The complete bipartite graphs which have exactly two orientably edge-transitive embeddings[*]

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Abstract

In 2018, Fan and Li classified the complete bipartite graph $\mathbf{K}_{m,n}$ that has a unique orientably edge-transitive embedding. In this paper, we extend this to give a complete classification of $\mathbf{K}_{m,n}$ which have exactly two orientably edge-transitive embeddings.

Keywords: Bipartite graphs, edge-transitive embeddings.

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1   Introduction

Let $\mathcal{M} = (V, E, F)$ be an orientable map with vertex set V, edge set E and face set F, that is, $\mathcal{M}$ is a 2-cell embedding of the underlying graph $\Gamma = (V, E)$ in an orientable surface. A permutation of $V \cup E \cup F$ which preserves V, E, F, and their incidence relations is called an *automorphism* of $\mathcal{M}$. All automorphisms of $\mathcal{M}$ form the *automorphism group* Aut $\mathcal{M}$ under composition.

A map $\mathcal{M} = (V, E, F)$ is said to be *G-edge-transitive* if $G \leq$ Aut $\mathcal{M}$ is transitive on E; if in addition G also preserves the orientation of the supporting surface, then $\mathcal{M}$ is called *orientably edge-transitive*. Similarly, *orientably arc-transitive* maps are defined.

It is a main aim of topological graph theory to determine and enumerate all the 2-cell embeddings of a given class of graphs, see [2, 3, 7, 10, 11, 12] for arc-transitive maps, and [5, 8, 9, 13] for edge-transitive maps.

Although each map has a unique underlying graph, a graph may have many non-isomorphic 2-cell embeddings usually. For example, $\mathbf{K}_{3,2}$ has two edge transitive embeddings that have automorphism groups $\mathbb{Z}_6$ and S_3, respectively. As a special case, the complete bipartite graphs that has a unique edge-transitive embedding has been received much attention. For instance, Jones, Nedela and Škoviera [10] proved that $\mathbf{K}_{n,n}$ has a unique orientably arc-transitive embedding if and only if $\gcd(n, \phi(n)) = 1$, where $\phi(n)$ is the Euler *phi-function*. Fan and Li [4] showed that $\mathbf{K}_{m,n}$ have a unique edge-transitive embedding if and only if $\gcd(m, \phi(n)) = 1 = \gcd(n, \phi(m))$. For convenience, we call the pair (m, n) *singular* if $\gcd(m, \phi(n)) = 1 = \gcd(n, \phi(m))$ in the following.

The aim of this paper is to consider the analogous problem for the complete bipartite graph $\mathbf{K}_{m,n}$, and we give a complete classification of $\mathbf{K}_{m,n}$ which have exactly two orientably edge-transitive embeddings. To state the theorem, we need some notations. For an integer n and a prime p, let $n = n_p n_{p'}$ such that n_p is a p-power and $\gcd(n_p, n_{p'}) = 1$. The main theorem of this paper is now stated as follows.

Theorem 1.1. *A complete bipartite graph $\mathbf{K}_{m,n}$ has exactly two orientably edge-transitive embeddings if and only if, interchanging m and n if necessary, one of the following holds:*

(i) $(m, n) = (4, 2)$;

(ii) $m = p^e$ *with p odd, $n = 2n_{2'}$, and $(m, n_{2'})$ is a singular pair;*

(iii) $m = p^e$ *with $p \equiv 3 \pmod 4$, $n = 2^f n_{2'}$ with $f \geq 2$, and $(m, n_{2'})$ is a singular pair;*

(iv) $m = 2p^e$ *with p odd, and $n = 2$.*

This solved the problem in [4] to determine complete bipartite graphs which have exactly two non-isomorphic orientably edge-transitive embeddings.

Particularly, the following corollary about $\mathbf{K}_{n,n}$ is easily observed.

Corollary 1.2. *There exists no complete bipartite graph $\mathbf{K}_{n,n}$ ($n \geq 2$) that has exactly two non-isomorphic orientably edge-transitive embeddings.*

2   Complete bipartite edge-transitive maps

Let m, n be positive integers, and let $\Gamma = (V, E)$ be a complete bipartite graph $\mathbf{K}_{m,n}$. Let $\mathcal{M}$ be an orientable map with underlying graph $\Gamma = \mathbf{K}_{m,n}$. Let $\mathrm{Aut}^+ \mathcal{M}$ consist of

automorphisms of $\mathcal{M}$ which preserves the biparts of Γ, and let $\mathrm{Aut}^{\mathrm{O}}\mathcal{M}$ be the subgroup of $\mathrm{Aut}\,\mathcal{M}$ which preserves the orientation of the supporting surface. Let

$$\mathrm{Aut}^{\oplus}\mathcal{M} = \mathrm{Aut}^{+}\mathcal{M} \cap \mathrm{Aut}^{\mathrm{O}}\mathcal{M}.$$

Then $\mathrm{Aut}^{\oplus}\mathcal{M}$ contains all elements of $\mathrm{Aut}\,\mathcal{M}$ which preserve the orientation of the supporting surface, and fixes the biparts of the underlying graph. It is clear that isomorphic embeddings of $\mathbf{K}_{m,n}$ have isomorphic automorphism groups.

