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New results on modular Golomb rulers, optical orthogonal codes and related structures*

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Abstract

We prove new existence and nonexistence results for modular Golomb rulers in this paper. We completely determine which modular Golomb rulers of order k exist, for all $k \leq 11$, and we present a general existence result that holds for all $k \geq 3$. We also derive new nonexistence results for infinite classes of modular Golomb rulers and related structures such as difference packings, optical orthogonal codes, cyclic Steiner systems and relative difference families.

Keywords: Golomb ruler, optical orthogonal code, difference family.

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1 Introduction and definitions

A *Golomb ruler* of *order* k is a set of k distinct integers, say $x_1 < x_2 < \cdots < x_k$, such that all the differences $x_j - x_i$ ($i \neq j$) are distinct. To avoid trivial cases, we assume $k \geq 3$. The *length* of the ruler is $x_k - x_1$. For a survey of constructions of Golomb rulers, see [12].

A (v, k)-*modular Golomb ruler* (or (v, k)-MGR) is a set of k distinct integers,

$$0 \leq x_1 < x_2 < \cdots < x_k \leq v - 1,$$

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such that all the differences $x_j - x_i \bmod v$ ($i \neq j$) are distinct elements of $\mathbb{Z}_v$. We define length and order as before. It is obvious that a modular Golomb ruler is automatically a Golomb ruler. We can assume without loss of generality that $x_1 = 0$.

Known results on modular Golomb rulers are summarized in [9, §VI.19.3]. We state a few basic results and standard constructions now.

Theorem 1.1. *If there exists a (v, k)-MGR, then $v \geq k^2 - k + 1$. Further, a $(k^2 - k + 1, k)$-MGR is equivalent to a cyclic $(k^2 - k + 1, k, 1)$-difference set.*

Of course a $(q^2 + q + 1, q + 1, 1)$-difference set (i.e., a Singer difference set) is known to exist if q is a prime power. So we have the following Corollary.

Corollary 1.2. *There exists a $(k^2 - k + 1, k, 1)$-MGR if $k - 1$ is a prime power.*

It is widely conjectured that a $(q^2 + q + 1, q + 1, 1)$-difference set exists only if q is a prime power, and this conjecture has been verified for all $q < 2,000,000$; see [14].

Theorem 1.3 (Bose [3])**.** *For any prime power q, there is a $(q^2 - 1, q)$-MGR.*

Theorem 1.4 (Rusza [21])**.** *For any prime p, there is a $(p^2 - p, p - 1)$-MGR.*

A $(v, k; n)$*-difference packing* is a set of n k-element subsets of $\mathbb{Z}_v$, say $X_1, \ldots, X_n$, such that all the differences in the multiset

$$\{x - y : x, y \in X_i, x \neq y, 1 \leq i \leq n\}$$

are nonzero and distinct. The following result is obvious.

Theorem 1.5. *A (v, k)-MGR is equivalent to a $(v, k; 1)$-difference packing.*

A (v, b, r, k)*-configuration* is a set system $(V, \mathcal{B})$, where V is a set of v *points* and $\mathcal{B}$ is a set of b *blocks*, each of which contains exactly k points, such that the following properties hold:

1. no pair of points occurs in more than one block, and
2. every point occurs in exactly r blocks.

It is easy to see that the parameters of a (v, b, r, k)-configuration satisfy the equation $bk = vr$. For basic results on configurations, see [9, §VI.7]. A (v, b, r, k)-configuration is *symmetric* if $v = b$, which of course implies $r = k$. In this case we speak of it as a *symmetric* (v, k)*-configuration.* A symmetric (v, k)-configuration is *cyclic* if there is a cyclic permutation of the v points that maps every block to a block.

We state the following easy result without proof.

Theorem 1.6. *A (v, k)-MGR is equivalent to a cyclic symmetric (v, k)-configuration.*

For additional connections between Golomb rulers and symmetric configurations, see [7, 10].

A $(v, k, \lambda_a, \lambda_c)$*-optical orthogonal code of size* n is a set $\mathcal{C}$ of n $(0, 1)$-vectors of length v, which satisfies the following properties:

1. the Hamming weight of $\mathbf{x}$ is equal to k, for all $\mathbf{x} \in \mathcal{C}$,

2. *autocorrelation:* for all $\mathbf{x} = (x_0, \ldots, x_{v-1}) \in \mathcal{C}$, the following holds for all integers τ such that $0 < \tau < v$:
$$\sum_{i=0}^{v-1} x_i x_{i+\tau} \leq \lambda_a,$$
where subscripts are reduced modulo v.
3. *cross-correlation:* for all $\mathbf{x} = (x_0, \ldots, x_{v-1}) \in \mathcal{C}$ and all $\mathbf{y} = (y_0, \ldots, y_{v-1}) \in \mathcal{C}$ with $\mathbf{x} \neq \mathbf{y}$, the following holds for all integers τ such that $0 \leq \tau < v$:
$$\sum_{i=0}^{v-1} x_i y_{i+\tau} \leq \lambda_c,$$
where subscripts are reduced modulo v.

We sometimes abbreviate the phrase "optical orthogonal code" to "OOC." If $\lambda_a = \lambda_c = \lambda$, then the optical orthogonal code is denoted as a (v, k, λ)-optical orthogonal code.

Optical orthogonal codes were introduced by Chung, Salehi and Wei [8] in 1989 and have been studied by numerous authors since then. The following result establishes the equivalence of OOC and difference packings.

Theorem 1.7 ([8])**.** *A $(v, k; n)$-difference packing is equivalent to a $(v, k, 1)$-optical orthogonal code of size n.*

The following result is proven in [8] by a simple counting argument.

Theorem 1.8. *If there exists a $(v, k, 1)$-optical orthogonal code of size n, then*
$$n \leq \left\lfloor \frac{v-1}{k(k-1)} \right\rfloor .$$

A $(v, k, 1)$-optical orthogonal code is *optimal* if the relevant inequality in Theorem 1.8 is met with equality.

Relative difference families have been introduced in [5] as a natural generalization of relative difference sets. We define them now. Let H be a subgroup of a finite additive group G, and let k, λ be positive integers. A (G, H, k, λ)*-relative difference family*, or (G, H, k, λ)-RDF for short, is a collection X of k-subsets of G (called *base blocks*) whose list of differences has no element in H and covers all elements of $G \setminus H$ exactly λ times. If G has order v and H has order w, we say that X is a (v, w, k, λ)-RDF in G *relative to* H. If X consists of n base blocks, it is evident that
$$\lambda(v - w) = k(k-1)n. \tag{1.1}$$

When $H = \{0\}$ (or, equivalently, if $w = 1$), one usually speaks of an *ordinary* (v, k, λ)-difference family or (v, k, λ)-difference family ((v, k, λ)-DF, for short), in G. If $n = 1$, then we refer to a (G, H, k, λ)-relative difference family as a (G, H, k, λ)*-relative difference set*. Analogously, a (v, k, λ)-difference family of size $n = 1$ is a (v, k, λ)*-difference set* ((v, k, λ)-DS, for short).

1.1 Number-theoretic background

In this section, we record some number-theoretic results that we will be using later in the paper.

Theorem 1.9.

1. *A positive integer can be written as a sum of two squares if and only if its prime decomposition contains no prime* $p \equiv 3 \pmod 4$ *raised to an odd power.*
2. *A positive integer can be written as a sum of three squares if and only if it is not of the form* $4^a(8b+7)$*, where* a *and* b *are nonnegative integers.*
3. *Any positive integer can be written as a sum of four squares.*

Proof. Statement 1. is proven in many textbook on elementary number theory, e.g., [20, Theorem 13.6]. The result 2. is known as Legendre's Three-square Theorem (for a proof of it, see, e.g., [18, Chapter 20, Theorem 1]). Finally, 3. is Lagrange's Four-square Theorem. □

Lemma 1.10. *For any positive integer* t*, there exist* t *consecutive positive integers, none of which is a sum of two squares.*

Proof. Take t distinct primes $p_1, \dots, p_t$ all of which are $\equiv 3 \pmod 4$ (they exist by the Dirichlet's Theorem on primes in an arithmetic progression). By the Chinese Remainder Theorem, the system of t congruences

$$x + i \equiv p_i \pmod{p_i^2} \qquad (1 \le i \le t)$$

has a solution s.

Since $s + i \equiv p_i \pmod{p_i^2}$, it is clear that $s+i$ is divisible by p_i, but not by p_i^2. Since $p_i \equiv 3 \pmod 4$, it follows from Theorem 1.9 that $s+i$ is not a sum of two squares. This holds for $1 \le i \le t$. □

Lemma 1.11. *Two consecutive integers, say* n *and* $n+1$*, are both not expressible as a sum of three squares if and only if* $n = 4^a(8b+7) - 1$*, where* $a \ge 2$ *and* $b \ge 0$*.*

Proof. This is a consequence of Legendre's Three-square Theorem (Theorem 1.9). If n is not expressible as a sum of three squares, then $n \equiv 0, 4$ or $7 \pmod 8$. Therefore, if n and $n+1$ are both not expressible as a sum of three squares, then $n \equiv 7 \pmod 8$. It follows from Legendre's Three-square Theorem that n and $n+1$ are both not expressible as a sum of three squares if and only if $n + 1 = 4^a(8b+7)$ where $a \ge 2$ and $b \ge 0$. □

1.2 Our contributions

Section 2 gives existence results for modular Golomb rulers. We summarize exhaustive searches that we have carried out for all $k \le 11$, and we present a general existence result that holds for all $k \ge 3$. Section 3 proves nonexistence results for various infinite classes of modular Golomb rulers. Many of our new results are based on counting even and odd differences and then applying some classical results from number theory which establish which integers can be expressed as a sum of a two or three squares. Section 4 studies optical orthogonal codes and provides nonexistence results for certain optimal OOCs. In Section 5, we consider cyclic Steiner systems and relative difference families and we present additional nonexistence results using the techniques we have developed. Finally, Section 6 is a brief summary.

2 Existence results for (v, k)-MGR

In this section, we report the results of exhaustive searches for (v, k)-MGR with $k \leq 11$. We also prove a general existence result that holds for all integers $k \geq 3$. First, we discuss a few preliminary results..

Given a positive integer $k \geq 3$, define

$$\mathsf{MGR}(k) = \{v : \text{there exists a } (v, k)\text{-MGR}\}.$$

We are interested in the set $\mathsf{MGR}(k)$. In particular, it is natural to try to determine the minimum integer in $\mathsf{MGR}(k)$ as well as the maximum integer not in $\mathsf{MGR}(k)$.

Another parameter of interest is the length of a Golomb ruler. There has been considerable research done on finding the minimum length of a Golomb ruler of specified order k, which we denote by $L^*(k)$. In the modular case, we will define $L^*_m(k)$ to be the minimum L such that there exists a (v, k)-MGR of length L for some v.

The following basic lemma is well-known.

Lemma 2.1. *Suppose there is a Golomb ruler of order k and length L, and suppose $v \geq 2L + 1$. Then there is a (v, k)-MGR.*

Proof. We have a Golomb ruler consisting of k integers

$$0 = x_1 < x_2 < \cdots < x_k = L.$$

Consider these as residues modulo v, where $v \geq 2L+1$. Clearly all the "positive residues" $x_j - x_i \bmod v$ $(i < j)$ are nonzero and distinct, as are all the "negative residues" $x_j - x_i \bmod v$ $(j < i)$. The largest positive residue is L and the smallest negative residue is $v - L$. Since $v > 2L$, no positive residue is equal to a negative residue. □

The following is an immediate consequence of Lemma 2.1.

Lemma 2.2. *For any positive integer $k \geq 2$, $L^*(k) = L^*_m(k)$.*

Given a positive integer $k \geq 3$, define

$$\mathsf{MGR}(k) = \{v : \text{there exists a } (v, k)\text{-MGR}\}.$$

We have performed exhaustive backtracking searches in order to determine the sets $\mathsf{MGR}(k)$ for $3 \leq k \leq 11$. For each value of k, once we have constructed a sufficient number of "small" (v, k)-MGR, we can apply Lemma 2.1 to conclude that all (v, k)-MGR exist for larger values of v. To this end, when we compute all the (v, k)-MGR for given values of v and k, we keep track of the ruler having the smallest possible length. This facilitates the application of Lemma 2.1

Our computational results are summarized as follows.

Theorem 2.3.

1. $\mathsf{MGR}(3) = \{v : v \geq 7\}$.
2. $\mathsf{MGR}(4) = \{v : v \geq 13\}$.
3. $\mathsf{MGR}(5) = \{21\} \cup \{v : v \geq 23\}$.
4. $\mathsf{MGR}(6) = \{31\} \cup \{v : v \geq 35\}$.

5. $\mathsf{MGR}(7) = \{v : v \geq 48\}$.
6. $\mathsf{MGR}(8) = \{57\} \cup \{v : v \geq 63\}$.
7. $\mathsf{MGR}(9) = \{73, 80\} \cup \{v : v \geq 85\}$.
8. $\mathsf{MGR}(10) = \{91\} \cup \{v : v \geq 107\}$.
9. $\mathsf{MGR}(11) = \{120, 133\} \cup \{v : v \geq 135\}$.

Proof. Proof details are in Table 1. □

Table 1: (v, k)-modular Golomb rulers for $3 \leq k \leq 11$.

v	k	ruler
$v = 7$	3	$0, 1, 3$
$v \geq 8$	3	Lemma 2.1, $v = 7$, $L = 3$
$v = 13$	4	$0, 1, 4, 6$
$v \geq 14$	4	Lemma 2.1, $v = 13$, $L = 6$
$v = 21$	5	$0, 2, 7, 8, 11$
$v = 22$	5	does not exist
$v \geq 23$	5	Lemma 2.1, $v = 21$, $L = 11$
$v = 31$	6	$0, 1, 4, 10, 12, 17$
$32 \leq v \leq 34$	6	does not exist
$v \geq 35$	6	Lemma 2.1, $v = 31$, $L = 17$
$43 \leq v \leq 47$	7	does not exist
$v = 48$	7	$0, 5, 7, 18, 19, 22, 28$
$v = 49$	7	$0, 2, 3, 10, 16, 21, 25$
$v = 50$	7	$0, 1, 5, 7, 15, 18, 27$
$v \geq 51$	7	Lemma 2.1, $v = 49$, $L = 25$
$v = 57, 64, 68$	8	$0, 4, 5, 17, 19, 25, 28, 35$
$58 \leq v \leq 62$	8	does not exist
$v = 63, 67$	8	$0, 1, 8, 20, 22, 25, 31, 35$
$v = 65$	8	$0, 2, 10, 11, 16, 28, 31, 35$
$v = 66$	8	$0, 2, 10, 21, 24, 25, 30, 37$
$v = 69$	8	$0, 1, 4, 9, 15, 22, 32, 34$
$v \geq 70$	8	Lemma 2.1, $v = 69$, $L = 34$
$v = 73$	9	$0, 2, 10, 24, 25, 29, 36, 42, 45$
$74 \leq v \leq 79$	9	does not exist
$v = 80$	9	$0, 1, 12, 16, 18, 25, 39, 44, 47$
$81 \leq v \leq 84$	9	does not exist
$v = 85$	9	$0, 1, 7, 12, 21, 29, 31, 44, 47$
$v = 86, 88$	9	$0, 2, 5, 13, 17, 31, 37, 38, 47$
$v = 87$	9	$0, 1, 4, 13, 24, 30, 38, 40, 45$
$v = 89$	9	$0, 1, 5, 12, 25, 27, 35, 41, 44$
$v \geq 90$	9	Lemma 2.1, $v = 89$, $L = 44$
$v = 91$	10	$0, 1, 6, 10, 23, 26, 34, 41, 53, 55$
$92 \leq v \leq 106$	10	does not exist
$v = 107$	10	$0, 2, 15, 21, 22, 32, 46, 50, 55, 58$

Table 1: (v, k)-modular Golomb rulers for $3 \leq k \leq 11$ (cont.)

v	k	ruler
$v = 108$	10	$0, 2, 8, 27, 32, 36, 39, 49, 50, 65$
$v = 109$	10	$0, 4, 11, 16, 25, 35, 38, 53, 55, 61$
$v = 110$	10	$0, 3, 14, 16, 36, 37, 42, 46, 54, 61$
$v \geq 111$	10	Lemma 2.1, $v = 91$, $L = 55$
$111 \leq v \leq 119$	11	does not exist
$v = 120$	11	$0, 1, 4, 9, 23, 30, 41, 43, 58, 68, 74$
$121 \leq v \leq 132$	11	does not exist
$v = 133$	11	$0, 1, 9, 19, 24, 31, 52, 56, 58, 69, 72$
$v = 134$	11	does not exist
$v = 135$	11	$0, 5, 7, 11, 31, 41, 49, 50, 63, 66, 78$
$v = 136$	11	$0, 2, 11, 27, 37, 42, 45, 59, 65, 66, 78$
$v = 137$	11	$0, 1, 16, 21, 24, 33, 43, 61, 68, 72, 74$
$v = 138$	11	$0, 4, 5, 23, 25, 37, 52, 59, 65, 68, 76$
$v = 139$	11	$0, 1, 3, 11, 25, 41, 45, 54, 60, 72, 77$
$v = 140$	11	$0, 4, 10, 24, 25, 27, 36, 43, 65, 73, 78$
$v = 141$	11	$0, 2, 3, 7, 20, 29, 41, 52, 60, 66, 76$
$v = 142$	11	$0, 1, 13, 16, 22, 33, 47, 51, 70, 75, 77$
$v = 143, 144$	11	$0, 3, 7, 22, 27, 43, 56, 57, 66, 68, 74$
$v \geq 145$	11	Lemma 2.1, $v = 133$, $L = 72$

Remark 2.4. Existence of a $(110, 10)$-MGR also follows from Theorem 1.4, and existence of a $(48, 7)$-MGR and a $(120, 11)$-MGR follow from Theorem 1.3.

The rulers that are presented in Table 1 provide upper bounds on $L^*_m(k)$ for $3 \leq k \leq 11$. However, it turns out that all these values are in fact exact. This is because the exact values of $L^*(k)$ are known for small k (see, for example, [11, Table 2.2]) and they match the minimum lengths of the modular Golomb rulers that we have recorded in Table 1. Thus we have the following result.

Theorem 2.5. *$L^*_m(3) = 3$; $L^*_m(4) = 6$; $L^*_m(5) = 11$; $L^*_m(6) = 17$; $L^*_m(7) = 25$; $L^*_m(8) = 34$; $L^*_m(9) = 44$; $L^*_m(10) = 55$; and $L^*_m(11) = 72$.*

Now we state and prove two general existence results that hold for all $k \geq 3$.

Theorem 2.6. *For any integer $k \geq 3$, there is a (v, k)-MGR for some integer $v \leq 3k^2/2$.*

Proof. For $3 \leq k \leq 11$, we refer to the results in Table 1. Indeed, for these values of k, there is a (v, k)-MGR for some integer $v \leq k^2 - 1$.

For $12 \leq k \leq 24$, we use Corollary 1.2. There is a $(p^2 + p + 1, p + 1, 1)$-difference set in $\mathbb{Z}_{p^2+p+1}$ for $p = 11, 13, 16, 17, 19$ and 23. If we delete $\delta = p + 1 - k$ elements from such a difference set, we obtain a $(p^2 + p + 1, k)$-MGR. For $k = 12$, we have $p = 11$ and $\delta = 0$; for $k = 13, 14$, we have $p = 13$ and $\delta \leq 1$; for $15 \leq k \leq 17$, we have $p = 16$ and $\delta \leq 2$; for $k = 18$, we have $p = 17$ and $\delta = 0$; for $k = 19, 20$, we have $p = 19$ and $\delta \leq 1$; and for $21 \leq k \leq 24$, we have $p = 23$ and $\delta \leq 3$. So, for $12 \leq k \leq 24$, there is a

(v,k)-MGR for some integer

$$\begin{aligned} v &\le (k+\delta-1)^2+(k+\delta-1)+1 \\ &\le (k+2)^2+k+3 \\ &= k^2+5k+7. \end{aligned}$$

It is easy to verify that $k^2+5k+7 \le 3k^2/2$ if $k \ge 12$.

Finally, suppose $k \ge 25$. Let p be the smallest prime such that $p \ge k-1$. By a result of Nagura [19], we have $p \le 6(k-1)/5 < 6k/5$. From Corollary 1.2, there exists a $(p^2+p+1, p+1, 1)$-difference set in $\mathbb{Z}_{p^2+p+1}$. Delete $p+1-k$ elements from this difference set to obtain a (p^2+p+1,k)-MGR. We have

$$\begin{aligned} p^2+p+1 &< \left(\frac{6k}{5}\right)^2+\frac{6k}{5}+1 \\ &< \frac{3k^2}{2}, \end{aligned}$$

where the last inequality holds for $k \ge 21$. □

Theorem 2.7. *For any integer $k \ge 3$ and any integer $v \ge 3k^2-1$, there is a (v,k)-MGR.*

Proof. From Theorem 2.6, there exists a (v,k)-MGR for some integer $v \le 3k^2/2$. This ruler has length $L \le 3k^2/2-1$. Applying Theorem 2.1, there is a (v,k)-MGR for all $v \ge 2(3k^2/2-1)+1 = 3k^2-1$. □

Remark 2.8. Of course there are stronger results known on gaps between consecutive primes that hold for larger integers. For example, it was shown by Dusart [13] that, if $k \ge 89693$, then there is at least one prime p such that

$$k < p \le \left(1+\frac{1}{\ln^3 k}\right)k.$$

So improved versions of Theorems 2.6 and 2.7 could be proven that hold for sufficiently large values of k.

3 Nonexistence results for (v,k)-MGR

We present several nonexistence results for infinite classes of modular Golomb rulers in this section.

3.1 (k^2-k+2,k)-MGR

We have noted that $v \ge k^2-k+1$ if a (v,k)-MGR exists, and the (k^2-k+1,k)-MGR are equivalent to cyclic difference sets with $\lambda = 1$. There has been considerable study of these difference sets and various nonexistence results are known. We do not discuss this case further here, but we refer to [16, §8] for a good summary of known results.

The next case is $v = k^2-k+2$. First, we note that there are two small examples of (k^2-k+2,k)-MGR, namely, an $(8,3)$-MGR and a $(14,4)$-MGR. These are found in Table 1. In fact, these are the only examples that are known to exist. We now discuss some nonexistence results for (k^2-k+2,k)-MGR.

We next observe that (k^2-k+2,k)-MGR are equivalent to certain relative difference sets in the cyclic group $\mathbb{Z}_{k^2-k+2}$. The proof of this easy result is left to the reader.

Theorem 3.1. *A (k^2-k+2,k)-MGR is equivalent to a $(\mathbb{Z}_{k^2-k+2},H,k,1)$-relative difference set, where H is the unique subgroup of order 2 in $\mathbb{Z}_{k^2-k+2}$, i.e., $H=\{0,(k^2-k+2)/2\}$.*

It is well-known that relative difference sets give rise to certain square divisible designs, which we define now. A $(w,u,k,\lambda_1,\lambda_2)$-*divisible design* is a set system (actually, a type of group-divisible design) on $v=uw$ points and having blocks of size k, such that the following conditions are satisfied:

1. the points are partitioned into u groups of size w,
2. two points in the same group occur together in exactly λ_1 blocks, and
3. two points in different groups occur together in exactly λ_2 blocks.

If the number of blocks is the same as the number of points, then we have a *square divisible design*.

The following result is a consequence of Theorem 3.1, since a square divisible design is obtained by developing a relative difference set through the relevant cyclic group.

Theorem 3.2. *If there exists a (k^2-k+2,k)-MGR, then there exists a square divisible design with parameters $w=2$, $u=(k^2-k+2)/2$, $\lambda_1=0$ and $\lambda_2=1$.*

We will make use of some results due to Bose and Connor [4], as stated in [15, Proposition 1.8].

Theorem 3.3 (Bose and Connor)**.** *Suppose there exists a square divisible design with parameters w, u, k, λ_1 and λ_2. Denote $v=uw$. Then the following hold.*

1. *If u is even, then $k^2-\lambda_2 v$ is a perfect square. If furthermore $u\equiv 2 \pmod 4$, then $k-\lambda_1$ is the sum of two squares.*
2. *If u is odd and w is even, then $k-\lambda_1$ is a perfect square and the equation*

$$(k^2-\lambda_2 v)x^2+(-1)^{u(u-1)/2}\lambda_2 w y^2=z^2$$

has a nontrivial solution in integers x, y and z.

We can use Theorem 3.3 to obtain necessary conditions for the existence of (k^2-k+2,k)-MGR.

Corollary 3.4. *Suppose there exists a (k^2-k+2,k)-MGR. Then the following hold.*

1. $k\not\equiv 7 \pmod 8$.
2. *If $k\equiv 2 \pmod 8$, then $k-2$ is a perfect square and k is the sum of two squares.*
3. *If $k\equiv 3,6 \pmod 8$, then $k-2$ is a perfect square.*
4. *If $k\equiv 0,1 \pmod 8$, then k is a perfect square and the equation*

$$(k-2)x^2+2y^2=z^2$$

has a nontrivial solution in integers x, y and z.

5. *If* $k \equiv 4, 5 \pmod 8$*, then* k *is a perfect square and the equation*

$$(k-2)x^2 - 2y^2 = z^2$$

has a nontrivial solution in integers x*,* y *and* z*.*

Proof. Suppose there exists a $(k^2 - k + 2, k)$-MGR. Then, from Theorem 3.2, there is a square divisible design with parameters $w = 2$, $u = (k^2 - k + 2)/2$, $v = k^2 - k + 2$, $\lambda_1 = 0$ and $\lambda_2 = 1$. We apply Theorem 3.3, making use of the fact that $k^2 - \lambda_2 v = k - 2$.

First, we observe that u is even if and only if $k \equiv 2, 3 \pmod 4$. Further, $u \equiv 2 \pmod 4$ if and only if $k \equiv 2, 7 \pmod 8$.

If $k \equiv 7 \pmod 8$, then $k^2 - \lambda_2 v = k - 2 \equiv 5 \pmod 8$, so $k^2 - \lambda_2 v$ is not a perfect square. Therefore, from part 1. of Theorem 3.3, a $(k^2 - k + 2, k)$-MGR does not exist if $k \equiv 7 \pmod 8$.

If $k \equiv 2 \pmod 8$, then part 1. of Theorem 3.3 says that $k - 2$ is a perfect square and k is the sum of two squares.

If $k \equiv 3, 6 \pmod 8$, then part 1. of Theorem 3.3 says that $k - 2$ is a perfect square.

When $k \equiv 0, 1 \pmod 8$, we have $u \equiv 1 \pmod 4$ and hence $(-1)^{u(u-1)/2} = 1$. When $k \equiv 4, 5 \pmod 8$, we have $u \equiv 3 \pmod 4$ and hence $(-1)^{u(u-1)/2} = -1$. The stated results then follow immediately from part 2. of Theorem 3.3. □

3.2 $(k^2 - k + 2\ell, k)$-MGR

For $v > k^2 - k + 2$, a (v, k)-MGR is not necessarily a relative difference set and it does not necessarily imply the existence of a square divisible design. So, in general, we cannot apply the results in Theorem 3.3. However, we can derive some nice necessary conditions for the existence of certain (v, k)-MGR using elementary counting arguments. These arguments are in the spirit of techniques introduced in [6, §2]; see also [17]. Before studying MGR, we present a simple example to illustrate the basic idea.

Example 3.5. Suppose we have a (v, k, λ)-difference set in $\mathbb{Z}_v$ when v is even. There are $v/2 - 1$ nonzero even differences and $v/2$ odd differences, each of which occurs λ times. Suppose the difference set consists of a even elements and b odd elements. Then $a + b = k$ and $2ab = \lambda v/2$. So a and b are the solutions of the quadratic equation

$$x^2 - kx + \frac{\lambda v}{4} = 0.$$

Since a and b are integers, the discriminant must be a perfect square. Therefore, $k^2 - \lambda v$ is a square. However, $k(k-1) = \lambda(v-1)$, so $k^2 - \lambda v = k - \lambda$ must be a perfect square. (Of course, this condition is the same as in the Bruck-Ryser-Chowla Theorem for v even, which holds for any symmetric BIBD.)

In the next theorem, we will use this counting technique to obtain necessary conditions for the existence of a $(k^2 - k + 2\ell, k)$-MGR for a given integer $\ell \geq 1$. First, we give a couple of definitions that will be useful in the rest of the paper.

Suppose X is a (v, k)-MGR. Define

$$\Delta X = \{x - y \bmod v : x, y \in X, x \neq y\}$$

and

$$L(X) = \mathbb{Z}_v \setminus \Delta X.$$

Note that ΔX consists of all the differences obtained from pairs of distinct elements in X and $L(X)$ is the complement of ΔX. The set $L(X)$ is called the *leave* of X. For $i = 0, 1$, define $L_i(X)$ to consist of the elements of $L(X)$ that are congruent to i modulo 2.

The following lemma is straightforward but useful.

Lemma 3.6. *Suppose X is a (v, k)-MGR where v is even. Then $\{0, v/2\} \subseteq L(X)$. If $v \equiv 0 \pmod 4$, then $|L_0(X)|$ and $|L_1(X)|$ are both even. If $v \equiv 2 \pmod 4$, then $|L_0(X)|$ and $|L_1(X)|$ are both odd.*

Proof. It is evident that $0 \in L(X)$. Also, if we have $x - y = v/2$ for some pair $(x, y) \in X \times X$, then we have $y - x = v/2$ as well. This would imply that $v/2$ appears at least twice as a difference, which is not allowed. Hence $\{0, v/2\} \subset L(X)$.

Now note that if $d \in \Delta X$, then $v - d \in \Delta X$ as well. Consequently, if $d \in L(X)$, then $v - d \in L(X)$. Of course $d = v - d$ if and only if $d = 0$ or $d = v/2$. The remaining elements of $\mathbb{Z}_v$ can be matched into pairs $(d, v - d)$ having the same parity. Thus, considering that $v/2$ is even or odd according to whether $v \equiv 0$ or 2 modulo 4, respectively, it is clear that $|L_1(X)|$ and $|L_2(X)|$ are both even in the first case and both odd in the second. $\square$

Theorem 3.7. *Suppose $v = k^2 - k + 2\ell$, where $\ell \geq 1$, and suppose there is a (v, k)-MGR. Then the following hold.*

1. *If $v \equiv 2 \pmod 4$, then $k - 2\ell + 2 + 4i$ is a perfect square for some integer $i \in \{0, \ldots, \ell - 1\}$.*
2. *If $v \equiv 0 \pmod 4$, then $k - 2\ell + 4i$ is a perfect square for some integer $i \in \{0, \ldots, \ell - 1\}$.*

Proof. Let X be a (v, k)-MGR. Since $|X| = k$, we have

$$|L(X)| = v - (k^2 - k) = 2\ell.$$

Suppose X contains a even elements and b odd elements; then $a + b = k$.

Suppose first that $v \equiv 2 \pmod 4$, so $v/2$ is odd. From Lemma 3.6, $|L_1(X)|$ is odd, say $|L_1(X)| = 2i + 1$, and $v/2 \in L_1(X)$. Therefore, $0 \leq i \leq \ell - 1$.

The quantity $2ab$ is equal to the number of odd differences in ΔX, so

$$2ab = \frac{v}{2} - (2i + 1) = \frac{v - 2 - 4i}{2}.$$

It follows that a and b are the solutions of the quadratic equation

$$x^2 - kx + \frac{v - 2 - 4i}{4} = 0.$$

The solutions a and b must be integers, which can happen only if the discriminant is a perfect square. Hence,

$$k^2 - (v - 2 - 4i) = k - 2\ell + 2 + 4i$$

is a perfect square. Hence, $k - 2\ell + 2 + 4i$ is a perfect square for some integer $i \in \{0, \ldots, \ell - 1\}$.

The proof is similar when $v \equiv 0 \pmod 4$. Here, from Lemma 3.6, $|L_1(X)|$ is even, say $|L_1(X)| = 2i$ and $\{0, v/2\} \subseteq L_0(X)$. Since $\{0, v/2\} \subseteq L_0(X)$, we have $|L_1(X)| \leq 2\ell - 2$. Hence $i \in \{0, \ldots, \ell - 1\}$.

We have

$$2ab = \frac{v}{2} - 2i = \frac{v - 4i}{2}.$$

It follows that a and b are the solutions of the quadratic equation

$$x^2 - kx + \frac{v - 4i}{4} = 0.$$

The solutions a and b must be integers, which can happen only if the discriminant is a perfect square. Hence,

$$k^2 - (v - 4i) = k - 2\ell + 4i$$

is a perfect square. Hence, $k - 2\ell + 4i$ is a perfect square for some integer $i \in \{0, \ldots, \ell - 1\}$. □

Example 3.8. Suppose $k = 10$ and $v = 94 = 10 \times 9 + 2 \times 2$, $\ell = 2$. Here $v \equiv 2 \pmod 4$. Then we compute

$$10 - 2 \times 2 + 2 + 4i = 8 + 4i$$

for $i = 0, 1$, obtaining 8 and 12. Neither of these is a perfect square, so we conclude that a $(94, 10)$-MGR does not exist.

It is interesting to see what Theorem 3.7 tells us when $\ell = 1$.

Corollary 3.9. *Suppose there is a $(k^2 - k + 2, k)$-MGR. Then the following hold.*

1. *If $k \equiv 2, 3 \pmod 4$, then $k - 2$ is a perfect square.*
2. *If $k \equiv 0, 1 \pmod 4$, then k is a perfect square.*

Proof. Take $\ell = 1$ in Theorem 3.7; then $v = k^2 - k + 2$ and we have $i = 0$. We note that $v \equiv 0 \pmod 4$ if $k \equiv 2, 3 \pmod 4$ and $v \equiv 2 \pmod 4$ if $k \equiv 0, 1 \pmod 4$, so the stated results follow immediately. □

Remark 3.10. We observe that Theorem 3.2 and Corollary 3.4 provide stronger necessary conditions for the existence of $(k^2 - k + 2, k)$-MGR than those stated in Corollary 3.9.

For certain values of k, we are able to find "intervals" in which MGR cannot exist. Define

$$S_{k,\ell} = \{k - 2\ell + 2 + 4i : 0 \leq i \leq \ell - 1\}$$

and define

$$T_{k,\ell} = \{k - 2\ell + 4i : 0 \leq i \leq \ell - 1\}.$$

Lemma 3.11. *Suppose $v = k^2 - k + 2\ell$.*

1. *If $v \equiv 2 \pmod 4$, then all elements of $S_{k,\ell}$ are $\equiv 0, 1 \pmod 4$.*
2. *If $v \equiv 0 \pmod 4$, then all elements of $T_{k,\ell}$ are $\equiv 0, 1 \pmod 4$.*

Proof. We prove 1. Suppose $v = 2\ell + k^2 - k \equiv 2 \pmod 4$. Then $2\ell + k^2 - k - 2 - 4i \equiv 0 \pmod 4$. It follows that $k^2 \equiv k - 2\ell + 2 + 4i \pmod 4$. Since $k^2 \equiv 0, 1 \pmod 4$ for all integers k, the result follows.

The proof of 2. is similar. □

Theorem 3.12. *Let t be a positive integer.*

1. *If $k = 4t^2 + 4t + 4$, then there does not exist a $(k^2 - k + 4s, k)$-MGR for all s such that $1 \leq s \leq t$.*
2. *If $k = 4t^2 + 4t + 2$, then there does not exist a $(k^2 - k + 4s, k)$-MGR for all s such that $1 \leq s \leq t$.*
3. *If $k = 4t^2 + 3$, then there does not exist a $(k^2 - k + 4s - 2, k)$-MGR for all s such that $1 \leq s \leq t$.*
4. *If $k = 4t^2 + 1$, then there does not exist a $(k^2 - k + 4s - 2, k)$-MGR for all s such that $1 \leq s \leq t$.*

Proof. We prove 1. Denote $\ell = 2s$, where $1 \leq s \leq t$ and let $v = k^2 - k + 4s$. Since $k \equiv 0 \pmod 4$, we have $v \equiv 0 \pmod 4$. So we examine the elements in $T_{k,\ell}$, which are all congruent to 0 modulo 4 by Lemma 3.11. For the smallest element of $T_{k,\ell}$, which is $k - 2\ell$, we have

$$\begin{aligned} k - 2\ell &\geq k - 4t \\ &= 4(t^2 + t + 1) - 4t \\ &= 4t^2 + 4 \\ &> (2t)^2. \end{aligned}$$

Similarly, for the largest element of $T_{k,\ell}$, which is $k - 2\ell + 4(\ell - 1)$, we have

$$\begin{aligned} k - 2\ell + 4(\ell - 1) &\leq k + 4t - 4 \\ &= 4(t^2 + t + 1) + 4t - 4 \\ &= 4t^2 + 8t \\ &< (2t + 2)^2. \end{aligned}$$

Since all the elements of $T_{k,\ell}$ are congruent to 0 modulo 4 and they are between two consecutive even squares, there cannot be any perfect squares in the set $T_{k,\ell}$.

The proofs of 2., 3. and 4. are similar. □

Example 3.13. If we take $t = 3$ in Theorem 3.12, we see that there does not exist a $(k^2 - k + 4, k)$-MGR, a $(k^2 - k + 8, k)$-MGR or a $(k^2 - k + 12, k)$-MGR when $k = 50, 52$. Further, there does not exist a $(k^2 - k + 2, k)$-MGR, a $(k^2 - k + 6, k)$-MGR or a $(k^2 - k + 10, k)$-MGR when $k = 37, 39$.

We will show that we can improve Theorem 3.7 when $v \equiv 0 \pmod 4$. First we state and prove a simple numerical lemma.

Lemma 3.14. *Let a be a positive integer. Then*

$$\left\{h(a-h) : 0 \le h \le \left\lfloor \frac{a}{2} \right\rfloor\right\} = \left\{\left(\frac{a}{2}\right)^2 - \left(\frac{a}{2} - h\right)^2 : 0 \le h \le \left\lfloor \frac{a}{2} \right\rfloor\right\}. \tag{3.1}$$

Further, if a is even, then

$$\left\{h(a-h) : 0 \le h \le \frac{a}{2}\right\} = \left\{\left(\frac{a}{2}\right)^2 - h^2 : 0 \le h \le \frac{a}{2}\right\}. \tag{3.2}$$

Proof. Clearly we have

$$h(a-h) = \left(\frac{a}{2}\right)^2 - \left(\frac{a}{2} - h\right)^2.$$

Therefore (3.1) holds. If a is even, then

$$\left\{\left(\frac{a}{2}\right)^2 - \left(\frac{a}{2} - h\right)^2 : 0 \le h \le \frac{a}{2}\right\} = \left\{\left(\frac{a}{2}\right)^2 - h^2 : 0 \le h \le \frac{a}{2}\right\},$$

and (3.2) holds. □

Theorem 3.15. *Suppose that X is a (v, k)-MGR with $v = k^2 - k + 2\ell$. Then the following hold.*

1. *If $v \equiv 0 \pmod 8$, then there exist integers $i \in \{0, 1, \dots, \ell - 1\}$ and $j \in \{0, 1, \dots, \ell - 1 - i\}$ such that $k - 2\ell + 4i$ is a perfect square and $k - 2\ell + 2i + 4j$ is a sum of two squares.*
2. *If $v \equiv 4 \pmod 8$, then there exist integers $i \in \{0, 1, \dots, \ell - 1\}$ and $j \in \{0, 1, \dots, \ell - 1 - i\}$ such that that $k - 2\ell + 4i$ is a perfect square and $k - 2\ell + 2i + 4j + 2$ is a sum of two squares.*

Proof. Suppose $v \equiv 0 \pmod 8$; then $v/2 \equiv 0 \pmod 4$. From Lemma 3.6 and the proof of Theorem 3.7, there are an even number, say $2i$, of odd elements in $L(X)$, where $0 \le i \le \ell - 1$. The number of elements $\equiv 2 \pmod 4$ that are in $L(X)$ is also even, say $2j$, and we must have $0 \le j \le \ell - 1 - i$.

Let a and b be the number of even and odd elements in X, respectively. We showed in the proof of Theorem 3.7 that a and b are the solutions to the quadratic equation

$$x^2 - kx + \frac{v}{4} - i = 0,$$

and hence $a + b = k$, $ab = \frac{v}{4} - i$, and $k^2 - v + 4i = k - 2\ell + 4i$ is a perfect square.

Let n_α be the number of elements of X that are congruent to α modulo 4, for $\alpha = 0, 1, 2, 3$. It is evident that $n_0 + n_2 = a$ and that $n_1 + n_3 = b$. Thus, from (3.1) in Lemma 3.14, we have

$$n_0 n_2 \in \left\{h(a-h) : 0 \le h \le \left\lfloor \frac{a}{2} \right\rfloor\right\} = \left\{\left(\frac{a}{2}\right)^2 - \left(\frac{a}{2} - h\right)^2 : 0 \le h \le \left\lfloor \frac{a}{2} \right\rfloor\right\}$$

and

$$n_1 n_3 \in \left\{h(b-h) : 0 \le h \le \left\lfloor \frac{b}{2} \right\rfloor\right\} = \left\{\left(\frac{b}{2}\right)^2 - \left(\frac{b}{2} - h\right)^2 : 0 \le h \le \left\lfloor \frac{b}{2} \right\rfloor\right\}.$$

Multiplying by four, we get:

$$4n_0n_2 \in \left\{a^2 - (a-2h)^2 : 0 \le h \le \left\lfloor \frac{a}{2} \right\rfloor \right\} \tag{3.3}$$

and

$$4n_1n_3 \in \left\{b^2 - (b-2h)^2 : 0 \le h \le \left\lfloor \frac{b}{2} \right\rfloor \right\}. \tag{3.4}$$

Now note that $2n_0n_2 + 2n_1n_3$ is the number of differences in ΔX that are congruent to 2 modulo 4, which of course is also equal to $\frac{v}{4} - 2j$. Thus, from (3.3) and (3.4), there are integers h_1, h_2 such that $0 \le h_1 \le \lfloor \frac{a}{2} \rfloor$, $0 \le h_2 \le \lfloor \frac{b}{2} \rfloor$ and

$$a^2 - (a-2h_1)^2 + b^2 - (b-2h_2)^2 = \frac{v}{2} - 4j. \tag{3.5}$$

Using the facts that

$$a + b = k \quad \text{and}$$
$$ab = \frac{v}{4} - i,$$

we have

$$a^2 + b^2 = (a+b)^2 - 2ab$$
$$= k^2 - \frac{v}{2} + 2i.$$

Substituting this into (3.5), we have

$$k^2 - \frac{v}{2} + 2i - (a-2h_1)^2 - (b-2h_2)^2 = \frac{v}{2} - 4j,$$

or

$$k^2 - v + 2i + 4j = (a-2h_1)^2 + (b-2h_2)^2.$$

Since $v = k^2 - k + 2\ell$, we obtain

$$k - 2\ell + 2i + 4j = (a-2h_1)^2 + (b-2h_2)^2.$$

We conclude that $k - 2\ell + 2i + 4j$ is a sum of two squares.

Suppose $v \equiv 4 \pmod 8$. As before, there are an even number, say $2i$, of odd elements in $L(X)$, where $0 \le i \le \ell - 1$. However, $v/2 \equiv 2 \pmod 4$, so the number of elements $\equiv 2 \pmod 4$ that are not in ΔX is an odd number, say $2j + 1$, where $0 \le j \le \ell - 1 - i$.

Reasoning exactly as in the case where $v \equiv 0 \pmod 8$, we find that $k - 2\ell + 4i$ is a perfect square and that $k - 2\ell + 2i + 4j + 2$ is a sum of two squares. □

We now give an application of Theorem 3.15.

Corollary 3.16. *Suppose that $k = n^2 - 2\ell + 4$ where $\ell \ge 1$ and $n \ge \ell + 1$. Let $v = k^2 - k + 2\ell$.*

1. *If $v \equiv 0 \pmod 8$ and $k - 2$ is not the sum of two squares, then a (v, k)-MGR does not exist.*

2. *If* $v \equiv 4 \pmod 8$ *and* k *is not the sum of two squares, then a* (v,k)*-MGR does not exist.*

Proof. We note that $k - 2\ell + 4(\ell - 1) = n^2$ is a perfect square. We claim there are no squares of the form $k - 2\ell + 4i$ where $0 \leq i \leq \ell - 2$. This is because the smallest such integer is

$$\begin{aligned} k - 2\ell &= n^2 - 4\ell + 4 \\ &\geq n^2 - 4(n-1) + 4 \\ &= n^2 - 4n + 8 \\ &= (n-2)^2 + 4. \end{aligned}$$

Since all these integers have the same parity as n^2 and they are not larger than $k - 2\ell + 4(\ell - 1) = n^2$, the result follows. Therefore $i = \ell - 1$ is the only value in $[0, \ell - 1]$ such that $k - 2\ell + 4i$ is a perfect square.

Now, in applying Theorem 3.15, we need to check that a certain condition holds for $0 \leq j \leq \ell - 1 - i$. Since $i = \ell - 1$, we only need to consider $j = 0$. Theorem 3.15 then states that a (v,k)-MGR does not exist if $v \equiv 0 \pmod 8$ and $k - 2\ell + 2(\ell - 1) = k - 2$ is not a sum of two squares; or if $v \equiv 4 \pmod 8$ and $k - 2\ell + 2(\ell - 1) + 2 = k$ is not a sum of two squares. (It is not hard to verify that $v \equiv 0 \pmod 4$, so either $v \equiv 0 \pmod 8$ or $v \equiv 4 \pmod 8$.) □

We give some examples to illustrate results that can be obtained using Corollary 3.16.

Example 3.17. Suppose we take $n = 4t + 2$ and $\ell = 5$ in Corollary 3.16. Then $v = k^2 - k + 10 \equiv 0 \pmod 8$. Here we have

$$k - 2 = (4t + 2)^2 - 10 + 4 - 2 = 4(4t^2 + 4t - 1).$$

This integer is not the sum of two squares because $4t^2 + 4t - 1 \equiv 3 \pmod 4$. Hence, no $(k^2 - k + 10, k)$-MGR exists if $k = 4(2t + 1)^2 - 6$. The first values of k covered by this result are $k = 30, 94, 190, 318, 478, 670, 894, 1150, 1438, 1758$.

Example 3.18. Suppose we take $n = 4t + 2$ and $\ell = 3$ in Corollary 3.16. Then $v = k^2 - k + 6 \equiv 0 \pmod 8$. Here we have

$$k - 2 = (4t + 2)^2 - 6 + 4 - 2 = 16t^2 + 16t.$$

This integer is the sum of two squares if and only if $t^2 + t$ is the sum of two squares. Hence, no $(k^2 - k + 10, k)$-MGR exists if $k = 4(2t + 1)^2 - 2$ and $t^2 + t$ is not the sum of two squares. The first values of k covered by this result are $k = 98$, 194, 482, 674, 898, 1762, 2114, 2498, 2914 and 3362.

Example 3.19. Suppose we take $n = 4t$ and $\ell = 5$ in Corollary 3.16. Then $v = k^2 - k + 10 \equiv 4 \pmod 8$. Here we have

$$k = 16t^2 - 6 = 2(8t^2 - 3).$$

This integer is the sum of two squares if and only if $8t^2 - 3$ is the sum of two squares. Hence, no $(k^2 - k + 10, k)$-MGR exists if $k = (4t)^2 - 6$ and $8t^2 - 3$ is not the sum of two squares. The first values of k covered by this result are $k = 138$, 5704, 1290, 2298, 2698, 3594, 5178, 6394, 7050 and 9210.

4 Nonexistence results for $(v, k, 1)$-OOC

In this section, we prove nonexistence results for some optimal $(v, k, 1)$-optical orthogonal codes of size $n > 1$. Note that we are investigating the cases where v is even in this section.

Lemma 4.1. *Suppose* $1 \le \ell \le \binom{k}{2}$ *and* $v = k(k-1)n + 2\ell$. *Then a* $(v, k, 1)$*-OOC of size* n *is optimal.*

Proof. For v as given, we have

$$\left\lfloor \frac{v-1}{k(k-1)} \right\rfloor = n + \left\lfloor \frac{2\ell - 1}{k(k-1)} \right\rfloor .$$

However, $2\ell - 1 < k(k-1)$ because $\ell \le \binom{k}{2}$, so

$$\left\lfloor \frac{v-1}{k(k-1)} \right\rfloor = n. \qquad \square$$

Suppose $X = \{X_1, \ldots, X_n\}$ is a $(v, k, 1)$-optical orthogonal code. We define ΔX and the leave, $L(X)$, in the obvious way:

$$\Delta X = \bigcup_{i=1}^{n} \{x - y \bmod v : x, y \in X_i, x \ne y\}$$

and

$$L(X) = \mathbb{Z}_v \setminus \Delta X.$$

The following lemma is a straightforward generalization of Lemma 3.6.

Lemma 4.2. *Suppose* X *is a* $(v, k, 1)$*-optical orthogonal code where* v *is even. Then* $\{0, v/2\} \subseteq L(X)$. *If* $v \equiv 0 \pmod 4$, *then* $|L_0(X)|$ *and* $|L_1(X)|$ *are both even. If* $v \equiv 2 \pmod 4$, *then* $|L_0(X)|$ *and* $|L_1(X)|$ *are both odd.*

Theorem 4.3. *Given* $v = k(k-1)n + 2\ell$ *with* $1 \le \ell \le \binom{k}{2}$, *define the two sets*

$$S = \left\{ \left\lfloor \frac{v}{4} \right\rfloor - h : 0 \le h \le \ell - 1 \right\}.$$

and

$$T = \left\{ h(k-h) : 0 \le h \le \left\lfloor \frac{k}{2} \right\rfloor \right\}.$$

Then a necessary condition for the existence of an optimal $(v, k, 1)$*-OOC is that at least one element of* S *is representable as a sum of* n *integers of* T.

Proof. Note than an optimal $(v, k, 1)$-OOC will have size n, from Lemma 4.1. Assume that $X = \{X_1, \ldots, X_n\}$ is an (optimal) $(v, k, 1)$-OOC.

From Lemma 4.2, we see that $v/2 \in L(X)$ and $|L_1(X)|$ has the same parity as $\frac{v}{2}$. Also, as in the proof of Lemma 3.7, $0 \le |L_1(X)| \le 2\ell - 2$. Thus, considering that the number of odd elements in $\mathbb{Z}_v$ is $v/2$, we see that the number of odd differences in $\bigcup_{i=1}^{n} \Delta X_i$ is twice an element of S.

Suppose that X_i contains exactly a_i even elements, so $k - a_i$ is the number of odd elements in X_i. Then the number of odd elements in ΔX_i is $2a_i(k - a_i)$, that is, twice an element of T. It follows that at least one element of S is representable as a sum of n integers belonging to T. $\square$

Let us see some consequences of Theorem 4.3. As a first example, we consider the cases where $k = 3$.

Corollary 4.4. *An optimal* $(v, 3, 1)$*-OOC does not exist if* $v \equiv 14, 20 \pmod{24}$.

Proof. When we take $k = 3$ in Theorem 4.3, we have $T = \{0, 2\}$. Suppose v is even and we write $v = 24t + 2w$, where $1 \leq w \leq 12$. We express v in the form $v = 6n + 2\ell$, where $1 \leq \ell \leq 3$, obtaining the values of n and ℓ and the sets S that are shown in Table 2.

Table 2: Applications of Theorem 4.3 when $k = 3$.

v	n	ℓ	S
$24t + 2$	$4t$	1	$\{6t\}$
$24t + 4$	$4t$	2	$\{6t, 6t + 1\}$
$24t + 6$	$4t$	3	$\{6t - 1, 6t, 6t + 1\}$
$24t + 8$	$4t + 1$	1	$\{6t + 2\}$
$24t + 10$	$4t + 1$	2	$\{6t + 1, 6t + 2\}$
$24t + 12$	$4t + 1$	3	$\{6t + 1, 6t + 2, 6t + 3\}$
$24t + 14$	$4t + 2$	1	$\{6t + 3\}$
$24t + 16$	$4t + 2$	2	$\{6t + 3, 6t + 4\}$
$24t + 18$	$4t + 2$	3	$\{6t + 2, 6t + 3, 6t + 4\}$
$24t + 20$	$4t + 3$	1	$\{6t + 5\}$
$24t + 22$	$4t + 3$	2	$\{6t + 4, 6t + 5\}$
$24t + 24$	$4t + 3$	3	$\{6t + 4, 6t + 5, 6t + 6\}$

When $v \equiv 14, 20 \pmod{24}$, the set S consists of a single element, which is an odd integer. Clearly it is not a sum of even integers, so we conclude from Theorem 4.3 that an optimal $(v, 3, 1)$-OOC does not exist if $v \equiv 14, 20 \pmod{24}$. □

Remark 4.5. It is well-known that an optimal $(v, 3, 1)$-OOC exists if and only if $v \not\equiv 14, 20 \pmod{24}$ (e.g., see [1, 2] for discussion about this result).

We adapt the argument used in Corollary 4.4 to prove a generalization that works for odd integers $k \not\equiv 1 \pmod 8$. First, we observe that, if k is odd, then all the elements of T are even. So we obviously get a contradiction in Theorem 4.3 if the set S consists of a single odd integer. This happens if $\ell = 1$ (so $v = nk(k - 1) + 2$) and one of the following two conditions hold:

1. $nk(k - 1) \equiv 2 \pmod 8$ ($v \equiv 0 \pmod 4$ in this case) or
2. $nk(k - 1) \equiv 4 \pmod 8$ ($v \equiv 2 \pmod 4$ in this case).

Since k is odd, we have $k \equiv 1, 3, 5, 7 \pmod 8$. We consider each case separately.

$k \equiv 1 \pmod 8$:

Here $k(k - 1) \equiv 0 \pmod 8$, neither of 1. or 2. can hold.

$k \equiv 3 \pmod 8$:

Here $k(k-1) \equiv 6 \pmod 8$. For 1., we obtain $6n \equiv 2 \pmod 8$, so $n \equiv 3 \pmod 4$ and $v = (4t + 3)k(k - 1) + 2$ for some integer t. It follows that

$$v \equiv 3k(k - 1) + 2 \pmod{4k(k - 1)}.$$

For 2., we obtain $6n \equiv 4 \pmod 8$, so $n \equiv 2 \pmod 4$ and $v = (4t+2)k(k-1)+2$ for some integer t. It follows that

$$v \equiv 2k(k-1)+2 \pmod{4k(k-1)}.$$

$\boldsymbol{k \equiv 5 \pmod 8}$**:**

Here $k(k-1) \equiv 4 \pmod 8$. For 1., we obtain $4n \equiv 2 \pmod 8$, which is impossible. For 2., we obtain $4n \equiv 4 \pmod 8$, so $n \equiv 1 \pmod 2$ and $v = (2t+1)k(k-1)+2$ for some integer t. It follows that

$$v \equiv k(k-1)+2 \pmod{2k(k-1)}.$$

$\boldsymbol{k \equiv 7 \pmod 8}$**:**

Here $k(k-1) \equiv 2 \pmod 8$. For 1., we obtain $2n \equiv 2 \pmod 8$, so $n \equiv 1 \pmod 4$ and $v = (4t+1)k(k-1)+2$ for some integer t. It follows that

$$v \equiv k(k-1)+2 \pmod{4k(k-1)}.$$

For 2., we obtain $2n \equiv 4 \pmod 8$, so $n \equiv 2 \pmod 4$ and $v = (4t+2)k(k-1)+2$ for some integer t. It follows that

$$v \equiv 2k(k-1)+2 \pmod{4k(k-1)}.$$

Summarizing the above discussion, we have the following theorem.

Theorem 4.6. *There does not exist an optimal* $(v, k, 1)$*-OOC whenever one of the following conditions hold:*

- $k \equiv 3 \pmod 8$ *and* $v \equiv 3k(k-1)+2 \pmod{4k(k-1)}$.
- $k \equiv 3 \pmod 8$ *and* $v \equiv 2k(k-1)+2 \pmod{4k(k-1)}$.
- $k \equiv 5 \pmod 8$ *and* $v \equiv k(k-1)+2 \pmod{2k(k-1)}$.
- $k \equiv 7 \pmod 8$ *and* $v \equiv k(k-1)+2 \pmod{4k(k-1)}$.
- $k \equiv 7 \pmod 8$ *and* $v \equiv 2k(k-1)+2 \pmod{4k(k-1)}$.

The following results are immediate corollaries of Theorem 4.6.

Corollary 4.7. *An optimal* $(v, 3, 1)$*-OOC does not exist if* $v \equiv 14, 20 \pmod{24}$*; an optimal* $(v, 5, 1)$*-OOC does not exist if* $v \equiv 22 \pmod{40}$*; and an optimal* $(v, 7, 1)$*-OOC does not exist if* $v \equiv 44, 86 \pmod{168}$.

Example 4.8. As an example where Theorem 4.3 can be applied to an even value of k, consider the case of an optimal $(62, 6, 1)$-OOC. Here we have $62 = 2 \times 6 \times 5 + 2 \times 1$, so $n = 2$ and $\ell = 1$. The set $S = \{15\}$ and $T = \{0, 5, 8, 9\}$. It is impossible to express 15 as the sum of two numbers from T, so we conclude that an optimal $(62, 6, 1)$-OOC does not exist.

We now prove some general nonexistence results.

Theorem 4.9. *Suppose* $1 \le \ell \le \binom{k}{2}$*, and suppose an optimal* $(2k(k-1)+2\ell, k, 1)$*-OOC exists. Define the set* R *as follows:*

$$R = \begin{cases} \left\{\left\lfloor \frac{k-\ell+1}{2} \right\rfloor + h : 0 \le h \le \ell - 1\right\} & \text{if } k \text{ is even;} \\ \{k - \ell + 2h : 0 \le h \le \ell - 1\} & \text{if } k \text{ is odd and } \ell \text{ is even;} \\ \{k - \ell + 2h + 1 : 0 \le h \le \ell - 1\} & \text{if } k \text{ and } \ell \text{ are both odd.} \end{cases}$$

Then at least one integer in the set R *can be expressed as the sum of two squares.*

Proof. First, suppose k is even. Apply Theorem 4.3. We have $v = 2k(k-1)+2\ell$ and thus we have

$$S = \left\{ \frac{k(k-1)}{2} + \left\lfloor \frac{\ell}{2} \right\rfloor - h : 0 \le h \le \ell - 1 \right\}.$$

From Lemma 3.14, we have

$$T = \left\{ \left(\frac{k}{2}\right)^2 - h^2 \ : \ 0 \le h \le \frac{k}{2} \right\}.$$

From Theorem 4.3, we have

$$\frac{k(k-1)}{2} + \left\lfloor \frac{\ell}{2} \right\rfloor - h = \left(\frac{k}{2}\right)^2 - i^2 + \left(\frac{k}{2}\right)^2 - j^2$$

for integers h, i, j where $0 \le h \le \ell - 1$ and $0 \le i, j \le k/2$. Simplifying, we obtain

$$\frac{k}{2} - \left\lfloor \frac{\ell}{2} \right\rfloor + h = i^2 + j^2.$$

The result follows by noting that

$$\frac{k}{2} - \left\lfloor \frac{\ell}{2} \right\rfloor = \left\lfloor \frac{k - \ell + 1}{2} \right\rfloor$$

since k is even.

Next, suppose k is odd and ℓ is even. Here $v \equiv 0 \pmod 4$. We again apply Theorem 4.3. Here we have

$$S = \left\{ \frac{k(k-1)+\ell}{2} - h : 0 \le h \le \ell - 1 \right\}$$

and, from Lemma 3.14, we have

$$T = \left\{ \left(\frac{k}{2}\right)^2 - \left(\frac{k}{2} - h\right)^2 : 0 \le h \le \frac{k-1}{2} \right\}.$$

From Theorem 4.3, we get

$$\frac{k(k-1)+\ell}{2} - h = \left(\frac{k}{2}\right)^2 - \left(\frac{k}{2} - i\right)^2 + \left(\frac{k}{2}\right)^2 - \left(\frac{k}{2} - j\right)^2$$

for integers h, i, j where $0 \leq h \leq \ell - 1$ and $0 \leq i, j \leq (k-1)/2$. Simplifying, we have

$$2k(k-1) + 2\ell - 4h = 2k^2 - (k-2i)^2 - (k-2j)^2.$$

Therefore,

$$(k-2i)^2 + (k-2j)^2 = 2(k - \ell + 2h),$$

and the result follows.

The final case is when k and ℓ are both odd. The proof for this case is very similar to previous case. □

Corollary 4.10. *Suppose that k has prime decomposition that contains a prime $p \equiv 3 \pmod 4$ raised to an odd power. Then an optimal $(2k(k-1)+2, k, 1)$-OOC does not exist.*

Proof. Suppose an optimal $(2k(k-1)+2, k, 1)$-OOC exists. Take $\ell = 1$ in Theorem 4.9; then $h = 0$ in the definition of the set R. It follows that, if k is even, then $k/2$ is the sum of two squares; and if k is odd, then k is the sum of two squares. The desired result then follows from Theorem 1.9. □

Remark 4.11. The smallest applications of Corollary 4.10 are when $k = 3$ and $k = 6$. We conclude that optimal $(14, 3, 1)$-OOC and optimal $(62, 6, 1)$-OOC do not exist. We note that Corollary 4.4 also shows that an optimal $(14, 3, 1)$-OOC does not exist. Also, Example 4.8 proved the nonexistence of an optimal $(62, 6, 1)$-OOC using a slightly different argument. The next values of k covered by Corollary 4.10 are $k = 7$, 11, 12, 14, 15, 19, 21, 22, 23 and 24.

Now we prove a nonexistence result that holds for arbitrarily large values of ℓ.

Theorem 4.12. *For any positive integer ℓ, there are infinitely many even integers k such that an optimal $(2k(k-1)+2\ell, k, 1)$-OOC does not exist.*

Proof. Using Lemma 1.10, choose an even integer k such that $\lfloor \frac{k-\ell+1}{2} \rfloor + h$ is not the sum of two squares, for $0 \leq h \leq \ell - 1$. Then apply Theorem 4.9. □

We next prove the nonexistence of certain optimal $(3k(k-1)+2, k, 1)$-OOC with k even.

Theorem 4.13. *There does not exist an optimal $(3k(k-1)+2, k, 1)$-OOC if $k = (4^{a+1}(24c+7)+2)/3$ with $a, c \geq 0$ or if $k = 4^{a+1}(8c+5)$ with $a, c \geq 0$.*

Proof. Assume that X is an optimal $(3k(k-1)+2, k, 1)$-OOC with k even. We apply Theorem 4.3 with $n = 3$. Here, with the usual notation, we have

$$S = \left\{ \frac{3k^2 - 3k + 2}{4} \right\}$$

if $k \equiv 2 \pmod 4$, and

$$S = \left\{ \frac{3k^2 - 3k}{4} \right\}$$

if $k \equiv 0 \pmod 4$. Also, as in the proof of Theorem 4.9, we have

$$T = \left\{ \left(\frac{k}{2}\right)^2 - h^2 : 0 \leq h \leq \frac{k}{2} \right\}.$$

It follows that the unique element in the set S must be a sum of three elements of T.

For $k \equiv 2 \pmod 4$, we have

$$\frac{3k^2 - 3k + 2}{4} = 3\left(\frac{k}{2}\right)^2 - (h_1^2 + h_2^2 + h_3^2)$$

for integers h_1, h_2, h_3. It follows that $(3k - 2)/4$ is a sum of three squares, and hence $(3k - 2)/4$ is not of the form $4^a(8b + 7)$ where $a, b \geq 0$. Thus, if

$$\frac{3k - 2}{4} = 4^a(8b + 7), \tag{4.1}$$

an optimal $(3k(k - 1) + 2, k, 1)$-OOC does not exist. (4.1) holds if and only if

$$k = \frac{4^{a+1}(8b + 7) + 2}{3}.$$

In order for k to be an integer, b must be divisible by 3, say $b = 3c$. Therefore, if

$$k = \frac{4^{a+1}(24c + 7) + 2}{3},$$

where $a, c \geq 0$, an optimal $(3k(k - 1) + 2, k, 1)$-OOC does not exist.

The case $k \equiv 0 \pmod 4$ is similar. Here, $3k/4$ must be a sum of three squares, and hence $3k/4$ is not of the form $4^a(8b+7)$. Therefore an optimal $(3k(k-1)+2, k, 1)$-OOC does not exist if

$$k = \frac{4^{a+1}(8b + 7)}{3}.$$

In order for k to be an integer, we must have $b \equiv 1 \pmod 3$, say $b = 3c + 1$. Then $(8b + 7)/3 = 8c + 5$. We conclude that an optimal $(3k(k - 1) + 2, k, 1)$-OOC does not exist if

$$k = 4^{a+1}(8c + 5),$$

where $a, c \geq 0$. □

Finally, we prove the nonexistence of certain optimal $(3k(k - 1) + 4, k, 1)$-OOC with k even.

Theorem 4.14. *There does not exist an optimal* $(3k(k - 1) + 4, k, 1)$*-OOC if* $k = (4^{a+3}(24c + 23) - 2)/3$ *with* $a, c \geq 0$ *or if* $k = 4^{a+3}(8c + 5)$ *with* $a, c \geq 0$.

Proof. We proceed as in the proof of Theorem 4.13, by applying Theorem 4.3 with $n = 3$. Assume that X is an optimal $(3k(k - 1) + 4, k, 1)$-OOC with k even. We have

$$S = \left\{ \frac{3k^2 - 3k - 2}{4}, \frac{3k^2 - 3k - 2}{4} + 1 \right\}$$

if $k \equiv 2 \pmod 4$, and

$$S = \left\{ \frac{3k^2 - 3k}{4}, \frac{3k^2 - 3k}{4} + 1 \right\}$$

if $k \equiv 0 \pmod 4$. Also,

$$T = \left\{ \left(\frac{k}{2}\right)^2 - h^2 : 0 \le h \le \frac{k}{2} \right\}.$$

At least one element in the set S must be a sum of three elements of T.

Suppose $k \equiv 2 \pmod 4$ and let $n = (3k+2)/4 - 1$. Proceeding as in the proof of Theorem 4.13, we see that one of n or $n+1$ is the sum of three squares. However, if $n + 1 = 4^a(8b + 7)$ where $a \ge 2$, then Lemma 1.11 implies that neither n nor $n+1$ is the sum of three squares. In this case, optimal $(3k(k-1)+4, k, 1)$-OOC does not exist. This occurs when

$$\frac{3k+2}{4} = 4^a(8b+7),$$

with $a \ge 2$, or

$$k = \frac{4^{a+1}(8b+7) - 2}{3}.$$

Since k is an integer, $b \equiv 2 \pmod 3$, say $b = 3c + 2$, and then

$$k = \frac{4^{a+1}(24c+23) - 2}{3},$$

where $a \ge 2$. For k of this form, an optimal $(3k(k-1)+4, k, 1)$-OOC does not exist.

Suppose $k \equiv 0 \pmod 4$ and let $n = 3k/4 - 1$. Here, by the same logic as above, an optimal $(3k(k-1)+4, k, 1)$-OOC does not exist when

$$\frac{3k}{4} = 4^a(8b+7),$$

or

$$k = \frac{4^{a+1}(8b+7)}{3},$$

where $a \ge 2$. Here, $b \equiv 1 \pmod 3$, say $b = 3c + 1$, and then

$$k = 4^{a+1}(8c+5),$$

where $a \ge 2$. For k of this form, an optimal $(3k(k-1)+4, k, 1)$-OOC does not exist. □

5 Other types of designs

In this section, we obtain necessary conditions for the existence of certain cyclic Steiner 2-designs and relative difference families using the techniques we have developed.

5.1 Cyclic Steiner 2-designs

A *Steiner* 2*-design of order* v *and block-size* k, denoted as $\mathrm{S}(2,k,v)$, consists of a set of k-subsets (called *blocks*) of a v-set (whose elements are called *points*) such that every pair of points occurs in a unique block. An $\mathrm{S}(2,k,v)$ is *cyclic* if there is a cyclic permutation of the v points that maps every block to a block.

It is well-known that a cyclic $\mathrm{S}(2,k,v)$ exists only if $v \equiv 1$ or $k \pmod{k(k-1)}$. A cyclic $\mathrm{S}(2,k,v)$ with $v \equiv 1 \pmod{k(k-1)}$ is equivalent to a $(v,k,1)$-OOC of size n; in this case the leave is $\{0\}$. Further, a cyclic $\mathrm{S}(2,k,v)$ with $v \equiv k \pmod{k(k-1)}$ is equivalent to a $(v,k,1)$-OOC of size n whose leave is the subgroup of $\mathbb{Z}_v$ of order k.

Assume that $X = \{X_1,\dots,X_n\}$ is an $(k(k-1)n+k,k,1)$-OOC of size n that is obtained from a cyclic $\mathrm{S}(2,k,k(k-1)n+k)$ with both k and n even. The leave $L(X)$ has exactly $k/2$ odd elements and therefore the number of odd differences in $\bigcup_{i=1}^{n} \Delta X_i$ is $k(k-1)n/2$.

Reasoning as in the proof of Theorem 4.9, we see that $k(k-1)n/4$ is the sum of n integers in the set

$$T = \left\{ \left(\frac{k}{2}\right)^2 - h^2 : 0 \le h \le \frac{k}{2} \right\}.$$

Thus we have

$$\frac{kn}{4} = h_1^2 + h_2^2 + \cdots + h_n^2$$

for a suitable n-tuple $(h_1,\dots,h_n)$ of nonnegative integers, each of which does not exceed $k/2$. Using Lagrange's Four-square Theorem (Theorem 1.9), it is an easy exercise to see that such an n-tuple certainly exists for $n \ge 4$.

However, if $n = 2$, this is not always the case. Here we require

$$\frac{k}{2} = h_1^2 + h_2^2$$

for nonnegative integers $h_1, h_2 \le k/2$. As stated in Theorem 1.9, a positive integer can be written as the as a sum of two squares if and only if its prime decomposition contains no prime $p \equiv 3 \pmod 4$ raised to an odd power. So we obtain the following result.

