is no way to satisfy the angle-folding relation around vertex v_2.

(ii) $\alpha + \delta + \delta = \pi$; as we can observe in Figure 11a, an incompatibility between sides at vertex v_3 cannot be avoided.

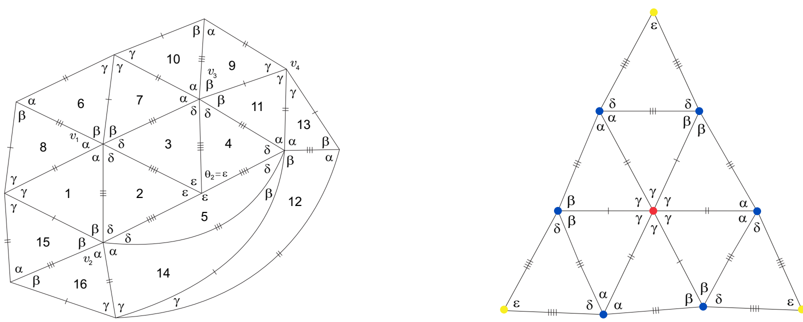
(iii) $\alpha + \delta + \gamma = \pi$; in this case (Figure 11b), there is no way to satisfy the angle-folding relation around vertex v_4.



a b

Figure 11: Local configurations.

(iv) $\alpha+\delta+\beta=\pi$; in this situation, the last configuration extends to the one illustrated in Figure 12a. Now, at vertex v_4 we have necessarily $\gamma+\gamma+\rho=\pi$, with $\rho\in\{\alpha,\beta,\gamma\}$. It is easy to verify that the two first cases lead to impossibilities. The last case ($\rho=\gamma$) leads to a continuous family of f-tilings formed by 72 tiles. Due to the large dimension of the corresponding planar representation, we only illustrate its eighth fundamental region in Figure 12b.



a

b A eighth fundamental region of $\mathcal{L}_\beta$

Figure 12: Local configurations.

We denote this continuous family of f-tilings by $\mathcal{L}_\beta$, where

$$\alpha+\delta+\beta=\pi,\quad 2\varepsilon=\pi \quad \textit{and} \quad 3\gamma=\pi.$$

Using Equation (3.1), we get

$$\sin^2(\alpha+\beta)\,(1+2\cos(\alpha-\beta))=2\sin\alpha\sin\beta,\quad \text{with}\quad \frac{\pi}{3}<\beta<\arccos\frac{\sqrt{6}}{6}.$$

3D representations of $\mathcal{L}_\beta$ are illustrated in Figure 4a.

2. Suppose now that $\alpha + \delta = \pi$. We have $\alpha > \delta$ and $\alpha > \frac{\pi}{2}$. In fact, if $\alpha \leq \delta$, we would have $\varepsilon > \delta \geq \alpha > \beta > \gamma$, with $\delta \geq \frac{\pi}{2}$, and consequently $\varepsilon + \theta_2 > \pi$, $\theta_2 \in \{\varepsilon, \delta\}$.

2.1 If $\theta_2 = \delta$ (Figure 5b), then it is a straightforward exercise to prove that $\gamma = \frac{\pi}{k}$, for some $k \geq 4$, and the complete planar representation derives uniquely as illustrated in Figure 13.

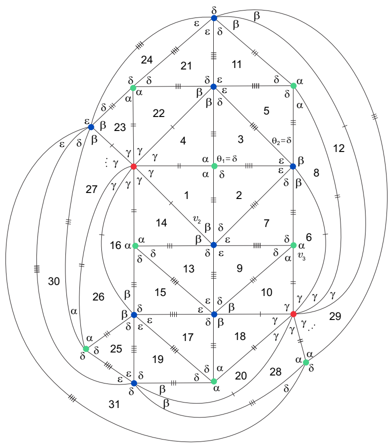


Figure 13: Planar representation of $\mathcal{D}_{\varepsilon}^{k}$, $k \geq 4$.

This family of f-tilings is denoted by $\mathcal{D}_{\varepsilon}^{k}$, where $\alpha + \delta = \pi$, $\delta + \beta + \varepsilon = \pi$ and $k\gamma = \pi$, with $k \geq 4$. Using (3.1), we get

$$\delta = \delta_k(\varepsilon) = \arctan \frac{2 \sin \varepsilon \cos^2 \frac{\varepsilon}{2}}{\cos \frac{\pi}{k} - \cos^2 \varepsilon}, \quad k \geq 4,$$

with $\varepsilon \in \left(\varepsilon_{\min}, \frac{(k-1)\pi}{k}\right)$, where $\varepsilon_{\min} = \arccos \frac{\sqrt{1+8\cos\frac{\pi}{k}}-1}{4}$. 3D representations of $\mathcal{D}_{\varepsilon}^{4}$ and $\mathcal{D}_{\varepsilon}^{5}$ are given in Figures 4c – 4d.

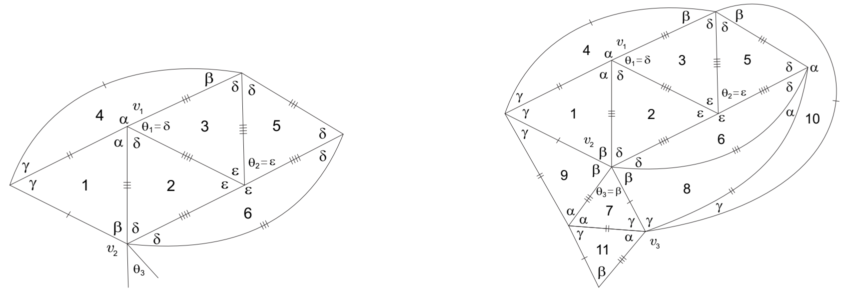
2.2 If $\theta_2 = \varepsilon$, we have $\beta \geq \delta$ or $\beta < \delta$.

2.2.1 If $\beta \geq \delta$, we have $\alpha > \frac{\pi}{2} = \varepsilon > \delta > \frac{\pi}{4}$ and the last configuration extends to the one illustrated in Figure 14a. Now, we have $\theta_3 \in \{\beta, \delta, \gamma\}$.

2.2.1.1 If $\theta_3 = \beta$, at vertex v_2 we must have $\delta + \beta + \beta + k\gamma = \pi$, with $k \geq 0$. It is easy to verify that k has to be zero, giving rise to the configuration of Figure 14b. At vertex v_3 we obtain $\alpha + \bar{k}\gamma = \pi$, with $\bar{k} \geq 2$. Taking into account Equation (3.1) and the relations between angles, we get $2\cos\frac{\delta}{k} = \cos\delta \csc\frac{\delta}{2}$. Consequently, we obtain $\sin\delta \leq \cos\delta$ and $\delta \leq \frac{\pi}{4}$, which is not possible.

2.2.1.2 If $\theta_3 = \delta$, at vertex v_2 we have $\delta + \beta + \delta + k\gamma = \pi$, with $k \geq 0$. It is a straightforward exercise to show that (i) if $k = 0$, although a complete configuration is achieved, it leads to $\beta = \gamma = \frac{\pi}{3}$, which is not possible; (ii) if $k = 1$, again a complete configuration is achieved, with $\delta = \frac{\pi}{3} = \beta + \gamma$, which is a contradiction; (iii) the case $k > 1$ leads to an incompatibility between sides.

2.2.1.3 If $\theta_3 = \gamma$, at vertex v_2 we must have one of the following situations:



a b

Figure 14: Local configurations.

(i) $\beta+\delta+k\gamma=\pi$, $k\geq 1$; the case $k>1$ leads to $\delta=\beta=\frac{\pi}{3}$, which is not possible, and so $k=1$. In this case we obtain the planar representation of Figure 15. We denote this family of f-tilings by $\mathcal{M}_\gamma$, where $\alpha+\delta=\pi$, $\beta+\delta+\gamma=\pi$ and $\gamma\in\left(\frac{\pi}{4},\frac{\pi}{3}\right)$. Using Equation (3.1), we get $\delta=\gamma$.

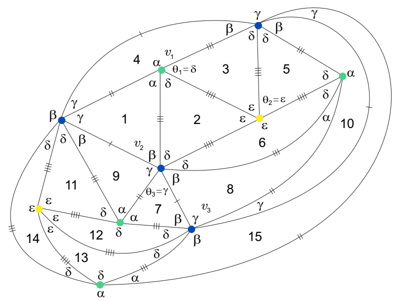
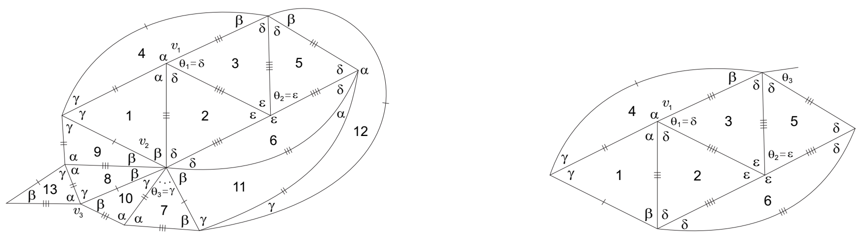


Figure 15: Planar representation of $\mathcal{M}_\gamma$.

3D representations of $\mathcal{M}_\gamma$, $\gamma\in\left(\frac{\pi}{4},\frac{\pi}{3}\right)$, are illustrated in Figure 4b.

(ii) $\beta+\delta+\beta+k\gamma=\pi$, $k\geq 1$; as we can observe in Figure 16a, we reach an impossibility as there is no way to complete the sum of alternate angles around vertex v_3.



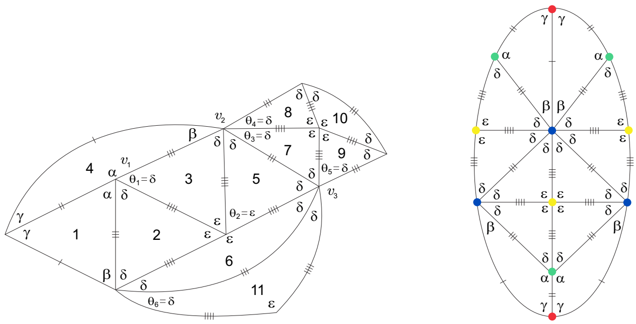
a b

Figure 16: Local configurations.

(iii) $\beta+\delta+\delta+k\gamma=\pi$, $k\geq 1$; in this case there is no way to satisfy the angle-folding relation around vertex v_2.

2.2.2 If $\beta<\delta$ (Figure 5b), it is easy to conclude that $\alpha>\frac{\pi}{2}=\varepsilon>\delta>\beta>\gamma$. Now, with the labeling of Figure 16b we have $\theta_3\in\{\delta,\beta\}$.

2.2.2.1 If $\theta_3=\delta$, we obtain the configuration illustrated in Figure 17a. Note that θ_4 cannot be ε, otherwise there is no way to satisfy the angle-folding relation around vertex v_2. Also, θ_5 cannot be ε, as it implies $\theta_6=\alpha$. Now, at vertex v_3, we have necessarily $3\delta+\beta=\pi$. In fact, if $3\delta=\pi$, at vertex v_2 we obtain $\delta+\delta+\beta<\pi$ and $\delta+\delta+\beta+\rho>\pi$, $\rho\in\{\beta,\gamma\}$, as $\beta+\gamma>\delta$. Then, the last configuration extends uniquely to a complete



a

b A kth fundamental region of $\mathcal{N}^k$, $k\geq 6$.

Figure 17: Local configurations.

planar representation formed by $12k$ tiles. Due to its large dimension, we only illustrate the kth fundamental region in Figure 17b. As $\beta>\gamma$, using Equation (3.1), we must have $k\gamma=\pi$, with $k\geq 6$. We denote this family of f-tilings by $\mathcal{N}^k$, where $\alpha+\delta=\pi$, $3\delta+\beta=\pi$ and $k\gamma=\pi$, $k\geq 6$. Moreover,

$$\delta=\delta_k=\arccos\sqrt{\frac{1}{2}\cos\frac{\pi}{k}},\quad k\geq 6.$$

3D representations of $\mathcal{N}^k$, for $k=6,7$, are illustrated in Figures 4e – 4f.

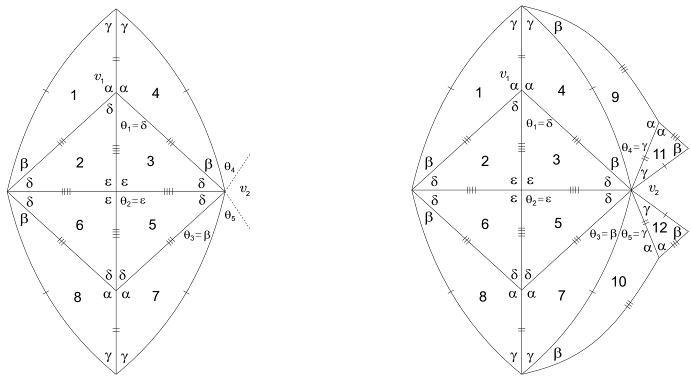
2.2.2.2 If $\theta_3=\beta$, we obtain the configuration of Figure 18a.

It is a straightforward exercise to prove that if vertex v_2 has valency six, we obtain $\alpha+k\gamma=\pi$, $k\geq 2$, or $\beta+\delta+\gamma=\pi$, and in either cases Equation (3.1) has no solution. Moreover, this equation also has no solution if there is a vertex with a sum of alternate angles of the form $\beta+\delta+\delta=\pi$. Now, we consider separately the cases $\theta_4=\theta_5=\gamma$, $\theta_4=\theta_5=\beta$, and $\theta_4=\beta$ and $\theta_5=\gamma$.

2.2.2.2.1 If $\theta_4=\theta_5=\gamma$, vertex v_2 must have valency greater than eight. In fact, valency eight implies the existence of a vertex with a sum of alternate angles of the form $\beta+\delta+\delta=\pi$.

Now, with the labeling of Figure 18b, if v_2 has valency greater or equal to ten and

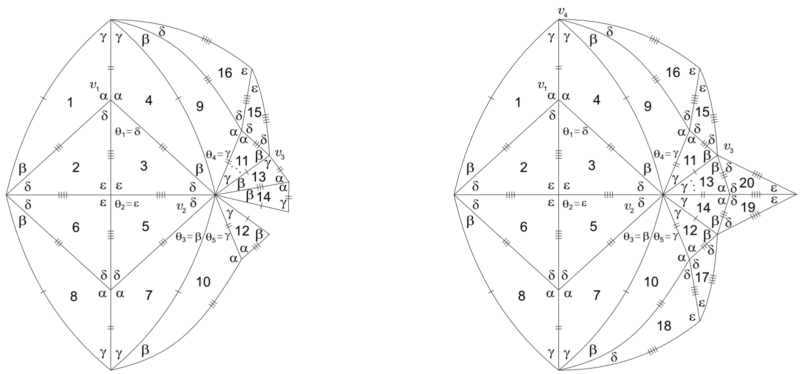
- there is an additional angle β in the sum of alternate angles (note that is not possible to have an additional angle δ, as $2\delta+2(\beta+\gamma)>\pi$), the last configuration extends



a b

Figure 18: Local configurations.

to the one illustrated in Figure 19a and there is no way to satisfy the angle-folding relation around vertex v_3.



a b

Figure 19: Local configurations.

- $\beta + \delta + k\gamma = \pi$, with $k \geq 3$, we obtain the configuration illustrated in Figure 19b. At vertex v_3 we must have one of the following situations:
  - (i) $\beta + \delta + \varepsilon = \pi$; this condition leads to a sum of alternate angles at vertex v_4 containing $\varepsilon + \delta + \beta + \gamma > \pi$, which is not possible;
  - (ii) $\beta + \delta + \delta + \beta = \pi$; in this case we obtain a complete planar representation formed by $16k$ tiles. Due to its dimension, we only illustrate one octant of the sphere (fundamental region) in Figure 20. Observe that one of the hemispheres is obtained from the other through a 90 degree rotation. Note that if $\theta_6 = \delta$, we would obtain $\beta + 3\delta = \pi$ and consequently no solution would exist for Equation (3.1). We have $\delta = \arctan\left(\sec\frac{\pi}{2k}\right)$, $\beta = \frac{\pi}{2} - \delta$, $\gamma = \frac{\pi}{2k}$, $\alpha =$

$\pi - \delta$, $\varepsilon = \frac{\pi}{2}$ and $k \geq 3$. We denote this f-tiling by $\mathcal{P}^k$, $k \geq 3$, whose 3D representations, for $k = 3, 4$, are presented in Figures 4g – 4h.

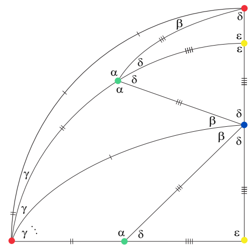


Figure 20: A eighth fundamental region of $\mathcal{P}^k$, $k \geq 3$.

(iii) $\beta + \delta + \delta + \beta + \beta = \pi$; in this case we have necessarily $k \geq 4$ and it gives rise to a sum of alternate angles of the form $\alpha + \bar{k}\gamma = \pi$, with $\bar{k} \geq 2$. Due to the angles relations, we have $\bar{k} = 2$. Nevertheless, under these conditions, Equation (3.1) has no solution.

(iv) $\beta + \delta + \delta + \beta + \gamma = \pi$; in this case we also have $k \geq 4$ and we obtain a complete planar representation formed by $28k$ tiles. Due to its large dimension, we only illustrate one hemisphere in Figure 21. The other hemisphere is obtained through a 180 degree rotation along the x axis and a reflection. We have $\delta = \arctan\left(\sin \frac{(k-1)\pi}{2k-1} \sec \frac{\pi}{2k-1}\right)$, $\beta = \frac{(k-1)\pi}{2k-1} - \delta$, $\gamma = \frac{\pi}{2k-1}$, $\alpha = \pi - \delta$ and $\varepsilon = \frac{\pi}{2}$. We denote this f-tiling by $\mathcal{Q}^k$, $k \geq 4$, whose 3D representations, for $k = 4, 5$, are presented in Figures 4i – 4j.

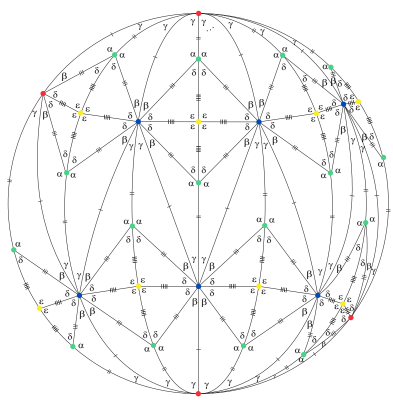


Figure 21: One hemisphere of $\mathcal{Q}^k$, $k \geq 4$.

(v) $\beta + \delta + \delta + \delta = \pi$; under this condition, it is easy to verify that we achieve at vertex v_4 (see Figure 19b) a sum of alternate angles containing $\delta + \delta + \beta + \gamma$, but $\delta + \delta + \beta + \gamma < \delta + \delta + \beta + \delta = \pi$ and $\delta + \delta + \beta + \gamma + \rho > \pi$, for all $\rho \in \{\alpha, \beta, \delta, \varepsilon\}$.

2.2.2.2.2 If $\theta_4 = \theta_5 = \beta$ (Figure 18a), it is easy to observe that vertex v_2 cannot be surrounded by six consecutive angles β, as we obtain a vertex with a sum of alternate angles of the form $\alpha+\gamma+\rho$, with $\rho \in \{\alpha, \beta, \delta\}$, which is not possible. Moreover, it is not possible to have angles γ surrounding v_2, as it gives rise to a vertex with a sum of alternate angles containing α and β. Taking into account these restrictions and analyzing the angles relations and side lengths, at vertex v_2 we must have one of the following cases:

(i) $\beta+\delta+\beta+\delta=\pi$; in this case we obtain the configuration illustrated in Figure 22a.

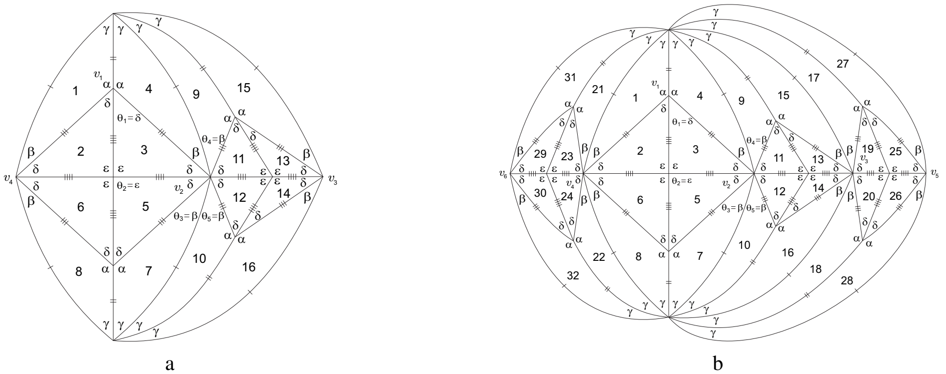


Figure 22: Local configurations.

Given the sums of alternate angles $S_1 : \beta+\delta+\beta+\delta=\pi$ and $S_2 : \beta+\delta+k\gamma=\pi$, $k \geq 3$, it is a straightforward exercise to prove that at vertices v_3 and v_4 we must have only S_1 or a combination of S_1 and S_2 (note that we have symmetry, so order does not matter). If we have a combination of S_1 and S_2, we obtain a complete representation of f-tiling $\mathcal{P}^k$, $k \geq 3$, previously achieved. On the other hand, if we have only S_1, the last configuration extends to the one illustrated in Figure 22b. At vertices v_5 and v_6 we must have only S_1 or S_2. In the last case, as before we obtain the f-tiling $\mathcal{P}^k$, with $k \geq 4$. If S_1 is the sum of alternate angles at vertices v_5 and v_6, then we obtain a complete representation formed by $8k$ tiles. A fundamental region is illustrated in Figure 23. For each $k \geq 6$, we have $\alpha = 2\arctan\left(\cos\frac{\pi}{k} + \sqrt{1+\cos^2\frac{\pi}{k}}\right)$, $\beta = \frac{\pi}{2} - \delta$, $\gamma = \frac{\pi}{k}$, $\delta = \pi - \alpha$ and $\varepsilon = \frac{\pi}{2}$. We denote this f-tiling by $\mathcal{R}^k$, $k \geq 6$, whose 3D representations, for $k = 6, 7$, are presented in Figures 4k – 4l.

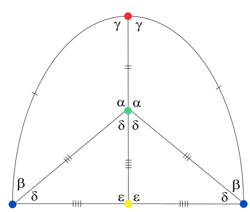


Figure 23: A $2k$th fundamental region of $\mathcal{R}^k$, $k \geq 6$.

(ii) $\beta+\delta+\beta+\delta+\beta=\pi$; this case leads to the following additional relations between angles: $\alpha+\gamma+\gamma=\pi$ and $k\gamma=\pi$, with $k \geq 8$. Nevertheless, under these conditions, (3.1) has no solution.

2.2.2.2.3 If $\theta_4 = \beta$ and $\theta_5 = \gamma$ (Figure 18a), it is easy to observe that vertex v_2 cannot be surrounded by the sequence $(\ldots, \beta, \beta, \gamma, \gamma, \ldots)$, as we achieve a vertex with a sum of alternate angles containing $\alpha + \beta$, which is not possible as $\alpha + \beta + \rho > \pi$, for all ρ. As the sum of alternate angles surrounding v_2 must contain at least one angle γ, taking into account the previous restriction and analyzing angles relations and side lengths, at vertex v_2 we have necessarily $\beta + \delta + \beta + \delta + \gamma = \pi$, as illustrated in Figure 24.

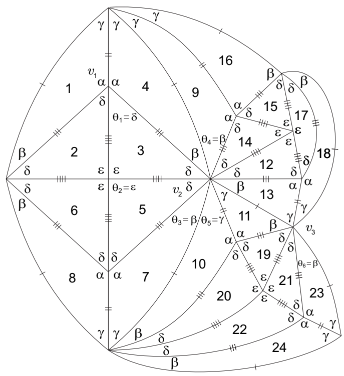


Figure 24: Local configuration.

Note that θ_6 must be β (tile 23), as $\theta_6 = \delta$ immediately leads to an impossibility. It is a straightforward exercise to verify that at vertex v_3 we must have $\beta + \delta + k\gamma = \pi$, with $k \geq 4$, or $\beta + \delta + \beta + \delta + \gamma = \pi$. In the first case, analyzing the symmetry of the figure and all possible combinations of angles surrounding specific vertices, we obtain the f-tiling $\mathcal{Q}^k$, $k \geq 4$, formerly achieved. In the last case, beside this family of f-tilings, we also obtain a complete planar representation formed by $16k$ tiles. Due to its dimension, we only illustrate one hemisphere in Figure 25. the other hemisphere is obtained through a 180 degree rotation along the x axis and a reflection.