Orientable edge-transitive embeddings of $\mathbf{K}_{m,n}$ have automorphism groups being *bicyclic*, which is defined as follows.

Definition 2.1. A group G is called *bicyclic* if $G = \langle a \rangle \langle b \rangle$ for some elements $a, b \in G$. If $|a| = m$ and $|b| = n$, then G is said to be of *order* $\{m, n\}$. If in addition $\langle a \rangle \cap \langle b \rangle = 1$, then G is called an *exact bicyclic group*, and $\{a, b\}$ is called an *exact bicyclic pair* of order $\{m, n\}$.

It is known that orientable edge-transitive embeddings of $\mathbf{K}_{m,n}$ precisely correspond to exact bicyclic pairs of order $\{m, n\}$. We denote by

$$\mathcal{M}(G, a, b)$$

the edge-transitive embedding of $\mathbf{K}_{m,n}$ corresponding to a bicyclic group G associated with a bicyclic pair $\{a, b\}$. For convenience, (a, b) is called an *edge-regular pair* for G. Moreover, $\mathcal{M}(G, a, b)$ is called an *abelian embedding* if G is abelian, and *non-abelian embedding* otherwise.

The following lemma is well-known and easy to prove, see [11] or [6].

Lemma 2.2. *Let G be an exact bicyclic group of order $\{m, n\}$, and let $a, b \in G$ be a bicyclic pair. Then there is an edge-transitive orientable embedding $\mathcal{M} = \mathcal{M}(G, a, b)$ of $\mathbf{K}_{m,n}$ such that $\mathrm{Aut}^{\oplus}\mathcal{M} = G$ is edge-regular on $\mathcal{M}$, and for any bicyclic pair $x, y \in G$, $\mathcal{M}(G, a, b) \cong \mathcal{M}(G, x, y)$ if and only if there is an automorphism σ of $\mathrm{Aut}(G)$ such that $(a, b)^{\sigma} = (x, y)$.*

Since there exists an abelian bicyclic group $\mathbb{Z}_m \times \mathbb{Z}_n$ for any positive integers m and n, the graph $\Gamma = \mathbf{K}_{m,n}$ has a unique orientable edge-regular embedding $\mathcal{M}$ such that $\mathrm{Aut}^{\oplus}\mathcal{M} \cong \mathbb{Z}_m \times \mathbb{Z}_n$, see [4, Lemma 2.3]. Moreover, it is known that if $\{m, n\}$ is a singular pair of integers then each exact bicyclic group of order $\{m, n\}$ is abelian, see [4, Lemma 3.3]. This leads to the following observation.

Lemma 2.3. *Let m, n be positive integers for which $\mathbf{K}_{m,n}$ has exactly two non-isomorphic edge-transitive embeddings. Then there exists a unique non-abelian exact bicyclic group of order $\{m, n\}$.*

In the next section, we work out a classification of integer pairs $\{m, n\}$ for which there is only one non-abelian exact bicyclic group.

3 Uniqueness of groups

Let (m, n) be a pair of integers such that there is a unique non-abelian exact bicyclic group of order $\{m, n\}$. Then (m, n) is not a singular pair. So there exist divisors $m_p \mid m$, and $n_q \mid n$ such that (m_p, n_q) is not a singular pair.

The first lemma determines (m_p, n_p) for the same prime p.

Lemma 3.1. *If a prime* $p \mid \gcd(m, n)$, *then*

(i) $\{m_p, n_p\} = \{4, 2\}$, *or*

(ii) $\{m_p, n_p\} = \{p^e, p\}$ *with* p *an odd prime and* $e \geq 2$, *or*

(iii) $\{m_p, n_p\} = \{p^2, p^2\}$ *with* p *an odd prime.*

Proof. To prove the lemma, we may assume that $m_p \geq n_p$ and $m_p = p^e$ with $e \geq 2$.

