

On local properties of 1-planar graphs with high minimum degree

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Abstract

A graph is called 1-planar if there exists its drawing in the plane such that each edge contains at most one crossing. We prove that each 1-planar graph of minimum degree 7 contains a pair of adjacent vertices of degree 7 as well as several small graphs whose vertices have small degrees; we also prove the existence of a 4-cycle with relatively small degree vertices in 1-planar graphs of minimum degree at least 6.

Keywords: 1-planar graph, light graph.

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1 Introduction

Throughout this paper, we consider connected graphs without loops or multiple edges; we use the standard graph terminology by [8].

A graph G is called *planar* if there exists its drawing $D(G)$ in the plane such that no two edges of $D(G)$ have an internal point (a *crossing*) in common; the drawing $D(G)$ with this property is called a *plane graph*. Planar graphs are one of the most studied graph families, possessing a wide variety of applications.

There are several different approaches generalizing the concept of planarity. One of them allows, in a drawing of a graph, a constant number of crossings per edge. In particular, if there exists a drawing $D(G)$ of a graph G in the plane such that each edge of $D(G)$ contains at most one crossing, then G is called *1-planar*. These graphs were introduced by Ringel [19] in connection with the simultaneous vertex/face colouring of plane graphs (note

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that the graph of adjacency/incidence of vertices and faces of a plane graph is 1-planar); in [19], it was proved that 1-planar graphs are 7-colourable (a linear 7-colouring algorithm was presented in [7]), and Borodin ([1, 3]) proved Ringel's conjecture that these graphs are 6-colourable.

Comparing to the family of planar graphs, the family of 1-planar graphs is still only little explored (for example, except the above mentioned colouring results, there is only one other paper [5] on acyclic colourings of 1-planar graphs). In general, 1-planar graphs differ from planar ones in several fundamental aspects – they are not minor closed, their recognition is NP-complete (see [16]) and, for sufficiently large n , there are exponentially many nonisomorphic critical non-1-planar graphs on n vertices ([15, 16]).

On the other hand, concerning the local structure, 1-planar graphs show a similar behaviour as planar graphs. Recently, the local structure of these graphs was studied in [11], [4], [12]; it was shown that, under certain conditions (like prescribed minimum degree or girth), 1-planar graphs contain various small subgraphs whose vertices have small degrees (that is, bounded above by a constant depending only on the type of the subgraph). In this paper, we continue this research by studying the local properties of 1-planar graphs which have (or are close to have) the maximum possible minimum degree (by [19], this value is equal to 7). Our motivation comes from analogical results for planar graphs of minimum degree 5 which contain a large variety of subgraphs with vertices of small degrees. For example, Borodin in [2] proved that each planar graph of minimum degree 5 contains a 3-cycle with the weight (that is, the sum of degrees of its vertices) at most 17; similar results hold also for short cycles and stars (see [13, 6, 14, 17, 18]). Note that, for planar graphs of minimum degree 4, analogical results do not hold (for details, see [9] and [10]). Therefore, one may expect that, with the increasing minimum degree of a 1-planar graph, there will appear many small subgraphs with similar properties.

In the following, let $K_{2,3}^*$ denote a graph $K_{2,3}$ with an extra edge between two vertices of the smaller bipartition, and let W_5 denote the 5-wheel (the 4-sided pyramid graph).

We prove

Theorem 1.1. *Each 1-planar graph of minimum degree 7 contains*

- a) *a pair of adjacent 7-vertices,*
- b) *a copy of K_4 with all vertices of degree at most 13,*
- c) *a copy of $K_{2,3}^*$ with all vertices of degree at most 13,*
- d) *a copy of W_5 with all vertices of degree at most 11.*

In addition, we prove

Theorem 1.2. *Each 1-planar graph of minimum degree 6 contains a copy of C_4 with all vertices of degree at most 47.*

The proofs of all these results follow the same strategy – we assume the existence of a hypothetical counterexample G to a particular theorem, and its 1-planar drawing $D = D(G)$. The drawing D is then transformed into *associated plane graph* $G^\times = (V^\times, E^\times, F^\times)$ of G in such a way that all crossings of D become new 4-vertices. These 4-vertices are called *false*, the vertices of G^\times which correspond to the original vertices of D are called *true*; a face of G^\times is a *false face* if it is incident with at least one false vertex, otherwise it is a *true face*. Next, we proceed by Discharging Method. Each vertex and

each face of G^\times is assigned a quantity c called *initial charge*; this can be done in several different ways, of which we use the following two ones:

$$\begin{aligned} c(v) &= \deg_{G^\times}(v) - 4 \text{ for each vertex } v \in V^\times, \\ c(\alpha) &= \deg_{G^\times}(\alpha) - 4 \text{ for each face } \alpha \in F^\times, \end{aligned} \tag{1}$$

$$\begin{aligned} c(v) &= \deg_{G^\times}(v) - 6 \text{ for each vertex } v \in V^\times, \\ c(\alpha) &= 2 \deg_{G^\times}(\alpha) - 6 \text{ for each face } \alpha \in F^\times, \end{aligned} \tag{2}$$