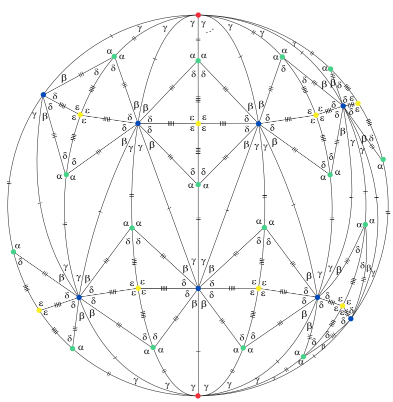


Figure 25: One hemisphere of $\mathcal{S}^k$, $k \geq 7$.

We have $\delta = \arctan\left(\sin \frac{(k-1)\pi}{2k} \sec \frac{\pi}{2k}\right)$, $\beta = \frac{(k-1)\pi}{2k} - \delta$, $\gamma = \frac{\pi}{k}$, $\alpha = \pi - \delta$ and

$\varepsilon = \frac{\pi}{2}$. We denote this f-tiling by $\mathcal{S}^k$, $k \geq 7$, whose 3D representations, for $k = 7, 8$, are presented in Figures 4m – 4n. □

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Configured polytopes and extremal configurations

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Abstract

We examine a class of involutory self-dual convex polytopes with a specified sets of diameters, compare their vertex sets to extremal Lenz configurations, and present some of their realizations.

Keywords: Involutory self-dual polytopes, configured polytopes, Lenz configurations, extremal configurations.

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1 Introduction

We describe points in $\mathbb{R}^d$ by standard coordinates $(x_1, x_2, \ldots, x_d)$. For $3 \leq i \leq d$, let $H_i(b_i)$ denote the hyperplane $x_i = b_i$, and $L_e(b_{e+1}, \ldots, b_d) = \cap_{i=e+1}^{d} H_i(b_i)$, $e = 2, \ldots, d-1$. $L_e(b_{e+1}, \ldots, b_d)$ is an e-flat, and denote the $(e-1)$-sphere with centre c and radius t in $L_e(b_{e+1}, \ldots, b_d)$ by $\mathbb{S}^{e-1}(c, t)$. We denote the origin of $\mathbb{R}^d$ by c_d, and let $(\lambda w, p) := \lambda w + (0, \ldots, 0, p)$, for a point $w \in H_d(0) = L_{d-1}(0)$ and $\{\lambda, p\} \subset \mathbb{R}$.

Let Y be a set of points in $\mathbb{R}^d$. Then $\operatorname{conv}(Y)$ and $\operatorname{aff}(Y)$ denote, respectively, the convex hull and the affine hull of Y. For sets $Y_1, Y_2, \ldots Y_n$, let $[Y_1, Y_2, \ldots Y_n] = \operatorname{conv}(\cup_{i=1}^{n} Y_i)$ and $\langle Y_1, Y_2, \ldots Y_n \rangle = \operatorname{aff}(\cup_{i=1}^{n} Y_i)$. If $Y = \{y_1, y_2, \ldots, y_n\}$ is finite, we let $[y_1, y_2, \ldots, y_n] = \operatorname{conv}(Y)$ and $\langle y_1, y_2, \ldots, y_n \rangle = \operatorname{aff}(Y)$.

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Let $P \subset \mathbb{R}^d$ denote a convex $d-$polytope with $\mathcal{L}(P)$ and $\mathcal{F}_i(P)$, $0 \leq i \leq d-1$, denoting the face lattice and the set of $i-$faces of P. We let $f_i(P) = |\mathcal{F}_i(P)|$, $V(P) = \mathcal{F}_0(P)$ and $\mathcal{F}(\mathcal{P}) = \mathcal{F}_{d-1}(P)$, assume familiarity with the basic notions of convex polytopes, and refer to [3, 6] and [18] for basic terminology and definitions. Specifically, two polytopes P_1 and P_2 are *combinatorially equivalent* ($P_1 \cong P_2$) if there is an isomorphism (inclusion preserving) from $\mathcal{L}(P_1)$ to $\mathcal{L}(P_2)$, and are *dual* if there is an anti-isomorphism (inclusion reversing) from $\mathcal{L}(P_1)$ to $\mathcal{L}(P_2)$. If there is an anti-isomorphism Φ from $\mathcal{L}(P)$ to $\mathcal{L}(P)$ then P is *self-dual*, moreover, if $\Phi^2 = id$ then P is *involutory self-dual*.

Let $P \subset \mathbb{R}^d$ be involutory self-dual via the anti-isomorphism on $\mathcal{L}(P)$ induced by the map $v \to v^*$ with $v \in V(P)$, $v^* \in \mathcal{F}(P)$ and $v \notin v^*$. A segment $[v, w]$, with end-points v and w, both vertices of P and with $w \in v^*$, is called a *principal diagonal* of P and let $\mathcal{D}(P)$ denote the set of principal diagonals of P. Finally, we say that P is *configured* if each principal diagonal in P has length $\operatorname{diam}(P)$, and that P is *strictly configured* if it is configured and only principal diagonals of P have length $\operatorname{diam}(P)$. We note that odd regular polygons are strictly configured.

Let $X_n \subset \mathbb{R}^d$ be a set of $n > d \geq 2$ points and $M_d(X_n)$ be the number of pairs $\{x, y\} \subset X_n$ such that $\operatorname{diam}(X_n) = \|x - y\|$, the distance between x and y. Let $M(d, n)$ be the maximum of $M_d(X_n)$ over all $X_n \subset \mathbb{R}^d$. Then X_n is an *extremal configuration* if $M_d(X_n) = M(d, n)$.

The problem of determining $M(d, n)$ is due to Erdős in [4]. We list contributions to the problem in the References, with specific mention of [11, 12] and [17] and the following results:

(1) $M(2, n) = n$, and $X_n \subset \mathbb{R}^2$ is extremal if and only if $V(P) \subseteq X_n \subseteq \operatorname{bd}(P)$ for some Reuleaux polygon P.

(2) $M(3, n) = 2n - 2$ and $X_n \subset \mathbb{R}^3$ is extremal if and only if X_n is the vertex set of certain types of polytopal (Reuleaux) ball polytopes.

(3) $M(d, n)$, $d \geq 4$, grows quadratically with n, and extremal X_n are attained only by Lenz Constructions.

In this last regard, we note (cf. [17]) that an (even dimensional) *Lenz Configuration* in $\mathbb{R}^d$, $d = 2p \geq 2$, is any translate of a finite subset of $\cup_{i=1}^{p} C_i$ where C_i is a circle with centre at the origin O and radius r_i, so that $r_j^2 + r_k^2 = 1$ for all j, k and the subspaces U_i, spanned by C_i, yield the orthogonal decomposition $\mathbb{R}^d = U_1 \oplus U_2 \oplus \ldots \oplus U_p$. For odd dimensions $d = 2p + 1$, C_1 is replaced by a 2-sphere with centre O and radius $r = \frac{1}{\sqrt{2}}$.

Theorem 1.1 (K. Swanepoel). *For each $d \geq 4$, there exists a number $N(d)$ such that all extremal configurations X_n, with $n \geq N(d)$, are Lenz Configurations.*

We note that in [17], Swanepoel also determines $M(d, n)$ for sufficiently large n.

Our interests in this paper are realizations (constructions) of strictly configured d-polytopes P, $d \geq 3$, and values of $M_d(P)$ (number of principal diagonals of P). In Section 2, we will show that for strictly configured 4-polytopes there is a formula similar to 1) and 2) that depends on the number of vertices and edges; furthermore we show the convex hull of vertices of an extremal Lenz configuration is never a configured d-polytope. The former raises the question of whether in dimension $d \geq 4$ the situation for $M(d, n)$ may have at least another possible scenario, if the points are not in Lenz configurations. In

Section 3 we will give constructions of configured d-polytopes P for $d \geq 3$ such that for $d = 4$, $M_4(P) \leq 4n$. These constructions consist of two steps: determining self-dual polytopes so that all principal diagonals have length (say 1), and then showing that the diameter of the polytope is 1.

2 Principal diagonals

In this section, we assume that $P \subset \mathbb{R}^d$ is an involutory self-dual d-polytope via the anti-isomorphism on $\mathcal{L}(P)$ induced by $v \in V(P) \to v^* \in \mathcal{F}(P)$, and recall that $\mathcal{D}(P)$ denotes the set of principal diagonals of P.

Theorem 2.1. *Let $P \subset \mathbb{R}^3$ be a configured 3-polytope. Then P is strictly configured and extremal, that is, $|\mathcal{D}(P)| = 2f_0(P) - 2$.*

Proof. Since P is self-dual, we have that $f_0(P) = f_2(P)$ and so, $f_1(P) = 2f_0(P) - 2$ by Euler's Theorem.

Let $v \in V(P)$. Then $v^* \in \mathcal{F}_2(P)$ is a polygon and $f_0(v^*) = f_1(v^*)$. On the one hand, $f_0(v^*) = |\{g \in \mathcal{D}(P) \mid v \in g\}|$ by definition. On the other hand, $v \in e \in \mathcal{F}_1(P)$ iff $e^* \in \mathcal{F}_1(v^*)$, and so, $f_1(v^*) = |\{e \in \mathcal{F}_1(P) \mid v \in e\}|$. Thus $|\{g \in \mathcal{D}(P) \mid v \in g\}| = |\{e \in \mathcal{F}_1(P) \mid v \in e\}|$ and $|\mathcal{D}(P)| = |\mathcal{F}_1(P)|$. □

Theorem 2.2. *Let $P \subset \mathbb{R}^4$ be a strictly configured 4-polytope. Then $|\mathcal{D}(P)| \leq 2f_1(P) - 2f_0(P)$.*

Proof. Let $V(P) = \{v_1, ..., v_n\}$ and $\mathcal{F}_1(P) = \{e_1, ..., e_m\}$. Then $\mathcal{F}_2(P) = \{e_1^*, ..., e_m^*\}$ and $\mathcal{F}(P) = \{v_1^*, ..., v_n^*\}$ by the self-duality of P.

We recall from [1] that $f_{jk}(P)$, $0 \leq j < k \leq 3$, is the number of pairs of j-faces G_j and k-faces G_k such that $G_j \subset G_k$, and that $f_{02}(P) \leq 6f_1(P) - 6f_0(P)$. By the self-duality of P, we have also that

$$\sum_{i=1}^{n} f_1(v_i^*) = f_{13}(P) = f_{02}(P),$$

$$\sum_{i=1}^{n} f_2(v_i^*) = f_{23}(P) = f_{01}(P) \text{ and}$$

$$f_{01}(P) = \sum_{j=1}^{m} f_0(e_j) = 2f_1(P)$$

Finally, let $v \in V(P)$ and $e \in \mathcal{D}(P)$ of a configured $P \subset \mathbb{R}^4$.

Then $v \in e$ if, and only if, $e = [v, w]$ and $w \in \mathcal{F}_0(v^*)$. Thus, $f_0(v^*)$ is the number of principal diagonals of P that contain v, and $\sum_{i=1}^{n} f_0(v_i^*) = 2|\mathcal{D}(P)|$. Then by Euler's Theorem,

$$\begin{aligned} |\mathcal{D}(P)| &= \frac{1}{2}\sum_{i=1}^{n}(2 + f_1(v_i^*) - f_2(v_i^*)) \\ &= n + \frac{1}{2}\sum_{i=1}^{n} f_1(v_i^*) - \frac{1}{2}\sum_{i=1}^{n} f_2(v_i^*) \qquad (2.1) \\ &= f_0(P) + \frac{1}{2}f_{02}(P) - \frac{1}{2}f_{01}(P) \\ &\leq f_0(P) + [3f_1(P) - 3f_0(P)] - f_1(P). \end{aligned}$$

End of Theorem 2.2. □

We let $M_d(Q) = M_d(V(Q))$ for a $d-$polytope Q, and observe that if $P \subset \mathbb{R}^4$ is strictly configured then $M_4(P)$ is linear in $f_1(P)$ and $f_0(P)$. This raises the following question: Is there a set of n vertices of a strictly configured polytope in Lenz Configuration? We show below that the answer is no if $f_0(P) > 5$; in fact, we present in Section 3 a subfamily of such $P \subset \mathbb{R}^4$ with $f_1(P) \leq 3f_0(P)$ and $M_4(P) \leq 4f_0(P)$.

If $n = 5$ and $d = 4$, it is easy to prove that the polytope with vertices $(0, 0, \frac{\sqrt{6}}{12}, \frac{\sqrt{10}}{4})$, $(0, 0, \frac{\sqrt{2}}{3}, 0)$, $\frac{1}{\sqrt{3}}(\cos\frac{\pi}{3}, \sin\frac{\pi}{3}, 0, 0)$, $\frac{1}{\sqrt{3}}(\cos\frac{2\pi}{3}, \sin\frac{2\pi}{3}, 0, 0)$ and $\frac{1}{\sqrt{3}}(1, 0, 0, 0)$ is a Lenz Configuration and that it is strictly configured. This is the only case with $d = 4$ where the vertices of a strictly configured polytope is a Lenz Configuration.

Theorem 2.3. *Let $X \subset \mathbb{R}^4$ be a 4-dimensional extremal Lenz Configuration with $|X| \geq 6$. Then $P = \operatorname{conv}(X)$ is not configured.*

Proof. We assume $X \subset \mathbb{R}^4$ is a 4-dimensional Lenz Configuration with $X \subset C_1 \cup C_2$, $C_i \subset U_i$, where $\mathbb{R}^4 = U_1 \oplus U_2$. It is clear that P is a 4-polytope with $V(P) = X$ and diameter 1. Let $X \cap C_1 = \{w_1, ..., w_a\}$, $X \cap C_2 = \{z_1, ..., z_b\}$ and note that for $i = 1, 2$, $G_i := U_i \cap P \in \mathcal{F}_2(P)$.

From [17], we have that $M_4(X) = M(4, n)$ with $|X \cap C_1| = \lceil \frac{n}{2} \rceil$ and $|X \cap C_2| = \lfloor \frac{n}{2} \rfloor$, say. Furthermore, $M(4, 6) = t_2(6) + 4$, $M(4, 7) = t_2(7) + 4$ and $M(4, n) \leq t_2(n) + \lceil \frac{n}{2} \rceil + 1$ for $n \geq 8$ where $t_2(n)$ is the number of pairs $\{w_j, z_k\}$ such that $\|w_j - z_k\| = 1$. Accordingly, there are $M(4, n) - t_2(n)$ diameters of X that have end points in either C_1 or C_2.

We suppose that P is configured via the anti-isomorphism induced by $v \to v^*$, $v \notin v^*$, and seek a contradiction. Then $a \geq 3$, $b \geq 3$, $v \notin v^*$ and $\mathcal{F}(P) = \{w_1^*, ..., w_a^*, z_1^*, ..., z_b^*\}$ yield that $v^* \cap C_1 \neq \emptyset \neq v^* \cap C_2$ for $v \in X \cap C_1$, and $G_1 = z_1^* \cap z_2^*$ and $G_2 = w_1^* \cap w_2^*$ say: Thus, $w_j^* \cap G_1 \in \mathcal{F}_1(w_j^*)$ and $z_k^* \cap G_2 \in \mathcal{F}_1(z_k^*)$ for $3 \leq j \leq a$ and $3 \leq k \leq b$.

It now follows that the number of principal diagonals of P in G_1 and G_2 is:

- two through each w_j and z_k with $j \geq 3$, $k \geq 3$ and
- at least one through each of w_1, w_2, z_1 and z_2;

that is, at least $\frac{1}{2}(2(a-2)+2(b-2)+4) = a+b-2 = n-2$ and $n-2 \leq \lceil \frac{n}{2} \rceil \neq 1$. Then $n = 6$, $w_3^* \cap G_1 = [w_1, w_2]$ and so, $w_3 \in w_1^* \cap w_2^*$, $[G_1, w_3] \subset w_1^* \cap w_2^*$, and $w_1^* = w_2^*$; a contradiction. □

We note that the arguments and the result in Theorem 2.3 extend to $d \geq 5$ for extremal Lenz Configuration X with sufficiently large $|X|$. This raises the issue of how to realize configured polytopes with a large number of vertices in higher dimensions.

3 Constructions of strictly configured polytopes

In this section, we present realizations of strictly configured polytopes that are $(d-2)$-fold d-pyramids or "stratified" d-polytopes. We note that configured polytopes play an important part in the study of, among others, graphs, hypergraphs, and bodies of constant width.

3.1 Prismoids

Let $m \geq d \geq 3$ and $\mathcal{Q} \subset H_d(0)$ be a $(d-1)$-polytope with $V(\mathcal{Q}) = \{w_1, w_2, \ldots, w_m\}$ and c_d as a relative interior point.

We consider translated homothetic copies (homotheties) $\mathcal{Q}_{jm}$ of $\mathcal{Q}$. For $k \geq 2$ and $1 \leq j \leq k$, let $\mathcal{Q}_{jm} = [y_{j1}, y_{j2}, \ldots, y_{jm}]$ with $y_{jr} = (\lambda_{jr} w_r, p_j)$, $p_k < p_{k-1} < \cdots < p_1$ and $\lambda_j > 0$. We let $R_{km} = [\mathcal{Q}_{1m}, \mathcal{Q}_{2m}, \ldots, \mathcal{Q}_{km}]$, and say that R_{km} is a *k-layered d-prismoid* if $|V(R_{km})| = km$ and for $r = 1, \ldots, m$, $[y_{(j-1)r}, y_{jr}]$ are the edges of R_{km} that intersect $\mathcal{Q}_{(j-1)m}$ and $\mathcal{Q}_{jm}$.

Then $[\mathcal{Q}_{im}, \mathcal{Q}_{jm}]$ is a d-prismoid for $1 \leq i \leq j \leq m$, $\{\mathcal{Q}_{1m}, \mathcal{Q}_{km}\} \subset \mathcal{F}(R_{km})$ and we let $P_{km} = [y_{00}, R_{km}]$ for some point $y_{00} = (0, \ldots, 0, q) \in \mathbb{R}^d$. We say that P_{km} is a *stratified* d-polytope if y_{00} is beyond either $\mathcal{Q}_{1m}$ or $\mathcal{Q}_{km}$, and beneath all other facets of R_{km} (cf. [6] p. 78), and hence, $|V(P_{km})| = km + 1$.

In what follows, we assume that $P_{km} = [y_{00}, R_{km}] \subset \mathbb{R}^d$ is stratified with R_{km} as above and y_{00} beyond exactly $\mathcal{Q}_{1m}$. It is clear that P_{km} is dependent upon the $(d-1)$-polytope $\mathcal{Q} = [w_1, w_2, \ldots, w_m] \subset H_d(0)$, and we examine properties of P_{km} that are inherited from $\mathcal{Q}$.

As a point of reference, $P_{2m} \subset \mathbb{R}^3$ is called an apexed 3-prism in [11].

3.1.1

Let $\mathcal{Q} = [w_1, w_2, \ldots, w_m] \subset H_d(0)$ be involutary self-dual via the anti-isomorphism on $\mathcal{L}(\mathcal{Q})$ induced by $w_r \to \tilde{w}_r \in \mathcal{F}(\mathcal{Q})$. Then $\mathcal{F}(\mathcal{Q}) = \{\tilde{w}_1, \tilde{w}_2, \ldots, \tilde{w}_m\}$ and we have that

- $\mathcal{Q}_{jm}$ is involutary self-dual via the anti-isomorphism of $\mathcal{L}(\mathcal{Q}_{jm})$ that sends $y_{jr} \to \tilde{y}_{jr}$, and $y_{js} \in \tilde{y}_{jr}$ if, and only if, $w_s \in \tilde{w}_r$,
- $\mathcal{F}(\mathcal{Q}_{jm}) = \{\tilde{y}_{j1}, \tilde{y}_{j2}, \ldots, \tilde{y}_{jm}\}$,
- $\mathcal{F}(R_{km}) = \{\mathcal{Q}_{1m}, \mathcal{Q}_{km}\} \cup \{\tilde{y}_{(j-1)r}, \tilde{y}_{jr}\} | 2 \leq j \leq k, 1 \leq r \leq m\}$ and
- $\mathcal{F}(P_{km}) = (\mathcal{F}(R_{km}) \setminus \{\mathcal{Q}_{1m}\}) \cup \{[y_{00}, \tilde{y}_{1r}] | 1 \leq r \leq m\}$.

Then (cf. [2], Theorem 2.1) P_{km} is involutary self-dual via the anti-isomorphism on $\mathcal{L}(P_{km})$ induced by the map $y_{jr} \to Y_{jr}$ with $Y_{00} = \mathcal{Q}_{km}$, $Y_{kr} = [y_{00}, \tilde{y}_{1r}]$ and $Y_{jr} = [\tilde{y}_{(k-j)r}, \tilde{y}_{(k-j+1)r}]$ for $j = 1, \ldots, k-1$ and $r = 1, \ldots, m$. □

3.1.2

With $\mathcal{Q}$ as in 3.1.1, let $V(\mathcal{Q}) \subset \mathbb{S}^{d-2}(c_d, t) \subset H_d(0)$ and $\|w_r - w_s\| = 1$ for each $w_r \in V(\mathcal{Q})$ and $w_s \in \tilde{w}_r$. We say that P_{km} is metrically embedded in $\mathbb{R}^d$ if $\|y - y'\| = 1$ for every $\{y, y'\} \subset V(P_{km})$ such that $[y, y']$ is a principal diagonal of P_{km}. Thus, a metrically embedded P_{km} of diameter 1 is configured.

From Theorem 4.1 in [2]; if $y_{00} = (0, 0, \ldots, 0, q)$, then there are $0 < \lambda_k \leq \lambda_1 < \cdots < \lambda_j \leq \lambda_{k-j} < \cdots < \lambda_{[\frac{k+1}{2}]} = 1$ that yield $0 = p_k < p_{k-1} < \cdots < p_1 < q$ so that for every $y_{jr} \in V(P_{km})$: if $y_{is} \in Y_{jr}$ then $\|y_{jr} - y_{is}\| = 1$. Specifically, we note that $q^2 = 1 - \lambda_k^2 t^2$, $p_1^2 = 1 - \|\lambda_k w_r - \lambda_1 w_s\|^2$ and $p_{k-1} = p_1 - \sqrt{\beta}$ with $\beta = 1 - \|\lambda_{k-1} w_r - \lambda_1 w_s\|^2$. □

Our present interest is to determine involutary self-dual $P_{km} \subset \mathbb{R}^d$ of, say, diameter 1 and then to characterize its diameters. To that end, we seek involutary self-dual $\mathcal{Q} \subset H_d(0)$ of diameter 1 and with vertices on a $(d-2)$-sphere.

3.2 Pyramids with polygonal bases

With the a_i's to be specified, let $d \geq 3$ and $\mathcal{Q} \subset L_2(-a_3, \ldots, -a_d)$ be a regular m-gon with cyclically labeled vertices $w_1, w_2, \ldots, w_m$, the circumradius t, the diameter 1 and $m = 2u + 1 \geq 3$. Then it is well known that $1 = \|w_r - w_{r+u}\| = \|w_r - w_{r+u+1}\|$ for each w_r, and that $\mathcal{Q}$ has $2m$ diameters.

As a simplification, we write $w_r = (x_1, x_2, -a_3, \ldots, -a_d)$ as $w_r = (x_1, x_2)$ in relation to the plane $L_2(-a_3, \ldots, -a_d)$.

3.2.1

With $\theta = \frac{2\pi}{m}$ and $w_r = t(\cos(r\theta), \sin(r\theta))$ for $r = 1, \ldots, m$, we note that $w_m = (t, 0), w_{m+u} = w_u$ and $1 = \|w_m - w_u\|^2 = 2t^2(1 - \cos(u\theta)) = 2t^2(1 + \cos(\frac{\pi}{m}))$ from $m = 2u + 1$.

3.2.2

With $m = 2u + 1 \geq 5$ and $\lambda > 0$, we claim that $\|\lambda w_r - w_j\| < \|\lambda w_r - w_{r+u}\|$ for $w_j \in V(\mathcal{Q}) \setminus \{w_r, w_{r+u}, w_{r+u+1}\}$.

