Characterization of graphs with exactly two non-negative eigenvalues

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Abstract

In this paper we characterize all graphs with exactly two non-negative eigenvalues. As a consequence we obtain all graphs G such that $\lambda_3(G) < 0$, where $\lambda_3(G)$ is the third largest eigenvalue of G.

Keywords: Spectrum and eigenvalues of graphs, graphs with exactly two non-negative eigenvalues.

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1 Introduction

Throughout this paper all graphs are simple, that is finite and undirected without loops and multiple edges. Let G be a graph with vertex set $\{v_1, \ldots, v_n\}$. The adjacency matrix of G, $A(G) = [a_{ij}]$, is an $n \times n$ matrix such that $a_{ij} = 1$ if v_i and v_j are adjacent, and $a_{ij} = 0$, otherwise. Thus $A(G)$ is a symmetric matrix with zeros on the diagonal and all the eigenvalues of $A(G)$ are real. By the eigenvalues of G we mean those of its adjacency matrix. We denote the eigenvalues of G by $\lambda_1(G) \geq \cdots \geq \lambda_n(G)$. By the spectrum of G that is denoted by $Spec(G)$, we mean the multiset of eigenvalues of G. The characteristic polynomial of G, $det(\lambda I - A(G))$, is denoted by $P(G, \lambda)$. Studying the eigenvalues of graphs, the roots of characteristic polynomials of graphs, has always been of great interest to researchers, for instance see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10] and the references therein.

It is well known that $\lambda_1(G) + \cdots + \lambda_n(G) = 0$ and $\lambda_1^2(G) + \cdots + \lambda_n^2(G) = 2m$, where m is the number of edges of G. Thus if G has at least one edge, then G has at least one positive eigenvalue. One of the attractive problems is the characterization of graphs with a

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few non-zero eigenvalues. In [5] all bipartite graphs with at most six non-zero eigenvalues have been characterized. The another interesting problem is the characterization of graphs with a few positive eigenvalues. In [10] Smith characterized all graphs with exactly one positive eigenvalue. In fact, a graph has exactly one positive eigenvalue if and only if its non-isolated vertices form a complete multipartite graph. In [9] Petrović has studied the characterization of graphs with exactly two non-negative eigenvalues. In this paper with a different proof we state a new characterization of all graphs G with exactly two non-negative eigenvalues. In other words we find the graphs G with $\lambda_1(G) \geq 0$, $\lambda_2(G) \geq 0$ and $\lambda_3(G) < 0$.

For a graph G, $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively; $\overline{G}$ denotes the complement of G. The *order* of G denotes the number of vertices of G. The *closed neighborhood* of a vertex v of G which is denoted by $N[v]$, is the set $\{u \in V(G) : uv \in E(G)\} \cup \{v\}$. For every vertex $v \in V(G)$, the *degree* of v is the number of edges incident with v and is denoted by $deg_G(v)$ (for simplicity we use $deg(v)$ instead of $deg_G(v)$). By $\delta(G)$ we mean the minimum degree of vertices of G. A set $S \subseteq V(G)$ is an *independent set* if there is no edge between the vertices of S. The *independence number* of G, $\alpha(G)$, is the maximum cardinality of an independent set of G. For two graphs G and H with disjoint vertex sets, $G + H$ denotes the graph with the vertex set $V(G) \cup V(H)$ and the edge set $E(G) \cup E(H)$, i.e. the disjoint union of two graphs G and H. In particular, nG denotes the disjoint union of n copies of G. The complete product (join) $G \vee H$ of graphs G and H is the graph obtained from $G + H$ by joining every vertex of G with every vertex of H. For positive integers $n_1, \ldots, n_\ell$, $K_{n_1, \ldots, n_\ell}$ denotes the complete multipartite graph with ℓ parts of sizes $n_1, \ldots, n_\ell$. Let K_n, $nK_1 = \overline{K_n}$, C_n and P_n be the complete graph, the null graph, the cycle and the path on n vertices, respectively.

2 The structure of graphs with exactly two positive eigenvalues

In this section we obtain a characterization of graphs that have exactly two positive eigenvalues. We need the Interlacing Theorem.

Theorem 2.1. *([4, Theorem 9.1.1]) Let G be a graph of order n and H be an induced subgraph of G with order m. Suppose that $\lambda_1(G) \geq \cdots \geq \lambda_n(G)$ and $\lambda_1(H) \geq \cdots \geq \lambda_m(H)$ are the eigenvalues of G and H, respectively. Then for every i, $1 \leq i \leq m$, $\lambda_i(G) \geq \lambda_i(H) \geq \lambda_{n-m+i}(G)$.*

Theorem 2.2. *([10], see also [3, Theorem 6.7]) A graph has exactly one positive eigenvalue if and only if its non-isolated vertices form a complete multipartite graph.*

First we characterize all graphs with exactly one non-negative eigenvalue.

Theorem 2.3. *Let G be a graph of order $n \geq 2$ with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$. Then $\lambda_2 < 0$ if and only if $G \cong K_n$.*

Proof. If $G \cong K_n$ and $n \geq 2$, then $\lambda_2 = -1$. Now suppose that $\lambda_2 < 0$. We show that $G \cong K_n$. Suppose that $G \not\cong K_n$. Thus $2K_1$ is an induced subgraph of G. So by Interlacing Theorem 2.1, $\lambda_2 \geq \lambda_2(2K_1) = 0$, a contradiction. Hence $G \cong K_n$.   □

Lemma 2.4. *Let G be a graph of order $n \geq 3$ with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$. Suppose that $\lambda_1 \geq 0$, $\lambda_2 \geq 0$ and $\lambda_3 < 0$. Then the following hold:*

1. *If G is disconnected, then $G \cong K_r + K_{n-r}$, for some positive integer r, where $r \leq n - 1$.*

2. *If G is connected and $\lambda_2 = 0$, then $G \cong K_n \setminus e$ for an edge e of K_n.*

Proof. 1. Let G be disconnected. Assume that $G_1, \ldots, G_k$ are the connected components of G, where $k \geq 2$. Since $\lambda_1(G_1) \geq 0, \ldots, \lambda_1(G_k) \geq 0$ are k eigenvalues of G and $\lambda_3 < 0$ we obtain that $k = 2$. In other words G has exactly two connected components. Thus $G = G_1 + G_2$. We prove that G_1 and G_2 are complete graphs. First we show that G_1 is a complete graph. If $G_1 \cong K_1$, there is nothing to prove. Assume that $|V(G_1)| \geq 2$ (equivalently $G_1 \not\cong K_1$). We claim that $\lambda_2(G_1) < 0$. By contradiction suppose that $\lambda_2(G_1) \geq 0$. Since $\lambda_1(G_1) \geq 0$, $\lambda_2(G_1) \geq 0$ and $\lambda_1(G_2) \geq 0$ are three eigenvalues of G we obtain that $\lambda_3 \geq 0$, a contradiction (since $\lambda_3 < 0$). Hence the claim is proved. In other words $\lambda_2(G_1) < 0$. So by Theorem 2.3, G_1 is a complete graph. Similarly we obtain that G_2 is a complete graph. Hence G is a disjoint union of two complete graphs.

2. Suppose that G is connected and $\lambda_2 = 0$. Since $\lambda_3 < 0$, $G \not\cong nK_1$. Thus $\lambda_1 > 0$. Hence G has exactly one positive eigenvalue. By Theorem 2.2 there are some positive integers t and $n_1 \geq \cdots \geq n_t \geq 1$, so that $n_1 + \cdots + n_t = n$ and $G \cong K_{n_1, \ldots, n_t}$. If $t = 1$, then $G \cong nK_1$, a contradiction (since G is connected). Thus $t \geq 2$. If $n_1 = 1$, then $G \cong K_n$ and so $\lambda_2 = -1$, a contradiction. Therefore $n_1 \geq 2$. If $n_2 \geq 2$, then C_4 is an induced subgraph of G. Using Interlacing Theorem 2.1 we get $\lambda_3 \geq \lambda_3(C_4) = 0$, a contradiction. Thus $n_2 = \cdots = n_t = 1$. Now if $n_1 \geq 3$, then $K_{1,3}$ is an induced subgraph of G. Similarly by Interlacing Theorem 2.1 we obtain $\lambda_3 \geq \lambda_3(K_{1,3}) = 0$, a contradiction. So $n_1 = 2$. Thus $G \cong K_{2,1,\ldots,1}$. In other words $G \cong K_n \setminus e$, for an edge e of K_n. We note that

$$Spec(K_n \setminus e) = \{\frac{n - 3 + \sqrt{n^2 + 2n - 7}}{2}, 0, \underbrace{-1, \ldots, -1}_{n-3}, \frac{n - 3 - \sqrt{n^2 + 2n - 7}}{2}\}.$$

The proof is complete. □

In [2] all graphs G with $\lambda_1 > 0$, $\lambda_2 \leq 0$ and $\lambda_3 < 0$ have been characterized.