Assume first that $n_p = p$. If $p = 2$ and $e \geq 3$, then there are 3 non-isomorphic groups $G_i = \mathbb{Z}_m{:}\mathbb{Z}_n = \mathbb{Z}_{m_{2'}} \times \mathbb{Z}_{n_{2'}} \times P_i$, where $i = 1, 2$ or 3, and $P_i = \langle x_2 \rangle \langle y_2 \rangle = \mathbb{Z}_{2^e}{:}\mathbb{Z}_2$ as below:

$$P_1 = \langle x_2, y_2 \mid x_2^{y_2} = x_2^{-1} \rangle,$$
$$P_2 = \langle x_2, y_2 \mid x_2^{y_2} = x_2^{2^{e-1}+1} \rangle,$$
$$P_3 = \langle x_2, y_2 \mid x_2^{y_2} = x_2^{2^{e-1}-1} \rangle.$$

This contradiction shows that either $p = 2$ and $e = 2$, that is $(m_p, n_p) = (4, 2)$, or p is odd and $(m_p, n_p) = (p^e, p)$ with $e \geq 2$.

Next, assume that $n_p = p^f$ with $f \geq 2$. Suppose further that $e \geq 3$. Then there exist at least 2 non-isomorphic groups $G_i = \mathbb{Z}_m{:}\mathbb{Z}_n = \mathbb{Z}_{m_{p'}} \times \mathbb{Z}_{n_{p'}} \times P_i$, where $i = 1$ or 2, and $P_i = \langle x_p \rangle \langle y_p \rangle = \mathbb{Z}_{p^e}{:}\mathbb{Z}_{p^f}$ as below:

$$P_1 = \langle x_p, y_p \mid x_p^{y_p} = x_p^{1+p^{e-1}} \rangle,$$
$$P_2 = \langle x_p, y_p \mid x_p^{y_p} = x_p^{1+p^{e-2}} \rangle.$$

This is a contradiction. Thus $e = 2$, and $f = 2$.

Suppose further that $p = 2$, there are two non-isomorphic groups $G_i = \mathbb{Z}_m \mathbb{Z}_n = \mathbb{Z}_{m_{2'}} \times \mathbb{Z}_{n_{2'}} \times P_i$, where $i = 1$ or 2, and $P_i = \langle x_2 \rangle \langle y_2 \rangle = \mathbb{Z}_4 \mathbb{Z}_4$ is non-abelian as below:

$$P_1 = \langle x_2, y_2 \mid x_2^4 = y_2^4 = 1, x_2^{y_2} = x_2^{-1} \rangle,$$
$$P_2 = \langle x_2, y_2 \mid x_2^4 = y_2^4 = [x_2^2, y_2] = [x_2, y_2^2] = 1, [y_2, x_2] = x_2^2 y_2^2 \rangle.$$

So $\{m_p, n_p\} = \{p^2, p^2\}$ with p an odd prime. $\qquad\square$

The next lemma determines the relation m_p and n_q for distinct primes p, q.

Lemma 3.2. *Assume that* $m_p = p^e$ *and* $n_q = q^f$, *where* $q \mid (p-1)$. *Then either* $f = 1$, *or* $q^2 \nmid (p-1)$; *equivalently,* $\gcd(n_q, \phi(m_p)) = q$.

Proof. Suppose that $f > 1$ and q^2 divides $p - 1$. Then there exist at least 2 non-abelian groups

$$G_i = \mathbb{Z}_m{:}\mathbb{Z}_n = \mathbb{Z}_{m_{p'}} \times \mathbb{Z}_{n_{q'}} \times H_i,$$

where $i = 1$ or 2, and $H_i = \langle x_p \rangle{:}\langle y_q \rangle = \mathbb{Z}_{p^e}{:}\mathbb{Z}_{q^f}$ are as below:

$$H_1 = \langle x_p, y_q \mid x_p^{y_q} = x_p^i \rangle, \text{ where } i \neq 1 \text{ and } i^q \equiv 1 \pmod{p^e};$$
$$H_2 = \langle x_p, y_q \mid x_p^{y_q} = x_p^j \rangle, \text{ where } j^q \not\equiv 1 \pmod{p^e}.$$

This contradiction shows that either $f = 1$, or $q^2 \nmid (p-1)$, as stated. $\qquad\square$

Remark on Lemma 3.2. Interchange m and n, if $p \mid (q-1)$, then either $e = 1$, or $p^2 \nmid (q-1)$; equivalently, $\gcd(m_p, \phi(n_q)) = p$.

Now we are ready to state the main result of this section.