With coordinates as in 3.2.1, we may assume that $w_r = w_m$ and that w_j is in the upper half-plane. Then $0 < j\theta < u\theta < \pi$ and $\cos(u\theta) < \cos(j\theta)$ and $\|\lambda w_m - w_u\|^2 - \|\lambda w_m - w_j\|^2 = 2\lambda t^2(\cos(j\theta) - \cos(u\theta))$. □

3.2.3

For $\lambda > \mu > 0$ and $w_s \in \{w_{r+u}, w_{r+u+1}\}$, we have that $[\lambda w_r, \mu w_r, \mu w_s, \lambda w_s]$ is an isosceles trapezoid of side lengths λ, μ and $(\lambda - \mu)t$ and $\|\lambda w_r - \mu w_s\|^2 = \lambda\mu + (\lambda - \mu)^2 t^2 = \|\lambda w_s - \mu w_r\|^2$. □

3.2.4

From $1 = \|w_m - w_u\|^2 = 2t^2(1 + \cos(\frac{\pi}{m}))$ and $m \geq 3$, we obtain that $\frac{1}{4} < t^2 \leq \frac{1}{3}$ and $\frac{1}{3} < \frac{1}{4(1-t^2)} \leq \frac{3}{8}$. We let $t_2 = t$, $t_d^2 = \frac{1}{4(1-t_{d-1}^2)}$ for $d \geq 3$ and note that $\frac{1}{3} < t_3^2 \leq \frac{3}{8} < t_4^2 \leq \frac{2}{5} < t_5^2 \leq \frac{5}{12} < t_6^3 \leq \frac{3}{7} < t_7^2 \leq \frac{7}{16} < t_d^2 < \frac{1}{2}$ with $d \geq 8$. □

3.2.5

With $d \geq 4$ and $\mathcal{Q} \subset L_2(-a_3, \ldots, -a_d) \subset L_3(-a_4, \ldots, -a_d)$ as above, we write $w_r = (t_2 \cos(r\theta), t_2 \sin(r\theta), -a_3)$ in relation to $L_3(-a_4, \ldots, -a_d)$. We consider the 2-sphere $\mathbb{S}^2 := \mathbb{S}^2((0,0,0), t_3) \subset L_3(-a_4, \ldots, -a_d)$ with $t_3^2 = \frac{1}{4(1-t_2^2)}$, and let $a_3 = \sqrt{t_3^2 - t_2^2}$. Then $V(\mathcal{Q}) \subset \mathbb{S}^2$ and with $w_{m+1} = (0, 0, t_3)$, we claim that $\|w_{m+1} - w_r\| = 1$ for $r = 1, 2, \ldots, m$.

As $\mathcal{Q}$ is symmetric about the x_3-axis, we verify the claim with $w_r = w_m = (t_2, 0, -a_3)$.

From $t_3^2 = \|w_m\|^2 = t_2^2 + a_3^2$ and $t_2^2 = \frac{4t_3^2 - 1}{4t_3^2}$, it follows that $\|w_{m+1} - w_m\|^2 =$

$$t_2^2 + (t_3 + a_3)^2 = 2t_3^2 + 2t_3\sqrt{t_3^2 - t_2^2} = 2t_3^2 + 2t_3\left(\frac{(1-2t_3^2)^2}{4t_3^2}\right)^{\frac{1}{2}} = 1.$$

Theorem 3.1. *Let $d \geq 3$ and $\mathcal{Q}^2 = [w_1, \ldots, w_m] \subset L_2(-a_3, \ldots, -a_d)$ be a regular m-gon of diameter 1 and circumradius t_2; $m = 2u + 1 \geq 3$. Then for $e = 3, \ldots, d$, $t_e^2 = \frac{1}{4\left(1-t_{e-1}^2\right)}$, $a_e^2 = t_e^2 - t_{e-1}^2$ and $c_e = (0, \ldots, -a_{e+1}, \ldots, -a_d)$ if $e \neq d$, there is an involutary self-dual $(e-2)$-fold e-pyramid $\mathcal{Q}^e = [w_1, \ldots, w_m, \ldots, w_{m+e-2}]$ of diameter 1 and basis $\mathcal{Q}^2$ such that*

(i) $\mathcal{Q}^e \subset L_e(-a_{e+1}, \ldots, -a_d)$ *if* $e \neq d$,

(ii) $V(\mathcal{Q}^e) \subset \mathbb{S}^{e-1}(c_e, t_e)$ *and*

(iii) $\mathcal{Q}^e$ *is strictly configured.*

Proof. With reference to Subsections 3.2.1, 3.2.2, 3.2.3, 3.2.4 and 3.2.5, we let:

- $w_i = (t_2 \cos(i\theta), t_2 \sin(i\theta), -a_3, \ldots, -a_d)$ for $i = 1, \ldots, m$
- $w_{m+i} = (0, \ldots, 0, t_{i+2}, -a_{i+3}, \ldots, -a_d)$ for $i = 1, \ldots, d-3$ and
- $w_{m+d-2} = (0, \ldots, 0, t_d)$.

We observe first that for $2 \leq i < j \leq d$, $t_i^2 + a_{i+1}^2 = t_{i+1}^2$ and so, $t_i^2 + a_{a+1}^2 + \cdots + a_j^2 = t_j^2$. From this it follows that $\|w_i - c_e\|^2 = t_2^2 + a_3^2 + \cdots + a_e^2 = t_e^2$ for $w_i \in V(\mathcal{Q}^2)$, $3 \leq e \leq d$ $\|w_{m+i} - c_e\|^2 = t_{i+2}^2 + a_{i+3}^2 + \cdots + a_e^2 = t_e^2$ for $i + 2 \leq e \leq d - 1$ and $\|w_j - c_d\|^2 = \|w_j\|^2 = t_d^2$ for $w_j \in V(\mathcal{Q}^d)$.

Next, with $w_r = (t_2 \cos(r\theta), t_2 \sin(r\theta), -a_3, \ldots, -a_d)$ and $w'_r = (t_2 \cos(r+u)\theta$, $t_2 \sin(r+u)\theta, -a_3, \cdots - a_d)$, we note that $\mathcal{Q}^2$ is involutary self-dual via the anti-isomorphism of $\mathcal{L}(\mathcal{Q}^2)$ induced by $w_r \to \bar{w}_r = [w'_r, w'_{r+1}]$. Then for $e = 3, \ldots, d$,

$\mathcal{F}(\mathcal{Q}^e) = \{[\bar{w}_r, w_{m+1}, \ldots w_{m+e-2}] | r = 1, \ldots m\} \cup \{[V(\mathcal{Q}^e) \setminus \{w_r\}] | r = m+1, \ldots, m+e-2\}$

and $\mathcal{Q}^e$ is involutary self-dual via the anti-isomorphism on $\mathcal{L}(\mathcal{Q}^e)$ induced by $w_r \to \tilde{w}_r$ where

$$\tilde{w}_r = \begin{cases} [\bar{w}_r, w_{m+1}, \ldots w_{m+e-2}], & r = 1, \ldots, m; \\ [V(\mathcal{Q}^e) \setminus \{w_r\}], & r = m+1, \ldots, m+e-2. \end{cases}$$

Finally, we observe that for $1 \leq j \leq m + i$, $\|w_{m+i} - w_j\|^2 = t_{i+1}^2 + (t_{i+2} + a_{i+2})^2$. Then, as in 3.2.5, $t_{i+1}^2 = \frac{4t_{i+2}^2 - 1}{4t^2}$ yields that $\|w_{m+i} - w_j\| = 1$. From this and $t_2^2 = \frac{1}{2(1+\cos(\frac{\pi}{m}))}$, we obtain that $\|w_r - w_s\| = 1$ for $w_s \in \tilde{w}_r$; furthermore, if $\{w_r, w_z\} \subset V(\mathcal{Q}^2)$ and $w_z \notin \tilde{w}_r$ then $\|w_r - w_z\| < 1$. □

We note that $M_e(\mathcal{Q}^e) = 2M_2(\mathcal{Q}^2) + \sum_{m+1}^{m+e-3} j = (e-1)m + \binom{e-2}{2}$ and that $\mathcal{Q}^3$ is extremal.

Theorem 3.2. *Let $d \geq 3$, $m = 2u + 1$, $n = m + d - 3$ and $k \in \{2, 3\}$. Then there is an involutary self-dual stratified $P_{kn} = [y_{00}, R_{kn}] \subset \mathbb{R}^d$ of diameter 1 that is strictly configured.*

Proof. With reference to Subsection 3.1 and Theorem 3.1 with $e = d - 1$ and $a_d = 0$, we consider P_{kn} with the property that:

- y_{00} is beyond exactly $\mathcal{Q}_{1n}$.

- $\mathcal{Q} = [w_1, ..., w_n] \subset L_{d-1}(-a_d) = H_d(0)$,
- $\mathcal{Q}^{d-1}$ is a involutary self-dual $(d-3)$-fold $(d-1)$-pyramid with diameter 1 and basis $\mathcal{Q}^2$, and
- $\mathcal{Q}^2 = [w_1, ..., w_m] \subset L_2(-a_3, ..., -a_d)$ is a regular m-gon of diameter 1.

Then $c_{d-1} = (0, \dots, 0, -a_d) = c_d$ and with $t_2, \dots t_{d-1}$ as in 3.2.4, we simplify notation and let $t = t_{d-1}$.

We now apply 3.1.2 with $y_{00} = (0, \dots, 0, q)$ and $p_k < p_{k-1} < \cdots < p_1 < q$.

Case 1: $k = 2$ and hence, $\lambda_1 = 1$ and $p_2 = 0$.

With $0 < \lambda_2 < 1$: P_{2n} is stratified, $Y_{00} = Q_{2n}$, $Y_{1r} = [\tilde{y}_{1r}, \tilde{y}_{2r}]$ and $Y_{2r} = [y_{00}, \tilde{y}_{1r}]$. With $q^2 = 1 - \lambda_2 t^2$ and $p_1^2 = 1 - \|\lambda_2 w_r - w_s\|^2 = 1 - (\lambda_2 + (1-\lambda_2)^2 t^2)$ (cf. 3.2.3), we have that $\|y_{jr} - y_{is}\| = 1$ for $y_{is} \in Y_{jr}$.

With $\lambda_2 = \frac{1}{2}$; we have $q^2 = \frac{4-t^2}{4}$, $p_1^2 = \frac{2-t^2}{4}$ and claim that $\|y_{jr} - y_{iz}\| < 1$ for $y_{iz} \notin Y_{jr}$. From $\frac{1}{3} < t^2 < \frac{1}{2}$, we obtain that

$$\begin{aligned}
\|y_{00} - y_{1r}\|^2 &= \|(0,q) - (w_r, p_1)\|^2 = \|w_r\|^2 + (q - p_1)^2 \\
&= t^2 + q^2 + p_1^2 - 2qp_1 \\
&= \frac{1}{4}(6 - 2t^2 - 2\sqrt{4-t^2}\sqrt{2-t^2}) \\
&\leq \frac{1}{4}\left(6 + 2\left(\frac{1}{2}\right) - 2\sqrt{4 - \frac{1}{3}}\sqrt{2 - \frac{1}{3}}\right) < 1
\end{aligned} \tag{3.1}$$

Let $y_{iz} \neq y_{00} \neq y_{jr}$ and $y_{iz} \notin Y_{jr}$. Then $y_{iz} = (\lambda_i w_z, p_i)$, $y_{jr} = (\lambda_j w_r, p_j)$ and $w_z \notin \tilde{w}_r$ (cf. 3.1.1). Since Q_{1n} and Q_{2n} are homothets of Q, we may assume by Theorem 3.1(iii) that $j = 1$ and $i = 2$, say. Since $w_z \notin \tilde{w}_r$, it follows as in the proof of Theorem 3.1 that $w_z = w_r$ or $\{w_z, w_r\} \subset V(Q^2)$. If $w_z = w_r$, then $\|y_{1r} - y_{2r}\|^2 = \frac{t^2}{4} + p_1^2 = \frac{1}{2}$. If $\{w_z, w_r\} \subset V(Q^2)$, then it follows from 3.2.2 that $\|w_r - \frac{1}{2}w_z\| < \|w_r - \frac{1}{2}w_s\|$ with $w_s \in \tilde{w}_r \cap V(Q^2)$. Hence, $\|y_{1r} - y_{2z}\| < \|y_{1r} - y_{2s}\| = 1$.

Case 2: $k = 3$ and hence, $\lambda_2 = 1$ and $p_3 = 0$.

Let $Y_{00} = Q_{3n}$, $Y_{1r} = [\tilde{y}_{2r}, \tilde{y}_{3r}]$, $Y_{2r} = [\tilde{y}_{1r}, \tilde{y}_{2r}]$ and $Y_{3r} = [y_{00}, \tilde{y}_{1r}]$. With $\lambda = \lambda_1 = \lambda_3 = \frac{1}{2}$ and $q^2 = 1 - \lambda t^2 = \frac{4-t}{4}$, $p_1^2 = 1 - \|\lambda w_r - \lambda w_s\|^2 = 1 - \lambda^2 = \frac{3}{4}$ (cf. 3.1.2 and 3.2.3), $\beta = 1 - \|\lambda_2 w_r - \lambda_1 w_s\|^2 = 1 - \|w_r - \lambda w_s\|^2 = 1 - \lambda + (1-\lambda)^2 t^2 = \frac{2-t^2}{4}$ and $p_2 = p_1 - \sqrt{\beta}$, we obtain that $\|y_{jr} - y_{is}\| = 1$ for $y_{is} \in Y_{jr}$.

Let $y_{iz} \notin Y_{jr}$. We claim that $\|y_{jr} - y_{iz}\| < 1$ and then it follows that each Y_{jr} is a facet of P_{3n}; that is, R_{3n} is a 3-layered prismoid and P_{3n} is stratified.

We observe that if $a < t^2 \leq b$ then

$$\begin{aligned}
\|y_{00} - y_{2r}\|^2 &= \|(0,q) - (w_r, p_2)\|^2 = \|w_r\|^2 + (q - p_2)^2 \\
&= t^2 + q^2 + p_1^2 + \beta + 2q\sqrt{\beta} - 2p_1\left(q + \sqrt{\beta}\right) \\
&= \frac{1}{4}\left(9 + 2t^2 + 2\sqrt{(4-t^2)(2-t^2)} - 2\sqrt{3}(\sqrt{4-t^2} + \sqrt{2-t^2}\right) \\
&< \frac{1}{4}\left(9 + 2b + 2\sqrt{(4-a)(2-a)} - 2\sqrt{3}(\sqrt{4-b} + \sqrt{2-b}\right)
\end{aligned} \tag{3.2}$$

and $\|y_{00} - y_{2r}\| < 1$ for $(a, b) \in \{(\frac{1}{3}, \frac{3}{8}), (\frac{3}{8}, \frac{2}{5}), (\frac{2}{5}, \frac{5}{12}), (\frac{5}{12}, \frac{3}{7}), (\frac{3}{7}, \frac{7}{16}), (\frac{7}{16}, \frac{1}{2})\}$, that is, for each $d \geq 3$ (cf. 3.2.4).

It is clear that $\|y_{00} - y_{1r}\| < \|y_{00} - y_{2r}\|$, and hence, we may assume that $y_{iz} = (\lambda_i w_z, p_i)$, $y_{jr} = (\lambda_j w_r, p_j)$ and $w_z \notin \tilde{w}_r$. Then $\|w_r - w_z\| < \|w_r - w_s\|$ for $w_s \notin \tilde{w}_r$, and $\|y_{1r} - y_{3z}\| < \|y_{1r} - y_{3s}\| = 1$ for $y_{3s} \in \tilde{y}_{1r} \subset Y_{1r}$.

From $t^2 < \frac{1}{2}$, we obtain that $\beta > \frac{3}{16} = \frac{p_1^2}{4}$, $p_2 = p_1 - \sqrt{\beta} < \frac{p_1}{2}$ and $p_2 < p_1 - p_2$. Thus, $\|y_{3r} - y_{2z}\| < \|y_{1r} - y_{2z}\|$ and we argue as above that $\|y_{1r} - y_{2z}\| < 1$.

In summary; $\|y_{jr} - y'\| < 1$ for $\{y_{jr}, y'\} \subset \{y_{00}\} \cup \{y_{jr} | j = 1, \ldots, k$ and $r = 1, \ldots, n\}$, and with equality if and only if $y' \in Y_{jr}$. Thus

$$\mathcal{F}(P_{kn}) = \{Y_{00}\} \cup \{Y_{jr} | j = 1, \ldots, k, r = 1, \ldots, n\},$$
$$V(P_{kn}) = \{y_{00}\} \cup \{y_{jr} | j = 1, \ldots, k, r = 1, \ldots, n\}$$

and P_{kn} is involutary self-dual under the anti-isomorphism on $\mathcal{L}(P_{kn})$ induced by $y_{jr} \to Y_{jr}$. □

Theorem 3.3. *Let $P_{km} \subset \mathbb{R}^3$ be an involutary self-dual stratified 3-polytope that is configured with diameter 1; $k \geq 2$ and $m = 2u + 1 \geq 3$. Then there is an involutary self-dual stratified $P_{(k+1)m} \subset \mathbb{R}^3$ that is configured with diameter 1.*

Proof. We let $l = k + 1$ and denote P_{km} as in 3.1.1 and 3.1.2 with $d = 3$. Specifically,

- $Q = [w_1, \ldots, w_m] \subset H_3(0)$ is a regular m-gon of diameter 1 and circumcentre $c_3 = (0, 0, 0)$ as in 3.2.1,
- $Q_{jm} = [y_{j1}, \ldots, y_{jm}]$ with $y_{jr} = (\lambda_j w_r, p_j)$ and $0 < \lambda_k \leq \lambda_1 < \cdots < \lambda_j \leq \lambda_{k-j} < \cdots < \lambda_{[\frac{l}{2}]} = 1$, $0 < p_k < p_{k-1} < \cdots < p_1 < q \leq 1$ and $y_{00} = (0, 0, q)$,
- the anti-isomorphism on $\mathcal{L}(P_{km})$ is induced by $y_{jr} \to Y_{jr}$ with $Y_{00} = Q_{km}$, $Y_{km} = [y_{00}, \tilde{y}_{1r}]$, $Y_{jr} = [\tilde{y}_{(k-j)r}, \tilde{y}_{(l-j)r}]$, $1 \leq j \leq k - 1$, and $\tilde{y}_{jr} = [y_{j(r+u)}, y_{j(r+u+1)}]$, and
- $\|y_{jr} - y_{is}\| = 1$ if, and only if, $y_{is} \in Y_{jr}$.

Let $\mathbb{S}(y) := \mathbb{S}^2(y, 1)$ for $y \in \mathbb{R}^3$, and consider the homothets $Q_{0m} = [y_{01}, \ldots, y_{om}]$ of Q with $y_{0r} = (\lambda_0 w_r, p_0)$, $0 < \lambda_0 < \lambda_1$ and $p_1 < p_0 < q$. From $[y_{k(r+u)}, y_{k(r+u+1)}] = \tilde{y}_{kr} = Y_{00} \cap Y_{1r}$, it follows that $\|y_{00} - y_{ks}\| = 1 = \|y_{1r} - y_{rs}\|$ for $s \in \{r+u, r+u+1\}$, and so,

$$\{y_{00}, y_{1r}\} \subset C_{kr} := \mathbb{S}(y_{k(r+u)}) \cap \mathbb{S}(y_{k(r+u+1)}),$$

a circle with centre $\frac{1}{2}(y_{k(r+u)} + y_{k(r+u+1)})$. It is now clear that

(i) for each $p_1 < p_0 < q$, there is $0 < \lambda_0 < \lambda_1$ such that $y_{0r} \in C_{kr}$.

In fact, $y_{0r} \in \alpha_{kr}$, the shorter arc of C_{kr} with end points y_{00} and y_{1r}. We note also that $V(Q_{0m}) \cap V(P_{km}) = \emptyset$ for each such p_0. Let $V = V(P_{km})$, $B(y) = [\mathbb{S}(y)]$ and $B(V) = \cap_{y \in V} B(y)$. Since $\operatorname{diam}(P_{km}) = 1$, it follows that

(ii) $\alpha_{kr} \subset \operatorname{bd}(B(V))$ for $r = 1, \ldots m$.

Since P_{km} is involutary self-dual with no fixed points, it follows from Theorem 3.2 of [13] that $B(V)$ is polytopal and the face polyhedral structure of $B(V)$ is a lattice

isomorphic to $\mathcal{L}(P_{km})$. Accordingly, $B(V)$ is similarly self-dual and from Theorem 4.1 of [13], any surface $\Phi \subset \mathbb{R}^3$ obtained from $\mathrm{bd}(B(V))$ (by performing their surgery on one edge-arc of each pair of dual edge-arcs of $\mathrm{bd}(B(V))$) is the boundary of a body of constant width. In this case, $V \subset \Phi$ and $\mathrm{diam}(V) = 1$ yield Φ is of constant width 1.
We note that dual edge-arcs of $\mathrm{bd}(B(V))$ correspond to dual edges of $\mathcal{L}(P_{km})$. Thus, the duality $[y_{00}, y_{1r}] \longleftrightarrow Y_{00} \cap Y_{1r} = \tilde{y}_{kr}$
yields that α_{kr} is dual to the shorter edge-arc in $\mathbb{S}(y_{00}) \cap \mathbb{S}(y_{1r})$ with end point $y_{k(r+u)}$ and $y_{k(r+u+1)}$. We consider those Φ that contain each of $\alpha_{k1}, \alpha_{k2}, \dots, \alpha_{km}$. Then the symmetry of P_{km} about the x_3–axis and $i)$ yield that

(iii) $V' = V \cup V(Q_{0m}) \subset \Phi$ and $\mathrm{diam}(V') = 1$,

(iv) $\mathbb{S}(y_{00}) \cap V' = V(Q_{km})$ and the spherical region $\mathbb{S}(y_{00}) \cap \Phi$ is not empty and bounded in $H_3(0)$ by the circumcircle of Q_{km}, and

(v) $y'_{00} = (0, 0, q-1) \in \mathbb{S}(y_{00}) \cap \Phi$.

From $\mathrm{diam}(V) = 1$, $|V| = km + 1$, $M(3, km + 1) = 2km$ and Theorem 2.1, we have that $M_3(V) = 2km + 1$. From $\mathrm{diam}(V') = 1$, $|V'| = lm + 1$ and $i)$, we have that $M_3(V') \geq M_3(V) + 2m = 2lm$. Thus, $M_3(V') = 2lm$ and

(vi) $\|y_{0r} - y\| < 1$ for $y_{0r} \in V(Q_{0m})$ and $y \in V \setminus \{y_{k(r+u)}, y_{k(r+u+1)}\}$.

Let $V'' = V' \cup \{y'_{00}\}$. Then $\mathrm{diam}(V'') = 1$, $|V''| = lm + 2$, $\|y_{00} - y'_{00}\| = 1$ and $2|V''| - 2 = 2lm + 2 \geq M_3(V'') \geq 2lm + 1$. From the rotational symmetry of V'' and $\mathbb{S}(y'_{00})$ about the x_3–axis, it follows that

(vii) $\|y'_{00} - y\| < 1$ for $y \in V' \setminus \{y_0\}$, and

(viii) $\|y_\epsilon - y\| < 1$ for $y \in V' \setminus \{y_0\}$ for sufficiently small $\epsilon > 0$ and $y_\epsilon = (0, 0, q-1-\epsilon)$.

Let $p_0 = q - \epsilon$ and μ be the radius of the circle $H_3(p_0) \cap \mathbb{S}(y'_{00})$. Then $\{(0, 0, p_0)\} = H_3(p_0) \cap \mathbb{S}(y_\epsilon) \subset Q_{0m} \subset [H_3(p_0) \cap \mathbb{S}(y'_{00})]$ and with λ_0 chosen so that $0 < \lambda_0 < \lambda_1$ and $y_{0r} \in \alpha_{kr}$, we have that $0 < \lambda_0 t \leq \mu$. Accordingly, there is a point $z_{00} \in [y'_{00}, y_\epsilon]$ such that $\lambda_0 t$ is the radius of $H_3(p_0) \cap \mathbb{S}(z_{00})$; that is,

(ix) $\|z_{00} - y_{0r}\| = 1$ for $r = 1, 2, \dots, m$.

Finally, let $z_{jr} = y_{(l-j)r}$,$\tilde{z}_{jr} = \tilde{y}_{(l-j)r}$ and $Q'_{jm} = Q_{(l-j)m}$ for $j = 1, 2, \dots, l$ and $r = 1, 2, \dots, m$. In addition, let $Z_{00} = Q'_{lm} = Q_{0m}$, $Z_{lr} = [z_{00}, \tilde{z}_{1r}] = [z_{00}, \tilde{y}_{kr}]$ and $Z_{jr} = [\tilde{z}_{(l-j)r}, \tilde{z}_{(l-j+1)r}] = [\tilde{y}_{jr}, \tilde{y}_{(j-1)r}]$. From the preceding, we have that $P_{lm} = [z_{00}, Q'_{1m}, \dots, Q'_{lm}]$ is involutary self-dual via $z_{jr} \to Z_{jr}$, stratified and configured with diameter 1. □

Finally, we show that if a set of n points are the vertices of a configured 4-polytope P such as in Theorem 3.2 then $M_4(P) \leq 4n$.

Theorem 3.4. *Let $P_{km} = [y_{00}, R_{km}] \subset \mathbb{R}^4$ be a configured stratified 4-polytope, with $n = km + 1$ vertices. Then number of principal diagonals of P_{km} is at most $4n$.*

Proof. By Theorem 2.2, it is sufficient to prove that $f_1(P) \leq 3n$ for every configured stratified 4-polytope. By construction, $R_{km} = [\mathcal{Q}_{1m}, \mathcal{Q}_{2m}, \dots \mathcal{Q}_{km}]$ where each copy

$\mathcal{Q}_{im}$ is self-dual and contains m vertices, and thus, $f_1(\mathcal{Q}_{im}) = 2m-2$ by Euler's Theorem and self-duality.