Theorem 5.1. *If* k *is an even integer whose prime decomposition contains a prime* $p \equiv 3 \pmod 4$ *raised to an odd power, then there does not exists a cyclic* $\mathrm{S}(2,k,2k(k-1)+k)$.

We can apply Theorem 5.1 with $k = 6, 12, 14, 22, 24, 28$, etc.

Now assume that $X = \{X_1,\dots,X_n\}$ is a $(k(k-1)n+k,k,1)$-OOC that is obtained from a cyclic $\mathrm{S}(2,k,k(k-1)n+k)$ with k even and n odd. Here all the elements of the leave of X are odd, and hence all $(k(k-1)n+k)/2$ odd elements of $\mathbb{Z}_v$ have to appear in $\bigcup_{i=1}^{n} \Delta X_i$. Reasoning as above, we see that $\frac{k(k-1)n+k}{4}$ is the sum of n integers in the set

$$T = \left\{ \left(\frac{k}{2}\right)^2 - h^2 : 0 \le h \le \frac{k}{2} \right\},$$

i.e.,

$$\frac{k(n-1)}{4} = h_1^2 + h_2^2 + \cdots + h_n^2$$

for a suitable n-tuple $(h_1, \ldots, h_n)$ of nonnegative integers not exceeding $k/2$. Again, such a n-tuple exists by Lagrange's Four-square Theorem if $n \geq 5$.

But this is not always the case if $n = 3$. Here we require

$$\frac{k}{2} = h_1^2 + h_2^2 + h_3^2$$

for nonnegative integers $h_1, h_2, h_3 \leq k/2$.

Applying Legendre's Three-square Theorem (Theorem 1.9), we have the following result.

Theorem 5.2. *If $k = 2^a(8b+7)$ where a and b are nonnegative integers and a is odd, then there does not exist a cyclic $\mathrm{S}(2, k, 3k(k-1)+k)$.*

We can apply Theorem 5.1 with $k = 14, 46, 56, 62$, etc.

5.2 Relative difference families

When $G = \mathbb{Z}_v$ and the order of the subgroup H is equal to w, a $(G, H, k, 1)$-RDF is clearly a $(v, k, 1)$-OOC whose leave is the subgroup of $\mathbb{Z}_v$ of order w. In this case, some authors (e.g., [22]) speak of a w-*regular* $(v, k, 1)$-OOC. Note that a w-regular $(v, k, 1)$-OOC is optimal provided that $w \leq k(k-1)$. Also, note that a k-regular $(v, k, 1)$-OOC gives rise to a cyclic $\mathrm{S}(2, k, v)$.

Theorem 5.3. *Let G be a group with a subgroup S of index 2 and let X be a (G, H, k, λ)-relative difference family of size n, where $|H| = w$. If H is contained in S, then $kn - \lambda w$ is a sum of n squares. If H is not contained in S, then kn is a sum of n squares.*

Proof. Let us say that an element of G is *even* or *odd* according to whether it belongs to or does not belong to S, respectively. Set $X = \{X_1, \ldots, X_n\}$ and, for $i = 1, \ldots, n$, let a_i and b_i be the number of even and odd elements in X_i, respectively. The number of odd elements in ΔX_i is $2a_ib_i$ (note that here we are treating ΔX_i and ΔX as multisets since differences may be repeated). Also, by definition, the number of odd elements in ΔX is λ times the number of all odd elements of $G \setminus H$.

If H, S are subgroups of a group G with $|G : S| = 2$, then either $H \subseteq S$ or $|H \cap S| = |H|/2$. Hence, we have

$$\sum_{i=1}^{n} 2a_ib_i = \frac{\lambda v}{2} \quad \text{or} \quad \frac{\lambda(v-w)}{2},$$

according to whether H is contained or not contained in S. Thus we have:

$$\sum_{i=1}^{n} 4a_ib_i = \begin{cases} \lambda v & \text{if } H \subseteq S \\ \lambda(v-w) & \text{if } H \not\subseteq S. \end{cases}$$

Now, given that $a_i + b_i = k$, we have

$$4a_ib_i = 4a_i(k - a_i) = k^2 - (k - 2a_i)^2.$$

Replacing this in the above formula and taking account of (1.1), we get

$$\sum_{i=1}^{n} (k - 2a_i)^2 = \begin{cases} kn - \lambda w & \text{if } H \subseteq S \\ kn & \text{if } H \not\subseteq S. \end{cases}$$

and the assertion follows. □

Theorem 5.3 is trivial for $n \geq 4$ in view of Theorem 1.9. On the other hand, it gives some important information for $n = 1, 2, 3$. We now discuss several consequences of Theorem 1.9.

First, we point out a connection with the Bose-Connor Theorem (Theorem 3.3). Suppose we take $n = 1$ in Theorem 5.3 and suppose $H \subseteq S$. Recall that S is a subgroup of index 2. Denote $|G| = v = uw$, where $|H| = w$. Then Theorem 5.3 asserts that $k - \lambda w$ must be a perfect square. This result can also be obtained from Theorem 3.3, as follows. The development of the (G, H, k, λ)-relative difference family through the group G yields a divisible design with $\lambda_1 = 0$ and $\lambda_2 = \lambda$. Since H and S are subgroups of G and $H \subseteq S$, it must be the case that $w \mid \frac{v}{2}$, say $v/2 = tw$. Then $u = v/w = 2t$ is even. Therefore statement 1. of Theorem 3.3 applies, and $k^2 - \lambda v$ is a square. However, $k(k-1) - \lambda(v-w)$ from (1.1), so $k^2 - \lambda v = k - \lambda w$, so we obtain the same result.

In the special case of the preceding result where $w = 1$, we see that $k - \lambda$ is a square. This also follows from the Bruck-Ryser-Chowla Theorem (as we already discussed in Example 3.5 in the case where G is cyclic).

If we take $n = 2$ and $w = 1$, we see that, if X is a (v, k, λ)-DF with two base blocks in a group with a subgroup of index 2, then $2k - \lambda$ is a sum of two squares (this result was first shown in [17, Corollary 2.1]). Similarly, taking $n = 3$ and $w = 1$, we see that, if X is a (v, k, λ)-DF with three base blocks in a group with a subgroup of index 2, then $3k - \lambda$ is a sum of three squares (this result was first shown in [17, Corollary 2.2]).

Finally, $n \in \{2, 3\}$ and $w = k \equiv 0 \pmod 2$, then a cyclic $\mathrm{S}(2, k, k(k-1)n + k)$ exists only if k is a sum of n squares. This is equivalent to results obtained in Section 5.1.

6 Summary

We have proven a number of nonexistence results for infinite classes of modular Golomb rulers, optical orthogonal codes, cyclic Steiner systems and relative difference families. We note that very few results of this nature were previously known. Many of our new results are based on counting even and odd differences and then applying some classical results from number theory which establish which integers can be expressed as a sum of a two or three squares.

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A family of fractal non-contracting weakly branch groups

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Abstract

We construct a new example of an infinite family of groups acting on a d-adic tree, with $d \geq 2$ that is non-contracting and weakly regular branch over the derived subgroup.

Keywords: Groups of automorphisms of rooted trees, branch groups.

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1 Introduction

Weakly branch groups were first defined by Grigorchuk in 1997 as a generalization of the famous p-groups constructed by Grigorchuk himself [4, 5], and Gupta and Sidki [6]. These groups possess remarkable and exotic properties. For instance, the Grigorchuk group is the first example of a group of intermediate word growth, and amenable but not elementary amenable. Also, together with the Grigorchuk group, other subgroups of the group of automorphisms of rooted trees like the Gupta-Sidki p-groups and many groups in the family of the so-called Grigorchuk-Gupta-Sidki groups have been shown to be a counterexample to the General Burnside Problem.

For these reasons, (weakly) branch groups spread great interest among group theorists, who have actively investigated further properties of these in the recent years: just-infiniteness, fractalness, maximal subgroups, or contraction.

Roughly speaking, a group is said to be *contracting* if the sections of every element are "shorter" than the element itself, provided the element does not belong to a fixed finite set, called the nucleus (see the exact definition in Section 2).

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Even though in the literature there are many examples of weakly branch contracting groups, not much is known about weakly branch groups that are non-contracting. In 2005 Dahmani [2] provided the first example of a non-contracting weakly regular branch automaton group. Another example with similar properties was constructed by Mamaghani in 2011 [7]. Both are examples of groups acting on the binary tree. We also point out that in [9] Sidki and Wilson proved in particular that every group acting on the binary tree with finite abelianization (including non-contracting groups) embeds in a branch group. This provides more examples of non-contracting branch groups acting on the binary tree.

For $d \geq 3$, the Hanoi Towers group $\mathcal{H}(d) \leq \operatorname{Aut} \mathcal{T}_d$ (which represents the famous game of Hanoi Towers on d pegs) is non-contracting and only weakly branch. To the best of our knowledge if $d > 3$ it is not known if these groups can be branch. For more information on the topic, see [3] and [10].

In this paper we explicitly construct an example of an infinite family of non-contracting weakly branch groups acting on d-adic trees for any $d \geq 2$. This result gives a wealth of examples of groups with these properties. In the following we denote with $\operatorname{Aut} \mathcal{T}_d$ the group of automorphisms of a d-adic tree.

Theorem 1.1. *For any $d \geq 2$, there exists a group $\mathcal{M}(d) \leq \operatorname{Aut} \mathcal{T}_d$ that is weakly regular branch over its derived subgroup, non-contracting and fractal.*

1.1 Organization

In Section 2 we give some definitions of groups acting on regular rooted trees and of properties like fractalness, branchness and contraction. In Section 3 we introduce these groups and we prove the main theorem together with some additional results regarding the order of elements of $\mathcal{M}(d)$.

2 Preliminaries

In this section we fix some terminology regarding groups of automorphisms of d-adic (rooted) trees. For further information on the topic, see [1] or [8].

Let d be a positive integer, and $\mathcal{T}_d$ the d-adic tree. We denote with $\operatorname{Aut} \mathcal{T}_d$ the group of automorphisms of $\mathcal{T}_d$. We let $\mathcal{L}_n$ be the nth level of $\mathcal{T}_d$, and $\mathcal{L}_{\geq n}$ the levels of the tree from level n and below.

The stabilizer of a vertex u of the tree is denoted by $\operatorname{st}(u)$, and, more generally, the nth *level stabilizer* $\operatorname{st}(n)$ is the subgroup of $\operatorname{Aut} \mathcal{T}_d$ that fixes every vertex of $\mathcal{L}_n$. If $G \leq \operatorname{Aut} \mathcal{T}_d$, we define the nth level stabilizer of G as $\operatorname{st}_G(n) = \operatorname{st}(n) \cap G$. Notice that stabilizers are normal subgroups of the corresponding group. We let ψ be the isomorphism

$$\begin{aligned} \psi\colon \operatorname{st}(1) &\longrightarrow \operatorname{Aut} \mathcal{T}_d \times \overset{d}{\cdots} \times \operatorname{Aut} \mathcal{T}_d \\ g &\longmapsto (g_u)_{u \in \mathcal{L}_1}, \end{aligned}$$

where g_u is the section of g at the vertex u, i.e. the action of g on the subtree $\mathcal{T}_u$ that hangs from the vertex u. Let S_d be the symmetric group on d letters. An automorphism $a \in \operatorname{Aut} \mathcal{T}_d$ is called *rooted* if there exists a permutation $\sigma \in S_d$ such that a permutes rigidly the vertices of the subtrees hanging from the first level of the tree according to the permutation σ, i.e. if $v = xu \in V(\mathcal{T}_d)$, with $x \in \mathcal{L}_1$, then $a(xu) = \sigma(x)u$. We usually identify a and σ.

Notice that if $g \in \mathrm{st}(1)$ with $\psi(g) = (g_1, \dots, g_d)$, and σ is a rooted automorphism, then,

$$\psi(g^{\sigma}) = \left(g_{\sigma^{-1}(1)}, \dots, g_{\sigma^{-1}(d)}\right). \tag{2.1}$$

Any element $g \in G$ can be written uniquely in the form $g = h\sigma$, where $h \in \mathrm{st}(1)$ and σ is a rooted automorphism.

Notice also that the decomposition $g = h\sigma$, together with the action (2.1), yields isomorphisms

$$\begin{aligned}\operatorname{Aut}\mathcal{T}_d &\cong \mathrm{st}(1) \rtimes S_d \cong \left(\operatorname{Aut}\mathcal{T}_d \times \overset{d}{\dots} \times \operatorname{Aut}\mathcal{T}_d\right) \rtimes S_d \\ &\cong \operatorname{Aut}\mathcal{T}_d \wr S_d \cong ((\cdots \wr S_d) \wr S_d) \wr S_d.\end{aligned} \tag{2.2}$$

Throughout the paper, we will use the following shorthand notation: let $f \in \operatorname{Aut}\mathcal{T}$ of the form $f = gh$, where $g \in \mathrm{st}_G(1)$ and h is the rooted automorphism corresponding to the permutation $\sigma \in S_d$. If $\psi(g) = (g_1, \dots, g_d)$, we write $f = (g_1, \dots, g_d)\sigma$.

Definition 2.1. Let $G \leq \operatorname{Aut}\mathcal{T}_d$, and let $V(\mathcal{T}_d)$ be the set of vertices of $\mathcal{T}_d$. Then:

(a) The group G is said to be *self-similar* if for any $g \in G$ we have

$$\{g_u \mid g \in G, u \in V(\mathcal{T}_d)\} \subseteq G.$$

In other words, the sections of g at any vertex are still elements of G. For example, $\operatorname{Aut}\mathcal{T}_d$ is self-similar.

(b) A self-similar group G is said to be *fractal* if $\psi_u(\mathrm{st}_G(u)) = G$ for all $u \in V(\mathcal{T}_d)$, where ψ_u is the homomorphism sending $g \in \mathrm{st}(u)$ to its section g_u.

To prove that a group is self-similar it suffices to show that the condition above is satisfied by the vertices of the first level of the tree (see [3, Proposition 3.1]). The situation is similar in the case of fractal groups. More precisely, using Lemma 2.2, we deduce that to show that a group G is fractal, it is enough to check the vertices in the first level of $\mathcal{T}_d$. We recall that G is said to be *level transitive* if it acts transitively on every level of the tree.

Lemma 2.2 ([11, Lemma 2.7])**.** *If $G \leq \operatorname{Aut}\mathcal{T}_d$ is transitive on the first level and $\psi_x(\mathrm{st}_G(x)) = G$ for some $x \in \mathcal{L}_1$, then G is fractal and level transitive.*

Here we present a family of non-contracting weakly branch groups. To this end, in the following, we recall the corresponding two definitions.

Definition 2.3. A self-similar group $G \leq \operatorname{Aut}\mathcal{T}_d$ is *contracting* if there exists a finite subset $\mathcal{F} \subseteq G$ such that for every $g \in G$ there is n such that g_v belongs to $\mathcal{F}$ for all vertices v of $\mathcal{L}_{\geq n}$. Note that if you take two finite sets $\mathcal{F}_1$ and $\mathcal{F}_2$ satisfying the condition on the sections above, then also $\mathcal{F}_1 \cap \mathcal{F}_2$ will satisfy the condition. For this reason, one can consider the set that is intersection of such sets. This is called the *nucleus* of G and it is denoted by $\mathcal{N}$.

Definition 2.4. Let G be a self-similar subgroup of $\operatorname{Aut}\mathcal{T}_d$. We say that G is *weakly regular branch* over a subgroup $K \leq G$ if G is level transitive and we have

$$\psi(K \cap \mathrm{st}_G(1)) \geq K \times \cdots \times K.$$

If, additionally, K is of finite index in G, then G is said to be *regular branch* over K.

3 The groups $\mathcal{M}(d)$

Let $d \geq 2$, and let $\mathcal{T}_d$ be the d-adic tree. The group $\mathcal{M}(d) \leq \operatorname{Aut} \mathcal{T}_d$ is generated by d elements $m_1, \ldots, m_d$, where $m_1, \ldots, m_d$ are defined recursively as follows:

$$\begin{aligned} m_1 &= (1, \ldots, 1, m_1)(1 \ \ldots \ d) \\ m_2 &= (1, \ldots, 1, m_2, 1)(1 \ \ldots \ d-1) \\ m_3 &= (1, \ldots, m_3, 1, 1)(1 \ \ldots \ d-2) \\ &\ \vdots \\ m_{d-1} &= (1, m_{d-1}, 1, \ldots, 1)(1\ 2) \\ m_d &= (m_1, \ldots, m_d). \end{aligned}$$

For example, for $d = 3$, we have $\mathcal{M}(3) = \langle m_1, m_2, m_3 \rangle$, where

$$m_1 = (1, 1, m_1)(1\ 2\ 3), \quad m_2 = (1, m_2, 1)(1\ 2), \quad m_3 = (m_1, m_2, m_3).$$

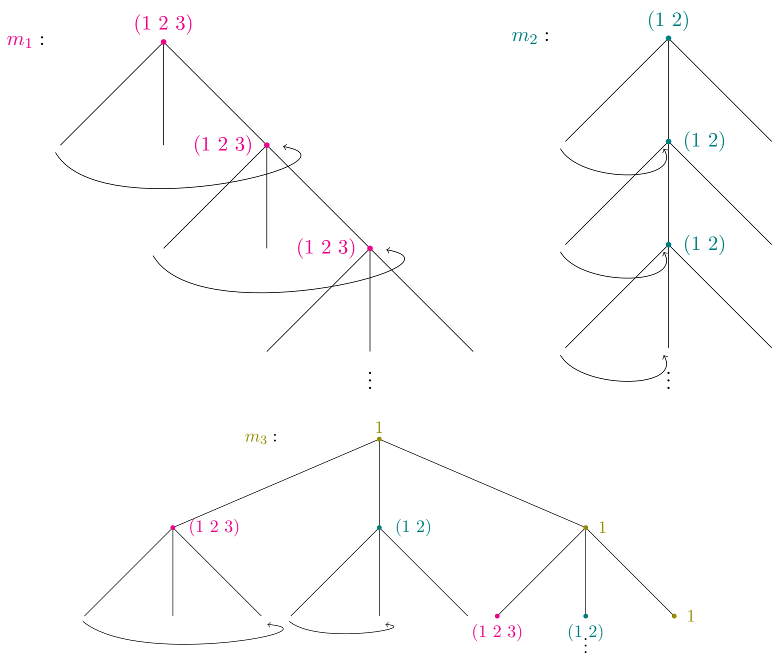

Figure 1: The generators of $\mathcal{M}(3)$.

3.1 Proof of the main theorem

In this section we prove the main result of the paper. In order to ease notation, and unless it is strictly necessary, we will simply write $\mathcal{M}$ to denote an arbitrary group $\mathcal{M}(d)$.

Proposition 3.1. *The group $\mathcal{M}$ is fractal and level transitive.*

Proof. Notice that the group is transitive on the first level because the rooted part of the generator m_1 is $(1\ 2\ \dots\ d)$. Also, it is straightforward to see that the group is self-similar, since the sections of every generator at the first level are generators of $\mathcal{M}$. To see that $\mathcal{M}$ is fractal, note that

$$\begin{aligned} m_1^d &= (m_1, \dots, m_1) \\ m_d^{m_1^{d-2}} &= (m_3^{m_1}, \dots, m_2) \\ &\vdots \\ m_d^{m_1^2} &= (m_{d-1}^{m_1}, \dots, m_{d-2}) \\ m_d^{m_1} &= (m_d^{m_1}, \dots, m_{d-1}) \\ m_d &= (m_1, \dots, m_d). \end{aligned}$$

Then in the last component of the elements above we obtain all the generators of $\mathcal{M}$. Using Lemma 2.2, we conclude that $\mathcal{M}$ is level transitive and fractal. □

Proposition 3.2. *Let $d \geq 2$. Then the group $\mathcal{M}(d)$ is weakly regular branch over its derived subgroup $\mathcal{M}'(d)$.*

Proof. We will distinguish the case $d = 2$, and $d \geq 3$ separately. Let $d = 2$. The element $[m_1, m_2]$ is non-trivial since

$$[m_1, m_2] = (m_1^{-1} m_2^{-1} m_1^2, m_1^{-1} m_2),$$

and $m_1^{-1} m_2 \notin \mathrm{st}_{\mathcal{M}}(1)$. Then $\mathcal{M}(2)'$ is non-trivial, and we have

$$[m_1^2, m_2] = (1, [m_1, m_2]). \tag{3.1}$$

From (3.1) and since $\mathcal{M}(2)' = \langle [m_1, m_2] \rangle^{\mathcal{M}(2)}$, we obtain that $\{1\} \times \mathcal{M}(2)' \leq \psi(\mathcal{M}(2)')$. As $\mathcal{M}(2)$ is level transitive, we conclude that $\mathcal{M}(2)' \times \mathcal{M}(2)' \leq \psi(\mathcal{M}(2)')$, as desired.

Let $d \geq 3$, and write $\mathcal{M}$ for $\mathcal{M}(d)$. First we show that $\mathcal{M}'$ is non-trivial. Let us denote $\sigma = (1\ 2\ \dots\ d)$ and $\tau = (1\ 2\ \dots\ d-1)$. We have

$$\begin{aligned} [m_1, m_2] &= \sigma^{-1}(1, \dots, 1, m_1^{-1})\tau^{-1}(1, \dots, 1, m_2^{-1}, m_1)\sigma(1, \dots, 1, m_2, 1)\tau \\ &= (1, \dots, 1, m_1^{-1})^{\sigma}(1, \dots, 1, m_2^{-1}, m_1)^{\tau\sigma}(1, \dots, 1, m_2, 1)^{\tau^{\sigma}}[\sigma, \tau] \\ &= (m_1^{-1}, 1, \dots, 1)(m_1, m_2^{-1}, 1, \dots, 1)(1, \dots, 1, m_2)(1\ 2\ d). \end{aligned}$$

Hence, we obtain that

$$[m_1, m_2] = (1, m_2^{-1}, 1, \dots, 1, m_2)(1\ 2\ d). \tag{3.2}$$

By (3.2), we have $[m_1, m_2] \notin \mathrm{st}_{\mathcal{M}}(1)$, thus $\mathcal{M}'$ is non-trivial.

Now, for $i = 1, \dots, d-2$, and $j = i+1, \dots, d-1$, we have

$$[m_i^{d+1-i}, m_j]^{m_1^{d-1}} = (1, \dots, 1, [m_i, m_j]). \tag{3.3}$$

Then in order to prove that $\{1\} \times \cdots \times \{1\} \times \mathcal{M}' \leq \psi(\mathcal{M}' \cap \mathrm{st}_{\mathcal{M}}(1))$, it only remains to show that for any $i = 1, \dots, d-1$, there exists $x(i) \in \mathcal{M}' \cap \mathrm{st}_{\mathcal{M}}(1)$ such that

$$x(i) = (1, \dots, 1, [m_i, m_d]).$$

To find such $x(i)$, we first observe that

$$[(m_i^{d+1-i})^{m_1^{i-1}}, m_d] = (1, \overset{i}{\ldots}, 1, [m_i, m_{i+1}], \ldots, [m_i, m_{d-1}], [m_i, m_d]).$$

In order to cancel all these commutators above except for the last component, we use (3.3), and we observe that since $\mathcal{M}$ is level transitive, if we conjugate with a suitable power of m_1, we get $[m_i, m_{i+1}]^{-1}, \ldots, [m_i, m_{d-1}]^{-1}$ in each component. For example, if $i = 2$, we have

$$[(m_2^{d-1})^{m_1}, m_d] = (1, 1, [m_2, m_3], [m_2, m_4], \ldots, [m_2, m_d]).$$

By using the considerations above, we obtain that $x(2)$ must be of the form

$$\begin{aligned} x(2) &= [m_3, m_2^{d-1}]^{m_1^2}[m_4, m_2^{d-1}]^{m_1^3} \ldots [m_{d-1}, m_2^{d-1}]^{m_1^{d-2}}[(m_2^{d-1})^{m_1}, m_d] \\ &= (1, \ldots, 1, [m_2, m_d]). \end{aligned}$$ □

To prove last part of the main theorem (that $\mathcal{M}(d)$ is non-contracting), we need some preliminary tools. Namely, we show some results regarding the order of elements of $\mathcal{M}(d)$. We will handle the case $d = 2$, and $d > 2$ separately. More precisely, we first prove that $\mathcal{M}(2)$ is torsion-free, and then, for $d > 2$, we show that the groups $\mathcal{M}(d)$ are neither torsion-free nor torsion, contrary to the case $d = 2$.

The following Remark 3.3 and Lemma 3.4 are key steps to prove that $\mathcal{M}(2)$ is torsion-free. We write $\mathcal{M}$ for $\mathcal{M}(2)$.

Remark 3.3. Let $h \in \mathcal{M}'$ with $h = (h_1, h_2)$. Then $h_1 h_2 \in \mathcal{M}'$.

Proof. Consider the following map ρ:

$$\begin{aligned} \rho\colon \mathrm{st}_{\mathcal{M}}(1) &\to \mathcal{M} &&\to \mathcal{M}/\mathcal{M}' \\ (h_1, h_2) &\mapsto h_1 h_2 &&\mapsto \overline{h_1 h_2}. \end{aligned}$$

Note that ρ is a homomorphism of groups since $\mathcal{M}/\mathcal{M}'$ is abelian. As $\mathrm{st}_{\mathcal{M}}(1)/\operatorname{Ker}\rho$ is abelian, $\mathcal{M}' \leq \operatorname{Ker}\rho$. This concludes the proof. □

In the proof of next lemma, for a prime p we denote with $\nu_p(m)$ the *p-adic valuation* of m, that is the highest power of p that divides m.

Lemma 3.4. *We have* $\mathcal{M}' = (\mathcal{M}' \times \mathcal{M}')\langle [m_1, m_2]\rangle$. *Furthermore*

$$\mathcal{M}/\mathcal{M}' \cong \langle m_1\mathcal{M}\rangle \times \langle m_2\mathcal{M}\rangle \cong \mathbb{Z} \times \mathbb{Z}.$$

Proof. Since $\mathcal{M}$ is weakly regular branch over $\mathcal{M}'$ by Proposition 3.2, and

$$[m_1, m_2] = ([m_1, m_2]m_2^{-1}m_1, m_1^{-1}m_2),$$

we deduce that $(m_2^{-1}m_1, m_1^{-1}m_2)$ is an element of $\mathcal{M}'$. Furthermore, we claim that the elements $[m_1, m_2]^y$ where $y \in \{m_1, m_2, m_1^{-1}, m_2^{-1}\}$ are in $\langle [m_1, m_2]\rangle$ modulo $\mathcal{M}' \times \mathcal{M}'$. Indeed, we have

$$\begin{aligned} [m_1, m_2]^{m_1} &= ({m_1}^{-2} m_2 m_1, {m_1}^{-1} m_2^{-1} {m_1}^2) \\ &= ([{m_1}^2, {m_2}^{-1}] m_2 {m_1}^{-1}, [m_1, m_2] {m_2}^{-1} m_1) \\ &\equiv ({m_1}^{-1} m_2, {m_2}^{-1} m_1) \pmod{\mathcal{M}' \times \mathcal{M}'}, \end{aligned}$$

and similarly for the other commutators. Thus $\mathcal{M}' = (\mathcal{M}' \times \mathcal{M}')\langle[m_1, m_2]\rangle$, as required.

Now we claim that m_1 is of infinite order. By way of contradiction suppose that, for some k, m_1 has order $n = 2k$, as m_1 has order 2 modulo the first level stabilizer. We have

$$m_1^n = (m_1^k, m_1^k) = (1, 1),$$

which yields a contradiction as $k < n$. This concludes the proof of the claim and implies that also m_2 is of infinite order, since $m_2 = (m_1, m_2)$.

Now we want to show that if $m_1^i m_2^j \in \mathcal{M}'$, then necessarily $i = j = 0$. As $m_1^i m_2^j \in \mathcal{M}' \leq \mathrm{st}_{\mathcal{M}}(1)$, then i must be even. By way of contradiction, we choose the element $m_1^i m_2^j \in \mathcal{M}'$ subject to the condition that i is divisible by the least possible positive power of 2, say 2^a, for some a. In other words, $\nu_2(i) = a$. Then if $m_1^r m_2^s \in \mathcal{M}'$, necessarily $2^a \mid r$. Note that it cannot happen that $r = 0$ and $s \neq 0$ as m_2 is of infinite order. Now, writing $i = 2i_1$ for some i_1, we have

$$m_1^i m_2^j = ({m_1}^{i_1+j}, {m_1}^{i_1}{m_2}^j) \equiv [m_1^k, m_2^k] \equiv ({m_1}^k {m_2}^{-k}, {m_1}^{-k}{m_2}^k) \pmod{\mathcal{M}' \times \mathcal{M}'}.$$

This implies that ${m_1}^{i_1+j-k}{m_2}^k \in \mathcal{M}'$ and ${m_1}^{i_1+k}{m_2}^{j-k} \in \mathcal{M}'$. As $2^a \mid i_1 + j - k$ and $2^a \mid i_1 + k$, then 2^a divides also j. This is because $2^a \mid 2i_1 + j = i + j$ and by hypothesis $2^a \mid i$. Finally, we also have ${m_1}^{i_1+k}{m_2}^{j-k} \in \mathcal{M}'$, from which we get

$${m_1}^{i_1+k}{m_2}^{j-k} = \left(m_1^{\frac{i_1+k}{2}+j-k}, m_1^{\frac{i_1+k}{2}} m_2^{j-k}\right).$$

By Remark 3.3, we have $m_1^{i_1+j} m_2^{j-k} \in \mathcal{M}'$ which implies that $2^a \mid i_1 + j$. As $\nu_2(i_1) = a - 1$ and $2^a \mid j$, then $\nu_2(i_1 + j) = a - 1$, a contradiction as $2^a \mid i_1 + j$. This completes the proof. □

As a consequence, we prove the following.

Proposition 3.5. *The group* $\mathcal{M}(2)$ *is torsion-free.*

Proof. Suppose by way of contradiction that there exists an element of finite order in $\mathcal{M}$. Since $\mathcal{M}/\mathcal{M}' \cong \mathbb{Z} \times \mathbb{Z}$ by Lemma 3.4, then this element must lie in $\mathcal{M}' \leq \mathrm{st}_{\mathcal{M}}(1)$. Suppose that among all elements of finite order, we take the element g that lies in $\mathrm{st}_{\mathcal{M}}(n) \setminus \mathrm{st}_{\mathcal{M}}(n+1)$, with n minimum with this property. Write $g = (g_1, g_2)$. As g is of finite order, then also g_1, g_2 must be of finite order. By our minimality assumption of n, the elements g_1, g_2 must lie at least in $\mathrm{st}_{\mathcal{M}}(n)$. This implies that $g = (g_1, g_2) \in \mathrm{st}_{\mathcal{M}}(n+1)$, a contradiction to the fact that $g \in \mathrm{st}_{\mathcal{M}}(n) \setminus \mathrm{st}_{\mathcal{M}}(n+1)$. □

In the following we determine the order of some elements of $\mathcal{M}(d)$, for $d > 2$.

Proposition 3.6. *Let* $d > 2$. *Then the group* $\mathcal{M}(d)$ *is neither torsion-free nor torsion.*

Proof. For ease of notation we write $\mathcal{M}$ for $\mathcal{M}(d)$. We start by proving that the given generators of $\mathcal{M}$ are of infinite order. Consider m_1, and suppose by way of contradiction that its order is n. Then if $m_1^n = 1$, we obtain that m_1^n must lie in $\mathrm{st}_{\mathcal{M}}(1)$. Also, its order must be a multiple of d, say $n = dk$ for some k, since m_1 has order d modulo the first level stabilizer. Since $m_1 = (1, \ldots, 1, m_1)(1\ 2\ \ldots\ d)$, we obtain

$$m_1^n = (m_1^k, \ldots, m_1^k) = (1, \ldots, 1).$$

This yields a contradiction since $m_1^k = 1$ and $k < n$. Similar arguments can be used for the generators $m_2, \dots, m_{d-1}$, and m_d has infinite order because $m_d = (m_1, \dots, m_d)$.

Furthermore, by (3.2), we have

$$[m_1, m_2] = (1, m_2^{-1}, 1, \dots, 1, m_2)(1\ 2\ d).$$

Thus it follows readily that $[m_1, m_2]^3 = 1$. Hence $\mathcal{M}$ is not torsion-free. □

We conclude the paper by proving the remaining part of the main theorem.

Proposition 3.7. *The group* $\mathcal{M}$ *is non-contracting.*

Proof. Suppose by way of contradiction that $\mathcal{M}$ is contracting with nucleus $\mathcal{N}$. Notice that the element $m_d^{m_1}$ stabilizes the vertex 1. As a consequence, by induction, $m_d^{m_1}$ fixes all the vertices of the path $v = 1.\overset{n}{\dots}.1$ for all $n \geq 1$. Also, $(m_d^{m_1})_v = m_d^{m_1}$. Clearly, this implies that $m_d^{m_1}$ lies in $\mathcal{N}$. Consider now a power k of $m_d^{m_1}$. Arguing as before, we obtain again that $(m_d^{m_1})^k$ fixes v and its section at v is $(m_d^{m_1})^k$. Thus, $(m_d^{m_1})^k \in \mathcal{N}$ for any $k \geq 1$. This concludes the proof since $m_d^{m_1}$ has infinite order. □

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From Farey fractions to the Klein quartic and beyond*

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Abstract

In a paper published in 1878/79 Klein produced his famous 14-sided polygon representing the Klein quartic, his Riemann surface of genus 3 which has $\mathrm{PSL}(2,7)$ as its automorphism group. The construction and method of side pairings are fairly complicated. By considering the Farey map modulo 7 we show how to obtain a fundamental polygon for Klein's surface using arithmetic. Now the side pairings are immediate and essentially the same as in Klein's paper. We also extend his work from 7 to 11 as Klein also did in a follow-up paper of 1879.

Keywords: Riemann surfaces, Klein quartic, regular maps, Farey tessellation, modular group, principal congruence subgroups.

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1 Introduction

The Klein quartic was introduced in one of Felix Klein's most famous papers, [5] of 1878/79. A slightly updated version appeared in Klein's Collected Works [7], while for a translation of this see the book *The Eightfold Way, the Beauty of Klein's Quartic Curve* edited by Silvio Levy [8]. This algebraic curve, whose equation is $x^3y + y^3z + z^3x = 0$, gives the compact Riemann surface of genus 3 with 168 automorphisms, the maximum number by the Hurwitz bound.

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Let $\mathbb{H}$ denote the upper-half complex plane and let $\mathbb{H}^* = \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$. Klein's surface is $\mathbb{H}^*/\Gamma(7)$, where $\Gamma(7)$ is the principal congruence subgroup mod 7 of the classical modular group $\Gamma = \mathrm{PSL}(2, \mathbb{Z})$. (For this concept see [4, p. 301].) Klein studies the Riemann surface of the Klein quartic by constructing his famous 14-sided fundamental region with its side identifications. See sections 11 and 12 of [5] for the construction and between pages 448 and 449 of [5], page 126 of [7], or page 320 of [8] for the figure itself.

Our approach is to construct a fundamental region for Klein's surface using the Farey tessellation $\mathcal{M}_3$ of $\mathbb{H}^*$, a triangular tessellation of $\mathbb{H}^*$ which we define in §2, and which was shown to be the universal triangular tessellation [10]. In §3 and §4, we study the *level n Farey map* $\mathcal{M}_3/\Gamma(n)$, through the correspondence of its directed edges with the elements of $\Gamma/\Gamma(n)$ and the correspondence of its vertices with the cosets of $\Gamma_1(n)$ in Γ. In §5 and §6, we study the level 7 Farey map $\mathcal{M}_3/\Gamma(7)$. As $\mathcal{M}_3 \subset \mathbb{H}^*$, $\mathcal{M}_3/\Gamma(7) \subset \mathbb{H}^*/\Gamma(7)$, this Farey map is embedded in the Klein surface. In a sense, we will show that this Farey map *is* the Klein surface.

In §7 and §8, we review Klein's original construction, computing Farey coordinates on Klein's 14-sided fundamental region and discussing the differences between the two approaches. In volume 15 of Mathematische Annalen in 1879 [6], Klein extended his work to study the surface $\mathbb{H}^*/\Gamma(11)$, which has $\mathrm{PSL}(2, 11)$ of order 660 as its automorphism group and is somewhat more complicated. He did not draw a fundamental region for the case $n = 11$ as he did for $n = 7$. However we are able to draw the corresponding Farey map in §9.

2 The Farey map

The vertices of the Farey map $\mathcal{M}_3$ are the extended rationals, i.e. $\mathbb{Q}\cup\{\infty\}$ and two rationals $\frac{a}{c}$ and $\frac{b}{d}$ are joined by an edge if and only if $ad - bc = \pm 1$. These edges are drawn as semicircles or vertical lines, perpendicular to the real axis, (i.e. hyperbolic lines). Here $\infty = \frac{1}{0}$. This map has the following properties.

(a) There is a triangle with vertices $\frac{1}{0}, \frac{1}{1}, \frac{0}{1}$, called the *principal triangle.*

(b) The modular group $\Gamma = \mathrm{PSL}(2, \mathbb{Z})$ acts as a group of automorphisms of $\mathcal{M}_3$.

(c) The general triangle has vertices $\frac{a}{c}, \frac{a+b}{c+d}, \frac{b}{d}$.

This forms a triangular tessellation of the upper half plane. Note that the triangle in (c) is just the image of the principal triangle under the Möbius transformation corresponding to the matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$.

In [10] it was shown that $\mathcal{M}_3$ is the *universal triangular map.* This means that if $\mathcal{M}$ is any triangular map on an orientable surface then $\mathcal{M}$ is the quotient of $\mathcal{M}_3$ by a subgroup Λ of the modular group. A map is regular if its orientation preserving automorphism group acts transitively on its *darts*, (i.e. directed edges) and $\mathcal{M}_3/\Lambda$ is regular if and only if Λ is a normal subgroup of Γ. The subgroup Λ here is called a *map subgroup*.

(In general if $\Delta(m, n)$ is the $(2, m, n)$ triangle group, then every map of type (m, n) has the form $\hat{\mathcal{M}}/M$ where $\hat{\mathcal{M}}$ is the universal map of type (m, n) and M is a subgroup of Γ. In our case we are thinking of the modular group Γ as being the $(2, 3, \infty)$ group. The infinity here means that we are not concerned with the vertex valencies; we just require the map to be triangular. For the general theory we refer to [3].)

We now consider the case when $\Lambda = \Gamma(n)$, the principal congruence subgroup mod n of the modular group Γ. The corresponding maps are denoted by $\mathcal{M}_3(n)$. As $\Gamma(n)$ is a

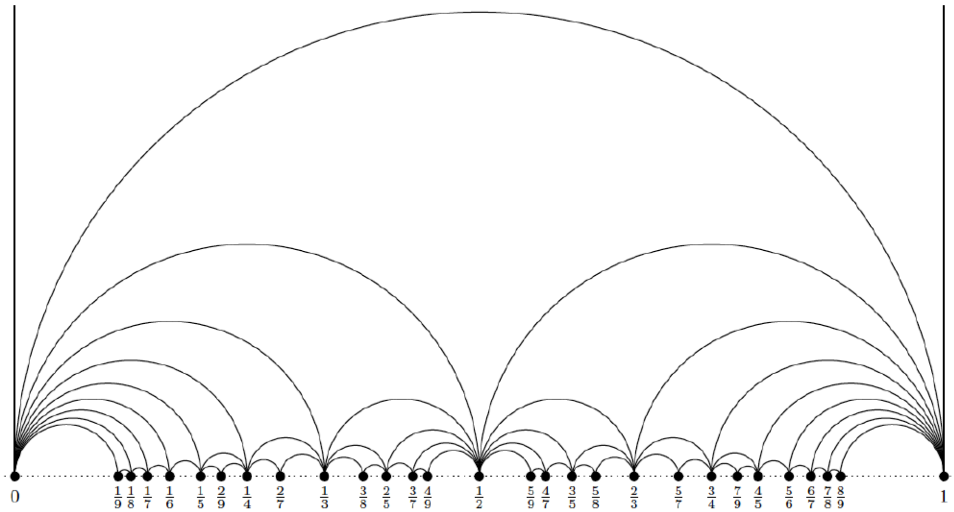

Figure 1: The Farey map, (drawn by Jan Karabaš).

normal subgroup of Γ these maps are regular.

3 The map $\mathcal{M}_3(n)$

The map $\mathcal{M}_3(n)$ is a regular map that lies on the Riemann surface $\mathbb{H}^*/\Gamma(n)$. The automorphism group of $\mathcal{M}_3(n)$ is $\Gamma/\Gamma(n) \cong \mathrm{PSL}(2, \mathbb{Z}_n)$ whose order $\mu(n)$ for $n > 2$ is

$$\mu(n) = \frac{n^3}{2} \prod_{p|n} \left(1 - \frac{1}{p^2}\right).$$

The product is taken over all prime factors of n, see [3, Chapter 6, Exercise 6L].

Also, $\mu(2) = 6$.

Now $\mu(n)$ is the number of darts of $\mathcal{M}_3(n)$ so the number of edges of this map is $\mu(n)/2$, and the number of faces is equal to $\mu(n)/3$. Note that $\frac{1}{0}$ is joined to $\frac{k}{1}$ for $k = 0, \ldots, n-1$ so that $\frac{1}{0}$ has valency n and by regularity every vertex has valency n. Thus the number of vertices is equal to $\mu(n)/n$. For example, $\mu(5) = 60, \mu(7) = 168, \mu(11) = 660$, so the numbers of vertices of $\mathcal{M}_3(n)$, for $n = 5, 7, 11$, are $12, 24, 60$, respectively. We can now use the Euler-Poincaré formula to find the well-known formula for the genus $g(n)$ of $\mathcal{M}_3(n)$;

$$g(n) = 1 + \frac{n^2}{24}(n-6) \prod_{p|n} \left(1 - \frac{1}{p^2}\right). \tag{3.1}$$

3.1 Farey coordinates for $\mathcal{M}_3(n)$

If $(a, c, n) = 1$ then the projection of $\frac{a}{c}$ from $\mathcal{M}_3$ to $\mathcal{M}_3(n)$ is denoted by $[\frac{a}{c}]$, or simply $\frac{a}{c}$ when there is no room for ambiguity, To be precise, a *Farey fraction* $\frac{a}{c}$ is an equivalence class of ordered pairs $(a, c) \in \mathbb{Z}_n^2$ with $(a, c, n) = 1$ under the equivalence relation $(a, c) \equiv$

(b, d) if $b = ua, d = uc \in \mathbb{Z}_n$ and $u = \pm 1 \in \mathbb{Z}_n$. This is sometimes referred to as a *Farey coordinate* of a vertex in $\mathcal{M}_3(n)$.

See §4.1 for the case $n = 5$, where we give the Farey coordinates for the icosahedron.

4 The quasi-icosahedral structure of Farey maps

We now show that every Farey map has a *quasi-icosahedral* structure. Let us give some definitions from [12].

1. The *(graph-theoretic) distance* $\delta(f_1, f_2)$ between two vertices f_1 and f_2 of a graph is the least number of edges joining these two vertices.
2. A *Farey circuit* is a sequence of Farey fractions $f_1, f_2, \ldots, f_k$ where f_i is joined by an edge to f_{i+1} with the indices taken mod k.
3. A *pole* of a Farey map is any vertex with coordinates $\frac{a}{0}$.

The following theorem was proved in [12].

Theorem 4.1. *Let $\frac{a}{c}, \frac{b}{d}$ be distinct vertices of $\mathcal{M}_3(p)$, where p is prime, and let $\Delta = ad - bc$. Then:*

$$\delta\left(\frac{a}{c}, \frac{b}{d}\right) = \begin{cases} 1 & \textit{if and only if } |\Delta| = 1, \\ 2 & \textit{if and only if } |\Delta| \neq 0, 1, \\ 3 & \textit{if and only if } \Delta = 0. \end{cases}$$

Now let us call $\frac{1}{0}$ the *north pole* N of $\mathcal{M}_3(p)$. Then by the above theorem $\delta(N, \frac{a}{c}) = 1$ if and only if $c = \pm 1$, $\delta(N, \frac{a}{c}) = 2$ if and only if $c \neq 0, \pm 1$, and $\delta(N, \frac{a}{c}) = 3$ if and only if $c = 0$. That is, the vertices of $\mathcal{M}_3(p)$ form four disjoint subsets: the north pole N at $\frac{1}{0}$, a set of size n consisting of vertices whose graph-theoretic distance from N is 1, another set of points at distance 2 from N, and other poles at distance 3 from N. In Theorem 4.2, we will show that these two sets are in fact circuits. As the icosahedron has this property we refer to these Farey maps as having a *quasi-icosahedral structure.*

(In [12] it was also shown that $\mathcal{M}_3(n)$ has diameter 3 for all $n \geq 5$.)

4.1 The icosahedron

$\mathcal{M}_3(5)$ is an icosahedron [12] with vertex set

$$\left\{\frac{1}{0}, \frac{2}{0}, \frac{0}{1}, \frac{1}{1}, \frac{2}{1}, \frac{3}{1}, \frac{4}{1}, \frac{0}{2}, \frac{1}{2}, \frac{2}{2}, \frac{3}{2}, \frac{4}{2}\right\};$$

see Figure 2. The north pole N at $\frac{1}{0}$, there is a Farey circuit of length 5 of points whose denominator is equal to 1 and have distance 1 from N and a second circuit of length 5 of points whose denominator is equal to 2 and have distance 2 from N. We also have the pole $\frac{2}{0}$ at distance 3 from N.

For a *quasi-icosahedral* structure on $\mathcal{M}_3(p)$ let $N = \frac{1}{0} \in \mathcal{M}_3(p)$. The circuit of points of distance 1 from N is

$$S_1(p) = \frac{0}{1}, \frac{1}{1}, \ldots, \frac{p-1}{1}.$$

The set of points at distance 2 from N is more complicated and we now construct it. To make the calculation clearer we start with the example $p = 7$. From Theorem 4.1, we see

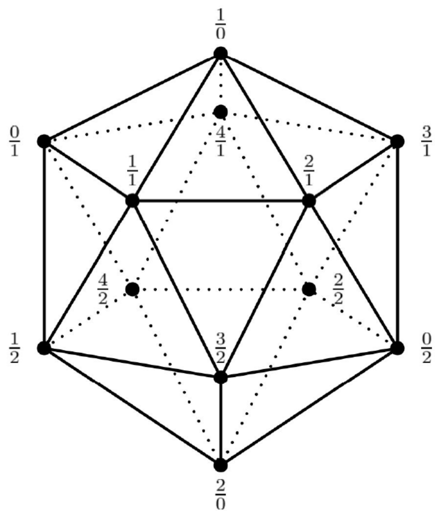

Figure 2: Drawing of $\mathcal{M}_3(5)$ with Farey coordinates.

that the points of distance 2 from $\frac{1}{0}$ have the form $\frac{b}{d}$ where $d = \pm 2$ or ± 3. Thus the points $\frac{1}{3}, \frac{1}{2}, \frac{2}{3} \in S_2(7)$ all have distance 2 from N. As the transformation $t \mapsto t+1$ fixes N and preserves distance, all points in $S(7)+k$ have distance 2 from N, for $k = 1, \ldots, 6$. Thus we find the set

$$S_2(7) = \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{4}{3}, \frac{3}{2}, \frac{5}{3}, \frac{0}{3}, \frac{5}{2}, \frac{1}{3}, \frac{3}{3}, \frac{0}{2}, \frac{4}{3}, \frac{6}{3}, \frac{2}{2}, \frac{0}{3}, \frac{2}{3}, \frac{4}{2}, \frac{3}{3}, \frac{5}{3}, \frac{6}{2}, \frac{6}{3}$$

consisting of points at distance 2 from N, see Figure 3. We now generalize this. Let $p \geq 5$ be a prime and let

$$S(p) = \frac{1}{(p-1)/2}, \frac{1}{(p-3)/2}, \ldots, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \ldots, \frac{(p-3)/2}{(p-1)/2}.$$

Then

Theorem 4.2. *The concatenation of sequences*

$$S_2(p) = S(S+1)(S+2)\ldots(S+p-1),$$

where $S = S(p)$, is the Farey circuit consisting of those points of distance 2 from N. The length of $S_1(p)$ is p and the length of $S_2(p)$ is $p(p-4)$. There are $(p-1)/2$ poles.

Proof. We first observe that the points in $S_2(p)$ do have distance 2 from N. Indeed, the points $\frac{1}{k}$ and $\frac{m-1}{m}$ for $2 \leq k, m \leq \frac{p-1}{2}$ have distance 2 from $N = \frac{1}{0}$ as $\frac{1}{k} \leftrightarrow \frac{0}{1}$ and $\frac{m-1}{m} \leftrightarrow \frac{1}{1}$ and none of these points have distance 1 from $\frac{1}{0}$. (The symbol $\leftrightarrow$ means adjacent to.)

The transformation $t \mapsto t+1$ fixes $\frac{1}{0}$ and preserves distance so that all points in $S+k$ have distance 2 from $N = \frac{1}{0}$. We now show that $S_2(p)$ is a Farey circuit. Clearly there are edges between $\frac{1}{k}$ and $\frac{1}{k+1}$ for $k \geq 2$ and between $\frac{k}{k+1}$ and $\frac{k+1}{k+2}$ for $k \geq 2$. So, we only need to show that there is an edge between the last vertex in $S+k$ and the first vertex in $S+k+1$. The last vertex of $S+k$ is

$$k+\frac{(p-3)/2}{(p-1)/2}=\frac{(p-3+kp-k)/2}{(p-1)/2}.$$

The first vertex of $S+k+1$ is

$$k+1+\frac{1}{(p-1)/2}=\frac{(kp-k+p+1)/2}{(p-1)/2}.$$

As

$$[(p-3+kp-k)/2][(p-1)/2]-[(kp-k+p+1)/2][(p-1)/2]=-p+1,$$

we see that the last vertex of $S+k$ is adjacent to the first vertex of $S+k+1$. Thus, $S_2(p)$ is a Farey circuit consisting of points of distance 2 from $\frac{1}{0}$.

Now $S_1(p)$ clearly has p points, and the set $S(p)$ has $p-4$ points, thus $S_2(p)$ has $p(p-4)$ points. The poles are $\frac{1}{0}, \frac{2}{0}, \dots$ with $\frac{k}{0}=\frac{-k}{0}$, and so the number of poles is $\frac{p-1}{2}$. □

5 Drawing $\mathcal{M}_3(7)$

The map $\mathcal{M}_3(7)$ has 24 vertices with Farey coordinates

$$\left\{\frac{1}{0},\frac{2}{0},\frac{3}{0},\frac{0}{1},\frac{1}{1},\frac{2}{1},\frac{3}{1},\frac{4}{1},\frac{5}{1},\frac{6}{1},\frac{0}{2},\frac{1}{2},\frac{2}{2},\frac{3}{2},\frac{4}{2},\frac{5}{2},\frac{6}{2},\frac{0}{3},\frac{1}{3},\frac{2}{3},\frac{3}{3},\frac{4}{3},\frac{5}{3},\frac{6}{3}\right\}.$$

The first circuit is $S_1(7)=\frac{0}{1},\frac{1}{1},\dots,\frac{6}{1}$, and we draw a polygonal curve $C_1(7)$, surrounding $\frac{1}{0}$, containing the points of $S_1(7)$. We draw a bigger simple closed curve $C_2(7)$, also surrounding $\frac{1}{0}$, containing the points of $S_2(7)$. In Figure 3, $C_2(7)$ passes through the points $\frac{1}{3},\frac{6}{3},\frac{6}{2},\frac{5}{3},\dots$.

Finally, we can draw a simple closed curve $C_3(7)$ exterior to both $C_1(7)$ and $C_2(7)$ which contains the poles $\frac{2}{0}$ and $\frac{3}{0}$, see the dotted line in Figure 3. The pole $\frac{2}{0}$ is a vertex of seven triangles whose base is on the second circuit. One of these triangles is $\frac{6}{3},\frac{2}{0},\frac{1}{3}$ and the others are found by adding $1,2,3,4,5,6$ to these three points. For example, adding 1 to $\frac{6}{3},\frac{2}{0},\frac{1}{3}$ gives $\frac{2}{3}(=\frac{9}{3}),\frac{2}{0},\frac{4}{3}$. (Adding the integer k has the geometric effect of rotating $\mathcal{M}_3(7)$ by $\frac{2\pi}{k}$.) The pole $\frac{3}{0}$ is a vertex of seven quadrilaterals which are unions of two Farey triangles, and also have one edge on $C_2(7)$. One of these is $\frac{1}{3},\frac{5}{2},\frac{3}{0},\frac{1}{2}$ and we get the other six by adding $1,2,3,4,5,6$. We end up with a 42-sided polygon pictured in Figure 3 (for now ignore the dashed curves). It is interesting that exactly the same polygon was obtained by E. Schulte and J. M. Wills in [9] by purely geometric methods.

6 The 14-sided polygon

We now show how to obtain a 14-sided polygon out of the Farey map $\mathcal{M}_3(7)$ with the same side-pairings as the Klein surface. As $\mathcal{M}_3(7)$ has 42 edges and we need a 14-sided polygon we define a *new-edge* to be a union of three consecutive edges which include vertices with Farey coordinates $\frac{2}{0}$ and $\frac{3}{0}$.

For example, our first new-edge goes from $\frac{2}{0}$ to $\frac{5}{3}$ to $\frac{3}{2}$ to $\frac{3}{0}$ and our second new edge goes from $\frac{3}{0}$ to $\frac{6}{2}$ to $\frac{6}{3}$ to $\frac{3}{0}$, see Figure 3. We now replace the new-edges by dashed lines.

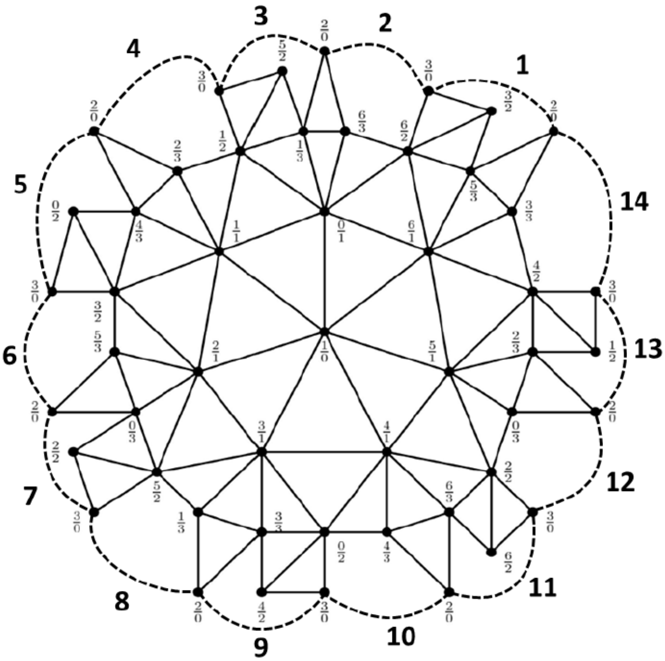

Figure 3: Drawing of $\mathcal{M}_3(7)$ with Farey coordinates.

In Figure 3 the dashed line labelled 1 goes from a vertex labelled $\frac{2}{0}$ to a vertex labelled $\frac{3}{0}$ surrounding the vertices $\frac{5}{3}$ and $\frac{3}{2}$ of the first new edge, and similarly the dashed line labelled 2 goes from $\frac{3}{0}$ to $\frac{2}{0}$ surrounding the vertices $\frac{6}{2}$ and $\frac{6}{3}$. Notice that the dashed lines are not part of the map $\mathcal{M}_3(7)$, they are just a convenient way of representing our 14-sided polygon. We can associate four Farey fractions to each dashed edge. For example, associated to the dashed edge 1 we have the Farey fractions $\frac{2}{0}, \frac{5}{3}, \frac{3}{2}, \frac{3}{0}$. We pair two new-edges if their associated Farey fractions are the same. For example, consider the new-edge labelled 6 in Figure 3. The associated Farey fractions are $\frac{3}{0}, \frac{3}{2}, \frac{5}{3}, \frac{2}{0}$. These are the same Farey fractions, but in reverse order as for the new-edge 1. This means we identify the new edges 1 and 6 orientably. Similarly we get the other six identifications. Thus the identifications are

$$1 \leftrightarrow 6, 3 \leftrightarrow 8, 5 \leftrightarrow 10, 7 \leftrightarrow 12, 9 \leftrightarrow 14, 11 \leftrightarrow 2, 13 \leftrightarrow 4.$$

This is exactly the same side-pairing as found by Klein from his 14-sided polygon which shows that our 14-sided polygon does give the Klein quartic. Our way of finding the side identifications is much more straightforward than the method used in Klein's paper, which we will summarize in §8.

7 Farey Coordinates for the Klein map

A regular map has type $\{m, n\}$ if every face has size m and every vertex has valency n. (We are following [1] here and not [3] where these numbers are reversed.) Now $\mathcal{M}_3(n)$ is a regular map of type $\{3, n\}$ because $\frac{1}{0}$ is adjacent to $\frac{0}{1}, \ldots, \frac{n-1}{1}$. Now $\mathcal{M}_3(7)$ is the

Klein map, or, in the standard notation in [1], the map $\{3, 7\}_8$. The '8' here is the length of a Petrie polygon. (For Petrie polygons and how we find the lengths of Petrie polygons using Farey fractions see [11].)

As noted in the introduction, the Klein map $\mathcal{M}_3(7)$ is embedded in the Klein surface $\mathbb{H}^*/\Gamma(7)$. The term "Klein map" comes from the drawing on page 320 of [7], or page 120 of [8], of Klein's 14-sided polygon. After the given side identifications this does give a map on a surface of genus 3. See Figure 4 (and just ignore the Farey coordinates in this diagram for now). This is *not* the Klein map, for it is not regular, having vertices of different valency. It consists of 336 triangles while the Klein map $\mathcal{M}_3(7)$ has 56 triangles. Nevertheless, we can easily obtain the Klein map from Figure 4. The vertices of the map are the vertices of valency 14. Before we describe the Klein map structure on this surface we show how to associate the 24 Farey fractions modulo 7 to the 24 vertices.

First, we assign the Farey coordinate $\frac{1}{0}$ to the centre point. We note that there are two circuits of seven vertices centred at $\frac{1}{0}$. We give the first circuit the Farey coordinates $\frac{0}{1}, \frac{1}{1}, \ldots, \frac{6}{1}$. If we extend the perpendicular bisector from $\frac{1}{0}$ to the hyperbolic line between $\frac{0}{1}$ and $\frac{1}{1}$ we get to another vertex of valency seven to which we assign the coordinate $\frac{0+1}{1+1} = \frac{1}{2}$. Similarly, we extend the perpendicular bisector from $\frac{1}{0}$ to the hyperbolic line between $\frac{1}{1}$ and $\frac{1}{2}$ to a vertex of valency seven which we give the Farey coordinate $\frac{3}{2}$. By continuing, we find all vertices with Farey coordinates $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{0}{2}, \frac{2}{2}, \frac{6}{2}$. Thus we have now found all vertices with Farey coordinates $\frac{x}{i}$ for $i = 1, 2$ and we just have to find the vertices with Farey ccordinates $\frac{x}{0}$ or $\frac{x}{3}$ which lie on the boundary of $\mathcal{K}$. After Klein's identifications shown in Figure 3, we see that the 14 corners of $\mathcal{K}$ belong to two classes, which we can label $\frac{2}{0}, \frac{3}{0}$. Between any two of these vertices there is precisely one more vertex of $\mathcal{M}_3(7)$. (After side identifications these vertices also have valency 14.) We can assign to them Farey coordinates of the form $\frac{x}{3}$ just by reading them off from Figure 3. In fact, each $\frac{x}{3}$ occurs exactly twice and we can now pair sides of $\mathcal{K}$ that have exactly the same value of x. Again, this gives exactly the same side pairing as Klein found. We thus have two methods, in sections 7 and 8, of using Farey coordinates to get Klein's pairings just by observation.

Figure 4 gives a description of Klein's work using Farey coordinates. We see that each of the 14 sides of the boundary of $\mathcal{K}$ consists of a Farey edge and a non-Farey edge. The segment from $\frac{2}{0}$ to $\frac{x}{3}$ is a Farey edge whilst the segment from $\frac{x}{3}$ to $\frac{3}{0}$ is not a Farey edge. There is no automorphism of $\mathcal{K}$ mapping one segment to the other since all elements of Γ map Farey edges to Farey edges. Note that by section 3, the Klein map has 24 vertices, 56 faces and 84 edges.

We now give the map structure. The vertices of the map are the points of valency 14 in Figure 4, that is, those points that have been given Farey coordinates. An edge joins points with Farey coordinates $\frac{a}{c}$ iand $\frac{b}{d}$ if and only if $ad - bc \equiv 1 \pmod 7$. Three vertices with Farey coordinates $\frac{a}{c}$, $\frac{b}{d}$ and $\frac{e}{f}$ form a triangular face if and only if $e \equiv a + b \pmod 7$, $f \equiv c + d \pmod 7$. For example, there is a triangle with vertices $\frac{4}{1}, \frac{4}{3}$ and $\frac{6}{3}$ for $\frac{6}{3}$ represents the same point as $\frac{1}{4}$, for $\frac{1}{4} = \frac{-6}{-3}$.

8 What Klein did

Here we review Klein's original construction of his fundamental domain of the congruence subgroup $\Gamma(7)$, and show how this construction can be interpreted in terms of the Farey machinery we described above.

By the end of section 10 of [5] Klein had obtained the equation of his quartic curve and

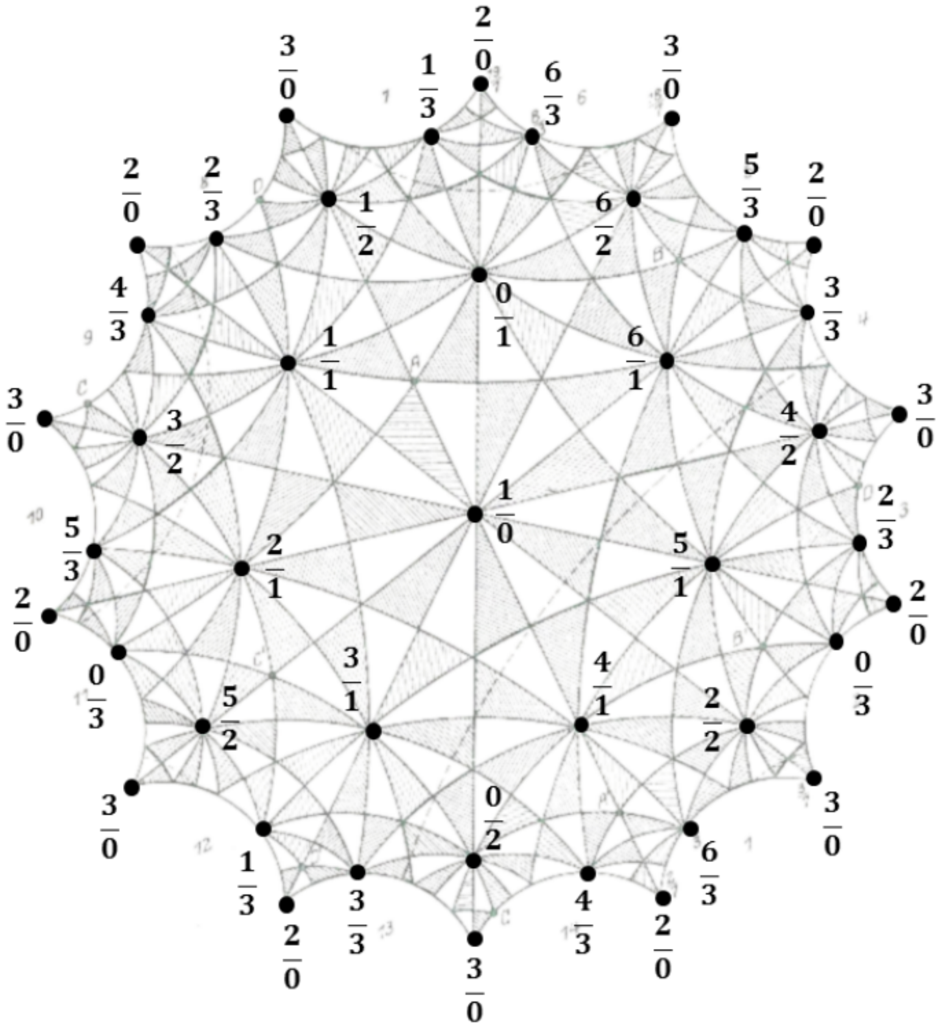

Figure 4: Farey coordinates on the Klein surface.

in section 11 he started to discuss the Riemann surface of this algebraic curve and also the corresponding map. In fact, this was one of the first publications to use maps (or in today's language *dessins d'enfants*) in a profound way, pointing up the deep correspondence between maps and algebraic curves. While this correspondence was not properly understood until Grothendieck's *Esquisse d'un programme* some 105 years later [2], we note that in an interesting anticipation of Grothendieck's programme, Figure 2 of Klein's follow-up paper [6] shows the ten planar dessins of type $(2, 3, 11)$ and degree 11.

Klein's quite complicated construction of his fundamental domain comes from considering fundamental regions for subgroups of indices 7 and 8 in the modular group. In section 12, he writes (in German) "In order not to make these considerations too abstract I will resort to the ω-plane"; this is the upper-half plane on which the modular group acts.

In Figure 6, he constructs a hyperbolic polygon corresponding to his 14-sided polygon describing his surface. Then, in Figure 7, he draws semicircles (hyperbolic lines) in the upper-half plane with rational vertices, which correspond to the edges of his 14-sided polygon. Now consider this polygon as being inscribed in the unit disc so the vertices all lie on the boundary circle. As the unit disc is conformally equivalent to the upper-half plane the boundary circle corresponds to the real axis and so, every point of the circle has some real coordinate. He starts with one edge (labelled 1) of his 14-sided polygon corresponding to two consecutive edges of the polygon in the upper-half plane with vertices $\frac{2}{7}, \frac{1}{3}$ and $\frac{1}{3}, \frac{3}{7}$. (As we already noted above, $\frac{2}{7}, \frac{1}{3}$ is a Farey edge while $\frac{1}{3}, \frac{3}{7}$ is not, therefore we cannot map one to the other by an element of Γ). A second edge (labelled 6) is given as the pair of consecutive edges $\frac{18}{7}, \frac{8}{3}$ and $\frac{8}{3}, \frac{19}{7}$. The Möbius transformation corresponding to the matrix

$$\begin{pmatrix} 113 & -35 \\ 42 & -13 \end{pmatrix}$$

in $\Gamma(7)$ maps edge 1 (i.e. $\frac{2}{7}, \frac{1}{3}, \frac{3}{7}$) to edge 6 (i.e. $\frac{18}{7}, \frac{8}{3}, \frac{19}{7}$), and one more explicit example of edge pairing is given. He states that in total seven such matrices can be found that give all the side pairings. We feel that our technique of just using Farey coordinates is much easier.

9 $\mathcal{M}_3(11)$

About a year after Klein wrote his paper [5] on the quartic curve, he wrote a further paper [6] with the same title but with 'siebenter' replaced with 'elfter', i.e. 'seventh' replaced with 'eleventh'; basically, he was considering $\mathbb{H}^*/\Gamma(11)$. In that paper he did not draw a diagram of the fundamental region equivalent to his drawing of $\mathcal{K}$ in [5]. Here we show how to draw the Farey map $\mathcal{M}_3(11)$ in a similar way to how we drew $\mathcal{M}_3(7)$. This Farey map will be embedded in the surface $\mathbb{H}^*/\Gamma(11)$.

The first circuit of vertices at distance 1 from $\frac{1}{0}$ is

$$S_1(11) = \frac{0}{1}, \frac{1}{1}, \dots, \frac{10}{1}.$$

Now consider the sequence of vertices

$$S(11) = \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5};$$

and then the second circuit is

$$\mathcal{S}_2(11) = S(11)\ (S(11) + 1) \dots (S(11) + 10).$$

The orientation-preserving automorphism group of $\mathcal{M}_3(11)$ is $\mathrm{PSL}(2, 11)$ of order 660 so the Farey map $\mathcal{M}_3(11)$ has $660/2 = 330$ edges, $660/3 = 220$ triangles and $660/11 = 60$ vertices. The Farey coordinates of the vertices are $\frac{1}{0}, \frac{2}{0}, \frac{3}{0}, \frac{4}{0}, \frac{5}{0}$ and all Farey fractions of the form $\frac{r}{s}$ for $r = 0$ to 10 and $s = 1$ to 5.

To draw the map we just need to find the 220 triangular faces. Because $z \mapsto z + 1$ is an automorphism of $\mathcal{M}_3(11)$, which acts as a rotation about the centre $\frac{1}{0}$ of the map, we see that this map is divided into eleven congruent sectors each containing $220/11 = 20$ triangles each. We construct one such sector W, shown in Figure 5, by starting from the

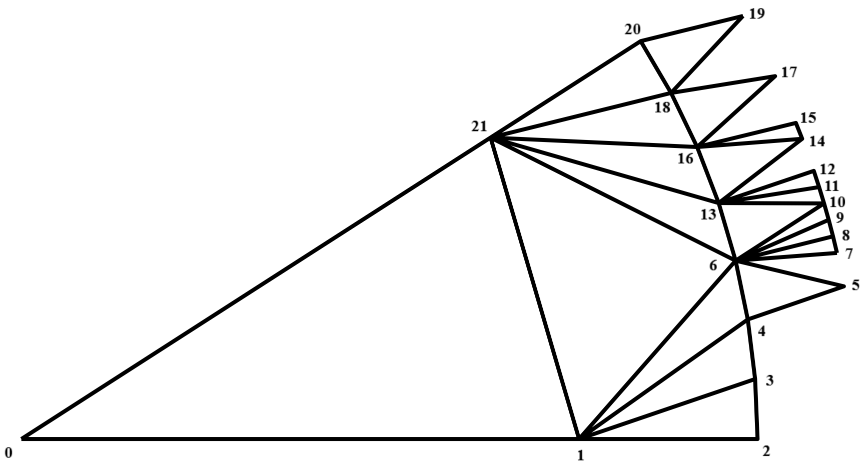

Figure 5: The sector W.

central triangle $\frac{1}{0}, \frac{0}{1}, \frac{1}{1}$ and adding 19 distinct triangles whose vertices lie in $S(11)$. Exactly 8 of these 19 triangles have a vertex on the first circuit $S_1(11)$ ($\frac{0}{1}$ or $\frac{1}{1}$ in particular) and are uniquely determined. For the remaining 11 triangles, which either have three vertices on $S_2(11)$, or two vertices on $S_2(11)$ and a pole vertex, there are several choices satisfying the condition that they are distinct under rotation about $\frac{1}{0}$.