Remark 2.5. Let $n_1, \ldots, n_t$ be some positive integers and $G = K_{n_1, \ldots, n_t}$. Similar to the proof of the second part of Lemma 2.4 by Interlacing Theorem 2.1 one can see that $\lambda_2(G) < 0$ if and only if $n_1 = \cdots = n_t = 1$. On the other hand by Theorem 2.2, $\lambda_2(K_{n_1, \ldots, n_t}) \leq 0$. Thus $\lambda_2(K_{n_1, \ldots, n_t}) = 0$ if and only if $n_k > 1$ for some k. In other words, the second largest eigenvalue of any complete multipartite graph except complete graph is zero.

Remark 2.6. Let G be a graph of order $n \geq 3$ with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$. Assume that G has exactly two non-negative eigenvalues. In other words, $\lambda_1 \geq 0$, $\lambda_2 \geq 0$ and $\lambda_3 < 0$. Since $\lambda_3 < 0$, $G \not\cong nK_1$. Thus $\lambda_1 > 0$. Hence $\lambda_1 > 0$, $\lambda_2 \geq 0$ and $\lambda_3 < 0$. If G is disconnected, then by the first part of Lemma 2.4, $G \cong K_r + K_{n-r}$ for some positive integer $r \leq n - 1$. If G is connected and $\lambda_2 = 0$, then by the second part of Lemma 2.4, $G \cong K_n \setminus e$, where e is an edge of K_n. Thus to characterize all graphs with exactly two non-negative eigenvalues it remains to find connected graphs G such that $\lambda_1(G) > 0$, $\lambda_2(G) > 0$ and $\lambda_3(G) < 0$. In sequel we find this characterization.

Definition 2.7. A graph G is called *semi-complete* if G is a disjoint union of two complete graphs or is obtained by adding some new edges to disjoint union of two complete graphs (see Figure 1).
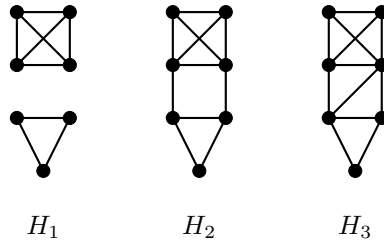

Figure 1: The graphs H_1, H_2 and H_3 are semi-complete that are obtained from $K_3 + K_4$.

Now we prove one of the main results of this section.

Lemma 2.8. *Let G be a connected graph of order $n \geq 3$ and with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$. If $\lambda_2 > 0$ and $\lambda_3 < 0$, then for every vertex $v \in V(G)$ with degree $\delta(G)$ we have $N[v] \cong K_{\delta(G)+1}$ and $G \setminus N[v] \cong K_{n-\delta(G)-1}$. In particular, G is semi-complete.*

Proof. Let $\lambda_2 > 0$ and $\lambda_3 < 0$. Since $\lambda_2 > 0$, G is not complete graph. Therefore $\alpha(G) \geq 2$. If $\alpha(G) \geq 3$, then $3K_1$ is an induced subgraph of G. Thus by Interlacing Theorem 2.1, $\lambda_3(G) \geq \lambda_3(3K_1) = 0$, a contradiction. Therefore $\alpha(G) = 2$. Thus for every vertex $u \in V(G)$, $G \setminus N[u]$ is a complete graph. In fact, $G \setminus N[u] \cong K_{n-deg(u)-1}$.

Let v_0 be a vertex of G with degree $\delta(G)$, that is v_0 has the minimum degree among all vertices of G. Since $G \not\cong K_n$, $deg(v_0) \leq n - 2$. Since $G \setminus N[v_0]$ is a complete graph, to complete the proof it is sufficient to show that the induced subgraph on the set $N[v_0]$ is a complete graph, that is every two vertices of $N[v_0]$ are adjacent. This also shows that G is obtained by adding some edges to the complete graphs $N[v_0]$ and $G \setminus N[v_0]$ and so G is semi-complete.

Now we show that $N[v_0]$ is a complete graph. By contradiction, suppose that w and z are two non-adjacent vertices of $N[v_0]$. Let a be an arbitrary vertex of $V(G) \setminus N[v_0]$. The induced subgraph on $\{v_0, w, z, a\}$ in G is one of the graphs, A_1, A_2, A_3 or A_4 (see Figure 2). Since $\lambda_3(A_1) = \lambda_3(A_4) = 0$ and $\lambda_3 < 0$, Interlacing Theorem 2.1 shows that
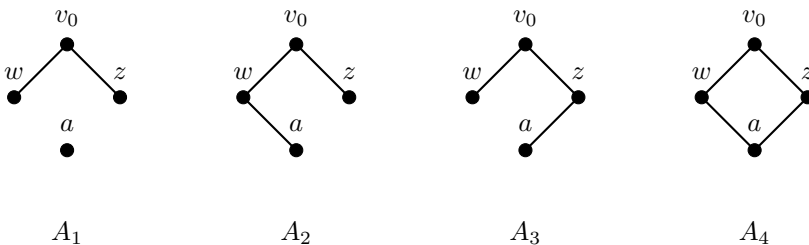

Figure 2: The subgraphs A_1, A_2, A_3 and A_4.

the induced subgraph on $\{v_0, w, z, a\}$ is A_2 or A_3. In other words any vertex of $G \setminus N[v_0]$

has exactly one neighbor in $\{w, z\}$. Without losing the generality assume that a is adjacent to w. Now we show that every vertex of $G \setminus N[v_0]$ is adjacent to w. By contradiction suppose that $b \neq a$ is a vertex of $G \setminus N[v_0]$ such that b is adjacent to z. Since $G \setminus N[v_0]$ is complete and $a, b \in V(G) \setminus N[v_0]$, the vertices a and b are adjacent. Thus the induced subgraph on $\{v_0, w, z, a, b\}$ is isomorphic to the cycle C_5. Since $\lambda_3(C_5) \simeq .618 > 0$, by Interlacing Theorem 2.1, we have $\lambda_3 > 0$, a contradiction. This contradiction shows that all vertices of $G \setminus N[v_0]$ are adjacent only to w. This implies that $deg(z) \leq deg(v_0) - 1$, a contradiction, since v_0 has minimum degree. This contradiction completes the proof. $\square$

Claim 2.9. *Let G be a connected graph of order $n \geq 3$ and with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$ such that $\lambda_2 > 0$ and $\lambda_3 < 0$. Let $X = N[v_0]$ and $Y = G \setminus N[v_0]$, where v_0 is a vertex of G with degree $\delta(G)$. Then for every two vertices a and b in X (also for a and b in Y) $N[a] \subseteq N[b]$ or $N[b] \subseteq N[a]$.*

Proof. Let a and b be two vertices of X. We show that $N[a] \subseteq N[b]$ or $N[b] \subseteq N[a]$. First note that by Lemma 2.8, X is a complete graph. This implies that $N[a] \cap X = N[b] \cap X = X$. Now by contradiction suppose that there are some vertices c and d in Y such that $c \in N[a] \setminus N[b]$ and $d \in N[b] \setminus N[a]$. Thus the induced subgraph on $\{a, b, c, d\}$ is isomorphic to C_4. Using Interlacing Theorem 2.1 we get $\lambda_3 \geq \lambda_3(C_4) = 0$, a contradiction (since $\lambda_3 < 0$). Thus the result follows. Similarly one can prove that for any two vertices v and w in Y, $N[v] \subseteq N[w]$ or $N[w] \subseteq N[v]$. $\square$

As an example we find an infinite family of connected graphs with positive second largest eigenvalue and negative third largest eigenvalue.