Theorem 3.3. *Given a pair of integers $\{m, n\}$. Then the following two statements are equivalent:*

(a) *there is a unique non-abelian exact bicyclic group (up to isomorphism) of order $\{m, n\}$,*

(b) *there exist exactly one prime $p \mid m$ and exactly one prime $q \mid n$ such that (m_p, n_q) is not a singular pair, and either*

(i) $\gcd(\phi(m_p), n_q) = q$, *and* $\gcd(m_p, \phi(n_q)) = 1$, *or*

(ii) $\gcd(m_p, \phi(n_q)) = p$, *and* $\gcd(\phi(m_p), n_q) = 1$.

If further $p = q$, then $\{m_p, n_p\} = \{4, 2\}$, or $\{p^2, p^2\}$ with p an odd prime, or $\{p^e, p\}$ with p a prime and $e \geq 3$.

Proof. First, assume (a) holds. Let $G = \langle a \rangle \langle b \rangle$ be the unique exact nonabelian bicyclic group of order $\{m, n\}$, where $|a| = m$ and $|b| = n$. Then (m, n) is not a singular pair. So there exist at least one prime $p \mid m$, and at least one prime $q \mid n$, such that (m_p, n_q) is not a singular pair.

Suppose that p_1, p_2 are prime divisors of m and q_1, q_2 are prime divisors of n such that $\gcd(n_{q_i}, \phi(m_{p_i})) \neq 1$ with $i = 1$ or 2, and either $p_1 \neq p_2$ or $q_1 \neq q_2$. Then there are 2 non-isomorphic nonabelian exact bicyclic groups of the form $G = \langle a \rangle{:}\langle b \rangle = \mathbb{Z}_m{:}\mathbb{Z}_n$:

$$\langle a \rangle{:}\langle b \rangle = \langle a_{p_1'} a_{p_1} \rangle{:}\langle b_{q_1'} b_{q_1} \rangle = \langle a_{p_1'} \rangle \times \langle b_{q_1'} \rangle \times (\langle a_{p_1} \rangle{:}\langle b_{q_1} \rangle),$$
$$\langle a \rangle{:}\langle b \rangle = \langle a_{p_2'} a_{p_2} \rangle{:}\langle b_{q_2'} b_{q_2} \rangle = \langle a_{p_2'} \rangle \times \langle b_{q_2'} \rangle \times (\langle a_{p_2} \rangle{:}\langle b_{q_2} \rangle).$$

This is a contradiction.

Similarly, interchanging m and n, suppose that p_1, p_2 are prime divisors of m and q_1, q_2 are prime divisors of n such that $\gcd(m_{p_i}, \phi(n_{q_i})) \neq 1$ with $i = 1$ or 2, and either $p_1 \neq p_2$ or $q_1 \neq q_2$. Then there are 2 non-isomorphic nonabelian exact bicyclic groups of the form $G = \langle b \rangle{:}\langle a \rangle = \mathbb{Z}_n{:}\mathbb{Z}_m$:

$$\langle b \rangle{:}\langle a \rangle = \langle b_{p_1'} b_{p_1} \rangle{:}\langle a_{q_1'} a_{q_1} \rangle = \langle a_{p_1'} \rangle \times \langle b_{q_1'} \rangle \times (\langle b_{q_1} \rangle{:}\langle a_{p_1} \rangle),$$
$$\langle b \rangle{:}\langle a \rangle = \langle b_{p_2'} b_{p_2} \rangle{:}\langle a_{q_2'} a_{q_2} \rangle = \langle a_{p_2'} \rangle \times \langle b_{q_2'} \rangle \times (\langle b_{q_2} \rangle{:}\langle a_{p_2} \rangle).$$

This is a contradiction.

Now, suppose that p_1, p_2 are prime divisors of m and q_1, q_2 are prime divisors of n such that $\gcd(n_{q_1}, \phi(m_{p_1})) \neq 1$ and $\gcd(m_{p_2}, \phi(n_{q_2})) \neq 1$. Then there are 2 non-isomorphic nonabelian exact bicyclic groups G of the form:

$$\langle a \rangle{:}\langle b \rangle = \langle a_{p_1'} a_{p_1} \rangle{:}\langle b_{q_1'} b_{q_1} \rangle = \langle a_{p_1'} \rangle \times \langle b_{q_1'} \rangle \times (\langle a_{p_1} \rangle{:}\langle b_{q_1} \rangle),$$
$$\langle b \rangle{:}\langle a \rangle = \langle b_{q_2'} b_{q_2} \rangle{:}\langle a_{p_2'} a_{p_2} \rangle = \langle b_{q_2'} \rangle \times \langle a_{p_2'} \rangle \times (\langle b_{q_2} \rangle{:}\langle a_{p_2} \rangle).$$

This is a contradiction.