Figure 5 shows one such solution as the union of 20 triangles. The actual Farey coordinates are

$P_0 = \frac{1}{0}$	$P_1 = \frac{0}{1}$	$P_2 = \frac{1}{5}$	$P_3 = \frac{1}{4}$	$P_4 = \frac{1}{3}$	$P_5 = \frac{2}{5}$
$P_6 = \frac{1}{2}$	$P_7 = \frac{5}{0}$	$P_8 = \frac{6}{2}$	$P_9 = \frac{7}{4}$	$P_{10} = \frac{3}{5}$	$P_{11} = \frac{6}{3}$
$P_{12} = \frac{4}{0}$	$P_{13} = \frac{2}{3}$	$P_{14} = \frac{6}{4}$	$P_{15} = \frac{3}{0}$	$P_{16} = \frac{3}{4}$	$P_{17} = \frac{4}{2}$
$P_{18} = \frac{4}{5}$	$P_{19} = \frac{2}{0}$	$P_{20} = \frac{6}{5}$	$P_{21} = \frac{1}{1}$		

Each point P_i is labelled i in Figure 5 to reduce clutter.

Now let

$$W^* = W \cup (W + 1) \cup \cdots \cup (W + 10)$$

where $W + k$ is defined as in Section 5, that is, geometrically, is the rotation of W by $\frac{2\pi}{k}$. Then W^* is the union of 220 triangles as required and its boundary is a polygon with $11 \times 18 = 198$ sides. A diagram of the map W^* is given in Figure 6.

Table 1 in the Appendix shows a list of the 198 boundary vertices of W^* arranged in 11 rows. The first row corresponds to W and the kth row is just the first row plus $(k-1)$. We now notice that we have an orientable side pairing. For example, the first edge in row 1 going from $\frac{1}{5}$ to $\frac{1}{4}$ is paired with the edge in row 5 going from $\frac{1}{4}$ to $\frac{1}{5}$, the next edge in row 1 going from $\frac{1}{4}$ to $\frac{1}{3}$ is paired with the edge in row 8 going from $\frac{1}{3}$ to $\frac{1}{4}$. Proceeding in this way we find that all the 198 edges of the polygon are paired orientably which shows that this polygon represents an orientable surface which must be $\mathcal{M}_3(11)$. As the map W^* has 60 vertices, 220 edges, and 330 triangles, by the Euler-Poincaré formula the genus of the surface is 26.

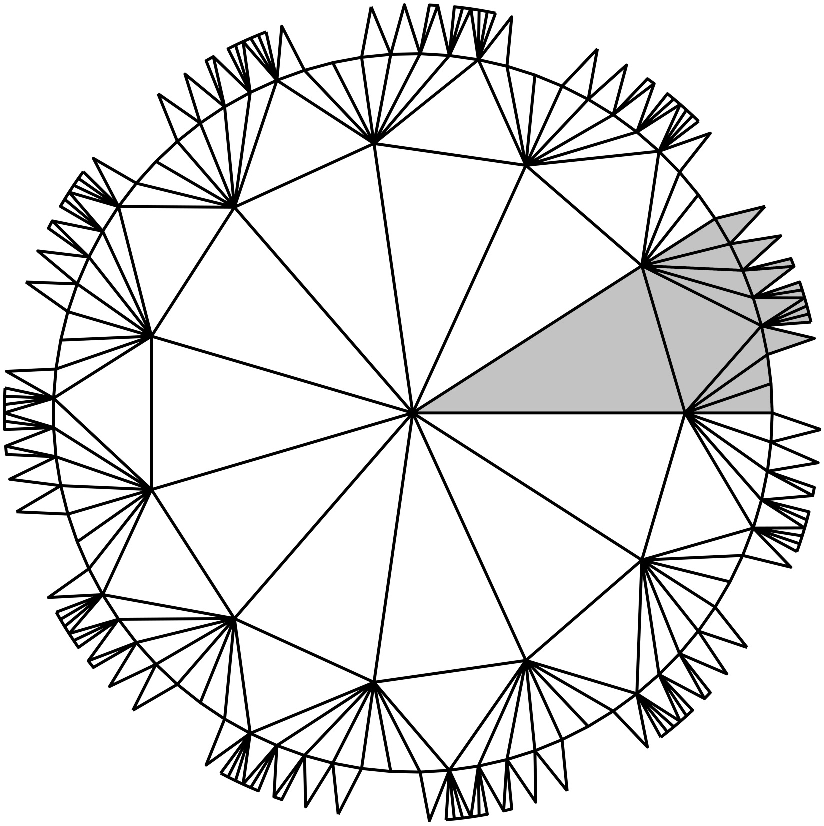

Figure 6: The map W^*.

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A Appendix

Table 1: The boundary vertices of W^*. The last vertex of a row is repeated as the first vertex of the row below. Each row represents a sector; the first row represents sector W in Figure 5. Vertices in bold belong to edges which are paired with edges in the first row W.

$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{2}{5}$	$\frac{1}{2}$	$\frac{5}{0}$	$\frac{6}{2}$	$\frac{7}{4}$	$\frac{3}{5}$	$\frac{6}{3}$	$\frac{4}{0}$	$\frac{2}{3}$	$\frac{6}{4}$	$\frac{3}{0}$	$\frac{3}{4}$	$\frac{4}{2}$	$\frac{4}{5}$	$\frac{2}{0}$	$\frac{6}{5}$
$\frac{6}{5}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{7}{5}$	$\frac{3}{2}$	$\frac{5}{0}$	$\frac{8}{2}$	$\frac{0}{4}$	$\frac{8}{5}$	$\frac{9}{3}$	$\frac{4}{0}$	$\frac{5}{3}$	$\frac{10}{4}$	$\frac{3}{0}$	$\mathbf{\frac{7}{4}}$	$\mathbf{\frac{6}{2}}$	$\frac{9}{5}$	$\frac{2}{0}$	$\frac{0}{5}$
$\frac{0}{5}$	$\frac{9}{4}$	$\frac{7}{3}$	$\frac{1}{5}$	$\frac{5}{2}$	$\frac{5}{0}$	$\frac{10}{2}$	$\frac{4}{4}$	$\mathbf{\frac{2}{5}}$	$\mathbf{\frac{1}{3}}$	$\frac{4}{0}$	$\frac{8}{3}$	$\mathbf{\frac{3}{4}}$	$\mathbf{\frac{3}{0}}$	$\frac{0}{4}$	$\frac{8}{2}$	$\frac{3}{5}$	$\frac{2}{0}$	$\frac{5}{5}$
$\frac{5}{5}$	$\frac{2}{4}$	$\frac{10}{3}$	$\frac{6}{5}$	$\frac{7}{2}$	$\mathbf{\frac{5}{0}}$	$\mathbf{\frac{1}{2}}$	$\frac{8}{4}$	$\frac{7}{5}$	$\frac{4}{3}$	$\frac{4}{0}$	$\frac{0}{3}$	$\frac{7}{4}$	$\frac{3}{0}$	$\frac{4}{4}$	$\frac{10}{2}$	$\frac{8}{5}$	$\frac{2}{0}$	$\frac{10}{5}$
$\frac{10}{5}$	$\mathbf{\frac{6}{4}}$	$\mathbf{\frac{2}{3}}$	$\frac{0}{5}$	$\frac{9}{2}$	$\frac{5}{0}$	$\frac{3}{2}$	$\mathbf{\frac{1}{4}}$	$\mathbf{\frac{1}{5}}$	$\frac{7}{3}$	$\frac{4}{0}$	$\frac{3}{3}$	$\frac{0}{4}$	$\frac{3}{0}$	$\frac{8}{4}$	$\mathbf{\frac{1}{2}}$	$\mathbf{\frac{2}{5}}$	$\mathbf{\frac{2}{0}}$	$\mathbf{\frac{4}{5}}$
$\frac{4}{5}$	$\frac{10}{4}$	$\frac{5}{3}$	$\frac{5}{5}$	$\frac{0}{2}$	$\frac{5}{0}$	$\frac{5}{2}$	$\frac{5}{4}$	$\frac{6}{5}$	$\frac{10}{3}$	$\mathbf{\frac{4}{0}}$	$\mathbf{\frac{6}{3}}$	$\frac{4}{4}$	$\frac{3}{0}$	$\frac{1}{4}$	$\frac{3}{2}$	$\frac{7}{5}$	$\frac{2}{0}$	$\frac{9}{5}$
$\frac{9}{5}$	$\frac{3}{4}$	$\frac{8}{3}$	$\frac{10}{5}$	$\frac{2}{2}$	$\frac{5}{0}$	$\frac{7}{2}$	$\frac{9}{4}$	$\frac{0}{5}$	$\mathbf{\frac{2}{3}}$	$\mathbf{\frac{4}{0}}$	$\frac{9}{3}$	$\frac{8}{4}$	$\frac{3}{0}$	$\frac{5}{4}$	$\frac{5}{2}$	$\frac{1}{5}$	$\frac{2}{0}$	$\frac{3}{5}$
$\mathbf{\frac{3}{5}}$	$\mathbf{\frac{7}{4}}$	$\frac{0}{3}$	$\mathbf{\frac{4}{5}}$	$\mathbf{\frac{4}{2}}$	$\frac{5}{0}$	$\frac{9}{2}$	$\frac{2}{4}$	$\frac{5}{5}$	$\frac{5}{3}$	$\frac{4}{0}$	$\mathbf{\frac{1}{3}}$	$\mathbf{\frac{1}{4}}$	$\frac{3}{0}$	$\frac{9}{4}$	$\frac{7}{2}$	$\mathbf{\frac{6}{5}}$	$\mathbf{\frac{2}{0}}$	$\frac{8}{5}$
$\frac{8}{5}$	$\frac{0}{4}$	$\frac{3}{3}$	$\frac{9}{5}$	$\mathbf{\frac{6}{2}}$	$\mathbf{\frac{5}{0}}$	$\frac{0}{2}$	$\frac{6}{4}$	$\frac{10}{5}$	$\frac{8}{3}$	$\frac{4}{0}$	$\frac{4}{3}$	$\frac{5}{4}$	$\frac{3}{0}$	$\frac{2}{4}$	$\frac{9}{2}$	$\frac{0}{5}$	$\frac{2}{0}$	$\frac{2}{5}$
$\frac{2}{5}$	$\frac{4}{4}$	$\mathbf{\frac{6}{3}}$	$\mathbf{\frac{3}{5}}$	$\frac{8}{2}$	$\frac{5}{0}$	$\frac{2}{2}$	$\frac{10}{4}$	$\frac{4}{5}$	$\frac{0}{3}$	$\frac{4}{0}$	$\frac{7}{3}$	$\frac{9}{4}$	$\mathbf{\frac{3}{0}}$	$\mathbf{\frac{6}{4}}$	$\frac{0}{2}$	$\frac{5}{5}$	$\frac{2}{0}$	$\frac{7}{5}$
$\frac{7}{5}$	$\frac{8}{4}$	$\frac{9}{3}$	$\frac{8}{5}$	$\frac{10}{2}$	$\frac{5}{0}$	$\mathbf{\frac{4}{2}}$	$\mathbf{\frac{3}{4}}$	$\frac{9}{5}$	$\frac{3}{3}$	$\frac{4}{0}$	$\frac{10}{3}$	$\frac{2}{4}$	$\frac{3}{0}$	$\frac{10}{4}$	$\frac{2}{2}$	$\frac{10}{5}$	$\frac{2}{0}$	$\frac{1}{5}$

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On the incidence map of incidence structures*

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Abstract

By using elementary linear algebra methods we exploit properties of the incidence map of certain incidence structures with finite block sizes. We give new and simple proofs of theorems of Kantor and Lehrer, and their infinitary version. Similar results are obtained also for diagrams geometries.

By mean of an extension of Block's Lemma on the number of orbits of an automorphism group of an incidence structure, we give informations on the number of orbits of: a permutation group (of possible infinite degree) on subsets of finite size; a collineation group of a projective and affine space (of possible infinite dimension) over a finite field on subspaces of finite dimension; a group of isometries of a classical polar space (of possible infinite rank) over a finite field on totally isotropic subspaces (or totally singular in case of a orthogonal space) of finite dimension.

Furthermore, when the structure is finite and the associated incidence matrix has full rank, we give an alternative proof of a result of Camina and Siemons. We then deduce that certain families of incidence structures have no sharply transitive sets of automorphisms acting on blocks.

Keywords: Incidence structure, incidence map, diagram geometry.

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1 Introduction

An *incidence structure* is a triple $\mathcal{I} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ where $\mathcal{P}$ and $\mathcal{B}$ are disjoint sets and I is a subset of $\mathcal{P} \times \mathcal{B}$. The elements of $\mathcal{P}$ are called *points*, those of $\mathcal{B}$ *blocks* and I defines the following *incidence relation*: the point P and the block B are *incident* if and only if $(P, B) \in \mathrm{I}$, and we will write $P \,\mathrm{I}\, B$. The incidence structure $\mathcal{I}$ has *finite block sizes* if $\{P \in \mathcal{P} : P \,\mathrm{I}\, B\}$ has finite size for all $B \in \mathcal{B}$; $\mathcal{I}$ is *finite* if $\mathcal{P}$ and $\mathcal{B}$, and hence also I, are finite sets. An *automorphism* of an incidence structure is a pair of permutations (π, β), with π acting on $\mathcal{P}$ and β on $\mathcal{B}$, such that $P \,\mathrm{I}\, B$ if and only if $P^{\pi} \,\mathrm{I}\, B^{\beta}$, for all $P \in \mathcal{P}$ and $B \in \mathcal{B}$. The group of all automorphisms is denoted by $\mathrm{Aut}\ \mathcal{I}$.

A finite incidence structure can be represented by a $(0, 1)$-matrix A with rows indexed by points and columns indexed by blocks, and with the (P, B)-entry equal to 1 if and only if P is incident with B. The incidence matrix A have been studied by many authors at least since the 1960s, and most of their investigations were on the rank of A. Dembowski in [12, p. 20] showed that the rank of the incidence matrix defined by the natural incidence relation of points versus i-dimensional subspaces of a finite d-dimensional projective or affine space is the number of points of the geometry. This result was generalized by Kantor in [14]. He showed that the incidence matrix defined by the incidence between the i-dimensional subspaces and the j-dimensional subspaces of a finite d-dimensional projective or affine space, with $0 \leq i < j \leq d - i - 1$, has full rank. Analogous results for the incidence matrices of all k-subsets versus all l-subsets of a m-set and for the incidence matrices arising from finite polar spaces were proved by Lehrer [16].

A *decomposition* of an incidence structure $\mathcal{I} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ is a partition of $\mathcal{P}$ into *point classes* together with a partition of $\mathcal{B}$ into *block classes*. A decomposition is said to be *block-tactical* if the number of points in a point class which lie in a block depends only on the class in which the block lies. When the incidence structure is finite then the fundamental *Block's Lemma* [2, Theorem 2.1] states that in a block-tactical decomposition the number of point classes differs from the number of block classes by at most the nullity of the incidence matrix of the structure. A principle example of block tactical decomposition is obtained by taking as the point and the block classes the orbits of any automorphism group of the structure. So, Block's Lemma naturally leads to consideration of the rank of the incidence matrix in order to study the number of orbits of an automorphism group of an incidence structure.

When $\mathcal{I} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ is finite, and both permutation representations of any automorphism of $\mathcal{I}$ are regarded as linear representations of the automorphism group, then the incidence matrix A of $\mathcal{I}$ is an intertwining operator between the linear representations of the automorphism group on $\mathcal{P}$ and $\mathcal{B}$. Using this fact, Camina and Siemons [11] showed that when A has maximum rank then the permutation representation on points is a subrepresentation of the permutation representation on blocks. This containment relation implies the non-existence of sharply 1-transitive sets of automorphisms on blocks unless the number of points divides the number of blocks [19].

The aim of this paper is to bring together all the previous questions by providing a unified treatment. Our approach is different from those adopted by the authors referred to above: the main idea is to exploit properties of the incidence map of incidence structures by using elementary linear algebra methods. We find a new and simpler proof of Kantor's and Lehrer's theorems, beside giving the infinitary version of these results. We also provide some geometric version of the main result in [9] on the number of orbits of a permutation group on unordered sets by mean of an extension of Block's Lemma [2] on the number of orbits of an automorphism group of an incidence structure. Furthermore, when the structure is finite and the associated incidence matrix has full rank, we give an alternative proof of the result of Camina and Siemons [11].

We now give a summary of the present paper. In Section 2 we prove that the incidence map of certain (possibly infinite) incidence structures is one-to-one. The keystone is a result (Lemma 2.6) about the kernel of the incidence map from i-dimensional subspaces to $(i+1)$-dimensional subspaces of a finite d-dimensional projective space, where incidence is the inclusion relation. By replacing the dimension with size of a set and the Gaussian coefficients with binomial coefficients, we get the analogous result for the incidence map from k-sets to $(k+1)$-sets of an m-set, where incidence is the inclusion relation. This leads to an alternative proof of both of Kantor's theorems, on the incidence structures arising from projective and affine spaces, and of Lehrer's theorem [16] on the incidence structures arising from subsets. These results are summarized in Theorem 2.7. In Section 3 we illustrate some applications of Theorem 2.7. Under the hypothesis that every block is incident with a finite number of points we prove the infinitary version of the above results. From Kantor's theorem for projective spaces, and because of its infinitary version, we prove that the Lehrer result about incidence structures in finite classical polar spaces [16] holds also in case of polar spaces of infinite rank. Similar results are obtained for diagram geometries associated to certain finite Chevalley groups. If Δ denotes the diagram of the geometry, then by using [7, Theorem 2] we show that the k-varieties give rise to full substructures of the incidence structure of i-varieties versus j-varieties of the geometry, provided i and k lie in distinct connected components of $\Delta - \{j\}$. This gives plenty of scope to apply the main result (Lemma 3.1) of this section. It is conceivable that the weak conclusion that there are as many j-varieties as i-varieties could be useful to diagram geometers. Section 4 is related with Block's Lemma. In the function space and incidence map setting we prove a slight extension of this fundamental result. We then apply it to obtain informations on the number of orbits of: a permutation group (of possible infinite degree) on subsets of finite size; a collineation group of a projective and affine space (of possible infinite dimension) over a finite field on subspaces of finite dimension; a group of isometries of a classical polar space (of possible infinite rank) over a finite field on totally isotropic subspaces (or totally singular in case of a orthogonal space) of finite dimension. We point out that the result on permutation groups was obtained by Cameron in [9], where the theorem of Livingstone and Wagner [17] is proved to hold also for permutation groups of infinite degree. Section 5 is all in the finite setting. We provide an alternative proof of the result of Camina and Siemons [11] which states that if the incidence map of a finite incidence structure is one-to-one, then the permutation representation on points of any given automorphism group is a subrepresentation of the representation on blocks with equal or greater multiplicity. We then deduce that certain families of incidence structures have no sharply transitive sets of automorphisms acting on blocks.

Although some of the results presented here have been obtained by other authors and appear scattered over a large number of papers, in our opinion it is difficult to find a convenient reference for this knowledge with a presentation that doesn't assume a lot of the reader. This work can be considered as an attempt to providing such a reference.

2 The rank of incidence maps

In order to treat our arguments by linear algebra methods, we introduce the incidence map of a finite incidence structure. Let $\mathcal{I} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be an incidence structure. The *point space* of $\mathcal{I}$ is the vector space $\mathbb{Q}^{\mathcal{P}}$ of all functions $\mathcal{P} \to \mathbb{Q}$; the *block space* of $\mathcal{I}$ is the vector space $\mathbb{Q}^{\mathcal{B}}$ of all functions $\mathcal{B} \to \mathbb{Q}$. When $\mathcal{I}$ has finite block sizes, we define the (linear) *incidence map* $\alpha\colon \mathbb{Q}^{\mathcal{P}} \to \mathbb{Q}^{\mathcal{B}}$ of $\mathcal{I}$ by the rule

$$(f\alpha)(B) = \sum_{P \,\mathrm{I}\, B} f(P),$$

for all $B \in \mathcal{B}$ and $f \in \mathbb{Q}^{\mathcal{P}}$.

For any subset Y of a given set X the *characteristic function* $\chi_Y \in \mathbb{Q}^X$ of Y is defined as follows:

$$\chi_Y(x) = \begin{cases} 1 & \text{for } x \in Y; \\ 0 & \text{for } x \in X \setminus Y. \end{cases}$$

With this notation, the set $\{\chi_{\{P\}} : P \in \mathcal{P}\}$ is a basis for $\mathbb{Q}^{\mathcal{P}}$ and $\{\chi_{\{B\}} : B \in \mathcal{B}\}$ is a basis for $\mathbb{Q}^{\mathcal{B}}$; we refer to each of these bases as the *natural basis* of the corresponding space. If $\mathcal{I}$ is finite the matrix of the map α with respect to these bases is precisely the incidence matrix of $\mathcal{I}$, with multiplication being on the right (i.e., vectors regarded as rows).

We now exhibit some properties of the incidence maps of the incidence structures arising from subspaces of a finite dimensional projective space over a finite field.

Let $\mathrm{PG}(d, q)$ be the projective space of dimension d over the finite field with q elements. For $0 \le i \le d-1$, let F_i denote the set of all i-dimensional subspaces (or i-subspaces, for short) of $\mathrm{PG}(d, q)$. For $i \ne j$ we consider the incidence structure $\mathcal{I} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ where $\mathcal{P} = F_i$, $\mathcal{B} = F_j$ and the incidence relation I is given by set-theoretic inclusion.

The following notation will be adopted in the rest of the paper:

- V_i denotes the vector space $\mathbb{Q}^{F_i}$ of functions from F_i to $\mathbb{Q}$;
- $\alpha_{i,j}$ denotes the incidence map from V_i to V_j, with $i \ne j$;
- $W_{-1} = V_{-1} = \{\emptyset\}$;
- W_i denotes the kernel of $\alpha_{i,i-1}$, for $i \ge 0$.

With the above notation, $\alpha_{i,i}$ is the identity map on V_i. For any $S_i \in F_i$, the coordinate array of $\chi_{\{S_i\}}\alpha_{i,j}$, whose entries are indexed by elements of V_j, is precisely the i-th row of the incidence matrix A of $\alpha_{i,j}$. In other words, if $i > j$ then the image under $\alpha_{i,j}$ of $\chi_{\{S_i\}}$ is the characteristic function of the set of j-subspaces contained in S_i. Similarly, if $i < j$ then the image under $\alpha_{i,j}$ of $\chi_{\{S_i\}}$ is the characteristic function of the pencil of j-subspaces passing through S_i.

In the following we need the q-analogs of binomial coefficients, which are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \prod_{i=0}^{k-1} (q^{n-i} - 1)/(q^{k-i} - 1)$$

for non-negative integers n, k with $n \ge k$. Note that $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is the number of $(k-1)$-subspaces of $\mathrm{PG}(n-1, q)$.

Lemma 2.1. *Let* $-1 \leq i \leq j \leq k \leq d-1$. *Then*

$$\alpha_{i,j}\alpha_{j,k} = \begin{bmatrix} k-i \\ j-i \end{bmatrix}_q \alpha_{i,k}.$$

Proof. By applying directly the definition of $\alpha_{i,j}$ we see that

$$(f\alpha_{i,j}\alpha_{j,k})(S_k) = \sum_{S_i \subseteq S_j \subseteq S_k} f(S_i)$$

holds for all $f \in V_i$. The result now follows by recalling that the number of j-subspaces in $\mathrm{PG}(d,q)$ through any given i-subspace which is in turn contained in a k-subspace is $\begin{bmatrix} k-i \\ j-i \end{bmatrix}_q$. □

Lemma 2.2. *For* $i = -1, \ldots, d$,

$$V_i = \bigoplus_{j=-1}^{i} W_j \alpha_{j,i}. \tag{2.1}$$

(Note that some of the summands may be 0).

Proof. For $i = -1$ the result is trivial. For $i = 0, \ldots, d-1$, we note that V_i is a vector space over a field of characteristic zero. Then the inner product defined by

$$\langle g, h \rangle_i = \sum_{S_i \in F_i} g(S_i) h(S_i), \tag{2.2}$$

for all $g, h \in V_i$, is a non-degenerate bilinear form. Since, in the natural bases of V_i and V_j, the matrix of $\alpha_{i-1,i}$ is the transpose of the matrix of $\alpha_{i,i-1}$, then $\langle f\alpha_{i-1,i}, g \rangle_i = \langle f, g\alpha_{i,i-1} \rangle_{i-1}$, for all $f \in V_{i-1}$ and $g \in V_i$, i.e. the incidence map $\alpha_{i-1,i}$ and the dual map $\alpha_{i,i-1}$ are adjoint.

We now show that $V_i = W_i \oplus \mathrm{Im}\,\alpha_{i-1,i}$. Let $\perp_i$ denote the polarity defined by the inner product $\langle -, - \rangle_i$. Since V_i is finite dimensional, then $V_i = \mathrm{Im}\,\alpha_{i-1,i} \oplus (\mathrm{Im}\,\alpha_{i-1,i})^{\perp_i}$. Furthermore, for all $g \in W_i$ and $f \in V_{i-1}$, $\langle f\alpha_{i-1,i}, g \rangle_i = \langle f, g\alpha_{i,i-1} \rangle_{i-1} = 0$ holds, giving $\mathrm{Im}\,\alpha_{i-1,i} \subseteq W_i^{\perp_i}$, or equivalently, $W_i \subseteq (\mathrm{Im}\,\alpha_{i-1,i})^{\perp_i}$. Conversely, if $g \in (\mathrm{Im}\,\alpha_{i-1,i})^{\perp_i}$, then $0 = \langle f\alpha_{i-1,i}, g \rangle_i = \langle f, g\alpha_{i,i-1} \rangle_{i-1}$, for all $f \in V_{i-1}$. By the non-degeneracy of $\langle -, - \rangle_{i-1}$, we get $g\alpha_{i,i-1} = 0$, and hence $g \in W_i$.

We now use induction on i. For $i = -1$ we have $V_{-1} = W_{-1}$. Assume the statement holds for V_{i-1}, that is $V_{i-1} = \bigoplus_{j=-1}^{i-1} W_j \alpha_{j,i-1}$. As $V_i = \mathrm{Im}\,\alpha_{i-1,i} \oplus W_i \alpha_{i,i}$, to conclude the proof we only need to prove that $\mathrm{Im}\,\alpha_{i-1,i} = \bigoplus_{j=-1}^{i-1} W_j \alpha_{j,i}$. But this easily follows from Lemma 2.1 since

$$\mathrm{Im}\,\alpha_{i-1,i} = V_{i-1}\alpha_{i-1,i} = \bigoplus_{j=-1}^{i-1} W_j \alpha_{j,i-1}\alpha_{i-1,i} = \bigoplus_{j=-1}^{i-1} W_j \alpha_{j,i}.$$ □

Remark 2.3. We point out that the bilinear form defined by (2.2) is an appropriate one for the permutation module V_i, in that permutations of the characteristic functions of singletons are isometries of the form. In the basis consisting of the characteristic functions of singletons, this is just a way of saying that permutation matrices are orthogonal in the usual sense of the term, that is $PP^T = I$.

Lemma 2.4. *For* $i = 0, \dots, d-1$,

$$\alpha_{i,i+1}\alpha_{i+1,i} = \alpha_{i,i-1}\alpha_{i-1,i} + \left(\begin{bmatrix} d-i \\ 1 \end{bmatrix}_q - \begin{bmatrix} i+1 \\ 1 \end{bmatrix}_q \right) \alpha_{i,i}.$$

Proof. Let $S_i, S_i' \in F_i$. For any given $S_{i+1} \in F_{i+1}$ we have

$$(\chi_{\{S_i\}}\alpha_{i,i+1})(S_{i+1}) = \begin{cases} 1 & \text{if } S_i \subset S_{i+1}; \\ 0 & \text{otherwise.} \end{cases}$$

It easily follows that

$$(\chi_{\{S_i\}}\alpha_{i,i+1}\alpha_{i+1,i})(S_i') = \sum_{S_{i+1} \supset S_i'} (\chi_{\{S_i\}}\alpha_{i,i+1})(S_{i+1})$$

is the number of $(i+1)$-subspaces containing both S_i and S_i'. This number equals

$$\begin{array}{ll} 0 & \text{if } \dim(S_i \cap S_i') < i-1; \\ 1 & \text{if } \dim(S_i \cap S_i') = i-1; \\ \left[\begin{smallmatrix} d-i \\ 1 \end{smallmatrix}\right]_q & \text{if } S_i' = S_i. \end{array}$$

Applying similar arguments we see that $(\chi_{\{S_i\}}\alpha_{i,i-1}\alpha_{i-1,i})(S_i')$ is the number of $(i-1)$-subspaces contained in both S_i and S_i'. This number is

$$\begin{array}{ll} 0 & \text{if } \dim(S_i \cap S_i') < i-1; \\ 1 & \text{if } \dim(S_i \cap S_i') = i-1; \\ \left[\begin{smallmatrix} i+1 \\ 1 \end{smallmatrix}\right]_q & \text{if } S_i' = S_i. \end{array}$$

The result then follows. □

Lemma 2.5. *For* $j = -1, \dots, i$,

$$(\alpha_{i,i+1}\alpha_{i+1,i})|_{W_j\alpha_{j,i}} = \sum_{k=j}^{i} \left(\begin{bmatrix} d-k \\ 1 \end{bmatrix}_q - \begin{bmatrix} k+1 \\ 1 \end{bmatrix}_q \right) \alpha_{i,i}.$$

Proof. We use induction on i. For $i = -1$ we have $W_{-1} = V_{-1} = \{\emptyset\}$ by definition. We also note that $\left[\begin{smallmatrix} d+1 \\ 1 \end{smallmatrix}\right]_q = (q^{d+1}-1)/(q-1)$ is the number of points in $\mathrm{PG}(d,q)$, that is the size of F_0. Then,

$$(\alpha_{-1,0}\alpha_{0,-1})|_{V_{-1}} = (q^{d+1}-1)/(q-1)\alpha_{-1,-1} = \begin{bmatrix} d+1 \\ 1 \end{bmatrix}_q \alpha_{-1,-1}.$$

Now let $i \geq 0$. For $j = i$, the result follows immediately from Lemma 2.4.

Let $j < i$. By Lemma 2.4 we have

$$(\alpha_{i,i+1}\alpha_{i+1,i})|_{W_j\alpha_{j,i}} = (\alpha_{i,i-1}\alpha_{i-1,i})|_{W_j\alpha_{j,i}} + \left(\begin{bmatrix} d-i \\ 1 \end{bmatrix}_q - \begin{bmatrix} i+1 \\ 1 \end{bmatrix}_q \right) \alpha_{i,i}|_{W_j\alpha_{j,i}}.$$

To conclude the proof it is enough to show that

$$(\alpha_{i,i-1}\alpha_{i-1,i})|_{W_j\alpha_{j,i}} = \sum_{k=j}^{i-1}\left(\begin{bmatrix} d-k \\ 1 \end{bmatrix}_q - \begin{bmatrix} k+1 \\ 1 \end{bmatrix}_q\right)\alpha_{i,i}.$$

By the inductive hypothesis

$$(\alpha_{i-1,i}\alpha_{i,i-1})|_{W_j\alpha_{j,i-1}} = \sum_{k=j}^{i-1}\left(\begin{bmatrix} d-k \\ 1 \end{bmatrix}_q - \begin{bmatrix} k+1 \\ 1 \end{bmatrix}_q\right)\alpha_{i-1,i-1},$$

and Lemma 2.1 gives $\alpha_{j,i-1}\alpha_{i-1,i} = \begin{bmatrix} i-j \\ i-j-1 \end{bmatrix}_q \alpha_{j,i} = \begin{bmatrix} i-j \\ 1 \end{bmatrix}_q \alpha_{j,i}$. Hence, we may write

$$\begin{aligned} w_j\alpha_{j,i}\alpha_{i,i-1}\alpha_{i-1,i} &= \begin{bmatrix} i-j \\ 1 \end{bmatrix}_q^{-1} w_j\alpha_{j,i-1}(\alpha_{i-1,i}\alpha_{i,i-1})\alpha_{i-1,i} \\ &= \sum_{k=j}^{i-1}\left(\begin{bmatrix} d-k \\ 1 \end{bmatrix}_q - \begin{bmatrix} k+1 \\ 1 \end{bmatrix}_q\right)\begin{bmatrix} i-j \\ 1 \end{bmatrix}_q^{-1} w_j\alpha_{j,i-1}\alpha_{i-1,i} \\ &= \sum_{k=j}^{i-1}\left(\begin{bmatrix} d-k \\ 1 \end{bmatrix}_q - \begin{bmatrix} k+1 \\ 1 \end{bmatrix}_q\right) w_j\alpha_{j,i}, \end{aligned}$$

for $w_j \in W_j$. This implies

$$(\alpha_{i,i-1}\alpha_{i-1,i})|_{W_j\alpha_{j,i}} = \sum_{k=j}^{i-1}\left(\begin{bmatrix} d-k \\ 1 \end{bmatrix}_q - \begin{bmatrix} k+1 \\ 1 \end{bmatrix}_q\right)\alpha_{i,i},$$

which is the desired result. □

Lemma 2.6. *Let* $i = 0, \ldots, d-1$. *Then*

$$\ker \alpha_{i,i+1} = \begin{cases} 0 & \text{for } i < \frac{d-1}{2}; \\ W_{d-i-1}\alpha_{d-i-1,i} & \text{for } i \geq \frac{d-1}{2}. \end{cases}$$

Proof. It is clear that $\ker \alpha_{i,i+1} \leq \ker(\alpha_{i,i+1}\alpha_{i+1,i})$. In addition,

$$\dim\ker(\alpha_{i,i+1}\alpha_{i+1,i}) = \dim\ker\alpha_{i,i+1} + \dim(\ker\alpha_{i+1,i} \cap \operatorname{Im}\alpha_{i,i+1}).$$

From the proof of Lemma 2.2, we get $\ker\alpha_{i+1,i} \cap \operatorname{Im}\alpha_{i,i+1} = 0$. Therefore $\ker\alpha_{i,i+1} = \ker(\alpha_{i,i+1}\alpha_{i+1,i})$.

From Lemmas 2.2 and 2.5, the eigenvalues of $\alpha_{i,i+1}\alpha_{i+1,i}$ are the integers

$$\sum_{k=j}^{i}\left(\begin{bmatrix} d-k \\ 1 \end{bmatrix}_q - \begin{bmatrix} k+1 \\ 1 \end{bmatrix}_q\right), \tag{2.3}$$

for $j = -1, \ldots, i$, with the j-th eigenvalue corresponding to the summand $W_j\alpha_{j,i}$ in the decomposition (2.1) of V_i. For $i < (d-1)/2$ all these integers are non-zero, and therefore $\ker\alpha_{i,i+1} = 0$.

Let $i \geq (d-1)/2$. Two cases are treated separately according as d is odd or even. Let d be odd and assume $i = (d-1)/2$. It is easily seen that the only zero eigenvalue of $\alpha_{i,i+1}\alpha_{i+1,i}$ is for $j = i = d-i-1$, as $d-(d-1)/2 = (d-1)/2+1$. Therefore,

$$\ker \alpha_{\frac{d-1}{2},\frac{d+1}{2}} = W_{\frac{d-1}{2}}\alpha_{\frac{d-1}{2},\frac{d-1}{2}}.$$

Now let $i = (d-1)/2+\delta$, for some integer $\delta > 0$. We note that the summand with $k = (d-1)/2$ in the expression (2.3) is zero. A straightforward calculation shows that for sufficiently small j, the summand with $k = (d-1)/2-l$ in (2.3) erases with the summand with $k = (d-1)/2+l$, for $1 \leq l \leq \delta$. This implies that the only zero eigenvalue of $\alpha_{i,i+1}\alpha_{i+1,i}$ is for $j = (d-1)/2-\delta = d-i-1$. Hence, the kernel of $\alpha_{i,i+1}\alpha_{i+1,i}$ is $W_{d-i-1}\alpha_{d-i-1,i}$.

For d even, the above approach still works up to some differences. For completeness, we give all details.

If d is even, we write $i = \lceil \frac{d-1}{2} \rceil + \delta$, for some integer $\delta \geq 0$. For sufficiently small j, the summand with $k = \lceil \frac{d-1}{2} \rceil - l - 1$ in the expression (2.3) erases with the summand with $k = \lceil \frac{d-1}{2} \rceil + l$, for $0 \leq l \leq \delta$. This implies that the only zero eigenvalue of $\alpha_{i,i+1}\alpha_{i+1,i}$ is for $j = \lceil \frac{d-1}{2} \rceil - \delta - 1 = d-i-1$. Hence the kernel of $\alpha_{i,i+1}\alpha_{i+1,i}$ is $W_{d-i-1}\alpha_{d-i-1,i}$. □

The above Lemmata lead to the following fundamental theorem whose proof is new and, in our opinion, more elementary than those provided in [14] and [16].

Theorem 2.7. *The incidence map of the following incidence structures is one-to-one:*

(i) *i-sets versus j-sets of a d-set, with $i < j$ and $i + j \leq d < \infty$.*

(ii) *i-spaces versus j-spaces of* $\mathrm{PG}(d, q)$*, with* $0 \leq i < j \leq d-1$ *and* $i + j < d < \infty$.

(iii) *i-flats versus j-flats of the affine space* $\mathrm{AG}(d, q)$ *of dimension d over the finite field with q elements, with* $0 \leq i < j \leq d-1$ *and* $i + j < d < \infty$.

Proof. We first give the proof of (ii). We need to prove that $\ker \alpha_{i,j} = 0$, for $0 \leq i < j \leq d-1$ and $i + j < d$. We use induction on $j - i$.

If $j - i = 1$ then $\ker \alpha_{i,i+1} = 0$, by Lemma 2.6 as $i < (d-1)/2$. Now let $j - i > 1$ and assume $\ker \alpha_{i',j'} = 0$ for any pair (i', j') with $0 \leq i' < j' \leq d-1$, $i' + j' < d$ and $j' - i' < j - i$. By Lemma 2.1, we have $\ker \alpha_{i,j} = \ker \alpha_{i,i+1}\alpha_{i+1,j}$. In addition $\dim \ker \alpha_{i,i+1}\alpha_{i+1,j} = \dim \ker \alpha_{i,i+1} + \dim (\mathrm{Im}\, \alpha_{i,i+1} \cap \ker \alpha_{i+1,j})$.

Assume $i + j < d-1$ so $i < (d-1)/2$ and $i + 1 + j < d$. Then $\ker \alpha_{i,i+1} = 0$ by Lemma 2.6, and $\ker \alpha_{i+1,j} = 0$ by inductive hypothesis. Hence $\ker \alpha_{i,j} = 0$ in this case.

Now assume $i + j = d - 1$. We will prove the result by calculating the dimension of $\mathrm{Im}\, \alpha_{i,d-i-1}$. By Lemma 2.1 and 2.2 we have

$$\mathrm{Im}\, \alpha_{i,d-i-1} = V_i\alpha_{i,d-i-1} = \bigoplus_{k=-1}^{i} W_k\alpha_{k,d-i-1}.$$

By the previous part, the map $\alpha_{k,d-i-1}$ is one-to-one for $k = -1, \ldots, i-1$ as $k+d-i-1 < d-1$. Then $\dim W_k\alpha_{k,d-i-1} = \dim W_k$, with $W_k = \ker \alpha_{k,k-1}$. By the arguments in the proof of Lemma 2.2 we get $\dim W_k = \dim V_k - \dim \mathrm{Im}\, \alpha_{k-1,k}$. By Lemma 2.6

the map $\alpha_{k-1,k}$ is one-to-one for $k = -1, \dots, i-1$ as $k-1 < (d-1)/2$. Therefore $\dim \operatorname{Im} \alpha_{k-1,k} = \dim V_{k-1}$. This implies

$$\begin{aligned}\dim\, W_k\alpha_{k,d-i-1} &= \dim\, W_k \\ &= \dim\, V_k - \dim V_{k-1} \\ &= \begin{bmatrix} d+1 \\ k+1 \end{bmatrix}_q - \begin{bmatrix} d+1 \\ k \end{bmatrix}_q,\end{aligned}$$

for $k = -1, \dots, i-1$. Therefore

$$\begin{aligned}\dim \operatorname{Im} \alpha_{i,d-i-1} &= \dim V_i\alpha_{i,d-i-1} \\ &= 1 + \sum_{k=0}^{i-1} \left(\begin{bmatrix} d+1 \\ k+1 \end{bmatrix}_q - \begin{bmatrix} d+1 \\ k \end{bmatrix}_q \right) + \dim W_i\alpha_{i,d-i-1} \\ &= \begin{bmatrix} d+1 \\ i \end{bmatrix}_q + \dim W_i\alpha_{i,d-i-1}.\end{aligned}$$

Still by the proof of Lemma 2.2, we may write $V_i = \operatorname{Im} \alpha_{i-1,i} \oplus W_i$, where $\alpha_{i-1,i}$ is one-to-one as $i < (d-1)/2$. Hence,

$$\dim W_i = \dim V_i - \dim V_{i-1} = \begin{bmatrix} d+1 \\ i+1 \end{bmatrix}_q - \begin{bmatrix} d+1 \\ i \end{bmatrix}_q.$$

This implies

$$\dim W_i\alpha_{i,d-i-1} = \begin{bmatrix} d+1 \\ i+1 \end{bmatrix}_q - \begin{bmatrix} d+1 \\ i \end{bmatrix}_q - \varepsilon,$$

for some $\varepsilon \geq 0$. Thus

$$\begin{aligned}\dim \operatorname{Im} \alpha_{i,d-i-1} &= \dim V_i\alpha_{i,d-i-1} \\ &= \begin{bmatrix} d+1 \\ i \end{bmatrix}_q + \dim W_i\alpha_{i,d-i-1} \\ &= \begin{bmatrix} d+1 \\ i \end{bmatrix}_q + \left(\begin{bmatrix} d+1 \\ i+1 \end{bmatrix}_q - \begin{bmatrix} d+1 \\ i \end{bmatrix}_q - \varepsilon \right) \\ &= \begin{bmatrix} d+1 \\ i+1 \end{bmatrix}_q - \varepsilon.\end{aligned}$$

As $\dim V_i = \left[\begin{smallmatrix} d+1 \\ i+1 \end{smallmatrix}\right]_q$, then $\dim \ker \alpha_{i,d-i-1} = \varepsilon$. At this point to finish the proof we need to evaluate $\dim W_i\alpha_{i,d-i-1}$. We have $\operatorname{Im} \alpha_{i,d-i-1} \leq V_{d-1-1}$, and $\dim V_{d-i-1} = \dim V_i$ by duality. Note that $W_i\alpha_{i,d-i-1}$ is a component of V_{d-i-1} by Lemma 2.1. Then

$$\dim V_{d-i-1} - \dim W_i\alpha_{i,d-i-1} = \dim V_i - \dim W_i\alpha_{i,i} = \begin{bmatrix} d+1 \\ i \end{bmatrix}_q.$$

Hence $\varepsilon = 0$ and this concludes the proof of (ii).

Similar arguments can be used to prove (i). We just need to replace the projective dimension with size of set minus one and the q-binomial coefficients with binomial coefficients.

We now prove (iii). Let $\alpha^A_{i,j}$ denote the incidence map of the i-flats versus the j-flats of $\mathrm{AG}(d,q)$. Embed $\mathrm{AG}(d,q)$ in $\mathrm{PG}(d,q)$ identifying every k-flat of $\mathrm{AG}(d,q)$ with the k-dimensional spaces of $\mathrm{PG}(d,q)$ it spans. Let H denote the hyperplane at infinity of $\mathrm{AG}(d,q)$. Let $f \in \ker \alpha^A_{i,j}$ and g be the extension of f on V_i defined as follows:

$$g(S_i) = \begin{cases} f(S_i) & \text{if } S_i \not\subseteq H; \\ 0 & \text{if } S_i \subseteq H. \end{cases}$$

Then

$$(g\alpha_{i,j})(S_j) = \sum_{S_i \subseteq S_j} g(S_i) = \begin{cases} (f\alpha^A_{i,j})(S_j) & \text{if } S_j \not\subseteq H; \\ 0 & \text{if } S_j \subseteq H. \end{cases}$$

Since $f \in \ker \alpha^A_{i,j}$, we get $g \in \ker \alpha_{i,j}$. By (ii) $g = 0$ and hence $f = 0$. □

Remark 2.8. For $2i + 1 \leq d$, the summands $W_j\alpha_{j,i}$ in the decomposition of V_i given in Lemma 2.2, are all the irreducible constituents of the permutation representation of $\mathrm{PGL}(d,q)$ on F_i. To see this, set $G = \mathrm{PGL}(d,q)$. From the proof of Lemma 2.1 we have $V_i = \mathrm{Im}\,\alpha_{i-1,i} \oplus W_i$. The map $\alpha_{i-1,i}$ is one-to-one, so the number of irreducible components in its image is precisely the number of the irreducible components of the permutation $\mathbb{Q}G$-module V_{i-1}. This number is $i+1$, being the dimension of the intersection of two $(i-1)$-subspaces a complete invariant. This shows that the modules in question are pairwise non-isomorphic, and irreducible. This was proved by Steinberg [22] using deeper representation theory.

An analogous result holds for the permutation $\mathbb{Q}G$-module defined by the symmetric group $\mathrm{Sym}(n)$ acting on the m-sets, with $2m \leq n$. Here the size of set minus one replaces the projective dimension, and binomial coefficients replace q-binomial coefficients.

Remark 2.9. For $2i+1 \leq d$, the summand $W_j\alpha_{j,i}$, for $j = 0,\ldots,i$, in the decomposition of V_i given in Lemma 2.2, is the restriction over the rationals of the $(j+1)$-th eigenspace of the Bose-Mesner algebra of the association scheme on F_i [13, Theorem 2.7]. For a thorough treatment on association schemes we refer the reader to [1, 4].

3 Some applications of Theorem 2.7

The incidence structure $\mathcal{J} = (\mathcal{Q},\mathcal{C},\mathrm{J})$ is said to be a *substructure* of $\mathcal{I} = (\mathcal{P},\mathcal{B},\mathrm{I})$ if $\mathcal{Q} \subseteq \mathcal{P}$, $\mathcal{C} \subseteq \mathcal{B}$ and $\mathrm{J} = \mathrm{I} \cap (\mathcal{Q} \times \mathcal{C})$. The substructure $\mathcal{J}$ of $\mathcal{I}$ is said to be *full* if $\{P \in \mathcal{P} : P\mathrm{I}C\} \subseteq \mathcal{Q}$, for all $C \in \mathcal{C}$.

Lemma 3.1. *Let $\mathcal{I} = (\mathcal{P},\mathcal{B},\mathrm{I})$ be an incidence structure with finite block sizes. Suppose that there is a set $\mathcal{F}$ of full substructures of $\mathcal{I}$, all of whose incidence maps are one-to-one, and such that, for any $P \in \mathcal{P}$ there exists $\mathcal{J} \in \mathcal{F}$ such that P is a point of $\mathcal{J}$. Then the incidence map of $\mathcal{I}$ is one-to-one.*

Proof. Let $\alpha_{\mathcal{I}}$ be the incidence map of $\mathcal{I}$ and $f \in \ker \alpha_{\mathcal{I}}$. For any given $P \in \mathcal{P}$, let $\mathcal{J} = (\mathcal{Q},\mathcal{C},\mathrm{J}) \in \mathcal{F}$ such that $P \in \mathcal{Q}$. Let $\alpha_{\mathcal{J}}$ be the incidence map of $\mathcal{J}$. Set $g = f|_{\mathcal{Q}}$. Since $\mathcal{J}$ is full we have

$$(g\alpha_{\mathcal{J}})(C) = \sum_{\substack{Q \in \mathcal{Q} \\ Q\,\mathrm{J}\,C}} g(Q) = \sum_{\substack{R \in \mathcal{P} \\ R\,\mathrm{I}\,C}} f(R) = (f\alpha_{\mathcal{I}})(C),$$

for all $C \in \mathcal{C}$. Since $f \in \ker(\alpha_{\mathcal{I}})$ we have $(g\alpha_{\mathcal{J}})(C) = 0$, for all $C \in \mathcal{C}$, that is $g \in \ker \alpha_{\mathcal{J}}$. Thus $g = 0$, and therefore $f(P) = g(P) = 0$. Since P is arbitrary, it follows that $f = 0$. □

The above Lemma allows to get the infinitary version of Theorem 2.7; this means that the incidence structures involved are over a set with infinite size (in case (i)), or a space with infinite dimension (in case (ii) and (iii)).

Theorem 3.2. *The incidence map of the following structures is one-to-one:*

(i) *i-sets versus j-sets of an infinite set, with $i < j < \infty$.*

(ii) *i-spaces versus j-spaces of a projective space of infinite dimension over a finite field, with $i < j < \infty$.*

(iii) *i-flats versus j-flats of an affine space of infinite dimension over a finite field, with $i < j < \infty$.*

Proof. We apply Lemma 3.1 and Theorem 2.7 to the above structures by taking the set $\mathcal{F}$ of full substructures as follows: all subsets of size $i + j$ for statement (i), all subspaces of dimension $i+j+1$ for statement (ii), all flats of dimension $i+j+1$ for statement (iii). □

Theorem 3.3. *Let $\mathcal{A}$ be a classical polar space of (possible infinite) rank m over a finite field. Then the incidence map of totally isotropic subspaces (or totally singular in case of a orthogonal space) of $\mathcal{A}$ of algebraic dimension k versus singular subspaces of algebraic dimension l is one-to-one, if $k < l < \infty$ and $k + l \leq m$.*

Proof. Let $\mathcal{I} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be the incidence structure defined by the subspaces of algebraic dimension k versus subspaces of algebraic dimension l of $\mathcal{A}$. Let $\mathcal{F}$ be the family of all the subspaces of $\mathcal{A}$ of algebraic dimension $k + l$. Since every element $\mathcal{J}$ of $\mathcal{F}$ is a full substructure of $\mathcal{I}$, we may apply Theorem 2.7 (ii), or Theorem 3.2 (ii) for the infinitary version, with $i = k-1$, $j = l-1$ and $d = k+l-1$. Thus we get that the incidence map $\alpha_{\mathcal{J}}$ of $\mathcal{J}$ is one-to-one. The result then follows by applying Lemma 3.1 to the family $\mathcal{F}$. □

Remark 3.4. For the case of finite rank the above theorem is due to Lehrer [16, Theorem 5.3]. Note that Lehrer mistakenly asserts that the incidence map of the incidence structure of singular 1-spaces versus singular $(n-1)$-spaces of the $O^+(2n, q)$ polar space is not one-to-one. This error is caused by confusing the $O^+(2n, q)$ polar space with the $D_n(q)$ building.

In the following we apply Lemma 3.1 to the incidence structures known as *diagram geometries*. For a thorough treatment on diagram geometries we refer the reader to [7, 8]; our notation is taken from [7].

Let $\Gamma = (S, \bar{\mathrm{I}}, \overline{\Delta}, \tau)$ be a diagram geometry of finite rank with diagram Δ, and $\mathcal{I} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be the incidence structure where $\mathcal{P}$ is the set of all i-varieties and $\mathcal{B}$ the set of all j-varieties of S; I is the restriction of $\bar{\mathrm{I}}$ on $\mathcal{P} \times \mathcal{B}$. Assume that blocks in $\mathcal{I}$ have finite size and let $k \in \overline{\Delta} \setminus \{j\}$ such that i and k lie in distinct components of the diagram $\Delta - \{j\}$. We now show that the set of k-varieties of S gives rise to a family $\mathcal{F}$ of full substructures of $\mathcal{I}$ with the property that for any point (i-variety) P of $\mathcal{I}$ there exists $\mathcal{J} \in \mathcal{F}$ such that P is a point of $\mathcal{J}$.

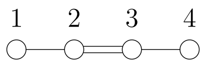
For any given k-variety Λ of S, set $\mathcal{J}_\Lambda = (\mathcal{P}_\Lambda, \mathcal{B}_\Lambda, \mathrm{I}_\Lambda)$ where $\mathcal{P}_\Lambda$ and $\mathcal{B}_\Lambda$ are the set of all i-varieties and j-varieties of S incident to Λ in Γ, respectively; I_Λ is the restriction of I on $\mathcal{P}_\Lambda \times \mathcal{B}_\Lambda$.

Let B be a j-variety in $\mathcal{B}_\Lambda$ and let Γ_B be the residue of B in Γ, that is the diagram geometry $(S', \mathrm{I}', \Delta', \tau')$ where S' is the set of all varieties of S of type $m \in \overline{\Delta} \setminus \{j\}$ which are incident with B, the incidence relation I' is the restriction of $\bar{\mathrm{I}}$ to S', $\Delta' = \tau(S')$ and τ' is the restriction of τ to S'. It is known that the diagram of Γ_B is $\Delta - \{j\}$ [7, Theorem 1]. Therefore the i-varieties of S' are precisely all elements (i-varieties) of $\mathcal{P}_\Lambda$ that are incident with B in $\mathcal{J}_\Lambda$. In addition, as Λ is incident with B, it is a k-variety of S'. Since i and k lie in distinct components of $\Delta - \{j\}$, by [7, Theorem 2] every i-variety of S' is incident with every k-variety, in particular every i-variety of S' is incident with Λ. This implies that $\{P \in \mathcal{P} : P\,\mathrm{I}\,\mathrm{B}\}$ is a subset of $\mathcal{P}_\Lambda$. From the arbitrariness of B in $\mathcal{B}_\Lambda$ it follows that $\mathcal{J}_\Lambda$ is a full substructures of $\mathcal{I}$.

Let $\mathcal{F}$ be the family of the substructures $\mathcal{J}_\Lambda$, for all k-varieties Λ of S. Since the type map τ take all values of Δ on every maximal flag of Γ then for every i-variety P of S there exists a k-variety Λ such that P is a point of $\mathcal{J}_\Lambda$. These considerations together with Lemma 3.1 led to the following result.

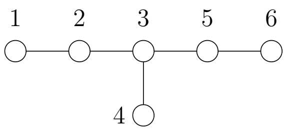
Theorem 3.5. *Let $\Gamma = (S, \bar{\mathrm{I}}, \overline{\Delta}, \tau)$ be the diagram geometry underlying the buildings of types F_4, E_6, E_7 and E_8. Then the incidence map of i-varieties versus j-varieties of Γ is one-to-one in the following cases:*

(i) F_4:



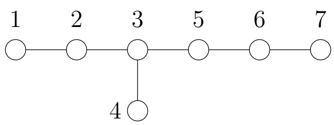
$(i, j) = (1, 2), (4, 3).$

(ii) E_6:



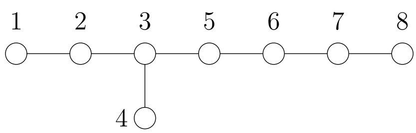
$(i, j) = (1, 2), (1, 3), (2, 3), (6, 5), (6, 3), (5, 3).$

(iii) E_7:



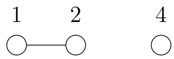
$(i, j) = (1, 2), (1, 3), (2, 3), (7, 6), (7, 5), (7, 3), (6, 5), (6, 3), (5, 3).$

(iv) E_8:



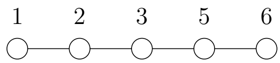
$(i, j) = (1, 2), (1, 3), (2, 3), (8, 7), (8, 6), (8, 5), (8, 3), (7, 6), (7, 5), (7, 3), (6, 5), (6, 3).$

Proof. Consider the diagram $\Gamma = (S, \bar{\mathrm{I}}, \overline{\Delta}, \tau)$ for F_4, and take $(i, j) = (1, 2)$, $k = 3$. Let $\mathcal{F}$ be the family of full substructures arising from the 3-varieties of S constructed as above. The points and blocks of any $\mathcal{J}_\Lambda \in \mathcal{F}$ are precisely the 1- and 2-varieties of S incident with Λ. By [7, Theorem 1], these are precisely the 1- and 2-varieties of the residue $R(\Lambda)$ of Λ in Γ, whose diagram is



Note that every 1- and 2-variety is incident with every 4-variety. This implies that the set of the 1- and 2-varieties of S incident with Λ form a finite projective plane, whose incidence map is injective by a result of Bruck and Ryser [5] and Bose [3]. Lemma 3.1 yields that the incidence map of 1-varieties versus 2-varieties of S is one-to-one in this case. Very similar argument can used with $(i, j) = (4, 3)$ and $k = 2$.

Now consider the diagram $\Gamma = (S, \bar{\mathrm{I}}, \overline{\Delta}, \tau)$ for E_6, and take $(i, j) = (1, 2)$, $k = 4$. As above the points and blocks of any $\mathcal{J}_\Lambda \in \mathcal{F}$ are precisely the 1- and 2-varieties of S incident with Λ, and these are precisely the 1- and 2-varieties of the residue $R(\Lambda)$ of Λ in Γ, whose diagram is



This implies that $R(\Lambda)$ has the geometry of a $\mathrm{PG}(5, q)$. We now apply Theorem 2.7 to conclude that the incidence map of $\mathcal{J}_\Lambda$ is incidence, and Lemma 3.1 yields that the incidence map of 1-varieties versus 2-varieties of S is one-to-one in this case. Very similar arguments apply for the remaining cases, and for the buildings E_7, E_8. $\square$

4 An extension of Block's Lemma

An *automorphism* of the incidence structure $\mathcal{I} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ is a mapping g of $\mathcal{P} \cup \mathcal{B}$ such that g defines permutations on $\mathcal{P}$ and $\mathcal{B}$ such that $P\mathrm{I}B$ if and only if $P^g\mathrm{I}B^g$. The group of all automorphisms of $\mathcal{I}$ is denoted by $\mathrm{Aut}\,\mathcal{I}$.

A *decomposition* of an incidence structure $\mathcal{I} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ is a pair $(\mathcal{X}, \mathcal{Y})$, with $\mathcal{X}$ a partition of $\mathcal{P}$ and $\mathcal{Y}$ a partition of $\mathcal{B}$. A decomposition $(\mathcal{X}, \mathcal{Y})$ of an incidence structure with finite block sizes is *block-tactical* if

$$|\{P \in X : P\,\mathrm{I}\,B_1\}| = |\{P \in X : P\,\mathrm{I}\,B_2\}|,$$

for all $X \in \mathcal{X}$, $Y \in \mathcal{Y}$, $B_1, B_2 \in Y$. An example of block tactical decomposition of an incidence structure $\mathcal{I}$ is obtained by taking the orbits on points and blocks of a subgroup of $\mathrm{Aut}\,\mathcal{I}$.

With a decomposition $(\mathcal{X}, \mathcal{Y})$ of $\mathcal{I} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ we associate the following subspaces of the point space $\mathbb{Q}^{\mathcal{P}}$ and the block space $\mathbb{Q}^{\mathcal{B}}$ of $\mathcal{I}$: the *point class space* $V_{\mathcal{X}}$ of all functions on $\mathcal{P}$ constant on each $X \in \mathcal{X}$, and the *block class space* $V_{\mathcal{Y}}$ of all functions on $\mathcal{B}$ constant on each $Y \in \mathcal{Y}$.

Lemma 4.1. *A decomposition $(\mathcal{X}, \mathcal{Y})$ of an incidence structure $\mathcal{I} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ with finite block sizes and incidence map α is block-tactical if and only if $V_{\mathcal{X}}\alpha \subseteq V_{\mathcal{Y}}$.*

Proof. Suppose $(\mathcal{X}, \mathcal{Y})$ is block-tactical and $f \in V_{\mathcal{X}}$. For each $X \in \mathcal{X}$, let P_X be a fixed chosen point in X. As f is constant on X, then $f(P) = f(P_X)$ for all $P \in X$. Let $Y \in \mathcal{Y}$

and $B_1, B_2 \in Y$. Then $|\{Q \in X : Q\mathrm{I}B_1\}| = |\{Q \in X : Q\mathrm{I}B_2\}|$ and therefore

$$\begin{aligned}(f\alpha)(B_1) &= \sum_{P\,\mathrm{I}\,B_1} f(P) = \sum_{X\in\mathcal{X}} \sum_{\substack{P\in X\\ P\mathrm{I}B_1}} f(P)\\ &= \sum_{X\in\mathcal{X}} |\{Q \in X : Q\mathrm{I}B_1\}| f(P_X)\\ &= \sum_{X\in\mathcal{X}} |\{Q \in X : Q\mathrm{I}B_2\}| f(P_X)\\ &= \sum_{X\in\mathcal{X}} \sum_{\substack{P\in X\\ P\mathrm{I}B_2}} f(P) = \sum_{P\mathrm{I}B_2} f(P) = (f\alpha)(B_2).\end{aligned}$$

Hence $f\alpha$ is constant on Y. So $f\alpha \in V_{\mathcal{Y}}$, giving $V_{\mathcal{X}}\alpha \subseteq V_{\mathcal{Y}}$.

Conversely, suppose that $V_{\mathcal{X}}\alpha \subseteq V_{\mathcal{Y}}$. Let $X \in \mathcal{X}$ and $\chi_X \in \mathbb{Q}^{\mathcal{P}}$ denote the characteristic function of X. Then, χ_X can be naturally considered as an element of $V_{\mathcal{X}}$, thus $\chi_X\alpha \in V_{\mathcal{Y}}$ by hypothesis. Therefore, we have

$$|\{P \in X : P\mathrm{I}B_1\}| = (\chi_X\alpha)(B_1) = (\chi_X\alpha)(B_2) = |\{P \in X : P\mathrm{I}B_2\}|,$$

for each $Y \in \mathcal{Y}$ and $B_1, B_2 \in \mathcal{Y}$. Hence $(\mathcal{X}, \mathcal{Y})$ is block-tactical. □

The following result is a slight extension of a fundamental result of R. E. Block [2, Theorem 2.1] often known as "Block's Lemma".

Lemma 4.2. *Let $\mathcal{I} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be an incidence structure with finite block sizes and $(\mathcal{X}, \mathcal{Y})$ a block-tactical decomposition of $\mathcal{I}$. Let α denote the incidence map of $\mathcal{I}$. Then*

$$\dim V_{\mathcal{X}} \leq \dim V_{\mathcal{Y}} + \dim(\ker\alpha).$$

Proof. By Lemma 4.1, we have $V_{\mathcal{X}}\alpha \subseteq V_{\mathcal{Y}}$, so $\dim(V_{\mathcal{X}}\alpha) \leq \dim V_{\mathcal{Y}}$. Now $\dim V_{\mathcal{X}} = \dim(V_{\mathcal{X}}\alpha) + \dim(V_{\mathcal{X}} \cap \ker\alpha) \leq \dim V_{\mathcal{Y}} + \dim(\ker\alpha)$. □

Theorem 4.3. *Let G be one of the following groups:*

(i) *a permutation group of finite degree d;*

(ii) *a group of collineations of $\mathrm{PG}(d, q)$, $d < \infty$;*

(iii) *a group of affine collineations of $\mathrm{AG}(d, q)$, $d < \infty$;*

(iv) *a group of semi-linear isometries of a classical polar space of finite rank d over a finite field.*

For any given non-negative integer $i < d$, let n_i denote the number of orbits on i-sets for (i)*, on subspaces of dimension i for* (ii)*, on flats of dimension i for* (iii)*, on totally isotropic subspaces (or totally singular in case of a orthogonal space) of dimension i for* (iv)*. Then $n_i \leq n_j$, for $i < j$ and $i + j < d$.*

Proof. Let $\mathcal{X}_i$ be the set of the orbits of G on the corresponding family of objects indexed by i. For any $i < j < d$, put $(\mathcal{X}, \mathcal{Y}) = (\mathcal{X}_i, \mathcal{X}_j)$. The set of all characteristic functions χ_X, $X \in \mathcal{X}$, is a basis for $V_{\mathcal{X}}$. Hence, $\dim V_{\mathcal{X}} = |\mathcal{X}| = n_i$. Similarly, $\dim V_{\mathcal{Y}} = |\mathcal{Y}| = n_j$, and Lemma 4.2 gives $|\mathcal{X}| \leq |\mathcal{Y}| + \dim(\ker \alpha)$ since the point- and block-orbits of any subgroup of the full automorphism group of an incidence structure form a block-tactical decomposition. The result is obtained by applying Theorems 2.7 and 3.3. □

The following is the infinite version of the previous result.

Theorem 4.4. *Let G be one of the following groups:*

(i) *a permutation group of infinite degree;*

(ii) *a group of collineations of a projective space of infinite dimension over a finite field;*

(iii) *a group of affine collineations of an affine space of infinite dimension over a finite field;*

(iv) *a group of semi-linear isometries of a classical polar space of infinite rank over a finite field.*

For any given non-negative integer i, let n_i denote the number of orbits on i-sets for (i)*, on subspaces of dimension i for* (ii)*, on flats of dimension i for* (iii)*, on totally isotropic subspaces (or totally singular in case of a orthogonal space) of dimension i for* (iv)*. Let l be the least index such that n_l is infinite. Then $n_0 \leq n_1 \leq \cdots \leq n_{l-1}$ and n_k is infinite for all $k \geq l$.*

Proof. Let $\mathcal{X}_i$ be the set of the orbits of G on the corresponding family of objects indexed by i.

Let $i < j \leq l - 1$. We apply very similar arguments as in the proof of Theorem 4.3 to the block-tactical decomposition $(\mathcal{X}, \mathcal{Y}) = (\mathcal{X}_i, \mathcal{X}_j)$. Then Theorems 3.2 and 3.3 give $n_i \leq n_j$.

Let $l \leq i < j < \infty$. Since the incidence map of the incidence structure associated with $(\mathcal{X}_i, \mathcal{X}_j)$ has trivial kernel by Theorems 3.2 and 3.3, we may apply Proposition 2.1 in [9] (where ρ is the incidence relation). □

Remark 4.5. Theorem 4.4 (i) is due to Cameron [9, Theorem 2.2].

Remark 4.6. By using the Generalized Continuum Hypothesis it is possible to give a slight improvement of the previous result when n_i and n_j, $i < j$, are infinite.

From Lemma 4.2 we get $\dim V_{\mathcal{X}_i} \leq \dim V_{\mathcal{X}_j} + \dim(\ker \alpha)$, and it is known that $\dim V = |V|$ when V is an infinite dimensional vector space over an infinite field F such that $|V| > |F|$.

Set $n_i = \aleph_{\beta_i}$, $\beta_i \geq 0$. Thus, $|V_{\mathcal{X}_i}| = |\mathbb{Q}^{\mathcal{X}_i}| = \aleph_0^{\aleph_{\beta_i}} = \aleph_{\beta_i+1} = 2^{\aleph_{\beta_i}} > \aleph_0 = |\mathbb{Q}|$ by the Generalized Continuum Hypothesis. Therefore, $\dim V_{\mathcal{X}_i} = 2^{\aleph_{\beta_i}}$, and similarly, $\dim V_{\mathcal{X}_j} = 2^{\aleph_{\beta_j}}$. Hence Lemma 4.2 yields

$$2^{\aleph_{\beta_i}} \leq 2^{\aleph_{\beta_j}} + \dim(\ker \alpha).$$

Theorems 3.2 and 3.3 yield $2^{\aleph_{\beta_i}} \leq 2^{\aleph_{\beta_j}}$, and the Generalized Continuum Hypothesis implies $\aleph_{\beta_i} \leq \aleph_{\beta_j}$, that is $n_i \leq n_j$.

Remark 4.7. In the paper [18], examples of infinite Desarguesian projective planes with collineation groups having three orbits on points and two on lines are provided, solving a problem posed by Cameron [10] and attributed to Kantor.

5 Incidence structures and permutation representations

Block's Lemma leads to consideration of $\ker\alpha$. It is particularly nice when $\ker\alpha$ is trivial, and the following lemma also emphasizes this case.

Lemma 5.1. *Let $\mathcal{I} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be a finite incidence structure whose incidence map is one-to-one. For any given automorphism group G of $\mathcal{I}$ the permutation representation of G on $\mathcal{P}$ is a subrepresentation of the permutation representation of G on $\mathcal{B}$ (considered as linear representation over a field of characteristic zero).*

Proof. The point space $\mathbb{Q}^{\mathcal{P}}$ is the permutation $\mathbb{Q}$-module for G on $\mathcal{P}$, and the block space $\mathbb{Q}^{\mathcal{B}}$ is the permutation $\mathbb{Q}$-module for G on $\mathcal{B}$. Since G preserves the incidence, we have

$$(f^g\alpha)(B) = \sum_{P\,\mathrm{I}\,B} f^g(P) = \sum_{P\,\mathrm{I}\,B} f(P^{g^{-1}}) = \sum_{P\,\mathrm{I}\,B^{g^{-1}}} f(P) = (f\alpha)(B^{g^{-1}}) = (f\alpha)^g(B),$$

for all $f \in \mathbb{Q}^{\mathcal{P}}$ and $g \in G$. Therefore, α is a $\mathbb{Q}G$-homomorphism from $\mathbb{Q}^{\mathcal{P}}$ to $\mathbb{Q}^{\mathcal{B}}$. As α is one-to-one, the permutation representation of G on $\mathcal{P}$ is a subrepresentation of the permutation representation of G on $\mathcal{B}$ (over $\mathbb{Q}$). For other fields of characteristic zero, we need only tensor up. □

Lemma 5.2. *Let G be a group acting as a transitive permutation group on a finite set X of size n. Let S be a subset of G such that $\sum_{s\in S} s$ is mapped to the 0-matrix under every irreducible non-principal representation. Then $|X|$ divides $|S|$.*

Proof. Let $\{x_1, \ldots, x_n\}$ be the natural basis of the permutation $\mathbb{Q}G$-module on X. The matrix representation with respect this basis of any element $s \in G$ on the trivial module is $1/|X|J$, where J is the all-one $n \times n$ matrix. This implies that the matrix representation of the endomorphism $\sum_{s\in S} s$ on the trivial module is $|S|/|X|J$.