Finally, there are m edges through y_{00} and $m(k-1)$ edges connecting the k homothets $\mathcal{Q}_{im}$, and so, $f_1(P_{km}) = k(2m-2)+m(k-1)+m = 3km-2k \leq 3km+3 = 3n$. $\square$

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Top-heavy phenomena for transformations*

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Abstract

Let S be a transformation semigroup acting on a set Ω. The action of S on Ω can be naturally extended to be an action on all subsets of Ω. We say that S is ℓ-homogeneous provided it can send A to B for any two (not necessarily distinct) ℓ-subsets A and B of Ω. On the condition that $k \leq \ell < k + \ell \leq |\Omega|$, we show that every ℓ-homogeneous transformation semigroup acting on Ω must be k-homogeneous. We report other variants of this result for Boolean semirings and affine/projective geometries. In general, any semigroup action on a poset gives rise to an automaton and we associate some sequences of integers with the phase space of this automaton. When this poset is a geometric lattice, we propose to investigate various possible regularity properties of these sequences, especially the so-called top-heavy property. In the course of this study, we are led to a conjecture about the injectivity of the incidence operator of a geometric lattice, generalizing a conjecture of Kung.

Keywords: Incidence operator, kernel space, rank, strong shape, valuated poset, weak shape.

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1 Introduction

1.1 Transformation and phase space

Let Γ be a *digraph*, namely a pair consisting of its vertex set $\mathrm{V}(\Gamma)$ and arc set $\mathrm{E}(\Gamma) \subseteq \mathrm{V}(\Gamma) \times \mathrm{V}(\Gamma)$. We call Γ *symmetric* if $(u, v) \in \mathrm{E}(\Gamma)$ holds if and only if so does $(v, u) \in \mathrm{E}(\Gamma)$. For any $A \subseteq \mathrm{V}(\Gamma)$, we adopt the notation $\Gamma[A]$ for the subdigraph of Γ induced by

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A which has vertex set A and arc set $\mathrm{E}(\Gamma) \cap (A \times A)$. The number of weakly connected components and the number of strongly connected components of Γ will be dubbed $\mathrm{wcc}(\Gamma)$ and $\mathrm{scc}(\Gamma)$, respectively.

For a set Ω, all maps from Ω to itself form the set Ω^Ω. For each $g \in \Omega^\Omega$ and $\alpha \in \Omega$, we write αg for the image of α under the map g. The composition of maps provides an associative product on the set Ω^Ω and thus turns it into a monoid, namely a semigroup with a multiplicative unit. We call this monoid the *full transformation monoid* on Ω and denote it by $\mathrm{T}(\Omega)$. A subset of $\mathrm{T}(\Omega)$ which is closed under map composition, whether or not it contains the identity map on Ω, is called a *transformation semigroup* acting on Ω. Let S be a transformation semigroup on Ω. We say that S is *transitive on a set* $A \subseteq \Omega$ if for every $\alpha, \beta \in A$ we can find $g \in S$ such that $\alpha g = \beta$; we call S *transitive* if S is transitive on Ω. If the transformation semigroup S is generated by a set $G \subseteq \Omega^\Omega$, namely S consists of products of elements of G of positive length, we call (S, G) a *deterministic automaton* on Ω [67, Section 1]. The *phase space* of an automaton (S, G) on Ω, denoted by $\Gamma(S, G)$, is the digraph with vertex set Ω and arc set $\{(\alpha, \alpha g) : \alpha \in \Omega, g \in G\}$. When Ω has at least two elements, the claim that S is transitive is equivalent to the claim that $\Gamma(S, G)$ is strongly connected for any generator set G of S. We write $\Gamma(S, S)$ simply as $\Gamma(S)$ and note that each strongly/weakly connected component of $\Gamma(S)$ coincides with a strongly/weakly connected component of $\Gamma(S, G)$ for any generator set G of S. For all work in this paper, we can simply focus on $\Gamma(S)$ instead of considering $\Gamma(S, G)$ for any specific generator set G. We emphasize $\Gamma(S, G)$ from the phase space viewpoint here to highlight the connection between semigroup theory and automata theory, and to indicate the role played by the choice of G in some problems related to various distance functions on the phase space, say the Černý conjecture. For any set Ω, a subset of $\mathrm{T}(\Omega)$ forms a *permutation group* on Ω whenever it is a transformation semigroup and each element has an inverse in it, namely it is a set of bijective transformations of Ω and is closed under compositions and taking inverses. Permutation groups correspond to reversible deterministic automata.

Let Ω be a set. We follow the common practice to use 2^Ω for the power set of Ω. For each $g \in \mathrm{T}(\Omega)$, let $\overline{g}$ be the element in $\mathrm{T}(2^\Omega)$ that sends each $A \in 2^\Omega$ to $A\overline{g} \doteq \{ag : a \in A\}$. More generally, for each $G \subseteq \mathrm{T}(\Omega)$, $\overline{G}$ refers to the set $\{\overline{g} : g \in G\}$. For any transformation semigroup S on Ω and any generator set G of S, $\overline{S}$, as a semigroup derived from S, is known to be the *powerset transformation semigroup of* S acting on 2^Ω and $(\overline{S}, \overline{G})$ is known to be the *powerset automaton* of (S, G). It may be interesting to iterate the powerset automaton construction and examine the evolution of the phase spaces of the resulting automata.

When discussing transformation semigroups, we may often be more interested in those which preserve some structures, say simplicial maps for simplicial complexes, continuous maps for topological spaces, ordering preserving maps for posets, or adjacency-preserving maps in matrix geometry [52, 66]. Unlike the work on group actions on posets [3, 59] and matroids [19], very little has been done on semigroup actions on these structures [62]. Moving from group actions to semigroup actions is just to consider general deterministic automata instead of reversible ones.

1.2 Valuated poset and its shape

For any two sets Ω and Ψ, if they are different or if we do not emphasize that they may be equal, the image of $\omega \in \Omega$ under a map $g \in \Psi^\Omega$ is denoted $g(\omega)$; note that we often write it as ωg when $\Omega = \Psi$.

A *poset* P consists of a set Ω and a binary relation $<_P$ on it which is transitive and acyclic, namely we require that $\alpha <_P \alpha$ never happens, and that $\alpha <_P \beta$ and $\beta <_P \gamma$ implies $\alpha <_P \gamma$ for all $\alpha, \beta, \gamma \in \Omega$. We often just write P for its ground set Ω and we say the poset P is *finite* if $|P|$ is finite. For each $\alpha \in P$, the *principal ideal* generated by α is the set $\{\beta : \beta <_P \alpha\} \cup \{\alpha\} \subseteq P$, which we denote by $P_\downarrow(\alpha)$; the *principal filter* generated by α is the set $\{\beta : \alpha <_P \beta\} \cup \{\alpha\} \subseteq P$, which we denote by $P_\uparrow(\alpha)$. An *ideal* (*filter*) is a union of principal ideals (filters). A map g from a poset P to a poset Q is *order-preserving* if $g(\beta) \in Q_\downarrow(g(\alpha))$ holds whenever $\beta \in P_\downarrow(\alpha)$. We use $\mathrm{End}(P)$ to denote the set of all order-preserving maps from P to itself.

Let $\mathbb{Z}_{\geq 0}$ be the set of nonnegative integers which carries a natural poset structure such that $a < b$ in $\mathbb{Z}_{\geq 0}$ if and only if $b - a$ is a positive integer. A *valuation* on a poset P is an order-preserving map r_P from P to the poset $\mathbb{Z}_{\geq 0}$; we call $\mathrm{r}_P(x)$ the rank of x in the valuated poset. When we say P is a *valuated poset*, we are considering the poset P together with a valuation r_P, though the valuation may be only implicitly indicated. The *rank* of a valuated poset P, denoted by $\mathrm{r}(P)$, is the maximum value of $\mathrm{r}_P(\alpha)$ for $\alpha \in P$ if it exists and is ∞ otherwise. For a poset P, the symbols like $<_P$ and r_P will often be abbreviated to $<$ and r when no confusion can arise. Let P be a valuated poset. For any $k \in \mathbb{Z}_{\geq 0}$, we write P_k for the set $\{\alpha \in P : \mathrm{r}(\alpha) = k\}$. We call the sequence $|P_0|, |P_1|, \ldots$ the *shape* of the valuated poset and refer to it by $\mathsf{S}(P)$. If $\mathrm{r}(P) < \infty$, $\mathsf{S}(P)$ is a sequence of $\mathrm{r}(P) + 1$ nonnegative integers.

Let P be a valuated poset and let S be a subsemigroup of $\mathrm{End}(P)$. The *weak shape of P under the action of S* is the sequence

$$\mathrm{wcc}(\Gamma(S)[P_0]), \mathrm{wcc}(\Gamma(S)[P_1]), \ldots$$

which we denote by $\mathsf{WS}(S, P)$; while the *strong shape of P under the action of S* is the sequence

$$\mathrm{scc}(\Gamma(S)[P_0]), \mathrm{scc}(\Gamma(S)[P_1]), \ldots$$

which we denote by $\mathsf{SS}(S, P)$. Note that

$$\mathsf{S}(P) = \mathsf{WS}(S, P) = \mathsf{SS}(S, P)$$

when the semigroup S consists of the identity transformation from $\mathrm{End}(P)$.

The main purpose of this note is to propose a study of the possible regularity in the strong/weak shape of a semigroup acting on a valuated poset.

1.3 Geometric lattice and top-heavy property

A matroid M consists of a ground set $\mathscr{E}_M$ and a rank function r_M from $2^{\mathscr{E}_M}$ to the set of nonnegative integers plus infinity such that the rank axioms are satisfied [13, Section 1.5]. The flats of a matroid M, ordered by inclusion, form a very pretty structure, called the *matroid lattice* of M and denoted by $\mathrm{F}(M)$. For each nonnegative integer t, let $\mathrm{F}_t(M)$ be the set of all rank-t flats of the matroid M. A *geometric lattice* is an atomic and semimodular lattice which does not have any infinite chain [64, page 305]. We mention that a geometric lattice is cryptomorphic to a natural object called combinatorial geometry [64, Theorem 23.1] and that finite geometric lattice is nothing but finite matroid lattice [36, page 163, Birkhoff's Theorem]. A geometric/matroid lattice has a natural valuated poset structure, where the valuation is given by its rank function. For example, for a matroid M, all elements in $\mathrm{F}_t(M)$ have rank t. In a geometric lattice, the elements of rank 1, 2 and 3 are viewed as points, lines and planes, respectively, thus giving geometric intuitions to many results about geometric lattices.

For each linear space V and each nonnegative integer k, we use $\mathrm{Gr}(k, V)$ for the set of all k-dimensional linear subspaces of V and we call $\bigcup_{k=0}^{\infty} \mathrm{Gr}(k, V)$ the *Grassmannian* of V, which is denoted by $\mathrm{Gr}(V)$. If V is finite dimensional, $\mathrm{Gr}(V)$ is surely a geometric lattice with elements from $\mathrm{Gr}(k, V)$ having rank k.

Example 1.1. Let n and k be two positive integers such that $k < n$. Fix a non-degenerate inner product on $\mathbb{Q}^n$, say $\langle , \rangle$. For each $g \in \mathrm{GL}_n(\mathbb{Q})$, let $g^\top$ stand for the adjoint of g, namely the element such that $\langle ug, v\rangle = \langle u, vg^\top\rangle$ for all $u, v \in \mathbb{Q}^n$, and we write $g_\#$ for $(g^{-1})^\top$. Let $S \leq \mathrm{GL}_n(\mathbb{Q})$ be a matrix group acting on $\mathbb{Q}^n$. If $\overline{S}$ is transitive on the set of all dimension-k subspaces and if $g_\# \in S$ for all $g \in S$, then $\overline{S}$ is transitive on the set of dimension-$(n-k)$ subspaces. To see this, fix a pair of subspaces (U, U') which are orthogonal complements to each other with respect to $\langle , \rangle$ and $(\dim U, \dim U') = (k, n-k)$. For each $g \in S$, we can see that $U\overline{g}$ and $U'\overline{g_\#}$ are orthogonal complements to each other with respect to the given inner product $\langle , \rangle$. Considering the set of pairs $\{(U\overline{g}, U'\overline{g_\#}) : g \in S\}$, we see that the transitivity on $\mathrm{Gr}(k, \mathbb{Q}^n)$ implies transitivity on $\mathrm{Gr}(n-k, \mathbb{Q}^n)$.

Motivated by Example 1.1, here is a very simple question on the very simple geometric lattice $\mathrm{Gr}(\mathbb{Q}^3)$. Surprisingly, we even could not find any discussion of it in the literature.

Question 1.2. If S is a general matrix group acting on $\mathbb{Q}^3$, can we draw the conclusion that $\overline{S}$ is transitive on $\mathrm{Gr}(1, \mathbb{Q}^3)$ from the assumption of its transitivity on $\mathrm{Gr}(2, \mathbb{Q}^3)$? What about only assuming that S is a matrix semigroup?

Some seemingly weird properties of sequences turn out to be ubiquitous when we are examining some interesting structures or processes [6, 10, 11, 29, 58, 61]. We review some of them below. Let $c_0, c_1, \ldots,$ be a sequence of $n+1$ real numbers, where n can be finite or infinite. We call it *t-top-heavy* if $c_k \leq t$ whenever there exists an integer ℓ such that $k \leq \ell \leq k+\ell \leq n$ and $c_\ell \leq t$; we call it *top-heavy* if it is t-top-heavy for all $t \in \mathbb{R}$, namely $c_k \leq c_\ell$ holds for all k, ℓ such that $k \leq \ell \leq k+\ell \leq n$; We call it *unimodal* if you cannot find three distinct integers i, j, k such that $0 \leq i < j < k \leq n$ and $c_i - c_j > 0 > c_j - c_k$; we call it *log-concave* if $c_i^2 \geq c_{i-1}c_{i+1}$ for all $i = 1, \ldots, n-1$. When n is finite, we call the sequence *real-rooted* provided the polynomial $c_0 + c_1x + \cdots + c_nx^n$ in the unknown x only has real roots and we call it *ultra-log-concave* provided $\frac{c_0}{\binom{n}{0}}, \ldots, \frac{c_n}{\binom{n}{n}}$ forms a log-concave sequence. Note that Question 1.2 is about the possible 1-top-heavy property of the strong shape of $\mathrm{Gr}(\mathbb{Q}^3)$ under a matrix semigroup action.

In the 1970s, two log-concavity conjectures [61, Conjecture 3] appeared in combinatorics community which claim that the sequences of Whitney numbers of both the first kind and the second kind of a finite matroid are log-concave. The first conjecture was verified by Adiprasito, Huh and Katz [1]. Mason [40] had made variants and stronger versions of the second conjecture; but even the original conjecture is still open. Dowling and Wilson [23] conjectured that the sequence of Whitney numbers of the second kind of a finite matroid is top-heavy. When restricted to finite realizable matroids, this top-heavy conjecture was proved by Huh and Wang [28]. The second log-concavity conjecture as described above, which is about the Whitney numbers of the second kind [50], simply says that the shape of every geometric lattice is log-concave. The above-mentioned Dowling-Wilson top-heavy conjecture says that the shape of every finite geometric lattice is top-heavy. On the condition that these two conjectures are both true, we know that the shape of a finite geometric lattice is both log-concave (and hence unimodal) and top-heavy. Can we draw this conclusion for the strong/weak shape of some semigroup actions on some geometric lattices?

Boolean lattices, partition lattices and projective/affine geometries are some most well-known geometric lattices. It is easy to see that their shapes are all ultra-log-concave (and hence real-rooted) and top-heavy [37]. The main result of this paper, Theorems 2.1 and 2.12, declare the top-heavy property for the strong/weak shape of some semigroups acting on Boolean lattices and projective/affine geometries. The semigroups considered by us are those derived from "simple" transformations. We also report our attempt at tackling the same problem for partition lattices and the Vámos matroid.

In Section 2, we will present our main results as well as pertinent problems, examples, and remarks. The first three subsections are devoted to Boolean lattices, partition lattices and projective/affine geometries. The last subsection is a simple discussion in the context of matroids. Before digging into the proofs of the main results, we develop some technical tools in Section 3. In the sequel, we provide in Sections 4 to 7 all the proofs missing from Sections 2.1 to 2.4. We conclude the paper in Section 8 with a brief discussion of the present work and some possible further research.

2 A top-heavy promenade

2.1 Boolean semiring and homogeneity

For any set Ω, the set $\mathrm{B}_\Omega \doteq \bigcup_{k=0}^{\infty} \binom{\Omega}{k}$ forms a poset under the inclusion relationship, which is often known as the *Boolean semiring over* Ω, and the set 2^Ω gives rise to the *Boolean algebra over* Ω. When we view B_Ω as a valuated poset, unless stated otherwise, the valuation will be $\mathrm{r}(A) = |A|$ for all $A \in \mathrm{B}_\Omega$. If Ω is a finite set, B_Ω coincides with 2^Ω and is referred to as a *Boolean lattice*.

Let A and Ω be two sets with $A \subseteq \Omega$. For any $g \in \Omega^\Omega$, write $g|_A$ for the restriction of g on A. Let S be a transformation semigroup on Ω. For any positive integer $k \leq |\Omega|$, we name S *k-homogeneous* if the transformation semigroup $\overline{S}$ is transitive on $\binom{\Omega}{k}$, that is, $\mathrm{scc}(\Gamma(\overline{S})[\binom{\Omega}{k}]) = 1$. The *stabiliser permutation group* of (S, A) is the permutation group $S_A \doteq \{g|_A : g \in S, A\overline{g} = A\}$ acting on A. The *relative transformation semigroup* of (S, A) is the transformation semigroup $\tilde{S}_A \doteq \{g|_A : g \in S, A\overline{g} \subseteq A\}$ acting on A. Note that the action of $\tilde{S}_A$ on A may not be transitive even if S acts on A transitively.

Theorem 2.1. *Let Ω be a set of size n. Let S be a transformation semigroup on Ω and let Γ be the phase space of $\overline{S}$.*

(1) $\mathsf{SS}(\overline{S}, \mathrm{B}_\Omega)$ *is 1-top-heavy.*

(2) *Both* $\mathsf{WS}(\overline{S}, \mathrm{B}_\Omega)$ *and* $\mathsf{SS}(\overline{S}, \mathrm{B}_\Omega)$ *are top-heavy.*

(3) *Let k and ℓ be two integers such that $0 \leq k \leq \ell \leq k + \ell \leq n + 1$. Let $A \in \binom{\Omega}{k}$ and $B \in \binom{\Omega}{\ell}$. If $n < \infty$ and S is ℓ-homogeneous, then* $\mathrm{scc}(\Gamma(S_A)) = \mathrm{wcc}(\Gamma(S_A)) \leq \mathrm{wcc}(\Gamma(S_B)) = \mathrm{scc}(\Gamma(S_B))$.

Question 2.2. Take a finite set Ω and two integers k and ℓ such that $k \leq \ell < k+\ell \leq |\Omega|+1$. Let S be an ℓ-homogeneous transformation semigroup acting on Ω. For any $A \in \binom{\Omega}{k}$ and $B \in \binom{\Omega}{\ell}$, does it always hold that $\mathrm{wcc}(\Gamma(\tilde{S}_A)) \leq \mathrm{wcc}(\Gamma(\tilde{S}_B))$?

When restricting to permutation groups, the results in Theorem 2.1 are all known more than 40 years ago: Claim (1) for an infinite set Ω was discovered by Brown [12, Corollary 1]; Claim (2) for a finite set Ω was derived by Livingstone and Wagner [38, Theorem 1]; Claim (3), as well as a positive answer to Question 2.2 for permutation groups, was proved by Cameron [15, Proposition 2.3] under the mild restriction of $k + \ell \leq |\Omega|$. Let G be a group acting on a finite set Ω. By Theorem 2.1(2), or more precisely Livingstone-Wagner Theorem [38, Theorem 1], we know that the strong/weak shape of 2^Ω under the action of $\overline{G}$ is a symmetric unimodal distribution. This means that, for any two integers k and ℓ such that $k \leq \ell < k + \ell \leq |\Omega|$, the number of $\overline{G}$-orbits on $\binom{\Omega}{\ell}$ is equal to the sum of a nonnegative integer c plus the number of $\overline{G}$-orbits on $\binom{\Omega}{k}$. As an improvement of this fact, Siemons [56, Corollary 4.3] found a natural linear space whose dimension equals this integer c and he [56, Theorem 4.2] even obtained an algorithm to reconstruct the $\overline{G}$-orbits on $\binom{\Omega}{k}$ from the information on the $\overline{G}$-orbits on $\binom{\Omega}{\ell}$ without reference to the group G.

Question 2.3. Let Ω be a finite set, and let k and ℓ be two integers such that $k \leq \ell < k + \ell \leq |\Omega|$. Let S be a transformation semigroup on Ω and let Γ be the phase space of $\overline{S}$.

(1) Is there a counterpart of [56, Corollary 4.3] which explains the nonnegativity constraint on the integer $\operatorname{wcc}(\Gamma[\binom{\Omega}{\ell}]) - \operatorname{wcc}(\Gamma[\binom{\Omega}{k}])$?

(2) If S is $(\ell + 1)$-homogeneous, is there a counterpart of [56, Corollary 4.3] which explains the nonnegativeness of the integer $\operatorname{scc}(\Gamma(S_B)) - \operatorname{scc}(\Gamma(S_A))$ for any $A \in \binom{\Omega}{k}$ and $B \in \binom{\Omega}{\ell+1}$?

(3) Is there any algorithm to determine the weakly connected components of $\Gamma[\binom{\Omega}{k}]$ from the weakly connected components of $\Gamma[\binom{\Omega}{\ell}]$ without reference to the transformation semigroup S?

Example 2.4. Let Ω be a set carrying a linear order $\prec$. A map $g \in \Omega^\Omega$ is order-preserving with respect to $\prec$ provided αg is not bigger than βg in $\prec$ whenever α is not bigger than β in $\prec$. Let S be the monoid consisting of all order-preserving maps on Ω with respect to the given linear order $\prec$. It is easy to see that S is ℓ-homogeneous for all $\ell \leq |\Omega|$ but it is even not 2-transitive; by contrast, this phenomenon never happens for permutation groups due to a result of Livingstone and Wagner [38, Theorem 2(b)]. Note that the only permutation contained in S is the identity map in case that Ω is a finite set. This suggests that you may not be able to read Theorem 2.1 or answer Question 2.3 directly from those known facts on permutation groups.

Example 2.5. Let $\Omega = \{1, \ldots, 6\}$. Let r and b be two maps in $\mathrm{T}(\Omega)$ such that

$$r(1) = r(2) = 3, \quad r(3) = r(4) = 5, \quad r(5) = r(6) = 1;$$
$$b(6) = b(1) = 2, \quad b(2) = b(3) = 4, \quad b(4) = b(5) = 6.$$

Let $S = \langle r, b\rangle$. On the left of Figure 1, we depict the phase space $\Gamma(S, \{r, b\})$; on the right of Figure 1, we display both the strong shape and the weak shape of 2^Ω under the action of $\overline{S}$. Both weak shape and strong shape are unimodal and top-heavy. But neither of them is log-concave. Note that the peak of the weak shape does not happen at the middle rank 3.

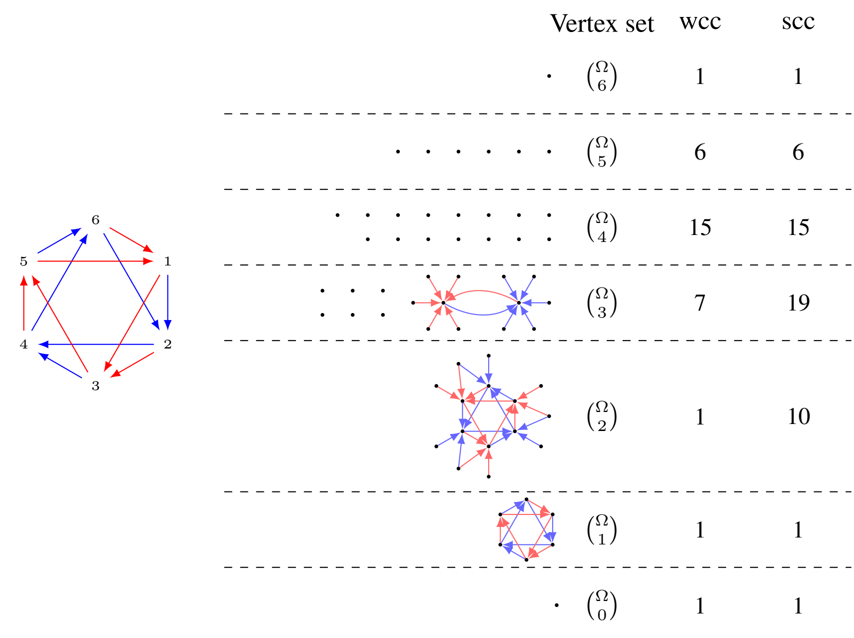

Figure 1: $\Gamma(S,\{r,b\})$ and $\Gamma(\overline{S},\{\overline{r},\overline{b}\})[\binom{\Omega}{k}]$, $k \in \{0,1,\dots,6\}$. See Example 2.5.