We thus conclude that there is exactly one prime $p \mid m$ and exactly one prime $q \mid n$ such that (m_p, n_q) is not a singular pair.

Assume that $\gcd(\phi(m_p), n_q) \neq 1$ and $\gcd(m_p, \phi(n_q)) \neq 1$ such that $p \neq q$. By Lemma 3.2, $\gcd(\phi(m_p), n_q) = q$ and $\gcd(m_p, \phi(n_q)) = p$, which implies that $q \mid (p-1)$ and $p \mid (q-1)$, this is not possible. Thus either part (b)(i) or part (b)(ii) holds. Moreover, if $p = q$, then by Lemma 3.1, we have $\gcd(m_p, \phi(n_p)) = p$, or $\gcd(\phi(m_p), n_p) = p$, which implies that $\{m_p, n_p\} = \{4, 2\}$, or $\{p^2, p^2\}$ with p an odd prime, $\{p^e, p\}$ with p a prime and $e \geq 3$.

Conversely, let m, n be integers satisfying condition (b). We claim that both $(m_{p'}, n)$ and $(m, n_{q'})$ are singular pairs. In fact, suppose to the contrary that one of $(m_{p'}, n)$ and $(m, n_{q'})$, say $(m_{p'}, n)$, is not singular. Then there is a prime $p_1 \neq p$ of m, and a prime q_1 of n, such that the pair (m_{p_1}, n_{q_1}) is not singular, which contradicts with the unique choice of the prime p.

Now let $G = \langle a \rangle \langle b \rangle$ such that $\langle a \rangle = \mathbb{Z}_m$, $\langle b \rangle = \mathbb{Z}_n$ and $\langle a \rangle \cap \langle b \rangle = 1$. Then G is supersoluble by [1]. Further let $G = \langle a \rangle \langle b \rangle = \langle a_p a_{p'} \rangle \langle b_q b_{q'} \rangle = G_p G_{p'}$, then $G_p = \langle a_p \rangle = \mathbb{Z}_{m_p}$ and $G_{p'} = \langle a_{p'} \rangle (\langle b_q \rangle \times \langle b_{q'} \rangle) = \mathbb{Z}_{m_{p'}} \mathbb{Z}_n$. As $\langle a_{p'} \rangle \cap \langle b \rangle = 1$, we have that $G_{p'}$ is an exact bicyclic group of order $\{m_{p'}, n\}$. By [4, Lemma 3.3], $G_{p'}$ is abelian. So $G_{p'} = (\langle a_{p'} \rangle \times \langle b_{q'} \rangle) \times \langle b_q \rangle$. Similarly, $G_{q'} = (\langle a_{p'} \rangle \times \langle b_{q'} \rangle) \times \langle a_p \rangle$. Thus a Hall subgroup $G_{\{p,q\}'}$ is abelian and centralizes both G_p and G_q, and so $G = G_{\{p,q\}'} \times G_{\{p,q\}}$, where $G_{\{p,q\}'} = \langle a_{p'} \rangle \times \langle b_{q'} \rangle = \mathbb{Z}_{m_p'} \times \mathbb{Z}_{n_{q'}}$ and $G_{\{p,q\}} = \langle a_p \rangle \langle b_q \rangle = \mathbb{Z}_{m_p} \mathbb{Z}_{n_q}$ is nonabelian. Moreover, assume that (b)(i) hold, that is $\gcd(\phi(m_p), n_q) = q$ and $\gcd(m_p, \phi(n_q)) = 1$, we have $a_p^{b_q} = a_p^\lambda$, where $\lambda \neq 1$ and $\lambda^q \equiv 1 \pmod{m_p}$. So the group $G_{\{p,q\}} = \langle a_p \rangle{:}\langle b_q \rangle$. We claim that the group $G_{\{p,q\}}$ is unique up to isomorphism. In fact, assume that $\mu \neq \lambda$ such that $H = \langle x, y \mid x^y = x^\mu, \mu \neq 1, \mu^q \equiv 1 \pmod{p^e} \rangle$. Then $\langle \lambda \rangle$ and $\langle \mu \rangle$ are both subgroups of order q in $\mathbb{Z}_{p^e}^*$, where $\mathbb{Z}_{p^e}^*$ is the multiplicative group consisting of all the unites in the ring $\mathbb{Z}_{p^e}$. Since $\mathbb{Z}_{p^e}^* \cong \mathbb{Z}_{p-1} \times \mathbb{Z}_{p^{e-1}}$, which has a unique subgroup of order q, we have $\langle \lambda \rangle = \langle \mu \rangle$. Thus $\lambda = \mu^k \pmod{p^e}$ for some integer k. Let $z = y^k$. Then $z \in H$ and $x^z = x^{u^k} = x^\lambda$. Hence $H \cong G_{\{p,q\}}$. Similarly, assume that (b)(ii) hold, we have the group $G_{\{p,q\}} = \langle b_q \rangle{:}\langle a_p \rangle$, $b_q^{a_p} = b_q^\lambda$, where $\lambda \neq 1$ and $\lambda^p \equiv 1 \pmod{n_q}$, which is unique up to isomorphism. Therefore, there is only one non-abelian exact bicyclic group of order $\{m, n\}$.