On the other hand, the matrix representation of $\sum_{s\in S} s$ in the basis $\{x_1, \ldots, x_n\}$ is $P_S = \sum_{s\in S} P(s)$, where $P(s)$ is the permutation matrix representing $s \in G$. Note that the entries in P_S are positive integers. Since $\sum_{s\in S} s$ is mapped to the 0-matrix under every irreducible non-principal representation, we have $P_S = |S|/|X|J$. The result then follows. □

Theorem 5.3 ([19])**.** *Let $\mathcal{I} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be a finite incidence structure with incidence map one-to-one. If the automorphism group of $\mathcal{I}$ contains a subset which is sharply transitive on blocks, then $|\mathcal{P}|$ divides $|\mathcal{B}|$.*

Proof. Set $G = \mathrm{Aut}\,\mathcal{I}$ and $S \subset G$ be sharply transitive on blocks. Hence, $|S| = |\mathcal{B}|$. By [19, Lemma 1], the endomorphism $\sum_{s\in S} s$ of the permutation $\mathbb{Q}G$-module $\mathbb{Q}^{\mathcal{B}}$ on blocks is mapped to the 0-matrix under every irreducible non-principal representation. By Lemma 5.1, every irreducible submodule of $\mathbb{Q}^{\mathcal{P}}$ is a submodule of $\mathbb{Q}^{\mathcal{B}}$ with less or equal multiplicity. Hence, $\sum_{s\in S} s$ acting on $\mathbb{Q}^{\mathcal{P}}$ is mapped to the 0-matrix under every irreducible non-principal representation in $\mathbb{Q}^{\mathcal{P}}$. By Lemma 5.2, we have $|\mathcal{P}|$ divides $|\mathcal{B}|$. □

Corollary 5.4. *Let $\mathcal{I} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be a finite incidence structure with incidence map one-to-one and automorphism group G acting transitively on blocks. If $|\mathcal{P}|$ does not divide $|\mathcal{B}|$, then G does not contain a subset acting sharply transitive on blocks.*

The above result can be restated as follows.

Corollary 5.5. *Let $\mathcal{I} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be a finite incidence structure with incidence incidence map one-to-one and automorphism group $\mathrm{Aut}\,\mathcal{I}$ acting transitively on blocks. Let H denote the one-block stabilizer in $\mathrm{Aut}\,\mathcal{I}$. If $|\mathcal{P}|$ does not divide $|\mathcal{B}|$, then the permutation representation of $\mathrm{Aut}\,\mathcal{I}$ on the cosets of H contains no sharply transitive subset.*

Remark 5.6. Corollary 5.4 applies to the following incidence structures as their incidence map is one-to-one: combinatorial designs, linear spaces and circular spaces (see [6]); incidence structures in projective and affine spaces (see [14] and Theorem 2.7); incidence structures in classical polar spaces (see [16] and Theorem 3.3); incidence structures on subsets ([9, 14, 15, 20] and Theorem 2.7); nonbipartite graphs (see [21]).

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On plane subgraphs of complete topological drawings*

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Abstract

Topological drawings are representations of graphs in the plane, where vertices are represented by points, and edges by simple curves connecting the points. A drawing is *simple* if two edges intersect at most in a single point, either at a common endpoint or at a proper crossing. In this paper we study properties of maximal plane subgraphs of simple drawings D_n of the complete graph K_n on n vertices. Our main structural result is that maximal plane subgraphs are 2-connected and what we call *essentially* 3*-edge-connected*. Besides, any maximal plane subgraph contains at least $\lceil 3n/2 \rceil$ edges. We also address the problem of obtaining a plane subgraph of D_n with the maximum number of edges, proving that this problem is NP-complete. However, given a plane spanning connected subgraph of D_n, a maximum plane augmentation of this subgraph can be found in $O(n^3)$ time. As a side result, we also show that the problem of finding a largest compatible plane straight-line graph of two labeled point sets is NP-complete.

Keywords: Graph, topological drawing, plane subgraph, NP-complete problem.

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1 Introduction

In a *topological drawing* (in the plane or on the sphere) of a graph, vertices are represented by points and edges by simple curves connecting the corresponding pairs of points.Usually, we only consider drawings satisfying some natural non-degeneracy conditions, in particular a drawing is called *simple* (or a *good drawing*) if two edges intersect at most in a single point, either at a common endpoint or at a crossing in their relative interior. When all the edges of a topological drawing are straight-line segments, then the drawing is called a *rectilinear drawing* or *geometric graph*.

In this paper we consider only simple topological drawings of the complete graph K_n on n vertices. Simple topological drawings of complete graphs have been studied extensively, mainly in the context of crossing number problems. It is well known that a drawing minimizing the number of crossings has to be simple, and besides, if $n \geq 8$, the drawings of K_n minimizing that crossing number are not rectilinear. We refer the reader to [1, 3, 4] for recent advances on the Harary-Hill conjecture on the minimum number of crossings of drawings of K_n, and to the survey [22] for some variants on this crossing number problem.

The problem of enumerating all the non-isomorphic drawings of K_n has been studied in [2, 12, 13, 18] (two drawings are isomorphic if there is a homeomorphism of the sphere that transforms one drawing into the other).

Let D_n be a simple topological drawing of K_n. Herein, we consider graphs in connection with their drawings, and in particular when addressing subgraphs of K_n we also consider the associated sub-drawing of D_n. We are interested in crossing-free edge sets F in D_n, and we will say that F is a plane subgraph of D_n. Crossing-free edge sets in D_n have attracted considerable attention, in part because problems on embedding graphs on a set of points usually generalize to finding plane subgraphs of D_n. For instance, the problem of computing the maximum number of plane Hamiltonian cycles that a simple drawings D_n can contain, is a generalization of the same problem considering only rectilinear drawings of K_n. And this last is the (open) problem of computing the maximum number of simple n-gons that can be formed on n points in the plane.

There are relatively few results on plane subgraphs of D_n. It is well known that in any drawing D_n of K_n, there are plane subgraphs with $2n-3$ edges, and that there are at most $2n-2$ edges uncrossed by any other edge [6, 8, 19]. Pach, Solymosi, and Tóth [14] showed that any D_n has $\Omega\left(\log^{1/6}(n)\right)$ pairwise disjoint edges. This bound was subsequently improved in [5, 16, 23]. The current best bound of $\Omega(n^{1/2-\epsilon})$ is by Ruiz-Vargas [20]. However, the much stronger conjecture that any simple drawing D_n of K_n contains a plane Hamiltonian cycle remains unproved, although it has been verified for $n \leq 9$, see [2].

In the course of their work on disjoint edges and empty triangles in D_n, Fulek and Ruiz-Vargas [6] showed the following lemma.[1]

Lemma 1.1 (Fulek and Ruiz-Vargas [6]). *Between any plane connected subgraph F of D_n and a vertex v not in F, there exist at least two edges from v to F that do not cross F.*

This result can be used to build large plane subgraphs. For instance, we can begin with F consisting of only one edge, then for each vertex v not in F, we add to F the edges from v to F not crossing F. In this way, we will obtain a maximal plane subgraph: a plane subgraph $\overline{F}$ such that any edge $e \notin \overline{F}$ crosses some edge of $\overline{F}$.

[1]Their lemma is actually more general. It does not require F and v to be elements of a drawing of K_n, but rather of a drawing that contains all edges from v to vertices of F.

In Section 2 of this work, we extend that Lemma 1.1 to arbitrary (not necessarily connected) plane subgraphs. Further, in Section 3, we prove that any plane subgraph of D_n can be augmented to a 2-connected plane subgraph of D_n. A consequence of this result is that maximal plane subgraphs contain at least $\min(\lceil 3n/2 \rceil, 2n-3)$ edges, and this bound is tight. Maximal plane subgraphs of D_n have other interesting properties. For example, we show that, when removing two edges from a maximal plane subgraph, it either stays connected or one of the two components is a single vertex. Another consequence of the previous results is that for every vertex v of a drawing D_n, there is a plane subgraph of D_n consisting of the n-vertex star of edges incident to v, plus the edges of a spanning tree on the $n-1$ vertices of $V \setminus \{v\}$.

The problem setting changes when we want our plane graphs not only to be maximal, but also to contain the maximum number of edges. While for geometric graphs, every maximal plane subgraph is a triangulation and thus also has a maximum number of edges, the situation is different for plane subgraphs of D_n. In Section 4, we will prove that computing a plane subgraph of D_n with maximum number of edges is an NP-complete problem. However, if a connected plane spanning subgraph F is given, we can adapt a classic algorithm from computational geometry to show that a maximum plane augmentation of F can be found in $O(n^3)$ time.

As a side result, we also show that the problem of finding a largest compatible plane graph on two labeled point sets is NP-complete.

Finally, going back to Lemma 1.1, we give an $O(n)$ algorithm to compute all the edges from a vertex v to a plane connected subgraph F that do not cross F.

2 Adding a single vertex

We now discuss a generalization of Lemma 1.1 to arbitrary plane subgraphs. This generalization will also follow independently from Theorem 3.1. Still, the following proposition gives further insight on the position of the uncrossed edges around the vertex v, which might help in the construction of algorithms.

We assume that a simple topological drawing D_n of K_n in the plane is given, with vertex set $V = \{v_1, \ldots, v_n\}$. If x_1, x_2 are two points on an edge e of D_n (not necessarily the endpoints of e), by line x_1x_2 we mean the portion of the curve e of the drawing placed between the points x_1 and x_2. For a vertex v, the *star graph* with center v is the subgraph formed by the edges connecting v to all the other vertices. We denote this set of edges by $S(v)$, usually call *rays* to these edges emanating from v, and we suppose that the rays of $S(v)$ are (circularly) clockwise ordered. By the clockwise range $[vp, vq]$ of $S(v)$ we mean the ordered set of rays placed clockwise between vp and vq, including rays vp and vq. When vp or vq or both are not included in that ordered set of rays, we will use $(vp, vq]$, $[vp, vq)$ or (vp, vq), respectively. In the same way, we can define counterclockwise ranges.

In the rest of this section, we suppose F is a given plane subgraph of D_n and v a vertex not in F. In the figures, we use red color for the edges of F, so we usually call them *red edges*. We say a ray vr of $S(v)$ is *uncrossed* if it does not cross any edge of F.

Suppose that the ray vr crosses some edge of F, let $e = pq$ be the first edge of F crossed by vr, and let x be the first crossing point. Without loss of generality, we can suppose that the rays vr, vp, and vq appear in this clockwise order in $S(v)$. See Figure 1.

We define the clockwise range R_{cw} of rays centered at v corresponding to the crossing

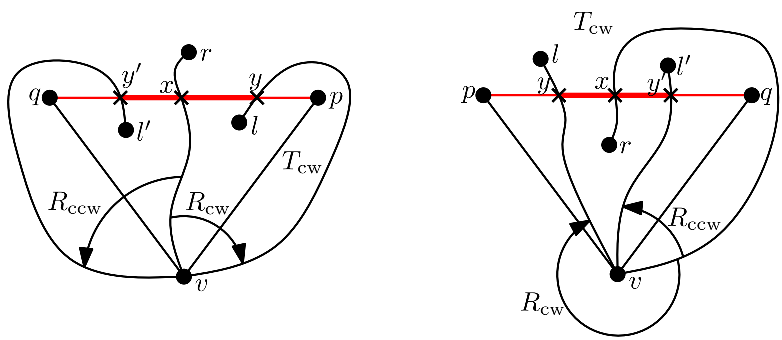


Figure 1: The clockwise and counterclockwise ranges of a first crossing.

x in the following way: if no ray in the clockwise range (vp, vq) crosses the edge pq between x and p, then R_{cw} is the range $(vr, vp]$; otherwise, (some rays in the clockwise range (vp, vq) cross the line xp), R_{cw} is the clockwise range $(vr, vl]$, where vl is the last ray in (vp, vq) crossing the line xp. That implies that if vl crosses xp at the point y, among the intersection points of rays in (vp, vq) with the line xp, the closest to x is y. See Figure 1. Analogously, the range R_{ccw} is defined either as the counterclockwise range $(vr, vq]$ if no edge in the counterclockwise range (vq, vp) crosses the line xq, or as the counterclockwise range $(vr, vl']$, where vl' is the ray in the counterclockwise range (vq, vp) crossing the line xq in a point y' closest to x. By definition, the rays vr, vp, vl, vl', vq appear clockwise in that order around v. Observe that R_{cw} and R_{ccw} are disjoint sets and they are also nonempty, as vp is in R_{cw} and vq is in R_{ccw}. The following result generalizes Lemma 1.1.[2]

Proposition 2.1. *Suppose the ray vr first crosses the edge e of F at the point x. Let R_{cw} and R_{ccw} be the clockwise and counterclockwise ranges of rays of v corresponding to that crossing. Then, each one of these two ranges contains an uncrossed ray. As a consequence, $S(v)$ contains at least two uncrossed rays.*

Proof. We prove the statement for R_{cw}, the proof for R_{ccw} is identical.

Observe that by definition, no red edge can cross the line vx, and a ray in the clockwise range $(vq, vr]$ cannot cross the line xp. Let y be the crossing point between the red edge $e = pq$ and the ray vl. When R_{cw} is $(vr, vp]$ (i.e., no ray in the clockwise range (vp, vq) crosses xp), then we identify the points p, l and y. The lines vx, xy and yv define a simple closed curve, that divides the plane into two regions $T_{\text{cw}}, \overline{T_{\text{cw}}}$, where T_{cw} is the region not containing the point q.

From the definition of T_{cw}, it follows that a ray containing a point placed in the interior of T_{cw} must be in the range R_{cw}. Besides, a red edge can cross the boundary of that region only through the line yv, and hence, if a red edge e crosses yv, one endpoint of e must be inside T_{cw} the other one in $\overline{T_{\text{cw}}}$.

The proof is done by induction on $|R_{\text{cw}}|$, the number of rays in that range. So, first suppose that the only ray in the range R_{cw} is the ray vp. In this case, T_{cw} is the region bounded by the closed curve vx, xp, pv not containing the point q. This region cannot contain any

[2]Like Lemma 1.1, this result is more general. It does not require F and v to be elements of a drawing of K_n, but rather of a drawing that contains all edges from v to vertices of F.

vertex r' of F, because then vr' would be in R_{cw}, therefore vp must be uncrossed. This proves the base case of the induction.

Now suppose that the proposition has been proved for any clockwise range containing less than $|R_{\text{cw}}|$ rays. Let vr' be the first ray of R_{cw}. Of course, if the ray vr' is uncrossed, the proof is done, so we can suppose that the ray vr' is first crossed by a red edge e' at a point x'. We are going to prove that the clockwise range R'_{cw} corresponding to the crossing x' is strictly contained in R_{cw}. Then, by induction, R'_{cw} contains an uncrossed ray, and thus also R_{cw}. To prove that $R'_{\text{cw}} \subset R_{\text{cw}}$ strictly, it is enough to prove that the clockwise last ray of R'_{cw} is contained in R_{cw}.

Let us first analyze Case A: when the edge e' is precisely the edge e. See Figure 2. In this case, if x' is between x and y, then the clockwise range R'_{cw} corresponding to the

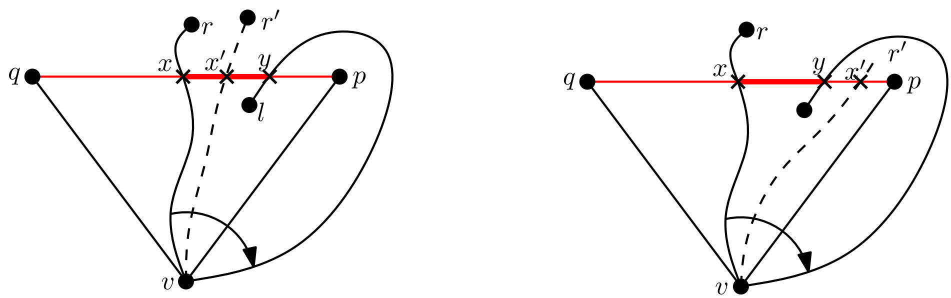
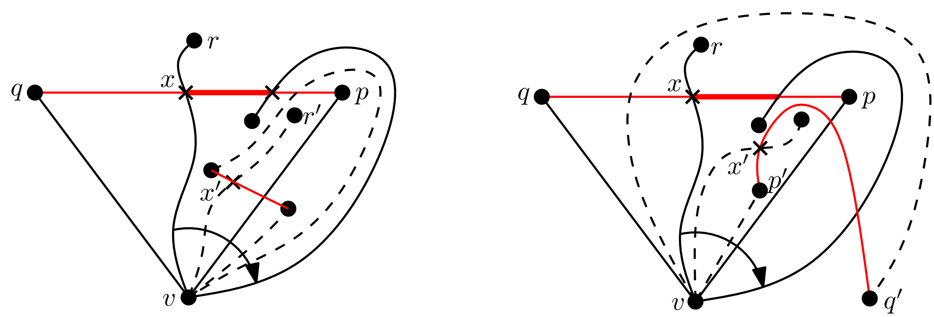


Figure 2: Case A, the ray vr' first crosses the edge $e = pq$.

new crossing point x' is precisely R_{cw} minus its first ray vr'. And if x' is between y and p, then all the points of the line $x'p$ (including point p) are in the interior of the region T_{cw}, therefore the corresponding last ray of R'_{cw} has to be in R_{cw}. Thus, in both subcases is $R'_{\text{cw}} \subset R_{\text{cw}}$ strictly.

Suppose now Case B: when $e' \neq e$. See Figure 3. Clearly, at least one endpoint of e' is



Subcase B1 Subcase B2

Figure 3: Case B, the ray vr' first crosses an edge $e' \neq e$.

in T_{cw}, as otherwise the ray vl would be crossed twice by e'. Hence, either both endpoints are in T_{cw}, subcase B1, or one of them is in T_{cw} and the other one in $\overline{T_{\text{cw}}}$, subcase B2. In subcase B1, the entire edge $e' = p'q'$ must be inside T_{cw}. Therefore, any ray containing a point of e' must be in $R_{\rm cw}$. In particular, the last ray of $R'_{\rm cw}$ must be in $R_{\rm cw}$, and hence, $R'_{\rm cw}$ is strictly contained in $R_{\rm cw}$.

In subcase B2, an endpoint, p', of e' is inside $T_{\rm cw}$ and the other, q', is in $\overline{T_{\rm cw}}$. Observe that the ray vp' must be in $R_{\rm cw}$, however the ray vq' cannot be in $R_{\rm cw}$ because the range (vr, vr') is empty, and a ray in $[vr', vl]$ finishing at q' has to cross the edge e'. See Figure 3. Therefore, the rays vr', vp', vq' appear clockwise around v in this order. Hence, the last ray of $R'_{\rm cw}$ is either vp' or a ray crossing the line $x'p'$. In any case, as the line $x'p'$ is inside $T_{\rm cw}$, this last ray of $R'_{\rm cw}$ has to be in $R_{\rm cw}$. This completes the proof. □

3 Structure of maximal plane subgraphs

Let D_n be an arbitrary simple drawing of K_n. In this section, we identify several structural properties of maximal plane subgraphs of D_n, using Lemma 1.1 or Proposition 2.1 as our main tool. Maximal plane subgraphs turn out to be 2-connected. While there are examples of maximal plane subgraphs that are not 3-connected, we elaborate further on the structure, showing that a maximal plane subgraph is either 3-edge-connected or has a vertex of degree 2.

Theorem 3.1. *A maximal plane subgraph of D_n is spanning and 2-connected.*

Proof. The proof is by induction on n. The result is obviously true for $n \leq 3$. For $n > 3$, assume there exists a maximal plane subgraph $\overline{F}$ that is not 2-connected, and let us see that a contradiction is reached.

We first claim that, under this assumption, $\overline{F}$ has no vertices of degree less than 3. Suppose the contrary, that the vertex v has degree ≤ 2. Let F' be the subgraph of $\overline{F}$ obtained after removing the vertex v, and let $\overline{F'}$ be a maximal plane subgraph (in the drawing $D_n - \{v\}$ of K_{n-1}) containing F'. By the induction hypothesis, $\overline{F'}$ is 2-connected. We observe that v cannot have (in $\overline{F}$) degree less than 2, since applying Lemma 1.1 to v and $\overline{F'}$ would give two edges at v not crossing $\overline{F}$, contradicting the maximality of $\overline{F}$. So suppose v has degree 2. As we assume that $\overline{F}$ is not 2-connected, F' cannot be 2-connected. However, $\overline{F'}$ is 2-connected, and hence there exists an edge e' in $\overline{F'} - F'$. By the maximality of $\overline{F}$, e' must cross at least one edge vw of $\overline{F}$ incident to v. But applying Lemma 1.1 to v and $\overline{F'}$ gives at least two edges incident to v not crossing $\overline{F'}$. These two edges and also vw do not cross $\overline{F}$, contradicting the maximality of $\overline{F}$. Therefore, the claim follows.

Assume now that $\overline{F}$ is not connected. Let C_1, C_2 be two connected components of $\overline{F}$. As all vertices have (in $\overline{F}$) degree at least 3, C_1 cannot be an outerplanar graph, and it has more than one face. Without loss of generality, we can suppose that C_2 is in the unbounded face of C_1. Let v_1 be an interior vertex of C_1, F' the graph obtained from $\overline{F}$ by removing v_1, and f_1 the face of F' containing v_1. The face containing C_2 remains unchanged by the removal of v_1. By induction, F' can be completed to a 2-connected plane graph $\overline{F'}$, and due to the maximality of $\overline{F}$, all the edges in $\overline{F'} - F'$ should be in the face f_1. But then, as C_2 is outside f_1, $\overline{F'}$ could not be connected, a contradiction. Thus, $\overline{F}$ has to be connected.

By a similar reasoning we arrive at our contradiction to $\overline{F}$ not being 2-connected. A *block* is a 2-connected component of a graph, and a *leaf block* is a block with only one cut vertex. Since $\overline{F}$ is not 2-connected, it has at least two leaf blocks B_1 and B_2. As all vertices have degree at least 3, B_1 cannot have all its vertices on the same face. Again, without loss of generality, we can suppose B_2 is in the outer face of B_1, and there is an interior vertex

v_1 of B_1. Removing v_1 from $\overline{F}$, we obtain a plane graph F' that has a face f_1 containing v_1, and F' is contained in a maximal plane graph $\overline{F'}$ that is 2-connected. Again, by the maximality of $\overline{F}$, all the edges in $\overline{F'} - F'$ must be in f_1, implying that B_2 is still a block of $\overline{F'}$, contradicting the fact that $\overline{F'}$ is 2-connected. Hence, $\overline{F}$ must be 2-connected. □

Theorem 3.1 can be used to obtain more properties of maximal plane subgraphs.

Lemma 3.2. *If a maximal plane subgraph $\overline{F}$ of D_n contains a vertex v of degree 2, then the subgraph of $\overline{F}$ obtained after removing v is also maximal in $D_n - \{v\}$.*

Proof. Suppose the contrary. Remove v from $\overline{F}$ to obtain F' and let $\overline{F'}$ be a maximal plane graph containing F'. As $\overline{F}$ is maximal but F' is not, $\overline{F'} - F'$ must contain an edge e' that crosses some edge vw of $\overline{F}$. But by Lemma 1.1 there are at least two edges from v to $\overline{F'}$. These two edges and also vw do not cross $\overline{F}$, contradicting the maximality of $\overline{F}$. □

Proposition 3.3. *Any maximal plane subgraph $\overline{F}$ of D_n with $n \geq 3$ must contain at least* $\min(\lceil 3n/2 \rceil, 2n-3)$ *edges. This bound is tight.*

Proof. Suppose that $n > 3$ and $\overline{F_0} = \overline{F}$ has a vertex v_0 with degree 2. By removing this vertex we obtain another maximal plane graph $\overline{F_1}$ (maximal on $n-1$ points), and if $\overline{F_1}$ is in the same conditions (with at least three vertices and a vertex v_1 of degree 2), by removing v_1 we obtain a new maximal plane graph $\overline{F_2}$, and so on. We finish this process in a step k because either $\overline{F_k}$ only has three points, or all the points of $\overline{F_k}$ have degree at least 3. In the first case, the original graph $\overline{F}$ contains $n = k+3$ vertices and $2k+3$ edges, so $2n-3$ edges. In the second case, $\overline{F}$ must contain at least $2k + \lceil 3(n-k)/2 \rceil$ edges, this amount reaching its minimum value when $k = 0$.

Finally, let us see that the bound is tight. If $2 \leq n \leq 6$, then a straight-line drawing on n points in convex position gives the bound $2n - 3 \leq \lceil 3n/2 \rceil$. If $n > 6$ and n is an even number, a drawing like the one shown in Figure 4 proves that the bound $\lceil 3n/2 \rceil$ is tight. The drawing is done on $n = 2(k+1)$ points in convex position, that clockwise are

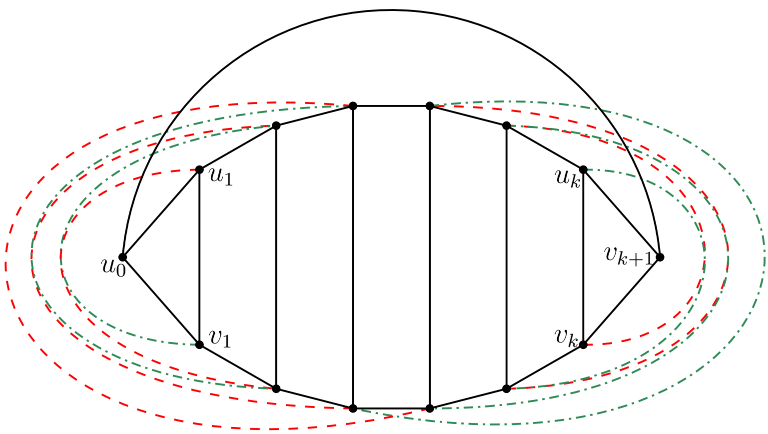


Figure 4: A drawing of K_n. The missing edges should be drawn as straight-line segments inside the convex hull of the set of points. The black edges form a maximal plane subgraph with $\lceil 3n/2 \rceil$ edges.

denoted by $u_0, u_1, u_2, \dots, u_k, v_{k+1}, v_k, \dots, v_2, v_1$. Let C denote the convex hull of that set of points. All the edges of D_n are drawn straight-line except for the $2(k-1)$ edges $u_iv_{i+1}, v_iu_{i+1}, i = 1, \dots, k-1$, and the edge u_0v_{k+1}, that are drawn outside C as shown in Figure 4. Observe that the $2(k-1)$ edges $u_iv_{i+1}, v_iu_{i+1}, i = 1, \dots, k-1$, are the diagonals of the $(k-1)$ quadrilaterals $u_iu_{i+1}v_{i+1}v_i$, with u_iv_{i+1} only crossing v_iu_{i+1} and u_0v_{k+1}, for $i = 1, \dots, k-1$. Clearly, straight-line edges can cross at most once, and the edges placed outside C, by construction, cross at most once. The graph $\overline{F}$ formed by the $2(k+1)$ edges on the boundary of C, the k edges $u_iv_i, i = 1, \dots, k$, and the edge u_0v_{k+1} is clearly plane and maximal, since the other straight-line edges cross at least one edge u_iv_i, and the non-straight-line edges cross the edge u_0v_{k+1}.

If n is odd, we can add to the previous set a point u_0' between u_0 and u_1, very close to segment u_0u_1, but keeping all the $2k+3$ points in convex position. By connecting u_0' with straight lines to the rest of the points, we obtain a simple topological drawing of K_n on this set of $n = 2k+3$ points, and a new maximal plane graph is obtained by adding the edges u_0u_0',$u_0'u_1$ to the above graph $\overline{F}$. This new maximal plane subgraph also has $\lceil 3n/2 \rceil$ edges. □

We mention another interesting implication of Theorem 3.1. For a vertex v, we can augment the star $S(v)$ to a 2-connected plane graph $\overline{F}$, and since $\overline{F} \setminus \{v\}$ is connected, it contains a spanning tree. So we have

Corollary 3.4. *For each vertex v there exists a spanning tree T_v of $V \setminus \{v\}$, such that the edges of $S(v) \cup T_v$ form a plane subgraph of D_n.*

Our next results are about diagonals on plane cycles. Let $C = (v_1, v_2, \dots, v_k)$ be a plane cycle of D_n. A diagonal of C is an edge of D_n connecting two non-consecutive vertices of C. It was previously known that, even for the case where there are diagonals intersecting both faces of C, there are at least $\lceil k/3 \rceil$ of them not crossing C (cf. [17, Corollary 6.6]). Proposition 3.3, applied to the subdrawing induced by the vertices of C, directly implies the following result.

Corollary 3.5. *Let $C = (v_1, v_2, \dots, v_k)$ be a plane cycle of D_n, with $k \geq 6$. Then, there exists a set D of diagonals, with $|D| \geq \lceil k/2 \rceil$, such that the subgraph $C \cup D$ is plane.*

It turns out that the structure of the diagonals of a cycle, as shown in the next lemma, is useful for our further results.

Lemma 3.6. *Let $C = (v_1, v_2, \dots, v_k)$ be a plane cycle of D_n, $k \geq 3$, dividing the plane into two faces f_1 and f_2. If there is no diagonal of C entirely in f_1, then all the diagonals of C are entirely in f_2.*

Proof. The proof is by induction on k. For $k < 5$ the statement is obvious, so suppose $k \geq 5$ and consider only the subdrawing D_k induced by the vertices of C. Suppose $C \cup D$ is a maximal plane graph of D_k, so necessarily, D consists of diagonals placed on f_2. Let d be a diagonal of D connecting two vertices at minimum distance on the graph C. Lemma 3.2 implies that in a maximal plane subgraph, vertices with degree 2 cannot be adjacent. Therefore, diagonal d has to connect two vertices at distance 2 on C. Without loss of generality, suppose $d = v_kv_2$ and let Δ be the triangle $v_kv_1v_2$. Then, the cycle $C_1 = (v_2, v_3, \dots, v_k)$ with $k-1$ vertices has the faces $f_1' = f_1 + \Delta$ and $f_2' = f_2 - \Delta$.

We claim that there cannot be diagonals of C_1 entirely in f'_1. Such a diagonal e entirely in f'_1 would have to intersect Δ. Then, adding e to $C \cup \{v_k v_2\}$ and removing all edges crossed by e, we would obtain a plane graph F in which v_1 has degree 0 or 1. By Lemma 1.1, there must be another edge between v_1 and C_1, and this edge would be a diagonal of C entirely in f_1, a contradiction. Thus, by induction, any diagonal $v_i v_j$ of C_1 is entirely in f'_2 and hence also in f_2.

It remains to see that the diagonals with endpoint v_1 are also in f_2. By our induction hypothesis, the diagonal $v_2 v_4$ is in f'_2 and thus also in f_2. Hence, arguing as before on the cycle $C_3 = (v_1, v_2, v_4, \ldots, v_k)$, we deduce that all the diagonals of C_3 incident to v_1 must be in f_2. So it remains to see that the diagonal $v_1 v_3$ is also in f_2. But $v_3 v_5$ is also in f'_2, so it is in f_2, and again applying the same reasoning on the cycle $(v_1, v_2, v_3, v_5, \ldots, v_k)$, all the diagonals of this cycle not incident to v_4 have to be in f_2. □

To prove the next result, we recall some definitions and properties of any 2-connected graph $G = (V, E)$. Two vertices v_1, v_2 are called a *separation pair* of G if the induced subgraph $G \setminus \{v_1, v_2\}$ on the vertices $V \setminus \{v_1, v_2\}$ is not connected. Let $G_1, \ldots, G_l$ be the connected components of $G \setminus \{v_1, v_2\}$, with $l \geq 2$. For each $i \in \{1, \ldots, l\}$, let G_i^* be the subgraph of G induced by $V(G_i) \cup \{v_1, v_2\}$. Observe that G_i^* contains at least one edge incident to v_1 and at least another edge incident to v_2.

Theorem 3.7. *Let $\overline{F}$ be a maximal plane subgraph of D_n, $n \geq 3$. Then, for each separation pair v_1, v_2 of $\overline{F}$, at least one of the subgraphs $\overline{F}_i^*$ must be 2-connected.*

Proof. Suppose that v_1, v_2 is a separation pair of $\overline{F}$, and let $\overline{F}_1, \overline{F}_2, \ldots, \overline{F}_l$ be the connected components of $\overline{F} \setminus \{v_1, v_2\}$, $l \geq 2$. Since $\overline{F}$ is 2-connected, the graph $\overline{F} \setminus \{v_2\}$ is connected with v_1 as a cut vertex. As $\overline{F}$ is plane, we can suppose that v_1 is in the outer face of $\overline{F} \setminus \{v_2\}$ (v_2 must be inside that face) and that clockwise around vertex v_1 first there appear the edges from v_1 to some vertices of the component $\overline{F}_1$, then edges connecting v_1 to some vertices of $\overline{F}_2$ and so on. See Figure 5.

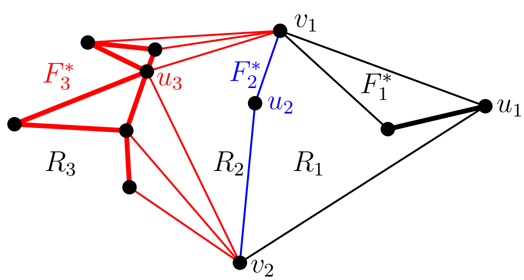


Figure 5: A plane graph with separating pair v_1, v_2 and three subgraphs F_i^*, none of them 2-connected. This plane graph cannot be maximal.

Now suppose that none of the subgraphs $\overline{F}_i^*$ is 2-connected. Then each subgraph $\overline{F}_i^*$ contains at least one cut vertex u_i. Since $\overline{F}_i$ is connected and there exist edges in $\overline{F}_i^*$ incident to v_1 and v_2, vertex u_i is different from v_1 and v_2. On the other hand, a connected component C of $\overline{F}_i^* \setminus \{u_i\}$ must contain at least one of v_1 or v_2 because otherwise, C would be a connected component of $\overline{F} \setminus \{u_i\}$, contradicting that $\overline{F}$ is 2-connected. Therefore, $\overline{F}_i^* \setminus \{u_i\}$ has exactly two components, one containing v_1, the other one containing v_2.

This also implies that the edge v_1v_2 of D_n cannot belong to $\overline{F}$, and that the cut-vertex u_i is in the outer face of $\overline{F}_i^*$ (and hence in the outer face of $\overline{F} \setminus \{v_2\}$) since v_1 and v_2 are in the outer face of $\overline{F} \setminus \{v_2\}$. See Figure 5.

In the graph $\overline{F} \setminus \{v_2\}$, around the vertex v_1, the edges to vertices of $\overline{F_1}$ first appear, then the edges to vertices of $\overline{F_2}$ and so on. Therefore, when we add to that graph the vertex v_2 and all the edges connecting v_2 to each component $\overline{F_i}$ to obtain $\overline{F}$, v_1 and v_2 must be in the faces R_i of $\overline{F}$ defined as the regions placed between the last edge from v_1 to $\overline{F}_i$ and the first edge from v_1 to $\overline{F}_{i+1}$, for $i = 1, \ldots, l$, and the vertex u_i must be in the faces R_i and R_{i-1}. However, by the maximality of $\overline{F}$, no edge of D_n is entirely in any of those faces R_i. Then, Lemma 3.6 implies that no point of the edge v_1v_2 of D_n can be inside any face R_i. See Figure 5. Thus, v_1v_2 must begin between two edges v_1v, v_1v' with both v and v' belonging to a common connected component $\overline{F}_i$. However, since u_i belongs to the faces R_{i-1} and R_i, any curve from v_1 to v_2 passes either through the point u_i or through the interior of R_{i-1} or R_i, which contradicts either the simplicity of D_n or Lemma 3.6. Therefore, if none of the subgraphs $\overline{F}_i^*$ is 2-connected, $\overline{F}$ cannot be maximal. □

We call a graph *essentially* 3*-edge-connected* if it stays connected after removing any two edges not sharing a vertex of degree 2 (i.e., the graph either stays connected or one component is a single vertex). Theorem 3.7 implies that a maximal plane subgraph is essentially 3-edge-connected:

Theorem 3.8. *Any maximal plane subgraph, $\overline{F}$, of a simple topological drawing of K_n is essentially* 3*-edge-connected.*

Proof. If the removal of two edges v_1v_2 and $v_1'v_2'$ from the plane subgraph $\overline{F}$ results in two non-trivial components C_1, C_2 (see Figure 6), then v_1, v_2' is a separation pair of $\overline{F}$, that has as induced subgraphs $C_1 \cup \{v_1'v_2'\}$ and $C_2 \cup \{v_1v_2\}$, neither of which is 2-connected. Then, by Theorem 3.7, $\overline{F}$ cannot be maximal. □

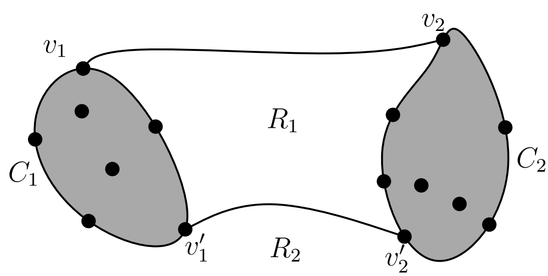


Figure 6: A graph that is not essentially 3-edge-connected. The induced subgraphs of the separation pair v_1, v_2' are subgraph C_1 plus edge $v_1'v_2'$ and subgraph C_2 plus edge v_1v_2. By Lemma 3.6, the edge v_1v_2' of D_n cannot enter either the R_1 face or the R_2 face, which is impossible in any good drawing.

4 Adding the maximum number of edges

Now, assume that a plane subgraph of D_n is given, and we want to add the maximum number of edges keeping plane the augmented graph. Clearly, the decision of adding one edge will in general block other edges from being added. We will see that the complexity of an algorithm solving this problem highly depends on whether the given subgraph is connected or not.

Before talking about algorithms and their complexity we have to talk about what information of the drawing D_n we will need to compute plane subgraphs. For each vertex v, the clockwise cyclic order of edges of $S(v)$ is usually given as a permutation of $V \setminus \{v\}$ (that is to be interpreted circularly) of the second vertices of all edges of $S(v)$. That permutation of $V \setminus \{v\}$ is called the *rotation* of v, and the *rotation system* of a drawing D_n consists of the collection of the rotations of each vertex v of D_n. It is well-known that from the information provided by the rotation system, one can determine whether two edges cross or not, and therefore, that information is enough to compute plane subgraphs. See [7, 11, 15]. From the rotation system, we can also compute (in $O(n^2)$ time) the *inverse rotation system* that, for each vertex v_i and index j, $j \neq i$, gives the position of v_j in the rotation of v_i.

When we say that a drawing D_n is given, we mean that we know the rotation system and the inverse rotation system of D_n. Using these two structures, one can determine whether two edges cross, in which direction an edge is crossed, and in which order two non-crossing edges cross a third one in constant time [11].

Theorem 4.1. *Let F be a connected spanning plane subgraph of D_n. Then there is an $O(n^3)$ time algorithm to augment F to a plane subgraph F' of D_n with the maximum number of edges.*

Proof. As F is plane and thus contains a linear number of edges, we can identify all the edges of D_n not crossed by F in $O(n^3)$ time. This also gives us, for each such edge, the face of F in which it is contained, and we can also compute for each face f of F the set Δ_f of edges of D_n entirely inside f. Clearly, each face of F can be considered independently, adding the maximum number of edges in it.

Let f be a face of F. For simplicity, we assume f to be bounded by a simple cycle $(v_1, \ldots, v_k)$. Other cases can be solved similarly by an appropriate "splitting" of edges having f on both sides. Disregarding D_n, consider the rectilinear drawing $\overline{D_k}$ obtained from k points $p_1, \ldots, p_k$ placed on a circle C, and assign to each edge $p_i p_j$ of $\overline{D_k}$ weight 0 if $v_i v_j$ is in Δ_f, weight 1 otherwise. Observe that two edges of Δ_f cross properly, if and only if, the corresponding 0-weight edges in circle C cross properly. It is well-known that a minimum-weight triangulation in $\overline{D_k}$ can be obtained in $O(k^3)$ time [9] by a dynamic programming algorithm, and this triangulation gives a plane set of 0-weight edges with maximum cardinality. Hence, the corresponding edges of Δ_f form a plane set of edges entirely inside face f with maximum cardinality. □

In contrast to this result, the problem becomes NP-complete when the subgraph F is not connected.

Theorem 4.2. *Given a simple topological drawing D_n of K_n and a cardinality k', it is NP-complete to decide whether there is a plane subgraph that has at least k' edges.*

Proof. We give a reduction from the independent set problem on segment intersection graphs ($\overline{SEG}$ problem), which is known to be NP-complete [10]: Given a set S of s segments in the plane that pairwise either are disjoint or intersect in a proper crossing, and an integer $k > 0$, is there a subset of k disjoint segments?

For each instance of a $\overline{SEG}$ problem, we are going to build, in polynomial time, a drawing D_n of K_n and an integer k' such that the instance of the $\overline{SEG}$ problem has a Yes answer, if and only if, the drawing D_n contains a plane subgraph with k' edges.

Let v_i, t_i be the endpoints of each segment $s_i, i = 1, \ldots, s$, of S. We can suppose that these endpoints are in general position and that their convex hull is a triangle. Thus, for each endpoint v_i, we can find a disc B_i centered at v_i, such that any straight line connecting two endpoints of S different from v_i does not cross B_i.

In each disc B_i, we place two points u_i, w_i very close to the segment $v_i t_i$, in such a way that when connecting the point v_i with straight-line segments to all the other points, the segments $v_i u_i, v_i t_i, v_i w_i$ are clockwise consecutive. In other words, the clockwise wedge defined by the half-lines $v_i u_i, v_i w_i$ only contains the endpoint t_i. See Figure 7.

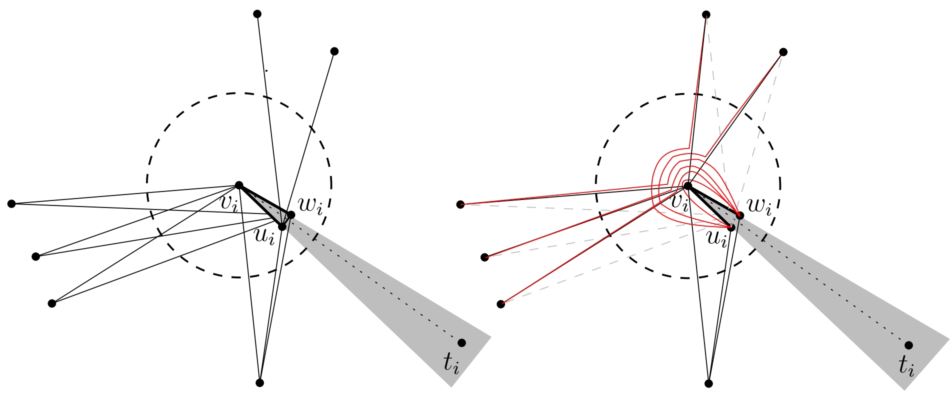


Figure 7: Drawings $\overline{D_n}$ (left) and D_n (right). The gray wedge only contains the endpoint t_i. In D_n, the dashed edges need to take a detour to avoid intersecting the edge $u_i w_i$ twice.

Consider the rectilinear drawing $\overline{D_n}$ obtained by connecting the $n = 4s$ points v_i, u_i, w_i, t_i. In $\overline{D_n}$, maximal plane graphs are triangulations, but we are going to consider only the family Γ of plane triangulations of $\overline{D_n}$ containing the $2s$ edges $u_i v_i, w_i v_i$. The *weight* of a triangulation of Γ is the number of edges $v_i t_i$ that it contains. Clearly, in the set S there are k disjoint segments, if and only if, there is a triangulation in Γ with weight k.

Now, consider the drawing D_n obtained from $\overline{D_n}$ doing the following changes:

For $i = 1, \ldots, s$, only the edges of the star $S(u_i)$ crossing $v_i w_i$, the edges of the star $S(w_i)$ crossing $u_i v_i$, and the edge $u_i w_i$ are modified.

Suppose that in $S(u_i)$ after $u_i v_i$ are clockwise the edges $u_i p_1, \ldots, u_i p_k, u_i w_i$, where each $u_i p_j$ has to cross $v_i w_i$. Let $v_i p_{i_1}, v_i p_{i_2} \ldots, v_i p_{i_k}$ be the clockwise ordered edges of $S(v_i)$ with endpoint one of the vertices p_i. Then, we modify $\overline{D_n}$ by redrawing $u_i p_{i_1}$ following first the line $u_i v_i$ until point v_i, then turning around v_i and following the line $v_i p_{i_1}$, in such a way that in the rotation of u_i the new edge $u_i p_{i_1}$ is placed just before $u_i v_i$. See Figure 7, right. The new drawing obtained is simple, because no edge crosses both $u_i v_i$ and $v_i p_{i_1}$, edges $u_i p_j$ cannot cross $v_i p_{i_1}$ and none edge of $S(p_{i_1})$ can cross $u_i v_i$. Moreover, the number of crossings in the edge $v_i w_i$ has decreased by one. We repeat the same process for the edge $u_i p_{i_2}$ (the new edge $u_i p_{i_2}$ is placed just before $u_i v_i$ in the rotation of u_i), then $u_i p_{i_3}$, and so on. The same process can be done with the edges $w_i q_j$ crossing $u_i v_i$. See Figure 7, right. Finally, we can redraw $u_i w_i$ in the same way, following the edge $u_i v_i$ then turning around v_i following edge v_iw_i. If we do this process for all the edges crossing v_iu_i or u_iw_i, $i = 1, \dots, s$, at the end we obtain the simple drawing D_n. By construction, in D_n, neither the edges v_iu_i nor the edges v_iw_i are crossed by any other edge.

Now, let us see that $\overline{D_n}$ has a triangulation of the family Γ of weight k, if and only if, D_n has a plane subgraph of size k', with $k' = 3n - 6 - (s - k) = 11s - 6 + k$. Suppose $\overline{D_n}$ has a triangulation F with weight k. This means that F contains $(s - k)$ edges u_iw_i. By removing from F these u_iw_i edges, we obtain a plane set F' of edges, where no edge of F' has been modified to obtain the drawing D_n. Therefore, the edges of F' also form a plane subgraph in D_n of size $3n - 6 - (s - k) = 11s - 6 + k$.

Conversely, suppose D_n contains a plane subgraph with $3n-6-(s-k)$ edges. Since the edges u_iv_i, w_iv_i are not crossed by any edge of D_n, they must belong to any maximal plane graph of D_n. Therefore, D_n has a plane subgraph F containing all the edges u_iv_i, v_iw_i and of size $k' \geq 3n - 6 - (s - k)$. As the wedge v_iw_i, v_iu_i only contains point t_i, if the edge v_it_i is not in F, then, the face of F containing the edges v_iu_i and v_iw_i cannot be a triangle. But, if a plane graph on n vertices contains more than $(s - k)$ non-triangular faces, its maximum number of edges is $< 3n - 6 - (s - k)$. As a consequence, v_it_i is not in F for at most $(s - k)$ indices i, or equivalently, the plane subgraph F contains at least k edges v_it_i. This means that we can obtain a triangulation of the family Γ of weight k by including k of these non-crossing edges. □

Note that in the straight-line setting, we can always draw a triangulation of the underlying point set, which contains the maximum number of edges. However, this is not the case for simple topological drawings. We were not able to come up with a reduction solving the following problem.

Problem 4.3. What is the complexity of deciding whether a given D_n contains a triangulation, i.e., a plane subgraph whose faces are all 3-cycles?

Our reduction can also be adapted for a related problem on compatible graphs. We leave the realm of general simple topological drawings and consider the following problem in the more specialized setting of geometric graphs (rectilinear drawings). Let $P = \{p_1, \dots, p_n\}$ and $P' = \{p'_1, \dots, p'_n\}$ be two sets of points in the plane. A planar graph is *compatible* if it can be embedded on both P and P' in a way that there is an edge p_ip_j if and only if there is an edge $p'_ip'_j$. Saalfeld [21] asked for the complexity of deciding whether two such point sets (with a given bijection between them) have a compatible triangulation. We will say that triangulations $\overline{F}$ of P and $\overline{F'}$ of P' have k' compatible edges when there exists a subset of k' edges p_ip_j of $\overline{F}$, such that their images, edges $p'_ip'_j$, are edges of $\overline{F'}$.

We can show the NP-completeness of the following optimization variant of the problem. (However, as the similar Open Problem 4.3, Saalfeld's problem remains unsolved.)

Theorem 4.4. *Given two point sets $P = \{p_1, \dots, p_n\}$ and $P' = \{p'_1, \dots, p'_n\}$ and the indicated bijection between them, as well as a cardinality k', the problem of deciding whether P and P' admit two triangulations with k' compatible edges is NP-complete.*

Proof. We follow the idea of the proof of Theorem 4.2, and use a reduction from the $\overline{SEG}$ problem. Suppose that an instance of the $\overline{SEG}$ problem is given: a set S of s segments in the plane that pairwise either are disjoint or intersect in a proper crossing, and an integer $k > 0$. We will build two sets of points $P = \{p_1, \dots, p_n\}$ and $P' = \{p'_1, \dots, p'_n\}$ and obtain an integer k' such that the $\overline{SEG}$ problem has answer Yes if and only if, P and P' admit two triangulations with k' compatible edges.

Let P be the set of $n = 5s$ points formed by the v_i, t_i, u_i, w_i ($i = 1, \ldots, s$) points obtained from S as in the above Theorem 4.2, plus s points $\tilde{v}_i$, where each point $\tilde{v}_i$ is placed inside the triangle $v_i u_i w_i$ very close to the point v_i, to the right of the oriented line $v_i t_i$, in such a way that in the wedge defined by the half-lines $\tilde{v}_i u_i, \tilde{v}_i w_i$ the only point of P is t_i, and the wedges $u_i v_i, u_i \tilde{v}_i$ and $w_i \tilde{v}_i, w_i v_i$ do not contain points of P. See Figure 8, left. By construction, any triangulation $\overline{F}$ of the set of points P must contain the edge $v_i \tilde{v}_i$. Observe that if the edge $t_i v_i$ is in $\overline{F}$, then the edge $t_i \tilde{v}_i$ has to be also in $\overline{F}$. Also note that $u_i w_i$ is only crossed by the edges $t_i v_i$ and $t_i \tilde{v}_i$.

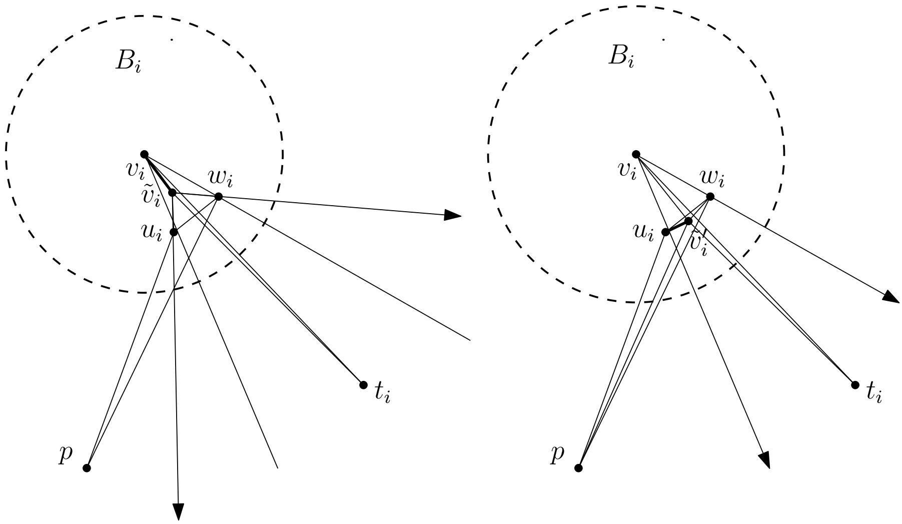


Figure 8: The point sets P (left) and P' (right).

In the same way, let P' be the set of $n = 5s$ points $v_i, t_i, u_i, w_i, \tilde{v}'_i$ ($i = 1, \ldots, s$), where now each point $\tilde{v}'_i$ is placed outside the triangle $v_i u_i w_i$, very close to the intersection point of $u_i w_i$ with $v_i t_i$, to the right of the line $v_i t_i$, and satisfying that any clockwise triangle $u_i w_i p$ contains inside the point $\tilde{v}'_i$. See Figure 8, right. The bijection between the points of P and P' is the obvious one, to each point $\tilde{v}_i$ of P corresponds point $\tilde{v}'_i$ of P', for any other point its image is itself.

To prove the statement of the theorem, it is enough to prove the following:

If in the set S there are k disjoint segments, then there are triangulations $\overline{F}$ and $\overline{F'}$ of the sets P and P', respectively, with $k' = 3n - 6 - (s - k)$ compatible edges. And reciprocally, if $\overline{F}$ and $\overline{F'}$ contain $k' = 3n - 6 - (s - k)$ compatible edges, then S contains k disjoint segments.

Suppose first that S contains a set D of k disjoint segments $v_i t_i$. Let P_0 be the set of $4s$ common points of P and P' (all the points v_i, u_i, w_i, t_i). We build a triangulation $\overline{F_0}$ of P_0 in the following way. If $v_i t_i$ is in D, then we include the edges $v_i t_i, v_i u_i, v_i w_i, t_i u_i, t_i w_i$ in $\overline{F_0}$. If $v_i t_i$ is not in D, then we include the edges $u_i v_i, v_i w_i, w_i u_i$ in $\overline{F_0}$. After that, we add edges in an arbitrary way until obtaining a triangulation $\overline{F_0}$ of P_0. Now, to obtain $\overline{F}$ and $\overline{F'}$, we add the points $\tilde{v}_i$ and $\tilde{v}'_i$ to $\overline{F_0}$ and retriangulate the triangular faces where they are. If the edge $v_i t_i$ is in D, then the points $\tilde{v}_i, \tilde{v}'_i$ are both in the triangle $u_i v_i t_i$. So, by adding the point $\tilde{v}_i$ and the three edges $\tilde{v}_i u_i, \tilde{v}_i v_i, \tilde{v}_i t_i$ to $\overline{F_0}$, or the point $\tilde{v}'_i$ and the three edges $\tilde{v}_i'u_i, \tilde{v}_i'v_i, \tilde{v}_i't_i$ we continue with all the edges being compatible. However, if the edge v_it_i is not in D, then the point $\tilde{v}_i$ is in the triangle $u_iv_iw_i$, but the point $\tilde{v}_i'$ is in a triangle $u_iw_ip_i$. Then, we obtain a triangulation $\overline{F}$ of P by adding the edges $\tilde{v}_iu_i, \tilde{v}_iv_i, \tilde{v}_iw_i$, and a triangulation $\overline{F'}$ of P' by adding the edges $\tilde{v}_i'u_i, \tilde{v}_i'p_i, \tilde{v}_iw_i$. Now, the images of the edges $\tilde{v}_iv_i$ of $\overline{F}$, edges $\tilde{v}_i'v_i$, are not in $\overline{F'}$ (there the edges $\tilde{v}_i'p_i$ appear instead). This situation occurs $(s-k)$ times, so the number of compatible edges between $\overline{F}$ and $\overline{F'}$ is $3n-6-(s-k)$.

Conversely, suppose P and P' contain triangulations $\overline{F}$ and $\overline{F'}$ with $k' = 3n-6-(s-k)$ compatible edges. If $\tilde{v}_it_i$ is not in $\overline{F}$, then the edges $\tilde{v}_iv_i$ and u_iw_i must be both in $\overline{F}$, because the edge u_iw_i can be crossed only by the edges t_iv_i and $t_i\tilde{v}_i$. However, in set P', always the edge $\tilde{v}_i'v_i$ is crossed by the edge u_iw_i. Then, for each index i such that $\tilde{v}_it_i$ is not in $\overline{F}$, one of the edges $\tilde{v}_iv_i$ or u_iw_i of $\overline{F}$ is not in $\overline{F'}$. Therefore, this situation can happen at most $(s-k)$ times, that is, the triangulation $\overline{F}$ must contain at least k edges $\tilde{v}_it_i$. But if k segments $\tilde{v}_it_i$ are disjoint, also their corresponding v_it_i edges are disjoint. Therefore, S has to contain at least k disjoint segments. □

Finally, let us analyze the complexity of augmenting a plane subgraph F of D_n until obtaining a maximal plane subgraph. Since F has $O(n)$ edges, the set of edges of $S(v)$ not crossing F can be trivially found in $O(n^2)$ time. This directly implies an $O(n^3)$ algorithm to obtain a maximal plane graph containing F: For $i = 1, \ldots, n$, update F by adding the edges of $S(v_i)$ non-crossing F, not in F. The following result implies that, if F is connected, finding a maximal plane subgraph containing F can be done in $O(n^2)$ time.

Theorem 4.5. *Given a simple topological drawing of K_n, a connected plane subgraph F, and a vertex v, we can find the edges from v to F not crossing F in $O(n)$ time.*

Proof. Notice that as F is a plane graph, we can compute in linear time, for each vertex w the clockwise order of the edges of F incident to w, the faces of F, and for each face f, the clockwise cyclic list of edges and vertices found along its boundary.

Suppose first that the vertex v is not in F, and let vw_1 be the first edge in the rotation of v with one endpoint in F. The algorithm runs in three stages. In the first stage, it starts by finding the edge of F, edge $e_1 = u_0u_1$, that intersects vw_1 closest to v along vw_1. When the first intersection point occurs precisely at the vertex w_1, we take e_1 as the first edge of F that follows, counterclockwise, to w_1v in $S(w_1)$. Using the rotation system and its inverse, this edge e_1 of F can be found in linear time, since $|F| \in O(n)$. It also gives us the face f of F containing the vertex v inside.

For simplicity, let us suppose that f is a bounded face and that the boundary of f is a simple cycle, formed by the edges $e_1 = u_0u_1, e_2 = u_1u_2, e_m = u_{m-1}u_0$. We will later discuss the general case. Notice that if the edges vw_i, vw_j, vw_k are in this clockwise order in $S(v)$, their corresponding first crossing points x_i, x_j, x_k with F are found in a clockwise walk of the boundary of f in that same clockwise order. See the right bottom drawing of Figure 9.

In the second stage, the algorithm simulates a clockwise walk $x_1u_1, u_1u_2, \ldots, u_{k-1}u_k$, $\ldots$ of the boundary of f starting at point x_1, the first crossing point of vw_1 with $e_1 = u_0u_1$, and simultaneously a clockwise walk $vw_1, vw_2, \ldots, vw_i, \ldots$ on the edges of the star $S(v)$, beginning with the edge vw_1. In each step, the algorithm makes progress in at least one of the two walks, by adding the following edge on the boundary to the boundary walk or passing to explore the following edge of $S(v)$. In this process the algorithm will keep a list σ with some of the explored edges of $S(v)$.

In a generic step, the edges $S_i = (vw_1, vw_2, \ldots, vw_i)$ of $S(v)$, and the portion of the boundary of f, $W_k = (x_1u_1, u_1u_2, \ldots, u_{k-1}u_k)$, have been visited, and the two following invariants hold:

(A) The first crossing of the edge vw_i is not on $W_{k-1} = (x_1u_1, u_1u_2, \ldots, u_{k-2}u_{k-1})$ (the walk W_k minus its last edge).

(B) The list σ contains an ordered list $(vu_{i_1}, vu_{i_2}, \ldots, vu_{i_s})$ of the explored edges of $S(v)$ finishing at some of the vertices $u_j, 1 \le j < k$, satisfying:

- (B1) All the explored edges of $S(v)$ not placed in σ cross the boundary of f.
- (B2) The first crossing point of each edge vu_j of σ with the boundary of f is either u_j or is placed clockwise after u_j.

Initially, if x_1 is an interior point of the edge $e_1 = u_0u_1$, then $W_k = (x_1u_1)$, $S_i = (vw_1, vw_2)$ and the list σ is empty. If x_1 coincides with the vertex u_0, then $W_k = (u_0u_1)$, $S_i = (vw_1, vw_2)$ and the list contains the edge vu_0. In both cases invariants (A) and (B) are satisfied (the walk W_{k-1} is empty or consists of only one vertex).

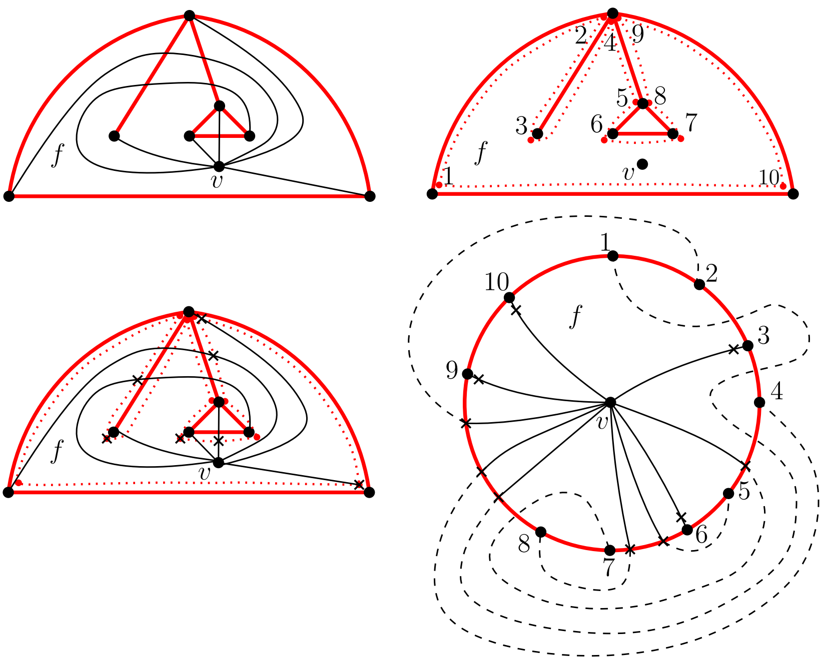


Figure 9: Top left: A vertex v inside the face f. Only the edges vu_i, with u_i incident to the face f, can be uncrossed by F. Top right: A clockwise walk along the boundary of the face f. Bottom left: In a walk along the boundary of f, the first crossing points of the edges of $S(v)$ are found in the same order as the edges of $S(v)$. Bottom right: An equivalent drawing to the top left figure with the boundary of f being a simple cycle. Some vertices, like $(2, 4, 9)$, can correspond to the same vertex of the first drawing.

In this second stage the algorithm proceeds as follows:

- If vw_i crosses the last edge of W_k, edge e_k, or if w_i is not a vertex of f, iterate considering the clockwise successor vw_{i+1} of vw_i in the rotation of v.

  As the first crossing of vw_i must be on the edge e_k or a posterior edge $e_t, t > k$, also the first crossing of vw_{i+1} must be on e_k or a posterior edge. Thus invariant (A) is kept. On the other hand, observe that σ does not change, vw_i must not be included in σ (it crosses f), and W_k is not modified. Therefore invariant (B) is also kept.

- If vw_i does not cross e_k and w_i is a vertex of f, $w_i \neq u_k$, then, add the following edge e_{k+1} on f to W_k, keeping the same edge vw_i of $S(v)$.

  Invariant (A) is kept, because the first crossing point of vw_i cannot be on W_k. Invariant (B) is also kept, because σ is not modified.

- If vw_i does not cross e_k and $w_i = u_k$, then, add vw_i to the list σ, pass to explore the following edge vw_{i+1} of $S(v)$ and add the following edge e_{k+1} on f to W_k.

  Again, invariant (A) is kept, because the first crossing of vw_{i+1} must be after u_k. On the other hand, the first crossing point of vw_i is either u_k or it is placed after u_k, hence property (B) is kept.

This second stage of the algorithm ends when all the edges of $S(v)$ and f have been explored. The last edge of the boundary of f being either u_0x_1 or $u_{m-1}u_0$. Therefore, at the end, invariant (B) implies that σ will contain the uncrossed edges of $S(v)$ plus some crossed edges vu_i of $S(v)$ satisfying that the first crossing (on the boundary of f) is placed after the endpoint u_i of that edge.

In each step of this stage, a new edge in the boundary of f, a new edge of $S(v)$, or both edges become explored. As the number of edges in f and in $S(v)$ is linear, this second stage of the algorithm runs in $O(n)$ time.

In the third stage, the algorithm repeats counterclockwise the above stage considering only the edges in σ. That means, it explores counterclockwise the boundary of f (in the order x_1u_0, $u_0u_{m-1}, \dots$, and counterclockwise the edges of $S(v)$ placed in σ (so, in the order $vu_{i_s}, vu_{i_{s-1}}, \dots$). In this third stage, in linear time, a new list $\overline{\sigma}$ is obtained. By invariant (B1), all the uncrossed edges of $S(v)$ have to be in $\overline{\sigma}$. And by invariant (B2), if vu_i is in $\overline{\sigma}$, its first crossing point cannot be clockwise nor counterclockwise before u_i, so it has to be u_i. Therefore, $\overline{\sigma}$ will contain the uncrossed edges of $S(v)$.

In general, the boundary of face f is not a simple cycle, some edges of f can be incident to f for both sides, so they appear twice in a walk along the boundary of f. However, this general case can be transformed to the previous case by standard techniques, as done in [6] in their proof of the general case of Lemma 1.1. In Figure 9, the bottom right figure shows how to transform the drawing of the top left figure, to obtain an equivalent drawing where the boundary of f is a simple cycle. When the face f is the unbounded face the algorithm is totally analogous.

Finally, let us consider the case when the vertex v is in F. Then, vertex v can be incident to several faces $f_1, \dots, f_l$, $l \geq 1$. Again, for simplicity, suppose that the boundary of each one of these faces is a simple cycle. For each face f_i, if vw_{i_1}, vw_{i_2} are the two edges incident to vertex v in f_i, we can compute by the above method the uncrossed edges of $S(v)$ placed inside f_i, using only the edges of $S(v)$ placed clockwise between vw_{i_1} and vw_{i_2}. □

5 Conclusion

In this paper, we considered maximal and maximum plane subgraphs of simple topological drawings of K_n. It turns out that maximal plane subgraphs have interesting structural properties. These insights could be useful in improving the bounds on the number of disjoint edges in any such drawing, continuing this long line of research.

Also, algorithmic questions arise. For example, Proposition 2.1 ensures that there are always two edges connecting a vertex v to a not necessarily connected plane graph F in D_n without crossings. Moreover, the set of edges of $S(v)$ not crossing F can be trivially found in $O(n^2)$ time. This leads to the following question.

Problem 5.1. Given a not necessarily connected plane graph F in D_n, plus a vertex v not in F, can the edges of $S(v)$ incident to but not crossing F be found in $o(n^2)$ time?

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Graphical Frobenius representations of non-abelian groups*

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Abstract

A group G has a Frobenius graphical representation (GFR) if there is a simple graph Γ whose full automorphism group is isomorphic to G acting on the vertices as a Frobenius group. In particular, any group G with a GFR is a Frobenius group and Γ is a Cayley graph. By very recent results of Spiga, there exists a function f such that if G is a finite Frobenius group with complement H and $|G| > f(|H|)$ then G admits a GFR. This paper provides an infinite family of graphs that admit GFRs despite not meeting Spiga's bound. In our construction, the group G is the Higman group $A(f, q_0)$ for an infinite sequence of f and q_0, having a nonabelian kernel and a complement of odd order.

Keywords: Cayley graph, Frobenius group, Suzuki 2*-group, Frobenius graphical representation.*

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1 Introduction

Graphs and their automorphism groups have intensively been investigated especially for vertex-transitive (and hence regular) graphs. Many contributions have concerned vertex-transitive graphs with large automorphism groups compared to the degree of the graph, and have in several cases relied upon deep results from group theory, such as the classification of primitive permutation groups.

On the other end, the smallest vertex-transitive automorphism groups of graphs occur

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when the group is regular on the vertex-set. A group is said to have a graphical regular representation (GRR) if there exists a graph whose (full) automorphism group is isomorphic to G acting regularly on the vertex-set. Actually, almost all finite groups have GRRs. In fact, all the few exceptions were found in the 1970-80s by a common effort of G. Sabidussi, W. Imrich, M. E. Watkins, L. A. Nowitz, D. Hetzel, C. D. Godsil, and L. Babai, see [2, Section 1]. Since regular automorphism groups of a graph are those which are vertex transitive but contain no non-trivial automorphism fixing a vertex, a natural next choice as a small vertex-transitive automorphism group of a graph may be a Frobenius group on the vertex-set: an automorphism group of a graph that is vertex-transitive but not regular and only the identity fixes more than one vertex. It is well known that any group may be a Frobenius group in at most one way. Furthermore, each graph Γ with a (sub)group G of automorphisms acting regularly on the vertex-set is a Cayley graph $\mathrm{Cay}(G,S)$.