Example 2.6. Let Ω be a set of size $n \geq 3$ and let S be a transformation semigroup acting on Ω. If $\mathsf{SS}(\overline{S},2^{\Omega})$ is not a sequence of all ones and has at least two ones at the beginning of it, then it cannot be log-concave. This happens when S is the alternating group of order $n \geq 4$ and when S is 2-homogeneous but not 3-homogeneous.

Example 2.7. Let n and k be two integers such that $1 \leq k \leq n$. Let Ω be a set of size n and take $X \in \binom{\Omega}{k}$. Let S be the set $\{f \in \mathrm{T}(\Omega) :\ f|_X = \mathrm{Id}\,|_X, \Omega\overline{f} = X\}$. Note that S is a transformation semigroup on 2^{Ω} satisfying

$$\mathrm{wcc}(\Gamma(\overline{S})[\binom{\Omega}{i}] = \begin{cases} 1, & \text{if } 0 \leq i \leq k; \\ \binom{n}{i}, & \text{if } k+1 \leq i \leq n. \end{cases}$$

This shows that the sequence $\mathsf{WS}(\overline{S},2^{\Omega})$ is unimodal and top-heavy and that it is not log-concave when $n \geq 2$. Note that $\mathsf{SS}(\overline{S},2^{\Omega})$ is a sequence of all ones.

Question 2.8. Let S be a transformation semigroup acting on an n-element set Ω. When can we conclude that the strong/weak shape of 2^{Ω} under the action of $\overline{S}$ is unimodal?

Neumann [44] asked whether every λ-homogeneous permutation group is θ-homogeneous for all cardinals $\lambda > \theta \geq \aleph_0$. Assuming Martin's Axiom, Shelah and Thomas [55] gave a negative answer to it. Hajnal [26] supplied an example to show that 2^{θ}-homogeneity does not imply θ-homogeneity. An observation in the same vein by Penttila and Siciliano [47, Remark 4.6] was based upon the Generalized Continuum Hyphothesis. For each statement in Theorem 2.1(3), Question 2.2 and 2.3, it is interesting to see whether or not it holds in the case that Ω is an infinite set. We are also wondering if the rich theory on oligomorphic permutation groups [16] should have a counterpart for transformation semigroups.

2.2 Partition lattice

Let Ω be a set. For any map $s \in \Omega^\Omega$, we define its *kernel map*, denoted by s^{-1}, to be the map from 2^Ω to 2^Ω that sends $X \in 2^\Omega$ to $Xs^{-1} = \{y \in \Omega : ys \in X\} \in 2^\Omega$. To illustrate the definition, we depict the phase space of a map s on the left of Figure 2 and part of the phase space of s^{-1} in the middle of Figure 2. A *partition* of Ω is a set of nonempty disjoint subsets of Ω whose union is Ω. We call these elements of a partition its *blocks*. The *rank* of a partition π is $\sum_{B\in\pi}(|B|-1)$. Write $\mathrm{P}(\Omega)$ for the set of all partitions of Ω of finite ranks. When $|\Omega| < \infty$, the set $\mathrm{P}(\Omega)$ together with the refinement relation forms a geometric lattice, which we call the *partition lattice* of Ω. Note that the rank of a partition in this geometric lattice is $|\Omega|$ minus the number of its blocks. Let $\mathrm{P}_k(\Omega)$ be the set of rank-k partitions of Ω, namely, those partitions of Ω of size $|\Omega| - k$. Each transformation $s \in \Omega^\Omega$ induces a transformation s^* of 2^Ω such that $\Pi s^* = \{\pi s^{-1} : \pi \in \Pi\} \setminus \{\emptyset\}$ for all $\Pi \in \mathrm{P}(\Omega)$. We demonstrate part of the phase space of s^* on the right of Figure 2 for the map s as shown on the left there. Let S be a transformation semigroup on Ω. We have a derived transformation semigroup $S^* := \{s^* : s \in S\}$ on $\mathrm{P}(\Omega)$, which we call the *kernel space* of S. We say that S is k*-kernel homogeneous* if for all $\Pi, \Pi' \in \mathrm{P}_k(\Omega)$ there exists $s \in S$ such that $\Pi s^* = \Pi'$, which surely implies $\mathrm{scc}(\Gamma(S^*)[\mathrm{P}_k(\Omega)]) \le 1$.

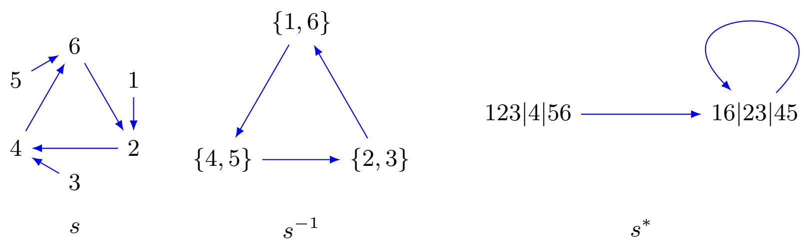

Figure 2: A map, its inverse and the derived action on partitions.

Example 2.9. On the left of Figure 3, we depict the so-called Černý automaton $\mathcal{C}_4 = \Gamma(S, G)$, where $G = \{a, b\}$ consists of two transformations on a four-element set Ω. On the right of Figure 3, we depict the automaton $\Gamma(S^*, G^*)$ where S^* is acting on $\mathrm{P}(\Omega)$. Observe that $\mathsf{WS}(S^*, \mathrm{P}(\Omega)) = (1, 1, 1, 1)$ and $\mathsf{SS}(S^*, \mathrm{P}(\Omega)) = (1, 2, 2, 1)$ are both unimodal and top-heavy.

For any finite set Ω, $A \in 2^\Omega$, $\pi \in \mathrm{P}(\Omega)$ and $s \in \Omega^\Omega$, it holds

$$\mathrm{r}(A) \ge \mathrm{r}(A\overline{s}) \quad \text{and} \quad \mathrm{r}(\pi) \le \mathrm{r}(\pi s^*).$$

This difference between Boolean lattice and partition lattice somehow hints at our difficulty of turning the following conjecture into a result like Theorem 2.1.

Conjecture 2.10. *Let Ω be a finite set and let S be a semigroup acting on Ω. Then both $\mathsf{WS}(S^*, \mathrm{P}(\Omega))$ and $\mathsf{SS}(S^*, \mathrm{P}(\Omega))$ are top-heavy.*

For each set Ω and each positive integer $k \le |\Omega|$, we use $\mathrm{P}(\Omega, k)$ for the set of partitions of Ω into k blocks.

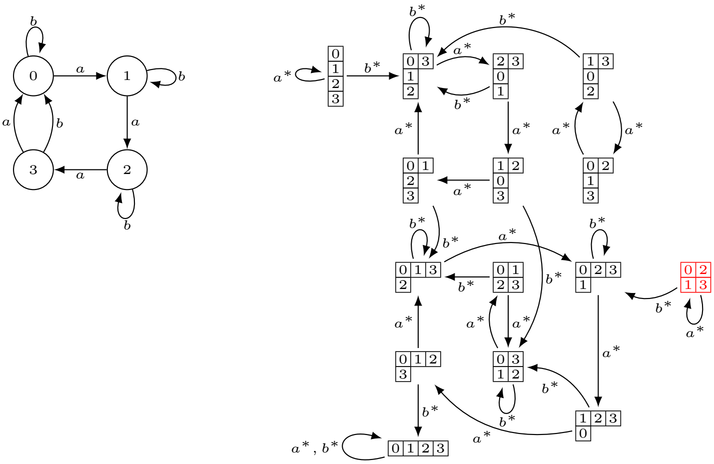

Figure 3: Černý automaton $\mathcal{C}_4$ and its kernel space. See Example 2.9.

Question 2.11. (1) Take two positive integers k and ℓ with $k < \ell$. Let Ω be an infinite set and let S be a semigroup S acting on Ω. If S^* is transitive on $\mathrm{P}(\Omega, \ell)$, is it true that S^* is transitive on $\mathrm{P}(\Omega, k)$?

(2) The shapes of all Dowling lattices, which include all partition lattices, are real-rooted [8]. What about the top-heavy property of the (strong/weak) shapes of Dowling lattices under a semigroup action?

There has been an active study of those permutation groups which are transitive on the set of all ordered or unordered partitions of a set of a given shape [2, 21, 39, 43]. But even when confining our attention to permutation groups, we are not aware of any work related to Conjecture 2.10 and Question 2.11.

2.3 Subspace lattice

Let Ω be a possibly infinite set of size n, let k be a nonnegative integer with $k \leq n$ and let F be a finite field. We mention that $\mathrm{Gr}(k, F^{\Omega})$ is a q-analogue of $\binom{\Omega}{k}$ and their relationship is like the one between Johnson graphs and Grassmann graphs [45]. For each prime power q, we write $\mathbb{F}_q$ for the q-element finite field. Write $\mathrm{Mat}_n(\mathbb{F}_q)$ for the multiplicative semigroup of all Ω by Ω matrices over $\mathbb{F}_q$ each row/column of which have finitely many nonzeros; and write $\mathrm{Aff}_n(\mathbb{F}_q)$ for the semigroup of all affine linear transformation on $\mathbb{F}_q^n$ equipped with the associated product of composition. We regard the empty set as the dimension-(-1) affine/linear subspace. The set of all nonempty finite linear subspaces of $\mathbb{F}_q^n$ is denoted by $\mathcal{P}_{q,n} \doteq \mathrm{Gr}(\mathbb{F}_q^n)$ and the set of all dimension-k linear subspaces of $\mathbb{F}_q^n$ is denoted by $\mathcal{P}_{q,n}^k \doteq \mathrm{Gr}(k, \mathbb{F}_q^n)$. By a finite affine subspace of a linear space V, we mean a translate of a finite linear subspace of V. The set of all finite affine subspaces of $\mathbb{F}_q^n$ is denoted by $\mathcal{A}_{q,n}$ and the set of all dimension-k affine subspaces of $\mathbb{F}_q^n$ is denoted by $\mathcal{A}_{q,n}^{k+1}$. Note that $\mathcal{P}_{q,n}$ and $\mathcal{A}_{q,n}$ are known as *projective geometry* and *affine geometry* over the field $\mathbb{F}_q$, respectively. For each nonnegative integer k, we put the rank of each element in $\mathcal{P}_{q,n}^k$ and the rank of each element in $\mathcal{A}_{q,n}^k$ to be k, thus getting two valuated posets $\mathcal{P}_{q,n}$ and $\mathcal{A}_{q,n}$, which are geometric lattices when $n < \infty$.

We are ready to display Theorem 2.12, a q-analogue of Theorem 2.1. Kantor [31, Theorem 1] deduced a q-analogue of the aforementioned result of Livingstone and Wagner [38, Theorem 1]. If the semigroup $S \leq \mathrm{Mat}_n(\mathbb{F}_q)$ is a subgroup of the general linear group $\mathrm{GL}_n(\mathbb{F}_q)$, Stanley [59, Corollary 9.9] found that $\mathsf{SS}(S^{\mathcal{P}}, \mathcal{P}_{q,n})$ and $\mathsf{WS}(S^{\mathcal{P}}, \mathcal{P}_{q,n})$ are both symmetric and unimodal for finite n. Penttila and Siciliano [47, Theorem 4.4(ii), (iii)] generalized this result of Stanley for groups to the case that n is infinite.

Theorem 2.12. *Let n be the size of a nonempty set, and let q be a prime power.*

(1) *Let $S \leq \mathrm{Mat}_n(\mathbb{F}_q)$ be a linear transformation semigroup acting on $\mathbb{F}_q^n$. For each $g \in S$, write $g^{\mathcal{P}}$ for $\overline{g}|_{\mathcal{P}_{q,n}}$. Let $S^{\mathcal{P}}$ be the transformation semigroup $\{g^{\mathcal{P}} : g \in S\}$ acting on $\mathcal{P}_{q,n}$. Then $\mathsf{SS}(S^{\mathcal{P}}, \mathcal{P}_{q,n})$ and $\mathsf{WS}(S^{\mathcal{P}}, \mathcal{P}_{q,n})$ are both top-heavy.*

(2) *Let $T \leq \mathrm{Aff}_{n-1}(\mathbb{F}_q)$ be an affine linear transformation semigroup acting on $\mathbb{F}_q^{n-1}$. For each $g \in T$, write $g^{\mathcal{A}}$ for $\overline{g}|_{\mathcal{A}_{q,n-1}}$. Let $T^{\mathcal{A}}$ be the transformation semigroup $\{g^{\mathcal{A}} : g \in T\}$ acting on $\mathcal{A}_{q,n-1}$. Then $\mathsf{SS}(T^{\mathcal{A}}, \mathcal{A}_{q,n-1})$ and $\mathsf{WS}(T^{\mathcal{A}}, \mathcal{A}_{q,n-1})$ are both top-heavy.*

Remark 2.13. When n is infinite, Theorems 2.1 and 2.12 in the original version of this paper, submitted on 19 July 2018, contains weaker results. Following the proof presented by Bercov and Hobby for [9, Corollary 1] and also the proof of Roy for [51, Theorem], we used the existence of Ramsey number [49, Theorem A] to derive Theorem 2.1(1) for infinite n. A similar argument based on Ramsey number shows that both $\mathsf{SS}(S^{\mathcal{P}}, \mathcal{P}_{q,n})$ and $\mathsf{SS}(T^{\mathcal{A}}, \mathcal{A}_{q,n-1})$ are 1-top-heavy for infinite n in the setting of Theorem 2.12. After the acceptance of this paper in 2022, we notice the work of Penttila and Siciliano [47, Lemma 3.1], which was submitted on 30 April 2019 and published in 2021, and thus arrive at the corresponding strengthening in Theorem 2.1(2) and Theorem 2.12 via an application of their idea. See Lemma 3.6.

Remark 2.14. Kantor [32, Theorem 2] determined all the ordered-basis-transitive finite geometric lattices of rank at least three: Roughly speaking, they are Boolean lattices, projective (affine) geometries, and four sporadic designs. Kantor's classification theorem along with Theorems 2.1 and 2.12 may be a basis for getting homogeneity results about ordered-basis-transitive matroids.

Question 2.15. A general projective geometry is defined to be a modular combinatorial geometry that is connected in the sense that the point set cannot be expressed as the union of two proper flats [64, page 313]. Can we establish a counterpart of Theorem 2.12 for general projective geometries?

In mathematics we encounter quite some nice duality phenomena, say Chow's Theorem [45, Corollary 3.1] and many duality concepts for matroids [13]. For projectie geometry, we have the following duality result of Stanley [59, Corollary 9.9].

Theorem 2.16 (Stanley)**.** *Let F be a finite field and let k and n be two positive integers with $k < n$. For any subgroup G of $\mathrm{GL}(n, F)$, the number of orbits of the action of G on $\mathrm{Gr}(k, F^n)$ must be the same with the number of orbits of G acting on $\mathrm{Gr}(n-k, F^n)$.*

Question 2.17. If n is the size of an infinite set, does Theorem 2.16 still hold? Here, we should first of all choose a good definition for infinite Grassmannians [46].

2.4 A glimpse of matroid

In previous subsections, we discuss those poset endomorphisms which are derived from either set transformations or linear transformations. Since finite geometric lattices just encode information of finite matroids, it is natural to ask why not directly consider matroids and morphisms among matroids, namely those transformations which preserve "independence structure".

Let M_1 and M_2 be two matroids and let f be a map from $\mathscr{E}_{M_1}$ to $\mathscr{E}_{M_2}$. We call f a *weak map from M_1 to M_2* provided

$$\mathrm{r}_{M_1}(A) \geq \mathrm{r}_{M_2}(A\overline{f})$$

holds for all $A \subseteq \mathscr{E}_{M_1}$, and we call f a *strong map from M_1 to M_2* provided the preimage of any flat in M_2 is a flat of M_1 [33, 35, 57]. It is known that all strong maps must be weak maps.

Let M be a matroid on the ground set $\mathscr{E}_M = \Omega$. Let $\mathrm{T}_M(\Omega)$ ($\mathrm{T}^*_M(\Omega)$) be the monoid consisting of all elements of $\mathrm{T}(\Omega)$ which are weak (strong) maps from M to itself. If we know that S is a subsemigroup of $\mathrm{T}_M(\Omega)$ ($\mathrm{T}^*_M(\Omega)$) acting on Ω, we can define a digraph $\Gamma_{M,t}(S)$ on $\mathrm{F}_t(M)$ as follows: for any $X, Y \in \mathrm{F}_t(M)$, there is an arc from X to Y if and only if there is $g \in S$ such that the minimum flat containing $X\overline{g}$ in M is Y. What is the relationship between the connectivity of $\Gamma_{M,t}(S)$ and $\Gamma_{M,r}(S)$ for different t and r? We can ask the same question by imposing the extra condition that every element $f \in S$ is a bijection on Ω. If the matroid is a very special uniform matroid, namely a matroid in which all sets are independent, one can see that what is discussed in Section 1.3 becomes a very special case of this general setting.

Vámos matroid, also known as Vámos cube, is a famous non-algebraic matroid [5, 22, 42, 54]; see [24, Example 6.30] for a description of this rank-4 matroid over a ground set of size eight.

Example 2.18. Let M be the Vámos matroid and let S be a subsemigroup of $\mathrm{T}^*_M(\mathscr{E}_M)$. It holds $\mathrm{wcc}(\Gamma_{M,1}(S)) \leq \mathrm{wcc}(\Gamma_{M,2}(S)) \leq \mathrm{wcc}(\Gamma_{M,3}(S))$ and $\mathrm{scc}(\Gamma_{M,1}(S)) \leq \mathrm{scc}(\Gamma_{M,2}(S)) \leq \mathrm{scc}(\Gamma_{M,3}(S))$.

Remark 2.19. Compared with the Fundamental Theorem of Projective (Affine) Geometry [17, 48], we think that weak/strong maps and bijective weak/strong maps for matroids are natural extensions of linear transformations and invertible linear transformations for linear spaces. We also mention the well-adopted viewpoint that the full permutation group and the full transformation semigroup can be interpreted as the general linear group and the linear transformation semigroup over the field with one element.

3 Valuated poset and incidence operator

3.1 Hereditary endomorphism and injective incidence operator

To prepare for a proof of our main results listed in Section 2, we will introduce a key property and then present a key lemma for our work. The key property is the so-called hereditary endomorphisms. The key lemma is Lemma 3.2, which gives us some information of the strong/weak shapes of a poset under some semigroup action, provided the semigroup consists of hereditary endomorphisms and that some linear map associated with the poset is injective.

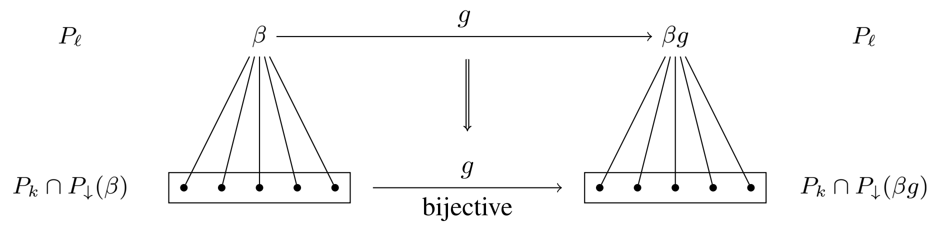

Figure 4: An (ℓ, k)-hereditary endomorphism.

Let P be a valuated poset. For any nonnegative integers $k \le \ell$, we call the poset P *(k,ℓ)-finite* provided $P_k \neq \emptyset$, $P_\ell \neq \emptyset$ and the set $P_\ell \cap P_\uparrow(\alpha)$ is finite for every $\alpha \in P_k$; we call P *(ℓ,k)-finite* provided $P_k \neq \emptyset$, $P_\ell \neq \emptyset$ and the set $P_\downarrow(\beta) \cap P_k$ is finite for every $\beta \in P_\ell$; we call $g \in \mathrm{End}(P)$ a *(k,ℓ)-hereditary endomorphism* if for all $\alpha \in P_k$ which satisfies $\mathrm{r}_P(g(\alpha)) = \mathrm{r}_P(\alpha) = k$ it happens that g induces a bijection from the set $P_\ell \cap P_\uparrow(\alpha)$ to $P_\ell \cap P_\uparrow(\alpha g)$; we call $g \in \mathrm{End}(P)$ an *(ℓ,k)-hereditary endomorphism* if for each $\beta \in P_\ell$, $\mathrm{r}_P(\beta g) = \mathrm{r}_P(\beta) = \ell$ ensures that g induces a bijection from the set $P_k \cap P_\downarrow(\beta)$ to $P_k \cap P_\downarrow(\beta g)$. See Figure 4 for an illustration. For any $k, \ell \in \mathbb{Z}_{\geq 0}$, we designate by $\mathrm{hEnd}_{k,\ell}(P)$ the set of all (k,ℓ)-hereditary endomorphisms of the valuated poset P.

Let S be a transformation semigroup on a valuated poset P and let G be a generating set of S. For any two nonnegative integers k and ℓ with $k \le \ell \le \mathrm{r}(P)$, we set $\Pi_{S,G}(k,\ell)$ to be the digraph with vertex set P_k and arc set

$$\{(\alpha,\alpha') \in P_k \times P_k : \exists g \in G, \beta \in P_\ell \text{ s.t. } \beta g \in P_\ell, \alpha' = \alpha g, \alpha \in P_\downarrow(\beta)\};$$

we set $\Pi_{S,G}(\ell,k)$ to be the digraph with vertex set P_ℓ and arc set

$$\{(\alpha,\alpha') \in P_\ell \times P_\ell : \exists g \in G, \beta \in P_k \text{ s.t. } \beta g \in P_k, \alpha' = \alpha g, \alpha \in P_\uparrow(\beta)\}.$$

We use the shorthand $\Pi_S(k,\ell)$ for $\Pi_{S,S}(k,\ell)$.

Lemma 3.1. *Let P be a valuated poset. Take two nonnegative integers k and ℓ such that $k, \ell \le \mathrm{r}(P)$ and that P is (ℓ,k)-finite. Let S be a sub-semigroup of $\mathrm{hEnd}_{\ell,k}(P)$, let G be a generator set of S, and let $\Gamma \doteq \Gamma(S,G)$. Let $\beta \in P_\ell$ and let $\alpha \in P_k$ be an element comparable with β. Assume that g and h are two elements of S such that $\beta g \in P_\ell$ and $\beta g h = \beta$. Then there exists $f \in S$ such that $\beta g f \in P_\ell$ and $\alpha g f = \alpha$. Especially, if every weakly connected component of $\Gamma[P_\ell]$ is strongly connected, then so is $\Pi_{S,G}(k,\ell)$.*

Proof. The second claim is immediate from the first and so our task is just to prove the first one. Without loss of generality, we assume that $k < \ell$. Since $\beta(gh) = \beta$ and $gh \in S \le \mathrm{hEnd}_{\ell,k}(P)$, it follows that gh induces a permutation on $P_k \cap P_\downarrow(\beta)$. But from the assumption that P is (ℓ,k)-finite, we see that $P_k \cap P_\downarrow(\beta)$ is a finite set, which contains α. This means that there exists a positive integer r such that $\alpha(gh)^r = \alpha$. Accordingly, for $f = (hg)^{r-1}h \in S$ it holds $(\beta g)f = (\beta g)(hg)^{r-1}h = \beta(gh)^r = \beta \in P_\ell$ and $(\alpha g)f = (\alpha g)(hg)^{r-1}h = \alpha(gh)^r = \alpha$, finishing the proof. □

For any set Ω, $\mathbb{Q}^\Omega$ refers to the linear space of all rational functions on Ω. If P is an (ℓ,k)-finite valuated poset, the *incidence operator* $\zeta_P^{k,\ell} \colon \mathbb{Q}^{P_k} \to \mathbb{Q}^{P_\ell}$ is the linear operator such that for all $f \in \mathbb{Q}^{P_k}$ and $\beta \in P_\ell$, we have

$$(\zeta_P^{k,\ell}(f))(\beta) = \begin{cases} \sum_{\alpha \in P_k \cap P_\downarrow(\beta)} f(\alpha), & \text{if } k \le \ell; \\ \sum_{\alpha \in P_k \cap P_\uparrow(\beta)} f(\alpha), & \text{if } k > \ell. \end{cases} \tag{3.1}$$

Lemma 3.2. *Let P be a valuated poset. Take two nonnegative integers k and ℓ not exceeding $\mathrm{r}(P)$ such that P is (ℓ, k)-finite, and hence $\zeta_P^{k,\ell}$ is well-defined. Let S be a subsemigroup of $\mathrm{hEnd}_{\ell,k}(P)$ and let Γ stand for $\Gamma(S)$. Assume that $\zeta_P^{k,\ell}$ is an injective linear map from $\mathbb{Q}^{P_k}$ to $\mathbb{Q}^{P_\ell}$.*

(1) $\mathrm{wcc}(\Gamma[P_k]) \le \mathrm{wcc}(\Pi_S(k,\ell)) \le \mathrm{wcc}(\Gamma[P_\ell])$.