In particular, assume that $p = q$. Then $G = \langle a \rangle \langle b \rangle = G_p G_{p'}$, where $G_p = \langle a_p \rangle \langle b_p \rangle \cong \mathbb{Z}_{m_p} \mathbb{Z}_{n_p}$ and $G_{p'} = \langle a_{p'} \rangle \langle b_{p'} \rangle \cong \mathbb{Z}_{m_{p'}} \mathbb{Z}_{n_{p'}}$. By the assumption, there exists exactly one prime $p \mid \gcd(m, n)$ such that (m_p, n_p) is not a singular pair. We conclude that $(m_{p'}, n_{p'})$, $(m_{p'}, n)$ and $(m, n_{p'})$ are all singular pairs. Since $\langle a_{p'} \rangle \cap \langle b_{p'} \rangle \subseteq \langle a \rangle \cap \langle b \rangle = 1$, by [4, Lemma 3.3], $G_{p'}$ is abelian. Similarly, from $\langle a_{p'} \rangle \cap \langle b \rangle = 1$ and $\langle a \rangle \cap \langle b_{p'} \rangle = 1$, it follows that $\langle a_{p'} \rangle \langle b \rangle = G_{p'} \langle b_p \rangle$ and $\langle a \rangle \langle b_{p'} \rangle = G_{p'} \langle a_p \rangle$ are abelian. That is to say $G_{p'}$ centralizes both $\langle a_p \rangle$ and $\langle b_p \rangle$. Thus $G = G_{p'} \times G_p$, where $G_{p'} = \langle a_{p'} \rangle \times \langle b_{p'} \rangle$ and $G_p = \langle a_p \rangle{:}\langle b_p \rangle$, which is unique discussed as above. These prove (a) holds. $\qquad \square$

4 Proof of the main theorem

Let m, n be integers for which $\mathbf{K}_{m,n}$ only has one non-abelian edge-transitive embedding. Then there is only one non-abelian exact bicyclic group $G = \mathbb{Z}_m \mathbb{Z}_n$. By Theorem 3.3, interchanging m and n if necessary, there are a unique prime divisor p of m and a unique prime divisor q of n such that $G = G_{\{p,q\}'} \times G_{\{p,q\}}$, and $G_{\{p,q\}} = \mathbb{Z}_{p^e}{:}\mathbb{Z}_{q^f}$ is nonabelian.

We give a basic fact at first which is used repeatedly in the following.

Lemma 4.1. *Suppose $H = \langle a \rangle{:}\langle b \rangle = \mathbb{Z}_k{:}\mathbb{Z}_l$ is a split extension such that $a^b = a^\lambda$, where $\lambda \neq 1$, $\lambda^l \equiv 1 \pmod{k}$ and l is odd. Then the following map of H:*

$$\sigma : a \mapsto a^i, \ b \mapsto b^{-1},$$

where $\gcd(i, k) = 1$, is not an automorphism of H.

Proof. Supppose to the contrary. Then $\sigma(a)^{\sigma(b)} = (a^i)^{b^{-1}} = ba^ib^{-1} = \sigma(a^\lambda) = a^{\lambda i} = b^{-1}a^ib$, and so $a^ib^2 = b^2a^i$. Since $o(b^2) = o(b)$ and $o(a^i) = o(a)$, it follows that $H = \langle a^i, b^2 \rangle$ is abelian, which is a contradiction. □

Now we determine the $\{p^e, q^f\}$ when $p = q$.

Lemma 4.2. *If $p = q$, then $\{p^e, p^f\} = \{4, 2\}$.*

Proof. Since a group of order p^2 is abelian, without loss of generality, we may assume that $e \geq 2$, and $e \geq f$.