All these give a motivation for the study of Frobenius groups G which have a graphical Frobenius representation (GFR), that is, there exists a graph whose (full) automorphism group is isomorphic to G acting on the vertex-set as a Frobenius group. The systematic study of the GFR problem was initiated by J. K. Doyle, T. W. Tucker and M. E. Watkins in their recent paper [2]. As pointed out by those authors, the GFR problem is largely not analogous to the GRR problem since all groups have a regular representation whereas Frobenius groups have highly restricted algebraic structures. Nevertheless, they conjectured that like the GRR-case, "all but finitely many Frobenius groups with a given Frobenius complement have a GFR". Very recently Spiga [9, 10] was able to prove that conjecture for Cayley graphs and Cayley digraphs (digraphical Frobenius representations, or DFRs). Spiga combined combinatorial properties of Cayley graphs with some deeper results on the 1-point stabilizers of primitive permutation groups to obtain a function $f\colon \mathbb{N} \to \mathbb{N}$ such that if G is a finite Frobenius group with complement H satisfying $|G| > f(|H|)$, then G admits a GFR; see [10, Theorem 1.1], [9, Theorem 1], and Section 9.

It is apparent from Spiga's work and from the results, examples and classification of smaller groups with GFRs in [2], see in particular [2, Theorem 5.3 and Remark 5.4], that an interesting open problem is the explicit constructions of GFR for Frobenius groups G which do not meet Spiga's bound and whose kernel H is a non-abelian 2-group.

In this paper we provide such a construction. Our choice of Frobenius groups is influenced by Higman's classification of Suzuki 2-groups [5], as we take for G the group $A(f,q_0)$ from Higman's list where q_0 and $q = 2^f$ are 2-powers. The group $A(f,q_0)$ is a subgroup of G of $\mathrm{GL}(3,\mathbb{F}_q)$ whose main properties are recalled in Section 2. We build a Cayley graph Γ_u on the Frobenius kernel K of G, with a certain inverse closed subset S of K as connecting set, constructed from an element $u \in \mathbb{F}_q$. We show that G has GFR on Γ_u provided that $q = 2^f$, q_0 and u are carefully chosen.

Our notation and terminology are standard. For the definitions and known results on Frobenius groups which play a role in the present paper, the reader is referred to [2].

2 The group $A(f, q_0)$

Let $\mathbb{F}_q$ be the finite field of order $q = 2^f$ with $f \geq 4$, and let $q_0 = 2^{f_0}$ be another power of 2 smaller than q. For $a, c \in \mathbb{F}_q$ and $\lambda \in \mathbb{F}_q^*$, we write

$$\Phi_{a,c} = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ c & a^{q_0} & 1 \end{bmatrix}, \qquad \Psi_\lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda^{q_0+1} \end{bmatrix}.$$

We define the groups

$$K = \{\Phi_{a,c} \mid a, c \in \mathbb{F}_q\},$$
$$H = \{\Psi_\lambda \mid \lambda \in \mathbb{F}_q^*\}.$$

Then, K is a 2-group of order q^2 and H is a cyclic group of order $q-1$. Moreover, H normalizes K, and its action fixes no nontrivial element in K. Their closure group is HK, and denoted by $A(f, q_0)$ in Higman's paper [5]. For brevity, we write G in place of $A(f, q_0)$. With this change $G = HK$. Since $H \cap H^g = 1$ holds for any $g \in G \setminus H$, G is a Frobenius group in its action on the set G/H of right cosets of H. The point stabilizer is H and K is a regular normal subgroup. It may be noticed that when $q = 2q_0^2$ then G is similar to the 1-point stabilizer of the Suzuki group $\mathrm{Sz}(q)$ in its double transitive action on q^2+1 points. A straightforward computation shows that the H-orbits on K are

$$\Omega_u = \{\Phi_{a,ua^{q_0+1}} \mid a \in \mathbb{F}_q^*\}, \qquad u \in \mathbb{F}_q, \tag{2.1}$$

and

$$\Omega_\infty = \{\Phi_{0,c} \mid c \in \mathbb{F}_q^*\}.$$

3 A Cayley graph arising from *G*

For every $u \in \mathbb{F}_q$, we may build a Cayley graph in the usual way:

$$\Gamma_u = \mathrm{Cay}(K, \Omega_u \cup \Omega_{u+1}).$$

Since $\Omega_u \cup \Omega_{u+1}$ is H-invariant, the group G induces automorphisms of Γ_u. This allows us to look at (the matrix group) G as a Frobenius group on $K = V(\Gamma_u)$. Our aim is to show that if q, q_0 and $u \in \mathbb{F}_q$ are carefully chosen then $\mathrm{Aut}(\Gamma_u)$ coincides with G. Define the set $\mathscr{U}_{q,q_0}$ of elements $u \in \mathbb{F}_q$ which satisfy both conditions:

(U1) $u = (1+\eta^{q_0})/(\eta+\eta^{q_0})$ for some primitive element η of $\mathbb{F}_q$;

(U2) the polynomial $X^{q_0+1} + uX^{q_0} + (u+1)X + 1$ has no roots in $\mathbb{F}_q$.

Then such an appropriate choice of the triple (q, q_0, u) is given in the following theorem.

Theorem 3.1. *Assume that $q-1$ and q_0^2-1 are relatively prime. Then*

(i) Γ_u is connected Cayley graph.

(ii) If, in addition, $u \in \mathscr{U}_{q,q_0}$, then $\mathrm{Aut}(\Gamma_u) = G$, that is, G has a graphical Frobenius representation on Γ_u.

The question whether Theorem 3.1 provides an infinite family is also answered positively.

Theorem 3.2. *For infinitely many 2-powers q it is true that whenever the 2-power q_0 satisfies $\gcd(q-1, q_0^2-1) = 1$, the set $\mathscr{U}_{q,q_0}$ is not empty.*

Computer calculations show that for many $u \notin \mathscr{U}_{q,q_0}$, the graph Γ_u is still a GFR for G. Hence, the conditions (U1) and (U2) are only needed for our proofs. However, $\mathscr{U}_{q,q_0} = \emptyset$ implies $\varphi(q)/q \leq 3$, which happens extremely rarely for $q = 2^f$, f odd; see Section 5.

4 Some more properties of the abstract structure of the group G

Lemma 4.1. *The following hold in K:*

(i) $\Phi_{a,c}^2 = \Phi_{0,a^{q_0+1}}$ *and* $\Phi_{a,c}^{-1} = \Phi_{a,c+a^{q_0+1}}$.

(ii) $\Phi_{a,c}^{-1}\Phi_{b,d}^{-1}\Phi_{a,c}\Phi_{b,d} = \Phi_{0,a^{q_0}b+ab^{q_0}}$.

(iii) Ω_∞ *consists of central involutions of* K.

(iv) For each $u \in \mathbb{F}_q$, *we have* $\Omega_u^{-1} = \Omega_{u+1}$.

Proof. Straightforward matrix computation. □

Lemma 4.2. *Assume that* $\gcd(q_0^2 - 1, q - 1) = 1$. *Then the following hold:*

(i) $K' = Z(K) = \{1\} \cup \Omega_\infty$.

(ii) K' *and* K/K' *are elementary abelian* 2*-groups of order* q.

(iii) For $u \in \mathbb{F}_q$, *the set* Ω_u *generates* K.

(iv) H *acts transitively (hence irreducibly) on the nontrivial elements of* K' *and* K/K'.

(v) The subgroup H *is maximal in* HK', *which is maximal in* G.

Proof. By the assumption, the map $a \mapsto a + a^{q_0}$ has kernel $\mathbb{F}_2$, and, $a \mapsto a^{q_0+1}$ is a bijection of $\mathbb{F}_q^*$. Hence, any element of $\mathbb{F}_q$ can be written in the form $a^{q_0}b + ab^{q_0}$, which implies (i). For $a \in \mathbb{F}_q^*$, we have $\Phi_{a,ua^{q_0+1}}^2 = \Phi_{0,a^{q_0+1}}$. Thus, $\Omega_\infty \subseteq \langle \Omega_u \rangle$ and (iii) follows. The rest is straightforward computation. □

Notice that Lemma 4.2(iii) yields Theorem 3.1(i).

5 On Conditions (U1) and (U2)

A natural key question regarding the applicability of Theorem 3.1 is the existence of some q such that $\mathscr{U}_{q,q_0}$ is not empty, that is, $\mathbb{F}_q$ contains an element u satisfying both Conditions (U1) and (U2). Theorem 3.2 states that infinitely many such q exist and we are going to show how to prove it using Euler's phi function and the Möbius function. For this purpose, we need some algebraic preparatory results stated in the next lemmas.

Lemma 5.1. *Let* $q = 2^f$ *be a power of* 2 *with odd exponent* f. *There exist at least* $2(q+1)/3$ *elements* $u \in \mathbb{F}_q$ *such that* $X^{q_0+1} + uX^{q_0} + (u+1)X + 1$ *has no roots in* $\mathbb{F}_q$.

Proof. Define the rational function

$$U(x) = \frac{x^{q_0+1} + x + 1}{x^{q_0} + x}.$$

Clearly, 0 and 1 are never roots of $X^{q_0+1} + uX^{q_0} + (u+1)X + 1$. Moreover, $X^{q_0+1} + uX^{q_0} + (u+1)X + 1$ has a root in $\mathbb{F}_q$ if and only if $u = U(x)$ for some $x \in \mathbb{F}_q \setminus \{0,1\}$. Since $U(0) = U(1) = \infty$ and

$$U(x) = U\left(\frac{x+1}{x}\right) = U\left(\frac{1}{x+1}\right)$$

identically, we have $|U(\mathbb{F}_q \setminus \{0,1\})| \leq (q-2)/3$. Here we use the fact that $\mathbb{F}_4$ is not a subfield of $\mathbb{F}_q$ and x, $(x+1)/x$, $1/(x+1)$ are distinct elements of $\mathbb{F}_q$. □

Lemma 5.2. *For infinitely many odd integers n, inequality $\varphi(2^n-1)/(2^n-1) > 1/3$ holds.*

Proof. The claim follows from the asymptotic formula of [7, Theorem 3]

$$\frac{1}{M}\sum_{1\le m\le M}\frac{\varphi(2^m-1)}{2^m-1} = \mu + O(M^{-1}\log M),$$

with μ is given by the absolute convergent series

$$\mu = \sum_{d \text{ odd}} \frac{\mu(d)}{dt_d} \approx 0.73192,$$

where t_d is the multiplicative order of 2 modulo d, and $\mu(d)$ is the Möbius function; see [11, Theorem 4.1].

We give a second, elementary proof based on Fermat's Little Theorem. We show that for primes p, $\varphi(2^p-1)/(2^p-1) \to 1$. Let $r_1,\dots,r_k$ be the different prime factors of 2^p-1. For $i=1,\dots,k$, let m_i be the order of 2 modulo r_i. Then $m_i \mid p$ and $p = m_i$. Moreover, $2^{r_i-1} \equiv 1 \pmod{r_i}$ implies $p \mid (r_i - 1)$. In fact, $p \mid (r_i-1)/2$ and $r_i = 2s_ip+1$ holds for some integer $s_i \ge 1$. This implies

$$k < \log_{2p}(2^p-1) < \frac{p}{\log_2 p}.$$

Hence,

$$1 > \frac{\varphi(2^p-1)}{2^p-1} = \prod_{i=1}^{k}\left(1-\frac{1}{r_i}\right) > \left(1-\frac{1}{2p}\right)^{\frac{p}{\log_2 p}},$$

where the latter term converges to 1. This proves our claim. □

Remark 5.3. As pointed out in [8], much more is true: [7] implies that given any $\varepsilon > 0$, there is a $c > 0$ such that $\varphi(2^n-1)/(2^n-1) > c$ apart from a set of n with upper density $< \varepsilon$.

We are in a position to prove Theorem 3.2. By Lemma 5.2, it suffices to show that for an arbitrary odd integer f with $\varphi(2^f-1)/(2^f-1) > 1/3$, $q = 2^f$ fulfills the conditions of Theorem 3.2. Fix such an f and choose an arbitrary integer f_0, coprime to f. Then $q_0 = 2^{f_0}$ satisfies $\gcd(q-1, q_0^2-1) = 1$. By the choice of f, $\mathbb{F}_q$ has more than $(q-1)/3$ primitive elements. In our case, $x \mapsto x^{q_0-1}$ is bijective in $\mathbb{F}_q$, hence the maps

$$\eta \mapsto \eta' = \frac{1+\eta^{q_0}}{\eta+\eta^{q_0}}, \qquad u \mapsto u' = 1 + \left(\frac{u}{u+1}\right)^{\frac{1}{q_0-1}}$$

are well-defined inverses to each other. Now, the claim follows from Lemma 5.1.

6 Incidences

Recall that Γ_u denotes the Cayley graph $\mathrm{Cay}(K, \Omega_u \cup \Omega_{u+1})$, where the vertices of Γ_u are the elements of K and Ω_u is defined in (2.1). The identity $\Phi_{0,0}$ of K will also be denoted by ε. The group $G = HK$ acts on K, the action is induced as follows: The elements of

K act in the right regular action and the elements of H act by conjugation. In the sequel, we identify G with its permutation action on K, whereby some caution is required since for a subset X of K, the point-wise stabilizer of X in G and the centralizer of X in G are in general different. As a permutation group, G is a subgroup of the automorphism group $\mathrm{Aut}(\Gamma_u)$, and H is its cyclic subgroup of order $q-1$, fixing ε and preserving both Ω_u and Ω_{u+1}. Formally, ε is viewed as an element of $\mathrm{Aut}(\Gamma_u)$; nevertheless, we will also use the notation id to denote the trivial automorphism of $\mathrm{Aut}(\Gamma_u)$.

For any two elements $\Phi_{a,c}, \Phi_{b,d} \in K$ with $\Phi_{a,c}\Phi_{b,d}^{-1} \in \Omega_u$, we introduce the directed edge notation $\Phi_{a,c} \xrightarrow{u} \Phi_{b,d}$ in Γ_u and we refer to it as a u-edge. It should be noticed that our notation is not the usual one for Cayley digraphs, where the arrows point in the opposite direction. An obvious observation is that the following are equivalent:

(i) $\Phi_{a,c} \xrightarrow{u} \Phi_{b,d}$,

(ii) $\Phi_{a,c}\Phi_{b,d}^{-1} \in \Omega_u$,

(iii) $c+d = (a+b)^{q_0}(ua+(u+1)b)$,

(iv) $c+d = u(a+b)^{q_0+1} + a^{q_0}b + b^{q_0+1}$.

Now we collect some incidences in Γ_u which play a role in our proof.

Lemma 6.1. *Assume* $\gcd(q-1, q_0^2-1) = 1$ *and define*

$$\eta = 1 + \left(\frac{u}{u+1}\right)^{\frac{1}{q_0-1}}$$

for $u \in \mathbb{F}_q \setminus \{0,1\}$. *Then the following hold in* Γ_u *for* $a, b \neq 0$*:*

$$\Phi_{a,ua^{q_0+1}} \xrightarrow{u} \Phi_{b,ub^{q_0+1}} \iff b = \frac{a}{\eta}, \tag{6.1a}$$

$$\Phi_{a,ua^{q_0+1}} \xrightarrow{u+1} \Phi_{b,ub^{q_0+1}} \iff b = a\eta, \tag{6.1b}$$

$$\Phi_{a,(u+1)a^{q_0+1}} \xrightarrow{u} \Phi_{b,(u+1)b^{q_0+1}} \iff b = a\cdot\frac{\eta}{1+\eta}, \tag{6.1c}$$

$$\Phi_{a,(u+1)a^{q_0+1}} \xrightarrow{u+1} \Phi_{b,(u+1)b^{q_0+1}} \iff b = a\cdot\frac{1+\eta}{\eta}, \tag{6.1d}$$

$$\Phi_{a,ua^{q_0+1}} \xrightarrow{u} \Phi_{b,(u+1)b^{q_0+1}} \iff b = \frac{a}{1+\eta}, \tag{6.1e}$$

$$\Phi_{a,ua^{q_0+1}} \xrightarrow{u+1} \Phi_{b,(u+1)b^{q_0+1}} \iff \left(\frac{a}{b}\right)^{q_0+1} + u\left(\frac{a}{b}\right)^{q_0} + (u+1)\left(\frac{a}{b}\right) + 1 = 0. \tag{6.1f}$$

Proof. (6.1a): Since Γ_u has no loops, we may assume $a \neq b$.

$$\begin{aligned}\Phi_{a,ua^{q_0+1}} \xrightarrow{u} \Phi_{b,ub^{q_0+1}} &\iff ua^{q_0+1} + ub^{q_0+1} = (a+b)^{q_0}(ua+(u+1)b)\\ &\iff 0 = (u+1)a^{q_0}b + uab^{q_0} + b^{q_0+1}\\ &\iff 0 = (u+1)\left(\frac{a}{b}\right)^{q_0} + u\left(\frac{a}{b}\right) + 1\\ &\iff 0 = (u+1)\left(\frac{a}{b}+1\right)^{q_0} + u\left(\frac{a}{b}+1\right)\end{aligned}$$

$$\iff \left(\frac{a}{b}+1\right)^{q_0-1} = \frac{u}{u+1} = (\eta+1)^{q_0-1}$$
$$\iff \frac{a}{b} = \eta.$$

Since $(u+1)$-edges are reversed u-edges, we obtain (6.1b) by switching a and b in the computation above. To show (6.1d), we replace u by $u+1$ and use the computation above to obtain

$$\Phi_{a,(u+1)a^{q_0+1}} \xrightarrow{u+1} \Phi_{b,(u+1)b^{q_0+1}} \iff \left(\frac{a}{b}+1\right)^{q_0-1} = \frac{u+1}{u} = \left(\frac{1}{1+\eta}\right)^{q_0-1}$$
$$\iff \frac{a}{b} = \frac{\eta}{1+\eta}.$$

This proves (6.1c) by switching a and b. For (6.1e):

$$\begin{aligned}\Phi_{a,ua^{q_0+1}} \xrightarrow{u} \Phi_{b,(u+1)b^{q_0+1}} &\iff ua^{q_0+1} + (u+1)b^{q_0+1} = (a+b)^{q_0}(ua+(u+1)b)\\ &\iff 0 = (u+1)a^{q_0}b + uab^{q_0}\\ &\iff \left(\frac{a}{b}\right)^{q_0-1} = \frac{u}{u+1} = (\eta+1)^{q_0-1}\\ &\iff \frac{a}{b} = 1+\eta.\end{aligned}$$

Finally,

$$\begin{aligned}\Phi_{a,ua^{q_0+1}} \xrightarrow{u+1} \Phi_{b,(u+1)b^{q_0+1}} &\iff ua^{q_0+1} + (u+1)b^{q_0+1} = (a+b)^{q_0}((u+1)a+ub)\\ &\iff 0 = a^{q_0+1} + ua^{q_0}b + (u+1)ab^{q_0} + b^{q_0+1}\\ &\iff 0 = \left(\frac{a}{b}\right)^{q_0+1} + u\left(\frac{a}{b}\right)^{q_0} + (u+1)\left(\frac{a}{b}\right) + 1,\end{aligned}$$

which shows (6.1f). □

Our next step is to describe the structure of the neighborhood of the vertex ε in Γ_u. For this purpose, we recall the concept of *generalized Petersen graphs* [3]. Let n and k be integers with $1 \le k < n/2$, the vertex set of $\mathrm{GPG}(n,k)$ is $\{c_1,\dots,c_n,c'_1,\dots,c'_n\}$ and the edge set consists of all pairs of the form

$$c_ic_{i+1}, \qquad c_ic'_i, \qquad c_ic'_{i+k}, \qquad i \in \{1,\dots,n\},$$

where all subscripts are to be read modulo n. In order to describe the automorphism group of $\mathrm{GPG}(n,k)$, define the permutations

$$\begin{aligned}\rho\colon c_i &\mapsto c_{i+1}, \quad c'_i \mapsto c'_{i+1},\\ \delta\colon c_i &\mapsto c_{-i}, \quad c'_i \mapsto c'_{-i},\\ \alpha\colon c_i &\mapsto c'_{ki}, \quad c'_i \mapsto c_{ki}\end{aligned}$$

for all $i \in \{1,\dots,n\}$. By [3, Theorem 1 and 2],

$$\langle\rho,\delta\rangle \le \mathrm{Aut}(\mathrm{GPG}(n,k)) \le \langle\rho,\delta,\alpha\rangle$$

provided that $n \notin \{4, 5, 8, 10, 12, 24\}$. Moreover, the generators ρ, δ, satisfy the relations $\rho^n = \delta^2 = \mathrm{id}$, $\delta\rho\delta = \rho^{-1}$, hence, $\langle\rho, \delta\rangle$ is isomorphic to the dihedral group of order $2n$. Also, $\alpha\delta = \delta\alpha$, $\alpha^2 \in \{\mathrm{id}, \delta\}$, and most importantly $\alpha^{-1}\rho\alpha = \rho^k$. This implies the following lemma:

Lemma 6.2. *Let n be an odd integer, $n \neq 5$, and $1 \leq k < n$. In $\mathrm{Aut}(\mathrm{GPG}(n, k))$, the following properties hold:*

(i) The elements of odd order form a unique cyclic normal subgroup of order n.

(ii) For $k \neq \pm 1$, no involution commutes with the cyclic normal subgroup of order n. □

Proposition 6.3. *Assume $\gcd(q-1, q_0^2-1) = 1$ and $u \in \mathscr{U}_{q,q_0}$. Then, the neighborhood $\Omega_u \cup \Omega_{u+1}$ of ε in Γ_u is isomorphic to the generalized Petersen graph $\mathrm{GPG}(q-1, k)$, where $u = (1+\eta^{q_0})/(\eta+\eta^{q_0})$ and the integer k is defined by $1+\eta = \eta^{k+1}$.*

Proof. By the choice of u, η is a primitive element of $\mathbb{F}_q$. Define

$$c_i = \Phi_{\eta^i, u\eta^{i(q_0+1)}}, \qquad c_i' = \Phi_{\eta^i/(1+\eta),(u+1)(\eta^i/(1+\eta))^{q_0+1}}.$$

From Lemma 6.1, c_ic_{i+1}, c_ic_i' are edges and there are no more edges in Ω_u and between Ω_u and Ω_{u+1}. In Ω_{u+1}, c_i' and c_j' are connected with an u-edge if and only if

$$\frac{\eta^j}{1+\eta} = \frac{\eta^i}{1+\eta} \cdot \frac{1+\eta}{\eta} \iff \eta^{j-i+1} = 1+\eta = \eta^{k+1} \iff j \equiv i+k \pmod{q-1}.$$

This finishes the proof. □

Notice that $k = \pm 1$ would imply $\eta = 0$ or $1+\eta+\eta^2 = 0$, which is not possible if $\gcd(q-1, q_0^2-1) = 1$ and η generates $\mathbb{F}_q^*$.

Corollary 6.4. *Assume $\gcd(q-1, q_0^2-1) = 1$ and $u \in \mathscr{U}_{q,q_0}$. Let A be the permutation group induced by the stabilizer $\mathrm{Aut}(\Gamma_u)_\varepsilon$ on $\Omega_u \cup \Omega_{u+1}$. Then A is solvable, its order is either $(q-1)$, $2(q-1)$ or $4(q-1)$, and it has a unique cyclic normal subgroup of odd order $q-1$. Moreover, $\mathrm{Aut}(\Gamma_u)_\varepsilon$ either preserves Ω_u and Ω_{u+1}, or it interchanges them.*

Proof. A contains the cyclic subgroup of order $q-1$ that is induced by H on $\Omega_u \cup \Omega_{u+1}$. Proposition 6.3 and Lemma 6.2 apply. □

We finish this section with another property of the stabilizer of ε in $\mathrm{Aut}(\Gamma_u)$.

Lemma 6.5. *Assume $\gcd(q-1, q_0^2-1) = 1$ and $u \in \mathscr{U}_{q,q_0}$.*

(i) Let A be the centralizer of the commutator subgroup K' in $\mathrm{Aut}(\Gamma_u)$. Then $K \leq A$ and $|A : K| \leq 2$. Moreover, any element of $A \setminus K$ interchanges the sets Ω_u and Ω_{u+1}.

(ii) Let $\alpha \in \mathrm{Aut}(\Gamma)$ be an involution which centralizes H. Then α fixes $\Omega_u \cup \Omega_{u+1}$ point-wise.

Proof. (i): Obvoiusly, $K \leq A$ and A is transitive. From the last sentence of Corollary 6.4, an element $\alpha \in A_\varepsilon$ either preserves Ω_u and Ω_{u+1}, or it interchanges them. We show that if α preserves Ω_u then $\alpha = \mathrm{id}$. This will imply $|A_\varepsilon| \leq 2$ and $|A| \leq 2q^2$. Since α commutes with K' and fixes ε, it fixes all points in the orbit $\varepsilon^{K'} = \{\varepsilon\} \cup \Omega_\infty$. The elements $\Phi_{a,ua^{q_0+1}} \in \Omega_u$ and $\Phi_{0,d} \in K'$ satisfy both relations

$$\Phi_{a,ua^{q_0+1}} \xrightarrow{u} \Phi_{0,d} \Longleftrightarrow d = 0,$$
$$\Phi_{a,ua^{q_0+1}} \xrightarrow{u+1} \Phi_{0,d} \Longleftrightarrow d = a^{q_0+1}.$$

This means that each element in Ω_u is connected with a unique element in Ω_∞. Hence, α fixes all elements in Ω_u. As each K'-orbit contains a unique element in Ω_u, we see that each K'-orbit is preserved. Once again, α commutes with K' and fixes an element in each K'-orbit. Therefore, α fixes all points in each K'-orbit.

(ii): As ε is the unique fixed point of H, $\varepsilon^\alpha = \varepsilon$ and α leaves the neighborhood $\Omega_u \cup \Omega_{u+1}$ of ε invariant. By Lemma 6.2(ii), the restriction of α to $\Omega_u \cup \Omega_{u+1}$ cannot have order 2, therefore, it must be trivial. □

7 Imprimitivity

In this section we show that an appropriate choice of $u \in \mathbb{F}_q$ ensures that $\mathrm{Aut}(\Gamma_u)$ cannot act primitively on the set of vertices of Γ_u. We recall that a primitive permutation group G is of *affine type* if it has an abelian regular normal subgroup, which is necessarily elementary abelian of order r^n for some prime r. In this case G is embedded in the affine group $\mathrm{AGL}(n, r)$ with the socle being the translation subgroup. Its stabiliser of $0 \in \mathbb{F}_r^n$ is a subgroup of $\mathrm{GL}(n, r)$ which acts irreducibly on $\mathbb{F}_r^n$. For our purpuse, a useful tool is the following result by Guralnick and Saxl.

Proposition 7.1 (Guralnick and Saxl [4])**.** *Let G be a primitive permutation group of degree 2^n. Then either G is of affine type, or G has a unique minimal normal subgroup $N = S \times \cdots \times S = S^t$, $t \geq 1$, S is a non-abelian simple group, and one of the following holds:*

- *(i) $S = A_m$, $m = 2^e \geq 8$, $n = te$, and the 1-point stabilizer in N is $N_1 = A_{m-1} \times \cdots \times A_{m-1}$, or*
- *(ii) $S = \mathrm{PSL}(2, p)$, $p = 2^e - 1 \geq 7$ is a Mersenne prime, $n = te$, and the 1-point stabilizer in N is the direct product of maximal parabolic subgroups each stabilizing a 1-space.*

Lemma 7.2. *Let G be a group acting transitively on the set X. For $x \in X$ and let $H = G_x$ be the stabilizer of x in G.*

- *(i) For $y \in X$, choose $g \in G$ such that $y = x^g$. Then the subgroup of H, fixing the H-orbit of y point-wise, coincides with $\cap_{h \in H} H^{gh}$.*
- *(ii) If G is 2-transitive on X then $\cap_{h \in H} H^{gh}$ is either H or $\{1\}$, depending upon whether $g \in H$ or $g \notin H$.*

Proof. If $y' \in y^H$, then $y' = y^h = x^{gh}$ for some $h \in H$. Hence, for the stabilizer we have $G_{y'} = G_x^{gh} = H^{gh}$. Therefore, the point-wise stabilizer of y^H is $\cap_{y' \in y^H} G_{y'} = \cap_{h \in H} H^{gh}$. This proves (i). Clearly, if $g \in H$ then $\cap_{h \in H} H^{gh} = H$. If $g \in G \setminus H$ then $x \neq y = x^g$ and $\cap_{h \in H} H^{gh}$ fixes all points in $\{x\} \cup y^H$. The latter set is X if G is 2-transitive. □

Lemma 7.3. *Assume* $\gcd(q-1, q_0^2-1) = 1$ *and* $u \in \mathscr{U}_{q,q_0}$. *If* $\mathrm{Aut}(\Gamma_u)$ *acts primitively on* Γ_u, *then its action is of affine type.*

Proof. Let us assume on the contrary that $\mathrm{Aut}(\Gamma_u)$ is not of affine type. Let N be its unique minimal normal subgroup. With the notation in Proposition 7.1, we have $N = S^t$ where either $S = A_m$, $m \geq 8$, or $S = \mathrm{PSL}(2,p)$, with a Mersenne prime $p = m-1 \geq 7$. In both cases, S has a 2-transitive action on m points. Moreover, if B is the 1-point stabilizer in S, then the point stabilizer of $\varepsilon = \Phi_{0,0}$ in N is $N_\varepsilon = B^t$. For $(g_1, \dots, g_t) \in S^t$ take a generic vertex $y = \varepsilon^{(g_1,\dots,g_t)}$ of Γ_u. Let Y be the B^t-orbit of y. By Lemma 7.2(i) the point-wise stabilizer of Y is

$$(\cap_{b\in B} B^{g_1 b}) \times \cdots \times (\cap_{b \in B} B^{g_t b}).$$

By Lemma 7.2(ii), each factor is either $\{1\}$ or B, depending upon whether $g_i \in B$ or not. Thus, the point-wise stabilizer of Y in B^t is B^{t_0}, where $0 \leq t_0 \leq t$, and $t_0 = t$ occurs if and only if $Y = \{\varepsilon\}$. Therefore, the B^t induces a permutation group on Y which is isomorphic to B^{t_1}, where $t_1 = t - t_0$. Furthermore, $t_1 = 0$ if and only if $Y = \{\varepsilon\}$.

The stabilizer N_ε acts on $\Omega_u \cup \Omega_{u+1}$. Let Y be a nontrivial N_ε-orbit contained in $\Omega_u \cup \Omega_{u+1}$. If $S = A_m$, then N_ε induces a nonsolvable group of automorphisms of $\Omega_u \cup \Omega_{u+1}$. If $S = \mathrm{PSL}(2,p)$, then $|B| = p(p-1)/2$, and N_ε induces a noncyclic group of odd order on $\Omega_u \cup \Omega_{u+1}$. Both possibilities are inconsistent with Corollary 6.4. □

We are now able to prove the imprimitivity of $\mathrm{Aut}(\Gamma_u)$.

Proposition 7.4. *Assume* $\gcd(q-1, q_0^2-1) = 1$ *and* $u \in \mathscr{U}_{q,q_0}$. *Then,* $\mathrm{Aut}(\Gamma_u)$ *acts imprimitively on* Γ_u.

Proof. As before, G is identified with its permutation action on Γ_u. In particular, we consider H, K as subgroups of $\mathrm{Aut}(\Gamma_u)$. At the same time, K is the set of vertices of Γ_u.

Assume on the contrary that $\mathrm{Aut}(\Gamma_u)$ is primitive, hence of affine type by Lemma 7.3. Let N be the unique minimal normal subgroup of $\mathrm{Aut}(\Gamma_u)$. Then N is a regular elementary abelian 2-group. Since H has odd order, N decomposes into the direct product of H-invariant subgroups. For any $1 \neq h \in H$ and $1 \neq n \in N$, h has a unique fixed point, while n has no fixed point. Hence $nh \neq hn$. Therefore $N = A_1 \times A_2$ where A_i is an elementary abelian group of order q and H acts regularly on $A_i \setminus \{1\}$, $i = 1, 2$. Consider the subgroup $M = N_{NK}(K)$. Since NK is nilpotent, we have $K \lneq M$ and $K' \lhd M$. The latter implies $K' \cap Z(M) \neq \{1\}$. Since both K' and $Z(M)$ are H-invariant while H acts regularly on $K' \setminus \{1\}$, we have $K' \leq Z(M)$. By Lemma 6.5, $|M : K| = 2$. On the one hand, $M = (M \cap N)K$. On the other hand, $N \cap K$ is an H-submodule of $M \cap N$. By Maschke's Theorem [1, (10.8)] applied to $M \cap N$, viewed as a $\mathbb{F}_2$-vector space, there is an H-invariant subgroup B in $M \cap N$ such that $M \cap N = B \times (N \cap K)$. Therefore,

$$B \cong (M \cap N)/(N \cap K) \cong (M \cap N)K/K \cong M/K \cong \mathbb{F}_2,$$

that is, the nontrivial element of $B \leq N$ commutes with H, a contradiction. □

8 Proof of the main result Theorem 3.1(ii)

In this section, we complete the proof of Theorem 3.1. As before, G is identified with its permutation action on Γ_u. From Proposition 7.4, we know that $A = \mathrm{Aut}(\Gamma_u)$ acts imprimitively on Γ_u. We claim that the only nontrivial blocks of imprimitivity of A are the cosets of the commutator subgroup K' of K. Or equivalently, K' is the only nontrivial block containing ε. Let B be an arbitrary nontrivial block of imprimitivity of A which contains ε. Then the stabilizer of the set B in G is a subgroup G_B of G, lying properly between H and G. By Lemma 4.2(v), $G_B = HK'$ and $B = K'$, which proves the claim. The next two lemmas describe the point-wise stabilizer of K' in A.

Lemma 8.1. *Let E be the point-wise stabilizer of $K' = \{\varepsilon\} \cup \Omega_\infty$ in $\mathrm{Aut}(\Gamma_u)$. Then E is either trivial or it is an elementary abelian 2-group which fixes all pairs $\{\Phi_{a,c}, \Phi_{a,c}^{-1}\}$.*

Proof. The observations made prior to Lemma 6.1 show

$$\Phi_{a,c} \xrightarrow{u} \Phi_{0,d} \iff d = c + ua^{q_0+1}$$

for all $a, c, d \in \mathbb{F}_q$. Thus, any vertex $\Phi_{a,c}$, $a \neq 0$, is u-connected to a unique element Φ_{0,d_1} of K' and $(u+1)$-connected to a unique element Φ_{0,d_2} of K', where $d_1 = c + ua^{q_0+1}$ and $d_2 = c + (u+1)a^{q_0+1}$. If d_1 and d_2 are distinct nonzero elements, then Φ_{0,d_1}, Φ_{0,d_2} are distinct vertices in Ω_∞, whose common neighbors are $\Phi_{a,c}$ and $\Phi_{a,c}^{-1} = \Phi_{a,c+a^{q_0+1}}$, where

$$a = (d_1 + d_2)^{\frac{1}{q_0+1}} \quad \text{and} \quad c \in \{d_1 + ua^{q_0+1}, d_2 + ua^{q_0+1}\}.$$

This shows that any automorphism of Γ_u, which fixes Ω_∞ point-wise, must leave the pair $\{\Phi_{a,c}, \Phi_{a,c}^{-1}\}$ invariant. It follows that E either trivial or has exponent 2 and in the latter case E is elementary abelian. □

Actually, E is trivial by the following lemma.

Lemma 8.2. *The only automorphism that fixes $\{\varepsilon\} \cup \Omega_\infty$ point-wise is the identity.*

Proof. Let E be defined as in Lemma 8.1. Since HK' preserves the set of vertices in K', HK' normalizes E. Assume on the contrary that $E \neq \{1\}$, then $C_E(K') \neq \{1\}$ is H-invariant. Since K' acts regularly on itself, $E \cap K' = \{1\}$. We apply Lemma 6.5(i) to conclude that $|C_E(K')| = 2$. This means that there is a unique involutory automorphism $\alpha \in A$ which centralizes both K' and H. Now, Lemma 6.5(ii) implies that α fixes $\Omega_u \cup \Omega_{u+1}$ point-wise. Finally, Lemma 6.5(i) yields $\alpha \in K$, a contradiction. □

Let us now focus on the point stabilizer A_ε of ε in $A = \mathrm{Aut}(\Gamma_u)$. Clearly, A_ε leaves $\Omega_u \cup \Omega_{u+1}$ invariant. Moreover, by the imprimitivity of A, A_ε preserves Ω_∞ as well. Since any element of Ω_u is connected with a unique element of Ω_∞, each automorphism fixing all points in $\{\varepsilon\} \cup \Omega_u \cup \Omega_{u+1}$ fixes all points in Ω_∞. Hence by Lemma 8.2, the action of A_ε on $\Omega_u \cup \Omega_{u+1}$ is faithful and the possibilities for $|A_\varepsilon|$ are $q-1$, $2(q-1)$ or $4(q-1)$ by Corollary 6.4.

Let S denote the stabilizer of the set K' in A. On the one hand, $HK' \leq S$, hence S is transitive on K'. On the other hand, $A_\varepsilon \leq S$ since K' is a block of imprimitivity. Therefore, $A_\varepsilon = S_\varepsilon$, and

$$|S| = q|A_\varepsilon| \in \{q(q-1), 2q(q-1), 4q(q-1)\}.$$

This implies that S induces a 2-transitive solvable permutation group $\bar{S}$ on K'. Since the order of K' is a power of 2, Huppert's Theorem [6, Theorem XII.7.3] yields that $\bar{S}$ is similar to a subgroup of the group $\mathrm{A\Gamma L}(1,q)$ of all semilinear mappings

$$z \mapsto az^\alpha + b, \qquad a, b \in \mathbb{F}_q, a \neq 0, \alpha \in \mathrm{Aut}(\mathbb{F}_q)$$

on $\mathbb{F}_q$. Here, $|\mathrm{A\Gamma L}(1,q)| = fq(q-1)$ for $q = 2^f$. Since $\gcd(q-1, q_0^2-1) = 1$, f is odd, and the only possibility for the cardinality of $\bar{S}$ is $q(q-1)$. We apply Lemma 8.2 once more to conclude that $|S| = q(q-1)$, which implies $A_\varepsilon = H$ and $A = HK = G$. This finishes the proof of Theorem 3.1(ii).

Remark 8.3. Since the proof of Lemma 8.2 depends on Huppert's classification of solvable 2-transitive groups of degree $2^h + 1$, a natural question is whether the possibility $|S| \in \{2q(q-1), 4q(q-1)\}$ can be ruled out by purely combinatorial arguments based on the structure of the graph Γ_u rather than by the use of Huppert's classification. We are likely to think that the answer is negative. In fact, the action of an extra-automorphism in case $|S| \in \{2q(q-1), 4q(q-1)\}$ does not seem to produce further useful constraint on the structure of the graph G_u in the case where the kernel K is nonabelian and the complement H has odd order. Actually, as Spiga himself pointed out in [10, Section 1.1], this case is by far the hardest in the GFR problem.

9 Spiga's bound

To state Spiga's bound we need some notation consistent with that used in [9]. For a Frobenius group $G = N \rtimes H$ with kernel N and complement H, let

$$\begin{aligned}
d_{11}(|N|,|H|) &= 1 + \frac{|N|-1}{|H|} - \left(\frac{\sqrt{|N|}}{4} - (\log_2 |N|)^2\right);\\
d_{12}(|N|,|H|) &= 1 + \frac{|N|-1}{|H|} - \left(\frac{\sqrt{|N|}-1-|H|}{|H|(1+2|H|)} \log_2 \tfrac{4}{3} - 2\log_2 \sqrt{|N|} + 1\right);\\
d_1(|N|,|H|) &= (\log_2 |N|)^2;\\
f_1(|N|,|H|) &= 2^{d_1(|N|,|H|)}(\tfrac{1}{2}|N|-1)\max\{2^{d_{11}(|N|,|H|)}, 2^{d_{12}(|N|,|H|)}\};\\
c_2(|N|,|H|) &= \frac{3}{4}\frac{|N|}{|H|} - \frac{1}{2|H|} + \frac{1}{2} + \sqrt{|N|}\frac{|H|-1}{2|H|} + (\log_2 |N|)^2;\\
f_2(|N|,|H|) &= 2^{c_2(|(N|,|H|)};\\
d_{31}(|N|,|H|) &= \left(\frac{2|N|}{|H|}\right)^{\frac{2}{3}} - 2|H|\sqrt{|N|};\\
d_{32}(|N|,|H|) &= \sqrt{N};\\
d_{33}(|N|,|H|) &= 4\sqrt{|N|} - 2|H|\sqrt[4]{|N|};\\
F_3(|N|,|H|) &= \frac{1}{4|H|}\min\{d_{31}(|N|,|H|), d_{32}(|N|,|H|), d_{33}(|N|,|H|)\};\\
c_3(|N|,|H|) &= \frac{3}{2} + \frac{|N|}{|H|} + \frac{1}{2|H|} - F_3(|N|,|H|) + (\log_2 |N|)^2;\\
f_3(|N|,|H|) &= 2^{c_3(|N|,|H|)}.
\end{aligned}$$

Theorem 9.1 (Spiga's Bound)**.** *If*

$$2^{1+(|N|-1)/|H|} > f_1(|N|,|H|) + f_2(|N|,|H|) + f_3(|N|,|H|)$$

then G admits a GFR.

As stated in [9, Theorem 2], Spiga's bound implies that when $|N|$ is large compared to $|H|$, then a random H-invariant subset S of N gives rise to GFR on $\mathrm{Cay}(N, S)$. Spiga also claimed on his strategy that "Theoretically this strategy is sound, in practice, even for relatively small groups H, the lower bound on $|N|$ is so large that it seems hopeless and infeasible to study the small groups N with a computer."

Proposition 9.2. *For a 2-power $q \geq 2$, let G be a Frobenius group of order $q^2(q-1)$ with nucleus of order q^2 and complement of order $q-1$. Then Spiga's bound does not hold for G.*

Proof. In our case, in the exponent on the left hand side in Spiga's bound we have $q + 2$. On the other hand, for $q = 2^m$,

$$c_2(|N|, |H|) > \frac{3}{4}\frac{q^2}{q-1} + \frac{q(q-2)}{2(q-1)} + 2m \geq \frac{q(5q-4)}{4(q-1)} + 2.$$

A straightforward computation shows that the last number is always bigger than $q+2$ when the claim follows. □

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On few-class Q-polynomial association schemes: feasible parameters and nonexistence results

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Abstract

We present the tables of feasible parameters of primitive 3-class Q-polynomial association schemes and 4- and 5-class Q-bipartite association schemes (on up to 2800, 10000, and 50000 vertices, respectively), accompanied by a number of nonexistence results for such schemes obtained by analysing triple intersection numbers of putative open cases.

Keywords: Association scheme, Q-polynomial, feasible parameters, distance-regular graph.

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1 Introduction

Much attention in literature on association schemes has been paid to distance-regular graphs, in particular to those of diameter 2, also known as strongly regular graphs – however, their complete classification is still a widely open problem. The tables of their feasible parameters, maintained by A. E. Brouwer [4, 5], are very helpful for the algebraic combinatorics community, in particular when one wants to check whether a certain example has already been proven (not) to exist, to be unique, etc. Compiling such a table can be a challenging problem, as, for example, some feasibility conditions require calculating roots of high degree polynomials.

The goal of this work is to present the tables of feasible parameters of Q-polynomial association schemes, compiled by the third author, and accompanied by a number of nonexistence results obtained by the first two authors.

Recall that Q-polynomial association schemes can be seen as a counterpart of distance-regular graphs, which, however, remains much less explored, although they have received considerable attention in the last few years [11, 25, 27, 28] due to their connection with some objects in quantum information theory such as equiangular lines and real mutually unbiased bases [24].

More precisely, let $A_0, \dots, A_D$ and $E_0, \dots, E_D$ denote the adjacency matrices and the primitive idempotents of an association scheme, respectively. An association scheme is *P-polynomial* (or *metric*) if, after suitably reordering the relations, there exist polynomials v_i of degree i such that $A_i = v_i(A_1)$ $(0 \le i \le D)$. If this is the case, the matrix A_i can be seen as the distance-i adjacency matrix of a distance-regular graph and vice-versa. Similarly, an association scheme is *Q-polynomial* (or *cometric*) if, after suitably reordering the eigenspaces, there exist polynomials v_j^* of degree j such that $E_j = v_j^*(E_1)$ $(0 \le j \le D)$, where the matrix multiplication is entrywise. These notions are due to Delsarte [15], who introduced the P-polynomial property as an algebraic definition of association schemes generated by distance-regular graphs, and then defined Q-polynomial association schemes as the dual concept to P-polynomial association schemes.

Many important examples of P-polynomial association schemes, which arise from classical algebraic objects such as dual polar spaces and forms over finite fields, also possess the Q-polynomial property. Bannai and Ito [1] posed the following conjecture.

Conjecture 1.1. *For D large enough, a primitive association scheme of D classes is P-polynomial if and only if it is Q-polynomial.*

We are not aware of any progress towards its proof. The discovery of a feasible set of parameters of hypothetical counter-examples (see [30]) casts some doubt on the conjecture, and in the very least shows that this will likely be difficult to prove (see the next section for the definition of feasible parameter sets). Moreover, the problem of classification of association schemes which are both P- and Q-polynomial (i.e., Q-polynomial distance-regular graphs) is still open. We refer the reader to [13] for its current state.

Recall that, for a P-polynomial association scheme defined on a set X, its intersection numbers p^k_{ij} satisfy the *triangle inequality*: $p^k_{ij} = 0$ if $|i - j| > k$ or $i + j < k$, which naturally gives rise to a graph structure on X. Perhaps, due to the lack of such an intuitive combinatorial characterization, much less is known about Q-polynomial association schemes when the P-polynomial property is absent (which also indicates that there should be much more left to discover). To date, only few examples of Q-polynomial schemes are known which are neither P-polynomial nor duals of P-polynomial schemes [28] – most of them are imprimitive and related to combinatorial designs. The first infinite family of primitive Q-polynomial schemes that are not also P-polynomial was recently constructed in [31]. Due to Conjecture 1.1, it seems that the most promising area for constructing new examples of Q-polynomial association schemes which are not P-polynomial includes those with few classes, say, in the range $3 \leq D \leq 6$. The tables of feasible parameters of primitive 3-class Q-polynomial association schemes and 4- and 5-class Q-bipartite association schemes presented in Section 3 may serve as a source for new constructions.

We note that imprimitive Q-polynomial 3-class schemes are either Taylor graphs (see [5, pp. 4–6]) or linked systems of symmetric designs (see [27]). For current research on Q-antipodal 4- and 5-class association schemes, see [24, 25] and [11]. Due to this recent work on Q-antipodal schemes, the third author has focused only on the less studied primitive and Q-bipartite cases in his tables. We note the primitive case is far more computationally demanding than the Q-bipartite case, and this is the reason the class number in the tables does not go to 4 or 5.

The parameters of P-polynomial association schemes are restricted by a number of conditions implied by the triangle inequality. On the other hand, the Q-polynomial property allows us to consider *triple intersection numbers* with respect to some triples of vertices, which can be thought of as a generalization of intersection numbers to triples of starting vertices instead of pairs. This technique has been previously used by various researchers [8, 10, 17, 21, 22, 23, 36, 37], mostly to prove nonexistence of some strongly regular and distance-regular graphs with equality in the so-called Krein conditions, in which case combining the restrictions implied by the triangle inequality with triple intersection numbers seems the most fruitful. Yet, while calculating triple intersection numbers when the P-polynomial property is absent is harder, we managed to rule out a number of open cases from the tables. This includes a putative Q-polynomial association scheme on 91 vertices whose existence has been open since 1999 [12].

The paper is organized as follows. In Section 2, we recall the basic theory of association schemes and their triple intersection numbers. In Section 3, we comment on the tables of feasible parameters of Q-polynomial association schemes and how they were generated. In Section 4, we explain in detail the analysis of triple intersection numbers of Q-polynomial association schemes and prove nonexistence for many open cases from the tables. Finally, in Section 5, we discuss the generalization of triple intersection numbers to quadruples of vertices.

2 Preliminaries

In this section we prepare the notions needed in subsequent sections.

2.1 Association schemes

Let X be a finite set of vertices and $\{R_0, R_1, \ldots, R_D\}$ be a set of non-empty subsets of $X \times X$. Let A_i denote the adjacency matrix of the (di-)graph (X, R_i) ($0 \leq i \leq D$). The pair $(X, \{R_i\}_{i=0}^{D})$ is called a *(symmetric) association scheme* of D classes (or a *D-class scheme* for short) if the following conditions hold:

(1) $A_0 = I_{|X|}$, which is the identity matrix of size $|X|$,

(2) $\sum_{i=0}^{D} A_i = J_{|X|}$, which is the square all-one matrix of size $|X|$,

(3) $A_i^\top = A_i$ ($1 \leq i \leq D$),

(4) $A_i A_j = \sum_{k=0}^{D} p_{ij}^k A_k$, where p_{ij}^k are nonnegative integers ($0 \leq i, j \leq D$).

The nonnegative integers p_{ij}^k are called *intersection numbers*: for a pair of vertices $x, y \in X$ with $(x, y) \in R_k$ and integers i, j ($0 \leq i, j, k \leq D$), p_{ij}^k equals the number of vertices $z \in X$ such that $(x, z) \in R_i$, $(y, z) \in R_j$.

The vector space $\mathcal{A}$ over $\mathbb{R}$ spanned by the matrices A_i forms an algebra. Since $\mathcal{A}$ is commutative and semisimple, there exists a unique basis of $\mathcal{A}$ consisting of primitive idempotents $E_0 = \frac{1}{|X|} J_{|X|}, E_1, \dots, E_D$ (i.e., projectors onto the maximal common eigenspaces of $A_0, \dots, A_D$). Since the algebra $\mathcal{A}$ is closed under the entry-wise multiplication denoted by $\circ$, we define the *Krein parameters* q_{ij}^k ($0 \leq i, j, k \leq D$) by

$$E_i \circ E_j = \frac{1}{|X|} \sum_{k=0}^{D} q_{ij}^k E_k. \tag{2.1}$$

It is known that the Krein parameters are nonnegative real numbers (see [15, Lemma 2.4]). Since both $\{A_0, A_1, \dots, A_D\}$ and $\{E_0, E_1, \dots, E_D\}$ form bases of $\mathcal{A}$, there exists matrices $P = (P_{ij})_{i,j=0}^D$ and $Q = (Q_{ij})_{i,j=0}^D$ defined by

$$A_i = \sum_{j=0}^{D} P_{ji} E_j \quad \text{and} \quad E_i = \frac{1}{|X|} \sum_{j=0}^{D} Q_{ji} A_j. \tag{2.2}$$

The matrices P and Q are called the *first* and *second eigenmatrix* of $(X, \{R_i\}_{i=0}^D)$.

Let n_i, $0 \leq i \leq D$, denote the *valency* of the graph (X, R_i), and m_j, $0 \leq j \leq D$, denote the *multiplicity* of the eigenspace of $A_0, \dots, A_D$ corresponding to E_j. Note that $n_i = p_{ii}^0$, while $m_j = q_{jj}^0$.

For an association scheme $(X, \{R_i\}_{i=0}^D)$, an ordering of $A_1, \dots, A_D$ such that for each i ($0 \leq i \leq D$), there exists a polynomial $v_i(x)$ of degree i with $P_{ji} = v_i(P_{j1})$ ($0 \leq j \leq D$), is called a *P-polynomial ordering* of relations. An association scheme is said to be *P-polynomial* if it admits a P-polynomial ordering of relations. The notion of an association scheme together with a P-polynomial ordering of relations is equivalent to the notion of a *distance-regular graph* – such a graph has adjacency matrix A_1, and A_i ($0 \leq i \leq D$) is the adjacency matrix of its distance-i graph (i.e., $(x, y) \in R_i$ precisely when x and y are at distance i in the graph), and the number of classes equals the diameter of the graph. It is also known that an ordering of relations is P-polynomial if and only if the matrix of intersection numbers L_1, where $L_i := (p_{ij}^k)_{k,j=0}^D$ ($0 \leq i \leq D$), is a tridiagonal matrix with nonzero superdiagonal and subdiagonal [1, p. 189] – then $p_{ij}^k = 0$ holds whenever the triple (i, j, k) does not satisfy the triangle inequality (i.e., when $|i - j| < k$ or $i + j > k$). For a P-polynomial ordering of relations of an association scheme, set $a_i = p_{1,i}^i$, $b_i = p_{1,i+1}^i$, and $c_i = p_{1,i-1}^i$. These intersection numbers are usually gathered in the *intersection array* $\{b_0, b_1, \dots, b_{D-1}; c_1, c_2, \dots, c_D\}$, as the remaining intersection numbers can be computed from them (in particular, $a_i = b_0 - b_i - c_i$ for all i, where $b_D = c_0 = 0$). For an association scheme with a P-polynomial ordering of relations, the ordering $E_1, \dots, E_D$ is called the *natural ordering* of eigenspaces if $(P_{i1})_{i=0}^D$ is a decreasing sequence.

Dually, for an association scheme $(X, \{R_i\}_{i=0}^D)$, an ordering of $E_1, \dots, E_D$ such that for each i ($0 \leq i \leq D$), there exists a polynomial $v_i^*(x)$ of degree i with $Q_{ji} = v_i^*(Q_{j1})$ ($0 \leq j \leq D$), is called a *Q-polynomial ordering* of eigenspaces. An association scheme is said to be *Q-polynomial* if it admits a Q-polynomial ordering of eigenspaces. Similarly as before, it is known that an ordering of eigenspaces is Q-polynomial if and only if the matrix of Krein parameters L_1^*, where $L_i^* := (q_{ij}^k)_{k,j=0}^D$ $(0 \le i \le D)$, is a tridiagonal matrix with nonzero superdiagonal and subdiagonal [1, p. 193] – then $q_{ij}^k = 0$ holds whenever the triple (i, j, k) does not satisfy the triangle inequality. For a Q-polynomial ordering of eigenspaces, set $a_i^* = q_{1,i}^i$, $b_i^* = q_{1,i+1}^i$, and $c_i^* = q_{1,i-1}^i$. Again, these Krein parameters are usually gathered in the *Krein array* $\{b_0^*, b_1^*, \dots, b_{D-1}^*; c_1^*, c_2^*, \dots, c_D^*\}$ containing all the information needed to compute the remaining Krein parameters (in particular, we have $a_i^* = b_0^* - b_i^* - c_i^*$ for all i, where $b_D^* = c_0^* = 0$). For an association scheme with a Q-polynomial ordering of eigenspaces, the ordering $A_1, \dots, A_D$ is called the *natural ordering* of relations if $(Q_{i1})_{i=0}^D$ is a decreasing sequence. Unlike for the P-polynomial association schemes, there is no known general combinatorial characterization of Q-polynomial association schemes.

An association scheme is called *primitive* if all of $A_1, \dots, A_D$ are adjacency matrices of connected graphs. It is known that a distance-regular graph is imprimitive precisely when it is a cycle of composite length, an antipodal graph, or a bipartite graph (possibly more than one of these), see [5, Thm. 4.2.1]. The last two properties can be recognised from the intersection array as $b_i = c_{D-i}$ $(0 \le i \le D,\ i \ne \lfloor D/2 \rfloor)$ and $a_i = 0$ $(0 \le i \le D)$, respectively. We may define dual properties for a Q-polynomial association scheme – we say that it is *Q-antipodal* if $b_i^* = c_{D-i}^*$ $(0 \le i \le D,\ i \ne \lfloor D/2 \rfloor)$, and *$Q$-bipartite* if $a_i^* = 0$ $(0 \le i \le D)$. All imprimitive Q-polynomial association schemes are schemes of cycles of composite length, Q-antipodal or Q-bipartite (again, possibly more than one of these). The original classification theorem by Suzuki [34] allowed two more cases, which have however been ruled out later [9, 35]. An association scheme that is both P- and Q-polynomial is Q-antipodal if and only if it is bipartite, and is Q-bipartite if and only if it is antipodal.

A *formal dual* of an association scheme with first and second eigenmatrices P and Q is an association scheme such that, for some orderings of its relations and eigenspaces, its first and second eigenmatrices are Q and P, respectively. Note that this duality occurs on the level of parameters – an association scheme might have several formal duals, or none at all (we can speak of duality when there exists a regular abelian group of automorphisms, see [5, §2.10B]). An association scheme with $P = Q$ for some orderings of its relations and eigenspaces is called *formally self-dual*. For such orderings, $p_{ij}^k = q_{ij}^k$ $(0 \le i, j, k \le D)$ holds – in particular, a formally self-dual association scheme is P-polynomial if and only if it is Q-polynomial, and then its intersection array matches its Krein array.

Any imprimitive association scheme with two classes is both P- and Q-polynomial for either of the two orderings of relations and eigenspaces. The graph with adjacency matrix A_1 of such a scheme is said to be *strongly regular* (an *SRG* for short) with parameters (n, k, λ, μ), where $n = |X|$ is the number of vertices, $k = p_{11}^0$ is the valency of each vertex, and each two distinct vertices have precisely $\lambda = p_{11}^1$ common neighbours if they are adjacent, and $\mu = p_{11}^2$ common neighbours if they are not adjacent. In the sequel, we will identify P-polynomial association schemes with their corresponding strongly regular or distance-regular graphs.

By a *parameter set* of an association scheme, we mean the full set of p_{ij}^k, q_{ij}^k, P_{ij} and Q_{ij} described in this section, which are real numbers satisfying the identities in [5, Lemma 2.2.1, Lemma 2.3.1]. We say that a parameter set for an association scheme is *feasible* if it passes all known conditions for the existence of a corresponding association scheme. For distance-regular graphs, there are many known feasibility conditions, see [5, 13, 37]. For Q-polynomial association schemes, much less is known – see Section 3 for the feasibility conditions we have used.

2.2 Triple intersection numbers

For a triple of vertices $x, y, z \in X$ and integers i, j, k ($0 \leq i, j, k \leq D$) we denote by $\left[\begin{smallmatrix} x & y & z \\ i & j & k \end{smallmatrix}\right]$ (or simply $[i\ j\ k]$ when it is clear which triple (x, y, z) we have in mind) the number of vertices $w \in X$ such that $(x, w) \in R_i$, $(y, w) \in R_j$ and $(z, w) \in R_k$. We call these numbers *triple intersection numbers*.

Unlike the intersection numbers, the triple intersection numbers depend, in general, on the particular choice of (x, y, z). Nevertheless, for a fixed triple (x, y, z), we may write down a system of $3D^2$ linear Diophantine equations with D^3 triple intersection numbers as variables taking nonnegative values, thus relating them to the intersection numbers, cf. [22]:

$$\sum_{\ell=0}^{D} [\ell\ j\ k] = p^t_{jk}, \qquad \sum_{\ell=0}^{D} [i\ \ell\ k] = p^s_{ik}, \qquad \sum_{\ell=0}^{D} [i\ j\ \ell] = p^r_{ij}, \qquad (1 \leq i, j, k \leq D) \tag{2.3}$$

where $(x, y) \in R_r$, $(x, z) \in R_s$, $(y, z) \in R_t$, and

$$[0\ j\ k] = \delta_{jr}\delta_{ks}, \qquad [i\ 0\ k] = \delta_{ir}\delta_{kt}, \qquad [i\ j\ 0] = \delta_{is}\delta_{jt} \qquad (0 \leq i, j, k \leq D)$$

are constants. Note that the equations (2.3) are not all linearly independent, so the system is underdetermined in general when $D \geq 3$. Moreover, the following theorem sometimes gives additional equations.

Theorem 2.1 ([10, Theorem 3], cf. [7], [5, Theorem 2.3.2]). *Let $(X, \{R_i\}_{i=0}^D)$ be an association scheme of D classes with second eigenmatrix Q and Krein parameters q^t_{rs} $(0 \leq r, s, t \leq D)$. Then,*

$$q^t_{rs} = 0 \quad \iff \quad \sum_{i,j,k=0}^{D} Q_{ir} Q_{js} Q_{kt} \left[\begin{matrix} x & y & z \\ i & j & k \end{matrix}\right] = 0 \quad \textit{for all } x, y, z \in X.$$

Note that in a Q-polynomial association scheme, many Krein parameters are zero, and we can use Theorem 2.1 to obtain an equation for each of them.

3 Tables of feasible parameters for Q-polynomial association schemes

In this section we will describe the tables of feasible parameter sets for primitive 3-class Q-polynomial schemes and 4- and 5-class Q-bipartite schemes.

These tables were all completed using the MAGMA programming language (see [2]). Any parameter set meeting the following conditions was included in the table:

(1) The parameters satisfy the Q-polynomial condition.

(2) All p^k_{ij} are nonnegative integers, all valencies p^0_{jj} are positive.

(3) For each $j > 0$ we have np^0_{jj} is even (the handshaking lemma applied to the graph (X, R_j)).

(4) For each $j, k > 0$ we have $p^0_{jj}p^j_{jk}$ is even (the handshaking lemma applied to the subconstituent $(Y, \{(y,z) \in Y \times Y \mid (y,z) \in R_k\})$, where $x \in X$ and $Y = \{y \in X \mid (x,y) \in R_j\}$).

(5) For each $j > 0$ we have $np^0_{jj}p^j_{jj}$ is divisible by 6 (the number of triangles in each graph (X, R_j) is integral).

(6) All q^k_{ij} are nonnegative and for each j the multiplicity q^0_{jj} (i.e., the dimension of the E_j-eigenspace) is a positive integer (see [5, Proposition 2.2.2]).

(7) For all i, j we have $\sum_{q^k_{ij} \neq 0} m_k \leq m_i m_j$ if $i \neq j$ and $\sum_{q^k_{ii} \neq 0} m_k \leq \frac{m_i(m_i-1)}{2}$ (the absolute bound, see [5, Theorem 2.3.3] and the references therein).

(8) The splitting field is at most a degree 2 extension of the rationals (see [29]).

We note that there are many other conditions known for the special case of distance-regular graphs. It was decided to apply these conditions after the construction of the table, and those not meeting these extra conditions were labelled as nonexistent with a note as to the condition not met. We leave as an open question whether if any of these conditions could be generalized to any cases beyond distance-regular graphs; this (perhaps faint) hope is the main reason that they are included in the table.

We begin with the tables for Q-bipartite schemes, since this case is somewhat simpler than the primitive case. Schemes which are Q-bipartite are formally dual to bipartite distance-regular graphs. As a consequence, the formal dual to [5, Theorem 4.2.2(i)] gives the Krein array for the quotient scheme of a Q-bipartite scheme (see [27]). Namely, if the scheme has Krein array $\{b^*_0, b^*_1, \ldots, b^*_{D-1}; c^*_1, \ldots, c^*_D\}$ and $q^2_{11} = \mu^*$, then the Krein array of the quotient is

$$\left\{\frac{b^*_0 b^*_1}{\mu^*}, \frac{b^*_2 b^*_3}{\mu^*}, \ldots, \frac{b^*_{2t-2} b^*_{2t-1}}{\mu^*}; \frac{c^*_1 c^*_2}{\mu^*}, \frac{c^*_3 c^*_4}{\mu^*}, \ldots, \frac{c^*_{2t-1} c^*_{2t}}{\mu^*}\right\},$$

where $t = \lfloor \frac{D}{2} \rfloor$. Note that the quotient scheme has multiplicities $1, m_2, m_4, \ldots, m_{2t}$, from which it follows that the condition $\sum_{i=0}^{t} m_{2i} = \sum_{i=1}^{D-t} m_{2i-1}$ must be satisfied for a D-class Q-bipartite scheme.

When $D = 4, 5$ we obtain $t = 2$, so the quotient structure is a strongly regular graph. A database of strongly regular graph parameters up to 5000 vertices can be generated very quickly. From there, we can use the above condition on the multiplicities. The following proposition shows that the multiplicities determine all the parameters of the scheme.

Proposition 3.1. *A D-class Q-bipartite Q-polynomial association scheme with $D \in \{4, 5\}$ and multiplicities $1, m_1, m_2, \ldots, m_D$ has the Krein array*

$$\left\{m_1, m_1 - 1, \frac{m_1(m_2 - m_1 + 1)}{m_2}, \frac{m_1(m_3 - m_2 + m_1 - 1)}{m_3}; 1, \frac{m_1(m_1 - 1)}{m_2}, \frac{m_1(m_2 - m_1 + 1)}{m_3}, m_1\right\} \qquad (D = 4)$$

or

$$\left\{m_1, m_1-1, \frac{m_1(m_2-m_1+1)}{m_2}, \frac{m_1(m_3-m_2+m_1-1)}{m_3}, \frac{m_1(m_4-m_3+m_2-m_1+1)}{m_4}; 1, \frac{m_1(m_1-1)}{m_2}, \frac{m_1(m_2-m_1+1)}{m_3}, \frac{m_1(m_3-m_2+m_1-1)}{m_4}, m_1\right\} \qquad (D = 5).$$

Proof. Follows easily from the identities of [5, Lemma 2.3.1]. □

In the 4-class case, the parameters are entirely determined by the quotient's multiplicities (with a chosen Q-polynomial ordering) and m_1. To search, we take a strongly regular graph parameter set, choose one of two possible orderings for its multiplicities, calling its multiplicities $m_0 = 1, m_2, m_4$. From the absolute bound, we have $1 + m_2 \leq \frac{m_1(m_1+1)}{2}$, and from the positivity of c_2^* we have $\frac{(m_2-m_1+1)m_1}{m_2} \geq 0$. We then search over all $\sqrt{2(1+m_2)} - \frac{1}{2} \leq m_1 \leq m_2$, checking the conditions above. Given that we are iterating over SRG parameters together with two orderings and one integer, this search is very fast. The limitation of the table to 10000 vertices is mainly readability and practicality. The third author has unpublished tables (without comments or details) to 100000 vertices.

We note that Q-bipartite schemes with 5 classes are very similar, except we must iterate over both m_1 and m_3. Again, this is a very quick search, but the relative scarcity of 5-class parameter sets makes listing up to 50000 vertices, with annotation, manageable. The table actually goes slightly higher, to 50520 vertices, because of the existence of an example on that number of vertices.

The trickiest search was the primitive 3-class Q-polynomial parameter sets. In this case, there is no non-trivial quotient scheme to build on.

We use the following observation.

Theorem 3.2. *A primitive Q-polynomial association scheme of 3 classes must have a matrix L_i with 4 distinct eigenvalues.*

Proof. Assume not. If a matrix A_i has only two distinct eigenvalues, it is either complete, contradicting the fact that it is a 3-class scheme, or a disjoint union of more than one complete graph, contradicting the fact the scheme is primitive. Therefore, the only case left to consider is when A_1, A_2, A_3 all have three distinct eigenvalues, meaning the graphs are all strongly regular. A 3-class scheme where every non-trivial relation is strongly-regular is amorphic, see [19] and [14] for a definition and details on amorphic schemes. It was shown in [20] that amorphic schemes are formally self-dual. This implies that no column of Q has 4 distinct entries. Therefore, the second eigenmatrix Q cannot be generated by one column via polynomials, thus the scheme cannot be Q-polynomial. □

We note that, in fact, all Q-polynomial D-class schemes must have a relation with $D+1$ distinct eigenvalues. However, the above theorem and its proof is sufficient for our needs.

From this we conclude that each 3-class primitive Q-polynomial scheme has an adjacency matrix, which we label A_1, which has four distinct eigenvalues. Then the corresponding 4×4 intersection matrix L_1 has four distinct eigenvalues. From this matrix, all of the other parameters may be determined. In particular, from [5, Proposition 2.2.2], the left-eigenvectors of L_1, normalized so their leftmost entry is 1, must be the rows of P.

The rest of the parameters can be derived from the equations:

$$L_j = P^{-1} \operatorname{diag}(P_{0j}, P_{1j}, \ldots, P_{Dj})\, P,$$
$$L_j^* = Q^{-1} \operatorname{diag}(Q_{0j}, Q_{1j}, \ldots, Q_{Dj})\, Q.$$

However, checking the Q-polynomial condition is done before the computation of all parameters. We use the following theorem, a proof of which can be found in [30].

Theorem 3.3. *Let L_i be an intersection matrix of a D-class association scheme, where L_i has exactly $D+1$ distinct eigenvalues. Then the scheme is Q-polynomial if and only if there is a Vandermonde matrix U such that $U^{-1}L_iU = T$ where T is upper triangular.*

It is not hard to show that, without loss of generality, we can take T_{01} to be 0, implying that the first column of U is an eigenvector of L_1. We only then need to iterate over the three (nontrivial) eigenvectors of L_1 to check this condition. If the Q-polynomial condition is met, the rest of the parameters are computed and checked for the above conditions.

The schemes are then split into types depending on whether there is a strongly regular graph as a relation, and whether the splitting field is rational or not. These are split in this manner to aid in computation (following the list of types we give details on how these were used):

(1) Diameter 3 distance-regular graphs (DRG for short).

(2) No diameter 3 DRG, there is a strongly regular graph as a relation, the splitting field is the rational field.

(3) No diameter 3 DRG, there is a strongly regular graph as a relation, the splitting field is a degree-2 extension of the rational field.

(4) No diameter 3 DRG, there is no strongly regular graph as a relation, the splitting field is the rational field.

(5) No diameter 3 DRG, there is no strongly regular graph as a relation, the splitting field is a degree-2 extension of the rational field.

We note that we do not have any examples of primitive 3-class Q-polynomial schemes with an irrational splitting field, but there are open parameter sets of such (for example, see entry $\langle 216, 20\rangle$ in the third author's primitive 3-class table at [39]). It would be interesting to determine if these exist. We also point out that all the feasible parameter sets known to us have rational Krein parameters.

Type 1. For DRG's, we iterated over the number of vertices, intersection array and valencies. The order was n, $b_0 = n_1$, b_1, n_2 (noting n_2 is a divisor of n_1b_1), then b_2 (noting b_2 must be a multiple of $\frac{n_3}{\gcd(n_2,n_3)}$, where $n_3 = n - n_1 - n_2$), from which the rest could be determined.

When there is no DRG, it is tempting to try to formally dualize the above process. However, the Krein parameters of a scheme do not have to be integral, or even rational. For this reason, it seemed more advantageous to iterate over parameters that needed to be integral, namely the parameters p^k_{ij}. All arithmetic was done in MAGMA using the rational field, or a splitting field of a degree two irreducible polynomial over the rationals. Floating point arithmetic was avoided to minimize numerical errors.

