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On generalized truncations of complete graphs*

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Abstract

For a k-regular graph Γ and a graph Υ of order k, a generalized truncation of Γ by Υ is constructed by replacing each vertex of Γ with a copy of Υ. E. Eiben, R. Jajcay and P. Šparl introduced a method for constructing vertex-transitive generalized truncations. For convenience, we call a graph obtained by using Eiben *et al.*'s method a *special generalized truncation*. In their paper, Eiben *et al.* proposed a problem to classify special generalized truncations of a complete graph $\mathbf{K}_n$ by a cycle of length $n-1$. In this paper, we completely solve this problem by demonstrating that with the exception of $n = 6$, every special generalized truncation of a complete graph $\mathbf{K}_n$ by a cycle of length $n-1$ is a Cayley graph of $\mathrm{AGL}(1, n)$ where n is a prime power. Moreover, the full automorphism groups of all these graphs and the isomorphisms among them are determined.

Keywords: Truncation, vertex-transitive, Cayley graph, automorphism group.

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1 Introduction

In [6], the symmetry properties of graphs constructed by using the generalized truncations was investigated. In particular, a method for constructing vertex-transitive generalized truncations was proposed (see [6, Construction 4.1 and Theorem 5.1]), and this method was used to construct vertex-transitive generalized truncations of a complete graph $\mathbf{K}_n$ by a cycle of length $n-1$ for some small values of n. The vertex-transitive generalized truncations of a complete graph $\mathbf{K}_n$ by a graph Υ in context of [6, Theorem 5.1] can be defined as follows.

Let $\mathbf{K}_n$ be a complete graph of order n with $n \geq 4$, and let $V(\mathbf{K}_n) = \{v_1, v_2, \ldots, v_n\}$. Let G be an arc-transitive group of automorphisms of $\mathbf{K}_n$. Then G acts 2-transitively

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on $V(\mathbf{K}_n)$. Let $v = v_1$, and let $\mathcal{O}_v$ be a union of orbits of the stabilizer G_v acting on $\{\{x, y\} \mid x \neq y, x, y \in V(\mathbf{K}_n) \setminus \{v\}\}$. Let Υ be the graph with vertex set $\{v_2, v_3, \dots, v_n\}$ and edge set $\mathcal{O}_v$. For each $u \in V(\mathbf{K}_n)$, let $V_u = \{(u, w) \mid w \in V(\mathbf{K}_n) \setminus \{u\}\}$. The *special generalized truncation of* $\mathbf{K}_n$ *by* Υ, denoted by $T(\mathbf{K}_n, G, \Upsilon)$, is the graph with the vertex set $\bigcup_{u \in V(\mathbf{K}_n)} V_u$, and the adjacency relation in which a vertex (u, w) is adjacent to the vertex (w, u) and to all the vertices (u, w') for which there exists a $g \in G$ with the property $u^g = v$ and $\{w, w'\}^g \in \mathcal{O}_v$.

Based on the analysis of special generalized truncations of a complete graph $\mathbf{K}_n$ by a cycle of length $n - 1$ for some small values of n, the authors of [6] proposed the following problem.

Problem 1.1 ([6, Problem 5.4])**.** Classify the special generalized truncations of $\mathbf{K}_n$ $(n \geq 4)$ by a cycle of length $n - 1$.

The main purpose of this paper is to give a solution of this problem. Before stating the main result of this paper, we first set some notation. For a positive integer n, we denote by $\mathbb{Z}_n$ the cyclic group of order n, and by D_{2n} the dihedral group of order $2n$. Let $\mathbb{Z}_n^*$ be the multiplicative group of units mod n ($\mathbb{Z}_n^*$ consists of all positive integers less than n and coprime to n). Also we use A_n and S_n respectively to denote the alternating and symmetric groups of degree n. For two groups M and N, $N \rtimes M$ denotes a semidirect product of N by M. For a group G, the automorphism group of G and the socle of G will be represented by $\mathrm{Aut}(G)$ and $\mathrm{soc}(G)$, respectively. For a graph Γ we denote by $V(\Gamma)$, $E(\Gamma)$, $A(\Gamma)$ and $\mathrm{Aut}(\Gamma)$ the vertex set, edge set, arc set and full automorphism group of Γ, respectively. A graph Γ is said to be *vertex-transitive* (resp. *arc-transitive* (or *symmetric*)) if $\mathrm{Aut}(\Gamma)$ acts transitively on $V(\Gamma)$ (resp. $A(\Gamma)$). Cayley graphs form an important class of vertex-transitive graphs. Given a finite group G and an inverse closed subset $S \subseteq G \setminus \{1\}$, the *Cayley graph* $\mathrm{Cay}(G, S)$ on G with respect to S is the graph with vertex set G and edge set $\{\{g, sg\} \mid g \in G, s \in S\}$. Finally, we use $\mathbf{K}_n$ and C_n respectively to denote the complete graph and cycle with n vertices.

Let p be a prime and e a positive integer. Let $\mathrm{GF}(p^e)$ be the Galois field of order p^e and let x be a primitive root of $\mathrm{GF}(p^e)$. Then

$$\mathrm{AGL}(1, p^e) = \{\alpha_{x^i, z'} \colon z \mapsto zx^i + z', \forall z \in \mathrm{GF}(p^e) \mid i \in \mathbb{Z}_{p^e-1}, z' \in \mathrm{GF}(p^e)\},$$

and $\mathrm{AGL}(1, p^e)$ is a 2-transitive permutation group on $\mathrm{GF}(p^e)$. Let

$$H = \{\alpha_{1, z'} \colon z \mapsto z + z', \forall z \in \mathrm{GF}(p^e) \mid z' \in \mathrm{GF}(p^e)\},$$
$$K = \{\alpha_{x^i, 0} \colon z \mapsto zx^i, \forall z \in \mathrm{GF}(p^e) \mid i \in \mathbb{Z}_{p^e-1}\}.$$

Then H is regular on $\mathrm{GF}(p^e)$ and the point stabilizer $\mathrm{AGL}(1, p^e)_0$ of the zero element 0 of $\mathrm{GF}(p^e)$ is K. So $\mathrm{AGL}(1, p^e) = H \rtimes K$.