By Euler polyhedral formula, we obtain that, under the first initial charge assignment, $\sum_{x \in V^\times \cup F^\times} c(x) = -8$ (or -12 under the second assignment).

Next, the initial charges of elements of G^\times are locally redistributed in such a way that the total sum of all charges remains the same (hence, negative). This redistribution is performed by a set of discharging rules which specify the way of transfer of a charge from one element to another one in specific configurations of vertices and faces of G^\times . Finally, by a case analysis, it is shown that, after discharging, each element of G^\times has a nonnegative final charge c^* ; thus, the total sum of final charges is also nonnegative, a contradiction.

For purposes of these proofs, we use more specialized notation. Given a d -vertex x of G^\times , x_1, \dots, x_d will denote its neighbours in clockwise order. By $f_i, i = 1, \dots, d$, we denote the face of G^\times which contains the facial subwalk $x_i x_{i+1}$ (index modulo d). If f_i is a 3-face, then f'_i denotes a face having the common edge $x_i x_{i+1}$ (index modulo d) with f_i ; further, if f'_i is a 3-face, then its third vertex (which is different from x_i and x_{i+1}) will be denoted by x'_i .

2 Proofs

Proof of Theorem 1.1a). By contradiction. Suppose that there exists a 1-planar graph G of minimum degree 7 such that each its 7-vertex is adjacent only with ≥ 8 -vertices. We proceed with the Discharging Method with the initial charge assignment (1); the initial charges are redistributed according to the following rules:

Rule 1: Each ≥ 7 -vertex $v \in V^\times$ redistributes its initial charge uniformly among incident 3-faces.

Rule 2: Each 3-face with a positive charge after application of Rule 1 sends $\frac{3}{7}$ to each incident 7-vertex.

Rule 3: Each 7-vertex with a positive charge after application of Rules 1 and 2 redistributes this charge uniformly among incident false 3-faces.

We check the nonnegativity of final charges of vertices and faces of G^\times . Since false vertices and ≥ 4 -faces of G^\times are not influenced by discharging, their final charge is equal to the initial one; hence, it is nonnegative. Also, from the formulation of discharging rules, it is easy to see that the final charge of ≥ 7 -vertices is nonnegative. Thus, it is enough to analyze just the final charge of 3-faces; let α be a 3-face of G^\times .

Case 1: Let α be true and incident only with ≥ 8 -vertices. Then, by Rule 1, $c^*(\alpha) \geq -1 + 3 \cdot \frac{8-4}{8} = -1 + \frac{3}{2} > 0$.

Case 2: Let α be true and incident with exactly one 7-vertex. By Rules 1 and 2, $c^*(\alpha) \geq -1 + 2 \cdot \frac{8-4}{8} + \frac{7-4}{7} - \frac{3}{7} = 0$.

(Note that after application of Rule 1, each true 3-face has a positive charge at least $\frac{3}{7}$.)

Case 3: Let $\alpha = [xyz]$ be false with a false vertex x .

Case 3.1: If both y, z are ≥ 8 -vertices, then, by Rule 1, $c^*(\alpha) \geq -1 + 2 \cdot \frac{8-4}{8} = -1 + 2 \cdot \frac{1}{2} = 0$.

Case 3.2: Suppose that, without loss of generality, y is a 7-vertex and z is an ≥ 8 -vertex. If y is incident with an ≥ 4 -face, then $c^*(\alpha) \geq -1 + \frac{7-4}{7-1} + \frac{8-4}{8} = 0$; similarly, if z is incident with an ≥ 4 -face, then $c^*(\alpha) \geq -1 + \frac{7-4}{7} + \frac{8-4}{8-1} = 0$. Hence, suppose that both y, z are incident with 3-faces only. Then there exists a true 3-face incident with y , which sends, by Rule 2, $\frac{3}{7}$ to y . Then, by Rule 3, y sends at least $\frac{3}{7-1} = \frac{1}{14}$ to α , and we obtain $c^*(\alpha) \geq -1 + \frac{7-4}{7} + \frac{8-4}{8} + \frac{1}{14} = 0$. □

Proof of Theorem 1.1b). By contradiction. Suppose that there exists a 1-planar graph G of minimum degree 7 such that each its subgraph K_4 contains at least one ≥ 14 -vertex, called *big*; vertices of degrees between 7 and 13 are called *intermediate*. We proceed with the Discharging Method with the initial charge assignment (2); the initial charges are redistributed according to following rules:

Rule 1: Each ≥ 4 -face $\alpha \in F^\times$ redistributes its initial charge uniformly among incident 4-vertices.