Suppose that p is odd. Then there exists a non-abelian metacyclic group $G_{\{p,q\}} = \langle x_p \rangle{:}\langle y_p \rangle$ such that $x_p^{y_p} = x_p^\lambda$ where $\lambda \neq 1$ and $\lambda^q \equiv 1 \pmod{p^e}$. Let

$$G = G_{\{p,q\}'} \times G_{\{p,q\}} = \langle x_{p'} \rangle \times \langle y_{p'} \rangle \times (\langle x_p \rangle{:}\langle y_p \rangle) = \langle x_{p'}x_p \rangle{:}\langle y_{p'}y_p \rangle \cong \mathbb{Z}_m{:}\mathbb{Z}_n.$$

Then the pairs $(x_{p'}x_p, y_{p'}y_p)$ and $(x_{p'}x_p, y_{p'}y_p^{-1})$ are not equivalent under $\mathrm{Aut}(G)$ by Lemma 4.1, and thus $\mathbf{K}_{m,n}$ has at least 3 non-isomorphic orientably edge-transitive embeddings, which is a contradiction.

We thus conclude that $p = 2$, and so by Theorem 3.3, $\{p^e, p^f\} = \{4, 2\}$. □

Next we determine the $\{p^e, q^f\}$ when $p \neq q$.

Lemma 4.3. *If $p \neq q$, then $q = 2$, and either $q^f = 2$, or $q^f \geq 4$ and $p \equiv 3 \pmod{4}$.*

Proof. Assume $p \neq q$. Since $G_{\{p,q\}} = \mathbb{Z}_{p^e}{:}\mathbb{Z}_{q^f}$ is nonabelian, q divides $p - 1$. Suppose that q is odd. Let $\langle x' \rangle = \mathbb{Z}_{m_{p'}}$ and $\langle y' \rangle = \mathbb{Z}_{n_{q'}}$, and let

$$G = \langle x' \rangle \times \langle y' \rangle \times (\langle x_p \rangle{:}\langle y_q \rangle) = \langle x_px' \rangle{:}\langle y_qy' \rangle = \mathbb{Z}_m{:}\mathbb{Z}_n.$$

Then (x_px', y_qy') and $(x_px', y_q^{-1}y')$ are not equivalent under $\mathrm{Aut}(G)$ by Lemma 4.1, and so $\mathbf{K}_{m,n}$ has at least 3 non-isomorphic orientably edge-transitive embeddings, which is a contradiction.

We thus have that $q = 2$, and $\gcd(n_2, \phi(m_p)) = 2$ by Lemma 3.2, that is, either $q^f = 2$, or $q^f \geq 4$ and $p \equiv 3 \pmod{4}$. □

Now we are ready to produce a list of groups for G.

Lemma 4.4. *The unique nonabelian exact bicyclic group $G = \mathbb{Z}_m\mathbb{Z}_n$ satisfies one of the following, where p is a prime:*

(i) $G = \mathrm{D}_8$;

(ii) $G = \mathrm{D}_{2p^e} \times \mathbb{Z}_{n_{2'}}$, where $(m, n_{2'})$ is a singular pair;

(iii) $G = (\mathbb{Z}_{p^e}{:}\mathbb{Z}_{2^f}) \times \mathbb{Z}_{n_{2'}}$, where $p \equiv 3 \pmod{4}$ and $(m, n_{2'})$ is a singular pair;

(iv) $G = \mathrm{D}_{4p^e}$ *with p odd.*

Proof. Noting that the group $G = G_{\{p,q\}'} \times G_{\{p,q\}}$, and $G_{\{p,q\}} = \mathbb{Z}_{p^e}{:}\mathbb{Z}_{q^f}$ is nonabelian. By Lemma 4.3, $q = 2$, and if $m_2 > 2$ is even, then $(m_2, n_2) = (4, 2)$ by Lemma 4.2. We conclude that $(m, n) = (4, 2)$, and the corresponding group $G = \mathrm{D}_8$, as in part (i).

Assume now that $m_2 = 1$. It follows that $m = p^e$ and either $q^f = 2$, or $q^f = 2^f \geq 4$ and $p \equiv 3 \pmod 4$ by Lemma 4.3. If $n_2 = 2$, then $G = \mathbb{Z}_{n_{2'}} \times (\mathbb{Z}_{p^e}{:}\mathbb{Z}_2) = \mathbb{Z}_{n_{2'}} \times \mathrm{D}_{2p^e}$ such that $(m, n_{2'})$ is a singular pair, as in part (ii). If $n_2 \geq 4$, then $G = (\mathbb{Z}_{p^e}{:}\mathbb{Z}_{2^f}) \times \mathbb{Z}_{n_{2'}}$, where $p \equiv 3 \pmod 4$ and $(m, n_{2'})$ is a singular pair, as in part (iii).

