ISSN 2590-9770
The Art of Discrete and Applied Mathematics 8 (2025) #P2.04
https://doi.org/10.26493/2590-9770.1799.53e
(Also available at http://adam-journal.eu)

All bipartite circulants are dispersable

Shannon Overbay*
Gonzaga University, Spokane, WA, U.S.A.

Samuel S. Joslin, Paul C. Kainen
Georgetown University, Washington DC, U.S.A.

Received 30 April 2024, accepted 26 August 2024, published online 12 March 2025

Abstract

We show that a cyclic vertex order due to Yu, Shao and Li gives a dispersable book embedding for any bipartite circulant.

Keywords: Edge-coloring, graph drawing, universal ordering.

Math. Subj. Class.: 05C10, 05C15, 05C78

It has been conjectured [1] that vertex transitive bipartite graphs can be laid out with vertices on the unit circle, with edges as chords, such that there is a Vizing type-1 edge-coloring where no crossings are monochromatic. Such a drawing and coloring is a *dispersable book embedding* [3].

We proposed this in 1979, omitting vertex transitivity [3], but in 2018, Alam et al [2] found two counterexamples and later Alam et al. [1] showed the existence of an infinite family of regular bipartite graphs, of fixed degree, which require an arbitrarily large number of edge colors to avoid monochrome edge-crossings.

Based on their examples, Alam et al [1] added the qualification of vertex transitivity, and we further conjectured in [11] that vertex transitive *non*-bipartite graphs have a type-2 edge coloring (i.e., are *nearly dispersable*). See [9, 10, 13, 14, 15] which support both conjectures.

In [16, Theorem 4.1], Yu, Shao and Li found what we call the YSL-order; see Figure 1. They used it only for bipartite degree 3 and 4 circulants. In contrast, we show that the following holds and give a short proof.

Theorem 1. Any bipartite circulant with the YSL-order is dispersable.

*Corresponding author.
E-mail address: overbay@gonzaga.edu (Shannon Overbay), ssj34@georgetown.edu (Samuel S. Joslin), kainen@georgetown.edu (Paul C. Kainen)

This work is licensed under https://creativecommons.org/licenses/by/4.0/

The theorem is additional evidence for the conjecture of [1]. Circulants have been one of the more widely used graph-types and so the theorem might have interesting applications. See, e.g., [5, 7, 8].

The YSL-order also gives dispersable embeddings for the Franklin and Heawood graphs but not for the Desargues graph.

We now briefly sketch some definitions for the reader's convenience.

A graph $G = (V, E)$ is a *circulant* if $V = \{1, \ldots, n\}$, $n \geq 3$, and there exists $S \subseteq \{1, 2, \ldots, \lfloor n/2 \rfloor\}$ such that $E = \{ij : j = i + s, s \in S\}$, where addition is mod n. Such a circulant is denoted $C(n, S)$ and the elements in S are the *jump-lengths*. The $\mathbf{C_n}$-*distance* between two vertices u, w of a circulant $C(n, S)$ is the graph-theoretic distance in the n-cycle C_n; for instance, the C_{16}-distance from 2 to 13 is 5.

A graph G is *dispersable* [3] if it has an outerplane drawing (crossings allowed) and an edge-coloring with $\Delta(G)$ colors (Δ = maximum degree) such that two edges of the same color neither cross nor share an endpoint. The color-classes (or *pages*) are *matchings* for this book embedding.

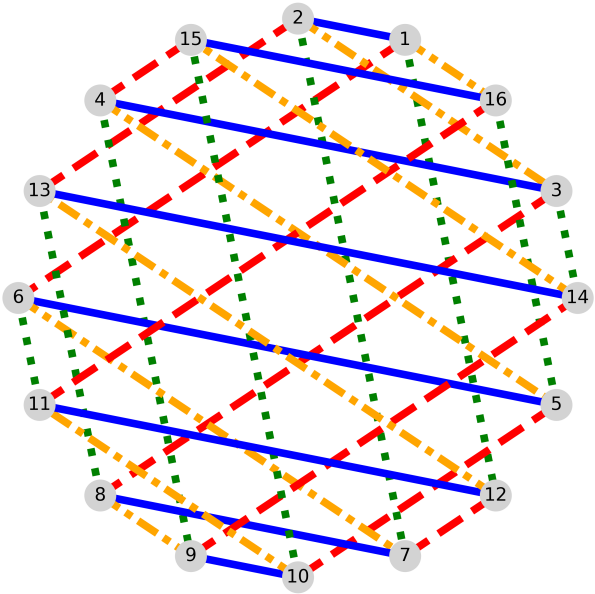


Figure 1: YSL-order for $C(16, \{1, 3, 5, 7\})$. Only 1 and 5 are shown.