Rule 2: Each intermediate vertex sends $\frac{1}{7}$ to each adjacent 4-vertex.

Rule 3: Let $[xyz]$ be a 3-face of G^\times , x be a 4-vertex and y be an intermediate vertex. Then y sends additional $\frac{1}{14}$ to x .

Rule 4: Each big vertex sends $\frac{4}{7}$ to each adjacent 4-vertex.

Rule 5: Let $[xyz]$ be a 3-face of G^\times , x be a 4-vertex and y be a big vertex. Then y sends additional $\frac{2}{7}$ to x .

We check the nonnegativity of final charges of vertices and faces of G^\times . From the formulation of discharging rules, it is easy to see that the final charge of all faces is nonnegative. Thus, it is enough to analyze just the final charge of vertices.

Case 1: Let x be a 4-vertex of G^\times . If x is incident with at least two ≥ 4 -faces, then, by Rule 1, $c^*(x) \geq -2 + 2 \cdot \frac{2 \cdot 4 - 6}{2} = -2 + 2 \cdot 1 = 0$. If x is incident with exactly one ≥ 4 -face, then, by Rules 1, 2 and 3 (or, eventually, 1, 4 and 5) we obtain the estimation $c^*(x) \geq -2 + \frac{2 \cdot 4 - 6}{2} + 4 \cdot \frac{1}{7} + 6 \cdot \frac{1}{14} = 0$. Finally, if x is incident only with 3-faces, then its neighbours in G^\times induce a K_4 , hence, one of them is big; then, by Rules 2, 3, 4 and 5, we obtain $c^*(x) \geq -2 + 3 \cdot \frac{1}{7} + 6 \cdot \frac{1}{14} + \frac{4}{7} + 2 \cdot \frac{2}{7} = 0$.

Case 2: Let x be an intermediate d -vertex of G^\times . Then $c^*(x) \geq d - 6 - d \cdot \frac{1}{7} \geq 0$ for $d \geq 7$.

Case 3: Let x be a big d -vertex of G^\times . Then $c^*(x) \geq d - 6 - d \cdot \frac{4}{7} \geq 0$ for $d \geq 14$. □

Proof of Theorem 1.1c). By contradiction. Suppose that there exists a 1-planar graph G of minimum degree 7 such that each its subgraph $K_{2,3}^*$ contains at least one ≥ 14 -vertex, called *big*; vertices of degrees between 7 and 13 are called *intermediate*. We proceed with the Discharging Method with the initial charge assignment (1); the initial charges are redistributed according to following rules (the first four rules are applied in the first phase of discharging):

Rule 1: Each intermediate vertex sends $\frac{3}{7}$ to each incident 3-face.

Rule 2: Let $\alpha = [xyz]$ be a 3-face of G^\times , β be an ≥ 4 -face having the common edge xy with α and let v be a true vertex of β which is incident to x or y . Then v sends $\frac{3}{14}$ to α .

Rule 3: Each big vertex sends $\frac{4}{7}$ to each incident 3-face.

Rule 4: Let $\alpha = [xyz], \beta = [yzv]$ be 3-faces of G^\times and v be a big vertex. Then v sends $\frac{1}{7}$ to α .

Rule 5: Each 3-face with positive charge after application of Rules 1 – 4 redistributes this charge uniformly among adjacent 3-faces that have negative charge after application of the above-mentioned rules.

Rule 6: Each ≥ 5 -face sends $\frac{1}{5}$ to each adjacent 3-face.

We check the nonnegativity of final charges of vertices and faces of G^\times . Firstly, all false vertices have nonnegative final charge as they are not involved in the discharging procedure. For a 4-face α , $c^*(\alpha) = c(\alpha) = 0$; if α is an r -face of G^\times with $r \geq 5$, then, by Rule 6, $c^*(\alpha) \geq r - 4 - r \cdot \frac{1}{5} = \frac{4}{5}r - 4 \geq 0$.