Construction 1.2. *Let* z' *be a non-zero element of* $\mathrm{GF}(p^e)$. *For each* $i \in \mathbb{Z}_{p^e-1}^*$ *with* $i < \frac{p^e-1}{2}$, *let*

$$\mathbf{K}_{p^e}^i = \mathrm{Cay}(\mathrm{AGL}(1, p^e), \{\alpha_{-1, z'}, \alpha_{x^i, 0}, \alpha_{x^{-i}, 0}\}) \quad (p > 2),$$
$$\mathbf{K}_{2^e}^i = \mathrm{Cay}(\mathrm{AGL}(1, 2^e), \{\alpha_{1, z'}, \alpha_{x^i, 0}, \alpha_{x^{-i}, 0}\}) \quad (p = 2).$$

Remark 1.3. Let z', z'' be two non-zero elements of $\mathrm{GF}(p^e)$. There exists $x^j \in \mathrm{GF}(p^e) \setminus \{0\}$ such that $z'x^j = z''$. So

$$\{\alpha_{-1,z'}, \alpha_{x^i,0}, \alpha_{x^{-i},0}\}^{\alpha_{x^j,0}} = \{\alpha_{-1,z''}, \alpha_{x^i,0}, \alpha_{x^{-i},0}\} \quad (p > 2),$$
$$\{\alpha_{1,z'}, \alpha_{x^i,0}, \alpha_{x^{-i},0}\}^{\alpha_{x^j,0}} = \{\alpha_{1,z''}, \alpha_{x^i,0}, \alpha_{x^{-i},0}\} \quad (p = 2).$$

It follows that

$$\mathrm{Cay}(\mathrm{AGL}(1,p^e), \{\alpha_{-1,z'}, \alpha_{x^i,0}, \alpha_{x^{-i},0}\}) \cong \mathrm{Cay}(\mathrm{AGL}(1,p^e), \{\alpha_{-1,z''}, \alpha_{x^i,0}, \alpha_{x^{-i},0}\}) \quad (p > 2),$$
$$\mathrm{Cay}(\mathrm{AGL}(1,2^e), \{\alpha_{1,z'}, \alpha_{x^i,0}, \alpha_{x^{-i},0}\}) \cong \mathrm{Cay}(\mathrm{AGL}(1,2^e), \{\alpha_{1,z''}, \alpha_{x^i,0}, \alpha_{x^{-i},0}\}) \quad (p = 2).$$

In view of this fact, up to graph isomorphism, $\mathbf{K}^i_{p^e}$ is independent of the choice of z'.

The following is the main result of this paper.

Theorem 1.4. *Let $\widetilde{\mathbf{K}}_n$ be a special generalized truncation of $\mathbf{K}_n$ $(n \geq 4)$ by C_{n-1}. Then $\widetilde{\mathbf{K}}_n$ is isomorphic to either $T(\mathbf{K}_6, A_5, C_5)$ (see Figure 1), or one of the graphs $\mathbf{K}^i_{p^e}$ $(i \in \mathbb{Z}^*_{p^e-1},\ i < \frac{p^e-1}{2})$. Conversely, each of the above graphs is indeed a special generalized truncation of $\mathbf{K}_n$ $(n \geq 4)$ by a cycle of length $n-1$, where $n = 6$ or a prime power.*

Furthermore, for any distinct $i, i' \in \mathbb{Z}^*_{p^e-1}$ with $i, i' < \frac{p^e-1}{2}$, $\mathbf{K}^i_{p^e} \cong \mathbf{K}^{i'}_{p^e}$ if and only if $i' \equiv ip^j$ or $-ip^j \pmod{p^e-1}$ for some $1 \leq j \leq e$. Moreover, the following hold:

(i) $\mathrm{Aut}(T(\mathbf{K}_6, A_5, C_5)) \cong A_5$;

(ii) $\mathrm{Aut}(\mathbf{K}^1_4) \cong S_4$;

(iii) $\mathrm{Aut}(\mathbf{K}^1_7) \cong D_{42} \rtimes \mathbb{Z}_3$;

(iv) $\mathrm{Aut}(\mathbf{K}^3_{11}) \cong \mathrm{PGL}_2(11)$;

(v) $\mathrm{Aut}(\mathbf{K}^7_{23}) \cong \mathrm{PGL}_2(23)$;

(vi) if $\mathbf{K}^i_{p^e}$ is not isomorphic to one of the graphs: $\mathbf{K}^1_4$, $\mathbf{K}^1_7$, $\mathbf{K}^3_{11}$ and $\mathbf{K}^7_{23}$, then $\mathrm{Aut}(\mathbf{K}^i_{p^e}) \cong \mathrm{AGL}(1, p^e)$.

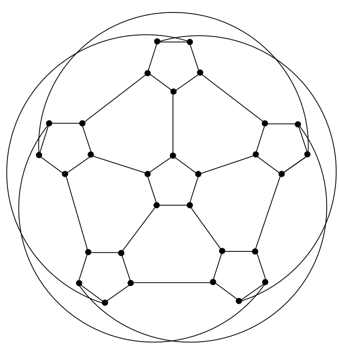

Figure 1: The graph $T(\mathbf{K}_6, A_5, C_5)$.

2 Preliminaries

All groups considered in this paper are finite and all graphs are finite, connected, simple and undirected. For the group-theoretic and graph-theoretic terminology not defined here we refer the reader to [3, 12].

Let $\Gamma = \mathrm{Cay}(G, S)$ be a Cayley graph on a group G relative to a subset S of G. It is easy to prove that Γ is connected if and only if S is a generating subset of G. For any $g \in G$, $R(g)$ is the permutation of G defined by $R(g)\colon x \mapsto xg$ for $x \in G$. Set $R(G) = \{R(g) \mid g \in G\}$. It is well-known that $R(G)$ is a subgroup of $\mathrm{Aut}(\Gamma)$. For briefness, we shall identify $R(G)$ with G in the following. In 1981, Godsil [7] proved that the normalizer of G in $\mathrm{Aut}(\Gamma)$ is $G \rtimes \mathrm{Aut}(G, S)$, where $\mathrm{Aut}(G, S)$ is the group of automorphisms of G fixing the set S set-wise. Clearly, $\mathrm{Aut}(G, S)$ is a subgroup of the stabilizer $\mathrm{Aut}(\Gamma)_1$ of the identity 1 of G in $\mathrm{Aut}(\Gamma)$. We say that the Cayley graph $\mathrm{Cay}(G, S)$ is *normal* if G is normal in $\mathrm{Aut}(\mathrm{Cay}(G, S))$ (see [13]). If $\Gamma = \mathrm{Cay}(G, S)$ is a normal Cayley graph on G, then we have $\mathrm{Aut}(G, S) = \mathrm{Aut}(\Gamma)_1$, and if, in addition, Γ is also arc-transitive, then $\mathrm{Aut}(G, S)$ is transitive on S. From this we can easily obtain the following lemma.