Corollary 2.10. *Let $n \geq 4$ be an integer. Let $K(n, t)$ be the graph obtained by deleting t edges incident to one vertex of K_n, where $2 \leq t \leq n - 2$. Then $\lambda_2(K(n, t)) > 0$ and $\lambda_3(K(n, t)) < 0$.*

Proof. Let $\lambda_1 \geq \cdots \geq \lambda_n$ be the eigenvalues of $K(n, t)$. Since K_{n-1} is an induced subgraph of $K(n, t)$, by Interlacing Theorem 2.1, $\lambda_1 \geq n - 2 \geq \lambda_2 \geq -1 \geq \lambda_3$. Thus $\lambda_3 < 0$. On the other, since $K(n, t)$ is not a complete multipartite, by Theorem 2.2, $\lambda_2 > 0$. $\square$

Definition 2.11. A graph G is called *quasi-reduced* if for every two vertices u and v of G, $N[u] \neq N[v]$.

As an example of quasi-reduced graphs, we define the graphs G_n that have important role for characterizing graphs with $\lambda_2 > 0$ and $\lambda_3 < 0$.

Definition 2.12. For every integer $n \geq 2$, let G_n be the graph of order n such that G_n is obtained from disjoint complete graphs $K_{\lceil \frac{n}{2} \rceil}$ and $K_{\lfloor \frac{n}{2} \rfloor}$ as following: Let $V(K_{\lceil \frac{n}{2} \rceil}) = \{v_1, \ldots, v_{\lceil \frac{n}{2} \rceil}\}$ and $V(K_{\lfloor \frac{n}{2} \rfloor}) = \{w_1, \ldots, w_{\lfloor \frac{n}{2} \rfloor}\}$. Then add some new edges to $K_{\lceil \frac{n}{2} \rceil} + K_{\lfloor \frac{n}{2} \rfloor}$ such that the following hold:

(i) $N[v_1] \subset \cdots \subset N[v_{\lceil \frac{n}{2} \rceil}]$ and $N[w_1] \subset \cdots \subset N[w_{\lfloor \frac{n}{2} \rfloor}]$.

(ii) $\left| N[v_i] \cap V(K_{\lfloor \frac{n}{2} \rfloor}) \right| = i - 1$ for $i = 1, \ldots, \lceil \frac{n}{2} \rceil$.

(iii) $\left| N[w_j] \cap V(K_{\lceil \frac{n}{2} \rceil}) \right| = \begin{cases} j - 1, & \text{if } n \text{ is even;} \\ j, & \text{if } n \text{ is odd} \end{cases}$ for $j = 1, \ldots, \lfloor \frac{n}{2} \rfloor$.

In Figure 3, the graphs G_2, G_3, G_4, G_5 and G_6 have been shown. In addition in Figure 4 one can see the complement of $G_7, \ldots, G_{12}$. We note that $G_{2k} = B_{2k}(1, \ldots, 1; 1, \ldots, 1)$ and $G_{2k+1} = B_{2k+1}(1, \ldots, 1; 1, \ldots, 1; 1)$, where B_{2k} and B_{2k+1} are the graphs that have been defined in [9].

Remark 2.13. For every $n \geq 2$, G_n is semi-complete and quasi-reduced. In addition if $n \geq 3$, then G_n is connected. We note that for every $n \geq 3$, G_n is an induced subgraph of G_{n+1}. In fact if n is even, then $G_{n+1} \cong K_1 \vee G_n$ and if n is odd, then G_{n+1} is obtained from G_n by adding a new vertex w such that w is adjacent to any vertex of $\{w_1, \ldots, w_{\lfloor \frac{n}{2} \rfloor}\} = V(K_{\lfloor \frac{n}{2} \rfloor})$, where $K_{\lfloor \frac{n}{2} \rfloor}$ is one of the parts of G_n (see Definition 2.12).

Remark 2.14. We note that for every $n \geq 2$, the group of all automorphisms of the graph G_n, $Aut(G_n)$, has exactly two elements.
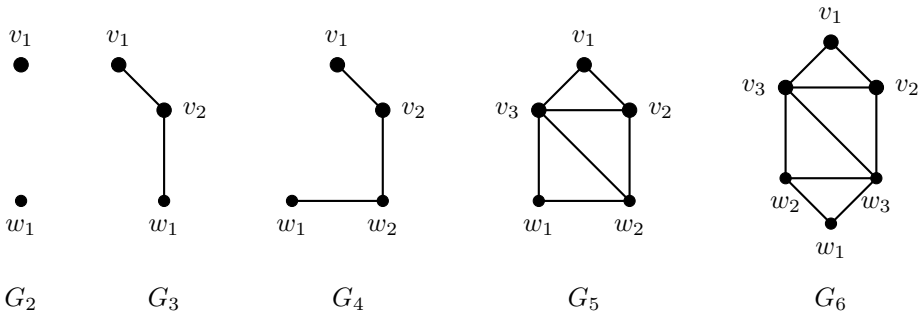

Figure 3: The graphs G_2, G_3, G_4, G_5 and G_6 are semi-complete and quasi-reduced.

The next result shows that there is only one connected quasi-reduced graph with $\lambda_2 > 0$ and $\lambda_3 < 0$.

Lemma 2.15. *Let G be a connected graph of order $n \geq 3$. If G is quasi-reduced and $\lambda_2(G) > 0$ and $\lambda_3(G) < 0$, then $G \cong G_n$.*

Proof. Assume that $\lambda_2(G) > 0$ and $\lambda_3(G) < 0$. Since G is connected, by Lemma 2.8, G is semi-complete. Let $\delta(G) = t$ and v_1' be a vertex of G with degree t. By Lemma 2.8, $N[v_1'] \cong K_{t+1}$ and $G \setminus N[v_1'] \cong K_{n-t-1}$. In fact G is obtained from the disjoint complete graphs K_{t+1} and K_{n-t-1} by adding some new edges (see the proof of Lemma 2.8). Let $V(K_{t+1}) = \{v_1', v_2', \ldots, v_{t+1}'\}$ and $V(K_{n-t-1}) = \{w_1', \ldots, w_{n-t-1}'\}$. By Claim 2.9, for every two vertices v_i' and v_j' in K_{t+1}, $N[v_i'] \subseteq N[v_j']$ or $N[v_j'] \subseteq N[v_i']$. Also for every two vertices w_i' and w_j' in K_{n-t-1}, $N[w_i'] \subseteq N[w_j']$ or $N[w_j'] \subseteq N[w_i']$. So without losing the generality assume that $N[v_1'] \subseteq \cdots \subseteq N[v_{t+1}']$ and $N[w_1'] \subseteq \cdots \subseteq N[w_{n-t-1}']$ (note that $N[v_1'] = V(K_{t+1})$ and for every $1 \leq i \leq t+1$, $N[v_1'] \subseteq N[v_i']$). Now suppose that G is quasi-reduced. Therefore we find that

$$0 = |N[v_1'] \cap V(K_{n-t-1})| < \cdots < |N[v_{t+1}'] \cap V(K_{n-t-1})| \leq n - t - 1, \qquad (2.1)$$

and

$$0 \leq |N[w_1'] \cap V(K_{t+1})| < \cdots < |N[w_{n-t-1}'] \cap V(K_{t+1})| \leq t. \qquad (2.2)$$
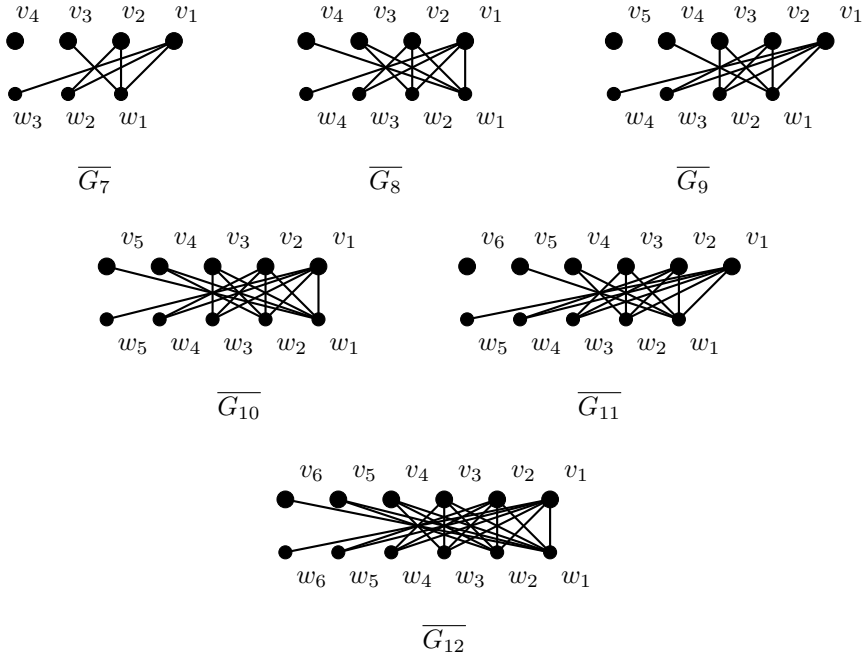
Figure 4: The complement graphs of G_7, G_8, G_9, G_{10}, G_{11} and G_{12}.