(2) $\mathrm{scc}(\Gamma[P_k]) \le \mathrm{scc}(\Pi_S(k,\ell)) \le \mathrm{scc}(\Gamma[P_\ell])$.

Proof. (1) The first inequality is a consequence of the fact that $\mathrm{E}(\Pi_S(k,\ell)) \subseteq \mathrm{E}(\Gamma[P_k])$.

Let $W \subseteq \mathbb{Q}^{P_\ell}$ be the subspace of all functions which are constant on each weakly connected component of $\Gamma[P_\ell]$; let $V \subseteq \mathbb{Q}^{P_k}$ be the subspace of all functions which are constant on each weakly connected component of $\Pi_S(k,\ell)$. Note that $\dim(V) = \mathrm{wcc}(\Pi_S(k,\ell))$ and $\dim(W) = \mathrm{wcc}(\Gamma[P_\ell])$ and so it suffices to demonstrate $\dim(V) \le \dim(W)$.

By symmetry, we only deal with the case of $k \le \ell$. For every $f \in V$ and every arc $(\beta, \beta g)$ of $\Gamma[P_\ell]$, we have

$$\begin{aligned}
(\zeta_P^{k,\ell}(f))(\beta g) &= \sum_{\alpha' \in P_k \cap P_\downarrow(\beta g)} f(\alpha') \\
&= \sum_{\alpha \in P_k \cap P_\downarrow(\beta)} f(\alpha g) && (g \in \mathrm{hEnd}_{\ell,k}(P)) \\
&= \sum_{\alpha \in P_k \cap P_\downarrow(\beta)} f(\alpha) && (f \in V) \\
&= (\zeta_P^{k,\ell}(f))(\beta).
\end{aligned}$$

This says that $\zeta_P^{k,\ell}(f) \in W$ for all $f \in V$. Hence, by the injectivity of $\zeta_P^{k,\ell}$, $\dim(V) \le \dim(W)$, as wanted.

(2) The first inequality is a consequence of the fact that $\mathrm{E}(\Pi_S(k,\ell)) \subseteq \mathrm{E}(\Gamma[P_k])$.

Let $W' \subseteq \mathbb{Q}^{P_\ell}$ be the subspace of all functions which are constant on each strongly connected component of $\Gamma[P_\ell]$; let $V' \subseteq \mathbb{Q}^{P_k}$ be the subspace of all functions which are constant on each strongly connected component of $\Pi_S(k,\ell)$. Note that $\dim(V') = \mathrm{scc}(\Pi_S(k,\ell))$ and $\dim(W') = \mathrm{scc}(\Gamma[P_\ell])$ and so it suffices to demonstrate $\dim(V') \le \dim(W')$. Take $f \in V'$. As $\zeta_P^{k,\ell}$ is injective, we aim to show that $\zeta_P^{k,\ell}(f) \in W'$.

By symmetry, we only deal with the case of $k \le \ell$. Assume that β and βg are from the same strongly connected component of $\Gamma[P_\ell]$, where $g \in S$. By the first claim of Lemma 3.1, for every $\alpha \in P_k \cap P_\downarrow(\beta)$, α and αg fall into the same strongly connected component of $\Gamma[P_k]$ and so, as $f \in V'$,

$$f(\alpha) = f(\alpha g). \tag{3.2}$$

This allows us to write

$$\begin{aligned}
(\zeta_P^{k,\ell}(f))(\beta g) &= \sum_{\alpha' \in P_k \cap P_{\downarrow}(\beta g)} f(\alpha') \\
&= \sum_{\alpha \in P_k \cap P_{\downarrow}(\beta)} f(\alpha g) \qquad (g \in \mathrm{hEnd}_{\ell,k}(P)) \\
&= \sum_{\alpha \in P_k \cap P_{\downarrow}(\beta)} f(\alpha) \qquad \text{(Equation (3.2))} \\
&= (\zeta_P^{k,\ell}(f))(\beta),
\end{aligned}$$

proving that $\zeta_P^{k,\ell}(V') \subseteq W'$, as desired. □

3.2 Injectivity

In order to apply Lemma 3.2, we may need to have some results to guarantee the injectivity of an incidence operator. In this regard, a good understanding of the incidence algebra of a poset may be valuable [36, 68]. We mention that Guiduli [4, Theorem 9.4] established an injectivity result for the so-called rank-regular semi-lattices. It may also be quite useful if the following conjecture [34, Conjecture 1.1] can be verified.

Conjecture 3.3 (Kung)**.** *Let P be a finite geometric lattice. Let k and ℓ be two positive integers such that $k \leq \ell \leq \frac{\mathrm{r}(P)}{2}$. Then* $\ker(\zeta_P^{k,\ell}) = \{0\}$.

We suggest a slight strengthening of Kung's Conjecture (Conjecture 3.3) as follows.

Conjecture 3.4. *Let P be a geometric lattice. Let k and ℓ be two nonnegative integers such that $k \leq \ell \leq k + \ell \leq \mathrm{r}(P)$. If P is (ℓ, k)-finite, then $\zeta_P^{k,\ell}$ is an injective map.*

Remark 3.5. Let M be a matroid of rank r. Let S be a subsemigroup of $T^*_M(\mathscr{E}_M)$. For every $f \in S$, let $f' \colon \mathrm{F}(M) \to \mathrm{F}(M)$ be the map sending a flat $X \in \mathrm{F}(M)$ to the minimum flat containing $\overline{Xf}$ in M. Assume that $f' \in \mathrm{hEnd}_{\ell,k}(\mathrm{F}(M))$ for every $f \in S$. In light of Lemma 3.2, if Conjecture 3.4 is valid for the lattice $\mathrm{F}(M)$, we will be able to conclude that both the sequence $(\mathrm{wcc}(\Gamma_{M,0}(S)), \dots, \mathrm{wcc}(\Gamma_{M,r}(S)))$ and the sequence $(\mathrm{scc}(\Gamma_{M,0}(S)), \dots, \mathrm{scc}(\Gamma_{M,r}(S)))$ are top-heavy.

Let P be a valuated poset which is (ℓ, k)-finite for all nonnegative integers $k \leq \ell$. We say that P has a *top-heavy injective incidence operator* provided $\zeta_P^{k,\ell}$ is an injective linear map from $\mathbb{Q}^{P_k}$ to $\mathbb{Q}^{P_\ell}$ for all nonnegative integers k and ℓ satisfying $k \leq \ell \leq k+\ell \leq \mathrm{r}(P)$.

Penttila and Siciliano [47, Lemma 3.1] pointed out a simple way to establish some injectivity result for linear operators between infinite-dimensional linear spaces whenever they fulfil certain finiteness characteristics. We reformulate their observation below for the convenience of our later usage.

Lemma 3.6. *Let P be a valuated poset. Let $k \leq \ell$ be two nonnegative integers such that P is (ℓ, k)-finite. Assume that for every $A \in P_k$, we can find a finite subset Y of $P_{k+\ell}$ such that the ideal generated by Y in P, denoted $Y^{\downarrow}$ and with the restriction of r_P as its rank function, contains A and possesses a top-heavy injective incidence operator. Then $\zeta_P^{k,\ell}$ is an injective linear map from $\mathbb{Q}^{P_k}$ to $\mathbb{Q}^{P_\ell}$.*

Proof. Take $f \in \ker \zeta_P^{k,\ell}$. Assume, for sake of contradiction, that $f(A) \neq 0$ for some $A \in P_k$. Choose $Y \subseteq P_{k+\ell}$ such that $A \in Y^{\downarrow} \cap P_k$ and $Y^{\downarrow}$ possesses a top-heavy injective incidence operator. Let Q represent the resulting valuated poset on $Y^{\downarrow}$. Let g be the restriction of f on Q_k and let h be the restriction of $\zeta_P^{k,\ell}(f) = 0$ on Y. We have $0 = h = \zeta_Q^{k,\ell}(g)$ but $g(A) = f(A) \neq 0$, violating the assumption that $Y^{\downarrow}$ has a top-heavy injective incidence operator. □

3.3 Incidence operator as an intertwiner

For $f \in \Psi^{\Omega}$, we sometimes need to talk about $f(\omega)$ for $\omega \notin \Omega$. Following the practice of those mathematics with natural multivalued operations [7, 14, 65], we create a universal "don't care" symbol $\star \notin \Psi$ and will set $f(\omega) = \star$. We often regard $\star$ as all possible values in Ψ and so, whenever we have some addition operation $+$ on Ψ, we extend it to $\Psi \cup \{\star\}$ by setting $\star + \psi = \star$ for all $\psi \in \Psi \cup \{\star\}$.

Let P be a valuated poset. Let k and ℓ be two nonnegative integers no greater than $\mathrm{r}(P)$. Let $g \in P^P$. For $f \in \mathbb{Q}^{P_k}$, we write $fg^{\dagger,k}$ for the element in $(\{\star\} \cup \mathbb{Q})^{P_k}$, where $\star$ stands for "don't care" and can be thought of as the whole set $\mathbb{Q}$, such that the following holds for all $\beta \in P_k$:

$$fg^{\dagger,k}(\beta) = \begin{cases} f(\beta g), & \text{if } \beta g \in P_k; \\ \star, & \text{if } \beta g \notin P_k. \end{cases}$$

Denote by $\operatorname{Fix} g^{\dagger,k}$ the set of $f \in \mathbb{Q}^{P_k}$ for which

$$fg^{\dagger,k}(\beta) \in \{f(\beta), \star\}$$

holds for all $\beta \in P_k$. If $g \in \mathrm{hEnd}_{\ell,k}(P)$, we say that it is a *good* (ℓ,k)*-hereditary endomorphism of* P provided that for any $\beta \in P_\ell$ with $\beta g \notin P_\ell$ it holds $\alpha g \notin P_k$ for some $\alpha \in P_k$ which is comparable to β in P. Assuming that g is a good (ℓ,k)-hereditary endomorphism of P, for any $\beta \in P_\ell$ and $f \in \mathbb{Q}^{P_k}$ we will have

$$\begin{aligned}
(\zeta_P^{k,\ell}(f)g^{\dagger,\ell})(\beta) &= (\zeta_P^{k,\ell}(f))(\beta g) \\
&= \sum_{\alpha' \in P_k \cap (P_{\downarrow}(\beta g) \cup P_{\uparrow}(\beta g))} f(\alpha') \\
&= \sum_{\alpha \in P_k \cap (P_{\downarrow}(\beta) \cup P_{\uparrow}(\beta))} f(\alpha g) \\
&= (\zeta_P^{k,\ell}(fg^{\dagger,k}))(\beta)
\end{aligned}$$

whenever $\beta g \in P_\ell$, and that

$$\begin{aligned}
(\zeta_P^{k,\ell}(f)g^{\dagger,\ell})(\beta) &= (\zeta_P^{k,\ell}(f))(\beta g) \\
&= \star \\
&= (\zeta_P^{k,\ell}(fg^{\dagger,k}))(\beta)
\end{aligned}$$

whenever $\beta g \notin P_\ell$. This observation can be summarized by the commutative diagram in Figure 5, which implies that $\operatorname{Fix} g^{\dagger,k}$ is mapped by $\zeta_P^{k,\ell}$ to $\operatorname{Fix} g^{\dagger,\ell}$ for all good (ℓ,k)-hereditary endomorphisms g of P.

$$\begin{array}{ccc} f & \xrightarrow{\ \zeta_P^{k,\ell}\ } & \zeta_P^{k,\ell}(f) \\ {\scriptstyle g^{\dagger,k}}\downarrow & & \downarrow{\scriptstyle g^{\dagger,\ell}} \\ fg^{\dagger,k} & \xrightarrow[\ \zeta_P^{k,\ell}\ ]{} & \zeta_P^{k,\ell}(fg^{\dagger,k}) \end{array}$$

Figure 5: The incidence operator intertwines with every good hereditary endomorphism.

Example 3.7. (1) Let Ω be a set of size n. Assume that $2 \leq k < \ell \leq n$. Here is an easy observation used often in the study of synchronizing automata: For any $g \in \Omega^\Omega$ and any $A \in \binom{\Omega}{\ell}$, we have $|A\overline{g}| = \ell$ if and only if $|B\overline{g}| = k$ for all $B \in \binom{A}{k}$. This conclusion is surely not valid any more when $k \leq 1$. Note that $\overline{g}$ is a good (ℓ, k)-hereditary endomorphism of the Boolean lattice 2^Ω for each $g \in \Omega^\Omega$.

(2) Take integers n, k and ℓ such that $2 \leq k < \ell \leq n$ and let q be a prime power. Let $P = \mathcal{P}_{q,n}$ or $P = \mathcal{A}_{q,n-1}$. Similar to the above claim on Boolean lattice, $\overline{M}$ is a good (ℓ, k)-hereditary endomorphism of P for each $M \in \mathrm{Mat}_n(\mathbb{F}_q)$ or $M \in \mathrm{Aff}_{n-1}(\mathbb{F}_q)$, respectively.

4 Boolean semiring

Let Ω be a set and let k and ℓ be two nonnegative integers such that $k < \ell \leq |\Omega|$. For the valuated poset $P = \mathrm{B}_\Omega$, we write the incidence operator $\zeta_P^{k,\ell}$ defined in Equation 3.1 as $\zeta_\Omega^{k,\ell}$. That is,

$$(\zeta_\Omega^{k,\ell}(f))(B) = \sum_{A \in \binom{B}{k}} f(A)$$

for all $f \in \mathbb{Q}^{\binom{\Omega}{k}}$ and $B \in \binom{\Omega}{\ell}$.

Following a common approach in establishing homogeneity of permutation groups [15, 41] [20, pages 20-22], we will make use of the ensuing result on the rank of the subset inclusion matrix. The result has been discovered independently by many but the earliest appearance of it dates back to the work of Gottlieb [25, Corollary 2]. Among many different proofs of this classical result, we refer the reader to [18, Corollary] and [56, Theorem 2.4]. Note that it gives a positive answer to Conjecture 3.4 for Boolean lattices.

Lemma 4.1 (Gottlieb)**.** *Let Ω be a nonempty finite set. Then $\ker \zeta_\Omega^{k,\ell} = \{0\}$ for any two integers k and ℓ satisfying $0 \leq k \leq \ell \leq k + \ell \leq |\Omega|$.*

Let Ω be a set and S be a transformation semigroup on Ω. Let $\Omega^\sharp \doteq \{(\omega, C) : \omega \in C \in 2^\Omega\}$ and, for each $g \in S$, let $g^\sharp$ be the transformation on $\Omega^\sharp$ which sends (ω, C) to $(\omega g, C\overline{g})$ for all $(\omega, C) \in \Omega^\sharp$. Let $S^\sharp$ stand for the transformation semigroup on $\Omega^\sharp$ consisting of all elements $g^\sharp$ for $g \in S$. For all positive integers ℓ, we use the following notation:

$$\Omega_\ell^\sharp \doteq \{(\omega, C) : \omega \in C \in \binom{\Omega}{\ell}\}$$

and

$$\Gamma^\sharp_\ell(S) \doteq \Gamma(S^\sharp)[\Omega^\sharp_\ell].$$

Here is a result analogous to Lemma 3.1.

Lemma 4.2. *Let m be a positive integer and let S be an m-homogeneous transformation semigroup acting on a set Ω. Then the digraph $\Gamma^\sharp_m(S)$ is symmetric. Especially,* $\mathrm{wcc}(\Gamma^\sharp_m(S)) = \mathrm{scc}(\Gamma^\sharp_m(S))$.

Proof. Take $(\omega, C) \in \Omega^\sharp_m$ and $g \in S$ such that $|C\overline{g}| = m$. Our task is to show the existence of $h \in S$ such that $(\omega g, C\overline{g})h^\sharp = (\omega, C)$. As S is m-homogeneous, we can find $f \in S$ such that $C\overline{gf} = (C\overline{g})\overline{f} = C$. Hence, the fact that $|C| = m < \infty$ allows us to obtain a positive integer r for which $(gf)^r|_C$ is the identity map on C. This means that we can choose h to be $f(gf)^{r-1}$. □

Lemma 4.3. *Let Ω be a set, let m be an integer satisfying $|\Omega| \geq m > 1$, and let S be a transformation semigroup on Ω. For every $X \in \binom{\Omega}{m}$, it holds*

$$\mathrm{scc}(\Gamma(S_X)) = \mathrm{wcc}(\Gamma(S_X)) \leq \mathrm{wcc}(\Gamma^\sharp_m(S)) \leq \mathrm{scc}(\Gamma^\sharp_m(S)). \tag{4.1}$$

Moreover, if S is m-homogeneous, then

$$\mathrm{scc}(\Gamma(S_X)) = \mathrm{wcc}(\Gamma(S_X)) = \mathrm{wcc}(\Gamma^\sharp_m(S)) = \mathrm{scc}(\Gamma^\sharp_m(S)). \tag{4.2}$$

Proof. It is trivial to see that $\mathrm{wcc}(\Gamma(S_X)) = \mathrm{scc}(\Gamma(S_X))$ and $\mathrm{wcc}(\Gamma^\sharp_m(S)) \leq \mathrm{scc}(\Gamma^\sharp_m(S))$. Let us call each strongly/weakly connected component of $\Gamma(S_X)$ a component. To prove Equation (4.1), let us find an injective map ψ from the set of components of $\Gamma(S_X)$ to the set of weakly connected components of $\Gamma^\sharp_m(S)$.

For each $\gamma \in X$, let the weakly connected component of $\Gamma^\sharp_m(S)$ containing (γ, X) be $\psi'(\gamma)$. Take γ_1, γ_2 from the same component of $\Gamma(S_X)$. We may assume that $\gamma_1 g = \gamma_2$ and $X\overline{g} = X$ for some $g \in S$. As $(\gamma_1, X)g^\sharp = (\gamma_1 g, X\overline{g}) = (\gamma_2, X)$, we see that $\psi'(\gamma_1) = \psi'(\gamma_2)$. For each component C of $\Gamma(S_X)$, we can now choose any $\gamma \in C$ and get a well-defined map ψ by setting $\psi(C) = \psi'(\gamma)$. For every weakly connected component $C^\sharp$ of $\Gamma^\sharp_m(S)$, let $\phi(C^\sharp)$ be the set $\{\gamma \in X : (\gamma, X) \in C^\sharp\}$. It is routine to check that $\phi\psi(C) = C$ for every component C of $\Gamma(S_X)$, proving that ψ is injective, as desired.

Assume now S is m-homogeneous. It follows from Lemma 4.2 that $\mathrm{wcc}(\Gamma^\sharp_m(S)) = \mathrm{scc}(\Gamma^\sharp_m(S))$. We thus call each strongly/weakly connected component of $\Gamma^\sharp_m(S)$ simply a component. Since S is m-homogeneous, for every component $C^\sharp$ of $\Gamma^\sharp_m(S)$, we have $\phi(C^\sharp) \neq \emptyset$. This verifies that ϕ and ψ are inverses of each other. We thus get Equation (4.2) and so finish the proof. □

Proof of Theorem 2.1. (1) This is a special case of (2).

(2) This is direct from Lemmas 3.2, 3.6 and 4.1.

(3) Since S is ℓ-homogeneous, it follows from Lemma 4.3 that

$$\mathrm{wcc}(\Gamma(S_A)) = \mathrm{scc}(\Gamma(S_A)) \leq \mathrm{wcc}(\Gamma^\sharp_k(S))$$

and

$$\mathrm{wcc}(\Gamma(S_B)) = \mathrm{scc}(\Gamma(S_B)) = \mathrm{wcc}(\Gamma^\sharp_\ell(S)).$$

It then remains to prove $\mathrm{wcc}(\Gamma^\sharp_\ell(S)) \geq \mathrm{wcc}(\Gamma^\sharp_k(S))$.

We regard $\Omega^\sharp$ as a valuated poset by putting $\mathrm{r}((\alpha, X)) = |X|$ and requiring $(\alpha, X) < (\beta, Y)$ if and only if $\alpha = \beta \in \Omega$ and $X \subsetneq Y \subseteq \Omega$. Note that $S^\sharp \subseteq \mathrm{hEnd}_{\ell,k}(\Omega^\sharp)$. In view of Lemma 3.2(1), it is sufficient to show that $\zeta^{k,\ell}_{\Omega^\sharp}$ is injective.

For each nonnegative integer m and each $\alpha \in \Omega$, let $\Omega^\sharp_{m,\alpha} \doteq \{(\alpha, A) : (\alpha, A) \in \Omega^\sharp_m\}$. Corresponding to the partition $\Omega^\sharp_k = \bigcup_{\alpha\in\Omega} \Omega^\sharp_{k,\alpha}$ and $\Omega^\sharp_\ell = \bigcup_{\beta\in\Omega} \Omega^\sharp_{\ell,\beta}$, the $\Omega^\sharp_k \times \Omega^\sharp_\ell$ matrix $\zeta^{k,\ell}_{\Omega^\sharp}$ is viewed as a partitioned matrix with blocks $\zeta_{\alpha,\beta}$, which are the submatrices with row index set $\Omega^\sharp_{k,\alpha}$ and column index set $\Omega^\sharp_{\ell,\beta}$, where $\alpha, \beta \in \Omega$. Observe that

$$\zeta_{\alpha,\beta} = \begin{cases} \zeta^{k-1,\ell-1}_{\Omega\setminus\{\alpha\}}, & \text{if } \alpha = \beta; \\ 0, & \text{otherwise.} \end{cases}$$

Since $(k - 1) + (\ell - 1) \leq |\Omega| - 1$, it follows from Lemma 4.1 that $\zeta_{\alpha,\alpha} = \zeta^{k-1,\ell-1}_{\Omega\setminus\{\alpha\}}$ is of full row rank for all $\alpha \in \Omega$. This implies that $\zeta^{k,\ell}_{\Omega^\sharp}$ is an injective linear map, as desired. □

Remark 4.4. Let Ω be a set, which is not necessarily finite. Let k and ℓ be two integers with $k \leq \ell \leq k + \ell \leq |\Omega|$. For all $f \in \mathbb{Q}^{\binom{\Omega}{\ell}}$ and $A \in \binom{\Omega}{k}$, we put

$$(\zeta^{\ell,k}_\Omega(f))(A) = \sum_{A\subseteq B} f(B).$$

Making use of Lemma 4.1, it is easy to see that the linear transformation $\zeta^{\ell,k}_\Omega : \mathbb{Q}^{\binom{\Omega}{\ell}}_{\text{fin}} \to \mathbb{Q}^{\binom{\Omega}{k}}_{\text{fin}}$ is always a surjective map. Unfortunately, we do not see if this observation is helpful for getting a possible counterpart of Theorem 2.1(3) for an infinite set Ω.

5 A graded Möbius algebra

Möbius algebra is a semigroup algebra which plays an important role in combinatorics [36, Section 3.6]. Huh and Wang [28] introduced a graded Möbius algebra for geometric lattices. Let L be a finite geometric lattice with rank function (valuation) r. Define a $\mathbb{Q}$-algebra $M(L, \mathbb{Q})$, called the *graded Möbius algebra* of L [28], to be the linear space with L as a $\mathbb{Q}$-basis together with a multiplication given by

$$xy = \begin{cases} x \vee y, & \text{if } \mathrm{r}(x) + \mathrm{r}(y) = \mathrm{r}(x \vee y) \\ 0, & \text{if } \mathrm{r}(x) + \mathrm{r}(y) > \mathrm{r}(x \vee y), \end{cases}$$

and extended by linearity and distributivity. For any nonnegative integers $k \leq \ell$, it is easy to see that the linear map $\xi^{k,\ell}_L$ as specified below is well-defined:

$$\begin{array}{rccl} \xi^{k,\ell}_L : & \mathbb{Q}^{L_k} & \to & \mathbb{Q}^{L_\ell} \\ & \phi & \mapsto & (\sum_{x\in L_1} x)^{\ell-k}\phi. \end{array}$$

We call a finite geometric lattice a *realizable lattice* if it is the matroid lattice of a finite realizable matroid. Here is the main result of Huh and Wang [28, Theorem 6] in their work on solving the realizable case of the top-heavy conjecture of Dowling-Wilson. Huh and Wang [28, Conjecture 7] conjectured that Theorem 5.1 holds without the assumption of realizability.

Theorem 5.1 (Huh and Wang)**.** *Let L be a finite realizable geometric lattice with rank r. For any integers k and ℓ such that $k \le \ell \le k+\ell \le r$, the linear map $\xi_L^{k,\ell}$ is injective.*

Remark 5.2. (1) The partition lattice $\mathrm{P}(\Omega)$ is isomorphic with the flat lattice of the graphic matroid of the complete graph on Ω. Note that a graphic matroid is regular, namely it is representable over every field. This means that finite partition lattices are realizable.