Let v be an intermediate d -vertex of G^\times . Observe that, in Rule 2, v just further redistributes (potentially into two $\frac{3}{14}$) the spared charge $\frac{3}{7}$ (as Rule 1 is not applied in this situation). Hence, $c^*(v) \geq d - 4 - \frac{3}{7}d \geq 0$ for $d \geq 7$.

Let v be a big d -vertex and α be a face incident with v . If α is a 3-face, then both Rule 3 and 4 may be involved with the total contribution $\frac{4}{7} + \frac{1}{7} = \frac{5}{7}$; if α is an ≥ 4 -face, then Rule 2 is possibly applied with the total contribution $2 \cdot \frac{3}{14} = \frac{3}{7}$. We conclude that the maximum charge transferred from v involving a face incident with v is $\frac{5}{7}$; therefore, $c^*(v) \geq d - 4 - \frac{5}{7}d \geq 0$ for $d \geq 14$.

In the following, we analyze the final charge of 3-faces in more detail.

Case 1: Let a 3-face α be incident with a big vertex. Then at least one of the remaining two vertices is true and, by Rules 3 and 1 or 3, $c^*(\alpha) \geq -1 + \frac{4}{7} + \frac{3}{7} = 0$.

Case 2: Let all vertices of a 3-face α be intermediate. Then $c^*(\alpha) \geq -1 + 3 \cdot \frac{3}{7} > 0$.

Case 3: Let a 3-face α be incident with a false vertex c and two other vertices x, y ; according to Case 1, x, y are intermediate. Let α_1, α_2 and α_3 be faces of G^\times incident with edges xy, xc and yc , respectively.

Case 3.1: Let some of $\alpha_i, i = 1, 2, 3$ be an ≥ 4 -face. If α_2 or α_3 is an ≥ 4 -face, then, by Rules 1 and 2, $c^*(\alpha) \geq -1 + 2 \cdot \frac{3}{7} + \frac{3}{14} = \frac{1}{14} > 0$. If α_1 is an ≥ 5 -face, then, by Rules 1 and 6, $c^*(\alpha) \geq -1 + 2 \cdot \frac{3}{7} + \frac{1}{5} = \frac{2}{35} > 0$. Suppose that α_1 is a 4-face. Then there exists a true vertex on α_1 that is incident to x or y ; hence, by Rules 2 and 1, $c^*(\alpha) \geq -1 + 2 \cdot \frac{3}{7} + \frac{3}{14} = \frac{1}{14} > 0$.

Case 3.2: Let all $\alpha_i, i = 1, 2, 3$, be 3-faces. Denote $\alpha_1 = [xyz_1], \alpha_2 = [xcz_2], \alpha_3 = [ycz_3]$ (observe that z_2 and z_3 are true vertices).

Case 3.2.1: Let z_1 be a true vertex. Then at least one of z_1, z_2, z_3 is big (to avoid a light copy of $K_{2,3}$); by Rules 1 and 4, we have $c^*(\alpha) \geq -1 + 2 \cdot \frac{3}{7} + \frac{1}{7} = 0$.

Case 3.2.2: Suppose that z_1 is false; further, we can suppose that both z_2, z_3 are intermediate (otherwise we argue similarly as in the Case 3.2.1). Let β_1, β_2 be faces that are adjacent to α_1 through the edge xz_1 and yz_1 , respectively. If both β_1, β_2 are ≥ 4 -faces, then, by Rules 1 and 2, the face α_1 receives at least $2 \cdot \frac{3}{7} + 2 \cdot \frac{3}{14} = \frac{9}{7}$ and (having initial charge -1) keeps at least $\frac{2}{7}$. Since α is the only 3-face adjacent to α_1 , the face α_1 sends at least

$\frac{2}{7}$ to α by Rule 5, so that $c^*(\alpha) \geq -1 + 2 \cdot \frac{3}{7} + \frac{2}{7} > 0$. If exactly one of β_1, β_2 , say β_1 , is a 3-face, then its third vertex (different from x, z_1) is big (otherwise we find a light copy of $K_{2,3}^*$). Using Rules 1, 4 and 2 (note that z_1 is adjacent to a true vertex incident with β_2), we obtain that α_1 receives at least $2 \cdot \frac{3}{7} + \frac{1}{7} + \frac{3}{14} = \frac{17}{14}$ and (subtracting the initial charge -1) keeps at least $\frac{3}{14}$. By Rule 5, this charge is sent to α , and so we obtain $c^*(\alpha) \geq -1 + 2 \cdot \frac{3}{7} + \frac{3}{14} > 0$. Finally, suppose that both β_1, β_2 are 3-faces. Then both these faces are incident with big vertices and, by Rules 1 and 3, they receive at least $\frac{3}{7} + \frac{4}{7} = 1$; we conclude that, after the first phase, β_1 and β_2 have nonnegative charge (hence, by Rule 5, α_1 sends a charge to at most one 3-face). Using Rules 1 and 4, we obtain that, in the first phase, α_1 receives at least $2 \cdot \frac{3}{7} + 2 \cdot \frac{1}{7}$; thus, by Rule 5, α_1 may send at least $\frac{1}{7}$ to α and we have $c^*(\alpha) \geq -1 + 2 \cdot \frac{3}{7} + \frac{1}{7} = 0$. □