Since $|N[v_1'] \cap V(K_{n-t-1})|, \ldots, |N[v_{t+1}'] \cap V(K_{n-t-1})|$ are $t+1$ distinct integers between 0 and $n - t - 1$, the Equation (2.1) shows that $t \leq n - t - 1$. Similarly, the Equation (2.2) implies that $n - t - 2 \leq t$. Hence $n - 2 \leq 2t \leq n - 1$. So $t = \lceil \frac{n}{2} \rceil - 1$.

If n is even, then the Equation (2.2) shows that $|N[w_j'] \cap V(K_{t+1})| = j - 1$, for $j = 1, \ldots, n - t - 1$. So w_1' has no neighbor in K_{t+1}. Thus for any $1 \leq i \leq t + 1$, $|N[v_i'] \cap V(K_{n-t-1})| \leq n - t - 2$. Using Equation (2.1) we conclude that $|N[v_i'] \cap V(K_{n-t-1})| = i - 1$, for $i = 1, \ldots, t + 1$. Hence $G \cong G_n$.

Similarly, for odd n we obtain that $|N[v_i'] \cap V(K_{n-t-1})| = i - 1$, for $i = 1, \ldots, t + 1$. Thus v_{t+1}' is adjacent to every vertex of $V(K_{n-t-1})$. Hence $1 \leq |N[w_1'] \cap V(K_{t+1})|$. Using inequality (2.2) we find that $|N[w_j'] \cap V(K_{t+1})| = j$, for $j = 1, \ldots, n - t - 1$. Thus $G \cong G_n$. $\qquad \square$

Lemma 2.16. *Let G_n be the semi-complete and quasi-reduced graph as mentioned above. Then $\lambda_2(G_n) > 0$ and $\lambda_3(G_n) < 0$ if and only if $4 \leq n \leq 12$.*

Proof. One can see that $\lambda_2(G_3) = 0$ and for every $4 \leq n \leq 12$, $\lambda_2(G_n) > 0$ and $\lambda_3(G_n) < 0$. Now assume that $n \geq 13$. Since $\lambda_3(G_{13}) = 0$ and G_{13} is an induced subgraph of G_n (by Remark 2.13), by Interlacing Theorem 2.1 we find that $\lambda_3(G_n) \geq \lambda_3(G_{13}) = 0$. This completes the proof. $\qquad \square$

Definition 2.17. Let G be a graph with vertex set $\{v_1, \ldots, v_n\}$. By $G[K_{t_1}, \ldots, K_{t_n}]$ we mean the graph obtained by replacing the vertex v_j by the complete graph K_{t_j} for

$1 \leq j \leq n$, where every vertex of K_{t_i} is adjacent to every vertex of K_{t_j} if and only if v_i is adjacent to v_j (in G). For example $K_2[K_p, K_q] \cong K_{p+q}$ and $\overline{K_2}[K_p, K_q] \cong K_p + K_q$.

Now we prove one of the main results of the paper.

Theorem 2.18. *Let G be a connected graph of order $n \geq 3$. If $\lambda_2(G) > 0$ and $\lambda_3(G) < 0$, then there exist some positive integers s and $t_1, \ldots, t_s$ so that $3 \leq s \leq 12$ and $t_1 + \cdots + t_s = n$ and $G \cong G_s[K_{t_1}, \ldots, K_{t_s}]$.*

Proof. Suppose that $\lambda_2(G) > 0$ and $\lambda_3(G) < 0$. By Lemma 2.8, G is semi-complete. If G is quasi-reduced, then by Lemma 2.15, $G \cong G_n \cong G_n[K_1, \ldots, K_1]$. So $\lambda_2(G_n) = \lambda_2(G) > 0$ and $\lambda_3(G_n) = \lambda_3(G) < 0$. Hence by Lemma 2.16, $4 \leq n \leq 12$.

Now assume that G is not quasi-reduced. Thus there exists a connected induced subgraph of G, say H, such that H is quasi-reduced and $G = H[K_{t_1}, \ldots, K_{t_s}]$, where s is the order of H and $t_1, \ldots, t_s$ are some positive integers. Thus $t_1 + \cdots + t_s = n$. If $H \cong K_s$, then $G \cong K_n$, a contradiction (since $\lambda_2(K_n) = -1 < 0$ while $\lambda_2(G) > 0$). Thus H is not a complete graph. On the other hand H is a connected graph of order s. Thus $s \geq 3$. Since H is obtained from G by removing some vertices and G is semi-complete, H is also semi-complete. Suppose that C_4 is an induced subgraph of H. Since H is an induced subgraph of G, by Interlacing Theorem 2.1 we conclude that $\lambda_3(G) \geq \lambda_3(H) \geq \lambda_3(C_4) = 0$, a contradiction. Thus H has no induced cycle C_4.

Now we show that $H \cong G_s$. Since H is semi-complete, H is obtained from the disjoint union of two complete graphs, say K_p and K_q, for some positive integers p and q. Let $X = K_p$ and $Y = K_q$. We claim that for every two vertices $a, b \in V(X)$, $N[a] \subseteq N[b]$ or $N[b] \subseteq N[a]$. By contradiction assume that $N[a] \nsubseteq N[b]$ and $N[b] \nsubseteq N[a]$. Thus there are two vertices c and d such that $c \in N[a] \setminus N[b]$ and $d \in N[b] \setminus N[a]$. Since $V(X) \subseteq N[a] \cap N[b]$, we find that c and d are two vertices of Y. Now we remark that the induced subgraph on the vertices a, b, c, d is isomorphic to C_4. It is a contradiction, since H has no induced cycle C_4. So the claim holds. Similarly for every two vertices $z, w \in V(Y)$, $N[z] \subseteq N[w]$ or $N[w] \subseteq N[z]$. On the other hand H is quasi-reduced, thus similar to the proof of Lemma 2.15 one can see that $H \cong G_s$.

If $s \geq 13$, then by Remark 2.13, G_{13} is an induced subgraph of H and so is an induced subgraph of G. Thus by Interlacing Theorem 2.1, $\lambda_3(G) \geq \lambda_3(G_{13}) = 0$, a contradiction, since $\lambda_3(G) < 0$. Hence $s \leq 12$. The proof is complete. □

We end this section by characterization the graphs with $\lambda_3 < 0$. We note that if G is a graph with $\lambda_3(G) < 0$, then G is not the null graph. Thus $\lambda_1(G) > 0$. Using Remark 2.5, the second part of Lemma 2.4 and Theorems 2.2, 2.3 and 2.18 we obtain this characterization.