For the rest of the types, L_1 and the valencies were iterated over. In particular, the parameters $a = p^1_{12}, b = p^1_{13}$ and $c = p^2_{13}$, together with n, n_1, n_2 determine the rest of L_1, noting that $a + b \le n_1 - 1$ and $c \le n_1 - \frac{n_1 a}{n_2}$. Any matrix without 4 distinct eigenvalues or with an irreducible cubic factor in its characteristic polynomial was discarded.

Types 2 and 3. For these types, we iterate over strongly regular graphs first, with parameters (n, k, λ, μ). We choose A_3 to be the adjacency matrix of the strongly regular graph relation, and L_1, L_2 to be fissions of the complement. Given this, the choice of n_1 will determine n_2. The possibilities for n_1 can be narrowed by observing that $p^1_{33} = \mu$, $n_3 = k$ and $p^1_{33}n_1 = p^3_{13}n_3$, implying that n_1 is divisible by $\frac{n_3}{\gcd(n_3,\mu)}$.

Using similar identities, we find b is divisible by $\frac{n_3}{\gcd(n_1,n_3)}$, a is divisible by $\frac{n_2}{\gcd(n_1,n_2)}$, and $c = \frac{n_1(n_3-b-\mu)}{n_2}$. After choosing these parameters all of L_1 follows.

Types 4 and 5. For these types, we know L_1, L_2 and L_3 all have 4 distinct eigenvalues. Therefore, we can assume n_1 is the smallest valency, and that $a \leq b$. Using a is divisible by $\frac{n_2}{\gcd(n_1,n_2)}$, b is divisible by $\frac{n_3}{\gcd(n_1,n_3)}$, and n_2 divides an_1, we choose n_1, a, n_2, b, c, from which the rest is determined. This is the slowest part of the search, and the reason the primitive table goes to 2800 vertices.

We close this section with some comments on the irrational splitting field types. The 2-class primitive Q-polynomial association schemes are equivalent to complementary pairs of primitive strongly regular graphs. The only case where strongly regular graphs have an irrational splitting field is the so-called “half-case”, when the graph has valency $\frac{n-1}{2}$. Such graphs do exist, for example the Paley graphs for non-square prime powers q with $q \equiv 1 \pmod 4$. We note that no primitive Q-polynomial schemes with more than 2 classes and a quadratic splitting field are known. All feasible parameter sets we know of are 3-class and have a strongly regular graph relation (type 3). The corresponding strongly regular graphs are also all unknown (see [4]). We have no feasible parameter set for type 5. However, one type 5 parameter set satisfied all criteria except the handshaking lemma. Given this, we expect feasible parameter sets for type 5 to exist, but may be quite large. This parameter set is listed below (including L_1^*, so it can be seen it is Q-polynomial, but not including the other L_i^* matrices), though this set is not included in the online table:

$$P=\begin{pmatrix}1 & 285 & 285 & 405\\ 1 & 19+8\sqrt{19} & -38+1\sqrt{19} & 18-9\sqrt{19}\\ 1 & -3 & 5 & -3\\ 1 & 19-8\sqrt{19} & -38-1\sqrt{19} & 18+9\sqrt{19}\end{pmatrix},\quad Q=\begin{pmatrix}1 & 60 & 855 & 60\\ 1 & \frac{76+32\sqrt{19}}{19} & -9 & \frac{76-32\sqrt{19}}{19}\\ 1 & \frac{-152+4\sqrt{19}}{19} & 15 & \frac{-152-4\sqrt{19}}{19}\\ 1 & \frac{8-4\sqrt{19}}{3} & \frac{-19}{3} & \frac{8+4\sqrt{19}}{3}\end{pmatrix},$$

$$L_1=\begin{pmatrix}0 & 285 & 0 & 0\\ 1 & 116 & 60 & 108\\ 0 & 60 & 90 & 135\\ 0 & 76 & 95 & 114\end{pmatrix},\quad L_2=\begin{pmatrix}0 & 0 & 285 & 0\\ 0 & 60 & 90 & 135\\ 1 & 90 & 59 & 135\\ 0 & 95 & 95 & 95\end{pmatrix},\quad L_3=\begin{pmatrix}0 & 0 & 0 & 405\\ 0 & 108 & 135 & 162\\ 0 & 135 & 135 & 135\\ 1 & 114 & 95 & 195\end{pmatrix},$$

$$L_1^* = \begin{pmatrix}0 & 60 & 0 & 0\\ 1 & \frac{400+32\sqrt{19}}{61} & \frac{3199-32\sqrt{19}}{61} & 0\\ 0 & \frac{12796-128\sqrt{19}}{3477} & \frac{181184+128\sqrt{19}}{3477} & \frac{80}{19}\\ 0 & 0 & 60 & 0\end{pmatrix}.$$

While feasible parameters may exist, the complete lack of examples elicits the following question:

Question 3.4. Do all 3-class primitive Q-polynomial schemes have a rational splitting field?

This is a special case of the so-called “Sensible Caveman” conjecture of William J. Martin:

Conjecture 3.5. *For Q-polynomial schemes of* 3 *or more classes that is not a polygon, if the scheme is primitive then its splitting field is rational.*

4 Nonexistence results

We derived our nonexistence results by analyzing triple intersection numbers of Q-polynomial association schemes. For some choice of relations R_r, R_s, R_t, the system of Diophantine equations derived from (2.3) and Theorem 2.1 may have multiple nonnegative solutions, each giving the possible values of the triple intersection numbers with respect to a triple (x, y, z) with $(x, y) \in R_r$, $(x, z) \in R_s$ and $(y, z) \in R_t$. However, in certain cases, there might be no nonnegative solutions – in this case, we may conclude that an association scheme with the given parameters does not exist.

Even when there are solutions for all choices of R_r, R_s, R_t such that $p^t_{rs} \neq 0$, sometimes nonexistence can be derived by other means. We may, for example, employ double counting.

Proposition 4.1. *Let x and y be vertices of an association scheme with $(x, y) \in R_r$. Suppose that $\alpha_1, \alpha_2, \dots, \alpha_m$ are distinct integers such that there are precisely κ_ℓ vertices z with $(x, z) \in R_s$, $(y, z) \in R_t$ and $\begin{bmatrix} x & y & z \\ i & j & k \end{bmatrix} = \alpha_\ell$ $(1 \leq \ell \leq m$, $\sum_{\ell=1}^m \kappa_\ell = p^r_{st})$, and $\beta_1, \beta_2, \dots, \beta_n$ are distinct integers such that there are precisely λ_ℓ vertices w with $(w, x) \in R_i$, $(w, y) \in R_j$ and $\begin{bmatrix} w & x & y \\ k & s & t \end{bmatrix} = \beta_\ell$ $(1 \leq \ell \leq n$, $\sum_{\ell=1}^n \lambda_\ell = p^r_{ij})$. Then,*

$$\sum_{\ell=1}^{m} \kappa_\ell \alpha_\ell = \sum_{\ell=1}^{n} \lambda_\ell \beta_\ell.$$

Proof. Count the number of pairs (w, z) with $(x, z) \in R_s$, $(y, z) \in R_t$, $(w, x) \in R_i$, $(w, y) \in R_j$ and $(w, z) \in R_k$. □

We consider the special case of Proposition 4.1 when a triple intersection number is zero for all triples of vertices in some given relations.

Corollary 4.2. *Suppose that for all vertices x, y, z of an association scheme with $(x, y) \in R_r$, $(x, z) \in R_s$, $(y, z) \in R_t$, $\begin{bmatrix} x & y & z \\ i & j & k \end{bmatrix} = 0$ holds. Then, $\begin{bmatrix} w & x & y \\ k & s & t \end{bmatrix} = 0$ holds for all vertices w, x, y with $(w, x) \in R_i$, $(w, y) \in R_j$ and $(x, y) \in R_r$.*

Proof. Apply Proposition 4.1 to all $(x, y) \in R_r$, with $m \leq 1$ and $\alpha_1 = 0$. Since β_ℓ and λ_ℓ $(1 \leq \ell \leq n)$ must be nonnegative, it follows that $n \leq 1$ and $\beta_1 = 0$. □

4.1 Computer search

The `sage-drg` package [38, 37] by the second author for the SageMath computer algebra system [32] has been used to perform computations of triple intersection numbers of Q-polynomial association schemes with Krein arrays that were marked as open in the tables of feasible parameter sets by the third author [39], see Section 3. The package was originally developed for the purposes of feasibility checking for intersection arrays of distance-regular graphs and included a routine to find general solutions to the system of equations for computing triple intersection numbers.

For the purposes of the current research, the package has been extended to support parameters of general association schemes, in particular, given as Krein arrays of Q-polynomial association schemes. Additionally, the package now supports generating integral solutions for systems of equations with constraints on the solutions (e.g., nonnegativity of triple intersection numbers) – these can also be added on-the-fly. The routine uses SageMath's mixed integer linear programming facilities, which support multiple solvers. We have used SageMath's default GLPK solver [26] and the CBC solver [16] in our computations – however, other solvers can also be used if they are available.

We have thus been able to implement an algorithm which tries to narrow down the possible solutions of the systems of equations for determining triple intersection numbers of an association scheme such that they satisfy Corollary 4.2, and conclude inequality if any of the systems of equations has no such feasible solutions.

(1) For each triple of relations (R_r, R_s, R_t) such that $p^t_{rs} > 0$, initialize an empty set of solutions, obtain a general (i.e., parametric) solution to the system of equations derived from (2.3) and Theorem 2.1, and initialize a generator of solutions with the constraint that the intersection numbers be integral and nonnegative. All generators (r, s, t) are initially marked as *active*, and all triple intersection numbers $(r, s, t; i, j, k)$ (representing $\left[\begin{smallmatrix} x & y & z \\ i & j & k \end{smallmatrix}\right]$ with $(x, y) \in R_r$, $(x, z) \in R_s$ and $(y, z) \in R_t$) are initially marked as *unknown*.

(2) For each *active* generator, generate one solution and add it to the corresponding set of solutions. If a generator does not return a new solution (i.e., it has exhausted all of them), then mark it as *inactive*.

(3) For each *inactive* generator, verify that the corresponding set of solutions is nonempty – otherwise, terminate and conclude nonexistence.

(4) Initialize an empty set Z.

(5) For each *unknown* triple intersection number $(r, s, t; i, j, k)$, mark it as *nonzero* if a solution has been found in which its value is not zero. If such a solution has not been found yet, make a copy of the generator (r, s, t) with the constraint that $(r, s, t; i, j, k)$ be nonzero, and generate one solution. If such a solution exists, add it to the set of solutions and mark $(r, s, t; i, j, k)$ as *nonzero*, otherwise mark $(r, s, t; i, j, k)$ as *zero* and add it to Z.

(6) If Z is empty, terminate without concluding nonexistence.

(7) For each triple intersection number $(r, s, t; i, j, k) \in Z$ and for each *nonzero* $(a, b, c; d, e, f) \in \{(r, i, j; s, t, k), (s, i, k; r, t, j), (t, j, k; r, s, i)\}$, remove all solutions from the corresponding set in which the value of the latter is nonzero, mark $(a, b, c; d, e, f)$ as *zero*, mark all *nonzero* $(a, b, c; \ell, m, n)$ with $(\ell, m, n) \neq (d, e, f)$ as *unknown*, and add a constraint that $(a, b, c; d, e, f)$ be zero to the generator (a, b, c) if it is *active*.

(8) Go to (2).

Note that generators and triple intersection numbers are considered equivalent under permutation of vertices, i.e., under actions $(r, s, t) \mapsto (r, s, t)^\pi$ and $(r, s, t; i, j, k) \mapsto ((r, s, t)^\pi; (i, j, k)^{\pi^{(1\ 3)}})$ for $\pi \in S_3$.

The above algorithm is available as the `check_quadruples` method of `sage-drg`'s `ASParameters` class. We ran it for all open cases in the tables from Section 3, and obtained 29 nonexistence results for primitive 3-class schemes, 92 nonexistence results for Q-bipartite 4-class schemes, and 11 nonexistence results for Q-bipartite 5-class schemes. The results are summarized in the following theorem and in the tables in Appendix A.

Theorem 4.3. *A Q-polynomial association scheme with Krein array listed in one of Tables 1, 2 and 3 does not exist.*

Proof. In all but two cases, it suffices to observe that for some triple of relations R_r, R_s, R_t, the system of equations derived from (2.3) and Theorem 2.1 has no integral nonnegative solutions – Tables 1 and 2 list the triple (r, s, t), while for all examples in Table 3, this is true for $(r, s, t) = (1, 1, 1)$. Note that the natural ordering of the relations is used.

Let us now consider the cases $\langle 225, 24\rangle$ and $\langle 1470, 104\rangle$ from Table 1. In the first case, the Krein array is $\{24, 20, 36/11; 1, 30/11, 24\}$. Such an association scheme has two Q-polynomial orderings, so we can augment the system of equations (2.3) with six equations derived from Theorem 2.1. Let w, x, y, z be vertices such that $(x, z), (y, z) \in R_1$ and $(w, x), (w, y), (x, y) \in R_3$. Since $p^3_{11} = 22$ and $p^3_{33} = 3$, such vertices must exist. We first compute the triple intersection numbers with respect to x, y, z. There are two integral nonnegative solutions, both having $[3\ 3\ 1] = 0$. On the other hand, there is a single solution for the triple intersection numbers with respect to w, x, y, giving $[1\ 1\ 1] = 3$. However, this contradicts Corollary 4.2, so such an association scheme does not exist.

In the second case, the Krein array is $\{104, 70, 25; 1, 7, 80\}$. Let w, x, y, z be vertices such that $(x, y), (x, z) \in R_1$, $(w, y), (y, z) \in R_2$ and $(w, x) \in R_3$. Since $p^1_{12} = 70$ and $p^1_{32} = 250$, such vertices must exist. There is a single solution for the triple intersection numbers with respect to x, y, z, giving $[3\ 2\ 3] = 0$. On the other hand, there are four solutions for the triple intersection numbers with respect to w, x, y, from which we obtain $[3\ 1\ 2] \in \{15, 16, 17, 18\}$. Again, this contradicts Corollary 4.2, so such an association scheme does not exist. This completes the proof. □

Remark 4.4. The `sage-drg` package repository provides two Jupyter notebooks containing the computation details in the proofs of nonexistence of two cases from Table 1:

- `QPoly-24-20-36_11-1-30_11-24.ipynb` for the case $\langle 225, 24\rangle$, and
- `DRG-104-70-25-1-7-80.ipynb` for the case $\langle 1470, 104\rangle$.

Remark 4.5. The parameter set $\langle 91, 12\rangle$ from Table 1 was listed by Van Dam [12] as the smallest feasible Q-polynomial parameter set for which no scheme is known. The next such open case is now the Krein array $\{14, 108/11, 15/4; 1, 24/11, 45/4\}$ for a primitive 3-class Q-polynomial association scheme with 99 vertices, which was also listed by Van Dam.

Since some of the parameters from Table 1 also admit a P-polynomial ordering, we can derive nonexistence of distance-regular graphs with certain intersection arrays. We have also found an intersection array for a primitive Q-polynomial distance-regular graph of diameter 4, which is listed in [5] and [3], and for which, to the best of our knowledge, nonexistence has not been previously known.

Theorem 4.6. *There is no distance-regular graph with intersection array*

$$\{83, 54, 21; 1, 6, 63\},$$
$$\{104, 70, 25; 1, 7, 80\},$$
$$\{195, 160, 28; 1, 20, 168\},$$
$$\{125, 108, 24; 1, 9, 75\},$$
$$\{126, 90, 10; 1, 6, 105\}, \textit{or}$$
$$\{203, 160, 34; 1, 16, 170\}.$$

Proof. The cases $\langle 1080, 83\rangle$, $\langle 1470, 104\rangle$, $\langle 2016, 195\rangle$ and $\langle 2640, 203\rangle$ from Table 1 are formally self-dual for the natural ordering of relations, while $\langle 2197, 126\rangle$ is formally self-dual with ordering of relations A_2, A_3, A_1 relative to the natural ordering. In each case, the corresponding association scheme is P-polynomial with intersection array equal to the Krein array. The case $\langle 2106, 65\rangle$ is not formally self-dual, yet the natural ordering of relations is P-polynomial with intersection array $\{125, 108, 24; 1, 9, 75\}$. In all of the above cases, Theorem 4.3 implies nonexistence of the corresponding association scheme, so a distance-regular graph with such an intersection array does not exist. □

Theorem 4.7. *There is no distance-regular graph with intersection array*

$$\{53, 40, 28, 16; 1, 4, 10, 28\}.$$

Proof. Consider a distance-regular graph with intersection array $\{53, 40, 28, 16; 1, 4, 10, 28\}$. Such a graph is formally self-dual for the natural ordering of eigenspaces and therefore also Q-polynomial. Augmenting the system of equations (2.3) with twelve equations derived from Theorem 2.1 gives a two parameter solution for triple intersection numbers with respect to three vertices mutually at distances $1, 3, 3$. However, it turns out that there is no integral solution, leading to nonexistence of the graph. □

Remark 4.8. The non-existence of a distance-regular graph with intersection array $\{53, 40, 28, 16; 1, 4, 10, 28\}$ also follows by applying the Terwilliger polynomial [17]. Recall that this polynomial, say $T_\Gamma(x)$, which depends only on the intersection numbers of a Q-polynomial distance-regular graph Γ and its Q-polynomial ordering, satisfies:

$$T_\Gamma(\eta) \geq 0, \tag{4.1}$$

where η is any non-principal eigenvalue of the local graph of an arbitrary vertex x of Γ. Furthermore, by [5, Theorem 4.4.3(i)], η satisfies

$$-1 - \frac{b_1}{\theta_1 + 1} \leq \eta \leq -1 - \frac{b_1}{\theta_D + 1}, \tag{4.2}$$

where $b_0 = \theta_0 > \theta_1 > \cdots > \theta_D$ are the $D + 1$ distinct eigenvalues of Γ.

For the above-mentioned intersection array, $T_\Gamma(x)$ is a polynomial of degree 4 with a negative leading term and the following roots: $-\frac{7}{3}$ $(= -1 - \frac{b_1}{\theta_1+1})$, $\frac{9-\sqrt{249}}{4} \approx -1.695$, $\frac{17}{3}$ $(= -1 - \frac{b_1}{\theta_D+1})$, $\frac{9+\sqrt{249}}{9} \approx 6.195$.

Thus, combining (4.1) and (4.2), we obtain

$$-\frac{7}{3} \leq \eta \leq \frac{9 - \sqrt{249}}{4} \text{ or } \eta = \frac{17}{3},$$

and one can finally obtain a contradiction as in [18, Claim 4.3].

4.2 Infinite families

The data from Tables 1, 2 and 3 allows us to look for infinite families of Krein arrays for which we can show nonexistence of corresponding Q-polynomial association schemes. We find three families, one for each number of classes.

The first family of Krein arrays is given by

$$\{2r^2 - 1, 2r^2 - 2, r^2 + 1; 1, 2, r^2 - 1\}. \tag{4.3}$$

This Krein array is feasible for all integers $r \geq 2$. A Q-polynomial association scheme with Krein array (4.3) has 3 classes and $4r^4$ vertices. Examples exist when r is a power of 2 – they are realized by duals of Kasami codes with minimum distance 5, see [5, §11.2].

Theorem 4.9. *A Q-polynomial association scheme with Krein array* (4.3) *and r odd does not exist.*

Proof. Consider a Q-polynomial association scheme with Krein array (4.3). Besides the Krein parameters failing the triangle inequality, q^1_{11} is also zero. Therefore, in order to compute triple intersection numbers, the system of equations (2.3) can be augmented with four equations derived from Theorem 2.1. We compute triple intersection numbers with respect to vertices x, y, z such that $(x, y), (x, z) \in R_1$ and $(y, z) \in R_2$. Since $p^2_{11} = r(r+2)(r^2-1)/4 > 0$, such vertices must exist. We obtain a four parameter solution (see the notebook `QPoly-d3-1param-odd.ipynb` on the `sage-drg` package repository for computation details). Then we may express

$$[1\ 2\ 3] = -\frac{r^4}{2} + 2r^2 + [1\ 3\ 1] + 3 \cdot [2\ 3\ 3] - [3\ 1\ 1] + 4 \cdot [3\ 3\ 3].$$

Clearly, the above triple intersection number can only be integral when r is even. Therefore, we conclude that a Q-polynomial association scheme with Krein array (4.3) and r odd does not exist. □

The next family is a two parameter family of Krein arrays

$$\{m, m-1, m(r^2-1)/r^2, m - r^2 + 1; 1, m/r^2, r^2 - 1, m\}. \tag{4.4}$$

This Krein array is feasible for all integers m and r such that $0 < 2(r^2-1) \leq m \leq r(r-1)(r+2)$ and $m(r+1)$ is even. A Q-polynomial association scheme with Krein array (4.4) is Q-bipartite with 4 classes and $2m^2$ vertices. One may take the Q-bipartite quotient of such a scheme (i.e., identify vertices in relation R_4) to obtain a strongly regular graph with parameters $(n, k, \lambda, \mu) = (m^2, (m-1)r^2, m + r^2(r^2-3), r^2(r^2-1))$, i.e., a pseudo-Latin square graph. Therefore, we say that a scheme with Krein array (4.4) is of *Latin square type*.

There are several examples of Q-polynomial association schemes with Krein array (4.4) for some r and m. For $(r, m) = (2, 6)$ and $(r, m) = (3, 16)$, this Krein array is realized by the schemes of shortest vectors of the E_6 lattice and an overlattice of the Barnes-Wall lattice in $\mathbb{R}^{16}$ [28], respectively. For $(r, m) = (2^{ij}, 2^{i(2j+1)})$, there are examples arising from duals of extended Kasami codes [5, §11.2] for each choice of positive integers i and j. In particular, the Krein array obtained by setting $i = j = 1$ uniquely determines the halved 8-cube.

In the case when r is a prime power and $m = r^3$, the formal dual of this parameter set (i.e., a distance-regular graph with the corresponding intersection array) is realized by a Pasechnik graph [6].

Theorem 4.10. *A Q-polynomial association scheme with Krein array* (4.4) *and m odd does not exist.*

Proof. Consider a Q-polynomial association scheme with Krein array (4.4). Since the scheme is Q-bipartite, we have $q_{ij}^k = 0$ whenever $i+j+k$ is odd or the triple (i,j,k) does not satisfy the triangle inequality. This allows us to augment the system of equations (2.3) with many equations derived from Theorem 2.1. We compute triple intersection numbers with respect to vertices x, y, z such that $(x,y), (x,z) \in R_1$ and $(y,z) \in R_2$. Since $p_{11}^2 = r^2(r^2-1)/2 > 0$, such vertices must exist. We obtain a one parameter solution (see the notebook `QPoly-d4-LS-odd.ipynb` on the `sage-drg` package repository for computation details) which allows us to express

$$[1\ 1\ 3] = r + \frac{r^2(1-r)}{2} - \frac{m}{2} + [1\ 1\ 1].$$

Clearly, the above triple intersection number can only be integral when m is even. Therefore, we conclude that a Q-polynomial association scheme with Krein array (4.4) and m odd does not exist. □

The last family is given by the Krein array

$$\left\{ \frac{r^2+1}{2}, \frac{r^2-1}{2}, \frac{(r^2+1)^2}{2r(r+1)}, \frac{(r-1)(r^2+1)}{4r}, \frac{r^2+1}{2r}; \right. \\ \left. 1, \frac{(r-1)(r^2+1)}{2r(r+1)}, \frac{(r+1)(r^2+1)}{4r}, \frac{(r-1)(r^2+1)}{2r}, \frac{r^2+1}{2} \right\}. \tag{4.5}$$

This Krein array is feasible for all odd $r \geq 5$. A Q-polynomial association scheme with Krein array (4.5) is Q-bipartite with 5 classes and $2(r+1)(r^2+1)$ vertices. One may take the Q-bipartite quotient of such a scheme to obtain a strongly regular graph with parameters $(n, k, \lambda, \mu) = ((r+1)(r^2+1), r(r+1), r-1, r+1)$ – these are precisely the parameters of collinearity graphs of generalized quadrangles $\mathrm{GQ}(r, r)$. The scheme also has a second Q-polynomial ordering of eigenspaces, namely the ordering E_5, E_2, E_3, E_4, E_1 relative to the ordering implied by the Krein array. For $r \equiv 1 \pmod 4$ a prime power, the Krein array (4.5) is realized by a scheme derived by Moorhouse and Williford [30] from a double cover of the $C_2(r)$ dual polar graph.

Theorem 4.11. *A Q-polynomial association scheme with Krein array* (4.5) *and* $r \equiv 3 \pmod 4$ *does not exist.*

Proof. Consider a Q-polynomial association scheme with Krein array (4.5). Since the scheme is Q-bipartite, we have $q_{ij}^k = 0$ whenever $i+j+k$ is odd or the triple (i,j,k) does not satisfy the triangle inequality. This allows us to augment the system of equations (2.3) with many equations derived from Theorem 2.1. We compute triple intersection numbers with respect to vertices x, y, z that are mutually in relation R_1. Since $p_{11}^1 = (r-1)/2 > 0$, such vertices must exist. We obtain a single solution (see the notebook `QPoly-d5-1param-3mod4.ipynb` on the `sage-drg` package repository for computation details) with

$$[1\ 1\ 1] = \frac{r-5}{4}.$$

Clearly, the above triple intersection number can only be integral when $r \equiv 1 \pmod 4$. Therefore, we conclude that a Q-polynomial association scheme with Krein array (4.5) and $r \equiv 3 \pmod 4$ does not exist. □

5 Quadruple intersection numbers

The argument of the proof of Theorem 2.1 ([5, Theorem 2.3.2]) can be further extended to s-tuples of vertices (see Remark (iii) in [5, §2.3]; cf. [34, Lemma 4(2)]). In particular, we may consider *quadruple intersection numbers* with respect to a quadruple of vertices $w, x, y, z \in X$. For integers h, i, j, k ($0 \leq h, i, j, k \leq D$), denote by $\begin{bmatrix} w & x & y & z \\ h & i & j & k \end{bmatrix}$ (or simply $[h\ i\ j\ k]$ when it is clear which quadruple (w, x, y, z) we have in mind) the number of vertices $u \in X$ such that $(u, w) \in R_h$, $(u, x) \in R_i$, $(u, y) \in R_j$, and $(u, z) \in R_k$.

For a fixed quadruple (w, x, y, z), one can obtain a system of linear Diophantine equations with quadruple intersection numbers as variables which relates them to the intersection numbers (or to the triple intersection numbers).

The following analogue of Theorem 2.1 allows us to obtain some additional equations.

Theorem 5.1. *Let $(X, \{R_i\}_{i=0}^D)$ be an association scheme of D classes with second eigenmatrix Q and Krein parameters q_{ij}^k ($0 \leq i, j, k \leq D$). Then, for fixed indices $\iota_1, \iota_2, \iota_3, \iota_4$ ($0 \leq \iota_1, \iota_2, \iota_3, \iota_4 \leq D$) and any permutation p, r, s, t of $\iota_1, \iota_2, \iota_3, \iota_4$,*

$$\sum_{\ell=0}^{D} q_{pr}^{\ell} q_{st}^{\ell} = 0 \quad \Longleftrightarrow \quad \sum_{h,i,j,k=0}^{D} Q_{hp} Q_{ir} Q_{js} Q_{kt} \begin{bmatrix} w & x & y & z \\ h & i & j & k \end{bmatrix} = 0 \quad \textit{for all } w, x, y, z \in X.$$

Proof. Since E_i is a symmetric idempotent matrix, one has

$$\sum_{w \in X} E_i(u, w) E_i(v, w) = E_i(u, v). \tag{5.1}$$

Let $\Sigma(M)$ denote the sum of all entries of a matrix M. Then, by (5.1),

$$\begin{aligned} \Sigma(E_p \circ E_r \circ E_s \circ E_t) &= \sum_{u,v \in X} E_p(u, v) E_r(u, v) E_s(u, v) E_t(u, v) \\ &= \sum_{w,x,y,z \in X} \left(\sum_{u \in X} E_p(u, w) E_r(u, x) E_s(u, y) E_t(u, z) \right) \cdot \\ &\qquad \left(\sum_{v \in X} E_p(v, w) E_r(v, x) E_s(v, y) E_t(v, z) \right) \\ &= \sum_{w,x,y,z \in X} \sigma(w, x, y, z)^2 \geq 0, \end{aligned} \tag{5.2}$$

where $\sigma(w, x, y, z) = \sum_{u \in X} E_p(u, w) E_r(u, x) E_s(u, y) E_t(u, z)$.

On the other hand, by (2.1),

$$\begin{aligned} |X|^2\, \Sigma(E_p \circ E_r \circ E_s \circ E_t) &= |X|^2\ \mathrm{Tr}((E_p \circ E_r) \cdot (E_s \circ E_t)) \\ &= \mathrm{Tr}\left(\left(\sum_{\ell=0}^{D} q_{pr}^{\ell} E_\ell \right) \cdot \left(\sum_{\ell=0}^{D} q_{st}^{\ell} E_\ell \right) \right) \\ &= \sum_{\ell=0}^{D} m_\ell q_{pr}^{\ell} q_{st}^{\ell}, \end{aligned} \tag{5.3}$$

where m_ℓ is the rank of E_ℓ (i.e., the multiplicity of the corresponding eigenspace), and by (2.2),

$$|X|^3\,\Sigma(E_p\circ E_r\circ E_s\circ E_t)=\frac{1}{|X|}\sum_{\ell=0}^{D}Q_{\ell p}Q_{\ell r}Q_{\ell s}Q_{\ell t}\Sigma(A_\ell)$$
$$=\sum_{\ell=0}^{D}n_\ell Q_{\ell p}Q_{\ell r}Q_{\ell s}Q_{\ell t},\tag{5.4}$$

where n_ℓ is the valency of (X,R_ℓ).

Since the multiplicities m_ℓ are positive numbers and the Krein parameters are non-negative numbers, by (5.2), (5.3), (5.4), we have $\Sigma(E_p\circ E_r\circ E_s\circ E_t)=0$ if and only if $q^\ell_{pr}q^\ell_{st}=0$ (with fixed p,r,s,t) for all $\ell=0,\dots,D$. In this case, we have $\sigma(w,x,y,z)=0$ for all quadruples (w,x,y,z), which implies

$$0=|X|^4\,\sigma(w,x,y,z)=|X|^4\sum_{u\in X}E_p(u,w)E_r(u,x)E_s(u,y)E_t(u,z)$$
$$=\sum_{h,i,j,k=0}^{D}Q_{hp}Q_{ir}Q_{js}Q_{kt}\begin{bmatrix}w&x&y&z\\h&i&j&k\end{bmatrix},$$

which completes the proof. □

The condition of Theorem 5.1 is satisfied when, for example, an association scheme is Q-bipartite, i.e., $q^k_{ij}=0$ whenever $i+j+k$ is odd (take $p+r$ and $s+t$ of different parity).

Suda [33] lists several families of association schemes which are known to be *triply regular*, i.e., their triple intersection numbers $\begin{bmatrix}x&y&z\\i&j&k\end{bmatrix}$ only depend on i,j,k and the mutual distances between x,y,z, and not on the choices of the vertices themselves:

- strongly regular graphs with $q^1_{11}=0$ (cf. [8]),
- Taylor graphs (antipodal Q-bipartite schemes of 3 classes),
- linked systems of symmetric designs (certain Q-antipodal schemes of 3 classes) with $a^*_1=0$,
- tight spherical 7-designs (certain Q-bipartite schemes of 4 classes), and
- collections of real mutually unbiased bases (Q-antipodal Q-bipartite schemes of 4 classes).

Schemes belonging to the above families seem natural candidates for the computations of their quadruple intersection numbers. However, the condition of Theorem 5.1 is never satisfied for primitive strongly regular graphs, while for Taylor graphs the obtained equations do not give any information that could not be obtained through relating the quadruple intersection numbers to the triple intersection numbers. This was also the case for the examples of triply regular linked systems of symmetric designs that we have checked. However, in the cases of tight spherical 7-designs and mutually unbiased bases, we do get new restrictions on quadruple intersection numbers. So far, we have not succeeded in using this new information for either new constructions or proofs of nonexistence.

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Appendix A Tables of nonexistence results

Here, we give the tables of nonexistence results obtained by running the algorithm from Subsection 4.1 on the open cases in the tables from Section 3. Tables 1, 2 and 3 give nonexistence results for Q-polynomial schemes which are primitive of 3 classes, and Q-bipartite (but not Q-antipodal) of 4 and 5 classes, respectively.

Label	Krein array	DRG	Nonexistence	Family
$\langle 91, 12\rangle$	$\{12, \frac{338}{35}, \frac{39}{25}; 1, \frac{312}{175}, \frac{39}{5}\}$		$(3,3,3)$	
$\langle 225, 24\rangle$	$\{24, 20, \frac{36}{11}; 1, \frac{30}{11}, 24\}$		$(3,1,1;3,3,1)$	
$\langle 324, 17\rangle$	$\{17, 16, 10; 1, 2, 8\}$		$(1,1,2)$	(4.3)
$\langle 324, 19\rangle$	$\{19, \frac{128}{9}, 10; 1, \frac{16}{9}, 10\}$		$(1,1,3)$	
$\langle 441, 20\rangle$	$\{20, \frac{378}{25}, 12; 1, \frac{42}{25}, 9\}$		$(1,1,3)$	
$\langle 540, 33\rangle$	$\{33, 20, \frac{63}{5}; 1, \frac{12}{5}, 15\}$		$(1,1,3)$	
$\langle 540, 35\rangle$	$\{35, \frac{243}{10}, \frac{27}{2}; 1, \frac{27}{10}, \frac{45}{2}\}$		$(1,1,3)$	
$\langle 576, 23\rangle$	$\{23, \frac{432}{25}, 15; 1, \frac{48}{25}, 9\}$		$(1,1,3)$	
$\langle 729, 26\rangle$	$\{26, \frac{486}{25}, 18; 1, \frac{54}{25}, 9\}$		$(1,1,3)$	
$\langle 1000, 37\rangle$	$\{37, 24, 14; 1, 2, 12\}$		$(1,1,3)$	
$\langle 1015, 28\rangle$	$\{28, \frac{2523}{130}, \frac{4263}{338}; 1, \frac{1218}{845}, \frac{203}{26}\}$		$(1,1,3)$	
$\langle 1080, 83\rangle$	$\{83, 54, 21; 1, 6, 63\}$	FSD	$(1,1,2)$	
$\langle 1134, 49\rangle$	$\{49, 48, \frac{644}{75}; 1, \frac{196}{75}, 42\}$		$(1,1,1)$	
$\langle 1189, 40\rangle$	$\{40, \frac{5043}{203}, \frac{123}{7}; 1, \frac{615}{406}, \frac{164}{7}\}$		$(1,1,2)$	
$\langle 1470, 104\rangle$	$\{104, 70, 25; 1, 7, 80\}$	FSD	$(1,1,2;3,2,3)$	
$\langle 1548, 35\rangle$	$\{35, \frac{2187}{86}, \frac{45}{2}; 1, \frac{135}{86}, \frac{27}{2}\}$		$(1,1,3)$	
$\langle 1680, 69\rangle$a	$\{69, 42, 7; 1, 2, 63\}$		$(1,1,2)$	
$\langle 1702, 45\rangle$	$\{45, \frac{4761}{148}, \frac{115}{4}; 1, \frac{345}{148}, \frac{69}{4}\}$		$(1,1,2)$	
$\langle 1944, 29\rangle$	$\{29, 22, 25; 1, 2, 5\}$		$(1,1,2)$	
$\langle 2016, 195\rangle$	$\{195, 160, 28; 1, 20, 168\}$	FSD	$(1,2,2)$	
$\langle 2106, 65\rangle$	$\{65, 64, \frac{676}{25}; 1, \frac{104}{25}, 26\}$	{125, 108, 24; 1, 9, 75}	$(1,1,1)$	
$\langle 2185, 114\rangle$	$\{114, \frac{4761}{65}, \frac{58121}{1521}; 1, \frac{11799}{1690}, \frac{6118}{117}\}$		$(1,1,3)$	
$\langle 2197, 36\rangle$	$\{36, \frac{45}{2}, \frac{45}{2}; 1, \frac{3}{2}, \frac{15}{2}\}$		$(1,1,3)$	
$\langle 2197, 126\rangle$	$\{126, 90, 10; 1, 6, 105\}$	FSD (0231)	$(2,2,3)$	
$\langle 2304, 47\rangle$	$\{47, \frac{135}{4}, 33; 1, \frac{9}{4}, 15\}$		$(1,1,3)$	
$\langle 2376, 95\rangle$	$\{95, 63, 12; 1, 3, 84\}$		$(1,1,3)$	
$\langle 2401, 48\rangle$	$\{48, 30, 29; 1, \frac{3}{2}, 20\}$		$(1,1,2)$	
$\langle 2500, 49\rangle$a	$\{49, 48, 26; 1, 2, 24\}$		$(1,1,2)$	(4.3)
$\langle 2640, 203\rangle$	$\{203, 160, 34; 1, 16, 170\}$	FSD	$(1,2,2)$	

Table 1: Nonexistence results for feasible Krein arrays of primitive 3-class Q-polynomial association schemes on up to 2800 vertices. For P-polynomial parameters (for the natural ordering of relations, unless otherwise indicated), the DRG column indicates whether the parameters are formally self-dual (FSD), or the intersection array is given. The Nonexistence column gives either the triple of relation indices for which there is no solution for triple intersection numbers, or the 6-tuple of relation indices $(r, s, t; i, j, k)$ for which Corollary 4.2 is not satisfied. The Family column specifies the infinite family from Subsection 4.2 that the parameter set is part of.

Label	Krein array	Nonexistence	Family
$\langle 200, 12\rangle$	$\{12, 11, \frac{256}{25}, \frac{36}{11}; 1, \frac{44}{25}, \frac{96}{11}, 12\}$	$(1, 1, 2)$	
$\langle 462, 21\rangle$	$\{21, 20, \frac{196}{11}, \frac{49}{5}; 1, \frac{35}{11}, \frac{56}{5}, 21\}$	$(1, 1, 2)$	
$\langle 486, 45\rangle$	$\{45, 44, 36, 5; 1, 9, 40, 45\}$	$(1, 1, 2)$	
$\langle 578, 17\rangle$	$\{17, 16, \frac{136}{9}, 9; 1, \frac{17}{9}, 8, 17\}$	$(1, 1, 2)$	(4.4)
$\langle 686, 28\rangle$	$\{28, 27, 25, 8; 1, 3, 20, 28\}$	$(1, 2, 2)$	
$\langle 702, 36\rangle$	$\{36, 35, \frac{405}{13}, \frac{72}{7}; 1, \frac{63}{13}, \frac{180}{7}, 36\}$	$(1, 2, 2)$	
$\langle 722, 19\rangle$	$\{19, 18, \frac{152}{9}, 11; 1, \frac{19}{9}, 8, 19\}$	$(1, 1, 2)$	(4.4)
$\langle 882, 21\rangle$	$\{21, 20, \frac{56}{3}, 13; 1, \frac{7}{3}, 8, 21\}$	$(1, 1, 2)$	(4.4)
$\langle 990, 66\rangle$	$\{66, 65, \frac{847}{15}, \frac{88}{13}; 1, \frac{143}{15}, \frac{770}{13}, 66\}$	$(1, 2, 2)$	
$\langle 1014, 78\rangle$	$\{78, 77, 65, 8; 1, 13, 70, 78\}$	$(1, 2, 2)$	
$\langle 1058, 23\rangle$	$\{23, 22, \frac{184}{9}, 15; 1, \frac{23}{9}, 8, 23\}$	$(1, 1, 2)$	(4.4)
$\langle 1250, 25\rangle$	$\{25, 24, \frac{200}{9}, 17; 1, \frac{25}{9}, 8, 25\}$	$(1, 1, 2)$	(4.4)
$\langle 1458, 27\rangle$	$\{27, 26, 24, 19; 1, 3, 8, 27\}$	$(1, 1, 2)$	(4.4)
$\langle 1458, 36\rangle$	$\{36, 35, 33, 16; 1, 3, 20, 36\}$	$(1, 2, 2)$	
$\langle 1482, 38\rangle$	$\{38, 37, \frac{12635}{351}, \frac{76}{37}; 1, \frac{703}{351}, \frac{1330}{37}, 38\}$	$(1, 2, 2)$	
$\langle 1674, 45\rangle$	$\{45, 44, \frac{1296}{31}, \frac{135}{11}; 1, \frac{99}{31}, \frac{360}{11}, 45\}$	$(1, 1, 2)$	
$\langle 1682, 29\rangle$	$\{29, 28, \frac{232}{9}, 21; 1, \frac{29}{9}, 8, 29\}$	$(1, 1, 2)$	(4.4)
$\langle 1694, 55\rangle$	$\{55, 54, \frac{352}{7}, 15; 1, \frac{33}{7}, 40, 55\}$	$(1, 1, 2)$	
$\langle 1862, 21\rangle$	$\{21, 20, \frac{364}{19}, \frac{81}{5}; 1, \frac{35}{19}, \frac{24}{5}, 21\}$	$(1, 1, 2)$	
$\langle 2058, 49\rangle$	$\{49, 48, \frac{686}{15}, \frac{77}{5}; 1, \frac{49}{15}, \frac{168}{5}, 49\}$	$(1, 1, 2)$	
$\langle 2060, 50\rangle$	$\{50, 49, \frac{4800}{103}, \frac{110}{7}; 1, \frac{350}{103}, \frac{240}{7}, 50\}$	$(1, 1, 2)$	
$\langle 2394, 27\rangle$	$\{27, 26, \frac{3240}{133}, \frac{279}{13}; 1, \frac{351}{133}, \frac{72}{13}, 27\}$	$(1, 1, 2)$	
$\langle 2466, 36\rangle$	$\{36, 35, \frac{4617}{137}, \frac{144}{7}; 1, \frac{315}{137}, \frac{108}{7}, 36\}$	$(1, 2, 2)$	
$\langle 2550, 85\rangle$	$\{85, 84, \frac{1156}{15}, \frac{187}{7}; 1, \frac{119}{15}, \frac{408}{7}, 85\}$	$(1, 1, 2)$	
$\langle 2662, 121\rangle$	$\{121, 120, \frac{5324}{49}, \frac{77}{5}; 1, \frac{605}{49}, \frac{528}{5}, 121\}$	$(1, 1, 2)$	
$\langle 2706, 66\rangle$	$\{66, 65, \frac{2541}{41}, \frac{44}{3}; 1, \frac{165}{41}, \frac{154}{3}, 66\}$	$(1, 2, 2)$	
$\langle 2730, 78\rangle$	$\{78, 77, \frac{507}{7}, \frac{52}{3}; 1, \frac{39}{7}, \frac{182}{3}, 78\}$	$(1, 2, 2)$	
$\langle 2750, 25\rangle$	$\{25, 24, \frac{250}{11}, \frac{185}{9}; 1, \frac{25}{11}, \frac{40}{9}, 25\}$	$(1, 1, 2)$	
$\langle 2862, 53\rangle$	$\{53, 52, \frac{11236}{225}, \frac{265}{13}; 1, \frac{689}{225}, \frac{424}{13}, 53\}$	$(1, 1, 2)$	
$\langle 2890, 153\rangle$	$\{153, 152, 136, 9; 1, 17, 144, 153\}$	$(1, 1, 2)$	
$\langle 2926, 171\rangle$	$\{171, 170, \frac{11552}{77}, \frac{171}{17}; 1, \frac{1615}{77}, \frac{2736}{17}, 171\}$	$(1, 1, 2)$	
$\langle 2970, 54\rangle$	$\{54, 53, \frac{567}{11}, 12; 1, \frac{27}{11}, 42, 54\}$	$(1, 2, 2)$	
$\langle 3042, 65\rangle$	$\{65, 64, \frac{182}{3}, 25; 1, \frac{13}{3}, 40, 65\}$	$(1, 1, 2)$	
$\langle 3074, 106\rangle$	$\{106, 105, \frac{2809}{29}, \frac{212}{9}; 1, \frac{265}{29}, \frac{742}{9}, 106\}$	$(1, 2, 2)$	
$\langle 3174, 184\rangle$	$\{184, 183, 161, 16; 1, 23, 168, 184\}$	$(1, 2, 2)$	
$\langle 3250, 50\rangle$	$\{50, 49, \frac{625}{13}, \frac{100}{9}; 1, \frac{25}{13}, \frac{350}{9}, 50\}$	$(1, 2, 2)$	
$\langle 3402, 126\rangle$	$\{126, 125, \frac{343}{3}, 28; 1, \frac{35}{3}, 98, 126\}$	$(1, 2, 2)$	
$\langle 3498, 77\rangle$	$\{77, 76, \frac{3872}{53}, \frac{231}{19}; 1, \frac{209}{53}, \frac{1232}{19}, 77\}$	$(1, 1, 2)$	
$\langle 3610, 133\rangle$	$\{133, 132, \frac{608}{5}, 21; 1, \frac{57}{5}, 112, 133\}$	$(1, 1, 2)$	
$\langle 3726, 36\rangle$	$\{36, 35, \frac{783}{23}, 24; 1, \frac{45}{23}, 12, 36\}$	$(1, 2, 2)$	
$\langle 4070, 55\rangle$	$\{55, 54, \frac{1936}{37}, \frac{77}{3}; 1, \frac{99}{37}, \frac{88}{3}, 55\}$	$(1, 1, 2)$	
$\langle 4250, 119\rangle$	$\{119, 118, \frac{13872}{125}, \frac{1309}{59}; 1, \frac{1003}{125}, \frac{5712}{59}, 119\}$	$(1, 1, 2)$	
$\langle 4370, 190\rangle$	$\{190, 189, \frac{3971}{23}, \frac{76}{7}; 1, \frac{399}{23}, \frac{1254}{7}, 190\}$	$(1, 2, 2)$	
$\langle 4410, 210\rangle$	$\{210, 209, 189, 12; 1, 21, 198, 210\}$	$(1, 2, 2)$	
$\langle 4464, 24\rangle$	$\{24, 23, \frac{2048}{93}, \frac{488}{23}; 1, \frac{184}{93}, \frac{64}{23}, 24\}$	$(1, 1, 2)$	
$\langle 4526, 73\rangle$	$\{73, 72, \frac{10658}{155}, \frac{511}{15}; 1, \frac{657}{155}, \frac{584}{15}, 73\}$	$(1, 1, 2)$	
$\langle 4558, 86\rangle$	$\{86, 85, \frac{12943}{159}, \frac{1376}{51}; 1, \frac{731}{159}, \frac{3010}{51}, 86\}$	$(1, 2, 2)$	
$\langle 4590, 75\rangle$	$\{75, 74, \frac{1200}{17}, 35; 1, \frac{75}{17}, 40, 75\}$	$(1, 1, 2)$	

(Continued on next page.)

(Continued.)

Label	Krein array	Nonexistence	Family
$\langle 4758, 117\rangle$	$\{117, 116, \frac{6760}{61}, \frac{273}{29}; 1, \frac{377}{61}, \frac{3120}{29}, 117\}$	$(1, 1, 2)$	
$\langle 4802, 49\rangle$	$\{49, 48, \frac{1176}{25}, 25; 1, \frac{49}{25}, 24, 49\}$	$(1, 1, 2)$	(4.4)
$\langle 5046, 261\rangle$	$\{261, 260, 232, 21; 1, 29, 240, 261\}$	$(1, 1, 2)$	
$\langle 5202, 51\rangle$	$\{51, 50, \frac{1224}{25}, 27; 1, \frac{51}{25}, 24, 51\}$	$(1, 1, 2)$	(4.4)
$\langle 5480, 100\rangle$	$\{100, 99, \frac{12800}{137}, \frac{140}{3}; 1, \frac{900}{137}, \frac{160}{3}, 100\}$	$(1, 1, 2)$	
$\langle 5566, 66\rangle$	$\{66, 65, \frac{1463}{23}, 24; 1, \frac{55}{23}, 42, 66\}$	$(1, 2, 2)$	
$\langle 5590, 78\rangle$	$\{78, 77, \frac{3211}{43}, \frac{312}{11}; 1, \frac{143}{43}, \frac{546}{11}, 78\}$	$(1, 2, 2)$	
$\langle 5618, 53\rangle$	$\{53, 52, \frac{1272}{25}, 29; 1, \frac{53}{25}, 24, 53\}$	$(1, 1, 2)$	(4.4)
$\langle 5618, 106\rangle$	$\{106, 105, \frac{901}{9}, 36; 1, \frac{53}{9}, 70, 106\}$	$(1, 2, 2)$	
$\langle 5642, 91\rangle$	$\{91, 90, \frac{2704}{31}, \frac{65}{3}; 1, \frac{117}{31}, \frac{208}{3}, 91\}$	$(1, 1, 2)$	
$\langle 5670, 105\rangle$	$\{105, 104, 98, 49; 1, 7, 56, 105\}$	$(1, 1, 2)$	
$\langle 5670, 105\rangle$a	$\{105, 104, 100, 25; 1, 5, 80, 105\}$	$(1, 1, 2)$	
$\langle 6050, 55\rangle$	$\{55, 54, \frac{264}{5}, 31; 1, \frac{11}{5}, 24, 55\}$	$(1, 1, 2)$	(4.4)
$\langle 6278, 73\rangle$	$\{73, 72, \frac{21316}{301}, \frac{365}{21}; 1, \frac{657}{301}, \frac{1168}{21}, 73\}$	$(1, 1, 2)$	
$\langle 6358, 85\rangle$	$\{85, 84, \frac{884}{11}, 45; 1, \frac{51}{11}, 40, 85\}$	$(1, 1, 2)$	
$\langle 6422, 91\rangle$	$\{91, 90, \frac{1664}{19}, \frac{119}{5}; 1, \frac{65}{19}, \frac{336}{5}, 91\}$	$(1, 1, 2)$	
$\langle 6426, 147\rangle$	$\{147, 146, \frac{2352}{17}, 35; 1, \frac{147}{17}, 112, 147\}$	$(1, 1, 2)$	
$\langle 6450, 105\rangle$	$\{105, 104, \frac{4320}{43}, \frac{357}{13}; 1, \frac{195}{43}, \frac{1008}{13}, 105\}$	$(1, 1, 2)$	
$\langle 6498, 57\rangle$	$\{57, 56, \frac{1368}{25}, 33; 1, \frac{57}{25}, 24, 57\}$	$(1, 1, 2)$	(4.4)
$\langle 6962, 59\rangle$	$\{59, 58, \frac{1416}{25}, 35; 1, \frac{59}{25}, 24, 59\}$	$(1, 1, 2)$	(4.4)
$\langle 7210, 103\rangle$	$\{103, 102, \frac{84872}{875}, \frac{927}{17}; 1, \frac{5253}{875}, \frac{824}{17}, 103\}$	$(1, 1, 2)$	
$\langle 7442, 61\rangle$	$\{61, 60, \frac{1464}{25}, 37; 1, \frac{61}{25}, 24, 61\}$	$(1, 1, 2)$	(4.4)
$\langle 7854, 66\rangle$	$\{66, 65, \frac{1089}{17}, \frac{88}{3}; 1, \frac{33}{17}, \frac{110}{3}, 66\}$	$(1, 2, 2)$	
$\langle 7878, 78\rangle$	$\{78, 77, \frac{7605}{101}, \frac{104}{3}; 1, \frac{273}{101}, \frac{130}{3}, 78\}$	$(1, 2, 2)$	
$\langle 7906, 134\rangle$	$\{134, 133, \frac{22445}{177}, \frac{2948}{57}; 1, \frac{1273}{177}, \frac{4690}{57}, 134\}$	$(1, 2, 2)$	
$\langle 7938, 63\rangle$	$\{63, 62, \frac{1512}{25}, 39; 1, \frac{63}{25}, 24, 63\}$	$(1, 1, 2)$	(4.4)
$\langle 8120, 100\rangle$	$\{100, 99, \frac{19200}{203}, \frac{620}{11}; 1, \frac{1100}{203}, \frac{480}{11}, 100\}$	$(1, 1, 2)$	
$\langle 8190, 90\rangle$	$\{90, 89, \frac{1125}{13}, 40; 1, \frac{45}{13}, 50, 90\}$	$(1, 2, 2)$	
$\langle 8246, 217\rangle$	$\{217, 216, \frac{3844}{19}, \frac{155}{3}; 1, \frac{279}{19}, \frac{496}{3}, 217\}$	$(1, 1, 2)$	
$\langle 8450, 65\rangle$	$\{65, 64, \frac{312}{5}, 41; 1, \frac{13}{5}, 24, 65\}$	$(1, 1, 2)$	(4.4)
$\langle 8450, 78\rangle$	$\{78, 77, \frac{377}{5}, 36; 1, \frac{13}{5}, 42, 78\}$	$(1, 2, 2)$	
$\langle 8470, 88\rangle$	$\{88, 87, \frac{429}{5}, 16; 1, \frac{11}{5}, 72, 88\}$	$(1, 2, 2)$	
$\langle 8478, 27\rangle$	$\{27, 26, \frac{3888}{157}, \frac{327}{13}; 1, \frac{351}{157}, \frac{24}{13}, 27\}$	$(1, 1, 2)$	
$\langle 8750, 325\rangle$	$\{325, 324, 300, 13; 1, 25, 312, 325\}$	$(1, 1, 2)$	
$\langle 8758, 232\rangle$	$\{232, 231, \frac{32799}{151}, \frac{464}{11}; 1, \frac{2233}{151}, \frac{2088}{11}, 232\}$	$(1, 2, 2)$	
$\langle 8798, 106\rangle$	$\{106, 105, \frac{8427}{83}, \frac{424}{9}; 1, \frac{371}{83}, \frac{530}{9}, 106\}$	$(1, 2, 2)$	
$\langle 8802, 351\rangle$	$\{351, 350, \frac{52488}{163}, \frac{351}{25}; 1, \frac{4725}{163}, \frac{8424}{25}, 351\}$	$(1, 1, 2)$	
$\langle 8978, 67\rangle$	$\{67, 66, \frac{1608}{25}, 43; 1, \frac{67}{25}, 24, 67\}$	$(1, 1, 2)$	(4.4)
$\langle 9310, 105\rangle$	$\{105, 104, \frac{17500}{171}, \frac{165}{13}; 1, \frac{455}{171}, \frac{1200}{13}, 105\}$	$(1, 1, 2)$	
$\langle 9350, 153\rangle$	$\{153, 152, \frac{8092}{55}, \frac{459}{19}; 1, \frac{323}{55}, \frac{2448}{19}, 153\}$	$(1, 1, 2)$	
$\langle 9386, 171\rangle$	$\{171, 170, \frac{2128}{13}, 27; 1, \frac{95}{13}, 144, 171\}$	$(1, 1, 2)$	
$\langle 9522, 69\rangle$	$\{69, 68, \frac{1656}{25}, 45; 1, \frac{69}{25}, 24, 69\}$	$(1, 1, 2)$	(4.4)
$\langle 9522, 161\rangle$	$\{161, 160, \frac{460}{3}, 49; 1, \frac{23}{3}, 112, 161\}$	$(1, 1, 2)$	
$\langle 9702, 126\rangle$	$\{126, 125, \frac{1323}{11}, 56; 1, \frac{63}{11}, 70, 126\}$	$(1, 2, 2)$	

Table 2: Nonexistence results for feasible Krein arrays of Q-bipartite (but not Q-antipodal) 4-class Q-polynomial association schemes on up to 10000 vertices. The Nonexistence column gives either the triple of relation indices for which there is no solution for triple intersection numbers. The Family column specifies the infinite family from Subsection 4.2 that the parameter set is part of.

Label	Krein array	Family
$\langle 576, 21\rangle$	$\{21, 20, 18, \frac{21}{2}, \frac{27}{7}; 1, 3, \frac{21}{2}, \frac{120}{7}, 21\}$	
$\langle 800, 25\rangle$	$\{25, 24, \frac{625}{28}, \frac{75}{7}, \frac{25}{7}; 1, \frac{75}{28}, \frac{100}{7}, \frac{150}{7}, 25\}$	(4.5)
$\langle 2000, 25\rangle$	$\{25, 24, \frac{625}{27}, \frac{50}{3}, \frac{25}{9}; 1, \frac{50}{27}, \frac{25}{3}, \frac{200}{9}, 25\}$	
$\langle 2400, 22\rangle$	$\{22, 21, 20, \frac{88}{5}, \frac{32}{11}; 1, 2, \frac{22}{5}, \frac{210}{11}, 22\}$	
$\langle 2928, 61\rangle$	$\{61, 60, \frac{3721}{66}, \frac{305}{11}, \frac{61}{11}; 1, \frac{305}{66}, \frac{366}{11}, \frac{610}{11}, 61\}$	(4.5)
$\langle 7232, 113\rangle$	$\{113, 112, \frac{12769}{120}, \frac{791}{15}, \frac{113}{15}; 1, \frac{791}{120}, \frac{904}{15}, \frac{1582}{15}, 113\}$	(4.5)
$\langle 14480, 181\rangle$	$\{181, 180, \frac{32761}{190}, \frac{1629}{19}, \frac{181}{19}; 1, \frac{1629}{190}, \frac{1810}{19}, \frac{3258}{19}, 181\}$	(4.5)
$\langle 25440, 265\rangle$	$\{265, 264, \frac{70225}{276}, \frac{2915}{23}, \frac{265}{23}; 1, \frac{2915}{276}, \frac{3180}{23}, \frac{5830}{23}, 265\}$	(4.5)
$\langle 37752, 121\rangle$	$\{121, 120, \frac{14641}{125}, \frac{484}{5}, \frac{121}{25}; 1, \frac{484}{125}, \frac{121}{5}, \frac{2904}{25}, 121\}$	
$\langle 40880, 365\rangle$	$\{365, 364, \frac{133225}{378}, \frac{4745}{27}, \frac{365}{27}; 1, \frac{4745}{378}, \frac{5110}{27}, \frac{9490}{27}, 365\}$	(4.5)
$\langle 47040, 116\rangle$	$\{116, 115, 112, \frac{696}{7}, \frac{144}{29}; 1, 4, \frac{116}{7}, \frac{3220}{29}, 116\}$	

Table 3: Nonexistence results for feasible Krein arrays of Q-bipartite (but not Q-antipodal) 5-class Q-polynomial association schemes on up to 50000 vertices. In all cases, there is no solution for triple intersection numbers for a triple of vertices mutually in relation R_1. The Family column specifies the infinite family from Subsection 4.2 that the parameter set is part of.

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The enclaveless competition game

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Abstract

For a subset S of vertices in a graph G, a vertex $v \in S$ is an enclave of S if v and all of its neighbors are in S, where a neighbor of v is a vertex adjacent to v. A set S is enclaveless if it does not contain any enclaves. The enclaveless number $\Psi(G)$ of G is the maximum cardinality of an enclaveless set in G. As first observed in 1997 by Slater, if G is a graph with n vertices, then $\gamma(G) + \Psi(G) = n$ where $\gamma(G)$ is the well-studied domination number of G. In this paper, we continue the study of the competition-enclaveless game introduced in 2001 by Philips and Slater and defined as follows. Two players take turns in constructing a maximal enclaveless set S, where one player, Maximizer, tries to maximize $|S|$ and one player, Minimizer, tries to minimize $|S|$. The competition-enclaveless game number $\Psi_g^+(G)$ of G is the number of vertices played when Maximizer starts the game and both players play optimally. We study among other problems the conjecture that if G is an isolate-free graph of order n, then $\Psi_g^+(G) \geq \frac{1}{2}n$. We prove this conjecture for regular graphs and for claw-free graphs.

Keywords: Competition-enclaveless game, domination game.

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1 Introduction

In 2010 Brešar, Klavžar, and Rall [2] published the seminal paper on the domination game which belongs to the growing family of competitive optimization graph games. Domination games played on graphs are now very well studied in the literature. Indeed, the

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subsequent rapid growth by the scientific community of research on domination games played on graphs resulted in several dozen papers to date on the domination-type games (see, for example, [3, 4, 10, 11, 12, 14, 17]). A recent book entitled “Domination games played on graphs” by Brešar, Henning, Klavžar, and Rall [1] presents the state of the art results to date, and shows that the area is rich for further research. In this paper, we continue the study of domination games played on graphs, and investigate in more depth the competition-enclaveless game birthed by Philips and Slater [15, 16].

A *neighbor* of a vertex v in G is a vertex that is adjacent to v. A vertex *dominates* itself and its neighbors. A *dominating set* of a graph G is a set S of vertices of G such that every vertex in G is dominated by a vertex in S. The *domination number* of G, denoted $\gamma(G)$, is the minimum cardinality of a dominating set in G, while the *upper domination number* of G, denoted $\Gamma(G)$, is the maximum cardinality of a minimal dominating set in G. A minimal dominating set of cardinality $\Gamma(G)$ we call a Γ-*set of* G.

The *open neighborhood* of a vertex v in G is the set of neighbors of v, denoted $N_G(v)$. Thus, $N_G(v) = \{u \in V \mid uv \in E(G)\}$. The *closed neighborhood of* v is the set $N_G[v] = \{v\} \cup N_G(v)$. If the graph G is clear from context, we simply write $N(v)$ and $N[v]$ rather than $N_G(v)$ and $N_G[v]$, respectively.

As defined by Alan Goldman and introduced by Slater in [19], for a subset S of vertices in a graph G, a vertex $v \in S$ is an *enclave* of S if it and all of its neighbors are also in S; that is, if $N[v] \subseteq S$. A set S is *enclaveless* if it does not contain any enclaves. We note that a set S is a dominating set of a graph G if and only if the set $V(G) \setminus S$ is enclaveless. The *enclaveless number* of G, denoted $\Psi(G)$, is the maximum cardinality of an enclaveless set in G, and the *lower enclaveless number* of G, denoted by $\psi(G)$, is the minimum cardinality of a maximal enclaveless set. The domination and enclaveless numbers of a graph G are related by the following equations.

Observation 1.1. *If G is a graph of order n, then $\gamma(G) + \Psi(G) = n = \Gamma(G) + \psi(G)$.*

The domination game on a graph G consists of two players, *Dominator* and *Staller*, who take turns choosing a vertex from G. Each vertex chosen must dominate at least one vertex not dominated by the vertices previously chosen. Upon completion of the game, the set of chosen (played) vertices is a dominating set in G. The goal of Dominator is to end the game with a minimum number of vertices chosen, while Staller has the opposite goal and wishes to end the game with as many vertices chosen as possible.

The Dominator-start domination game and the Staller-start domination game is the domination game when Dominator and Staller, respectively, choose the first vertex. We refer to these simply as the D-game and S-game, respectively. The *game domination number*, $\gamma_{\rm g}(G)$, of G is the number of moves in a D-game when both players play optimal strategies consistent with their goals. The *Staller-start game domination number*, $\gamma'_{\rm g}(G)$, of G is defined analogously for the S-game.

Philips and Slater [15, 16] introduced what they called the *competition-enclaveless game*. The game is played by two players, Maximizer and Minimizer, on some graph G. They take turns in constructing a maximal enclaveless set S of G. That is, in each turn a player plays a vertex v that is not in the set S of the vertices already chosen and such that $S \cup \{v\}$ does not contain an enclave, until there is no such vertex. We call such a vertex a *playable vertex*. The goal of Maximizer is to make the final set S as large as possible and for Minimizer to make the final set S as small as possible.

The *competition-enclaveless game number*, or simply the *enclaveless game number*,

$\Psi_g^+(G)$, of G is the number of vertices chosen when Maximizer starts the game and both players play an optimal strategy according to the rules. The *Minimizer-start competition-enclaveless game number*, or simply the *Minimizer-start enclaveless game number*, $\Psi_g^-(G)$, of G is the number of vertices chosen when Minimizer starts the game and both players play an optimal strategy according to the rules. The competition-enclaveless game, which has been studied for example in [8, 9, 15, 16, 18], has not yet been explored in as much depth as the domination game. In this paper we continue the study of the competition-enclaveless game. Our main motivation for our study are the following conjectures that have yet to be settled, where an isolate-free graph is a graph that does not contain an isolated vertex.

Conjecture 1.2. *If G is an isolate-free graph of order n, then $\Psi_g^+(G) \geq \frac{1}{2}n$.*

Conjecture 1.2 was first posed as a question by Peter Slater to the 2nd author on 8th May 2015, and subsequently posed as a conjecture in [9]. We refer to Conjecture 1.2 for general isolate-free graphs as the **$\frac{1}{2}$-Enclaveless Game Conjecture**. We also pose the following conjecture for the Minimizer-start enclaveless game, where $\delta(G)$ denotes the minimum degree of the graph G.

Conjecture 1.3. *If G is a graph of order n with $\delta(G) \geq 2$, then $\Psi_g^-(G) \geq \frac{1}{2}n$.*

We proceed as follows. By Observation 1.1, if the domination number of a graph is known, then we immediately know the enclaveless number, and vice versa. In contrast, we show in Section 2 that the game domination number and the enclaveless game number are very different and are not related in the same way that the domination and enclaveless numbers are related. Indeed, knowledge of the game domination number gives no information of the enclaveless game number, and vice versa. We show that the domination game and the enclaveless game are intrinsically different. In Section 3, we present fundamental bounds on the enclaveless game number and the Minimizer-start enclaveless game number. In Sections 4 and 5, we show that the $\frac{1}{2}$-Enclaveless Game Conjecture holds for regular graphs and claw-free graphs, respectively. We use the standard notation $[k] = \{1, \ldots, k\}$.

2 Game domination versus enclaveless game

Although the domination and enclaveless numbers of a graph G of order n are related by the equation $\gamma(G) + \Psi(G) = n$ (see Observation 1.1), as remarked in [9] the competition-enclaveless game is very different from the domination game. To illustrate this, we present two simple examples showing that the sum $\gamma_{\rm g}(G) + \Psi_g^+(G)$ on the class of graphs of a fixed order can differ greatly even when restricted to trees.

For the first example, for $k \geq 3$, let G be a tree with exactly two non-leaf vertices both of which have k leaf neighbors, that is, G is a double star $S(k, k)$. In this case, $\Psi_g^+(G) = \Psi_g^-(G) = k + 1$ and $\gamma_{\rm g}(G) = 3$ and $\gamma_{\rm g}'(G) = 4$. Thus if G is a double star of order n, then $\gamma_{\rm g}(G) + \Psi_g^+(G) = \frac{1}{2}n + 3$.

For the second example, we consider the class of paths; P_n denotes the path on n vertices. For $n \geq 1$, Košmrlj [14] showed that $\gamma_{\rm g}'(P_n) = \lceil \frac{n}{2} \rceil$ and that $\gamma_{\rm g}(P_n) = \lceil \frac{n}{2} \rceil - 1$ if $n \equiv 3 \pmod 4$ and $\gamma_{\rm g}(P_n) = \lceil \frac{n}{2} \rceil$, otherwise. For $n \geq 2$, Phillips and Slater [16] showed that $\Psi_g^+(P_n) = \lfloor \frac{3n+1}{5} \rfloor$ and $\Psi_g^-(P_n) = \lfloor \frac{3n}{5} \rfloor$. Thus if G is a path P_n, then $\gamma_{\rm g}(G) + \Psi_g^+(G) \approx n + \frac{1}{10}n$.

The most important general fact in the domination game is the so-called *Continuation Principle*, which provides a much-used monotonicity property of the game domination number and allows us to assume that each optimal move of Dominator (and of Staller) is taken from a restricted subset of the unchosen vertices. Due to its importance in the domination game, we recall this well-studied Continuation Principle. A *partially dominated graph* is a graph together with a declaration that some vertices are already dominated and need not be dominated, but can be played, in the rest of the game. Given a graph G and a subset S of vertices of G, we denote by $G|S$ the partially dominated graph with S as the set of declared vertices already dominated. We use $\gamma_{\rm g}(G|S)$ (resp. $\gamma_{\rm g}'(G|S)$) to denote the number of moves remaining in the game on $G|S$ under optimal play when Dominator (resp. Staller) has the next move. We are now in a position to state the *Continuation Principle* presented by Kinnersley, West, and Zamani in [12, Lemma 2.1].

Lemma 2.1 (Continuation Principle)**.** *If G is a graph and $A, B \subseteq V(G)$ with $B \subseteq A$, then $\gamma_{\rm g}(G|A) \le \gamma_{\rm g}(G|B)$ and $\gamma_{\rm g}'(G|A) \le \gamma_{\rm g}'(G|B)$.*

The Continuation Principle is one of the most important proof techniques to obtain results on the domination game and its variants. It yields, for example, the following fundamental monotonicity property of the domination game; see [2, Theorem 6] and [12, Corollary 4.1].

Theorem 2.2. *The Dominator-start game domination number and the Staller-start game domination number can differ by at most* 1*, that is, for any graph G, we have*

$$|\gamma_{\rm g}(G) - \gamma_{\rm g}'(G)| \le 1.$$

The most significant difference between the competition-enclaveless game and the domination game is that the Continuation Principle holds for the domination game but does not hold for the competition-enclaveless game. If the Continuation Principle were to hold for the competition-enclaveless game, then this would imply that the Maximizer-start enclaveless game number and the Minimizer-start enclaveless game number can differ by at most 1. However, this is not the case, and these two game numbers can differ significantly. For example, if $n \ge 1$ and G is a star $K_{1,n}$, then $\Psi_g^+(G) = n$ while $\Psi_g^-(G) = 1$. Thus, the numbers $\Psi_g^+(G)$ and $\Psi_g^-(G)$ can vary greatly.

Without the powerful proof method of the Continuation Principle at our disposal, the competition-enclaveless game is raised to a greater level of difficulty than other domination games played on graphs. Indeed, this suggests that there may exist a graph G (or infinite classes of graphs) for which $\Psi_g^-(G) > \Psi_g^+(G)$ is possible, and such that the difference $\Psi_g^-(G) - \Psi_g^+(G)$ can possibly be made arbitrarily large. However, we are unable at this time to construct such examples, if they exist. Moreover, we are also unable to prove at this time that $\Psi_g^-(G) \le \Psi_g^+(G)$ is always true.