Proof of Theorem 1.1d. By contradiction. Suppose that there exists a 1-planar graph G of minimum degree 7 such that each its subgraph W_5 contains a *big* vertex of degree at least 12. We proceed with the Discharging Method with the initial charge assignment (1); the initial charges are redistributed according to following rules (which are applied sequentially; by $c_i(x)$, we denote the charge of an element $x \in V^\times \cup F^\times$ after application of the first i rules):

Rule 1: Each ≥ 7 -vertex sends $\frac{4}{9}$ to each incident false 3-face and $\frac{1}{3}$ to each incident true 3-face.

Let $k(v)$ be the number of false 3-faces incident with a vertex v .

Rule 2: If v is a vertex of G^\times with $c_1(v) \geq \frac{1}{18}k(v)$, then v sends $\frac{1}{18}$ to each incident false 3-face; if $c_1(v) < \frac{1}{18}k(v)$ or $k(v) = 0$, no charge is transferred.

Let $n(v)$ be the number of 3-faces $[xyz]$ such that x is a 7-vertex and y is a false vertex of G^\times (such faces will be called *awkward*).

Rule 3: If v is a non-big vertex of G^\times with $c_2(v) > 0$, then v shares its charge equally among all incident awkward faces; further, if $c_2(v) \geq \frac{1}{18}n(v)$, then v sends $\frac{1}{18}$ to each incident awkward face.

Rule 4: Each big vertex sends $\frac{1}{3}$ to each adjacent 7-vertex.

Rule 5: If a 7-vertex v has received a contribution from a big vertex, then this contribution is further redistributed equally among all false 3-faces incident with v .

We check the nonnegativity of final charges of vertices and faces of G^\times . As ≥ 4 -faces and 4-vertices are not influenced by the discharging, it is enough to analyze 3-faces and ≥ 7 -vertices of G^\times .

Case 1: Let $\alpha = [xyz]$ be a 3-face.

Case 1.1: If α is a true 3-face, then, by Rule 1, $c^*(\alpha) = c_1(\alpha) \geq -1 + 3 \cdot \frac{1}{3} = 0$.

Case 1.2: Suppose that α is a non-awkward false 3-face and z is its false vertex. Let x be a d -vertex, $d \geq 8$. Then $c_1(x) \geq d - 4 - d \cdot \frac{4}{9} = \frac{5}{9}d - 4 \geq \frac{1}{18}d$. Thus $c_1(x) \geq \frac{1}{18}k(x)$ and, by Rule 2, x sends $\frac{1}{18}$ to α . The same consideration applies to y ; hence, we obtain that $c^*(\alpha) \geq -1 + 2 \cdot \frac{4}{9} + 2 \cdot \frac{1}{18} = 0$.

Case 1.3: Let α be an awkward face with 7-vertex x .

Case 1.3.1: Let y be a big d -vertex. Then $c_1(y) \geq d - 4 - \frac{4}{9}d = \frac{5}{9}d - 4 \geq \frac{1}{18}d \geq \frac{1}{18}k(y)$; hence, by Rule 2, y sends $\frac{1}{18}$ to α . Further, x receives $\frac{1}{3}$ from y by Rule 4. By parity argument, $k(x) \leq 6$, and x sends, by Rule 5, at least $\frac{1}{6} \cdot \frac{1}{3} = \frac{1}{18}$ to each incident false 3-face. Therefore, we obtain $c^*(\alpha) \geq -1 + 2 \cdot \frac{4}{9} + \frac{1}{18} + \frac{1}{18} = 0$.