Theorem 2.19. *Let G be a graph with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$. Assume that $\lambda_3 < 0$. Then the following hold:*

1. *If $\lambda_1 > 0$ and $\lambda_2 > 0$, then $G \cong K_p + K_q$ for some integers $p, q \geq 2$ or there exist some positive integers s and $t_1, \ldots, t_s$ so that $3 \leq s \leq 12$ and $t_1 + \cdots + t_s = n$ and $G \cong G_s[K_{t_1}, \ldots, K_{t_s}]$.*

2. *If $\lambda_1 > 0$ and $\lambda_2 = 0$, then $G \cong K_1 + K_{n-1}$ or $G \cong K_n \setminus e$, where e is an edge of K_n.*

3. *If $\lambda_1 > 0$ and $\lambda_2 < 0$, then $G \cong K_n$.*

Since $K_n \setminus e \cong G_3[K_1, K_1, K_{n-2}]$ and $K_p + K_q \cong \overline{K_2}[K_p, K_q]$, we can rewrite Theorem 2.19 as following:

Theorem 2.20. *Let G be a graph. If $\lambda_3(G) < 0$, then $G \cong K_n$ or there exist some positive integers s and $t_1, \ldots, t_s$ such that $2 \leq s \leq 12$ and $t_1 + \cdots + t_s = n$ and $G \cong G_s[K_{t_1}, \ldots, K_{t_s}]$.*

In the next section we investigate the converse of Theorem 2.20. In other words we obtain all values of $t_1, \ldots, t_s$ (for $2 \leq s \leq 12$) such that $\lambda_3(G_s[K_{t_1}, \ldots, K_{t_s}]) < 0$. We need the following important result for computing the characteristic polynomial of $G_s[K_{t_1}, \ldots, K_{t_s}]$, the polynomial $P(G_n[K_{t_1}, \ldots, K_{t_n}], \lambda)$.

Theorem 2.21. *[7] Let $n \geq 2$. Suppose that $\{v_1, \ldots, v_n\}$ is the vertex set of G_n and $A = [a_{ij}]$ is the adjacency matrix of G_n with respect to $\{v_1, \ldots, v_n\}$ ($a_{ij} = 1$ if and only if v_i and v_j are adjacent and $a_{ij} = 0$, otherwise). Let $t_1, \ldots, t_n$ be some positive integers and $M = [m_{ij}]$ be a $n \times n$ matrix, where*

$$m_{ij} := \left\{ \begin{array}{ll} t_i - 1, & \text{if } i = j; \\ a_{ij}t_j, & \text{if } i \neq j. \end{array} \right.$$

Then

$$P(G_n[K_{t_1}, \ldots, K_{t_n}], \lambda) = (\lambda + 1)^{t_1 + \cdots + t_n - n}\, g(\lambda),$$

where $g(\lambda) = det(\lambda I - M)$. In addition, the multiplicity of -1 as an eigenvalue of $G_n[K_{t_1}, \ldots, K_{t_n}]$ is equal to $t_1 + \cdots + t_n - n$.

3 The list of all connected graphs with $\lambda_2 > 0$ and $\lambda_3 < 0$

In this section we investigate the converse of Theorem 2.20. We use Petrović's notation [9] that is very similar to the notation of Definition 2.17. We note that in Definition 2.17, the graph $G[H_1, \ldots, H_n]$ is dependent to the labeling of the vertices of G while in the next definition first we fix a labeling for the vertices of G_n (see Definition 2.12), and then use the operation of Definition 2.17. For instance we consider the labeling $v_1, \ldots, v_s$ and $w_1, \ldots, w_s$ for the vertices of G_{2s} and then apply the operation of Definition 2.17.

Definition 3.1. Let $s \geq 1$ be an integer and $n_1, \ldots, n_{2s+1}$ be some positive integers. Let $B_{2s}(n_1, \ldots, n_s; n_{s+1}, \ldots, n_{2s})$ denote the graph obtained from G_{2s} by replacing the vertices v_1 by K_{n_1}, v_2 by K_{n_2},..., and v_s by K_{n_s} and w_1 by $K_{n_{s+1}}$, w_2 by $K_{n_{s+2}}$, ..., and w_s by $K_{n_{2s}}$ (see Definition 2.12). In other words

$$B_{2s}(n_1, \ldots, n_s; n_{s+1}, \ldots, n_{2s}) = G_{2s}[K_{n_1}, \ldots, K_{n_{2s}}],$$

where the ordering of the vertices of G_{2s} is $V(G_{2s}) = \{v_1, \ldots, v_s, w_1, \ldots, w_s\}$.

Similarly, by $B_{2s+1}(n_1, \ldots, n_s; n_{s+1}, \ldots, n_{2s}; n_{2s+1})$ we mean

$$B_{2s+1}(n_1, \ldots, n_s; n_{s+1}, \ldots, n_{2s}; n_{2s+1}) = G_{2s+1}[K_{n_1}, \ldots, K_{n_{2s+1}}],$$

where the ordering of the vertices of G_{2s+1} is $V(G_{2s+1}) = \{v_1, \ldots, v_s, w_1, \ldots, w_s, v_{s+1}\}$, (see Figure 5).

Remark 3.2. For every positive integers s and $n_1, \ldots, n_{2s+1}$, one can easily see that

$$B_{2s}(n_1, \ldots, n_s; n_{s+1}, \ldots, n_{2s}) \cong B_{2s}(n_{s+1}, \ldots, n_{2s}; n_1, \ldots, n_s),$$

and

$$B_{2s+1}(n_1, \ldots, n_s; n_{s+1}, \ldots, n_{2s}; n_{2s+1}) \cong B_{2s+1}(n_{s+1}, \ldots, n_{2s}; n_1, \ldots, n_s; n_{2s+1}).$$

For avoiding the repeating, using the dictionary ordering on $(n_1, \ldots, n_s)$ and $(n_{s+1}, \ldots, n_{2s})$ we just cite one of the graphs $B_{2s}(n_1, \ldots, n_s; n_{s+1}, \ldots, n_{2s})$ or $B_{2s}(n_{s+1}, \ldots, n_{2s}; n_1, \ldots, n_s)$ in our characterization. Similarly for the graphs $B_{2s+1}(n_1, \ldots, n_s; n_{s+1}, \ldots, n_{2s}; n_{2s+1})$ and $B_{2s+1}(n_{s+1}, \ldots, n_{2s}; n_1, \ldots, n_s; n_{2s+1})$ we only consider one of them. For example since by dictionary ordering $(4, 3, 2) \geq (4, 3, 1)$ we use $B_6(4, 3, 2; 4, 3, 1)$ instead of $B_6(4, 3, 1; 4, 3, 2)$. As another example we use $B_7(5, 3, 2; 5, 2, 4; 8)$ instead of $B_7(5, 2, 4; 5, 3, 2; 8)$, since $(5, 3, 2) \geq (5, 2, 4)$.
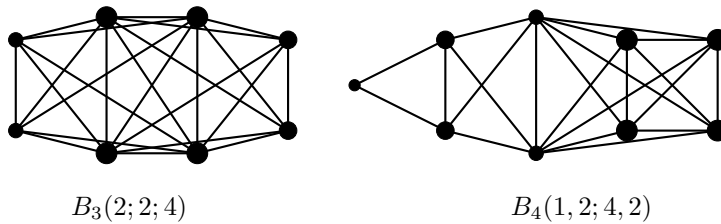

$$B_3(2; 2; 4) \qquad\qquad\qquad B_4(1, 2; 4, 2)$$

Figure 5: The graphs $B_3(2; 2; 4)$ and $B_4(1, 2; 4, 2)$.

The following theorem is the main result of [9].

Theorem 3.3. *[9] Graph G has the property $\lambda_3 < 0$ if and only if G is an induced subgraph of one of the following graphs:*

1. $B_4(3, 2; 2, r)$,

2. $B_5(1, r; 2, 3; 1)$,

3. $B_5(r, 1; 2, 3; 1)$,

4. $B_5(3, 2; 2, 1; r)$,

5. $B_5(r, 2; 1, 2; 2)$,

6. $B_6(r, 1, s; 1, 2, 2)$,

7. $B_6(2, 1, r; 2, 1, s)$,

8. $B_6(1, 2, 2; 1, r, 1)$,

9. $B_6(2, 2, 1; 1, 1, r)$,

10. $B_7(2, 1, 1; 2, 1, 1; r)$,

11. $B_7(r, 1, 2; 1, s, 1; 1)$,

12. $B_7(r, 1, 1; 1, 1, 2; 1)$,

13. $B_7(2, 2, 1; 1, r, 1; s)$,

14. $B_8(r, 1, 1, s; 1, 1, t, 1)$,

15. $B_9(1, r, 1, 1; 1, s, 1, 1; t)$,

where r, s and t are some positive integers or G is an induced subgraph of one of the 323 graphs with 12 vertices belonging respectively to the classes B_4 (10 graphs), B_5 (25 graphs), B_6 (69 graphs), B_7 (74 graphs), B_8 (80 graphs), B_9 (40 graphs), B_{10} (20 graphs), B_{11} (4 graphs) and B_{12} (1 graph).