In a *YSL-order*, the natural clockwise cyclic order $1, 2, \ldots, 2k$ is permuted so that the odd vertices remain, in clockwise order, at the odd positions, while the even vertices are placed at the even-indexed positions but in *counterclockwise* order. Our argument below shows that it doesn't matter where 2 is placed. In [16], Yu et al put 2 immediately counterclockwise of 1, so in clockwise direction, their order is

$$\ldots, 2k-3, 4, 2k-1, 2, 1, 2k, 3, 2k-2, 5, \ldots$$

Using characterizations of bipartite circulants by Heuberger [6] and of connected circulants by Boesch and Tindell [4], and the fact that a disjoint union of identical subgraphs is dispersable if the subgraph is dispersable, one sees that to prove Theorem 1, it suffices to show the following.

Theorem 2. Let $n = 2k$, $k \geq 2$, and let $\mu(k)$ be the largest odd number not exceeding k. Then $C = C(n, \{1, 3, \ldots, \mu(k)\})$ is dispersable under a YSL order, and the edges in each page correspond to a single jump-length.

Proof. Each page consists of a maximal parallel family of edges which we enumerate $1, \ldots, k$ in the order they intersect some orthogonal line. We show that

(i) the C_n-distance between the endpoints of edge 1 and edge 2 are equal and

(ii) the same holds for edge j and edge $j + 2$, $1 \leq j \leq n - 2$.

For (i), suppose that ab and $a'b'$ are edges 1 and 2, with (a', a, b, b') occurring consecutively clockwise. Then $a' = b \pm 2$ and $b' = a \pm 2$, where the signs agree because vertices alternate odd/even and the ordering is opposite for the two parities. Hence, the C_n distance from a to b is the same as the C_n distance from b' to a'. But distance is symmetric.

For (ii), suppose that ab and $a'b'$ are edges j and $j + 2$, with

$$(a', x, a, \ldots, b, y, b')$$

occurring consecutively clockwise but a and b can be nonconsecutive. Then $a' = a \pm 2$ and $b' = b \pm 2$, where again both signs must be equal. □

For the reader's convenience, here are the cited theorems.

Theorem 3 (Heuberger [6])**.** Let $C := C(n, \{a_1, \ldots, a_m\})$. Then C is bipartite if and only if there exists $\ell \in \mathbb{N}$ such that $2^\ell | a_1, \ldots, a_m$, $2^{\ell+1} | n$, but $2^{\ell+1}$ does *not* divide any of the a_j.

For t a positive integer, let tH denote t disjoint copies of graph H.

Theorem 4 (Boesch & Tindell [4])**.** Let $C := C(n, \{a_1, \ldots, a_m\})$ be a circulant. Then $C = r\, C(n/r, \{a_1/r, \ldots, a_m/r\})$, $r := \gcd(n, a_1, \ldots, a_m)$.

If C is bipartite, then $r = 2^\ell u$ for u odd, with ℓ as in Theorem 3.

Each maximal family of parallel edges, under the YSL ordering, has constant jump-length which could represent delay, allowing delay to depend on direction. In contrast, for Overbay's ordering [12, page 82], for a family of parallel edges, jump-lengths vary palindromically and unimodally $1, 3, \ldots, \mu(k), \ldots, 3, 1$, with $\mu(k)$ repeated if and only if k is even, as one traverses them with respect to a perpendicular line. See Figure 2.

In applications, the bipartition could be input and output vertices.

Postscript

In the interim, Yu, Shao, and Li have realized that the ordering they introduced has the universal property of applying to *all* bipartite circulants [17] (in parallel as it were). They only note the case where 2 is in specified position and their proof depends on the modularity of n, the order of the circulant. In contrast, our argument is direct.

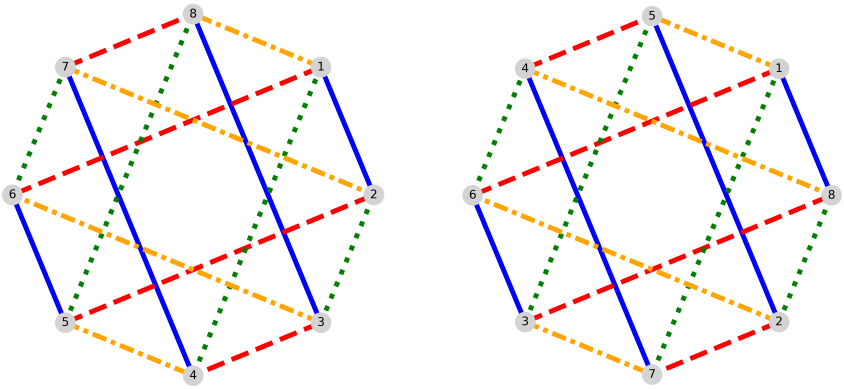


Figure 2: Overbay and YSL orders for $C(8, \{1, 3\})$.