Lemma 2.1. *There does not exist an arc-transitive normal Cayley graph of odd valency at least three on a cyclic group.*

A Cayley graph $\mathrm{Cay}(G, S)$ on a group G relative to a subset S of G is called a *CI-graph* of G, if for any Cayley graph $\mathrm{Cay}(G, T)$, whenever $\mathrm{Cay}(G, S) \cong \mathrm{Cay}(G, T)$ we have $T = S^\alpha$ for some $\alpha \in \mathrm{Aut}(G)$. The following proposition is a criterion for a Cayley graph to be a CI-graph.

Proposition 2.2 ([1, Lemma 3.1])**.** *Let Γ be a Cayley graph on a finite group G. Then Γ is a CI-graph of G if and only if all regular subgroups of $\mathrm{Aut}(\Gamma)$ isomorphic to G are conjugate.*

Let Γ be a connected vertex-transitive graph, and let $G \leq \mathrm{Aut}(\Gamma)$ be vertex-transitive on Γ. For a G-invariant partition $\mathcal{B}$ of $V(\Gamma)$, the *quotient graph* $\Gamma_{\mathcal{B}}$ is defined as the graph with vertex set $\mathcal{B}$ such that, for any two different vertices $B, C \in \mathcal{B}$, B is adjacent to C if and only if there exist $u \in B$ and $v \in C$ which are adjacent in Γ. Let N be a normal subgroup of G. Then the set $\mathcal{B}$ of orbits of N in $V(\Gamma)$ is a G-invariant partition of $V(\Gamma)$. In this case, the symbol $\Gamma_{\mathcal{B}}$ will be replaced by Γ_N.

In view of [11, Theorem 9], we have the following proposition.

Proposition 2.3. *Suppose that Γ is a connected trivalent graph with an arc-transitive group G of automorphisms. If $N \trianglelefteq G$ has more than two orbits in $V(\Gamma)$, then N is semiregular on $V(\Gamma)$, and Γ_N is a trivalent symmetric graph with G/N as an arc-transitive group of automorphisms.*

3 Proof of Theorem 1.4

3.1 Special generalized truncations of $\mathbf{K}_n$ by C_{n-1}

In this subsection, we shall prove the first part of Theorem 1.4 by determining all special generalized truncations of $\mathbf{K}_n$ ($n \geq 4$) by C_{n-1}. Throughout this subsection, we shall use the following assumptions and notations.

Assumption 3.1.

(1) Let $\mathbf{K}_n$ be a complete graph of order n with $n \geq 4$, and let $V(\mathbf{K}_n) = \{v_1, v_2, \ldots, v_n\}$.

(2) Let $G \leq \mathrm{Aut}(\mathbf{K}_n)$ be an arc-transitive group of automorphisms.

(3) Let $v = v_1$, and let $\mathcal{O}_v$ be a union of orbits of the stabilizer G_v acting on $\{\{x, y\} \mid x \neq y,\ x, y \in V(\mathbf{K}_n) \setminus \{v\}\}$. Let Υ be the graph with vertex set $\{v_2, v_3, \ldots, v_n\}$ and edge set $\mathcal{O}_v$.

(4) For each $u \in V(\mathbf{K}_n)$, let $V_u = \{(u, w) \mid w \in V(\mathbf{K}_n) \setminus \{u\}\}$.

(5) Let $\widetilde{\mathbf{K}}_n = T(\mathbf{K}_n, G, \Upsilon)$ be the graph with the vertex set $\bigcup_{u \in V(\mathbf{K}_n)} V_u$, and the adjacency relation in which a vertex (u, w) is adjacent to the vertex (w, u) and to all the vertices (u, w') for which there exists a $g \in G$ with the property $u^g = v$ and $\{w, w'\}^g \in \mathcal{O}_v$.

In view of [6, Theorem 5.1], we have the following proposition.

Proposition 3.2. *Use the notations in Assumption 3.1. Then $\mathrm{Aut}(\widetilde{\mathbf{K}}_n)$ has a vertex-transitive subgroup $\widetilde{G}$ such that $\mathcal{P} = \{V_u \mid u \in V(\mathbf{K}_n)\}$ is an imprimitivity block system for $\widetilde{G}$. Furthermore, the following hold.*

(1) The quotient graph of $\widetilde{\mathbf{K}}_n$ relative to $\mathcal{P}$ is isomorphic to $\mathbf{K}_n$.

(2) $\widetilde{G} \cong G$.

(3) $\widetilde{G}$ acts faithfully on $\mathcal{P}$.

For the two groups $\widetilde{G}, G$ in the above proposition, we shall follow [6] to say that $\widetilde{G}$ is the *lift* of G. The next lemma shows that if $\Upsilon \cong C_{n-1}$ then $\widetilde{G}$ is a 2-transitive permutation group on $\mathcal{P}$ and the point stabilizer $\widetilde{G}_{V_u}$ is either cyclic or dihedral.

Lemma 3.3. *Use the notations in Assumption 3.1. Let $\Upsilon \cong C_{n-1}$ and let $\widetilde{G}$ be the lift of G. Then for each $u \in V(\mathbf{K}_n)$, the subgraph of $\widetilde{\mathbf{K}}_n$ induced by V_u is a cycle of length $n-1$, and the subgroup $\widetilde{G}_{V_u}$ of $\widetilde{G}$ fixing V_u set-wise acts faithfully and transitively on V_u. In particular, $\widetilde{G}$ acts faithfully and 2-transitively on $\mathcal{P}$, and $\widetilde{G}_{V_u} \cong \mathbb{Z}_{n-1}$, or D_{n-1} (if n is odd), or $D_{2(n-1)}$.*

Proof. By Assumption 3.1 (3) and (5), the subgraph of $\widetilde{\mathbf{K}}_n$ induced by V_v is isomorphic to Υ. By Proposition 3.2, $\mathcal{P} = \{V_u \mid u \in V(\mathbf{K}_n)\}$ is an imprimitivity block system for $\widetilde{G}$, and so for each $u \in V(\mathbf{K}_n)$, the subgraph of $\widetilde{\mathbf{K}}_n$ induced by V_u is a cycle of length $n-1$.

For any two vertices u, w of $\mathbf{K}_n$, by Assumption 3.1 (5), $\{(u, w), (w, u)\}$ is the unique edge of $\widetilde{\mathbf{K}}_n$ connecting V_u and V_w. This implies that the subgroup K of $\widetilde{G}_{V_u}$ fixing V_u point-wise will fix every block in $\mathcal{P}$. It then follows from Proposition 3.2 (3) that $K = 1$, and so $\widetilde{G}_{V_u}$ acts faithfully on V_u. Since $\widetilde{G}$ is transitive on $V(\widetilde{\mathbf{K}}_n)$, $\widetilde{G}_{V_u}$ is transitive on V_u. Since the subgraph of $\widetilde{\mathbf{K}}_n$ induced by V_u is a cycle of length $n-1$, one has $\widetilde{G}_{V_u} \cong \mathbb{Z}_{n-1}$, or D_{n-1} (if n is odd), or $D_{2(n-1)}$.