Another significant difference between the domination game and the competition-enclaveless game is that upon completion of the domination game, the set of played vertices is a dominating set although not necessarily a minimal dominating set, while upon completion of the competition-enclaveless game, the set of played vertices is always a maximal enclaveless set. Thus, the enclaveless game numbers of a graph G are always squeezed between the lower enclaveless number $\psi(G)$ of G and the enclaveless number $\Psi(G)$ of G. We state this formally as follows.

Observation 2.3. *If G is a graph, then*

$$\psi(G) \le \Psi_g^-(G) \le \Psi(G) \qquad \text{and} \qquad \psi(G) \le \Psi_g^+(G) \le \Psi(G).$$

A graph G is *well-dominated* if all the minimal dominating sets of G have the same cardinality. Examples of well-dominated graphs include, for example, the complete graph K_n, C_7, P_4, the corona of any graph, and the graph formed from two vertex disjoint cycles of order 5 joined by a single edge. Finbow, Hartnell and Nowakowski [7] characterized the well-dominated graphs having no 3-cycle nor 4-cycle. As observed earlier, upon completion of the enclaveless game, the set of played vertices is always a maximal enclaveless set. Hence, any sequence of legal moves by Maximizer and Minimizer (regardless of strategy) in the enclaveless game played in a well-dominated graph of order n will always lead to the game ending in $n - \gamma(G)$ moves. Thus as a consequence of Observation 2.3, we have the following interesting connection between the enclaveless game and the class of well-dominated graphs.

Observation 2.4. *If G is a well-dominated graph of order n, then*

$$\Psi_g^-(G) = \Psi_g^+(G) = n - \gamma(G).$$

It is well-known that if G is an isolate-free graph of order n, then $\gamma(G) \le \frac{1}{2}n$, implying by Observation 1.1 that $\Psi(G) = n - \gamma(G) \ge \frac{1}{2}n$. Hence one might think that $\gamma_g(G) \le \Psi_g^+(G)$ for such a graph G with no isolated vertex. We now provide an infinite class of graphs to show that the ratio γ_g/Ψ_g^+ of these two graphical invariants can be strictly larger than, and bounded away from, 1. The *corona* $\mathrm{cor}(G)$ of a graph G, also denoted $G \circ K_1$ in the literature, is the graph obtained from G by adding for each vertex v of G a new vertex v' and the edge vv' (and so, the vertex v' has degree 1 in $\mathrm{cor}(G)$). The edge vv' is called a *pendant edge*. We shall need the following 2014 result due to Košmrlj [13].

Theorem 2.5 ([13, Theorem 4.1])**.** *For $k \ge 1$, if $G = \mathrm{cor}(P_k)$, then $\gamma_{\rm g}(G) = k + \lceil \frac{k-7}{10} \rceil$.*

Let $\mathcal{G}$ be the (infinite) family of coronas of paths P_k where $k \ge 8$ and $k \bmod 10 \in \{8, 9\}$, that is,

$$\mathcal{G} = \{\mathrm{cor}(P_k) : k \bmod 10 \in \{8, 9\}\}.$$

As a consequence of Theorem 2.5, we have the following result.

Theorem 2.6. *For every graph $G \in \mathcal{G}$, we have*

$$\frac{\gamma_g(G)}{\Psi_g^+(G)} > \frac{11}{10}.$$

Proof. Let $G \in \mathcal{G}$, and so $G = \mathrm{cor}(P_k)$ for some positive integer k where $k \bmod 10 \in \{8, 9\}$. Every minimal dominating set of G has cardinality k, which implies by Observation 1.1 that every maximal enclaveless set of G also has cardinality k; that is, $\psi(G) = \Psi(G) = k$ where we recall that $\psi(G)$ denotes the cardinality of the smallest maximal enclaveless set in G and $\Psi(G)$ is the cardinality of a largest enclaveless set in G. Hence by Observation 2.3, $\Psi_g^+(G) = k$. Consequently, by Theorem 2.5 and since $k \bmod 10 \in \{8, 9\}$ we have

$$\frac{\gamma_g(G)}{\Psi_g^+(G)} = \frac{k + \lceil \frac{k-7}{10} \rceil}{k} > \frac{11}{10}. \qquad \square$$

Hence, by Theorem 2.6, the difference $\gamma_g(G) - \Psi_g^+(G)$ can be made arbitrarily large for an infinite family of graphs.

3 Fundamental bounds

In this section, we establish some fundamental bounds on the (Maximizer-start) enclaveless game number and the Minimizer-start enclaveless game number. We establish next a lower and upper bound on the enclaveless number of a graph in terms of the maximum degree and order of the graph.

Proposition 3.1. *If G is an isolate-free graph of order n with maximum degree $\Delta(G) = \Delta$, then*

$$\left(\frac{1}{\Delta+1}\right) n \le \psi(G) \le \Psi(G) \le \left(\frac{\Delta}{\Delta+1}\right) n.$$

Proof. If G is any graph of order n and maximum degree Δ, then $\gamma(G) \ge \frac{n}{\Delta+1}$. Hence, by Observation 1.1,

$$\Psi(G) = n - \gamma(G) \le n - \frac{n}{\Delta+1} = \left(\frac{\Delta}{\Delta+1}\right) n.$$

On the other hand, let D be a minimal dominating set of maximum cardinality, and so $|D| = \Gamma(G)$. Let $\overline{D} = V(G) \setminus D$, and so $|\overline{D}| = n - |D|$. Let ℓ be the number of edges between D and $\overline{D}$. Since G is an isolate-free graph and D is a minimal dominating set, every vertex in D has at least one neighbor in $\overline{D}$, and so $\ell \ge |D|$. Since G has maximum degree Δ, every vertex in $\overline{D}$ has at most Δ neighbors in D, and so $\ell \le \Delta \cdot |\overline{D}| = \Delta(n - |D|)$. Hence, $|D| \le \Delta(n - |D|)$, implying that $\Gamma(G) = |D| \le \Delta n/(\Delta + 1)$. Thus by Observation 1.1,

$$\psi(G) = n - \Gamma(G) \ge n - \left(\frac{\Delta}{\Delta+1}\right) n = \left(\frac{1}{\Delta+1}\right) n.$$

This completes the proof of Proposition 3.1. □

By Observation 2.3, the set of played vertices in either the Maximizer-start enclaveless game or the Minimizer-start enclaveless game is a maximal enclaveless set of G. Thus as an immediate consequence of Proposition 3.1, we have the following result.

Corollary 3.2. *If G is an isolate-free graph of order n with maximum degree $\Delta(G) = \Delta$, then*

$$\left(\frac{1}{\Delta+1}\right) n \le \Psi_g^-(G) \le \left(\frac{\Delta}{\Delta+1}\right) n \quad \textit{and} \quad \left(\frac{1}{\Delta+1}\right) n \le \Psi_g^+(G) \le \left(\frac{\Delta}{\Delta+1}\right) n.$$

We show that the upper bounds in Corollary 3.2 are realized for infinitely many connected graphs.

Proposition 3.3. *There exist infinitely many positive integers n along with a connected graph G of order n satisfying*

$$\Psi_g^-(G) = \Psi_g^+(G) = \left(\frac{\Delta(G)}{\Delta(G)+1}\right) n.$$

Proof. Let r be an integer such that $r \ge 4$ and let m be any positive integer. For each $i \in [m]$, let H_i be a graph obtained from a complete graph of order $r + 1$ by removing the edge x_iy_i for two distinguished vertices x_i and y_i. The graph F_m is obtained from the disjoint union of $H_1, \ldots, H_m$ by adding the edges y_ix_{i+1} for each $i \in [m]$ where the subscripts are computed modulo m. The vertices x_i and y_i are called connectors in F_m, and each of the $r-1$ vertices in the set $V(H_i) \setminus \{x_i, y_i\}$ is called a hidden vertex of H_i. Note that F_m is r-regular and has order $n = m(r+1)$.

We first show that $\Psi_g^-(F_m) = (\frac{r}{r+1})n$. Suppose the Minimizer-start enclaveless game is played on F_m. We provide a strategy for Maximizer that forces exactly rm vertices to be played. Maximizer's strategy is to make sure that all the connector vertices in the graph are played. If he can accomplish this, then exactly rm vertices will be played when the game ends because of the structure of F_m. Suppose that at some point in the game Minimizer plays a vertex from some H_j. If one of the connector vertices, say x_j, is playable, then Maximizer responds by playing x_j. If both connector vertices have already been played and some hidden vertex, say w, in H_j is playable, then Maximizer plays w. If no vertex of H_j is playable, then Maximizer plays a connector vertex from H_i for some $i \neq j$ if one is playable and otherwise plays any playable vertex. Since H_k contains at least 3 hidden vertices for each $k \in [m]$, it follows that Maximizer can guarantee that all the connector vertices are played by following this strategy. This implies that for each $i \in [m]$, exactly one hidden vertex of H_i is not played during the course of the game. That is, the set of played vertices has cardinality

$$rm = \left(\frac{r}{r+1}\right) m(r+1) = \left(\frac{\Delta(F_m)}{\Delta(F_m)+1}\right) n\,,$$

where we recall that $\Delta(F_m) = r$. Thus,

$$\Psi_g^-(F_m) \geq \left(\frac{\Delta(F_m)}{\Delta(F_m)+1}\right) n.$$

By Corollary 3.2,

$$\Psi_g^-((F_m)) \leq \left(\frac{\Delta(F_m)}{\Delta(F_m)+1}\right) n.$$

Consequently, $\Psi_g^-(F_m) = \left(\frac{\Delta(F_m)}{\Delta(F_m)+1}\right) n$.

If the Maximizer-start enclaveless game is played on F_m, then the same strategy as above for Maximizer forces rm vertices to be played (even with the relaxed condition that r be an integer larger than 2). Thus as before, $\Psi_g^+(F_m) = \left(\frac{\Delta(F_m)}{\Delta(F_m)+1}\right) n$. □

The lower bound in Corollary 3.2 on $\Psi_g^-(G)$ is achieved, for example, by taking $G = K_{1,\Delta}$ for any given $\Delta \geq 1$ in which case $\Psi_g^-(G) = 1 = (\frac{1}{\Delta+1})n$ where $n = n(G) = \Delta + 1$.

The lower bound in Corollary 3.2 on $\Psi_g^+(G)$ is trivially achieved when $\Delta = 1$, in which case G is the disjoint union of copies of K_2. However, as remarked in the introductory section, the main open problem in the competition-enclaveless game is the $\frac{1}{2}$-*Enclaveless Game Conjecture* (stated formally in Conjecture 1.2) that claims that if G is an isolate-free graph of order n, then $\Psi_g^+(G) \geq \frac{1}{2}n$. If the conjecture is true, then, from our earlier examples such as the double star, the bound is achieved for isolate-free graphs with arbitrarily large maximum degree Δ.

4 Regular graphs

In this section, we show that $\frac{1}{2}$-Enclaveless Game Conjecture (see Conjecture 1.2) holds for the class of regular graphs, as does Conjecture 1.3 for the Minimizer-start enclaveless game. For a set $S \subset V(G)$ of vertices in a graph G and a vertex $v \in S$, we define the *S-external private neighborhood* of a vertex v, abbreviated $\mathrm{epn}_G(v,S)$, as the set of all vertices outside S that are adjacent to v but to no other vertex of S; that is,

$$\mathrm{epn}_G(v,S) = \{w \in V(G) \setminus S : N_G(w) \cap S = \{v\}\}.$$

As remarked in the introduction, if the graph G is clear from the context, we omit the subscript G in the above definitions. We define an *S-external private neighbor* of v to be a vertex in $\mathrm{epn}(v,S)$.

We establish next a $\frac{1}{2}$-lower bound on $\Psi_g^+(G)$ and $\Psi_g^-(G)$ by forbidding induced stars of a certain size. We remark that the proof of the following result uses similar counting techniques to those employed by Southey and Henning in [20].

Proposition 4.1. *If G is a graph with order n, minimum degree δ and with no induced $K_{1,\delta+1}$, then $\psi(G) \ge \frac{1}{2}n$.*

Proof. Let D be an arbitrary minimal dominating set of G. Denote by D_1 the set of vertices in D that have a D-external private neighbor. That is, $D_1 = \{x \in D : \mathrm{epn}(x,D) \neq \emptyset\}$. In addition, let $D_2 = D \setminus D_1$. Since D is a minimal dominating set, the set D_2 consists of those vertices in D that are isolated in the subgraph $G[D]$ of G induced by D. Let

$$C_1 = \bigcup_{x \in D_1} \mathrm{epn}(x,D) \qquad \text{and} \qquad C_2 = V(G) \setminus (D \cup C_1).$$

We note that by definition, there are no edges in G joining a vertex in D_2 and a vertex in C_1. That is, each vertex in D_2 has at least δ neighbors in C_2. Since the set D_2 is independent and G has no induced $K_{1,\delta+1}$, each vertex of C_2 has at most δ neighbors in D_2. Denote by ℓ the number of edges of the form uv where $u \in D_2$ and $v \in C_2$. It now follows that $\delta|D_2| \le \ell \le \delta|C_2|$, that is, $|D_2| \le |C_2|$. Now,

$$|D| = |D_1| + |D_2| \le |C_1| + |C_2| = n - |D|,$$

which shows that $\Gamma(G) \le \frac{1}{2}n$. Hence by Observation 1.1, we have $\psi(G) \ge \frac{1}{2}n$. □

Observation 2.3 and Proposition 4.1 now yield the following result.

Corollary 4.2. *If G is a graph with order n, minimum degree δ and with no induced $K_{1,\delta+1}$, then $\Psi_g^+(G) \ge \frac{1}{2}n$ and $\Psi_g^-(G) \ge \frac{1}{2}n$.*

As a special case of Corollary 4.2, we have the desired $\frac{1}{2}$-lower bound on $\Psi_g^+(G)$ and $\Psi_g^-(G)$ for regular graphs G without isolated vertices.

Corollary 4.3. *If G is a regular graph of order n without isolated vertices, then $\Psi_g^+(G) \ge \frac{1}{2}n$ and $\Psi_g^-(G) \ge \frac{1}{2}n$.*

We remark that if G is a graph of order n that is a disjoint union of copies of K_2, then $\Psi_g^+(G) = \frac{1}{2}n$ and $\Psi_g^-(G) = \frac{1}{2}n$. The same conclusion holds if G is a disjoint union of copies of C_4. Hence, for $k \in \{1,2\}$ there are k-regular graphs G that achieve equality in the lower bound in Corollary 4.3. However, it remains an open problem to characterize the graphs achieving equality in Corollary 4.3 for each value of $k \ge 3$.

5 Claw-free graphs

A graph is *claw-free* if it does not contain the star $K_{1,3}$ as an induced subgraph. In this section, we show that $\frac{1}{2}$-Enclaveless Game Conjecture (see Conjecture 1.2) holds for the class of claw-free graphs with no isolated vertex, as does Conjecture 1.3 for the Minimizer-start enclaveless game. For this purpose, we recall the definition of an irredundant set. For a set S of vertices in a graph G and a vertex $v \in S$, the *S-private neighborhood* of v is the set

$$\mathrm{pn}_G[v, S] = \{w \in V : N[w] \cap S = \{v\}\}.$$

If the graph G is clear from context, we simply write $\mathrm{pn}[v, S]$ rather than $\mathrm{pn}_G[v, S]$. We note that $\mathrm{epn}(v, S) \subseteq \mathrm{pn}[v, S] \subseteq \mathrm{epn}(v, S) \cup \{v\}$ and $v \in \mathrm{pn}[v, S]$ if and only if v is isolated in $G[S]$. A vertex in the set $\mathrm{pn}[v, S]$ is called an *S-private neighbor* of v. The set S is an *irredundant set* if every vertex of S has an S-private neighbor. The *upper irredundance number* $\mathrm{IR}(G)$ is the maximum cardinality of an irredundant set in G.

The *independence number* $\alpha(G)$ of G is the maximal cardinality of an independent set of vertices in G. An independent set of vertices of G of cardinality $\alpha(G)$ is called an *α-set of G*. Every maximum independent set in a graph is a minimal dominating set, and every minimal dominating set is an irredundant set. Hence we have the following inequality chain.

Observation 5.1 ([5])**.** *For every graph G, we have $\alpha(G) \le \Gamma(G) \le \mathrm{IR}(G)$.*

The inequality chain in Observation 5.1 is part of the canonical domination chain which was first observed by Cockayne, Hedetniemi, and Miller [5] in 1978.

In 2004, Favaron [6] established the following upper bound on the upper irredundance number of a claw-free graph.

Theorem 5.2 ([6])**.** *If G is a connected, claw-free graph of order n, then $\mathrm{IR}(G) \le \frac{1}{2}(n+1)$. Moreover, if $\mathrm{IR}(G) = \frac{1}{2}(n+1)$, then $\alpha(G) = \Gamma(G) = \mathrm{IR}(G)$.*

In addition, she proved the following stronger upper bound for the upper irredundance number of claw-free graphs with minimum degree at least 2.

Corollary 5.3 ([6])**.** *If G is a connected, claw-free graph of order n and minimum degree at least 2, then $\mathrm{IR}(G) \le \frac{1}{2}n$.*

Suppose that G is a claw-free graph of order n and minimum degree $\delta \ge 2$. By Corollary 5.3, $\mathrm{IR}(G) \le \frac{1}{2}n$, and thus by Observations 1.1 and 5.1, we have

$$\psi(G) = n - \Gamma(G) \ge n - \mathrm{IR}(G) \ge n - \frac{1}{2}n = \frac{1}{2}n.$$

By Observation 2.3, we therefore have the following result.

Theorem 5.4. *If G is a claw-free graph of order n and $\delta(G) \ge 2$, then*

$$\Psi_g^+(G) \ge \frac{1}{2}n \qquad \textit{and} \qquad \Psi_g^-(G) \ge \frac{1}{2}n.$$

By Theorem 5.4, we note that Conjecture 1.3 holds for connected claw-free graphs. In order to prove that Conjecture 1.2 holds for connected claw-free graphs, we need the characterization due to Favaron [6] of the graphs achieving equality in the bound of Theorem 5.2. For this purpose, we recall that a vertex v of a graph G is a *simplicial vertex* if its open neighborhood $N(v)$ induces a complete subgraph of G. A *clique* of a graph G is a maximal complete subgraph of G. The *clique graph* of G has the set of cliques of G as its vertex set, and two vertices in the clique graph are adjacent if and only if they intersect as cliques of G. A *non-trivial tree* is a tree of order at least 2.

Favaron [6] defined the family $\mathcal{F}$ of claw-free graphs G as follows. Let $T_1, \dots, T_q$ be $q \geq 1$ non-trivial trees. Let L_i be the line graph of the corona $\mathrm{cor}(T_i)$ of the tree T_i for $i \in [q]$. If $q = 1$, let $G = L_1$. If $q \geq 2$, let G be the graph constructed from the line graphs $L_1, L_2, \dots, L_q$ by choosing $q - 1$ pairs $\{x_{ij}, x_{ji}\}$ such that the following holds.

- x_{ij} and x_{ji} are simplicial vertices of L_i and L_j, respectively, where $i \neq j$.
- The $2(q-1)$ vertices from the $q - 1$ pairs $\{x_{ij}, x_{ji}\}$ are all distinct vertices.
- Contracting each pair of vertices x_{ij} and x_{ji} into one common vertex c_{ij} results in a graph whose clique graph is a tree.

To illustrate the above construction of a graph G in the family $\mathcal{F}$ consider, for example, such a construction when $q = 3$ and the trees T_1, T_2, T_3 are given in Figure 1.

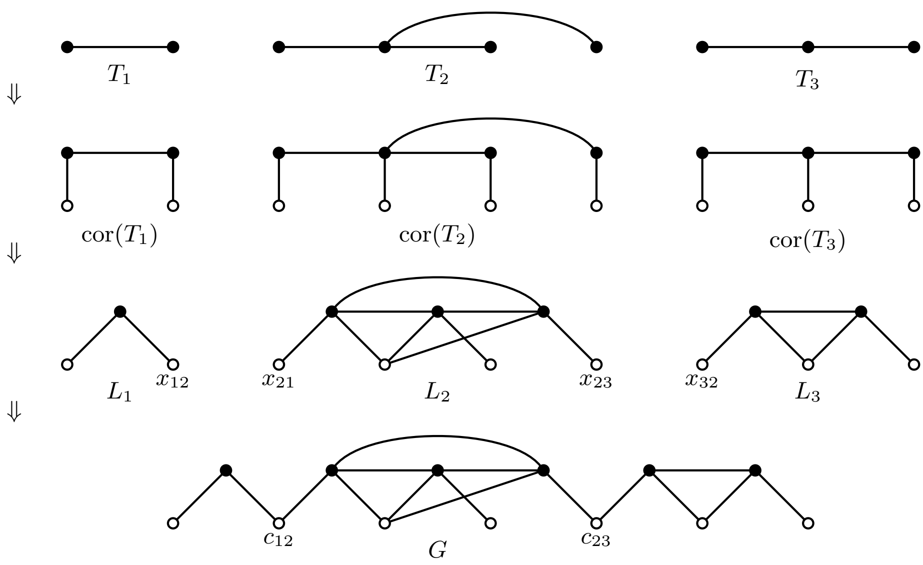

Figure 1: An illustration of the construction of a graph G in the family $\mathcal{F}$.

We note that if G is an arbitrary graph of order n in the family $\mathcal{F}$, then $n \geq 3$ is odd and the vertex set of G can be partitioned into two sets A and B such that the following holds.

- $|A| = \frac{1}{2}(n-1)$ and $|B| = \frac{1}{2}(n+1)$.
- The set B is an independent set.

- Each vertex in A has exactly two neighbors in B.

We refer to the partition (A, B) as the partition associated with G. For the graph $G \in \mathcal{F}$ illustrated in Figure 1, the set A consists of the darkened vertices and the set B consists of the white vertices.

We are now in a position to state the characterization due to Favaron [6] of the graphs achieving equality in the bound of Theorem 5.2.

Theorem 5.5 ([6]). *If G is a connected, claw-free graph of order $n \geq 3$, then* $\mathrm{IR}(G) \leq \frac{1}{2}(n+1)$*, with equality if and only if $G \in \mathcal{F}$.*

We prove next the following property of graphs in the family $\mathcal{F}$.

Lemma 5.6. *If $G \in \mathcal{F}$ and (A, B) is the partition associated with G, then the set B is the unique* IR*-set of G.*

Proof. We proceed by induction on the order $n \geq 3$ of $G \in \mathcal{F}$. If $n = 3$, then $G = P_3$. In this case, the set B consists of the two leaves of G, and the desired result is immediate. This establishes the base case. Suppose that $n \geq 5$ and that the result holds for all graphs $G' \in \mathcal{F}$ of order n', where $3 \leq n' < n$. Let Q be an IR-set of G.

By construction of the graph G, the set B contains at least two vertices of degree 1 in G. Let v be an arbitrary vertex in B of degree 1 in G, and let u be its neighbor. We note that $u \in A$. Let $G' = G - \{u, v\}$ and let G' have order n', and so $n' = n - 2$. Let $A' = A \setminus \{u\}$ and $B' = B \setminus \{v\}$. By construction of the graph G and our choice of the vertex v, we note that $G' \in \mathcal{F}$ and that (A', B') is the partition associated with G'. Applying the inductive hypothesis to G', the set B' is the unique IR-set of G'. Let w be the second neighbor of u in G that belongs to the set B, and so $N(u) \cap B = \{v, w\}$. By the structure of the graph $G \in \mathcal{F}$, we note that $N[w] \subset N[u]$ and that the subgraph of G induced by $N[w]$ is a clique.

Suppose, to the contrary, that $Q \neq B$. Let Q' be the restriction of Q to the graph G', and so $Q' = Q \cap V(G')$. Suppose that $u \in Q$. Since Q is an irredundant set, this implies that $v \notin Q$. If $w \in Q$, then $\mathrm{pn}[w, Q] = \emptyset$, contradicting the fact that Q is an irredundant set. Hence, $w \notin Q$, and so $Q' \neq B'$. By the inductive hypothesis, the set Q' is therefore not an IR-set of G', and so $|Q'| < \mathrm{IR}(G')$. Thus, $\mathrm{IR}(G) = |Q| = |Q'| + 1 \leq (\mathrm{IR}(G') - 1) + 1 = \frac{1}{2}(n' + 1) = \frac{1}{2}(n - 1) < \mathrm{IR}(G)$, a contradiction. Hence, $u \notin Q$. In this case, $\mathrm{IR}(G) = |Q| \leq |Q'| + 1 \leq \mathrm{IR}(G') + 1 = \frac{1}{2}(n' + 1) + 1 = \frac{1}{2}(n + 1) = \mathrm{IR}(G)$. Hence, we must have equality throughout this inequality chain. This implies that $v \in Q$ and $|Q'| = \mathrm{IR}(G')$. By the inductive hypothesis, we therefore have $Q' = B'$. Hence, $Q = Q' \cup \{v\} = B' \cup \{v\} = B$. Thus, the set B is the unique IR-set of G. □

Corollary 5.7. *If $G \in \mathcal{F}$ and (A, B) is the partition associated with G, then the set B is the unique α-set of G and the unique Γ-set of G.*

Proof. By Theorem 5.2, $\alpha(G) = \Gamma(G) = \mathrm{IR}(G) = \frac{1}{2}(n + 1)$. By Lemma 5.6, the set B is the unique IR-set of G. Since every α-set of G is an IR-set of G and $\alpha(G) = \mathrm{IR}(G)$, this implies that B is the unique α-set of G. Since every Γ-set of G is an IR-set of G and $\Gamma(G) = \mathrm{IR}(G)$, this implies that B is the unique Γ-set of G. □

We show next that Conjecture 1.2 holds for connected claw-free graphs.

Theorem 5.8. *If G is a connected, claw-free graph of order $n \geq 2$, then the following holds.*

(a) $\Psi_g^+(G) \geq \frac{1}{2}n$.

(b) *If* $G \neq P_3$*, then* $\Psi_g^-(G) \geq \frac{1}{2}n$.

Proof. Let G be a connected, claw-free graph of order $n \geq 2$. If $\mathrm{IR}(G) \leq \frac{1}{2}n$, then

$$\min\{\Psi_g^+(G), \Psi_g^-(G)\} \geq \psi(G) \geq n - \mathrm{IR}(G) \geq \frac{1}{2}n.$$

Therefore, by Theorem 5.5, we can assume that $\mathrm{IR}(G) = \frac{1}{2}(n+1)$ and $G \in \mathcal{F}$. Let (A, B) be the partition associated with G. We show in this case we have $\min\{\Psi_g^+(G), \Psi_g^-(G)\} \geq \psi(G) + 1$.

By Observation 1.1, $\Gamma(G) + \psi(G) = n$. Moreover, the complement of every Γ-set of G is a maximal enclaveless set, and the complement of every ψ-set of G is a minimal dominating set. By Corollary 5.7, the set B is the unique Γ-set of G. These observations imply that the complement of the set B, namely the set A, is the unique ψ-set of G. Thus every maximal enclaveless set of G of cardinality $\psi(G)$ is precisely the set A.

In the Maximizer-start enclaveless game played on G, Maximizer plays as his first move any vertex from the set B. In the Minimizer-start enclaveless game played on G, by supposition we have $G \neq P_3$, implying that there are at least two vertices in the set B at distance at least 3 apart in G. Thus, whatever the first move is played by Minimizer, Maximizer can always respond by playing as his first move a vertex chosen from the set B. Hence, no matter who starts the game, Maximizer can play a vertex in B as his first move. Thus if S denotes the set of all vertices played when the game ends (in either game), then the set S is a maximal enclaveless set in G. By our earlier observations, such a set S contains a vertex of B and is therefore different from the set A. Since the set A is the unique ψ-set of G, this implies that $|S| > \psi(G)$. Therefore,

$$|S| \geq \psi(G) + 1 = (n - \Gamma(G)) + 1 = n - \frac{1}{2}(n+1) + 1 = \frac{1}{2}(n+1).$$

Since the first move of Maximizer from the set B may not be an optimal move, we have that $\Psi_g^+(G) \geq \psi(G) + 1$ if we are playing the Maximizer-start enclaveless game and $\Psi_g^-(G) \geq \psi(G) + 1$ if we are playing the Minimizer-start enclaveless game. Thus, in both games Maximizer has a strategy to finish the game in at least $\frac{1}{2}(n+1)$ moves. Hence, if we assume that $\mathrm{IR}(G) = \frac{1}{2}(n+1)$, then $\Psi_g^+(G) \geq \min\{\Psi_g^+(G), \Psi_g^-(G)\} \geq \frac{1}{2}(n+1)$. □

By Theorem 5.8(a), we note that Conjecture 1.2 holds for connected claw-free graphs. Moreover by Theorem 5.8(b), we note that Conjecture 1.3 holds for connected claw-free graphs even if we relax the minimum degree two condition and replace it with the requirement that the graph is isolate-free and different from the path P_3.

6 Open problems and conjectures

In this paper, we have shown that the $\frac{1}{2}$-*Enclaveless Game Conjecture* (see, Conjecture 1.2) is true for special classes of graphs, such as regular graphs and claw-free graphs. However, the conjecture has yet to be solved in general. It would be very interesting to prove or disprove the conjecture, or at least prove the conjecture for certain other important classes of graphs. We have also shown that the related conjecture for the Minimizer-start enclaveless game (see, Conjecture 1.3) holds for special classes. Again, it would be interesting to make further inroads into the conjecture. We close with the following two questions that we have yet to settle.

Question 6.1. Do there exist graphs G such that $\Psi_g^-(G) > \Psi_g^+(G)$? If so, how large can the difference $\Psi_g^-(G) - \Psi_g^+(G)$ be made? Or is $\Psi_g^-(G) \leq \Psi_g^+(G)$ always true?

Question 6.2. Is it possible to characterize graphs G such that $\Psi_g^-(G) = \Psi_g^+(G)$?

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Strongly involutive self-dual polyhedra

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Abstract

A polyhedron is a graph which is simple, planar and 3-connected. In this note, we classify the family of *strongly involutive* self-dual polyhedra. The latter is done by using a well-known result due to Tutte characterizing 3-connected graphs. We also show that in this special class of polyhedra self-duality behaves topologically as the antipodal mapping. These self-dual polyhedra are related with several problems in convex and discrete geometry including the Vázsonyi problem.

Keywords: Polyhedra, graphs, duality, self-dual, antipodal.

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1 Introduction

A planar and 3-connected graph $\mathcal{G} = (V, E)$ can be drawn in essentially one way on the sphere or the plane. This fundamental fact is a result of the work of Whitney [14]. It tells us that we not only have the sets V and E defined, but that the set F of faces is also determined, and furthermore the dual graph $\mathcal{G}^*$ is well defined. The dual graph $\mathcal{G}^*$ is the graph whose vertex set V^* is the set of faces F of $\mathcal{G}$, and two new vertices in $\mathcal{G}^*$ are connected by an edge if and only if the faces that define them are adjacent in $\mathcal{G}$.

In this class of graphs, each face f is determined by its *boundary walk*, that is, a cyclically ordered sequence $(v_1, v_2, \dots, v_k)$ consisting of the vertices (and the edges) that are in the closure of the region defining the face f (see [4]). In this sense we can say that u *incides* on f, if it is any of the elements $v_1, v_2, \dots, v_k$ of the cycle defining the face f. We denote this situation simply by $u \in f$. From Steinitz's theorem [12] we know that it is the same to talk about polyhedra in the sense of convex polytopes and to talk about these graphs, so we will refer to them as *polyhedra*. A *polyhedron* is a graph $\mathcal{G}$ that is simple (without loops and multiple edges), planar and 3-connected.

A polyhedron $\mathcal{P}$ is said to be *self-dual* if there exists an isomorphism of graphs $\tau\colon \mathcal{P} \to \mathcal{P}^*$. This isomorphism is called a *duality isomorphism*. There may be several of these duality isomorphisms and each of them is a bijection between vertices and faces of $\mathcal{P}$, such that adjacent vertices correspond to adjacent faces. We are interested in such an isomorphism that satisfies two more properties:

(1) For each pair u, v of vertices, $u \in \tau(v)$ if and only if $v \in \tau(u)$.

(2) For every vertex v, we have that $v \notin \tau(v)$.

Such an isomorphism will be called a *strong involution*. If $\mathcal{P}$ is a self-dual polyhedron admitting a strong involution τ, we will say that $\mathcal{P}$ is a *strongly involutive polyhedron*.

Strongly involutive self-dual polyhedra are very common, like for example wheels on n-cycles with n odd and hyperwheels on n-cycles with n-even (see [11]). In fact the relevance of strongly involutive self-dual polyhedra is partially related with the famous Vázsonyi problem. Let T be a finite set of points of diameter h in Euclidean d-space. Characterize those sets T for which the diameter is attained a maximal number of times as a segment of length h with both endpoints in T. Y. S. Kupitz *et al.* [5], call these sets *extremal configurations*. For $d = 3$, if T is an extremal configuration and V is the intersection of balls of radius h with centers in points of T, a facial structure can be defined on the boundary of V that is *strongly self-dual* in the sense that it admits an duality isomorphism that is involution and is fixed-point free when acting as an automorphism of the first barycentric subdivision of the boundary cell complex of V (see [5]). Indeed, this unusual connection between discrete and convex geometry attracted the attention of several mathematicians to this and other related problems. See, for example L. Lóvasz [6], L. Montejano and E. Roldán-Pensado [9], L. Montejano *et al.* [8] and the work of Bezdek *et al.* [2]. For more about the Vázsonyi problem see [7].

In order to have a good understanding of strongly involutive self-dual polyhedra, we will use a result due to Tutte [13] establishing that every 3-connected graph is either a wheel (a cycle where every vertex is also connected with a central vertex o) or it can be obtained from a wheel by a finite sequence of two operations: adding an edge between any pair of vertices and splitting a given vertex v, with degree $\delta(v) \geq 4$, into two new adjacent vertices v' and v'' in such a way that the new graph obtained is still 3-connected.

In the following section we briefly summarize the notions and notation in relation with the above Tutte's result restricted to the case of simple and planar graphs. In [1], Grünbaum and Barnette used this idea for giving two proofs of Steinitz's Theorem. In Section 3, we show our main results that classify the strongly involutive self-dual polyhedra. Finally, in Section 4, we give a geometric interpretation of strong involutions by proving that such a duality is topologically equivalent to the antipodal mapping on the sphere.

2 Tutte's theorem for polyhedra

In this section we summarize the main ideas and terminology of a recursive classification of spherical polyhedra. These results are deduced from Tutte's work and the details can be found in [10]. Let $\mathcal{G}$ be a polyhedron and $e = (uv)$ any edge of $\mathcal{G}$. We write $\mathcal{G}\backslash e$ for the graph obtained from $\mathcal{G}$ by deleting e. We write $\mathcal{G}/e$ for the graph obtained from $\mathcal{G}\backslash e$ by identifying its endpoints u and v in a single vertex uv. In the same way, given any subset X of V, we write $\mathcal{G}\backslash X$ for the graph obtained from $\mathcal{G}$ by ommiting the elements of X and any edge such that one of its endpoints is an element of X. We will say that $e = (uv)$ *can be deleted* if $\mathcal{G}\backslash e$ is a polyhedron and we say that $e = (uv)$ *can be contracted* if $\mathcal{G}/e$ is a polyhedron. We will say that X is an *n-cutting set* if it has n vertices and $\mathcal{G}\backslash X$ is not connected. According to Tutte's terminology, we will say that an edge e is *essential* if neither $\mathcal{G}\backslash e$ nor $\mathcal{G}/e$ are polyhedra. In other words, e is essential if it cannot be deleted and it cannot be contracted.

Theorem 2.1 ([10])**.** *The following statements are equivalent.*

(1) $\mathcal{G}$ is a wheel.

(2) Every edge is essential.

(3) Every edge is on a triangular face and has one of its endpoints of degree 3*.*

This result can be rephrased as follows.

Remark 2.2. Every polyhedron is either a wheel or it can be obtained by a wheel by adding new edges within faces of the polyhedron or its dual's. Equivalently: if a polyhedron is not a wheel there is always a not essential edge, this means, an edge we can delete or contract in order to obtain a new polyhedron with one fewer edge.

In this way we can *reduce* any polyhedron by a finite sequence of this operations until we obtain a wheel. It happens that one can obtain different wheels from a given polyhedron by selecting different sequences of non essential edges.

3 Strongly involutive polyhedra

Throughout this section, we let $\mathcal{P} = (V, E, F, \tau)$ be a strongly involutive self-dual polyhedron and $(ab) \in E$ any edge of $\mathcal{P}$. By definition $\tau(a)$ and $\tau(b)$ are adjacent faces of $\mathcal{P}$, thus there must be an edge $(xy) \in E$ such that $\tau(a) \cap \tau(b) = (xy)$ and condition (1) of strong involution implies that $\tau(x) \cap \tau(y) = (ab)$. We will write $\tau(ab)$ for the edge (xy). We will say that (ab) is a *diameter* if and only if $a \in \tau(b)$ (and therefore $b \in \tau(a)$).

Lemma 3.1. *If (ab) and (xy) are both diameters, then $\mathcal{P}$ is the tetrahedron K_4.*

Proof. From the hypotheses we deduce $a \in \tau(x) \cap \tau(y) \cap \tau(b)$ and $x \in \tau(a) \cap \tau(b) \cap \tau(y)$, then $\{a, x\} \subset \tau(y) \cap \tau(b)$ but from the 3-connectivity, the intersection of any two faces must be empty, a single vertex or a single edge, thus (ax) is an edge, otherwise $\{a, x\}$ is a 2-cutting set. Analogously, (bx) is an edge. In the same way, (ya) and (yb) are edges. It follows that the induced graph on these four vertices is K_4. In addition, we have the faces $\tau(a), \tau(b), \tau(x)$ and $\tau(y)$ are triangles. Suppose there exist additional vertices. Take any vertex $v \in V \setminus \{a, b, x, y\}$ such that v is connected to some vertex in $\{a, b, x, y\}$ by an edge. Assume, without loss of generality, that (av) is an edge. This is a contradiction because in that case face $\tau(a)$ should form a cycle with at least four edges. □

Lemma 3.2. *If (ab) is a diameter and (xy) is not, then $\{a, b, x\}$ and $\{a, b, y\}$ are 3-cutting sets of $\mathcal{P}$.*

Proof. From the hypotheses we can deduce that $(\tau(a) \cup \tau(b)) \setminus (xy)$ and $(\tau(x) \cup \tau(y)) \setminus (ab)$ are cycles whose intersection is the set $\{a, b\}$, then we can observe that $(\tau(a) \cup \tau(b)) \setminus (xy)) \cup (ab)$ is the union of two cycles γ_1, γ_2 whose intersection is the edge (ab) and thus $\mathcal{P} \setminus \gamma_1$ consists of two connected pieces R_1, R_2 and also $\mathcal{P} \setminus \gamma_2$ consists of two connected pieces S_1, S_2. Since $\tau(x) \cap \tau(y) = (ab) = \gamma_1 \cap \gamma_2$, we may assume $\tau(x) \setminus (ab) \subset R_1 \cap S_1$ and $\tau(y) \setminus (ab) \subset R_2 \cap S_2$. Let be $w \in (\tau(y) \setminus (ab)) \setminus \gamma_1 \subset R_2 \cap S_2$. It exists because otherwise $\tau(y) = \gamma_1$, and therefore $x \in \tau(y)$, a contradiction since (xy) is not a diameter. Analogoulsy, let be $u \in (\tau(x) \setminus (ab)) \setminus \gamma_2 \subset S_1 \cap R_1$. Then in the graphs $\mathcal{P} \setminus \{a, b, y\}$ and $\mathcal{P} \setminus \{a, b, x\}$, the vertices u and w are disconnected. □

Theorem 3.3. *If $\mathcal{P}$ is not a wheel, then there exists an edge e satisfying the three following conditions:*

(1) e is not on a triangular face,

(2) e is not in a 3-cutting set and

(3) e is not a diameter.

Proof. Since $\mathcal{P}$ is not a wheel then, by Theorem 2.1, there is a not essential edge, say e that can be either deleted or contracted. In fact we ensure, since $\mathcal{P}$ is self-dual, there is an edge that can be contracted, otherwise we can take an edge e that can be deleted and the corresponding dual edge e^* can be contracted in $\mathcal{P}^*$ which is isomorphic to $\mathcal{P}$. Without loss of generality we may assume e can be contracted in $\mathcal{P}$. We may now check that e verifies the three desired conditions:

(1) e is not in a triangle. Otherwise, if e were contracted then $\mathcal{P}/e$ would have parallel edges (which is not possible since $\mathcal{P}/e$ is simple).

(2) e is not in any 3-cutting set. Otherwise, if e were contracted then $\mathcal{P}/e$ would have a 2-cutting set (which is not possible since $\mathcal{P}/e$ is 3-connected).

(3) e is not a diameter. Indeed, if e were a diameter then we would have that edge $\tau(e)$ cannot be a diameter (otherwise, by Lemma 3.1, $\mathcal{P}$ must be a tetrahedron, that is, a 3-wheel, which is not the case) and thus, by Lemma 3.2, e would be in a 3-cutting set, which is not possible. □

Theorem 3.4. *Let $e = (ab)$ be an edge which is neither on a triangular face nor in a 3-cutting set nor a diameter. Then, the graph $[\mathcal{P}/(ab)]\backslash\tau(ab)$, denoted by $\mathcal{P}^\diamond = \mathcal{P}^\diamond_{ab}$, is a strongly involutive self-dual polyhedron.*

Proof. Since (ab) satisfies the three properties of last theorem, then $\mathcal{P}/(ab)$ is a polyhedron, and therefore its dual $\mathcal{P}\backslash\tau(ab)$ is also a polyhedron. We will show that $\mathcal{P}^\diamond$ is a polyhedron. Indeed, it is simple and planar. We need it to be 3-connected. If it were not, then it would have a 2-cutting set $\{m, n\}$. Since $\tau(a)$ and $\tau(b)$ are the faces such that $\tau(a) \cap \tau(b) = \tau(ab)$ we may observe that one of the elements in $\{m, n\}$ is in $\tau(a)$ and the other is in $\tau(b)$. Let's supose $m \in \tau(a)$ and $n \in \tau(b)$. Furthermore the vertex $a = b$, denoted by ab must be one of the elements in $\{m, n\}$, otherwise $\{m, n\}$ would be a 2-cutting set of $\mathcal{P}\backslash\tau(ab)$, a contradiction. This implies that in $\mathcal{P}$, $a \in \tau(b)$ (and therefore $b \in \tau(a)$), so (ab) would be a diameter, which is not by hypothesis. Finally, by definition, $\mathcal{P}^\diamond$ is self-dual and it is strongly involutive with isomorphism $\tau^\diamond(u) = \tau(u)$ for every $u \notin \{a, b\}$ and with $\tau^\diamond(a = b)$ the face obtained by the union of $\tau(a)$ and $\tau(b)$ when edge (xy) is deleted. □

By the above theorem, we can define the *remove-contract* operation in any strongly involutive polyhedron which is not a wheel: there is at least one edge (ab) that we can contract and at the same time remove the edge $\tau(ab)$ in order to obtain a new strongly involutive polyhedron. We can apply this operation repeatedly in order to finish with a strongly involutive wheel (with odd number of vertices in the main cycle). Conversely, we can start with such a wheel and then diagonalizing faces and splitting their corresponding vertices carefully in order to *expand* a strongly involutive polyhedron. By diagonalizing we mean that given a face that is not a triangle, we add a new edge within the face joining two non-consecutive vertices.

In the above terms, Theorem 3.4 gives the following.

Corollary 3.5. *Every strongly involutive self-dual polyhedra is either a wheel or it can be obtained from an odd wheel by a finite sequence of operations consisting in diagonalizing faces of the polyhedron and its dual's simultaneously.*

4 Topological interpretation

In this section we are going to consider topological embeddings of a given graph $\mathcal{G}$ on the surface $\mathbb{S}^2$. By Whitney's Theorem we know that if $\mathcal{G}$ is simple, planar and 3-connected, then any two such embeddings are equivalent in the sense that the set of faces (and their adjacencies) is fully determined only by the graph (they are independent of the embedding). It is an interesting fact that with these conditions we can choose one of these embeddings in such a way that any automorphism of the graph of $\mathcal{P}$ acts as an isometry of $\mathbb{S}^2$. We will write this important fact as follows.

Lemma 4.1 (**Isometric embedding lemma** [11, Lemma 1]). *There exists an embedding $i\colon \mathcal{G} \to \mathbb{S}^2$ such that for every $\sigma \in Aut(\mathcal{G})$ there exists an isometry $\tilde{\sigma}$ of $\mathbb{S}^2$ satisfying $i \circ \sigma = \tilde{\sigma} \circ i$.*

Our goal for now is to interpret geometrically the strong involutions. In the rest of the section $\mathcal{G}$ is the underlying graph (simple, planar and 3-connected) of a strongly involutive self-dual polyhedron $\mathcal{P}$.

Let us define $\mathcal{G}_\square$ the *graph of squares of* $\mathcal{G}$ as follows:

$$V(\mathcal{G}_\square) = V(\mathcal{G}) \cup F(\mathcal{G}) \cup E(\mathcal{G}) \text{ and}$$
$$E(\mathcal{G}_\square) = \{(ve) : v \in V(\mathcal{G}), e \in E(\mathcal{G}), v \in e\} \cup \{(ec) : e \in E(\mathcal{G}), f \in F(\mathcal{G}), e \in f\}.$$

It is easy to observe that $\mathcal{G}_\square$ is a 3-connected simple planar graph and therefore it can be drawn on the sphere in such a way that any automorphism of $\mathcal{G}_\square$ is an isometry. We can suppose $\mathcal{G}_\square$ is embedded in that way and we will abuse of notation making no distintion between $\mathcal{G}_\square$ and its image under the embedding. By definition, the faces of $\mathcal{G}_\square$ are all quadrilaterals of the form $(vafb)$, where $v \in V(\mathcal{G}), a, b \in E(\mathcal{G})$ and $f \in F(\mathcal{G})$.

Theorem 4.2. *Let τ be a strong involution of $\mathcal{P}$. Then $\tilde{\tau}$ is the antipodal mapping $\alpha \colon \mathbb{S}^2 \to \mathbb{S}^2$, $\alpha(x) = -x$.*

Proof. First we can observe that τ is an automorphism of $\mathcal{G}_\square$ and condition (1) of strong involution implies $\tau^2 = id$. Therefore, $\tilde{\tau}$ (given in Lemma 4.1) must be an involution as isometry. There are three possible involutive isometries of the sphere: a reflection through a line (a spherical line), a rotation by $\frac{\pi}{2}$ and the antipodal mapping (a good reference is [3]). Only the antipodal mapping has no fixed points, so we will show that $\tilde{\tau}$ cannot have fixed points. We will proceed by contradiction, supposing $\tilde{\tau}$ has a fixed point and then we will conclude there exists a vertex v such that $v \in \tau(v)$.

If τ is a reflection through a plane H, let us consider $v \in V(\mathcal{G})$, $a, b \in E(\mathcal{G})$ and $f \in F(\mathcal{G})$ such that H intersects the quadrilateral $\mathcal{Q} = (vafb)$ in its interior. The only points of the edges of quadrilateral $\mathcal{Q}$ can intersect H are a and b, so $H \cap \mathbb{S}^2 = l$ where l is the spherical line through a and b, thus we must have $\tau(v) = f$ that means $v \in \tau(v)$.

If τ is a rotation in a line PP' (P, P' antipodal points on the sphere), let $\mathcal{Q} = (vafb)$ be a quadrilateral containing P. If P is the center (the barycenter) of the quadrilateral, then since τ is a duality, it must send v into f, but then $\tau(v) = f$, which means $v \in \tau(v)$. If P is a or b, say $P = a$, then the edge (va) is sent to an edge (af') where f' is a face of $\mathcal{G}$, distinct from f and containing v, but then the quadrilateral $\mathcal{Q}'$ corresponding to v and f' we have $\tau(v) = f'$, which means $v \in \tau(v)$. This concludes the proof. □

As a consequence of Theorem 4.2 we obtain the following.

Corollary 4.3. *For a strongly involutive self-dual polyhedron there is only one duality which is a strong involution.*

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Geometry of the parallelism in polar spine spaces and their line reducts

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Abstract

The concept of the spine geometry over a polar Grassmann space belongs to a wide family of partial affine line spaces. It is known that the geometry of a spine space over a projective Grassmann space can be developed in terms of points, so called affine lines, and their parallelism (in this case the parallelism is not intrinsically definable as it is not Veblenian). This paper aims to prove an analogous result for the polar spine spaces. As a by-product we obtain several other results on primitive notions for the geometry of polar spine spaces.

Keywords: Grassmann space, projective space, polar space, spine space, coplanarity, pencil of lines.

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Introduction

Some properties of the polar spine spaces were already established in [8], where the class of such spaces was originally introduced. Its definition resembles the definition of a spine space defined within a (projective) Grassmann space (= the space $\mathbf{P}_k(\mathbb{V})$ of pencils of k-subspaces in a fixed vector space $\mathbb{V}$), cf. [12, 13]. In every case, a spine space is a fragment of a (projective) Grassmannian whose points are subspaces which intersect a fixed subspace W in a fixed dimension m. In case of *polar spine spaces* we consider a two-step construction, in fact: we consider the subspaces of $\mathbb{V}$ that are totally isotropic (self conjugate, singular) under a fixed nondegenerate reflexive bilinear form ξ on $\mathbb{V}$, and then we restrict this class to the subspaces which touch W in dimension m.

It is a picture which is seen from the view of $\mathbb{V}$. Clearly, W can be extended to a subspace M of $\mathbb{V}$ with codimension 1 and then M yields a hyperplane $\mathcal{M}$ of the polar space

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Q_0 determined by ξ in $\mathbb{V}$. In other words, the projective points on W that are points of Q_0 yield a subspace $\mathcal{W}$ of Q_0 extendable to a hyperplane. Recall that situation of this sort was already investigated in [10]. The isotropic k-subspaces of $\mathbb{V}$ are the $(k-1)$-dimensional linear subspaces of Q_0, and first-step restriction yields the so called polar Grassmann space $\mathrm{Q}_{k-1} = \mathbf{P}_{k-1}(\mathrm{Q})$. The points of Q_{k-1} which touch W in dimension m are – from view of Q – the elements of Q_{k-1} which touch $\mathcal{W}$ in dimension $m-1$. So, *a polar spine space* is also the fragment of a polar Grassmannian which consists of subspaces which touch a fixed subspace extendable to a hyperplane in a fixed dimension. The analogy seems full.

In particular, when W is a hyperplane of $\mathbb{V}$ i.e. $\mathcal{W}$ is a hyperplane of Q_0 then a k-subspace of Q either is contained in $\mathcal{W}$ or it touches it in dimension $k-1$. It is seen that in this case the only reasonable value of m is $m = k-1$ and the obtained structure is the Grassmannian of subspaces of the affine polar space obtained from Q_0 by deleting $\mathcal{W}$ (cf. [3, 11]). So, the class of polar spine spaces contain Grassmannians of k-subspaces of arbitrary polar slit space: of a polar space with a subspace (extendable to a hyperplane) removed, see [10]. An interesting case appears, in particular, when we assume that W is isotropic.

1 Generalities

This section is quoted after [8] with slight modifications.

1.1 Point-line spaces and their fragments

A point-line structure $\mathfrak{B} = \langle S, \mathcal{L}\rangle$, where the elements of S are called *points*, the elements of $\mathcal{L}$ are called *lines*, and where $\mathcal{L} \subset 2^S$, is said to be a *partial linear space*, or a *point-line space*, if two distinct lines share at most one point and every line is of size (cardinality) at least 2 (cf. [2]).

A *subspace* of $\mathfrak{B}$ is any set $X \subseteq S$ with the property that every line which shares with X two or more points is entirely contained in X. We say that a subspace X of $\mathfrak{B}$ is *strong* if any two points in X are collinear. If S is strong, then $\mathfrak{B}$ is said to be a *linear space*.

Let us fix a nonempty subset $\mathcal{H} \subset S$ and consider the set

$$\mathcal{L}|_{\mathcal{H}} := \big\{k \cap \mathcal{H} : k \in \mathcal{L} \text{ and } |k \cap \mathcal{H}| \geq 2\big\}. \tag{1.1}$$

The structure

$$\mathfrak{M} := \mathfrak{B} \upharpoonright \mathcal{H} = \langle \mathcal{H}, \mathcal{L}|_{\mathcal{H}}\rangle$$

is a *fragment* of $\mathfrak{B}$ induced by $\mathcal{H}$ and itself it is a partial linear space. The incidence relation in $\mathfrak{M}$ is again $\in$, inherited from $\mathfrak{B}$, but limited to the new point set and line set. Following a standard convention we call the points of $\mathfrak{M}$ *proper*, and the points in $S \setminus \mathcal{H}$ *improper*. The set $S \setminus \mathcal{H}$ will be called the *horizon of* $\mathfrak{M}$. To every line $L \in \mathcal{L}|_{\mathcal{H}}$ we can assign uniquely the line $\overline{L} \in \mathcal{L}$, the *closure* of L, such that $L \subseteq \overline{L}$. For a subspace $X \subseteq \mathcal{H}$ the closure of X is the minimal subspace $\overline{X}$ of $\mathfrak{B}$ containing X. A line $L \in \mathcal{L}|_{\mathcal{H}}$ is said to be a *projective line* if $L = \overline{L}$, and it is said to be an *affine line* if $|\overline{L} \setminus L| = 1$. With every affine line L one can correlate the point $L^\infty \in S \setminus \mathcal{H}$ by the condition $L^\infty \in \overline{L} \setminus L$. We write $\mathcal{A}$ for the class of affine lines. In what follows we consider sets $\mathcal{H}$ which satisfy the following

$$|L \setminus \mathcal{H}| \leq 1 \text{ or } |L \cap \mathcal{H}| \leq 1 \text{ for all } L \in \mathcal{L}.$$

Note that the above holds when $\mathcal{H}$ or $S \setminus \mathcal{H}$ is a subspace of $\mathfrak{B}$, but the above does not force $\mathcal{H}$ or $S \setminus \mathcal{H}$ to be a subspace of $\mathfrak{B}$. In any case, under this assumption every line is either projective or affine. In case $\mathcal{L}|_{\mathcal{H}}$ contains projective or affine lines only, then $\mathfrak{M}$ is a *semiaffine* geometry (for details on terminology and axiom systems see [18]). In this approach an affine space is a particular case of a semiaffine space. For affine lines $L_1, L_2 \in \mathcal{L}|_{\mathcal{H}}$ we can define a parallelism in a natural way:

$$L_1, L_2 \text{ are } \textit{parallel } (L_1 \parallel L_2) \quad \text{iff} \quad \overline{L_1} \cap \overline{L_2} \cap (S \setminus \mathcal{H}) \neq \emptyset.$$

In what follows we assume that the notion of 'a plane' (= 2-dimensional strong subspace) is meaningful in $\mathfrak{B}$: e.g. $\mathfrak{B}$ is an exchange space, or a dimension function is defined on its strong subspaces. In the article in most parts we consider $\mathfrak{B}$ such that its planes are, up to an isomorphism, projective planes. We say that E is a plane in $\mathfrak{M}$ if $\overline{E}$ is a plane in $\mathfrak{B}$. Observe that there are two types of planes in $\mathfrak{M}$: projective and semiaffine. A semiaffine plane E arises from $\overline{E}$ by removing a point or a line. In result we get a punctured plane or an affine plane respectively. For lines $L_1, L_2 \in \mathcal{L}|_{\mathcal{H}}$ we say that they are *coplanar* and write

$$L_1 \,\boldsymbol{\pi}\, L_2 \quad \text{iff} \quad \text{there is a plane } E \text{ such that } L_1, L_2 \subset E. \tag{1.2}$$

Let E be a plane in $\mathfrak{M}$ and $U \in \overline{E}$. The set

$$\mathbf{p}(U, E) := \left\{ L \in \mathcal{L}|_{\mathcal{H}} : U \in \overline{L} \subseteq \overline{E} \right\} \tag{1.3}$$

will be called a *pencil of lines* if U is a proper point, or a *parallel pencil* otherwise. The point U is said to be the *vertex* and the plane E is said to be the *base plane* of that pencil. We write

$$L_1 \,\boldsymbol{\rho}\, L_2 \quad \text{iff} \quad \text{there is a pencil } p \text{ such that } L_1, L_2 \in p. \tag{1.4}$$

1.2 Cliques

Let ϱ be a binary symmetric relation defined on a set $\mathcal{X}$. A subset of $\mathcal{X}$ is said to be a *ϱ-clique* iff every two elements of this set are ϱ-related.

For any $x_1, x_2, \ldots, x_s$ in $\mathcal{X}$ we introduce

$$\begin{aligned} \boldsymbol{\Delta}^s_\varrho(x_1, x_2, \ldots, x_s) \quad \text{iff} \quad & \neq (x_1, x_2, \ldots, x_s) \text{ and } x_i \;\varrho\; x_j \text{ for all } i, j = 1, \ldots, s \\ & \text{and for all } y_1, y_2 \in \mathcal{X} \text{ if } y_1, y_2 \;\varrho\; x_1, x_2, \ldots, x_s \text{ then } y_1 \;\varrho\; y_2, \end{aligned} \tag{1.5}$$

cf. analogous definition of $\boldsymbol{\Delta}^\varrho_s$ in [9]. For short we will frequently write $\boldsymbol{\Delta}_\varrho$ instead of $\boldsymbol{\Delta}^s_\varrho$. Next, we define

$$[x_1, x_2, \ldots, x_s]_\varrho := \left\{ y \in \mathcal{X} : y \;\varrho\; x_1, x_2, \ldots x_s \right\}. \tag{1.6}$$

It is evident that if $\boldsymbol{\Delta}_\varrho(x_1, \ldots, x_s)$ holds (and ϱ is reflexive) then $[x_1, \ldots, x_s]_\varrho$ is the (unique) maximal ϱ-clique which contains $\{x_1, \ldots, x_s\}$. Finally, for an arbitrary integer $s \geq 3$ we put

$$\mathcal{K}^s_\varrho = \left\{ [x_1, x_2, \ldots, x_s]_\varrho : x_1, x_2, \ldots, x_s \in \mathcal{X} \text{ and } \boldsymbol{\Delta}_\varrho(x_1, x_2, \ldots, x_s) \right\}. \tag{1.7}$$

Then we write

$$\mathcal{K}_\varrho := \bigcup_{s=3}^{\infty} \mathcal{K}^s_\varrho.$$

In most of the interesting situations there is an integer $s_{\max}$ such that $\mathcal{K}_\varrho = \bigcup_{s=3}^{s_{\max}} \mathcal{K}^s_\varrho = \mathcal{K}^*(\varrho)$, where

$$\mathcal{K}^*(\varrho) \text{ is the set of maximal } \varrho\text{-cliques.}$$

1.3 Grassmann spaces and spine spaces

We start with some constructions of a general character. Let X be a nonempty set and let $\mathcal{P}$ be a family of subsets of X. Assume that there is a dimension function $\dim\colon \mathcal{P} \to \{0,\dots,n\}$ such that $\mathfrak{B} = \langle \mathcal{P}, \subset, \dim\rangle$ is an incidence geometry, cf. e.g. [1]. Write $\mathcal{P}_k$ for the set of all $U \in \mathcal{P}$ with $\dim(U) = k$.

Given $H \in \mathcal{P}_{k-1}$ and $B \in \mathcal{P}_{k+1}$ with $H \subset B$, a k-*pencil* over $\mathfrak{B}$ is a set of the form

$$\mathbf{p}(H, B) = \{U \in \mathcal{P}_k : H \subset U \subset B\}.$$

The idea behind this concept is the same as in (1.3), though this definition is more general. The family of all such k-pencils over $\mathfrak{B}$ will be denoted by $\mathcal{P}_k$. Then, the structure

$$\mathbf{P}_k(\mathfrak{B}) = \langle \mathcal{P}_k, \mathcal{P}_k\rangle$$

will be called a *Grassmann space* over $\mathfrak{B}$ (cf. [5, Section 2.1.3]). It is a partial linear space for $0 < k < n$.

Let us fix $W \in \mathcal{P}$ and an integer m. We will write

$$\mathcal{F}_{k,m}(\mathfrak{B}, W) := \{U \in \mathcal{P}_k : \dim(U \cap W) = m\}.$$

The fragment

$$\mathbf{A}_{k,m}(\mathfrak{B}, W) := \mathbf{P}_k(\mathfrak{B}) \restriction \mathcal{F}_{k,m}(\mathfrak{B}, W)$$

will be called a *spine space* over $\mathfrak{B}$ determined by W. It will be convenient to have an additional symbol for the line set of a spine space, which is

$$\mathcal{G}_{k,m}(\mathfrak{B}, W) := \mathcal{P}_k|_{\mathcal{F}_{k,m}(\mathfrak{B},W)}.$$

What follows are more specific examples of the above constructions that we actually investigate in our paper. Let $\mathbb{V}$ be a vector space and let $\mathrm{Sub}(\mathbb{V})$ be the set of all vector subspaces of $\mathbb{V}$. Then $\mathbf{P}_k(\mathbb{V})$ is a partial linear space called a *projective Grassmann space*. In particular $\mathbf{P}_1(\mathbb{V})$ is the projective space over $\mathbb{V}$. It is well known that $\mathbf{P}_k(\mathbb{V}) \cong \mathbf{P}_{k-1}(\mathbf{P}_1(\mathbb{V}))$.

Let $W \in \mathrm{Sub}(\mathbb{V})$. The spine space $\mathbf{A}_{k,m}(\mathbb{V}, W)$ was introduced in [12] and developed in [13, 14, 15, 16]. Note that $\mathbf{A}_{k,m}(\mathbb{V}, W) \cong \mathbf{A}_{k-1,m-1}(\mathbf{P}_1(\mathbb{V}), \mathrm{Sub}_1(W))$. The concept of a spine space makes a little sense without the assumption that

$$0, k - n + w \le m \le k, w, \tag{1.8}$$

where $w = \dim(W)$. It is a partial linear space when (1.8) is satisfied.

For possibly maximal values of m we get $\mathbf{A}_{k,k}(\mathbb{V}, W) = \mathbf{P}_k(W)$, where the points are basically vector subspaces of W, and $\mathbf{A}_{k,w}(\mathbb{V}, W) \cong \mathbf{P}_{k-w}(\mathbb{V}/W)$, where the points are those vector subspaces of $\mathbb{V}$ which contain W. Therefore, we assume that

$$m < k, w. \tag{1.9}$$

Now, let ξ be a nondegenerate reflexive bilinear form of index r on $\mathbb{V}$. For $U, W \in \mathrm{Sub}(\mathbb{V})$ we write $U \perp W$ iff $\xi(U, W) = 0$, meaning that $\xi(u, w) = 0$ for all $u \in U$, $w \in W$. Then the set of all totally isotropic subspaces of $\mathbb{V}$ w.r.t. ξ is

$$\mathrm{Q} := \{U \in \mathrm{Sub}(\mathbb{V}) : U \perp U\},$$

and $\mathrm{Q}_k := \mathrm{Q} \cap \mathrm{Sub}_k(\mathbb{V})$. The set Q_k is nonempty iff

$$k \leq r. \tag{1.10}$$

Provided that $2 \leq r$ the structure $\mathfrak{Q} = \mathbf{P}_1(\mathrm{Q})$ is a classical polar space embeddable into the projective space $\mathbf{P}_1(\mathbb{V})$. It is clear that $\mathfrak{Q} \cong \langle \mathrm{Q}_1, \mathrm{Q}_2, \subset \rangle$ and usually polar space is defined that way.

A *polar Grassmann space* is the structure $\mathbf{P}_k(\mathrm{Q})$. It is a partial linear space whenever

$$k < r. \tag{1.11}$$

Note that $\mathbf{P}_k(\mathrm{Q}) \cong \mathbf{P}_{k-1}(\mathbf{P}_1(\mathrm{Q}))$.

Finally,

$$\mathfrak{M} := \mathbf{A}_{k,m}(\mathrm{Q}, W),$$

a *polar spine space*, the main subject of our paper, arises. Note that we have $\mathfrak{M} \cong \mathbf{A}_{k-1,m-1}(\mathbf{P}_1(\mathrm{Q}), \mathrm{Sub}(W) \cap \mathrm{Q}_1)$.

Let $r_W = \mathrm{ind}(\xi \restriction W)$ be the index of the form ξ restricted to W. If $r_W < m$, then there is no totally isotropic subspace of $\mathbb{V}$, which meets W in some m-dimensional subspace. Every $U \in \mathrm{Q}$ can be extended to an $Y \in \mathrm{Q}_r$. Assume that $\dim(Y \cap W) > r - k + m$ for all $Y \in \mathrm{Q}_r$. This means that all totally isotropic subspaces of $\mathbb{V}$, which meet W in some m-dimensional subspace, are at most $(k-1)$-dimensional. On the other hand, this assumption implies $r_W > r - k + m$. Thus

$$m \leq r_W \leq r - k + m \tag{1.12}$$

is a sufficient condition for $\mathcal{F}_{k,m}(\mathrm{Q}, W) \neq \emptyset$.

Warning. The condition (1.12) is – in the context above – *only sufficient*. As we shall see there are sets W such that $r - k + m < r_W$ but $\mathcal{F}_{k,m}(\mathrm{Q}, W) \neq \emptyset$. Clearly, the condition $m \leq r_W$ *is necessary*.

Under (1.12) no point of $\mathfrak{M}$ is isolated and $\mathfrak{M}$ is a partial linear space. Now, let us have a look at the structure of strong subspaces of polar spine spaces. Following [13] they are called: α-stars, ω-stars, α-tops and ω-tops. For details see Table 2. Actually, this is an 'adaptation' of the classification of strong subspaces of $\mathbf{A}_{k,m}(\mathbb{V}, W)$ (consult [13]) to the case when we restrict $\mathbf{P}_k(\mathbb{V})$ to $\mathbf{P}_k(\mathrm{Q})$. With a slight abuse of language all sets of the type $\mathcal{T}^\alpha$ and $\mathcal{T}^\omega$ we call *tops*, and sets of the form $\mathcal{S}^\alpha$ and $\mathcal{S}^\omega$ *stars*. But note that due to some specific values of r, k, m and $\dim(Y \cap W)$ with $Y \in \mathrm{Q}_r$ families of some of these types may be empty. Moreover, stars and tops consist of strong subspaces of $\mathfrak{M}$, but stars or tops of some kind may be not maximal among strong. In general, $\mathcal{S}^\omega$ and $\mathcal{T}^\alpha$ consist of projective spaces, while the other consist of proper slit spaces (cf. [4, 18]), but if $\mathcal{F}_{k-1,m}(\mathrm{Q}, W) \ni H \subset Y \in \mathcal{F}_{r,m}(\mathrm{Q}, W)$ then $[H, Y]_k \cap \mathcal{F}_{k,m}(\mathrm{Q}, W) = [H, Y]_k \in \mathcal{S}^\alpha$ is a projective space as well.

Generally, $H \in \mathcal{P}_{k-1}$ determines a star and $B \in \mathcal{P}_{k+1}$ determines a top as follows

$$\mathrm{S}(H) = \{U \in \mathcal{P}_k : H \subset U\}, \qquad \mathrm{T}(B) = \{U \in \mathcal{P}_k : U \subset B\}.$$

Here, we occasionally make use of this convention in the context of polar spine spaces, where $\mathcal{P}_k = \mathcal{F}_{k,m}(\mathrm{Q}, W)$.

2 Lines classification and existence problems

In analogy to [13, 17] the lines of $\mathfrak{M}$ can be of three sorts: affine (in $\mathcal{A}$), α-projective (in $\mathcal{L}^\alpha$), and ω-projective (in $\mathcal{L}^\omega$). To be more concrete, comp. Table 1, these are pencils $L = \mathbf{p}(H, B) \cap \mathcal{F}_{k,m}(\mathrm{Q}, W)$ such that (we consider parameters k, m, Q, W as fixed)

$\mathcal{A}$: $H \in \mathcal{F}_{k-1,m}(\mathrm{Q}, W),\ B \in \mathcal{F}_{k+1,m+1}(\mathrm{Q}, W)$; in this case $L^\infty = H + (W \cap B) = (H + W) \cap B$. Note that $L^\infty \subset B \in \mathrm{Q}$ and therefore $L^\infty \in \mathcal{F}_{k,m+1}(\mathrm{Q}, W)$. In other words, L^∞ is a point of $\mathbf{A}_{k,m+1}(\mathrm{Q}, W)$.

$\mathcal{L}^\alpha$: $H \in \mathcal{F}_{k-1,m}(\mathrm{Q}, W),\ B \in \mathcal{F}_{k+1,m}(\mathrm{Q}, W)$.

$\mathcal{L}^\omega$: $H \in \mathcal{F}_{k-1,m-1}(\mathrm{Q}, W),\ B \in \mathcal{F}_{k+1,m+1}(\mathrm{Q}, W)$

(cf. Table 1).

Note that if $r_W < m + 1$ (in view of the global assumption $r_W \geq m$ this means $r_W = m$) then $\mathcal{A} \cup \mathcal{L}^\omega = \emptyset$. Looking at [8, Lemma 1.6] we see that in this case $\mathfrak{M}$ is disconnected as well or $m = w$. In the latter case also $\mathcal{A} \cup \mathcal{L}^\omega = \emptyset$. Besides, this also contradicts (1.9). Consequently, for $r_W < m+1$ the horizon $\mathbf{A}_{k,m+1}(\mathrm{Q}, W)$ of $\mathfrak{M}$ looses its sense.

The problem whether one of the three above classes of lines is nonempty reduces, in fact, to the problem whether the corresponding class of 'possible tops' of these lines is nonempty. More precisely, we have the following criterion.