ORCID iDs

Shannon Overbay https://orcid.org/0000-0002-8041-4187
Paul C. Kainen https://orcid.org/0000-0001-8035-0745

References

[1] J. M. Alam, M. A. Bekos, V. Dujmović, M. Gronemann, M. Kaufmann and S. Pupyrev, On dispersable book embeddings, *Theor. Comput. Sci.* **861** (2021), 1–22, doi:10.1016/j.tcs.2021.01.035, https://doi.org/10.1016/j.tcs.2021.01.035.

[2] J. M. Alam, M. A. Bekos, M. Gronemann, M. Kaufmann and S. Pupyrev, On dispersable book embeddings, in: *Graph-Theoretic Concepts in Computer Science. 44th International Workshop, WG 2018, Cottbus, Germany, June 27–29, 2018*, Springer, Cham, pp. 1–14, 2018, doi:10.1007/978-3-030-00256-5_1, https://doi.org/10.1007/978-3-030-00256-5_1.

[3] F. Bernhart and P. C. Kainen, The book thickness of a graph, *J. Comb. Theory, Ser. B* **27** (1979), 320–331, doi:10.1016/0095-8956(79)90021-2, https://doi.org/10.1016/0095-8956(79)90021-2.

[4] F. Boesch and R. Tindell, Circulants and their connectivities, *J. Graph Theory* **8** (1984), 487–499, doi:10.1002/jgt.3190080406, https://doi.org/10.1002/jgt.3190080406.

[5] F. Harary, The maximum connectivity of a graph, *Proc. Natl. Acad. Sci. USA* **48** (1962), 1142–1146, doi:10.1073/pnas.48.7.1142, https://doi.org/10.1073/pnas.48.7.1142.

[6] C. Heuberger, On planarity and colorability of circulant graphs, *Discrete Math.* **268** (2003), 153–169, doi:10.1016/S0012-365X(02)00685-4, https://doi.org/10.1016/S0012-365X(02)00685-4.

[7] X. Huang, A. F. Ramos and Y. Deng, Optimal circulant graphs as low-latency network topologies, *J. Supercomput.* **78** (2022), 13491–13510, doi:10.1007/s11227-022-04396-5, https://doi.org/10.1007/s11227-022-04396-5.

[8] F. K. Hwang, A survey on multi-loop networks., *Theor. Comput. Sci.* **299** (2003), 107–121, doi:10.1016/S0304-3975(01)00341-3, https://doi.org/10.1016/S0304-3975(01)00341-3.

[9] S. S. Joslin, P. C. Kainen and S. Overbay, On dispersability of some products of cycles, *Missouri J. Math. Sci.* **33** (2021), 206–213, doi:10.35834/2021/3302206, https://doi.org/10.35834/2021/3302206.

[10] P. C. Kainen, Complexity of products of even cycles, *Bull. Inst. Comb. Appl.* **62** (2011), 95–102.

[11] P. C. Kainen, S. Joslin and S. Overbay, On dispersability of some circulant graphs, *J. Graph Algorithms Appl.* **28** (2024), 225–241, doi:10.7155/jgaa.v28i1.2941, https://doi.org/10.7155/jgaa.v28i1.2941.

[12] S. Overbay, *Generalized book embeddings*, Ph.D. thesis, Colorado State University, Fort Collins, 1998.

[13] Z. Shao, H. Geng and Z. Li, Matching book thickness of generalized Petersen graphs, *Electron. J. Graph Theory Appl.* **10** (2022), 173–180, doi:10.5614/ejgta.2022.10.1.11, https://doi.org/10.5614/ejgta.2022.10.1.11.

[14] Z. Shao, Y. Liu and Z. Li, Matching book embedding of the Cartesian product of a complete graph and a cycle., *Ars Comb.* **153** (2020), 89–97.

[15] Z. Shao, X. Yu and Z. Li, On the dispersability of odd toroidal grids, *Appl. Math. Comput.* **453** (2023), Id/No 128087, 8 pp., doi:10.1016/j.amc.2023.128087, https://doi.org/10.1016/j.amc.2023.128087.

[16] X. Yu, Z. Shao and Z. Li, On the classification and dispersability of circulant graphs with two jump lengths, 2023, arXiv:2310.06612 [math.CO].

[17] X. Yu, Z. Shao and Z. Li, On the classification and dispersability of circulant graphs with two jump lengths, *Discrete Appl. Math.* **355** (2024), 268–286, doi:10.1016/j.dam.2024.05.012, https://doi.org/10.1016/j.dam.2024.05.012.