Again since $\{(u, w), (w, u)\}$ is the unique edge of $\widetilde{\mathbf{K}}_n$ connecting V_u and V_w, it follows that $\widetilde{G}_{V_u}$ also acts transitively on $\mathcal{P} \setminus \{V_u\}$. This implies that $\widetilde{G}$ acts 2-transitively on $\mathcal{P}$. By Proposition 3.2 (3), $\widetilde{G}$ acts faithfully on $\mathcal{P}$. □

The above lemma enables us to determine the structure of $\widetilde{G}$ in the case when $\Upsilon \cong C_{n-1}$.

Lemma 3.4. *Use the notations in Assumption 3.1. Let $\Upsilon \cong C_{n-1}$ and let $\widetilde{G}$ be the lift of G. Then one of the following holds:*

(1) $n = 6$ and $\mathrm{soc}(\widetilde{G}) = A_5$;

(2) $n = 4$ and $\widetilde{G} \cong \mathrm{AGL}(1, 2^2)$ or $\mathrm{A\Gamma L}(1, 2^2)$;

(3) $n = p^e \neq 4$ and $\widetilde{G} \cong \mathrm{AGL}(1, p^e)$, where p is a prime and e is a positive integer.

Proof. By Lemma 3.3, $\widetilde{G}$ can be viewed as a 2-transitive permutation group on $\mathcal{P}$ with point stabilizer isomorphic to $\mathbb{Z}_{n-1}$, or D_{n-1} (if n is odd), or $D_{2(n-1)}$. By [5, Propositon 5.2], $\mathrm{soc}(\widetilde{G})$ is either elementary abelian or non-abelian simple, and furthermore, if $\mathrm{soc}(\widetilde{G})$ is non-abelian simple, then by checking the list of the simple groups which can occur as socles of 2-transitive groups in [5, p. 8], we have $\mathrm{soc}(\widetilde{G}) = A_5$. In order to complete the proof of this lemma, it remains to deal with the case when $\mathrm{soc}(\widetilde{G})$ is elementary abelian.

In what follows, assume that $\mathrm{soc}(\widetilde{G}) \cong \mathbb{Z}_p^e$ for some prime p and positive integer e. View $\mathrm{soc}(\widetilde{G})$ as an e-dimensional vector space over a field of order p, and let 0 denote the zero vector of $\mathrm{soc}(\widetilde{G})$. Recall that $\widetilde{G}_0 \cong \mathbb{Z}_{p^e-1}$, D_{p^e-1} (p odd), or $D_{2(p^e-1)}$. By checking Hering's theorem on classification of 2-transitive affine permutation groups [8] (see also [10, Appendix 1]), we have $\widetilde{G} \leq \mathrm{A\Gamma L}(1, p^e)$ with point-stabilizer $\widetilde{G}_0 \leq \Gamma\mathrm{L}(1, p^e)$. As $\widetilde{G} = \mathrm{soc}(\widetilde{G}) \rtimes \widetilde{G}_0$, to determine $\widetilde{G}$, we only need to determine all possible subgroups of $\Gamma\mathrm{L}(1, p^e)$ which are isomorphic to $\mathbb{Z}_{p^e-1}$, D_{p^e-1} (p odd), or $D_{2(p^e-1)}$, and transitive on $\mathrm{soc}(\widetilde{G}) \setminus \{0\}$.

Note that $\Gamma\mathrm{L}(1, p^e)$ can be constructed in the following way. Let $\mathrm{GF}(p^e)$ be the Galois field of order p^e, and view $\mathrm{soc}(\widetilde{G})$ as the additive group of $\mathrm{GF}(p^e)$. It is well-known that the multiplicative group $\mathrm{GF}(p^e)^*$ of $\mathrm{GF}(p^e)$ is cyclic, and let x be a generator of $\mathrm{GF}(p^e)^*$. Then $\mathrm{GL}(1, p^e) = \langle x \rangle$. Let y be the Frobenius automorphism of $\mathrm{GF}(p^e)$ such that y maps every $g \in \mathrm{GF}(p^e)$ to g^p. Then we have

$$\Gamma\mathrm{L}(1, p^e) = \langle x, y \mid x^{p^e-1} = y^e = 1, y^{-1}xy = x^p \rangle.$$

In the following, we shall first determine all possible cyclic subgroups of $\Gamma\mathrm{L}(1, p^e)$ of order either $p^e - 1$ or $\frac{p^e-1}{2}$ (p odd) (Claim 1), and then this is used to determine all possible subgroups of $\Gamma\mathrm{L}(1, p^e)$ which are isomorphic to $\mathbb{Z}_{p^e-1}$, D_{p^e-1} (p odd), or $D_{2(p^e-1)}$, and transitive on $\mathrm{soc}(\widetilde{G}) \setminus \{0\}$.

Claim 1. *Let T be a cyclic subgroup of $\Gamma\mathrm{L}(1, p^e)$ of order $\frac{p^e-1}{r}$ with either $r = 1$ or $r = 2$ and p is odd. Then either $T = \langle x^r \rangle$, or $p^e = 3^2$, $T \cong \mathbb{Z}_{\frac{p^e-1}{2}}$ and $T = \langle xy \rangle$ or $\langle x^3 y \rangle$.*

Proof of Claim 1. Let $\ell = p^e - 1$ or $\frac{p^e-1}{2}$ (p odd). Since T is a cyclic subgroup of $\Gamma\mathrm{L}(1, p^e)$ of order ℓ, we may let $T = \langle x^j y^k \rangle$ with $0 \leq j \leq p^e - 2$ and $0 \leq k \leq e - 1$. If $k = 0$, then $T \leq \langle x \rangle$ and so $T = \langle x^r \rangle$ with either $r = 1$ or $r = 2$ and p is odd.

Assume now that $0 < k \leq e - 1$. Then $y^k \neq 1$. Since $y^{-1}xy = x^p$, one has $yx^p y^{-1} = x$, and hence $(yxy^{-1})^p = x$. Clearly, $p^e \equiv 1 \pmod{p^e - 1}$, so $yxy^{-1} = x^{p^{e-1}}$.