Lemma 2.1.

(i) *Let* $B \in \mathcal{F}_{k+1,m}(\mathrm{Q}, W)$*; then* $\mathrm{T}(B) \neq \emptyset$.

(ii) *Let* $B \in \mathcal{F}_{k+1,m}(\mathrm{Q}, W)$ *and* $U \in \mathrm{T}(B)$*; then there is an* $L = [H, B]_k \in \mathcal{L}^\alpha$ *such that* $U \in L$.

So, if $\mathcal{F}_{k+1,m}(\mathrm{Q}, W) \neq \emptyset$ *then* $\mathcal{L}^\alpha \neq \emptyset$.

(iii) *Let* $B \in \mathcal{F}_{k+1,m+1}(\mathrm{Q}, W)$*; then* $\mathrm{T}(B) \neq \emptyset$.

(iv) *Let* $B \in \mathcal{F}_{k+1,m+1}(\mathrm{Q}, W)$ *and* $U \in \mathrm{T}(B)$. *Then there are:*

- *an* $L' = [H', B]_k \in \mathcal{L}^\omega$ *(provided that* $m > 0$*) such that* $U \in L'$ *and*
- *an* $L'' = [H'', B]_k \in \mathcal{A}$ *such that* $U \in L''$.

Consequently, if $\mathcal{F}_{k+1,m+1}(\mathrm{Q}, W) \neq \emptyset$ *then* $\mathcal{A} \neq \emptyset$*, and* $\mathcal{L}^\omega \neq \emptyset$ *when* $m > 0$.

Proof. To justify (i) present B in the form $B = (B \cap W) \oplus D$, where $D \cap W = \Theta$ and $\dim(D) = k + 1 - m$. Let D' be a $(k - m)$-dimensional subspace of D and put $H := (B \cap W) + D'$. To justify (ii) we simply use (i) with B replaced by U to obtain the subspace H.

To justify (iii) we present B in the form $B = (B \cap W) \oplus D$ (now, $\dim(D) = k - m$) and proceed analogously to (i): $U = D + Z$, where Z is an m-dimensional subspace of $B \cap W$. To justify (iv) to get H' we apply (iii) with B replaced by U, and to get H'' we apply (i) with B replaced by U. □

Note that, sufficient conditions for the existence of the corresponding subspaces B in Lemma 2.1, i.e. for $\mathcal{F}_{k+1,m}(\mathrm{Q}, W), \mathcal{F}_{k+1,m+1}(\mathrm{Q}, W) \neq \emptyset$ are

$$m \leq r_W \leq r - (k+1) + m \quad \text{and} \quad m + 1 \leq r_W \leq r - k + m, \text{ respectively.}$$

We say that an $U \in \mathcal{F}_{k,m}(\mathrm{Q}, W)$ is an *α-point* iff each top containing U is of type α, i.e. each line through U is of type α. Similarly, an $U \in \mathcal{F}_{k,m}(\mathrm{Q}, W)$ is an *ω-point* iff each top containing U is of type ω, i.e. each line through U is either affine or of type ω.

Lemma 2.2. *Let $U \in \mathcal{F}_{k,m}(\mathrm{Q}, W)$.*

(i) *There is B such that $U \subset B \in \mathcal{F}_{k+1,m}(\mathrm{Q}, W) \cup \mathcal{F}_{k+1,m+1}(\mathrm{Q}, W)$.*

(ii) *U is an α-point iff $U^\perp \cap W \subset U$. In this case*

$$w \leq k + m. \tag{2.1}$$

Otherwise, if $U^\perp \cap W \not\subset U$ then there is a $B \in \mathcal{F}_{k+1,m+1}(\mathrm{Q}, W)$ such that $U \subset B$.

(iii) *U is an ω-point when $U^\perp \subset U + W$. In this case*

$$w \geq n + m - 2k. \tag{2.2}$$

Otherwise, if $U^\perp \not\subset U + W$ then there is a $B \in \mathcal{F}_{k+1,m}(\mathrm{Q}, W)$ such that $U \subset B$.

Proof. Clearly, U is not maximal isotropic, so there is a B such that $U \subset B \in \mathrm{Q}_{k+1}$. As in [12] we obtain $m \leq \dim(B \cap W) \leq m + 1$. This justifies (i).

To justify (ii) note that every $B \in \mathrm{Q}_{k+1}$ containing U belongs to $\mathcal{F}_{k+1,m}(\mathrm{Q}, W)$, and then $U \prec B \subset U^\perp$ and $U \cap W \subset B \cap W \subset U^\perp \cap W$. If we have $\dim(B \cap W) = m$ for all B, then $\dim(U^\perp \cap W) = m$ and $U \cap W = U^\perp \cap W$. As $U \subset U^\perp$ by definition of U, the obtained condition is equivalent to $U^\perp \cap W \subset U$.

In this case we have $W = (U \cap W) \oplus D$, where D is contained in a linear complement of $U^\perp$. D is at most $\mathrm{codim}(U^\perp) = k$-dimensional, so $\dim(W) \leq m + k$.

To justify (iii) note, first, that if $U \subset B \in \mathrm{Q}_{k+1}$ then $B \in \mathcal{F}_{k+1,m+1}(\mathrm{Q}, W)$. So, if $U \prec B \in \mathrm{Q}$, then $B = U \oplus \langle y \rangle$ with $y \in U^\perp \setminus U$. If $U^\perp \subset U + W$ then $y = u + w$ for some $u \in U$ and $w \in W \setminus U$ and then $B = U \oplus \langle w \rangle$. So, $B \cap W = (U \cap W) \oplus \langle w \rangle$. If there is $y \in U^\perp \setminus (U + W)$, then $U + \langle y \rangle$ intersects W in $U \cap W$.

If U is as required above then $n - k = \dim(U^\perp) \leq \dim(U + W) = w + k - m$. This gives $w \geq m + n - 2k$. □

From Lemmas 2.1 and 2.2(ii), 2.2(iii) we infer the following geometrical fact.

Corollary 2.3.

(i) *If $w > k + m$ then through* each *point of $\mathfrak{M}$ there passes an ω-line and an affine line.*

(ii) *If $w < n + m - 2k$ then through* every *point of $\mathfrak{M}$ there passes an α-line.*

Combining Lemmas 2.2(i) with 2.1(ii) and 2.1(iv) we obtain the following Corollary, a weakening of Corollary 2.3 but with more general assumptions.

Corollary 2.4. *If $U \in \mathcal{F}_{k,m}(\mathrm{Q}, W)$ then there is a line in $\mathcal{G}_{k,m}(\mathrm{Q}, W)$ through U. Consequently, if $\mathcal{F}_{k,m}(\mathrm{Q}, W) \neq \emptyset$, then $\mathcal{G}_{k,m}(\mathrm{Q}, W) \neq \emptyset$.*

Comments to Lemma 2.2.

ad (ii) Condition (2.1) is a *necessary* condition for the existence of an α-point.

By (1.12) and (2.1) we get $r_W \le r - k + m \le r - w$ (this implies $r - r_W \ge w$). This condition is not inconsistent. So, it may happen that $\mathfrak{M}$ contains both α-points and ω-tops.

One can note (it is, practically, proved in the proof of Lemma 2.2(ii) that if (2.1) is satisfied and $U \in \mathrm{Q}_k$ then there is a subspace W such that U is an α-point in $\mathbf{A}_{k,m}(\mathrm{Q}, W)$ and $\dim(W) = w$.

ad (iii) Analogously, condition (2.2) is a *necessary* condition for the existence of an ω-point.

It is seen that (under suitable assumption, obtained by (1.12) and (2.2): $r - r_W \ge k - m \ge n - k - w$) the space $\mathfrak{M}$ may contain both ω-points and α-tops.

And there do exist W for which associated spine spaces contain an ω-point.

As an immediate consequence of Lemma 2.2(iii) we obtain the following.

Corollary 2.5. *Assume that $w < n + m + 1 - 2k$. Then, for every $U \in \mathcal{F}_{k,m+1}(\mathrm{Q}, W)$ there is $L \in \mathcal{A}_{k,m}(\mathrm{Q}, W)$ such that $U = L^\infty$.*

3 Examples, particular cases

Let us examine in some detail polar spine spaces of some, particularly natural classes.

3.1 Grassmannians of affine polar spaces

Assume that W is a hyperplane of $\mathfrak{P}$; in turn this is equivalent to say that $\mathrm{Sub}_1(W)$ is a hyperplane in $\mathfrak{Q}$. In this case we have

$$m = k - 1 \text{ and} \tag{3.1}$$

$$\dim(W \cap Y) = \begin{cases} r & \text{when } Y \subset W \\ r - 1 & \text{when } Y \not\subset W \end{cases} \quad \text{for every } Y \in \mathrm{Q}_r. \tag{3.2}$$

It is clear that in this case

$$\mathcal{F}_{k,m}(\mathrm{Q}, W) \neq \emptyset; \text{ in view of Corollary 2.4, } \mathbf{A}_{k,m}(\mathrm{Q}, W) \text{ is nontrivial}$$

simply, because it is impossible to have $\mathrm{Q}_k \subset \mathrm{Sub}_k(W)$. However, this case raises several degenerations concerning the structure of strong subspaces of $\mathfrak{M}$.

Lemma 3.1.

(i) *Let $B \in \mathrm{Sub}_{k+1}(\mathbb{V})$. Then either* $\dim(B \cap W) = k + 1 = m + 2$ *(and then $B \subset W$) or* $\dim(B \cap W) = k = m + 1$. *Therefore, there is no strong subspace in $\mathcal{T}^\alpha$. Moreover, by the same reasons, $\mathcal{L}^\alpha = \emptyset$.*

(ii) *If $B \in \mathcal{F}_{k+1,m+1}(\mathrm{Q}, W)$ then* $\mathrm{T}(B) \cap \mathcal{F}_{k,m}(\mathrm{Q}, W) \in \mathcal{T}^\omega$ *is a k-dimensional punctured projective space.*

(iii) *Let* $X = [H, Y]_k \cap \mathcal{F}_{k,m}(\mathrm{Q}, W)$, $H \in \mathrm{Sub}_{k-1}(\mathbb{V})$, $Y \in \mathrm{Q}r$. *Assume that* $\dim(H \cap W) = m = k - 1$ *i.e.* $H \subset W$. *If* $Y \subset W$ *then, clearly,* $X = \emptyset$. *If* $Y \not\subset W$ *then* $X \in \mathcal{S}^\alpha$ *is a* $(r-k)$*-dimensional affine space.*

(iv) *Let* $X = [H, Y]_k \cap \mathcal{F}_{k,m}(\mathrm{Q}, W)$, $H \in \mathrm{Sub}_{k-1}(\mathbb{V})$, $Y \in \mathrm{Q}r$. *Assume that* $\dim(H \cap W) = m - 1 = k - 2$ *i.e.* $H \not\subset W$. *Then* $Y \not\subset W$ *and, consequently,* $\dim(Y \cap W) = r - 1$. *In this case* $X \in \mathcal{S}^\omega$ *is a* $(r-k)$*-dimensional projective space.*

Corollary 3.2. *If* $4 \le k + 2 \le r$ *then every line of* $\mathfrak{M}$ *has at least two extensions to a maximal at least* 2*-dimensional strong subspace: one to a top, and one to a star.*

3.2 Spine spaces with isotropic 'holes'

Next, let us assume that $W \in \mathrm{Q}$ i.e. W is isotropic. In this case we have

$$r_W = w. \tag{3.3}$$

So, let $m < w, k$; let us take arbitrary $D \in \mathrm{Sub}_m(W)$ and $Y \in \mathrm{Q}_r$ with $W \subset Y$. Then there is $Y_0 \in \mathrm{Q}_r$ such that $Y \cap Y_0 = D$. Consider any U such that $\dim(U) = k$ and $D \subset U \subset Y_0$; then $U \in \mathcal{F}_{k,m}(\mathrm{Q}, W)$. Thus we have proved that

$\mathcal{F}_{k,m}(\mathrm{Q}, W) \neq \emptyset$; in view of Corollary 2.4, $\mathbf{A}_{k,m}(\mathrm{Q}, W)$ is nontrivial.

Note that if we assume (1.10) then $k + w - m \le r + r - m \le n - m \le n$ follows, so (1.8) holds as well.

Next, let us pay attention to the problem of extending lines. Namely, let $L = \mathbf{p}(H, B) \in \mathcal{L}^\omega$. So, $\dim(B \cap W) = m + 1$. Suppose that $r = m + 1$; then we obtain contradictory $m < k < r = m + 1$. As above, we extend W to a maximal isotropic Y and find maximal isotropic Y' with $Y \cap Y' = B$. This proves

Lemma 3.3. *If* $k < r-1$, *then every line in* $\mathcal{L}^\omega$ *can be extended to an at least* 2*-dimensional star.*

4 Binary collinearity

Let us start with a Chow'like result concerning binary collinearity $\boldsymbol{\lambda}$ of points in a polar spine space $\mathfrak{M} = \mathbf{A}_{k,m}(\mathrm{Q}, W)$ defined for some integers k, m and a fixed subspace W of a vector space V equipped with a suitable form ξ. To this aim standard reasoning similar to this of [6, 7, 17] can be used:

> a line through two distinct points is the intersection of all the maximal $\boldsymbol{\lambda}$-cliques which contain these points.

In the sequel we intensively analyse Table 2. Let $U_1 \;\boldsymbol{\lambda}\; U_2$, $U_1 \neq U_2$. Put $L = \overline{U_1, U_2}$. Evidently, every line $L = \mathbf{p}(H, B)$ can be extended to a top $T = \mathrm{T}(B) \cap \mathcal{F}_{k,m}(\mathrm{Q}, W)$, which is a $(k-m)$-dimensional ($T \in \mathcal{T}^\alpha$) or a k-dimensional ($T \in \mathcal{T}^\omega$) slit space. We have assumed that $k > 1$. So, when $m < k - 1$ then T is greater than L. For any triangle $U_1, U_2, U_3 \in T$ we have $\boldsymbol{\Delta}_{\boldsymbol{\lambda}}(U_1, U_2, U_3)$ and $T = [U_1, U_2, U_3]_{\boldsymbol{\lambda}}$.

If L is an α-projective line or an affine line then it has at least one extension to a star S in $\mathcal{S}^\alpha$, which are $(r-k)$-dimensional slit spaces. Consequently, $L = T \cap S$. Assume that $k < r - 1$, so $L \subsetneq S$.

In this point we can choose one of the following two ways. Firstly, we notice that there is a finite system $U_1, U_2, \dots, U_t \in S$ such that $\boldsymbol{\Delta}_{\boldsymbol{\lambda}}(U_1, U_2, \dots, U_t)$, so $S' := S = [U_1, U_2, \dots, U_t]_{\boldsymbol{\lambda}}$. Secondly, we can extend U_1, U_2 to any triangle $U_1, U_2, U_3 \in S$ and note that $S' := [U_1, U_2, U_3]_{\boldsymbol{\lambda}}$ is the union of all the extensions of the plane spanned by U_1, U_2, U_3 to a maximal $\boldsymbol{\lambda}$-clique. In both cases $L = S' \cap T$ and thus L can be defined in terms of $\boldsymbol{\lambda}$.

A problem may arise when $L \in \mathcal{L}^\omega$. In this case each extension of L to a star S is contained in a segment $[H, Y]_k$ with a maximal totally isotropic extension Y of $B \supset H$ and it has dimension $\dim(W \cap Y) - m$. So, it may degenerate to the line L when $\dim(W \cap Y) = m + 1$. Is it possible that every such an extension Y intersects W in dimension $m+1$? Recall that the condition $L \in \mathcal{L}^\omega$ yields $\dim(B \cap W) = m + 1$ and therefore we obtain $W \cap Y = W \cap B$, for every $\mathrm{Q}_r \ni Y \supset B$. So, our problematic case reduces to the question: for which $B \in \mathcal{F}_{k+1,m+1}(\mathrm{Q}, W)$ there is no reasonable extension Y and: when each such a B has a required extension. Note that to find Y it suffices to find D such that $B \prec D \in \mathrm{Q}$ and $\dim(D \cap W) = m + 2$; then Y is an extension of D to a maximal totally isotropic subspace. On the other hand, the existence of D in question can be assured by a suitable substitution in Lemma 2.2(ii), which yields a sufficient condition for the existence of our Y:

$$w > k + m + 2. \tag{4.1}$$

As a consequence we can formulate the following result.

Theorem 4.1 (The Chow Theorem for $\mathfrak{M}$)**.** *If $m < k - 1$, $k < r - 1$, and each line in $\mathcal{L}^\omega$ can be extended to at least 2-dimensional star (which is assured, e.g. by* (4.1)*) then the structures $\mathfrak{M}$ and $\langle \mathcal{F}_{k,m}(\mathrm{Q}, W), \boldsymbol{\lambda} \rangle$ are definitionally equivalent.*

In particular, in view of Corollary 3.2 and Lemma 3.3, the Chow theorem holds in $\mathfrak{M}$ when W is an isotropic subspace and $k < r-1$, and it holds in $\mathfrak{M}$ when W is a hyperplane and $4 \le k + 2 \le r$.

One can continue these investigations in the fashion of [17] considering graphs of collinearity with some sorts of lines distinguished ($\boldsymbol{\lambda}^\alpha$, $\boldsymbol{\lambda}^\omega$, $\boldsymbol{\lambda}^{\alpha\vee\omega}$ etc.). Observing criteria in Lemma 2.2 and Corollary 2.5 we see that it may be a hard work: α-points and ω-points may appear, 'deep' improper points may appear as well.

5 Maximal cliques of $\boldsymbol{\lambda}^\sigma$

Let σ be a one of the symbols

$$\alpha, \omega, \alpha \vee \omega, \alpha^+, \omega^+.$$

The classes $\mathcal{L}^\sigma$ with $\sigma \in \{\alpha, \omega\}$ are already defined (usually, the arguments like k, m, $\mathbb{V}$, Q, W will be omitted, if unnecessary or fixed). Next, $\mathcal{L}^{\sigma^+} := \mathcal{L}^\sigma \cup \mathcal{A}$, and, finally $\mathcal{L}^{\alpha\vee\omega} = \mathcal{L}^\alpha \cup \mathcal{L}^\omega$. It is evident that

$$\mathfrak{M}^\sigma := \langle \mathcal{F}_{k,m}(\mathrm{Q}, W), \mathcal{L}^\sigma_{k,m}(\mathrm{Q}, W) \rangle \tag{5.1}$$

is a partial linear space for every admissible symbol σ as above, but it may be trivial for particular values of k, m, r, w etc.: *it may have a void line set*. Let us write $\boldsymbol{\lambda}^\sigma$ for the binary collinearity of points of $\mathfrak{M}^\sigma$. Let $\boldsymbol{\lambda}^{\mathsf{af}}$ be the binary collinearity in

$$\mathfrak{A} = \langle \mathcal{F}_{k,m}(\mathrm{Q}, W), \mathcal{A}_{k,m}, \parallel \rangle.$$

In the first part of this section we shall determine (maximal) cliques of $\boldsymbol{\lambda}^{\sigma}$ for particular values of σ as above. Clearly, each such a clique is a $\boldsymbol{\lambda}$-clique. So, it is contained in an appropriate strong subspace of $\mathfrak{M}$.

We begin with some results which state, generally, that the affine lines in many cases can be 'eliminated': they are definable in terms of other projective lines.

Proposition 5.1. *Assume that $m > 0$ or $w < r - k$. Then for arbitrary triple $U_1, U_2, U_3 \in \mathcal{F}_{k,m}(\mathrm{Q}, W)$ we have*

$$\begin{aligned} &\text{there is a line } L_0 \in \mathcal{A} \text{ s.t. } U_1, U_2, U_3 \in L_0 \iff \\ &\quad \text{there is a triangle } L_1, L_2, L_3 \in \mathcal{L}^{\alpha\vee\omega}_{k,m} \text{ s.t. } U_i \in L_i \text{ for } i = 1,2,3 \\ &\qquad \text{\& there is no } L \in \mathcal{L}^{\alpha\vee\omega} \text{ s.t. } U_i, U_j \in L \text{ for some } 1 \le i < j \le 3. \qquad (5.2) \end{aligned}$$

Proof. Let $U_1, U_2, U_3 \in L_0 \in \mathcal{A}$; then we can write $L_0 = \mathbf{p}(H, B) \cap \mathcal{F}_{k,m}(\mathrm{Q}, W)$ for suitable H, B. As in the proof of Lemma 6.6 we examine extensions of L_0 to maximal strong subspaces of $\mathfrak{M}$. First, let us have a look at $\mathrm{T}(B) \cap \mathcal{F}_{k,m}(\mathrm{Q}, W)$. It is an affine space only when $m = 0$; otherwise it contains a nonaffine semiaffine plane A which contains L_0. The lines on A are all in $\mathcal{L}^{\alpha\vee\omega}$ except the direction of L_0. It suffices to find adequate triangle on A to justify ($\Rightarrow$:) of (5.2).

Next, assume that $m = 0$ and take a look at extensions of L_0 of the form $[H, Y]_k \cap \mathcal{F}_{k,m}(\mathrm{Q}, W)$, then $B \subset Y \in \mathrm{Q}_r$. This extension is an affine space when $\dim(W \cap Y) = r - k$. If there is no such Y, which is assured by the condition assumed, our extension contains a plane A as above and ($\Rightarrow$:) of (5.2) is justified.

To prove ($\Leftarrow$:) it suffices to note that a triangle spans a plane A in $\mathfrak{M}$. Since this plane contains projective lines it is not affine, and since there are non projectively joinable points on A it contains just one direction of affine lines. The rest is evident. □

Thus we have proved the following result.

Proposition 5.2. *Under assumptions made in Proposition 5.1 the class $\mathcal{A}_{k,m}(\mathrm{Q}, W)$ is definable in $\mathfrak{M}^{\alpha\vee\omega}$. That means: $\mathfrak{M}$ is definable in $\mathfrak{M}^{\alpha\vee\omega}$.*

Remark 5.3. Analysing the proof of Proposition 5.1 one can note an even more detailed result:

(i) If $m > 0$ then $\mathcal{A}$ is definable in $\mathfrak{M}^{\omega}$ and therefore then $\mathfrak{M}^{\omega^+}$ is definable in $\mathfrak{M}^{\omega}$.

(ii) If every affine line $L = \mathbf{p}(H, B)$ can be extended to a non-affine star ($\dim(W \cap Y) \ge r - k + m - 3$ for some maximal isotropic Y containing B) then $\mathcal{A}$ is definable in $\mathfrak{M}^{\alpha}$. So, $\mathfrak{M}^{\alpha^+}$ is definable in $\mathfrak{M}^{\alpha}$.

For an arbitrary set X of points we write

$$\mathrm{L}(X) = \{L \in \mathcal{G}_{k,m}(\mathrm{Q}, W) : L \subset X\}.$$

Let us remind well known and fundamental classification of lines in strong subspaces of $\mathfrak{M}$.

Fact 5.4. *Let X be a strong subspace of $\mathfrak{M}$ and $\mathcal{X} = \mathrm{L}(X)$.*

$$\begin{aligned} &\text{If } X \in \mathcal{T}^{\alpha} \text{ then } \mathcal{X} \subset \mathcal{L}^{\alpha}, && \text{if } X \in \mathcal{S}^{\alpha} \text{ then } \mathcal{X} \subset \mathcal{L}^{\alpha^+}, \\ &\text{if } X \in \mathcal{T}^{\omega} \text{ then } \mathcal{X} \subset \mathcal{L}^{\omega^+}, && \text{if } X \in \mathcal{S}^{\omega} \text{ then } \mathcal{X} \subset \mathcal{L}^{\omega}. \end{aligned}$$

Let us note an elementary

Fact 5.5. *Let $\mathfrak{S}$ be a n_0-dimensional slit space with a w_0-dimensional hole i.e. let $\mathfrak{S}$ result from a n_0-dimensional projective space by deleting a w_0-dimensional subspace $\mathcal{D}$. Let $\mathcal{L}_0$ be the class of projective lines of $\mathfrak{S}$ and $\boldsymbol{\lambda}_0$ be the binary collinearity determined by $\mathcal{L}_0$. Then*

(i) *The maximal affine subspaces of $\mathfrak{S}$ (i.e. maximal strong subspace w.r.t. to the family of affine lines of $\mathfrak{S}$) are w_0+1 dimensional affine spaces. Two such subspaces either coincide or are disjoint.*

(ii) *The maximal projective subspaces of $\mathfrak{S}$ are $(n_0 - w_0 - 1)$-dimensional projective spaces. These are linear complements of D and the elements of $\mathcal{K}^*(\boldsymbol{\lambda}_0)$.*

(iii) *Let X be a maximal projective subspace of $\mathfrak{S}$; then $X \in \mathcal{K}^{n_0-w_0}_{\boldsymbol{\lambda}_0}$.*

If $w_0 \le n_0 - 3$ (i.e. every projective line of $\mathfrak{S}$ has two distinct extensions to maximal projective subspaces) then the Chow Theorem holds:

The class $\mathcal{L}_0$ is definable in terms of $\boldsymbol{\lambda}_0$.

Observing Table 2 and Fact 5.5 we conclude with the following.

Corollary 5.6.

(i) *The maximal $\boldsymbol{\lambda}^\alpha$-cliques are $(k - m)$-dimensional projective tops: elements of $\mathcal{T}^\alpha$, and $(r + m - k - \dim(W \cap Y))$-dimensional projective spaces of the form*

$$[H, E]_k, \text{ where } H \subset E \subset Y,\ E \cap ((W \cap Y) + H) = H$$

contained in a suitable element $[H, Y]_k \cap \mathcal{F}_{k,m}(\mathrm{Q}, W)$ of $\mathcal{S}^\alpha$.

(ii) *The maximal $\boldsymbol{\lambda}^\omega$-cliques are $(\dim(W \cap Y) - m)$-dimensional projective stars: elements of $\mathcal{S}^\omega$, and m-dimensional projective spaces of the form*

$$[G, B]_k, \text{ where } G \subset B,\ G \cap (B \cap W) = \Theta$$

contained in a suitable element $\mathrm{T}(B) \cap \mathcal{F}_{k,m}(\mathrm{Q}, W)$ of $\mathcal{T}^\omega$.

(iii) *The maximal $\boldsymbol{\lambda}^{\alpha^+}$-cliques are elements of $\mathcal{T}^\alpha \cup \mathcal{S}^\alpha$, and the maximal $\boldsymbol{\lambda}^{\omega^+}$-cliques are elements of $\mathcal{T}^\omega \cup \mathcal{S}^\omega$.*

(iv) $\mathcal{K}^*(\boldsymbol{\lambda}^{\alpha\vee\omega}) = \mathcal{K}^*(\boldsymbol{\lambda}^\alpha) \cup \mathcal{K}^*(\boldsymbol{\lambda}^\omega)$, *so the maximal $\boldsymbol{\lambda}^{\alpha\vee\omega}$-cliques are of the form* (i) *and of the form* (ii) *above.*

Corollary 5.7. *The following variants of the Chow Theorem hold in projective reducts of $\mathfrak{M}$.*

(i) *If $m > 1$ then $\mathfrak{M}^\omega$ is definable in $\langle \mathcal{F}_{k,m}(\mathrm{Q}, W), \boldsymbol{\lambda}^\omega \rangle$.*

(ii) *If every projective line $L = \mathbf{p}(H, B) \in \mathcal{L}^\alpha$ can be extended to a non-affine star ($\dim(W \cap Y) \le r - k + m - 2$ for some maximal isotropic Y containing B) then $\mathfrak{M}^\alpha$ is definable in $\langle \mathcal{F}_{k,m}(\mathrm{Q}, W), \boldsymbol{\lambda}^\alpha \rangle$.*

6 Parallelism, horizon, projective completion(s)

Let us summarize the following

(i) $\{L^\infty : L \in \mathcal{A}_{k,m}\} \subset \mathcal{F}_{k,m+1}(\mathrm{Q}, W)$.

(ii) by Lemma 2.2(ii) $\{L^\infty : L \in \mathcal{A}_{k,m}\} \supset \{U \in \mathcal{F}_{k,m+1}(\mathrm{Q}, W) : U^\perp \cap W \not\subset U\}$,

(iii) $\{L^\infty : L \in \mathcal{A}_{k,m}\} \supset \mathcal{F}_{k,m+1}(\mathrm{Q}, W)$, when $w < n + m + 1 - 2k$ by Corollary 2.5.

Note 6.1. The set $\{L^\infty : L \in \mathcal{A}_{k,m}\}$ will be frequently referred to as *the horizon of* $\mathfrak{M}$. We warn that, generally it does not coincide with the horizon $\mathrm{Q}_k \setminus \mathcal{F}_{k,m}(\mathrm{Q}, W)$ as defined in Section 1.

Note that the inequality in (iii) above is only sufficient. One can compute e.g.

Lemma 6.2. *Let $W \in \mathrm{Q}$. Then the claim of Corollary 2.5 holds i.e. for every $U \in \mathcal{F}_{k,m+1}(\mathrm{Q}, W)$ there is an $L \in \mathcal{A}_{k,m}(\mathrm{Q}, W)$ such that $U = L^\infty$. Consequently,*

$$\{L^\infty : L \in \mathcal{A}_{k,m}\} = \mathcal{F}_{k,m+1}(\mathrm{Q}, W).$$

Proof. By assumption, $\dim(U \cap W) = m + 1$. There are extensions $Y_1, Y_2 \in \mathrm{Q}_r$ such that $U \subset Y_1$, $W \subset Y_2$, and $Y_1 \cap Y_2 = U \cap W$. Take $B \in [U, Y_1]_{k+1}$; then $B \in \mathcal{F}_{k+1,m+1}(\mathrm{Q}, W)$ and we are through. □

For a subset X of $\mathcal{F}_{k,m}(\mathrm{Q}, W)$ we write

$$X^\infty := \{N^\infty : \mathcal{A}_{k,m} \ni N \subset X\}.$$

Lemma 6.3. *Let $L = \mathbf{p}(H, B) \in \mathcal{L}^\omega_{k,m+1} \cup \mathcal{L}^\alpha_{k,m+1}$.*

(i) *If $L \in \mathcal{L}^\alpha$ then there is in $\mathfrak{M}$ a plane $A = [G, B]_k \cap \mathcal{F}_{k,m}(\mathrm{Q}, W)$ with $G \in \mathcal{F}_{k-2,m}(\mathrm{Q}, W)$ such that $A^\infty = L$.*

(ii) *Assume that $w < n + m - 2k$. If $L \in \mathcal{L}^\omega$ then $A = [H, E]_k \cap \mathcal{F}_{k,m}(\mathrm{Q}, W)$ with some $E \in \mathcal{F}_{k+2,m+2}(\mathrm{Q}, W)$ is a plane in $\mathfrak{M}$ such that $A^\infty = L$.*

Proof. Ad (i): By assumption, $B \in \mathcal{F}_{k+1,m+1}(\mathrm{Q}, W)$ and $H \in \mathcal{F}_{k-1,m+1}(\mathrm{Q}, W)$. There is a point $U \in L$, so $U \in \mathcal{F}_{k,m+1}(\mathrm{Q}, W)$. By Lemma 2.1(iii) there is an H_0 such that $U \succ H_0 \in \mathcal{F}_{k-1,m}(\mathrm{Q}, W)$. Set $G = H_0 \cap H$; clearly, $\dim(G) = k - 2$, so $[G, B]_k$ is a plane in $\mathbf{P}_k(\mathrm{Q})$. Taking into account the fact that $H, H_0 \succ G$ we obtain $\dim(G \cap W) \in \{m+1, m\}$ and $\dim(G \cap W) \in \{m, m-1\}$. Thus $\dim(G \cap W) = m$. As $L \subset [G, B]_k$ and $[G, B]_k \supset [H_0, B]_k$ while $[H_0, B]_k \cap \mathcal{F}_{k,m}(\mathrm{Q}, W) \in \mathcal{A}_{k,m}$ we get that $A \cap \mathcal{F}_{k,m}(\mathrm{Q}, W)$ is a plane in $\mathfrak{M}$ with $A^\infty = L$.

Ad (ii): By assumption, $B \in \mathcal{F}_{k+1,m+2}(\mathrm{Q}, W)$ and $H \in \mathcal{F}_{k-1,m}(\mathrm{Q}, W)$. As above, we take any $U \in L$, so $U \in \mathcal{F}_{k,m+1}(\mathrm{Q}, W)$. By assumption of (ii) (they yield $w < n + (m+2) - 2(k+1)$) and Lemma 2.2(iii) there is an E such that $B \prec E \in \mathcal{F}_{k+2,m+2}(\mathrm{Q}, W)$. Next, there is $B_0 \in \mathcal{F}_{k+1,m+1}(\mathrm{Q}, W)$ with $U \subset E$: $B = U + \langle b \rangle$ with a $b \in W$ and $E = B + \langle e \rangle$ with an $e \notin W$; we take $B_0 = U + \langle e \rangle$. Clearly, $E = B + B_0$ and $[H, B_0]_k \cap \mathcal{F}_{k,m}(\mathrm{Q}, W) \in \mathcal{A}_{k,m}$. As above we argue that $A = [H, E]_k \cap \mathcal{F}_{k,m}(\mathrm{Q}, W)$ is a plane in $\mathfrak{M}$, and $L = A^\infty$. □

Roughly speaking, Lemma 6.3 gives sufficient condition under which a (projective) line L of $\mathbf{A}_{k,m+1}(\mathrm{Q},W)$ can be considered as a 'horizon' – the set of improper points of a plane in $\mathbf{A}_{k,m}(\mathrm{Q},W)$. On the other hand, considering classification of planes in $\mathbf{A}_{k,m}(\mathbb{V},W)$ presented in some details in [14] we easily conclude with the following

Lemma 6.4. *Let $X \subset \mathrm{Sub}_k(\mathbb{V})$ and $A = X \cap \mathcal{F}_{k,m}(\mathrm{Q},W)$ be a plane of $\mathfrak{M}$ such that A^∞ is a line of $\mathbf{A}_{k,m+1}(\mathrm{Q},W)$. Then one of the following holds:*

(i) *$X = [G,B]_k$ for some $G \in \mathcal{F}_{k-2,m}(\mathrm{Q},W)$, $B \in \mathcal{F}_{k+1,m+1}(\mathrm{Q},W)$.*

(ii) *$X = [H,E]_k$ for some $H \in \mathcal{F}_{k-1,m}(\mathrm{Q},W)$ and $E \in \mathcal{F}_{k+2,m+2}(\mathrm{Q},W)$.*

Conversely, if X is defined by (i) *then $X \cap \mathcal{F}_{k,m+1}(\mathrm{Q},W) = \big(X \cap \mathcal{F}_{k,m}(\mathrm{Q},W)\big)^\infty \in \mathcal{L}^\alpha_{k,m+1}$, and if* (ii) *holds, then $X \cap \mathcal{F}_{k,m+1}(\mathrm{Q},W) \in \mathcal{L}^\omega_{k,m+1}$.*

So, Lemma 6.4 states that the 'horizon' of any (affine) plane of $\mathcal{F}_{k,m}(\mathrm{Q},W)$ is a (projective) line of $\mathcal{F}_{k,m+1}(\mathrm{Q},W)$. As usually, the conditions of Lemma 6.3 are only sufficient. Dealing with concrete cases one should look for suitable extendability more or less 'by hand'. Let us quote an example:

Lemma 6.5. *Let $W \in \mathrm{Q}$. If $L = \mathbf{p}(H,B) \in \mathcal{L}^\omega_{k,m+1}$ then $A = [H,E]_k \cap \mathcal{F}_{k,m}(\mathrm{Q},W)$ with some $E \in \mathcal{F}_{k+2,m+2}(\mathrm{Q},W)$ is a plane in $\mathfrak{M}$ such that $A^\infty = L$.*

Hint. With the reasoning as in the proof of Lemma 6.3(ii) we look for an E such that $B \prec E \in \mathcal{F}_{k+2,m+2}(\mathrm{Q},W)$. It suffices to find an E such that $E \cap W = B \cap W$ just considering suitable maximal isotropic extensions of B and W. □

To accomplish this part of investigations on the parallelism let us check if directions are 'isolated': when for an affine line L of $\mathfrak{M}$ there are other lines parallel to L and coplanar with L; with the plane in question being affine in $\mathfrak{M}$.

Lemma 6.6. *Let $L = \mathbf{p}(H,B) \in \mathcal{A}_{k,m}$ and $U = L^\infty$.*

(i) *Assume that $k > m+1$. There is an $L_0 = \mathbf{p}(H_0,B) \in \mathcal{L}^\alpha_{k,m+1}$ such that $U \in L_0$ and $A = [H_0 \cap H, B]_k \cap \mathcal{F}_{k,m}(\mathrm{Q},W)$ is a plane in $\mathfrak{M}$ such that $A^\infty = L_0$. We have $\dim((H_0 \cap H) \cap W) = m-1$.*

(ii) *If B has an extension to a $Y \in \mathrm{Q}_r$ such that $\dim(W \cap Y) \ge m+2$ (this yields, necessarily, $m+2 \le r_W$) then there exists an $L_1 = \mathbf{p}(H,B_1) \in \mathcal{L}^\omega_{k,m+1}$ such that $U \in L_1$ and $A = [H, B+B_1]_k$ is a plane in $\mathfrak{M}$ such that $A^\infty = L_1$. We have $\dim((B_1+B) \cap W) = m+2$.*

Proof. Let us begin with a reminder: $H \in \mathcal{F}_{k-1,m}(\mathrm{Q},W)$, $B \in \mathcal{F}_{k+1,m+1}(\mathrm{Q},W)$. Have a look at the extension of L to a top $T = \mathrm{T}(B) \cap \mathcal{F}_{k,m}(\mathrm{Q},W)$ (an ω-top in this case). Since $k > m+1$, this is a semiaffine space, and its hole is at least 1-dimensional. Let L_0 be any line of $\mathbf{P}_k(\mathbb{V})$ contained in this hole and A be the plane spanned by $L \cup L_0$. That way we justify (i).

Next, let us look for appropriate extension of L to an α-star $S = [H,Y]_k \cap \mathcal{F}_{k,m}(\mathrm{Q},W)$. In general, it is a $(r-k)$-dimensional semiaffine space. Since $\mathrm{Q}_{k+1} \neq \emptyset$ we have $k+1 \le r$. So, S is at least a line. To assure that the hole of S contains at least a line of $\mathbf{P}_k(\mathbb{V})$ we must assume that $\dim(W \cap Y) \ge m+2$. That way we justify (ii). □

Let us remind that for distinct affine lines L_1, L_2 contained in a strong subspace of $\mathbf{A}_{k,m}(\mathbb{V}, W)$ their parallelism $\parallel$ can be characterized by the following formula (so called *Veblenian parallelism*).

$$L_1 \parallel_{\mathsf{V}} L_2 \iff \text{there are lines } L_1', L_2' \text{ s.t. } |L_1' \cap L_2'| = 1, \\ \text{and } L_1' \cap L_2' \cap L_i = \emptyset,\ |L_i' \cap L_j| = 1 \text{ for } i = 1, 2, \quad (6.1)$$

and then $L_1 \parallel L_2$ iff $L_1 \parallel_{\mathsf{V}} L_2$. It is easy to note that the same formula (6.1) characterizes parallelism of affine lines contained in a common strong subspace of $\mathfrak{M}$.

Let us begin with a special form of connectedness of the space of lines over $\mathfrak{M}$:

Lemma 6.7. *Let $U \in \mathcal{F}_{k,m+1}(\mathrm{Q}, W)$ and $L_1^\infty = U = L_2^\infty$ for $L_1, L_2 \in \mathcal{A}_{k,m}$. Moreover, assume that $k \leq r - 2$. Then there are lines $M_1, \dots, M_t \in \mathcal{A}_{k,m}$ ($t \leq r + 1$) such that $L_1 = M_1$, $L_2 = M_t$, and $M_i^\infty = U$, M_i, M_{i+1} are in a strong (semiaffine) subspace of $\mathfrak{M}$ or $M_i = M_{i+1}$, for $i = 1, \dots, t - 1$.*

Proof. Write $M_1 := L_1$. We have $H_1, H_2 \subset U \subset B_1, B_2, U \in \mathcal{F}_{k,m+1}(\mathrm{Q}, W)$ and $B_i \in \mathcal{F}_{k+1,m+1}(\mathrm{Q}, W)$, $H_i \in \mathcal{F}_{k-1,m}(\mathrm{Q}, W)$ for $i = 1, 2$. Put $N_1 := [H_1, B_2]_k \cap \mathcal{F}_{k,m}(\mathrm{Q}, W)$, $N_2 := [H_2, B_1]_k \cap \mathcal{F}_{k,m}(\mathrm{Q}, W)$. Then $N_1, N_2 \in \mathcal{A}_{k,m}$, $N_2^\infty = U = N_1^\infty$.

If $L_1 = N_2$ we set $M_2 := L_1$. Assume that $L_1 \neq N_2$. Note that $L_1, N_2 \in \mathrm{T}(B_1) \cap \mathcal{F}_{k,m}(\mathrm{Q}, W) \in \mathcal{T}^\omega$. So, we set $M_2 := N_2$.

Observe that $N_2, L_2 \subset [H_2, V]_k \cap \mathcal{F}_{k,m}(\mathrm{Q}, W)$. So, the problem reduces to find a required sequence of lines in the projective star $\mathrm{S}(H_2)$. Let $B_1 \subset Y' \in \mathrm{Q}_r$, $B_2 \subset Y'' \in \mathrm{Q}_r$. There is a sequence $Y_2, \dots, Y_t$ of elements of Q_r such that $Y' = Y_2$, $Y'' = Y_t$, and $U \subset Y_i$, $E_i := Y_i \cap Y_{i+1}$, $\dim(E_i) = r - 1$ for $i = 2, \dots, t - 1$, $t \leq r + 1$. Then $\dim(E_i \cap W) \geq m + 1$. From our assumption $k + 1 \leq r - 1 = \dim(E_i)$. So, for every $i = 3, \dots, t - 1$ one can find D_i such that $U \prec D_i \subset E_{i-1}$ and $\dim(D_i \cap W) = m + 1$. With $N_i = [H_2, D_i]_k$ we close our proof. □

Corollary 6.8. *Under assumptions of Lemma 6.7 the parallelism $\parallel$ in $\mathfrak{M}$ coincides with the transitive closure of $\parallel_{\mathsf{V}}$. Actually, it is the $(r + 1)$-th relational power $\underbrace{\parallel_{\mathsf{V}} \circ \cdots \circ \parallel_{\mathsf{V}}}_{(r+1) \text{ times}}$ of $\parallel_{\mathsf{V}}$, defined by* (6.1)*, and therefore $\parallel$ is definable in the incidence structure $\mathfrak{M}$.*

As an immediate corollary we conclude with the following theorem.

Theorem 6.9. *Assume the following*

(1) *$w < n + m - 2k$ to assure that every line in $\mathcal{L}^\omega_{k,m+1}$ can be extended to a nontrivial α-star of $\mathfrak{M}$ (cf. Lemma 6.3),*

(2) *$w < n + m + 1 - 2k$ to assure extendability of each improper point to an affine line (cf. Corollary 2.5),*

(3) *$m+1 > 0$ or $w < r-k$ to assure definability of $\mathcal{A}_{k,m+1}$ in $\mathbf{A}_{k,m+1}(\mathrm{Q}, W)$ in terms of its projective lines (cf. Proposition 5.2),*

(4) *$k \leq r - 2$, to assure definability of parallelism in $\mathfrak{M}$ (cf. Corollary 6.8).*

Then $\mathbf{A}_{k,m+1}(\mathrm{Q}, W)$ is definable within $\mathbf{A}_{k,m}(\mathrm{Q}, W)$.

In analogy to [17] in the fragment of $\mathbf{P}_k(\mathrm{Q})$ determined by $\mathcal{R} := \mathcal{F}_{k,m}(\mathrm{Q},W) \cup \mathcal{F}_{k,m+1}(\mathrm{Q},W)$ (i.e. the points of $\mathfrak{M}$ and the points of the "affine horizon" of $\mathfrak{M}$) we distinguish two substructures corresponding to two possible sorts of lines. Let us set $\mathcal{L}^{\tau}_{k,m} := \big\{[H,B]_k : H \in \mathcal{F}_{k-1,m}(\mathrm{Q},W),\ B \in \mathcal{F}_{k+1,m+1}(\mathrm{Q},W)\big\}$; it is seen that $\mathcal{L}^{\tau}_{k,m} = \{\bar{L} : L \in \mathcal{A}_{k,m}\}$. Note evident relation:

$$\big\{L : L \text{ is a line of } \mathbf{P}_k(\mathrm{Q}),\ L \subset \mathcal{R}\big\} = \mathcal{L}^{\alpha}_{k,m} \cup \mathcal{L}^{\omega}_{k,m} \cup \mathcal{L}^{\alpha}_{k,m+1} \cup \mathcal{L}^{\omega}_{k,m+1} \cup \mathcal{L}^{\tau}_{k,m}. \quad (6.2)$$

We define (write: $-\alpha = \omega,\ -\omega = \alpha$)

$$\mathfrak{N}^{\sigma} := \langle \mathcal{R}, \mathcal{L}^{\sigma}_{k,m} \cup \mathcal{L}^{\tau}_{k,m} \cup \mathcal{L}^{-\sigma}_{k,m+1}\rangle \text{ with } \sigma \in \{\alpha,\omega\}.$$

Evidently, $\mathfrak{M}^{\sigma}$ can be embedded into $\mathfrak{N}^{\sigma}$. Intuitively, while the structure

$$\big\langle \mathcal{R}, \{L : L \text{ is a line of } \mathbf{P}_k(\mathrm{Q}),\ L \subset \mathcal{R}\}\big\rangle$$

can be considered as a *projective completion of* $\mathfrak{M}$ and, under specific assumptions, it is definable in $\mathfrak{M}$, $\mathfrak{N}^{\sigma}$ is a projective completion of $\mathfrak{M}^{\sigma}$.

To close this part it is worth to note the following analogue of Remark 5.3 and, at the same time, an analogue of [17, Fact 3.1].

Remark 6.10. Assume (2) and (4) from Theorem 6.9.

(i) If $m > 0$ (cf. Remark 5.3) then the structure $\mathfrak{N}^{\omega}$ is definable in $\mathfrak{M}^{\omega}$.

(ii) If for each affine line $L = \mathbf{p}(H,B)$ there is a maximal isotropic Y such that $B \subset Y$ and $\dim(W \cap Y) \geq r - k + m - 3$ (cf. Remark 5.3) and $w < n + m - 2k$ (cf. Lemma 6.3), then the structure $\mathfrak{N}^{\alpha}$ is definable in $\mathfrak{M}^{\alpha}$.

According to Corollary 2.5 and Lemma 6.3, under condition $w < n + m - 2k$ each point of $\mathbf{A}_{k,m+1}(\mathrm{Q},W)$ is a direction of a line in $\mathfrak{M}$ and each line of $\mathbf{A}_{k,m+1}(\mathrm{Q},W)$ is a direction of a plane in $\mathfrak{M}$. This observation leads to the following.

Proposition 6.11. *If $w < n + m - 2k$, then the horizon $\mathbf{A}_{k,m+1}(\mathrm{Q},W)$ of $\mathfrak{M}$ can be defined in terms of $\mathfrak{A}$.*

Finally, the question arises whether the adjacency of $\mathfrak{M}$ is definable purely in terms of the geometry of $\mathfrak{A}$? Unfortunately, the answer is not straightforward. The reasoning for spine spaces that justifies [17, Proposition 4.12], based on the fact that two distinct stars or tops of $\mathfrak{A}$ share no line on the horizon, cannot be adopted here without significant alterations. Note that if $\mathcal{L}^{\omega} \cup \mathcal{L}^{\alpha} = \emptyset$, then practically $\mathfrak{A} = \mathfrak{M}$. Therefore we assume that $\mathcal{L}^{\omega} \cup \mathcal{L}^{\alpha} \neq \emptyset$.

Theorem 6.12. *If the ground field of $\mathbb{V}$ is of odd characteristic, then the structure $\mathfrak{M}$ can be defined in terms of $\mathfrak{A}$.*

Proof. The proof is divided into several steps. For distinct points U_1, U_2 of $\mathfrak{M}$ we define

$$\begin{aligned} U_1 \sim^{+} U_2 \ &:\Longleftrightarrow\ U_1, U_2 \subset B \text{ for some } B \in \mathcal{F}_{k+1,m+1}(\mathrm{Q},W),\\ U_1 \sim_{-} U_2 \ &:\Longleftrightarrow\ H \subset U_1, U_2 \text{ for some } H \in \mathcal{F}_{k-1,m}(\mathrm{Q},W),\\ U_1 \sim U_2 \ &:\Longleftrightarrow\ U_1 \sim^{+} U_2 \text{ or } U_1 \sim_{-} U_2. \end{aligned}$$

Note that $U_1 \sim^{+} U_2$ yields that either $U_1 \boldsymbol{\lambda}^{\mathrm{af}}\, U_2$ or $U_1 \boldsymbol{\lambda}^{\omega}\, U_2$, while $U_1 \sim_{-} U_2$ yields that either $U_1 \boldsymbol{\lambda}^{\mathrm{af}}\, U_2$, $U_1 \boldsymbol{\lambda}^{\alpha}\, U_2$, or U_1, U_2 are not collinear in $\mathfrak{M}$.

Step 1. The following conditions are equivalent.

(i) $U_1 \sim^+ U_2$ or $U_1 \sim_- U_2$.

(ii) There is a plane Π_1 through U_1 parallel to a plane Π_2 through U_2 in $\mathfrak{A}$.

Proof of Step 1. (i) $\Longrightarrow$ (ii): Assume that $U_1, U_2 \subset B \in \mathcal{F}_{k+1,m+1}(\mathrm{Q}, W)$, then $\mathrm{T}(B)$ is a semiaffine space (of the form $\mathcal{T}^\omega$) and one easily finds Π_1, Π_2 in it.

Next, assume that $U_1, U_2 \supset H \in \mathcal{F}_{k-1,m}(\mathrm{Q}, W)$. Set $B := U_1 + U_2$. If $B \in \mathrm{Q}$ then $L = \mathbf{p}(H, B)$ is a line of $\mathfrak{M}$. Applying analogous reasoning we find Π_1, Π_2 in an extension $[H, Y]_k$ of the type $\mathcal{S}^\alpha$. If $B \notin \mathrm{Q}$ then, in any case L is a line of the surrounding $\mathbf{A}_{k,m}(\mathbb{V}, W)$. Let us restrict to the subspaces around H; they form a spine space in the projective space $\mathbf{P}_1(\mathbb{V}/H)$ with the quadric $\mathrm{Q}(\xi/H)$ distinguished. Projective reasoning proves that required planes Π_1, Π_2 exist.

(ii) $\Longrightarrow$ (i): Let Π_i be parallel planes of $\mathfrak{A}$ with $U_i \in \Pi_i$, $i = 1, 2$. Let $L_0 = \Pi_1^\infty = \Pi_2^\infty$ be the improper line of Π_i. Then $L_0 \in \mathcal{L}^\alpha_{k,m+1}$ or $L_0 \in \mathcal{L}^\omega_{k,m+1}$. In the first case L_0, U_1, U_2 are contained in the (unique) extension to a top $\mathrm{T}(B)$ with $B \in \mathcal{F}_{k+1,m+1}(\mathrm{Q}, W)$ and therefore $U_1 \sim^+ U_2$. In the second case extensions of Π_i to maximal strong subspaces have form $[H, Y_i]_k$ (they have L_0 in common), where $H \in \mathcal{F}_{k-1,m}(\mathrm{Q}, W)$. So, $U_1 \sim_- U_2$. $\Diamond$

Let us write

$$\mathfrak{M}_0 := \mathbf{A}_{k,m}(\mathbb{V}, W) \restriction \mathcal{F}_{k,m}(\mathrm{Q}, W)$$

for the surrounding spine space with point set restricted to totally isotropic subspaces. Note that the distinction between $\mathfrak{M}$ and $\mathfrak{M}_0$ consists in the range of their line sets. More precisely, for a line $L = \mathbf{p}(H, B)$ of $\mathfrak{M}_0$ its base B needs not to be totally isotropic and

$$L \text{ is a line of } \mathfrak{M} \quad \text{iff} \quad |L| \geq 3.$$

Step 2. Let $U_1, U_2 \in \mathcal{F}_{k,m}(\mathrm{Q}, W)$ and $U_1 \neq U_2$. The following conditions are equivalent.

(i) $U_1 \sim U_2$.

(ii) U_1, U_2 are collinear in $\mathfrak{M}_0$ with exception when the line L of $\mathfrak{M}_0$ which joins them has form $L = \mathbf{p}(H, B)$ where $H \in \mathcal{F}_{k-1,m-1}(\mathrm{Q}, W)$, $B \in \mathcal{F}_{k+1,m+1}(\mathbb{V}, W)$, and $B \notin \mathrm{Q}$ (i.e. L is an ω-line in $\mathbf{A}_{k,m}(\mathbb{V}, W)$).

Proof of Step 2. (i) $\Longrightarrow$ (ii): It is clear that U_1, U_2 are collinear in the surrounding Grassmann space. If $U_1 \sim^+ U_2$, then they lie on an affine or ω-line in $\mathfrak{M}$ by Table 1, while if $U_1 \sim_- U_2$, then they lie on an affine or α-line in $\mathfrak{M}_0$.

(ii) $\Longrightarrow$ (i): Now, let U_1, U_2 be collinear in $\mathfrak{M}_0$. Hence $U_1, U_2 \in \mathbf{p}(H, B)$ for suitable H, B. If $\dim(B \cap W) = m$, then $\dim(H \cap W) = m$ and thus $U_1 \sim_- U_2$. If $\dim(B \cap W) = m + 1$, then two cases arise: $\dim(H \cap W) = m, m - 1$. In the former we have $U_1 \sim_- U_2$. In the later $H \in \mathcal{F}_{k-1,m-1}(\mathrm{Q}, W)$ and $B \in \mathcal{F}_{k+1,m+1}(\mathbb{V}, W)$. If $B \in \mathrm{Q}$, then $U_1 \sim^+ U_2$, otherwise we get the excluded case. $\Diamond$

Step 3. A set X of points of $\mathfrak{A}$ is a maximal at least 3-element $\sim$-clique iff X has one of the following forms:

(a) $X = \mathrm{T}(B)$ for some $B \in \mathcal{F}_{k+1,m+1}(\mathrm{Q}, W)$,

(b) $X = \mathrm{T}(B)$ for some $B \in \mathcal{F}_{k+1,m}(\mathrm{Q}, W)$,

(c) $X = \mathrm{S}(H)$ for some $H \in \mathcal{F}_{k-1,m}(\mathrm{Q}, W)$, or

(d) $X = [H, Y] \cap \mathcal{F}_{k,m}(\mathrm{Q}, W)$ for some $H \in \mathcal{F}_{k-1,m-1}(\mathrm{Q}, W)$ and $H \subset Y \in \mathrm{Q}_r$.

Proof of Step 3. It is easy to verify that sets defined in (a) – (d) are maximal $\sim$-cliques. Now, let X be a maximal at least 3-element $\sim$-clique. In view of Step 2, X is a subset of a clique in $\mathfrak{M}_0$. So, we need general tops $\mathrm{T}_0(B) = [\Theta, B]_k$ for $B \in \mathrm{Sub}_{k+1}(\mathbb{V})$ and stars $\mathrm{S}_0(H) = [H, V]_k$ for $H \in \mathrm{Sub}_{k-1}(\mathbb{V})$. Let us examine the following four cases:

$X \subseteq \mathrm{T}_0(B)$, $B \in \mathcal{F}_{k+1,m+1}(\mathbb{V}, W)$
: If $B \in \mathrm{Q}$, then any two points of $\mathfrak{M}_0$ in $\mathrm{T}(B)$ are $\sim^+$-adjacent and thus $X = \mathrm{T}(B) \cap \mathcal{F}_{k,m}(\mathrm{Q}, W)$ is a $\sim$-clique as in (a). If $B \notin \mathrm{Q}$, then $|X| \le 2$ by [19, Proposition 4.4], a contradiction.

$X \subseteq \mathrm{T}_0(B)$, $B \in \mathcal{F}_{k+1,m}(\mathbb{V}, W)$
: Since $|X| \ge 3$ we have $B \in \mathrm{Q}$ by [19, Proposition 4.4]. Any two points of $\mathfrak{M}_0$ in $\mathrm{T}(B)$ are $\sim_-$-adjacent, so $X = \mathrm{T}(B) \cap \mathcal{F}_{k,m}(\mathrm{Q}, W)$ has form (b).

$X \subseteq \mathrm{S}_0(H)$, $H \in \mathcal{F}_{k-1,m}(\mathbb{V}, W)$
: Note that $H \in \mathrm{Q}$ as X is nonempty. This implies that any two points of $\mathfrak{M}_0$ in $\mathrm{S}(H)$ are $\sim_-$-adjacent. Consequently, $X = \mathrm{S}(H) \cap \mathcal{F}_{k,m}(\mathrm{Q}, W)$ has form (c).

$X \subseteq \mathrm{S}_0(H)$, $H \in \mathcal{F}_{k-1,m-1}(\mathbb{V}, W)$
: As above $H \in \mathrm{Q}$. The points of $\mathfrak{M}_0$ in $\mathrm{S}_0(H)$ are $\sim$-adjacent iff they are $\sim^+$-adjacent i.e. they are collinear in the surrounding polar Grassmann space where the appropriate clique has form $[H, Y]_k$ for some $Y \in \mathrm{Q}_r$ (cf. [7, Section 3]). Hence $X = [H, Y]_k \cap \mathcal{F}_{k,m}(\mathrm{Q}, W)$ has form (d).

That way we obtain the desired list (a) – (d). ◊

Note that the $\boldsymbol{\lambda}^{\mathsf{af}}$-cliques are essentially smaller than $\sim$-cliques.

Step 4. At least 3-element minimal intersections of the maximal $\sim$-cliques are lines of $\mathfrak{M}$.

Proof of Step 4. Let $\mathcal{K}_{\mathrm{x}}$ be the family of cliques of the form (x) defined in Step 3. Let X_1, X_2 be two distinct $\sim$-cliques and $Z = X_1 \cap X_2$. If $X_1, X_2 \in \mathcal{K}_{\text{(a)}} \cup \mathcal{K}_{\text{(b)}}$, $X_1, X_2 \in \mathcal{K}_{\text{(c)}}$, or $X_1 \in \mathcal{K}_{\text{(b)}} \cup \mathcal{K}_{\text{(c)}}$ and $X_2 \in \mathcal{K}_{\text{(d)}}$, then Z contains at most a single point. If $X_1 \in \mathcal{K}_{\text{(a)}}$ and $X_2 \in \mathcal{K}_{\text{(c)}}$, then Z is an affine line of $\mathfrak{M}$. If $X_1 \in \mathcal{K}_{\text{(b)}}$ and $X_2 \in \mathcal{K}_{\text{(c)}}$, then Z is an α-line of $\mathfrak{M}$. If either $X_1 \in \mathcal{K}_{\text{(a)}}$ and $X_2 \in \mathcal{K}_{\text{(d)}}$ or $X_1, X_2 \in \mathcal{K}_{\text{(d)}}$, then at least 3-element minimal Z is an ω-line of $\mathfrak{M}$. ◊

It is evident that every projective line of $\mathfrak{M}$ can be presented as the intersection of cliques enumerated in Step 3. So, applying Step 4 we get the line set of $\mathfrak{M}$ recovered which makes the proof of Theorem 6.12 complete. □

Remark 6.13. The horizon of a star in $\mathfrak{M}$ may have strange properties. Assume that $W \in \mathrm{Q}$ and let $H \in \mathrm{Q}_{k-1}$, $H \subset W^{\perp}$. Set $m := \dim(H \cap W)$. This means that $k - 1 + w - m < r$. Then there is an $Y_0 \in \mathrm{Q}_r$ such that $H \cup W \subset Y_0$. So, $Y_0 \cap W = W$. Write $S_0 = [H, Y_0]_k \cap \mathcal{F}_{k,m}(\mathrm{Q}, W)$. Then $S_0^{\infty} = [H, H + W]_k$. Take any $S = [H, Y]_k \cap \mathcal{F}_{k,m}(\mathrm{Q}, W)$ contained in $\mathrm{S}(H)$. Then $S^{\infty} = [H, H + (W \cap Y)]_k \subset [H, H + W]_k = S_0^{\infty}$. So, in this case

$$\mathrm{S}(H)^{\infty} \text{ is the projective space } [H, H + W]_k \text{ contained in } \mathfrak{M}^{\infty}.$$

Nevertheless, $\mathrm{S}(H)$ contains affine subspaces of different dimensions.

Note that in this case $k-m-1 = \dim(\mathrm{T}(B)^\infty) = \dim(\mathrm{S}(H)^\infty) = w-m-1$ yields $w = k$, so horizons of stars and tops may have equal dimensions only when $\mathfrak{M}$ consists of points in Q_k that are at the fixed distance $k-m$ from the fixed point W.

Remark 6.14. Theorem 6.12 for polar spine spaces and its counterpart [17, Proposition 4.12] for spine spaces both say that the respective geometry depends only on affine lines together with parallelism, that is, projective lines can be recovered using affine line structure. However, the idea of the proof presented in this paper is more general than that in [17] because it does not rely on specific horizons and intersections of stars which are completely different in $\mathfrak{M}$ and in $\mathbf{A}_{k,m}(\mathbb{V}, W)$. As such it can be applied for spine spaces and is expected to give less complex reasonings.

7 Classifications

Table 1: The classification of lines in a polar spine space $\mathbf{A}_{k,m}(\mathrm{Q}, W)$.

Class	Representative line $g = \mathbf{p}(H,B) \cap \mathcal{F}_{k,m}(\mathrm{Q}, W)$	g^∞
$\mathcal{A}_{k,m}(\mathrm{Q}, W)$	$H \in \mathcal{F}_{k-1,m}(\mathrm{Q}, W),\ B \in \mathcal{F}_{k+1,m+1}(\mathrm{Q}, W)$	$H + (B \cap W)$
$\mathcal{L}^\alpha_{k,m}(\mathrm{Q}, W)$	$H \in \mathcal{F}_{k-1,m}(\mathrm{Q}, W),\ B \in \mathcal{F}_{k+1,m}(\mathrm{Q}, W)$	–
$\mathcal{L}^\omega_{k,m}(\mathrm{Q}, W)$	$H \in \mathcal{F}_{k-1,m-1}(\mathrm{Q}, W),\ B \in \mathcal{F}_{k+1,m+1}(\mathrm{Q}, W)$	–

Each strong subspace X of a polar spine space is a slit space, that is a projective space $\mathbf{P}$ with a subspace $\mathcal{D}$ removed. In the extremes $\mathcal{D}$ can be void, then X is basically a projective space, or a hyperplane, then X is an affine space.

Table 2: The classification of stars and tops in a polar spine space $\mathbf{A}_{k,m}(\mathrm{Q}, W)$.

Class	Representative subspace		
	$\dim(\mathbf{P})$	$\mathcal{D}$	$\dim(\mathcal{D})$
$\mathcal{S}^\omega_{k,m}(\mathrm{Q}, W)$	$[H, (H+W) \cap Y]_k : H \in \mathcal{F}_{k-1,m-1}(\mathrm{Q}, W), Y \in \mathrm{Q}_r, H \subset Y$		
	$\dim(W \cap Y) - m$	$\emptyset$	-1
$\mathcal{S}^\alpha_{k,m}(\mathrm{Q}, W)$	$[H, Y]_k \cap \mathcal{F}_{k,m}(\mathrm{Q}, W) : H \in \mathcal{F}_{k-1,m}(\mathrm{Q}, W), Y \in \mathrm{Q}_r, H \subset Y$		
	$r-k$	$[H, (H+W) \cap Y]_k$	$\dim(W \cap Y) - m - 1$
$\mathcal{T}^\alpha_{k,m}(\mathrm{Q}, W)$	$[B \cap W, B]_k : B \in \mathcal{F}_{k+1,m}(\mathrm{Q}, W)$		
	$k-m$	$\emptyset$	-1
$\mathcal{T}^\omega_{k,m}(\mathrm{Q}, W)$	$[\Theta, B]_k \cap \mathcal{F}_{k,m}(\mathrm{Q}, W) : B \in \mathcal{F}_{k+1,m+1}(\mathrm{Q}, W)$		
	k	$[B \cap W, B]_k$	$k-m-1$

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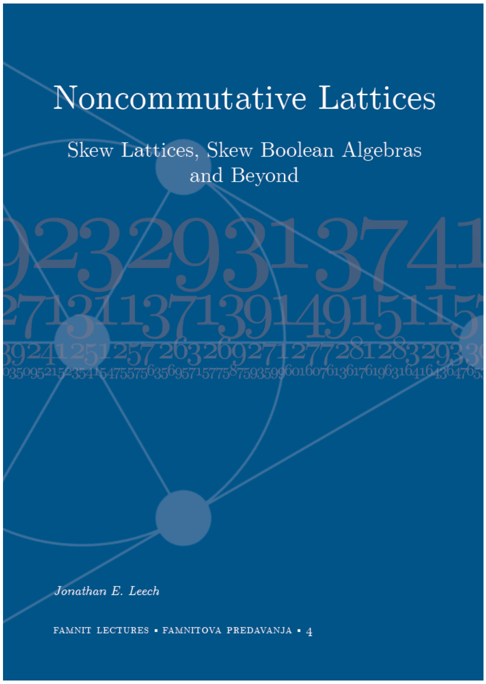
Jonathan E. Leech: Noncommutative Lattices: Skew Lattices, Skew Boolean Algebras and Beyond



About the book: The extended study of non-commutative lattices was begun in 1949 by Ernst Pascual Jordan, a theoretical and mathematical physicist and co-worker of Max Born and Werner Karl Heisenberg. Jordan introduced noncommutative lattices as algebraic structures potentially suitable to encompass the logic of the quantum world. The modern theory of noncommutative lattices began forty years later with Jonathan Leech's 1989 paper "Skew lattices in rings." Recently, noncommutative generalizations of lattices and related structures have seen an upsurge in interest, with new ideas and applications emerging, from quasilattices to skew Heyting algebras. Much of this activity is derived in some way from the initiation of Jonathan Leech's program of research in this area. The present book consists of seven chapters, mainly covering skew lattices, quasilattices and paralattices, skew lattices of idempotents in rings and skew Boolean algebras. As such, it is the first research monograph covering major results due to this renewed study of noncommutative lattices. It will serve as a valuable graduate textbook on the subject, as well as a handy reference to researchers of noncommutative algebras.

About the author: Jonathan Leech graduated from the University of Hawaii and earned a PhD at the University of California, Los Angeles. He has taught mathematics at the University of Tennessee, later at Missouri Western State University and finally at Westmont College in Santa Barbara, California. He has been a Visiting Professor at Case Western Reserve University, the Universidad de Granada in Spain and Universidade Mackenzie in Brazil, and a scholar in residence at both the University of Sidney and the University of Tasmania in Australia. Throughout his academic career Professor Leech has studied algebraic structures related to semigroups, with much of his emphasis being on the theory of noncommutative lattices, and of skew lattices in particular. He laid the foundations of the modern theory of noncommutative lattices in a number of (co)authored seminal publications. His work has inspired many mathematicians around the world to pursue research in this area.

J. E. Leech, *Noncommutative Lattices: Skew Lattices, Skew Boolean Algebras and Beyond*, volume 4 of *Famnit Lectures*, Slovenian Discrete and Applied Mathematics Society and University of Primorska Press, Koper, 2021, 284 pp., ISBN 978-961-95273-0-6.

The paperback edition of the book was published on March 5, 2021 by SDAMS, the Slovenian Discrete and Applied Mathematics Society. The cost of the book is 20.00 EUR + shipping. Society members have discount of 5.00 EUR. Orders should be sent to `info@sdams.si`. An invoice will be sent upon receipt of the order. The book will be shipped after payment is received.

Petra Šparl Award 2022: Call for Nominations

The Petra Šparl Award was established in 2017 to recognise in each even-numbered year the best paper published in the previous five years by a young woman mathematician in one of the two journals *Ars Mathematica Contemporanea* (*AMC*) and *The Art of Discrete and Applied Mathematics* (*ADAM*). It was named after Dr Petra Šparl, a talented woman mathematician who died mid-career in 2016.

The award consists of a certificate with the recipient's name, and invitations to give a lecture at the Mathematics Colloquium at the University of Primorska, and lectures at the University of Maribor and University of Ljubljana. The first award was made in 2018 to Dr Monika Pilśniak (AGH University, Poland) for a paper on the distinguishing index of graphs, and then two awards were made for 2020, to Dr Simona Bonvicini (Università di Modena e Reggio Emilia, Italy) for her contributions to a paper giving solutions to some Hamilton-Waterloo problems, and Dr Klavdija Kutnar (University of Primorska, Slovenia), for her contributions to a paper on odd automorphisms in vertex-transitive graphs.

The Petra Šparl Award Committee is now calling for nominations for the next award.

Eligibility: Each nominee must be a woman author or co-author of a paper published in either *AMC* or *ADAM* in the calendar years 2017 to 2021, who was at most 40 years old at the time of the paper's first submission.

Nomination Format: Each nomination should specify the following:
(a) the name, birth-date and affiliation of the candidate;
(b) the title and other bibliographic details of the paper for which the award is recommended;
(c) reasons why the candidate's contribution to the paper is worthy of the award, in at most 500 words; and
(d) names and email addresses of one or two referees who could be consulted with regard to the quality of the paper.

Procedure: Nominations should be submitted by email to any one of the three members of the Petra Šparl Award Committee (see below), **by 31 October 2021**.

Award Committee:
- Marston Conder, m.conder@auckland.ac.nz
- Asia Ivić Weiss, weiss@yorku.ca
- Aleksander Malnič, aleksander.malnic@guest.arnes.si

Marston Conder, Asia Ivić Weiss and Aleksander Malnič
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