Now we give a nicer characterization for graphs with $\lambda_3 < 0$. Note that the Petrović's result shows that any graph with exactly two non-negative eigenvalues is an induced subgraph of one the graphs described by Theorem 3.3. Since finding the structure of induced subgraphs of a graph is complicated, it is better to find the exact structure of all graphs with $\lambda_3 < 0$. In sequel we find this structure. To find our characterization, first we note that by Theorem 2.20 every graph with exactly two non-negative eigenvalues is isomorphic to $G_s[K_{t_1}, \ldots, K_{t_s}]$ for some positive integers $t_1, \ldots, t_s$, where $2 \leq s \leq 12$. In this section we find all values of $t_1, \ldots, t_s$ such that $G_s[K_{t_1}, \ldots, K_{t_s}]$ has exactly two non-negative eigenvalues. In other words we solve the converse of Theorem 2.20. We note that by Remark 2.6 it suffices to find all connected graphs with $\lambda_1 > 0$, $\lambda_2 > 0$ and $\lambda_3 < 0$. In other words we find all connected graphs $G_s[K_{t_1}, \ldots, K_{t_s}]$ such that $3 \leq s \leq 12$ and $\lambda_1 > 0$, $\lambda_2 > 0$ and $\lambda_3 < 0$. We obtain these graphs in ten theorems (for every s, $3 \leq s \leq 12$, we consider a theorem). First we prove the case $s = 3$. Since the cases $s = 4, \ldots, s = 9$ similarly are proved, we just prove the case $s = 6$. In addition the proofs of the cases $s = 10, 11, 12$ are similar and we only prove the case $s = 10$. Our proofs are based on three theorems, Theorem 2.21 for computing the characteristic polynomials of $G_s[K_{t_1}, \ldots, K_{t_s}]$, *Descartes' Sign Rule* for polynomials and Interlacing Theorem 2.1. Since $G_s[K_{t_1}, \ldots, K_{t_s}] = B_s(t_1, \ldots, t_s)$, in sequel we use $B_s(t_1, \ldots, t_s)$ instead of $G_s[K_{t_1}, \ldots, K_{t_s}]$.

Theorem 3.4. *Let $G = B_3(a; b; c)$, where a, b, c are some positive integers. Then $\lambda_2(G) > 0$ and $\lambda_3(G) < 0$ if and only if $ab \neq 1$.*

Proof. Let $g(\lambda) = P(B_3(a; b; c), \lambda)$. By Theorem 2.21 we obtain that

$$g(\lambda) = (\lambda + 1)^{a+b+c-3} f(\lambda), \tag{3.1}$$

where

$$f(\lambda) = \lambda^3 - (a+b+c-3)\lambda^2 + (ab - 2a - 2b - 2c + 3)\lambda + ac(b-1) + (a-1)(b+c-1).$$

If $ab = 1$, that is $a = b = 1$, then $g(\lambda) = \lambda(\lambda + 1)^{c-1}(\lambda^2 - (c-1)\lambda - 2c)$. This shows that $\lambda_1(G) > 0$ and $\lambda_2(G) = 0$. Now suppose that $ab \geq 2$. Let $z_1 \geq z_2 \geq z_3$ be all roots of f. Hence $f(\lambda) = (\lambda - z_1)(\lambda - z_2)(\lambda - z_3)$. Therefore $z_1 + z_2 + z_3 = a + b + c - 3 > 0$ and $z_1 z_2 z_3 = -f(0) = -(ac(b-1) + (a-1)(b+c-1)) < 0$. These equalities show that $z_1 > 0$, $z_2 > 0$ and $z_3 < 0$. On the other hand by the Equation (3.1), the eigenvalues

of $B_3(a; b; c)$ are $z_1, z_2, z_3, -1, \ldots, -1$ (the multiplicity of -1 is $a + b + c - 3$). Hence $\lambda_1(G) = z_1 > 0$, $\lambda_2(G) = z_2 > 0$ and $\lambda_3(G) = max\{z_3, -1\} < 0$. The proof is complete. $\qquad\square$

Theorem 3.5. *Let $G = B_4(a_1, a_2; a_3, a_4)$, where a_1, a_2, a_3, a_4 are some positive integers. Then $\lambda_2(G) > 0$ and $\lambda_3(G) < 0$ if and only if G is isomorphic to one of the following graphs:*

1. $B_4(a, b; 1, d)$, $B_4(a, x; y, 1)$, $B_4(a, 1; c, 1)$, $B_4(a, 1; w, x)$,

2. $B_4(a, 1; x, d)$, $B_4(w, b; x, 1)$, $B_4(w, x; y, d)$, $B_4(x, b; y, d)$,

3. 25 *specific graphs:* 5 *graphs of order* 10, 10 *graphs of order* 11, *and* 10 *graphs of order* 12,

where a, b, c, d, x, y, w are some positive integers such that $x \le 2$, $y \le 2$ and $w \le 3$.

Theorem 3.6. *Let $G = B_5(a_1, a_2; a_3, a_4; a_5)$, where a_1, a_2, a_3, a_4, a_5 are some positive integers. Then $\lambda_2(G) > 0$ and $\lambda_3(G) < 0$ if and only if G is isomorphic to one of the following graphs:*

1. $B_5(a, w; 1, 1; 1)$, $B_5(a, x; 1, d; 1)$, $B_5(a, x; 1, y; z)$, $B_5(a, x; 1, 1; e)$,

2. $B_5(a, 1; c, 1; e)$, $B_5(a, 1; x, w; 1)$, $B_5(a, 1; x, y; e)$, $B_5(a, 1; 1, d; e)$,

3. $B_5(w, x; y, 1; e)$, $B_5(x, b; 1, 1; 1)$, $B_5(x, w; 1, d; 1)$, $B_5(x, w; 1, 1; e)$,

4. $B_5(1, b; 1, d; 1)$, $B_5(1, b; 1, x; y)$, $B_5(1, x; 1, y; e)$,

5. 63 *specific graphs:* 13 *graphs of order* 10, 25 *graphs of order* 11, *and* 25 *graphs of order* 12,

where $a, b, c, d, e, x, y, z, w$ are some positive integers such that $x \le 2$, $y \le 2$, $z \le 2$ and $w \le 3$.

Theorem 3.7. *Let $G = B_6(a_1, a_2, a_3; a_4, a_5, a_6)$, where $a_1, \ldots, a_6$ are some positive integers. Then $\lambda_2(G) > 0$ and $\lambda_3(G) < 0$ if and only if G is isomorphic to one of the following graphs:*

1. $B_6(a, x, c; 1, 1, 1)$, $B_6(a, 1, c; 1, e, 1)$, $B_6(a, 1, c; 1, x, y)$, $B_6(a, 1, c; 1, 1, f)$,

2. $B_6(a, 1, 1; x, e, 1)$, $B_6(x, b, 1; y, 1, 1)$, $B_6(x, y, 1; 1, e, 1)$,

3. $B_6(x, y, 1; 1, 1, f)$, $B_6(x, 1, c; y, 1, f)$, $B_6(1, b, x; 1, 1, 1)$,

4. $B_6(1, b, 1; 1, e, 1)$, $B_6(1, b, 1; 1, x, y)$, $B_6(1, x, y; 1, 1, f)$,

5. 145 *specific graphs:* 22 *graphs of order* 10, 54 *graphs of order* 11, *and* 69 *graphs of order* 12,

where a, b, c, d, e, f, x, y are some positive integers such that $x \le 2$ and $y \le 2$.

Proof. Let Ω_6 be the set of all 13 above types of graphs. In other words,

$$\Omega_6 = \Big\{ B_6(a, x, c; 1, 1, 1), B_6(a, 1, c; 1, e, 1), \dots, B_6(1, b, 1; 1, x, y), B_6(1, x, y; 1, 1, f) \Big\},$$

where a, b, c, d, e, f are arbitrary positive integers and $x, y \in \{1, 2\}$. First we prove that every graph of Ω_6 has positive second largest eigenvalue and negative third largest eigenvalue. For instance we show that for any positive integers a, c and f, $\lambda_2(B_6(a, 1, c; 1, 1, f)) > 0$ and $\lambda_3(B_6(a, 1, c; 1, 1, f)) < 0$. The others are proved similarly.