It follows that $y^kx^jy^{-k} = x^{jp^{k(e-1)}}$, and so $y^kx^j = x^{jp^{k(e-1)}}y^k$. By this equality, we have for any positive integer m,

$$(x^jy^k)^m = x^{j(1+p^{k(e-1)}+p^{2k(e-1)}+\cdots+p^{(m-1)k(e-1)})}y^{mk} = x^{j\frac{p^{mk(e-1)}-1}{p^{k(e-1)}-1}}y^{mk}. \quad (3.1)$$

From Equation (3.1) it follows that $(x^jy^k)^e = x^{j\frac{p^{ek(e-1)}-1}{p^{k(e-1)}-1}}$. Since $p^e-1 \mid p^{ke(e-1)}-1$, one has

$$(x^jy^k)^{e(p^{k(e-1)}-1)} = x^{j(p^{ek(e-1)}-1)} = 1.$$

This implies that the order of x^jy^k divides $e(p^{k(e-1)}-1)$, namely, $\ell \mid e(p^{k(e-1)}-1)$. Since $\ell = p^e-1$ or $\frac{p^e-1}{2}$ (p odd), we have $p^e-1 \mid 2e(p^{k(e-1)}-1)$.

Suppose that $e \geq 3$. If $(p,e) = (2,6)$, then $\ell = p^e-1 = 63$. However, it is easy to check that $63 \nmid 6(2^{5k}-1)$ for any $k \leq 5$, contrary to $\ell \mid e(p^{k(e-1)}-1)$. Thus, $(p,e) \neq (2,6)$. Then by a result of Zsigmondy [14], there exists at least one prime q such that q divides p^e-1 but does not divide p^t-1 for any positive integer $t < e$. Clearly, $p \neq q$, so p is an element of $\mathbb{Z}_q^* \cong \mathbb{Z}_{q-1}$ of order e. In particular, we have $q > e$. Since $q \mid p^e-1$ and $p^e-1 \mid 2e(p^{k(e-1)}-1)$, we have $q \mid p^{k(e-1)}-1$, implying $k(e-1) > e$. Since $k \leq e-1$, we may let $k(e-1) = me+t$ for some positive integers m and $t < e$, and since $p^{me}(p^t-1) = (p^{k(e-1)}-1)-(p^{me}-1)$, we have $q \mid p^t-1$. However, this is impossible because it is assumed that $q \nmid p^t-1$ for any $t < e$.

Thus, $e < 3$. Since $0 < k \leq e-1$, one has $e = 2$ and $k = 1$, and then $p^2-1 \mid 4(p-1)$. It follows that $p+1 \mid 4$ and hence $p = 3$. Then $(x^jy)^2 = x^{4j}$ has order at most 2 since $\langle x\rangle \cong \mathbb{Z}_8$, and then x^jy has order dividing 4. This implies that $\ell = \frac{p^e-1}{2} = 4$ and $T = \langle xy\rangle$ or $\langle x^3y\rangle$. This completes the proof of Claim 1. □

By now, we have shown that Claim 1 is true. Recall that $\widetilde{G}_0 \leq \Gamma\mathrm{L}(1,p^e)$, $\widetilde{G}_0 \cong \mathbb{Z}_{p^e-1}$, D_{p^e-1} (p odd), or $D_{2(p^e-1)}$ and $\widetilde{G}_0$ is transitive on $\mathrm{soc}(\widetilde{G}) \setminus \{0\}$. We shall finish the proof by considering the following three cases.

Case 1. $\widetilde{G}_0 \cong \mathbb{Z}_{p^e-1}$.

In this case, by Claim 1, we must have $\widetilde{G}_0 = \langle x\rangle = \mathrm{GL}(1,p^e)$ and so $\widetilde{G} \cong \mathrm{AGL}(1,p^e)$.

Case 2. $\widetilde{G}_0 \cong D_{p^e-1}$ (p odd).

In this case, by Claim 1, either $x^2 \in \widetilde{G}_0$, or $p^e = 9$ and $\widetilde{G}_0$ contains xy or x^3y. For the former, we have $\widetilde{G}_0 = \langle x^2, f\rangle$, where f is an involution of $\Gamma\mathrm{L}(1,p^e)$ such that $fx^2f = x^{-2}$ and $f \notin \langle x\rangle$. Note that $\widetilde{G}_0$ is transitive on $\mathrm{soc}(\widetilde{G}) \setminus \{0\}$. We may let $f = xy^k$ and $0 < k \leq e-1$. By Equation (3.1), $f^2 = (xy^k)^2 = 1$ implies that e is even and $k = \frac{e}{2}$, and furthermore, $x^{p^{\frac{e(e-1)}{2}}+1} = 1$. It follows that $p^e-1 \mid p^{\frac{e(e-1)}{2}}+1$. However, since $p^{e\frac{(e-2)}{2}}(p^{\frac{e}{2}}+1) = (p^{\frac{e(e-1)}{2}}+1)+(p^{e\frac{(e-2)}{2}}-1)$, we would have $p^e-1 \mid p^{\frac{e}{2}}+1$, forcing that $p = 2$, a contradiction.

For the latter, we have $\widetilde{G}_0 \cong D_8$. However, it is easy to check that in $\Gamma\mathrm{L}(1,9) = \langle x, y \mid x^8 = y^2 = 1, y^{-1}xy = x^3\rangle$ there does not exist an involution inverting xy or x^3y, a contradiction.

Case 3. $\widetilde{G}_0 \cong D_{2(p^e-1)}$.

By Claim 1, we must have $\widetilde{G}_0 = \langle x\rangle \rtimes \langle y^{\frac{e}{2}}\rangle$ with $y^{\frac{e}{2}}xy^{\frac{e}{2}} = x^{-1}$. On the other hand, since $y^{-1}xy = x^p$, we have $y^{\frac{e}{2}}xy^{\frac{e}{2}} = x^{p^{\frac{e}{2}}}$ and hence $x^{p^{\frac{e}{2}}} = x^{-1}$. It follows that

$p^{\frac{e}{2}} \equiv -1 \pmod{p^e - 1}$ and hence $p^e - 1$ divides $p^{\frac{e}{2}} + 1$. Consequently, we have $p^e = 4$, $\widetilde{G}_0 = \langle x, y\rangle = \Gamma\mathrm{L}(1, 4)$, and $\widetilde{G} \cong \mathrm{A}\Gamma\mathrm{L}(1, 4) \cong S_4$. ☐

Now we are ready to determine all possible special generalized truncations of $\mathbf{K}_n$ by C_{n-1}.