Case 1.3.2: Let y be a d -vertex with $d \in \{9, 10, 11\}$. Note that $c_1(y) \geq \frac{1}{18}k(y)$ and y sends $\frac{1}{18}$ to α by Rule 2. Further, $c_2(y) \geq d - 4 - k(y) \cdot \frac{4}{9} - (d - k(y)) \cdot \frac{1}{3} - n(y) \cdot \frac{1}{18} = \frac{2}{3}d - 4 - \frac{1}{9}k(y) - \frac{1}{18}n(y) \geq \frac{2}{3}d - 4 - \frac{1}{9}k(y) - \frac{1}{18}k(y) = \frac{2}{3}d - 4 - \frac{3}{18}k(y) \geq \frac{2}{3}d - 4 - \frac{3}{18}d = \frac{9}{18}d - 4 \geq \frac{1}{18}d$ for $d \geq 9$. This implies that y sends additional $\frac{1}{18}$ to α by Rule 3; in total, we obtain $c^*(\alpha) \geq -1 + 2 \cdot \frac{4}{9} + \frac{1}{18} + \frac{1}{18} = 0$.

Case 1.3.3: Let y be an 8-vertex. Still, y sends $\frac{1}{18}$ to α by Rule 2. Suppose first that x is incident with at least one ≥ 4 -face. Then $c_1(x) \geq 7 - 4 - 6 \cdot \frac{4}{9} = \frac{3}{9} = \frac{1}{18} \cdot 6$; hence, x may send $\frac{1}{18}$ to α by Rule 2. Now, let x be incident only with 3-faces; note that, in this case, the number of false 3-faces containing x is even. If at least three of them are true 3-faces, then $c_1(x) \geq 7 - 4 - 4 \cdot \frac{4}{9} - 3 \cdot \frac{1}{3} = \frac{2}{9} = (7 - 3) \cdot \frac{1}{18}$ and again, x can send $\frac{1}{18}$ to α by Rule 2. So suppose that there is exactly one true 3-face incident with x . Then x and its four true neighbours induce a copy of W_5 in G . Thus, at least one of neighbours of x is big and sends, by Rule 4, $\frac{1}{3}$ to x ; by Rule 5, this charge is equally redistributed among six false 3-faces incident with x , which yields the contribution $\frac{1}{18}$ to α , and so $c^*(\alpha) \geq -1 + 2 \cdot \frac{4}{9} + \frac{1}{18} + \frac{1}{18} = 0$.

Case 1.3.4: Let y be a 7-vertex. Then we may use the arguments from Case 1.3.3 concluding that each of x, y sends $\frac{1}{18}$ to α either by Rule 2 or by Rule 5. Hence, we have $c^*(\alpha) \geq -1 + 2 \cdot \frac{4}{9} + 2 \cdot \frac{1}{18} = 0$.

Case 2: Let x be a non-big d -vertex. Note that transfers from x by Rules 2, 3 or 5 are performed only if x has sufficient charge after application of Rule 1; after these subsequent transfers, x is discharged to 0. Hence, it is enough to check the situation after using the Rule 1. Let t be number of transfers of $\frac{4}{9}$ by Rule 1 from x to incident false 3-faces, and q the number of transfers of $\frac{1}{3}$ from x to other incident 3-faces. Then $t + q \leq d$ and $t \leq 2 \cdot \lfloor \frac{d}{2} \rfloor$, and we obtain $c_1(x) \geq d - 4 - \frac{4}{9}t - \frac{1}{3}q = d - 4 - \frac{1}{3}(t + q) - \frac{1}{9}t \geq d - 4 - \frac{1}{3}d - \frac{1}{9} \cdot 2 \cdot \lfloor \frac{d}{2} \rfloor \geq 0$ for $d \geq 7$.

Case 3: Let x be a big d -vertex. Let n and p be numbers of false and true 3-faces incident with x and s be the number of 7-vertices adjacent to x . Then $p + n \leq d$ and $s \leq d$; moreover, since each pair of false 3-faces in the neighbourhood of x contains at least one vertex of degree $\neq 7$, it follows that $p + n + s \leq 2d - \frac{n}{2}$. Then, by Rules 1, 2 and 4, we obtain the estimation $c^*(x) \geq d - 4 - \frac{4}{9}n - \frac{1}{3}p - \frac{1}{18}n - \frac{1}{3}s = d - 4 - \frac{n}{2} - \frac{p}{3} - \frac{s}{3} \geq d - 4 - \frac{2}{3}d = \frac{d}{3} - 4 \geq 0$ for $d \geq 12$. □

Proof of Theorem 1.2. By contradiction. Suppose that there exists a 1-planar graph G of minimum degree 6 such that each its 4-cycle contains a *big* vertex of degree at least 48. We proceed with the Discharging Method with the initial charge assignment (2); the initial charges are redistributed according to the following rules :

Rule 1: Each big face sends $\frac{1}{2}$ to each incident 4-vertex.