First we note that G_6 is an induced subgraph of $B_6(a, 1, c; 1, 1, f)$. Thus by Interlacing Theorem 2.1, $\lambda_2(B_6(a, 1, c; 1, 1, f)) \geq \lambda_2(G_6) > 0$. On the other hand, if $m = \max\{a, c, f\}$, then $B_6(a, 1, c; 1, 1, f)$ is an induced subgraph of $B_6(m, 1, m; 1, 1, m)$. So by Interlacing Theorem 2.1, $\lambda_3(B_6(a, 1, c; 1, 1, f)) \leq \lambda_3(B_6(m, 1, m; 1, 1, m))$. Thus to show that the inequality $\lambda_3(B_6(a, 1, c; 1, 1, f)) < 0$, it is sufficient to prove that $\lambda_3(B_6(m, 1, m; 1, 1, m)) < 0$. Now we show that for every positive integer m, $\lambda_3(B_6(m, 1, m; 1, 1, m)) < 0$.

Let M and m be two positive integers. If $M \geq m$, then by Interlacing Theorem 2.1, $\lambda_3(B_6(M, 1, M; 1, 1, M)) \geq \lambda_3(B_6(m, 1, m; 1, 1, m))$. This shows that if for $m \geq 12$, $\lambda_3(B_6(m, 1, m; 1, 1, m)) < 0$, then for all m, $\lambda_3(B_6(m, 1, m; 1, 1, m)) < 0$. Hence suppose that $m \geq 12$. By Theorem 2.21 we can obtain the characteristic polynomial of $B_6(m, 1, m; 1, 1, m)$. Let $\Phi_m(\lambda) = P(B_6(m, 1, m; 1, 1, m), \lambda)$. By Theorem 2.21

$$\Phi_m(\lambda) = (\lambda + 1)^{3m-3}\, \Psi_m(\lambda), \tag{3.2}$$

where

$$\Psi_m(\lambda) = 3m + (6m^2 + 7m - 1)\lambda + (2m^3 + 12m^2 - m - 3)\lambda^2 + $$
$$(m^3 + 7m^2 - 14m - 2)\lambda^3 + (m^2 - 12m + 2)\lambda^4 + (3 - 3m)\lambda^5 + \lambda^6.$$

Since $m \geq 12$, all coefficients of $\Psi_m(\lambda)$ are positive except the coefficient of λ^5. In fact the coefficient of λ^5 is negative. Now by *Descartes' Sign Rule* we conclude that the number of positive roots of $\Psi_m(\lambda)$ is 0 or 2 and the number of negative roots is 0 or 2 or 4. Since $\Psi_m(0) = 3m \neq 0$, every root of $\Psi_m(\lambda)$ is non-zero. On the other hand by Equation (3.2) the roots of $\Psi_m(\lambda)$ with many numbers -1 are the eigenvalues of $B_6(m, 1, m; 1, 1, m)$. Hence every root of $\Psi_m(\lambda)$ is real. Since $B_6(1, 1, 1; 1, 1, 1) \cong G_6$ is an induced subgraph of $B_6(m, 1, m; 1, 1, m)$, by Interlacing Theorem 2.1 and Lemma 2.16 we find that $\lambda_1(B_6(m, 1, m; 1, 1, m)) \geq \lambda_1(G_6) > 0$ and $\lambda_2(B_6(m, 1, m; 1, 1, m)) \geq \lambda_2(G_6) > 0$. Therefore by Equation (3.2), $\lambda_1(B_6(m, 1, m; 1, 1, m))$ and $\lambda_2(B_6(m, 1, m; 1, 1, m))$ are two roots of $\Psi_m(\lambda)$. Hence $\Psi_m(\lambda)$ has exactly two positive roots. Since the degree of $\Psi_m(\lambda)$ is six and $\Psi_m(\lambda)$ has exactly two positive roots and $\Psi_m(0) \neq 0$, the number of negative roots of $\Psi_m(\lambda)$ is four. Therefore by Equation (3.2), $B_6(m, 1, m; 1, 1, m)$ has exactly two positive eigenvalues and $3m + 1$ negative eigenvalues. This shows that $\lambda_3(B_6(m, 1, m; 1, 1, m)) < 0$. Now we prove the necessity.

Claim 1. Let $H = B_6(a', b', c'; d', e', f')$ be a graph with at least 19 vertices, that is $a' + \dots + f' \geq 19$. If $H \notin \Omega_6$, then one of the graphs $H_1 = B_6(a' - 1, b', c'; d', e', f')$ or $H_2 = B_6(a', b' - 1, c'; d', e', f')$ or $H_3 = B_6(a', b', c' - 1; d', e', f')$ or $H_4 = B_6(a', b', c'; d' - 1, e', f')$ or $H_5 = B_6(a', b', c'; d', e' - 1, f')$ or $B_6(a', b', c'; d', e', f' - 1)$ is not in Ω_6. Note that these graphs are all induced subgraphs of H of order $|V(H)| - 1$.

Proof of Claim 1. Suppose that $H \notin \Omega_6$. By contradiction assume that all graphs $H_1, \ldots, H_6$ are in Ω_6. Now we consider H_1. If $H_1 = B_6(a, x, c; 1, 1, 1)$ for some positive integers a, c and $x \leq 2$, then $H \in \Omega_6$, a contradiction. If $H_1 = B_6(a, 1, c; 1, e, 1)$, for some positive integers a, c and e, then $H \in \Omega_6$, a contradiction. Similarly one can see that $H_1 \neq B_6(a, 1, c; 1, x, y), B_6(a, 1, c; 1, 1, f), B_6(a, 1, 1; x, e, 1)$. So $H_1 = B_6(x, b, 1; y, 1, 1)$ or $B_6(x, y, 1; 1, e, 1)$ or $B_6(x, y, 1; 1, 1, f)$ or $B_6(x, 1, c; y, 1, f)$ or $B_6(1, b, x; 1, 1, 1)$ or $B_6(1, b, 1; 1, e, 1)$ or $B_6(1, b, 1; 1, x, y)$ or $B_6(1, x, y; 1, 1, f)$, for some positive integers b, c, e, f and $x, y \leq 2$. Since $x \leq 2$, we find that $a' - 1 \leq 2$. Thus $a' \leq 3$. Similarly if $H_2, \ldots, H_6 \in \Omega_6$, we obtain that $b', \ldots, f' \leq 3$. Therefore $a' + \cdots + f' \leq 18$, a contradiction. Thus the claim is proved.

Claim 2. Let $K = B_6(a'', b'', c''; d'', e'', f'')$ be a graph with at least 13 vertices. If $K \notin \Omega_6$, then $\lambda_3(K) \geq 0$.

Proof of Claim 2. Assume that $K \notin \Omega_6$. We prove the claim by induction on $n = |V(K)|$. By computer one can check the validity for $n = 13, \ldots, 18$. Hence let $n \geq 19$. By Claim 1, K has an induced subgraph, say L, of order $n - 1$ such that $L \notin \Omega_6$. Since $n - 1 \geq 18$, by the induction hypothesis $\lambda_3(L) \geq 0$. Thus by Interlacing Theorem 2.1, $\lambda_3(K) \geq \lambda_3(L) \geq 0$. Thus the claim is proved.