Lemma 3.5. *Use the notations in Assumption 3.1. Let $\Upsilon \cong C_{n-1}$ and let $\widetilde{G}$ be the lift of G. Then $\widetilde{\mathbf{K}}_n = T(\mathbf{K}_n, G, \Upsilon)$ is isomorphic to either $T(\mathbf{K}_6, A_5, C_5)$ (see Figure 1), or one of the graphs $\mathbf{K}^i_{p^e}$ ($i \in \mathbb{Z}^*_{p^e-1}$, $i < \frac{p^e-1}{2}$) (see Construction 1.2 for the definition of these graphs).*

Proof. If $\mathrm{soc}(\widetilde{G}) \cong A_5$, then by [6, Example 5.3], we have $\widetilde{G} \cong A_5$ and up to graph isomorphism, there exists a unique graph, and so we may denote this graph by $T(\mathbf{K}_6, A_5, C_5)$ (see Figure 1).

In what follows, we assume that $\mathrm{soc}(\widetilde{G}) \ncong A_5$. Then from Lemma 3.4 we see that $\widetilde{G}$ has a subgroup, say $\widetilde{T}$ such that $\widetilde{T} \cong \mathrm{AGL}(1, p^e)$ and $\widetilde{T}$ acts regularly on $V(\widetilde{\mathbf{K}}_n)$, where p is a prime and e is a positive integer such that $p^e \geq 4$. It follows that $\widetilde{\mathbf{K}}_n$ is a Cayley graph on $\widetilde{T}$ ($\cong$ $\mathrm{AGL}(1, p^e)$) and $n = p^e$. For each $u \in V(\mathbf{K}_n)$, by Lemma 3.3, the subgraph of $\widetilde{\mathbf{K}}_n$ induced by V_u is a cycle of length $n - 1$, and the subgroup $\widetilde{G}_{V_u}$ of $\widetilde{G}$ fixing V_u set-wise acts faithfully and transitively on V_u. Furthermore, $\widetilde{G}$ acts faithfully and 2-transitively on $\mathcal{P}$. For convenience, we may identify $\mathcal{P}$ with $\mathrm{GF}(p^e)$, identify V_u with the zero element 0 of $\mathrm{GF}(p^e)$ and identify $\widetilde{T}$ with $\mathrm{AGL}(1, p^e)$. We shall use the notations for $\widetilde{T} = \mathrm{AGL}(1, p^e)$ as well as its elements and subgroups H and K introduced in the paragraph before Construction 1.2. Then $\widetilde{T}_{V_u} = K \cong \mathbb{Z}_{p^e-1}$.

Take $(u, w) \in V_u$, and assume that (u, w_1) and (u, w_2) are two vertices in V_u adjacent to (u, w). Since $\widetilde{T}_{V_u} = K \cong \mathbb{Z}_{p^e-1}$ is transitive on V_u, there exists a unique $\alpha_{x^i,0} \in \widetilde{T}_{V_u}$ such that $(u, w)^{\alpha_{x^i,0}} = (u, w_1)$ and $(u, w)^{\alpha_{x^{-i},0}} = (u, w_2)$, and since the subgraph of $\widetilde{\mathbf{K}}_n$ induced by V_u is a cycle of length $n - 1$, i is coprime to $p^e - 1$ ($n = p^e$). So we may let

$$\widetilde{\mathbf{K}}_n = \mathrm{Cay}(\mathrm{AGL}(1, p^e), \{\alpha_{x^i,0}, \alpha_{x^{-i},0}, \alpha_{x^j,z'}\}),$$

where $\alpha_{x^j,z'}$ is an involution. Since $\widetilde{\mathbf{K}}_n$ is connected, if p is odd, then we have $\alpha_{x^j,z'} = \alpha_{x^{\frac{p^e-1}{2}},z'} = \alpha_{-1,z'}$ and $z' \neq 0$, and if $p = 2$, then we have $\alpha_{x^j,z'} = \alpha_{1,z'}$ and $z' \neq 0$, and correspondingly, we obtain the two graphs $\mathbf{K}^j_{p^e}$ ($p > 2$) and $\mathbf{K}^j_{2^e}$ (see Construction 1.2). ☐

From Figure 1 it is easy to see that $T(\mathbf{K}_6, A_5, C_5)$ is a special generalized truncation of $\mathbf{K}_6$ by a cycle of length 5. The following lemma shows that each of the Cayley graphs $\mathbf{K}^i_{p^e}$ ($i \in \mathbb{Z}^*_{p^e-1}$, $i < \frac{p^e-1}{2}$) is also indeed a special generalized truncation of $\mathbf{K}_{p^e}$ by a cycle of length $p^e - 1$.

Lemma 3.6. *Each of the graphs $\mathbf{K}^i_{p^e}$ ($i \in \mathbb{Z}^*_{p^e-1}$, $i < \frac{p^e-1}{2}$) (see Construction 1.2) is a special generalized truncation of $\mathbf{K}_{p^e}$ by a cycle of length $p^e - 1$.*

Proof. Recall that each $\mathbf{K}^i_{p^e}$ ($i \in \mathbb{Z}^*_{p^e-1}$, $i < \frac{p^e-1}{2}$) is a trivalent Cayley graph on $\mathrm{AGL}(1, p^e)$ defined as follows:

$$\mathbf{K}^i_{p^e} = \mathrm{Cay}(\mathrm{AGL}(1, p^e), \{\alpha_{-1,z'}, \alpha_{x^i,0}, \alpha_{x^{-i},0}\}) \quad (z' \neq 0, p > 2),$$
$$\mathbf{K}^i_{2^e} = \mathrm{Cay}(\mathrm{AGL}(1, 2^e), \{\alpha_{1,z'}, \alpha_{x^i,0}, \alpha_{x^{-i},0}\}) \quad (z' \neq 0).$$

(Keep in mind we use the notations for $\mathrm{AGL}(1, p^e)$ as well as its elements and subgroups H and K introduced in the paragraph before Construction 1.2.) Note that $\mathrm{AGL}(1, p^e) = H \rtimes K$, where

$$H = \{\alpha_{1,z''} \colon z \mapsto z + z'', \forall z \in GF(p^e) \mid z'' \in GF(p^e)\},$$
$$K = \{\alpha_{x^j,0} \colon z \mapsto zx^j, \forall z \in GF(p^e) \mid j \in \mathbb{Z}_{p^e-1}\}.$$

Moreover, K is maximal in $\mathrm{AGL}(1, p^e)$ since $\mathrm{AGL}(1, p^e)$ is 2-transitive on $GF(p^e)$. As $i \in \mathbb{Z}^*_{p^e-1}$, one has $K = \langle \alpha_{x^i,0} \rangle$ and then the maximality of K implies that $\langle \alpha_{-1,z'}, \alpha_{x^i,0} \rangle = \mathrm{AGL}(1, p^e)$ for $p > 2$ and $\langle \alpha_{1,z'}, \alpha_{x^i,0} \rangle = \mathrm{AGL}(1, 2^e)$. Thus, every $\mathbf{K}^i_{p^e}$ ($i \in \mathbb{Z}^*_{p^e-1}, i < \frac{p^e-1}{2}$) is connected.