(2) Assume that L is a either a Boolean lattice, or a subspace lattice or a partition lattice. It is easy to see that $\xi_L^{k,\ell} = C_{L,k,\ell}\zeta_L^{k,\ell}$ for some positive integer $C_{L,k,\ell}$ which is determined by L, k and ℓ. Especially, $\xi_L^{k,k+1} = \zeta_L^{k,k+1}$. An important message here is that, $\zeta_L^{k,\ell}$ and $\xi_L^{k,\ell}$, as two $\mathbb{Q}$-linear maps, are either both injective or both non-injective.

Kung [34, Theorem 1.3] verified Conjecture 3.3 for partition lattices of finite sets. We can improve his result a little bit now. When Ω is finite, Lemma 5.3 claims that Conjecture 3.4 holds for partition lattices.

Lemma 5.3. *Let Ω be a set. Let k and ℓ be two integers such that $k \le \ell \le k+\ell \le |\Omega|$. Then* $\ker(\zeta_{\mathrm{P}(\Omega)}^{k,\ell}) = \{0\}$.

Proof. By Lemma 3.6, Theorem 5.1, and Remark 5.2. □

Let Ω be a finite set and let k and ℓ be two integers such that $0 \le k \le \ell \le k+\ell \le |\Omega|$. By virtue of Lemma 5.3, $\ker(\zeta_{\mathrm{P}(\Omega)}^{k,\ell}) = \{0\}$. So, to prove Conjecture 2.10 via Lemma 3.2, we want to have $s^* \in \mathrm{hEnd}_{\ell,k}(\mathrm{P}(\Omega))$ for all $s \in \Omega^\Omega$. It is a pity that what we can have instead is $s^* \in \mathrm{hEnd}_{k,\ell}(\mathrm{P}(\Omega))$ for all $s \in \Omega^\Omega$.

For any transformation g on a set Ω, we associate a partition $\ker_\Omega(g)$ of Ω in which two elements α and β fall into the same part provided $\alpha g = \beta g$, and we call $\ker_\Omega(g)$ the *kernel* of g. Note that $\ker_\Omega(g_1 g_2) = \ker_\Omega(g_2) g_1^*$ for all $g_1, g_2 \in \mathrm{T}(\Omega)$. For any transformation semigroup S on Ω, let $\mathrm{P}^S(\Omega)$ stand for the set $\{\ker_\Omega(s) : s \in S\} = \{\ker_\Omega(\mathrm{Id}_\Omega)s^* : s \in S\}$, and call it the *kernel partition subposet induced by* S. It is clear that $\mathrm{P}^S(\Omega)$ is invariant under the action of the kernel space S^*. Inheriting the rank function on P_Ω, $\mathrm{P}^S(\Omega)$ is still a valuated poset.

For a permutation group, all its elements have the same kernel. For a transformation semigroup, the existence of different kernels may make some arguments for permutation groups invalid. It looks interesting to study the action of the kernel space S^* on the kernel partition subposet $\mathrm{P}^S(\Omega)$.

Example 5.4. Consider the Černý automaton $\mathcal{C}_4 = \Gamma(S, G)$ as illustrated in Figure 3, where $G = \{a, b\}$. All partitions of $\{1, 2, 3, 4\}$, excepting $\{\{0, 2\}, \{1, 3\}\}$ which is displayed in red in Figure 3, belong to $\mathrm{P}^S(\Omega)$. One can check that

$$\mathsf{WS}(S^*|_{\mathrm{P}^S(\Omega)}, \mathrm{P}^S(\Omega)) = (1,1,1,1)\text{and } \mathsf{SS}(S^*|_{\mathrm{P}^S(\Omega)}, \mathrm{P}^S(\Omega)) = (1,2,1,1),$$

both of which being unimodal.

Example 5.5. Let $\Omega = \{1, \ldots, 6\}$ and let $S = \langle r, b\rangle$ be the transformation semigroup acting on Ω as defined in Example 2.5. Simple calculations shows that $\mathrm{P}^S(\Omega)$ is given by

$$\{\{\{1\},\{2\},\{3\},\{4\},\{5\},\{6\}\},\{\{1,2\},\{3,4\},\{5,6\}\},\{\{1,6\},\{2,3\},\{4,5\}\}\}.$$

One can further check that $\mathsf{WS}(S^*|_{\mathrm{P}^S(\Omega)}, \mathrm{P}^S(\Omega)) = \mathsf{SS}(S^*|_{\mathrm{P}^S(\Omega)}, \mathrm{P}^S(\Omega)) = (1, 0, 0, 1)$. If you delete those 0-entries (equivalently, adjusting the rank function for $\mathrm{P}^S(\Omega)$), the resulting sequence $(1, 1)$ is still unimodal.

6 Linear space

6.1 Top-heavy shape

Let n be the size of a nonempty set Ω. Let k and ℓ be two integers satisfying $0 \le k \le \ell \le n$. Let q be a prime power. As q-analogues of the set incidence operator specified in Equation (3.1), we define two linear transformations $M^{k,\ell}_{q,n}\colon \mathbb{Q}^{\mathcal{P}^k_{q,n}} \to \mathbb{Q}^{\mathcal{P}^\ell_{q,n}}$ and $N^{k,\ell}_{q,n}\colon \mathbb{Q}^{\mathcal{A}^k_{q,n-1}} \to \mathbb{Q}^{\mathcal{A}^\ell_{q,n-1}}$ as follows:

$$(M^{k,\ell}_{q,n}(f))(Y) \doteq \sum_{X \le Y, X \in \mathcal{P}^k_{q,n}} f(X),$$

and

$$(N^{k,\ell}_{q,n}(f'))(Y') \doteq \sum_{X' \le Y', X' \in \mathcal{A}^k_{q,n-1}} f(X'),$$

for all $f \in \mathbb{Q}^{\mathcal{P}^k_{q,n}}$, $Y \in \mathcal{P}^\ell_{q,n}$ and $f' \in \mathbb{Q}^{\mathcal{A}^k_{q,n-1}}$, $Y' \in \mathcal{A}^\ell_{q,n-1}$.

Kantor [30, Theorem] obtained a q-analogue of Gottlieb's Theorem [25, Corollary 2], which implies that Conjecture 3.4 holds for affine/projective geometries.

Lemma 6.1 (Kantor)**.** *Let n be a positive integer. Let k and ℓ be two nonnegative integers such that $k \le \ell \le k + \ell \le n$ and let q be any prime power. Then both $M^{k,\ell}_{q,n}$ and $N^{k,\ell}_{q,n-1}$ are injective.*

Proof of Theorem 2.12. Let k and ℓ be two integers such that $0 \le k \le \ell \le k + \ell \le n$. Note that $S^{\mathcal{P}} \subseteq \mathrm{hEnd}_{k,\ell}(\mathcal{P}_{q,n})$ and $T^{\mathcal{A}} \subseteq \mathrm{hEnd}_{k,\ell}(\mathcal{A}_{q,n-1})$. Since both $\mathcal{P}_{q,n}$ and $\mathcal{A}_{q,n-1}$ are (ℓ, k)-finite, the result follows readily from Lemmas 3.2, 3.6 and 6.1. □

6.2 Duality: A result of Stanley

First Proof of Theorem 2.16. Let F be a field and Ω be a set. For each linear subspace $U \le F^\Omega$, let $U^\perp$ be the subspace of F^Ω given by

$$U^\perp \doteq \{f \in F^\Omega : \sum_{\omega \in \Omega} f(\omega)g(\omega) = 0 \text{ for all } g \in U\}.$$

Take a matrix $A \in F^{\Omega\times\Omega}$ and record its transpose by $A^\top$. For any $f \in F^\Omega$, which can be thought of as a row vector indexed by Ω, the image of f under the action of A, written as fA, can be thought of as the product of the row vector f and the matrix A. The matrix A induces a transformation $\widehat{A}$ on $\mathrm{Gr}(F^\Omega)$ such that $U \in \mathrm{Gr}(F^\Omega)$ is sent to

$$\begin{array}{ccc} f & \xrightarrow{\;M_{q,n}^{k,\ell}\;} & M_{q,n}^{k,\ell}(f) \\ {\scriptstyle g^{\ddagger,k}}\downarrow & & \downarrow{\scriptstyle g^{\ddagger,\ell}} \\ fg^{\ddagger,k} & \xrightarrow[\;M_{q,n}^{k,\ell}\;]{} & M_{q,n}^{k,\ell}(fg^{\ddagger,k}) \end{array}$$

Figure 6: The incidence operator intertwines with every linear isomorphism g.

$U\widehat{A} \doteq \{fA : f \in U\}$. It is easy to see that for any $U, W \in \mathrm{Gr}(V)$ we have the implication

$$U\widehat{A} = W \Longrightarrow W^{\perp}\widehat{A^{\top}} \leq U^{\perp}; \tag{6.1}$$

especially, when $A \in \mathrm{GL}_n(F)$ it holds

$$U\widehat{A} = W \Longleftrightarrow W^{\perp}\widehat{A^{\top}} = U^{\perp}. \tag{6.2}$$

According to Taussky and Zassenhaus [63, Theorem 1], we can find $P \in \mathrm{GL}_n(F)$ such that $P = P^{\top}$ and $A^{\top} = PAP^{-1}$. This means that Equations (6.1), (6.2) become

$$U\widehat{A} = W \Longrightarrow (W^{\perp}\widehat{P})\widehat{A} \leq U^{\perp}\widehat{P}$$

and

$$U\widehat{A} = W \Longleftrightarrow (W^{\perp}\widehat{P})\widehat{A} = U^{\perp}\widehat{P}, \tag{6.3}$$

respectively. It is well-known that q-binomial coefficients (Gaussian coefficients) occur in pairs, namely in any n-dimensional linear space over a finite field, the number of k-dimensional subspaces is equal to the number of $(n-k)$-dimensional subspaces [24, Proposition 5.31] [60, Section 3]. In general, as a consequence of Equation 6.3, for any $A \in \mathrm{GL}_n(F)$, the number of k-dimension subspaces of F^n fixed by $\widehat{A}$ equals to the number of $(n-k)$-dimension subspaces of F^n fixed by $\widehat{A}$. If F is a finite field and G is a subgroup of $\mathrm{GL}_n(F)$, in view of the Orbit Counting Lemma (also known as Burnside's Lemma), the above discussion leads to a proof of Theorem 2.16. □

Second Proof of Theorem 2.16. Let $G \leq \mathrm{GL}_n(\mathbb{F}_q)$ and let k be a positive integer fulfilling $k \leq \frac{n}{2}$. The group G can be seen as a permutation group acting on both $\mathrm{Gr}(n-k, \mathbb{F}_q^n) = \mathcal{P}_{q,n}^{k}$ and $\mathrm{Gr}(n-k, \mathbb{F}_q^n) = \mathcal{P}_{q,n}^{n-k}$; we use W_k and W_{n-k} for the two permutation modules accordingly. From Lemma 6.1, we see that $M_{q,n}^{k,n-k}$ is an $\mathbb{F}_q$-linear isomorphism from $\mathcal{P}_{q,n}^{k}$ to $\mathcal{P}_{q,n}^{n-k}$. From Figure 5 and Example 3.7, we have the commutative diagram in Figure 6 for $2 \leq k \leq \frac{n}{2}$; assuming that g comes from the group G, clearly our deduction of Figure 5 shows that Figure 6 is also valid for $k = 1$. This then shows that W_k and W_{n-k} are isomorphic permutation modules for G. In particular, the number of orbits of G on $\mathcal{P}_{q,n}^{k}$ and the number of its orbits on $\mathcal{P}_{q,n}^{n-k}$ must be equal. □

By examining the proofs of Theorem 2.16, we intend to understand the challenge of extending some results on group actions to that on semigroup actions. The above two proofs apply to a set of invertible linear operators over finite linear spaces. If we have a single linear operator $A \in \mathrm{Mat}_n(F)$, by considering its action on the linear space obtained by "collapsing" the eventual kernel of A to zero, we may somehow still say something similar to above. When we have a subsemigroup S of the full linear transformation monoid acting on a finite linear space, different elements of S may have different eventual kernels and that makes it nontrivial to glean global information about the semigroup action.

7 Vámos matroid

Proof of Example 2.18. A simple calculation shows that $\ker(\zeta^{k,\ell}_{\mathrm{F}(M)}) = \{0\}$ for $(k,\ell) \in \{(1,2),(2,3)\}$. Let $f \in S$ and let $f'\colon \mathrm{F}(M) \to \mathrm{F}(M)$ be the map sending each flat $X \in \mathrm{F}(M)$ to the minimum flat containing $X\overline{f}$ in M. By Lemma 3.2, we will be done if we can show that $f' \in \mathrm{hEnd}_{\ell,k}(\mathrm{F}(M))$ for $(k,\ell) = (1,2),(2,3)$.

If we know that f is a bijection or that $|\mathscr{E}_M\overline{f}| \leq 2$, we can easily check that $f' \in \mathrm{hEnd}_{\ell,k}(\mathrm{F}(M))$, as wanted. We intend to find a contradiction under the hypothesis that neither of them holds.

By assumption, we can take three distinct elements x, y, z in $\mathscr{E}_M\overline{f}$ such that $|xf^{-1}| \geq 2$. Let A be the minimum flat containing $\{x,y,z\}$ and let $B = Af^{-1}$. Observe that $|A| \in \{3,4\}$. Since f is a strong map, B is a flat containing at least four elements and so $|B| \in \{4,8\}$.

Case 1: $|B| = 8$.

Take any $X \in \binom{A}{2}$. Note that X must be a flat and thus so is Xf^{-1}. Since $|\mathscr{E}_M\overline{f}| \geq 3$, we deduce that the flat Xf^{-1} is not equal to $\mathscr{E}_M$ and so $|Xf^{-1}| \leq 4$. Considering that $|A| \in \{3,4\}$, we find that $|A| = 4$ and each element in A has two perimages under f. Note that every element in $\binom{A}{2}$ is a flat. It follows that $\{Xf^{-1} : X \in \binom{A}{2}\}$ is a set of six distinct flats and each of them contains four elements, which cannot happen for the Vámos matroid M.

Case 2: $|B| = 4$.

Thanks to the assumption of $|B| = 4$, we see that $C = \{x,y\}$ is a flat in M satisfying $|Cf^{-1}| = 3$. Note that no three-elements subset of any four-elements flat in M can be a flat. This means that Cf^{-1} is not a flat, violating the assumption that f is a strong map. □

8 Concluding remarks

We have discussed some top-heavy phenomena for transformation semigroups acting on Boolean semirings, affine/projective geometries, and flat lattice of Vámos matroid; see Theorems 2.1 and 2.12 and Example 2.18. But some problems remain, say Question 2.2, 2.3 and 2.8, Conjecture 2.10 and Question 2.11, and Question 2.15. Our work relies on various injectivity results, say Lemmas 4.1, 5.3 and 6.1, which can all be deduced from Theorem 5.1 and Remark 5.2. We may think of Conjecture 3.4 as a natural companion to [28, Conjecture 7]. Since our results on comparing the number of components inside P_k and that of P_ℓ for various valuated posets P come from the injectivity of the relevant incidence operators (Lemma 3.2), we indeed have an injective map from components of P_k to that of P_ℓ which respects the poset structure. It is noteworthy that we do not find any general results on the unimodality of the strong/weak shape of a semigroup action on a valuated poset. We are wondering if the theory on the Lefschetz properties of commutative graded algebras [27] will be useful in this line of research.

Penttila and Siciliano [47, Lemma 3.1] suggested a machinery (Lemma 3.6) to remove certain finiteness assumption. But we do not see any way to solve Question 1.2 and 2.17 by this means. Since there are many other approaches to go from finite to infinite [53], it will not be a surprise if Question 1.2 has a positive solution as simple as that for Theorem 2.12. Here is another such question. By our definition, a valuated poset only has nonnegative integers as ranks of its elements. We may allow ranks to be any (not necessarily finite) cardinal number and then examine all the work in this paper again. A few results of this kinds from the literature have been addressed at the end of Section 2.1.

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The number of rooted forests in circulant graphs*

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Abstract

In this paper, we develop a new method to produce explicit formulas for the number $f_G(n)$ of rooted spanning forests in the circulant graphs $G = C_n(s_1, s_2, \ldots, s_k)$ and $G = C_{2n}(s_1, s_2, \ldots, s_k, n)$. These formulas are expressed through Chebyshev polynomials. We prove that in both cases the number of rooted spanning forests can be represented in the form $f_G(n) = p\, a(n)^2$, where $a(n)$ is an integer sequence and p is a certain natural number depending on the parity of n. Finally, we find an asymptotic formula for $f_G(n)$ through the Mahler measure of the associated Laurent polynomial $P(z) = 2k+1-\sum_{i=1}^{k}(z^{s_i}+z^{-s_i})$.

Keywords: Rooted tree, spanning forest, circulant graph, Laplacian matrix, Chebyshev polynomial, Mahler measure.

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1 Introduction

The famous Kirchhoff's Matrix Tree Theorem [15] states that the number of spanning trees in a graph can be expressed as the product of its non-zero Laplacian eigenvalues divided by the number of vertices. Since then, a lot of papers on enumeration of spanning trees for various classes of graphs were published. In particular, explicit formulae were derived for complete multipartite graphs [1, 5], almost complete graphs [33], wheels [3], fans [12], prisms [2], ladders [26], Moebius ladders [27], lattices [28], anti-prisms [31],

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complete prisms [25] and for many other families. For the circulant graphs some explicit and recursive formulae are given in [8, 23, 34, 35].

Along with the number of spanning trees in a given graph one can be interested in the number of rooted spanning forests in the graph. According to the classical result [14] (see also more recent paper [7, 16]) this value can be found with the use of determinant of the matrix $\det(I + L)$. Here L is the Laplacian matrix of the graph. At the same time, it is known very little about analytic formulas for the number of spanning forests. In particular, P. Chebotarev [6] enumerated the rooted spanning forests in path and cyclic graphs and O. Knill [16] proved that the number of rooted spanning forests in the complete graph K_n on n vertices is equal to $(n + 1)^{n-1}$. The rooted spanning forests in complete bipartite graphs were enumerated in [13]. Explicit formulas for the number of rooted spanning forests for cyclic, star, line and some other graphs were given by [16]. As for the number of unrooted forests, it has much more complicated structure [4, 19, 32].

Starting with Boesch and Prodinger [3] the idea to apply Chebyshev polynomials for counting various invariants of graphs arose. This idea provided a way to find complexity, that is the number of spanning tress, of circulant graphs and their natural generalisations in [8, 17, 23, 24, 35].

Recently, asymptotical behavior of complexity for some families of graphs was investigated from the point of view of so called Malher measure [11, 29]. Mahler measure of a polynomial $P(z)$, with complex coefficients, is the absolute value of the product of all the roots of $P(z)$ whose modulus is greater than 1 multiplied by the leading coefficient. For general properties of the Mahler measure see survey [30] and monograph [10].

The purpose of this paper is to present new formulas for the number of rooted spanning forests in circulant graphs and investigate their arithmetical properties and asymptotics.

We arrange the paper in the following way. First, in Sections 3 and 4, we present new explicit formulas for the number of spanning forests in the undirected circulant graphs $C_n(s_1, s_2, \ldots, s_k)$ and $C_{2n}(s_1, s_2, \ldots, s_k, n)$ of even and odd valency respectively. They will be given in terms of Chebyshev polynomials. Next, in Section 5, some arithmetic properties of the number of spanning forests are investigated. More precisely, it is shown that the number of spanning forests of the circulant graph G can be represented in the form $f_G(n) = p\, a(n)^2$, where $a(n)$ is an integer sequence and p is a certain natural number. At last, in Section 6, we use explicit formulas for $f_G(n)$ in order to find its asymptotics in terms of Mahler measure of the associated polynomials. For circulant graphs of even valency the associated polynomial is $P(z) = 2k + 1 - \sum_{j=1}^{k}(z^{s_j} + z^{-s_j})$. In this case (Theorem 6.1) we have $f_G(n) \sim A^n$, $n \to \infty$, where A is the Mahler measure of $P(z)$. For circulant graphs of odd valency we use the polynomial $R(z) = P(z)(P(z) + 2)$. Then the respective asymptotics (Theorem 6.2) is $f_G(n) \sim K^n$, $n \to \infty$, where $K = M(R)$. In the last Section 7, we illustrate the obtained results by a series of examples.

2 Basic definitions and preliminary facts

Consider a finite and not necessary connected graph G without loops. A *rooted tree* is a tree with one marked vertex called *root*. A *rooted forest* is a graph whose connected components are rooted trees. A *rooted spanning forest* F in the graph G is a subgraph that is a rooted forest containing all the vertices of G. We denote the vertex and edge set of G by $V(G)$ and $E(G)$, respectively. Given $u, v \in V(G)$, we set a_{uv} to be equal to the number of edges between vertices u and v. The matrix $A = A(G) = (a_{uv})_{u, v\in V(G)}$ is called *the* *adjacency matrix* of the graph G. The degree $d(v)$ of a vertex $v \in V(G)$ is defined by $d(v) = \sum_{u \in V(G)} a_{uv}$. Let $D = D(G)$ be the diagonal matrix indexed by the elements of $V(G)$ with $d_{vv} = d(v)$. The matrix $L = L(G) = D(G) - A(G)$ is called *the Laplacian matrix*, or simply *Laplacian*, of the graph G.

By I_n we denote the identity matrix of order n.

Denote by $\chi_G(\lambda) = \det(\lambda I_n - L(G))$ the characteristic polynomial of the Laplacian matrix of a graph G on n vertices. Its extended form is

$$\chi_G(\lambda) = c_1\lambda + \ldots + c_{n-1}\lambda^{n-1} + \lambda^n.$$

The theorem by Kelmans and Chelnokov [14] states that the absolute value of coefficient c_k of $\chi_G(\lambda)$ coincides with the number of rooted spanning k-forests in the graph G. Since all the Laplacian eigenvalues of G are non-negative, the sequence c_k is alternating. So, the number of rooted spanning forests of the graph G can be found by the formula

$$\begin{aligned} f_G(n) &= f_1 + f_2 + \ldots + f_n = |c_1 - c_2 + c_3 - \ldots + (-1)^{n-1}| \\ &= (-1)^n \chi_G(-1) = \det(I_n + L(G)). \end{aligned} \tag{2.1}$$

This result was independently obtained by many authors (P. Chebotarev and E. Shamis [7] O. Knill [16] and others).

Let $s_1, s_2, \ldots, s_k$ be integers such that $1 \le s_1 < s_2 < \ldots < s_k \le \frac{n}{2}$. The graph $C_n(s_1, s_2, \ldots, s_k)$ with n vertices $0, 1, 2, \ldots,\ n-1$ is called *circulant graph* if the vertex i, $0 \le i \le n-1$ is adjacent to the vertices $i \pm s_1, i \pm s_2, \ldots, i \pm s_k \pmod n$. When $s_k < \frac{n}{2}$ all vertices of the graph have even degree $2k$. If n is even and $s_k = \frac{n}{2}$, then all vertices have odd degree $2k-1$. Two circulant graphs $C_n(s_1, s_2, \ldots, s_k)$ and $C_n(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_k)$ of the same order are said to be conjugate by multiplier if there exists an integer r coprime to n such that $\{\tilde{s}_1, \tilde{s}_2, ..., \tilde{s}_k\} = \{rs_1, rs_2, \ldots, rs_k\}$ as subsets of $\mathbb{Z}_n$. In this case, the graphs are isomorphic, with multiplication by the unit $r \pmod n$ giving an isomorphism.

We call an $n \times n$ matrix *circulant,* and denote it by $\mathrm{circ}(a_0, a_1, \ldots, a_{n-1})$ if it is of the form

$$\mathrm{circ}(a_0, a_1, \ldots, a_{n-1}) = \begin{pmatrix} a_0 & a_1 & a_2 & \ldots & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \ldots & a_{n-2} \\ & \vdots & & \ddots & \vdots \\ a_1 & a_2 & a_3 & \ldots & a_0 \end{pmatrix}$$

It is easy to see that adjacency and Laplacian matrices of the circulant graph are circulant matrices. The converse is also true. If the Laplacian matrix of a graph is circulant then the graph is also circulant.

Recall [9] that the eigenvalues of matrix C $= \mathrm{circ}(a_0, a_1, \ldots, a_{n-1})$ are given by the following simple formulas $\lambda_j = P(\varepsilon_n^j)$, $j = 0, 1, \ldots, n-1$, where $P(x) = a_0 + a_1 x + \ldots + a_{n-1}x^{n-1}$ and ε_n is an order n primitive root of unity. Moreover, the circulant matrix T $= \mathrm{circ}(0, 1, 0, \ldots, 0)$ is the matrix representation of the shift operator $T\colon (x_0, x_1, \ldots, x_{n-2}, x_{n-1}) \to (x_1, x_2, \ldots, x_{n-1}, x_0)$.