Rule 2: Let $\beta = [xyz]$ be a 3-face having the common edge yz with a big face α , and let x be a 4-vertex. Then α sends $\frac{1}{4}$ to x .

Rule 3: Let $\beta = [xyz]$ be a 3-face having the common edge xy with a big face α , and let x be a 4-vertex. Then α sends $\frac{1}{8}$ to x .

Rule 4: Let $\beta = [xyz]$ be a 3-face adjacent with a 3-face $\gamma = [yzw]$ that has a common edge zw with a big face α , and let x be a 4-vertex. Then α sends $\frac{1}{8}$ to x .

Rule 5: Let α, β be two incident big faces with the common edge xy , and let x be a 4-vertex. Then α sends $\frac{1}{4}$ to x .

Rule 6: Let y be a big vertex adjacent to an 4-vertex x . Then y sends $\frac{5}{8}$ to x .

Rule 7: Let $\alpha = [xyz]$ be a 3-face, y be big, and x be a 4-vertex. Then y sends additional $\frac{5}{16}$ to x .

Rule 8: Let $\alpha = [xyz], \beta = [yzw]$ be 3-faces, w be big, and x be a 4-vertex. Then w sends $\frac{1}{4}$ to x .

Rule 9: Let $\alpha = [xyz], \beta = [xyw]$ be 3-faces, w be big, and x be a 4-vertex. Then w sends additional $\frac{1}{8}$ to x .

Rule 10: Let $\alpha = [xyz]$ be a 3-face having a common edge xy with a big face β , z be big, and x be a 4-vertex. Then z sends additional $\frac{1}{4}$ to x .

Rule 11: Let $\alpha = [xyz], \beta = [yzv]$, and $\gamma = [yvw]$ be 3-faces, w be big, and x be a 4-vertex. Then w sends additional $\frac{1}{8}$ to x .

We will analyze the final charges of vertices and faces of G^\times .

Case 1: Let x be a 4-vertex.

Case 1.1: If all $f_i, i = 1, \dots, 4$ are big, then, by Rule 1, $c^*(x) \geq -2 + 4 \cdot \frac{1}{2} = 0$.

Case 1.2: Let exactly one of f_i , say f_4 , be a 3-face. Then $c^*(x) \geq -2 + 3 \cdot \frac{1}{2} + 2 \cdot \frac{1}{8} + 4 \cdot \frac{1}{4} = \frac{3}{4} > 0$ by Rules 1 (applied three times), 3 (twice), and 5 (four times).

Case 1.3: Let two of $f_i, i = 1, \dots, 4$ be 3-faces. According to the symmetry, we distinguish two cases:

Case 1.3.1: Let f_2, f_4 be 3-faces. Then one of x_i , say x_1 , is big (otherwise $x_1x_3x_2x_4$ is a light 4-cycle). By Rules 1 (twice), 3 (four times), 6 (once) and 7 (once), $c^*(x) \geq -2 + 2 \cdot \frac{1}{2} + 4 \cdot \frac{1}{8} + \frac{5}{8} + \frac{5}{16} > 0$.

Case 1.3.2: Let f_1, f_4 be 3-faces. If one of x_1, \dots, x_4 is big, then $c^*(x) \geq -2 + 2 \cdot \frac{1}{2} + 2 \cdot \frac{1}{8} + 2 \cdot \frac{1}{4} + \frac{5}{8} > 0$ by Rules 1 (applied twice), 3 (twice), 5 (twice), and 6 (once).

So, assume that no neighbour of x is big. Consider the face f'_1 . If f'_1 is big, then it sends $\frac{1}{4}$ to x by Rule 2; otherwise it is a 3-face. If x'_1 is a true vertex, then it must be big (otherwise $x_1x'_1x_2x_4$ is a light 4-cycle). The big vertex x'_1 sends $\frac{1}{4}$ to x by Rule 8. It remains to resolve the case when f'_1 is a 3-face, but x'_1 is a false vertex. Then consider the faces f'^l_1, f'^r_1 that share with f'_1 common edge x'_1x_1 or x'_1x_2 , respectively. If some of them is a 3-face, then its third vertex $z \neq x'_1, x_1$ (or $z \neq x'_1, x_2$) must be big (otherwise $x_4x_1zx_2$ is a light 4-cycle). Hence, it contributes $\frac{1}{8}$ to x by Rule 11. If some of these faces is big, then it sends $\frac{1}{8}$ to x by Rule 4. Consequently, x receives $\geq \frac{1}{4}$ due to the transfers through edge x_1x_2 . The same consideration is applied on the transfers through edge x_4x_1 . Thus, we obtain $c^*(x) \geq -2 + 2 \cdot \frac{1}{2} + 2 \cdot \frac{1}{8} + 2 \cdot \frac{1}{4} + 2 \cdot \frac{1}{4} = \frac{1}{4} > 0$.