Now let $W = B_6(a''', b''', c'''; d''', e''', f''')$ be a graph of order n. Assume that $\lambda_2(W) > 0$ and $\lambda_3(W) < 0$. If $W \notin \Omega_6$, then by Claim 2, $n \leq 12$. By computer we find that there are only 145 graphs with this property. More precisely there are 22 graphs of order 10, 54 graphs of order 11 and 69 graphs of order 12 such that they are not in Ω_6 while their second eigenvalues are positive and third eigenvalues are negative. The proof is complete. □

Theorem 3.8. *Let $G = B_7(a_1, a_2, a_3; a_4, a_5, a_6; a_7)$, where $a_1, \ldots, a_7$ are some positive integers. Then $\lambda_2(G) > 0$ and $\lambda_3(G) < 0$ if and only if G is isomorphic to one of the following graphs:*

1. *$B_7(a, 1, x; 1, e, 1; 1)$, $B_7(a, 1, 1; 1, e, 1; g)$, $B_7(a, 1, 1; 1, 1, x; 1)$,*

2. *$B_7(x, y, 1; 1, e, 1; g)$, $B_7(x, 1, 1; y, 1, 1; g)$, $B_7(1, b, x; 1, 1, 1; g)$,*

3. *$B_7(1, b, 1; 1, e, 1; g)$, $B_7(1, 1, c; 1, 1, f; 1)$,*

4. *143 specific graphs: 18 graphs of order 10, 52 graphs of order 11, and 73 graphs of order 12,*

where $a, b, c, d, e, f, g, x, y$ are some positive integers such that $x \leq 2$ and $y \leq 2$.

Theorem 3.9. *Let $G = B_8(a_1, a_2, a_3, a_4; a_5, a_6, a_7, a_8)$, where $a_1, \ldots, a_8$ are some positive integers. Then $\lambda_2(G) > 0$ and $\lambda_3(G) < 0$ if and only if G is isomorphic to one of the following graphs:*

1. *$B_8(a, 1, 1, d; 1, 1, g, 1)$, $B_8(1, b, 1, 1; 1, f, 1, 1)$,*

2. *134 specific graphs: 12 graphs of order 10, 42 graphs of order 11, and 80 graphs of order 12,*

where a, b, d, f, g are some positive integers.

Theorem 3.10. *Let $G = B_9(a_1, a_2, a_3, a_4; a_5, a_6, a_7, a_8; a_9)$, where $a_1, \ldots, a_9$ are some positive integers. Then $\lambda_2(G) > 0$ and $\lambda_3(G) < 0$ if and only if G is isomorphic to one of the following graphs:*

1. *$B_9(1, b, 1, 1; 1, f, 1, 1; k)$,*

2. *59 specific graphs: 3 graphs of order 10, 17 graphs of order 11, and 39 graphs of order 12,*

where b, f, k are some positive integers.

Remark 3.11. The complete list of the mentioned 25 graphs in Theorem 3.5, 63 graphs in Theorem 3.6, 145 graphs in Theorem 3.7, 143 graphs in Theorem 3.8, 134 graphs in Theorem 3.9 and 59 graphs in Theorem 3.10 can be obtained from the author upon request.

Theorem 3.12. *Let $G = B_{10}(a_1, a_2, a_3, a_4, a_5; a_6, a_7, a_8, a_9, a_{10})$, where $a_1, \ldots, a_{10}$ are some positive integers. Then $\lambda_2(G) > 0$ and $\lambda_3(G) < 0$ if and only if G is isomorphic to one of the following 26 graphs:*

1. *$B_{10}(1, 1, 1, 1, 1; 1, 1, 1, 1, 1)$,*

2. *$B_{10}(1, 1, 1, 1, 2; 1, 1, 1, 1, 1)$, $B_{10}(1, 1, 1, 2, 1; 1, 1, 1, 1, 1)$,*

3. *$B_{10}(1, 1, 2, 1, 1; 1, 1, 1, 1, 1)$, $B_{10}(1, 2, 1, 1, 1; 1, 1, 1, 1, 1)$,*

4. *$B_{10}(2, 1, 1, 1, 1; 1, 1, 1, 1, 1)$,*

5. *$B_{10}(1, 1, 1, 1, 3; 1, 1, 1, 1, 1)$, $B_{10}(1, 1, 1, 2, 1; 1, 1, 1, 1, 2)$,*

6. *$B_{10}(1, 1, 1, 2, 1; 1, 1, 1, 2, 1)$, $B_{10}(1, 1, 1, 2, 2; 1, 1, 1, 1, 1)$,*

7. *$B_{10}(1, 1, 1, 3, 1; 1, 1, 1, 1, 1)$, $B_{10}(1, 1, 2, 1, 1; 1, 1, 1, 1, 2)$,*

8. *$B_{10}(1, 1, 2, 2, 1; 1, 1, 1, 1, 1)$, $B_{10}(1, 1, 3, 1, 1; 1, 1, 1, 1, 1)$,*

9. *$B_{10}(1, 2, 1, 1, 1; 1, 1, 2, 1, 1)$, $B_{10}(1, 2, 1, 1, 1; 1, 2, 1, 1, 1)$,*

10. *$B_{10}(1, 2, 1, 1, 2; 1, 1, 1, 1, 1)$, $B_{10}(1, 2, 2, 1, 1; 1, 1, 1, 1, 1)$,*

11. *$B_{10}(1, 3, 1, 1, 1; 1, 1, 1, 1, 1)$, $B_{10}(2, 1, 1, 1, 1; 1, 1, 1, 2, 1)$,*

12. *$B_{10}(2, 1, 1, 1, 1; 1, 1, 2, 1, 1)$, $B_{10}(2, 1, 1, 1, 1; 2, 1, 1, 1, 1)$,*

13. *$B_{10}(2, 1, 1, 1, 2; 1, 1, 1, 1, 1)$, $B_{10}(2, 1, 1, 2, 1; 1, 1, 1, 1, 1)$,*

14. *$B_{10}(2, 2, 1, 1, 1; 1, 1, 1, 1, 1)$, $B_{10}(3, 1, 1, 1, 1; 1, 1, 1, 1, 1)$.*

Proof. One can see that all of the above graphs have positive second largest eigenvalue and negative third largest eigenvalue. Now we prove the necessity. Let $G = B_{10}(a_1, \ldots, a_5; a_6, \ldots, a_{10})$ such that $\lambda_2(G) > 0$ and $\lambda_3(G) < 0$. We show that for $i = 1, \ldots, 10$, $a_i \leq 3$. For example by contradiction suppose that $a_1 \geq 4$. Thus $H = B_{10}(4, 1, 1, 1, 1; 1, 1, 1, 1, 1)$ is an induced subgraph of G. So by Interlacing Theorem 2.1, $\lambda_3(G) \geq \lambda_3(H)$. On the other hand $\lambda_3(H) > 0$, a contradiction. Similarly we obtain $a_2, \ldots, a_{10} \leq 3$. Also one can

see that at most one of the numbers $a_1, \ldots, a_{10}$ is 3. For example if $a_1 = 3$ and $a_2 = 3$, then $K = B_{10}(3, 3, 1, 1, 1; 1, 1, 1, 1, 1)$ is an induced subgraph of G. So by Interlacing Theorem 2.1, $\lambda_3(G) \geq \lambda_3(K)$ while $\lambda_3(K) > 0$, a contradiction. Since $a_1, \ldots, a_{10} \leq 3$ and at most one of them is 3, $a_1 + \cdots + a_{10} \leq 21$. Thus the order of G is at most 21. Now by computer one can check the result. $\square$

Theorem 3.13. *Let* $G = B_{11}(a_1, a_2, a_3, a_4, a_5; a_6, a_7, a_8, a_9, a_{10}; a_{11})$, *where* $a_1, \ldots, a_{11}$ *are some positive integers. Then* $\lambda_2(G) > 0$ *and* $\lambda_3(G) < 0$ *if and only if* G *is isomorphic to one of the following* 5 *graphs:*

 1. $B_{11}(1, 1, 1, 1, 1; 1, 1, 1, 1, 1; 1)$,

 2. $B_{11}(1, 1, 1, 1, 1; 1, 1, 1, 1, 1; 2)$, $B_{11}(1, 1, 1, 1, 2; 1, 1, 1, 1, 1; 1)$,

 3. $B_{11}(1, 2, 1, 1, 1; 1, 1, 1, 1, 1; 1)$, $B_{11}(1, 1, 2, 1, 1; 1, 1, 1, 1, 1; 1)$.

Theorem 3.14. *Let* $G = B_{12}(a_1, a_2, a_3, a_4, a_5, a_6; a_7, a_8, a_9, a_{10}, a_{11}, a_{12})$, *where* $a_1, \ldots,$ a_{12} *are some positive integers. Then* $\lambda_2(G) > 0$ *and* $\lambda_3(G) < 0$ *if and only if* $G \cong$ $B_{12}(1, 1, 1, 1, 1, 1; 1, 1, 1, 1, 1, 1)$.

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