Let $P(z) = a_0 z^s + \ldots + a_s = a_0 \prod_{i=1}^{s}(z - \alpha_i)$ be a nonconstant polynomial with complex coefficients. Then, following Mahler [21] its *Mahler measure* is defined to be

$$M(P) := \exp(\int_0^1 \log |P(e^{2\pi \mathrm{i} t})| dt), \tag{2.2}$$

the geometric mean of $|P(z)|$ for z on the unit circle. However, $M(P)$ had appeared earlier in a paper by Lehmer [18], in an alternative form

$$M(P) = |a_0| \prod_{|\alpha_i|>1} |\alpha_i|. \tag{2.3}$$

See, for example [10], for the proof of equivalence of these definitions.

The concept of Mahler measure can be naturally extended to the class of Laurent polynomials $P(z) = a_0 z^{p+s} + a_1 z^{p+s-1} + \ldots + a_{s-1} z^{s+1} + a_s z^s = a_0 z^p \prod_{i=1}^{s}(z-\alpha_i)$, where $a_0 \neq 0,$ s is a positive integer and p is an arbitrary and not necessarily positive integer.

Let $T_n(z) = \cos(n \arccos z)$ be the Chebyshev polynomial of the first kind. The following property of the Chebyshev polynomials is widely used in the paper $T_n(\frac{1}{2}(z+z^{-1})) = \frac{1}{2}(z^n + z^{-n})$. See [22] for more properties of Chebyshev polynomials.

3 The number of rooted spanning forests of circulant graphs of even valency

The aim of this section is to find new formulas for the numbers of rooted spanning forests of circulant graph $C_n(s_1, s_2, \ldots, s_k)$ in terms of Chebyshev polynomials. Here and below, we will use G to denote the circulant graph under consideration.

Theorem 3.1. *The number of rooted spanning forests $f_G(n)$ in the circulant graph $G = C_n(s_1, s_2, \ldots, s_k)$, $1 \leq s_1 < s_2 < \ldots < s_k < \frac{n}{2}$, is given by the formula*

$$f_G(n) = \prod_{p=1}^{s_k} |2T_n(w_p) - 2|,$$

where w_p, $p = 1, 2, \ldots, s_k$ are all the roots of the algebraic equation $\sum_{j=1}^{k}(2T_{s_j}(w) - 2) = 1$ and $T_s(w)$ is the Chebyshev polynomial of the first kind.

Proof. The number of rooted spanning forests of the graph G can be found by the formula $f_G(n) = \det(I_n + L(G))$. The latter value is equal to the product of all eigenvalues of the matrix $I_n + L(G)$. We denote by $T = \text{circ}(0, 1, \ldots, 0)$ the $n \times n$ cyclic shift operator. Consider the Laurent polynomial $P(z) = 2k + 1 - \sum_{i=1}^{k}(z^{s_i} + z^{-s_i})$. Then the matrix $I_n + L(G)$ has the following form

$$I_n + L(G) = P(T) = (2k+1)I_n - \sum_{i=1}^{k}(T^{s_i} + T^{-s_i}).$$

The eigenvalues of circulant matrix T are ε_n^j, $j = 0, 1, \ldots, n-1$, where $\varepsilon_n = e^{\frac{2\pi \mathrm{i}}{n}}$. Since all of them are distinct, the matrix T is similar to the matrix $\mathbb{T} = \text{diag}(1, \varepsilon_n, \ldots, \varepsilon_n^{n-1})$ with diagonal entries $1, \varepsilon_n, \ldots, \varepsilon_n^{n-1}$. So the matrix $I_n + L(G)$ is similar to the diagonal matrix $P(\mathbb{T})$. This essentially simplifies the problem of finding eigenvalues of $I_n + L(G)$. Indeed, let λ be an eigenvalue of $P(\mathbb{T})$ and x be the corresponding eigenvector. Then we have the following system of linear equations

$$((2k+1-\lambda)I_n - \sum_{i=1}^{k}(\mathbb{T}^{s_i} + \mathbb{T}^{-s_i}))x = 0.$$

Recall that the matrices under consideration are diagonal and the $(j+1, j+1)$-th entry of $\mathbb{T}$ is equal to ε_n^j, where $\varepsilon_n = e^{\frac{2\pi i}{n}}$. Then, for any $j = 0, \dots, n-1$, matrix $P(\mathbb{T})$ has an eigenvalue $\lambda_j = P(\varepsilon_n^j) = 2k + 1 - \sum_{i=1}^{k}(\varepsilon_n^{js_i} + \varepsilon_n^{-js_i})$. Hence we have

$$f_G(n) = \prod_{j=0}^{n-1} P(\varepsilon_n^j). \tag{3.1}$$

To continue the proof of the theorem we need the following lemma.

Lemma 3.2.

$$\prod_{j=0}^{n-1} P(\varepsilon_n^j) = \prod_{p=1}^{s_k} |2T_n(w_p) - 2|,$$

where w_p, $p = 1, \dots, s_k$ *are all the roots of the algebraic equation* $\sum_{j=1}^{k}(2T_{s_j}(w) - 2) = 1$.

To prove the above formula we introduce integer polynomial $\widetilde{P}(z) = -z^{s_k}P(z)$. This is a monic polynomial with the same roots as $P(z)$ and its degree is $2s_k$. As $P(z) = P(\frac{1}{z})$, its roots look like $z_1, \frac{1}{z_1}, \dots, z_{s_k}, \frac{1}{z_{s_k}}$.

We have $\prod_{j=0}^{n-1} P(\varepsilon_n^j) = \prod_{j=0}^{n-1}(-\varepsilon_n^{-s_k j}\widetilde{P}(\varepsilon_n^j)) = (-1)^{(s_k+1)(n+1)-1}\prod_{j=0}^{n-1}\widetilde{P}(\varepsilon_n^j)$. Recall one of the basic properties of resultants

$$\text{Res}\,(\widetilde{P}(z), \widetilde{Q}(z)) = (-1)^{\deg(\widetilde{P})\deg(\widetilde{Q})}\text{Res}\,(\widetilde{Q}(z), \widetilde{P}(z)),$$

where $\widetilde{P}(z)$ and $\widetilde{Q}(z)$ are monic polynomials of degree $\deg(\widetilde{P})$ and $\deg(\widetilde{Q})$ respectively. We set $\widetilde{Q}(z) = z^n - 1$ and note that $\deg(\widetilde{P}) = 2s_k$ is even. Then we obtain

$$\begin{aligned}\prod_{j=0}^{n-1}\widetilde{P}(\varepsilon_n^j) &= \text{Res}\,(\widetilde{P}(z), z^n - 1) = \text{Res}\,(z^n - 1, \widetilde{P}(z)) \\ &= \prod_{z:\widetilde{P}(z)=0}(z^n - 1) = \prod_{z:P(z)=0}(z^n - 1) \\ &= \prod_{p=1}^{s_k}(z_p^n - 1)(z_p^{-n} - 1) = (-1)^{s_k}\prod_{p=1}^{s_k}(2T_n(w_p) - 2).\end{aligned}$$

We used the identity $T_n(\frac{1}{2}(z + z^{-1})) = \frac{1}{2}(z^n + z^{-n})$. Here $w_p = \frac{1}{2}(z_p + \frac{1}{z_p})$, $p = 1, \dots, s_k$. These numbers are the roots of algebraic equation $\sum_{j=1}^{k}(2T_{s_j}(w) - 2) = 1$.

To finish the proof of the theorem we use Lemma 3.2 and take absolute value of the righthand side of the Equation 3.1. □

4 The number of rooted spanning forests in circulant graphs of odd valency

This section is devoted to investigation of the numbers of rooted spanning forests in circulant graph $C_{2n}(s_1, s_2, \dots, s_k, n)$ in terms of Chebyshev polynomials.

Theorem 4.1. *Let $C_{2n}(s_1, s_2, \ldots, s_k, n)$, $1 \le s_1 < s_2 < \ldots < s_k < n$, be a circulant graph of odd degree. Then the number $f_G(n)$ of rooted spanning forests in the graph $G = C_{2n}(s_1, s_2, \ldots, s_k, n)$ is given by the formula*

$$f_G(n) = \prod_{p=1}^{s_k}(2T_n(u_p) - 2)(2T_n(v_p) + 2),$$

where the numbers u_p and v_p, $p = 1, 2, \ldots, s_k$ are respectively the roots of the algebraic equations $Q(u) - 1 = 0$ and $Q(v) + 1 = 0$, $Q(w) = 2k + 2 - 2\sum_{i=1}^{k} T_{s_i}(w)$ and $T_s(w)$ is the Chebyshev polynomial of the first kind.

Proof. In order to find the number of rooted spanning forests $f_G(n)$ in the graph G we need to find the determinant $\det(I_{2n} + L(G))$. The matrix $I_{2n} + L(G)$ can be represented in the form

$$I_{2n} + L(G) = (2k + 2)I_{2n} - \sum_{j=1}^{k}(T^{s_j} + T^{-s_j}) - T^n,$$

where T is $2n \times 2n$ circulant matrix circ$(0, 1, 0, \ldots, 0)$. The eigenvalues of circulant matrix T are ε_{2n}^{j}, $j = 0, 1, \ldots, 2n - 1$, where $\varepsilon_{2n} = e^{\frac{2\pi i}{2n}}$. Since all of them are distinct, the matrix T is similar to the matrix $\mathbb{T} = \mathrm{diag}(1, \varepsilon_{2n}, \ldots, \varepsilon_{2n}^{2n-1})$ with diagonal entries $1, \varepsilon_{2n}, \ldots, \varepsilon_{2n}^{2n-1}$. To find the determinant $\det(I_{2n} + L(G))$ we use the product of all eigenvalues of matrix $I_{2n} + L(G)$. The matrix $I_{2n} + L(G)$ is similar to the diagonal matrix with eigenvalues

$$\lambda_j = 2k + 2 - \sum_{l=1}^{k}(\varepsilon_{2n}^{j\,s_l} + \varepsilon_{2n}^{-j\,s_l}) - \varepsilon_{2n}^{jn},\ j = 0, 1, \ldots, 2n - 1.$$

All of them are non-zero.

Consider the following Laurent polynomial $P(z) = 2k + 2 - \sum_{i=1}^{k}(z^{s_i} + z^{-s_i})$. Since $\varepsilon_{2n}^{n} = -1$, we can write $\lambda_j = P(\varepsilon_{2n}^{j}) - 1$ if j is even and $\lambda_j = P(\varepsilon_{2n}^{j}) + 1$ if j is odd. By Formula 2.1 we have

$$\begin{aligned} f_G(n) &= \prod_{j=0}^{2n-1} \lambda_j = \prod_{s=0}^{n-1}(P(\varepsilon_{2n}^{2s}) - 1)\prod_{s=0}^{n-1}(P(\varepsilon_{2n}^{2s+1}) + 1) \\ &= \prod_{s=0}^{n-1}(P(\varepsilon_{2n}^{2s}) - 1)\frac{\prod_{p=0}^{2n-1}(P(\varepsilon_{2n}^{p}) + 1)}{\prod_{s=0}^{n-1}(P(\varepsilon_{2n}^{2s}) + 1)} \\ &= \prod_{s=0}^{n-1}(P(\varepsilon_{n}^{s}) - 1)\frac{\prod_{p=0}^{2n-1}(P(\varepsilon_{2n}^{p}) + 1)}{\prod_{s=0}^{n-1}(P(\varepsilon_{n}^{s}) + 1)}. \end{aligned}$$

By making use of Lemma 3.2 and arguments from the proof of Theorem 3.1 we obtain

(i) $\prod_{s=0}^{n-1}(P(\varepsilon_{n}^{s}) - 1) = (-1)^{n\,(s_k+1)}\prod_{p=1}^{s_k}(2T_n(u_p) - 2)$,

(ii) $\prod_{s=0}^{n-1}(P(\varepsilon_{n}^{s}) + 1) = (-1)^{n\,(s_k+1)}\prod_{p=1}^{s_k}(2T_n(v_p) - 2)$, and

(iii) $\prod_{p=0}^{2n-1}(P(\varepsilon_{2n}^{p}) + 1) = \prod_{p=1}^{s_k}(2T_{2n}(v_p) - 2)$, where u_p and v_p are the same as in the statement of the theorem. Hence,

$$f_G(n) = \prod_{p=1}^{s_k}(2T_n(u_p) - 2)\prod_{p=1}^{s_k}\frac{T_{2n}(v_p) - 1}{T_n(v_p) - 1}.$$

Finally, taking into account the identity $T_{2n}(w)-1 = 2(T_n(w)-1)(T_n(w)+1)$ we obtain

$$f_G(n) = \prod_{p=1}^{s_k}(2T_n(u_p) - 2)(2T_n(v_p) + 2). \qquad \square$$

5 Arithmetic properties of the number of rooted spanning forests for circulant graphs

It has been proved in the paper [23] that the number of spanning trees $\tau(n)$ in circulant graph $C_n(s_1, s_2, \ldots, s_k)$ is given by the formula $\tau(n) = p\,n\,a(n)^2$, where $a(n)$ is an integer sequence and p is a natural number depending only on the parity of n. The aim of the next theorem is to find a similar property for the number of rooted spanning forests.

Recall that any positive integer p can be uniquely represented in the form $p = q\,r^2$, where p and q are positive integers and q is square-free. We will call q the *square-free part* of p.

Theorem 5.1. *Let $f_G(n)$ be the number of rooted spanning forests in the circulant graph*

$$C_n(s_1, s_2, \ldots, s_k),\ 1 \le s_1 < s_2 < \ldots < s_k < \frac{n}{2}.$$

Denote by p the number of odd elements in the sequence $s_1, s_2, \ldots, s_k$ and let q be the square-free part of $4p + 1$. Then there exists an integer sequence $a(n)$ such that

(1) $f_G(n) = a(n)^2$, *if n is odd;*

(2) $f_G(n) = q\,a(n)^2$, *if n is even.*

Proof. The number of odd elements in the sequence $s_1, s_2, \ldots, s_k$ is counted by the formula $p = \sum_{i=1}^{k} \frac{1-(-1)^{s_i}}{2}$.

We already know that all eigenvalues of the $I_n + L(G)$ are given by the formulas $\lambda_j = P(\varepsilon_n^j)$, $j = 0, \ldots, n-1$, where $P(z) = 2k+1-\sum_{i=1}^{k}(z^{s_i} + z^{-s_i})$ and $\varepsilon_n = e^{\frac{2\pi \mathrm{i}}{n}}$. We note that $\lambda_{n-j} = P(\varepsilon_n^{n-j}) = P(\varepsilon_n^j) = \lambda_j$.

Since $\lambda_0 = P(\varepsilon_n^0) = P(1) = 1$ by Formula 2.1 we have $f_G(n) = \prod_{j=1}^{n-1}\lambda_j$. Since $\lambda_{n-j} = \lambda_j$, we obtain $f_G(n) = (\prod_{j=1}^{\frac{n-1}{2}}\lambda_j)^2$ if n is odd and $f_G(n) = \lambda_{\frac{n}{2}}(\prod_{j=1}^{\frac{n}{2}-1}\lambda_j)^2$ if n is even. We note that each algebraic number λ_j comes with all its Galois conjugates [20]. So, the numbers $b(n) = \prod_{j=1}^{\frac{n-1}{2}}\lambda_j$ and $c(n) = \prod_{j=1}^{\frac{n}{2}-1}\lambda_j$ are integers. Also, for even n we have $\lambda_{\frac{n}{2}} = 2k + 1 - \sum_{i=1}^{k}((-1)^{s_i} + (-1)^{-s_i}) = 1 + 2\sum_{i=1}^{k}(1 - (-1)^{s_i}) = 4p + 1$. Hence, $f_G(n) = b(n)^2$ if n is odd and $f_G(n) = (4p + 1)\,c(n)^2$ if n is even. Let q be the square-free part of $4p + 1$ and $4p + 1 = q\,r^2$. Setting $a(n) = b(n)$ in the first case and $a(n) = r\,c(n)$ in the second, we conclude that number $a(n)$ is always integer which completes the proof. $\square$

The following theorem clarifies some number-theoretical properties of the number of rooted spanning forests $f_G(n)$ for circulant graphs of odd valency.

Theorem 5.2. *Let $f_G(n)$ be the number of rooted spanning forests in the circulant graph*

$$G = C_{2n}(s_1, s_2, \ldots, s_k, n),\ 1 \le s_1 < s_2 < \ldots < s_k < n.$$

Denote by p the number of odd elements in the sequence $s_1, s_2, \ldots, s_k$. Let q be the square-free part of $4p + 1$ and r be the square-free part of $4p + 3$. Then there exists an integer sequence $a(n)$ such that

(1) $f_G(n) = q\, a(n)^2$, *if n is even;*

(2) $f_G(n) = r\, a(n)^2$, *if n is odd.*

Proof. The number p of odd elements in the sequence $s_1, s_2, \ldots, s_k$ is equal to $\sum_{i=1}^{k} \frac{1-(-1)^{s_i}}{2}$. The eigenvalues of the matrix $I_{2n} + L(G)$ are given by the formulas

$$\lambda_j = P(\varepsilon_{2n}^j) - (-1)^j,\ j = 0, 1, \ldots, 2n - 1,$$

where $P(z) = 2k + 2 - \sum_{l=1}^{k}(z^{s_l} + z^{-s_l})$ and $\varepsilon_{2n} = e^{\frac{\pi \mathrm{i}}{n}}$.

Since $\lambda_0 = P(1) - 1 = 1$ by the Formula 2.1 we have $f_G(n) = \prod_{j=1}^{2n-1} \lambda_j$. Since $\lambda_{2n-j} = \lambda_j$, we obtain $f_G(n) = \lambda_n (\prod_{j=1}^{n-1} \lambda_j)^2$, where $\lambda_n = P(-1) - (-1)^n$. Now we have

$$\lambda_n = 2k + 2 - (-1)^n - 2\sum_{l=1}^{k}(-1)^{s_l} = 2 - (-1)^n + 4\sum_{l=1}^{k}\frac{1-(-1)^{s_l}}{2} = 4\,p + 2 - (-1)^n.$$

So, $\lambda_n = 4\,p + 1$, if n is even and $\lambda_n = 4\,p + 3$, if n is odd. We note that each algebraic number λ_j comes into the product $\prod_{j=1}^{n-1} \lambda_j$ together with all its Galois conjugates, so the number $c(n) = \prod_{j=1}^{n-1} \lambda_j$ is an integer [20].

Hence, $f_G(n) = (4\,p + 1)c(n)^2$, if n is even and $f_G(n) = (4\,p + 3)\,c(n)^2$, if n is odd. Let q and r be the square-free parts of $4\,p + 1$ and of $4p + 3$ respectively. Then for some integers x and y we have $4\,p + 1 = q\,x^2$ and $4\,p + 3 = r\,y^2$.

Now, the integer number $f_G(n)$ can be represented in the form

(1) $f_G(n) = q\,(x\,c(n))^2$ if n is even and

(2) $f_G(n) = r\,(y\,c(n))^2$ if n is odd.

Setting $a(n) = x\,c(n)$ in the first case and $a(n) = y\,c(n)$ in the second, we conclude that number $a(n)$ is always integer. The theorem is proved. □

6 Asymptotics for the number of spanning forests

In this section we give asymptotic formulas for the number of rooted spanning forests in circulant graphs.

Theorem 6.1. *The number of rooted spanning forests in the circulant graph*

$$G = C_n(s_1, s_2, \ldots, s_k),\ 1 \le s_1 < s_2 < \ldots < s_k < \frac{n}{2}$$

has the following asymptotics

$$f_G(n) \sim A^n, \text{ as } n \to \infty$$

where $A = \exp(\int_0^1 \log(P(e^{2\pi \mathrm{i} t}))dt)$ *is the Mahler measure of Laurent polynomial* $P(z) = 2k + 1 - \sum_{i=1}^{k}(z^{s_i} + z^{-s_i})$.

Proof. By Theorem 3.1, the number of rooted spanning forests $f_G(n)$ is given by

$$f_G(n) = \prod_{s=1}^{s_k} |2\,T_n(w_s) - 2|,$$

where $w_s = (z_s + z_s^{-1})/2$. We have $T_n(w_s) = \frac{1}{2}(z_s^n + z_s^{-n})$, where z_s and $1/z_s$, $s = 1, \ldots, s_k$ are all the roots of the polynomial $P(z)$. If $\varphi \in \mathbb{R}$ then $P(e^{\mathrm{i}\varphi}) = 2k + 1 - \sum_{j=1}^{k}(e^{s_j \mathrm{i}\varphi} + e^{-s_j \mathrm{i}\varphi}) = 2k + 1 - 2\sum_{j=1}^{k}\cos(s_j\varphi) \geq 1$, so $|z_s| \neq 1$ for all s. Replacing z_s by $1/z_s$, if it is necessary, we can assume that $|z_s| > 1$ for all s. Then $T_n(w_s) \sim \frac{1}{2}z_s^n$, as n tends to ∞. So, $|2T_n(w_s) - 2| \sim |z_s|^n,\ n \to \infty$. Hence

$$\prod_{s=1}^{s_k} |2\,T_n(w_s) - 2| \sim \prod_{s=1}^{s_k} |z_s|^n = \prod_{P(z)=0,\,|z|>1} |z|^n = A^n,$$

where $A = \prod_{P(z)=0,\,|z|>1} |z|$ is the Mahler measure of $P(z)$. By the results mentioned in the preliminary part, it can be found by the formula $A = \exp(\int_0^1 \log(P(e^{2\pi \mathrm{i} t}))dt)$.

Finally,

$$f_G(n) = \prod_{s=1}^{s_k} |2\,T_n(w_s) - 2| \sim A^n,\ n \to \infty. \qquad \square$$

The next theorem is a direct consequence of Theorem 4.1 and can be proved by the same arguments as Theorem 6.1.

Theorem 6.2. *The number of rooted spanning forests* $f_G(n)$ *in the circulant graph* $G = C_{2n}(s_1, s_2, \ldots, s_k, n)$, $1 \leq s_1 < s_2 < \ldots < s_k < n$ *has the following asymptotics*

$$f_G(n) \sim K^n, \text{ as } n \to \infty.$$

Here $K = \exp(\int_0^1 \log|P(e^{2\pi \mathrm{i} t})^2 - 1|dt)$ *is the Mahler measure of the Laurent polynomial* $P(z)^2 - 1$, *where* $P(z) = 2k + 2 - \sum_{i=1}^{k}(z^{s_i} + z^{-s_i})$.

7 Examples

(1) **Cycle graph** $G = C_n(1) = C_n$.

We need to solve the equation $1 + 2 - 2T_1(w) = 0$. We have $w = 3/2$. So, $f_G(n) = 2T_n(3/2) - 2$. Then $f_G(n) \underset{n\to\infty}{\sim} (\frac{3+\sqrt{5}}{2})^n$. Also, we have $f_G(n) = 5F_n^2$, if n is even, and $f_G(n) = L_n^2$, if n is odd, where F_n and L_n are the Fibonacci and Lucas numbers respectively. The latter result was independently obtained in [6].

(2) **Graph** $G = C_n(1,2)$.

We need to solve the equation $1 + 4 - 2T_1(w) - 2T_2(w) = 0$. Its roots are $w_1 = \frac{1}{4}(-1+\sqrt{29})$ and $w_2 = \frac{1}{4}(-1-\sqrt{29})$.

By Theorem 5.1, there exists an integer sequence $a(n)$ such that $f_G(n) = 5a(n)^2$, if n is even, and $f_G(n) = a(n)^2$, if n is odd.

(3) **Graph** $G = C_n(1,3)$.

Let w_1, w_2 and w_3 be the roots of the cubic equation $1+4-2T_1(w)-2T_3(w) = 0$. Then $f_G(n) = (2T_n(w_1)-2)(2T_n(w_2)-2)(2T_n(w_3)-2)$. In this case, $f_G(n) \sim A_{1,3}^n$, $n \to \infty$, where $A_{1,3} \approx 4.48461\ldots$ is the Mahler measure of the Laurent polynomial $5 - z - z^{-1} - z^3 - z^{-3}$. One can check that $A_{1,3}$ is a root of the equation $1 - x - 2x^2 - 4x^3 + x^4 = 0$. By Theorem 5.1, we have $f_G(n) = a(n)^2$, where $a(n)$ is an integer sequence.

(4) **Graph Möbius ladder** $G = C_{2n}(1,n)$.

We have to solve the equations $3 - 2T_1(w) = 0$ and $5 - 2T_1(w) = 0$. Their roots are $u_1 = 3/2$ and $v_1 = 5/2$ respectively. Then $f_G(n) = (2T_n(3/2)-2)(2T_n(5/2)+2) \sim K^n$, where $K = \frac{1}{4}(3+\sqrt{5})(5+\sqrt{21}) \approx 12.5438\ldots$. By Theorem 5.2, $f_G(n) = 5a(n)^2$, if n is even, and $f_G(n) = 7a(n)^2$, if n is odd for some integer sequence $a(n)$.

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