Finally assume that $m_2 = 2$. Then $m = 2p^e$ with p odd, and $n = 2$. So the corresponding group $G = \mathrm{D}_{4p^e}$, as in part (iv). This completes the proof. $\square$

To complete the proof of Theorem 1.1, we need to prove that for each group G listed in Lemma 4.4, all edge regular pairs are equivalent.

Lemma 4.5. *Let $G = \mathrm{D}_{2m}$ be dihedral. Then all edge-regular pairs for G on $\mathbf{K}_{m,2}$ are equivalent.*

Proof. Let (x, y) be an edge-regular pair for G acting on $\mathbf{K}_{m,2}$. Then $|x| = m$, $|y| = 2$, and $x^y = x^{-1}$. Let x', y' be another edge-regular pair such that $G = \langle x' \rangle \langle y' \rangle$. Then $|x'| = m$, $|y'| = 2$, and $(x')^{y'} = (x')^{-1}$. Clearly, there is an automorphism $\sigma \in \mathrm{Aut}(G)$ such that

$$\sigma \colon x \mapsto x', \ y \mapsto y',$$

so all regular pairs for G on $\mathbf{K}_{m,2}$ are equivalent. $\square$

We are now ready to prove the main theorem.

Proof of Theorem 1.1. The necessity is easily found from Lemma 4.4.

To prove the sufficienty, we need prove that for each group $G = \mathbb{Z}_m \mathbb{Z}_n$ listed in Lemma 4.4, there is exactly one non-abelian orientably edge-transitive embedding of $\mathbf{K}_{m,n}$. If G is a dihedral group, then the proof follows from Lemma 4.5. Thus we assume that G is not a dihedral group.

Assume that $m = p^e$ and $n = 2n_{2'}$. Then the only exact bicyclic group of order $\{m, n\}$ is $G = (\langle a \rangle{:}\langle b_2 \rangle) \times \langle b_{2'} \rangle$, where $\langle a \rangle{:}\langle b_2 \rangle = \mathrm{D}_{2p^e}$ and $|b_{2'}| = n_{2'}$. Let (x_1, y_1) and (x_2, y_2) be two edge-regular pairs from G. Then

$$x_1 = a^{i_1}, \qquad y_1 = b_2 b_{2'}^{j_1},$$
$$x_2 = a^{i_2}, \qquad y_2 = b_2 b_{2'}^{j_2},$$

where i_1, i_2 are coprime to p, and j_1, j_2 are coprime to $n_{2'}$. There is an automorphism $\sigma \in \mathrm{Aut}(\langle a \rangle{:}\langle b_2 \rangle)$ which sends a^{i_1} to a^{i_2}; there is an automorphism $\tau \in \mathrm{Aut}(\langle b_{2'} \rangle)$ which sends b^{j_1} to b^{j_2}. Then (σ, τ) is an automorphism of G which maps (x_1, y_1) to (x_2, y_2).

Assume that $G = (\mathbb{Z}_{p^e}{:}\mathbb{Z}_{2^f}) \times \mathbb{Z}_{n_{2'}}$, where $m = p^e$ with $p \equiv 3 \pmod 4$, and $\gcd(\phi(n), m) = 1$ and $\gcd(n, \phi(m)) = 2$. Then

$$G = (\langle a \rangle{:}\langle b_2 \rangle) \times \langle b_{2'} \rangle,$$

where $|a| = m = p^e$ and $n = 2^f n_{2'}$, and $a^{b_2} = a^{-1}$. Let (x, y) and (x', y') be edge-regular pairs for G on $\mathbf{K}_{m,n}$ such that $|x| = |x'| = m$ and $|y| = |y'| = n$. Then

$$x = a^i, \qquad y = b_2^j b_{2'}^k, \quad \text{and}$$
$$x' = a^{i'}, \qquad y' = b_2^{j'} b_{2'}^{k'},$$

where $p \nmid ii'$, jj' is odd and $\gcd(kk', n_{2'}) = 1$. It is easily shown that there are automorphisms $\sigma \in \operatorname{Aut}(\langle a \rangle{:}\langle b_2 \rangle)$ and $\tau \in \operatorname{Aut}(\langle b_{2'} \rangle)$ such that

$$\sigma \colon a^i \mapsto a^{i'}, \ b_2^j \mapsto b_2^{j'}$$
$$\tau \colon b_{2'}^k \mapsto b_{2'}^{k'}.$$

It follows that (σ, τ) is an automorphism of G which sends (x, y) to (x', y'). Thus all edge-regular pairs for G on $\mathbf{K}_{m,n}$ are equivalent. $\qquad\square$

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