Case 1.4: Let three of f_i , say f_1, f_2 , and f_4 , be 3-faces. Note that at least one of neighbours of x is big.

Case 1.4.1: If x has at least two big neighbours, then $c^*(x) \geq -2 + \frac{1}{2} + 2 \cdot \frac{1}{8} + 2 \cdot \frac{5}{8} + 2 \cdot \frac{5}{16} + 2 \cdot \frac{1}{8} > 0$ by Rules 1 (applied once), 3 (twice), 6 (at least twice), 7 (at least twice) and 9 (at least twice).

Case 1.4.2: Let x have exactly one big neighbour. If x_1 or x_2 is big, then $c^*(x) \geq -2 + \frac{1}{2} + 2 \cdot \frac{1}{8} + \frac{5}{8} + 2 \cdot \frac{5}{16} + \frac{1}{8} + \frac{1}{4} > 0$ by Rules 1 (applied once), 3 (twice), 6 (once), 7 (twice), 9 (once) and 10 (once).

Suppose, without loss of generality, that x_3 is big. Consider now the faces f'_1, f'_4 . We may use the same argumentation as in Case 1.3.2. The transfer through edges x_1x_2 and x_1x_4 is always at least $2 \cdot \frac{1}{4}$. Thus, in total, $c^*(x) \geq -2 + \frac{1}{2} + 2 \cdot \frac{1}{8} + \frac{5}{8} + \frac{5}{16} + \frac{1}{8} + 2 \cdot \frac{1}{4} > 0$ by Rules 1 (applied once), 3 (twice), 6 (once), 7 (once), 9 (once), and the above mentioned contribution.

Case 1.5: Let all faces incident with x be 3-faces. Note that at least one of neighbours of x is big. If at least two neighbours of x are big, then $c^*(x) \geq -2 + 2 \cdot \frac{5}{8} + 4 \cdot \frac{5}{16} + 2 \cdot \frac{1}{8} > 0$ by Rules 6 (twice), 7 (four times) and 9 (four times). Let exactly one of neighbours of x , say x_1 , be big. Then, using the same argumentation as in Case 1.3.2, x receives at least $2 \cdot \frac{1}{4}$ through edges x_2x_3 and x_3x_4 . In total, $c^*(x) \geq -2 + \frac{5}{8} + 2 \cdot \frac{5}{16} + 2 \cdot \frac{1}{8} + \frac{1}{2} = 0$ by Rules 6 (applied once), 7 (twice), 9 (twice), and the above mentioned contribution.

Case 2: Let α be a big face. Observe that Rules 3, 4 just halve the charge $\frac{1}{4}$ that is saved (since, in situation of Rules 3, 4, there is no such transfer as in Rule 2). Also, in Rule 5, the full transfer of $\frac{1}{4}$ is saved, because the situation of Rules 3 and 4 is not involved in this rule. Hence, we may roughly consider that, through each edge incident with α , $\frac{1}{4}$ of charge is transferred. Also, note that Rule 1 applies to at most $\lfloor \frac{\deg(\alpha)}{2} \rfloor$ vertices incident with α . Therefore, $c^*(\alpha) \geq c(\alpha) - \lfloor \frac{\deg(\alpha)}{2} \rfloor \cdot \frac{1}{2} - \deg(\alpha) \cdot \frac{1}{4} \geq 2 \cdot \deg(\alpha) - 6 - \frac{\deg(\alpha)}{4} - \frac{\deg(\alpha)}{4} = \frac{3}{2} \cdot \deg(\alpha) - 6 \geq 0$ if $\deg(\alpha) \geq 4$.

Case 3: Let x be a big vertex. Note that Rule 7 just halves the charge that is saved in the corresponding situation when Rule 6 is not used, Rules 9, 11 work in the same way in connection with Rule 8. The Rule 10 sends the full charge that was saved in Rule 8. We can use very rough estimation $\frac{5}{8} + \frac{1}{4}$ for a charge assigned to a neighbour of x , and to an edge of a 3-face incident with x . By this rough estimation, $c^*(x) \geq c(x) - (\frac{5}{8} + \frac{1}{4}) \cdot \deg(x) = \deg(x) - 6 - \frac{7}{8} \cdot \deg(x) = \frac{1}{8} \cdot \deg(x) - 6 \geq 0$ for $\deg(x) \geq 48$. □

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