It is easy to see that $\mathrm{Cay}(K, \{\alpha_{x^i,0}, \alpha_{x^{-i},0}\}) \cong C_{p^e-1}$ is a subgraph of $\mathbf{K}^i_{p^e}$ ($i \in \mathbb{Z}^*_{p^e-1}$, $i < \frac{p^e-1}{2}$). Since $\mathrm{AGL}(1, p^e)$ acts on $V(\mathbf{K}^i_{p^e})$ by right multiplication, the subgraph of $\mathbf{K}^i_{p^e}$ induced by Kg for any $g \in \mathrm{AGL}(1, p^e)$ is a cycle of length $p^e - 1$. As $\mathrm{AGL}(1, p^e)$ acts 2-transitively on $\mathcal{B} = \{Kg \mid g \in \mathrm{AGL}(1, p^e)\}$, the quotient graph of $\mathbf{K}^i_{p^e}$ relative to $\mathcal{B}$ is a complete graph $\mathbf{K}_{p^e}$. So we have $\mathbf{K}^i_{p^e} \cong T(\mathbf{K}_{p^e}, \mathrm{AGL}(1, p^e), \Upsilon_i)$, where Υ_i is the subgraph with vertex set $\mathcal{B} - \{K\}$ and edge set $\{\{K\gamma g, K\gamma\alpha_{x^i,0} g\} \mid g \in K\}$ where $\gamma = \alpha_{-1,z'}$ for $p > 2$ and $\gamma = \alpha_{1,z'}$ for $p = 2$. □

3.2 Automorphisms and isomorphisms

In this subsection, we shall determine the automorphism groups and isomorphisms of special generalized truncations of $\mathbf{K}_n$ by C_{n-1}, and thus prove the second part of Theorem 1.4. By checking [6, Table 1], we have the following lemma.

Lemma 3.7. $\mathrm{Aut}(T(\mathbf{K}_6, A_5, C_5)) \cong A_5$.

In the following two lemmas, we shall determine the automorphisms and isomorphisms of the graphs $\mathbf{K}^i_{p^e}$ ($i \in \mathbb{Z}^*_{p^e-1}$, $i < \frac{p^e-1}{2}$). We keep using the notations for $\mathrm{AGL}(1, p^e)$ as well as its elements and subgroups H and K introduced in the paragraph before Construction 1.2.

Lemma 3.8. *Let* Γ *be one of the graphs* $\mathbf{K}^i_{p^e}$ ($i \in \mathbb{Z}^*_{p^e-1}$, $i < \frac{p^e-1}{2}$) *(see Construction 1.2). Then Theorem 1.4 (ii)–(vi) hold.*

Proof. Recall that Γ is a connected trivalent Cayley graph on $X = \mathrm{AGL}(1, p^e)$. Let $A = \mathrm{Aut}(\Gamma)$. For convenience of the statement, we view X as a regular subgroup of A.

Suppose first that Γ is arc-transitive. Let $N = \bigcap_{g \in A} X^g$. If $N = 1$, then by [9, Theorem 1.1], we have $\mathrm{Aut}(\Gamma) \cong \mathrm{PGL}_2(p^e)$ with $p^e = 11$ or 23. If $p^e = 11$, then since $i \in \mathbb{Z}^*_{10}$ and $i < 5$, we have $i = 3$ and hence $\Gamma = \mathbf{K}^3_{11}$. If $p^e = 23$, then $i = 3, 5, 7$ or 9 as $i \in \mathbb{Z}^*_{22}$ and $i < 11$, and by Magma [4], $\mathrm{Aut}(\mathbf{K}^i_{23}) \cong \mathrm{PGL}_2(23)$ if and only if $i = 7$, and hence $\Gamma = \mathbf{K}^7_{23}$. If $N > 1$, then $N \trianglelefteq A$, and in particular, $N \trianglelefteq X$. Since $\mathrm{soc}(X) \cong \mathbb{Z}^e_p$ is the unique minimal normal subgroup of $X = \mathrm{AGL}(1, p^e)$, one has $\mathrm{soc}(X) \le N$. Clearly, $\mathrm{soc}(X)$ is a Sylow p-subgroup of N since $N \le X$. So $\mathrm{soc}(X)$ is characteristic in N and hence normal in A. Consider the quotient graph Σ of Γ relative to $\mathrm{soc}(X)$. Clearly, Σ has $p^e - 1$ vertices. Since $p^e - 1 > 2$, by Proposition 2.3, Σ would be a trivalent arc-transitive Cayley graph on $X/\mathrm{soc}(X) \cong \mathbb{Z}_{p^e-1}$. Furthermore, by [2, Corollary 1.3], either $\Sigma \cong \mathbf{K}_{3,3}$, or Σ is a trivalent normal arc-transitive Cayley graph on $X/\mathrm{soc}(X) \cong \mathbb{Z}_{p^e-1}$. However, the latter case cannot happen by Lemma 2.1. For the former, we have $p^e - 1 = 6$

and so $p = 7$ and $e = 1$. In this case, we have $i = 1$ and $\Gamma = \mathbf{K}_7^1$. By MAGMA [4], we have $\mathrm{Aut}(\mathbf{K}_7^1) \cong D_{42} \rtimes \mathbb{Z}_3$.

Suppose now that Γ is not arc-transitive. If $A > X$, then the vertex-stabilizer A_a is a 2-group for any $a \in V(\Gamma)$. Then A_a fixes one and only one neighbor of a. Assume that the neighbor of a fixed by A_a is b. Then $B = \{\{a,b\}^g \mid g \in A\}$ is a system of blocks of imprimitivity of A on $V(\Gamma)$. It follows that $\Gamma - B$ is a union of several cycles with equal lengths, and the set of vertex-sets of these cycles forms an A-invariant partition of $V(\Gamma)$. Let C be the cycle of Γ containing the identity 1 of X. Since Γ is a Cayley graph on X, X acts on $V(\Gamma) = X$ by right multiplication, and since $V(C)$ is a block of imprimitivity of A acting on $V(\Gamma)$, C is actually a subgroup of X. From the definition of $\Gamma = \mathbf{K}_{p^e}^i$, one may see that $V(C) = K = \{\alpha_{x^i,0} \colon z \mapsto zx^i, \forall z \in GF(p^e) \mid i \in \mathbb{Z}_{p^e-1}\}$, and the vertex set of every cycle of $\Gamma - B$ is just a right coset of K. Let $\mathcal{B} = \{Kg \mid g \in X\}$. Then $\mathcal{B}$ is an A-invariant partition of Γ. Clearly, X acts 2-transitively and faithfully on $\mathcal{B}$, so the quotient graph of Γ relative $\mathcal{B}$ is $\mathbf{K}_{p^e}$. Now it is easy to see that $\Gamma \cong T(\mathbf{K}_{p^e}, A, \Upsilon_i)$, where Υ_i is the subgraph with vertex set $\mathcal{B} - K$ and edge set $\{\{K\gamma g, K\gamma\alpha_{x^i,0}g\} \mid g \in A_K\}$ where $\gamma = \alpha_{-1,z'}$ for $p > 2$ and $\gamma = \alpha_{1,z'}$ for $p = 2$. Clearly, $\Upsilon_i \cong C_{p^e-1}$. From Lemma 3.4 it follows that either $p^e = 4$ and $A = \mathrm{A\Gamma L}(1,4) \cong S_4$, or $A = X = \mathrm{AGL}(1,p^e)$. □

Lemma 3.9. *For any distinct* $i, i' \in \mathbb{Z}_{p^e-1}^*$ *with* $i, i' < \frac{p^e-1}{2}$, $\mathbf{K}_{p^e}^i \cong \mathbf{K}_{p^e}^{i'}$ *if and only if there exists* $1 \le j \le e$ *such that* $i' \equiv ip^j$ *or* $-ip^j \pmod{p^e - 1}$.

Proof. If $p^e = 4$ or 7, then we must have $i = 1$, and so we have only one graph for each of these two cases. If $p^e = 11$ or 23, then by MAGMA [4], for any distinct $i, i' \in \mathbb{Z}_{p^e-1}^*$ with $i, i' < \frac{p^e-1}{2}$, one may check that $\mathbf{K}_{p^e}^i \cong \mathbf{K}_{p^e}^{i'}$ if and only if $i' \equiv ip^j$ or $-ip^j \pmod{p^e - 1}$.

Suppose that $\mathbf{K}_{p^e}^i$ is not isomorphic to one of the graphs: $\mathbf{K}_4^1, \mathbf{K}_7^1, \mathbf{K}_{11}^3$ and $\mathbf{K}_{23}^7$. By Lemma 3.8, $\mathrm{Aut}(\mathbf{K}_{p^e}^i) \cong \mathrm{AGL}(1,p^e)$ and by Proposition 2.2, $\mathbf{K}_{p^e}^i$ is a CI-graph. Recall that

$$\mathbf{K}_{p^e}^i = \mathrm{Cay}(\mathrm{AGL}(1,p^e), \{\alpha_{-1,z'}, \alpha_{x^i,0}, \alpha_{x^{-i},0}\}) \quad (p > 2),$$
$$\mathbf{K}_{2^e}^i = \mathrm{Cay}(\mathrm{AGL}(1,2^e), \{\alpha_{1,z'}, \alpha_{x^i,0}, \alpha_{x^{-i},0}\}) \quad (p = 2).$$

Since $\mathbf{K}_{p^e}^i$ is a CI-graph, for any distinct $i, i' \in \mathbb{Z}_{p^e-1}^*$ with $i, i' < \frac{p^e-1}{2}$, $\mathbf{K}_{p^e}^i \cong \mathbf{K}_{p^e}^{i'}$ if and only if there exists $\gamma \in \mathrm{Aut}(\mathrm{AGL}(1,p^e))$ such that $\{\alpha_{x^i,0}, \alpha_{x^{-i},0}\}^\gamma = \{\alpha_{x^{i'},0}, \alpha_{x^{-i'},0}\}$ and either $\alpha_{-1,z'}^\gamma = \alpha_{-1,z'}$ for $p > 2$ or $\alpha_{1,z'}^\gamma = \alpha_{1,z'}$ for $p = 2$.

Note that $\mathrm{Aut}(\mathrm{AGL}(1,p^e)) = \mathrm{A\Gamma L}(1,p^e) = \mathrm{AGL}(1,p^e) \rtimes \langle\eta\rangle$, where η is induced by the Frobenius automorphism of $\mathrm{GF}(p^e)$ such that $\alpha_{a,b}^\eta = \alpha_{a^p,b^p}$ for any $\alpha_{a,b} \in \mathrm{AGL}(1,p^e)$. Suppose first that $i' \equiv ip^j$ or $-ip^j \pmod{p^e - 1}$ for some $1 \le j \le e$. Then one may check that $\alpha_{\pm1,z'}^{\eta^j\alpha_{(z')^{-p^j}z',0}} = \alpha_{\pm1,z'}$ and $\{\alpha_{x^i,0}, \alpha_{x^{-i},0}\}^{\eta^j\alpha_{(z')^{-p^j}z',0}} = \{\alpha_{x^{i'},0}, \alpha_{x^{-i'},0}\}$. So $\mathbf{K}_{p^e}^i \cong \mathbf{K}_{p^e}^{i'}$. Conversely, if $\mathbf{K}_{p^e}^i \cong \mathbf{K}_{p^e}^{i'}$, then there exists $\gamma \in \mathrm{Aut}(\mathrm{AGL}(1,p^e))$ such that $\{\alpha_{x^i,0}, \alpha_{x^{-i},0}\}^\gamma = \{\alpha_{x^{i'},0}, \alpha_{x^{-i'},0}\}$ and either $\alpha_{-1,z'}^\gamma = \alpha_{-1,z'}$ for $p > 2$ or $\alpha_{1,z'}^\gamma = \alpha_{1,z'}$ for $p = 2$. Since $K = \langle\alpha_{x^i,0}\rangle$, γ normalizes K, and since $N_{\mathrm{A\Gamma L}(1,p^e)}(K) = K \rtimes \langle\eta\rangle$, one has $\gamma = \alpha_{x^k,0}\eta^j$, for some $k \in \mathbb{Z}_{p^e-1}^*$ and $1 \le j \le e$. Then

$$\alpha_{x^i,0}^\gamma = \alpha_{x^i,0}^{\alpha_{x^k,0}\eta^j} = \alpha_{x^i,0}^{\eta^j} = \alpha_{x^{ip^j},0} \in \{\alpha_{x^{i'},0}, \alpha_{x^{-i'},0}\}.$$

It follows that $i' \equiv ip^j$ or $-ip^j \pmod{p^e - 1}$. □

3.3 Proof of Theorem 1.4

From Lemmas 3.5 and 3.6 we can obtain the proof of the first part of Theorem 1.4, and from Lemmas 3.8 and 3.9, we obtain the proof of the second part of Theorem 1.4.

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