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Generalization of edge general position problem*

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Dedicated to Dragan Marušič on the occasion of his 70th birthday.

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Abstract

The edge geodesic cover problem of a graph G is to find a smallest number of geodesics that cover the edge set of G. The edge k-general position problem is introduced as the problem to find a largest set S of edges of G such that at most $k-1$ edges of S lie on a common geodesic. We show that these are dual min-max problems and connect them to an edge geodesic partition problem. Using these connections, exact values of the edge k-general position number is determined for different values of k and for various networks including torus networks, hypercubes, and Benes networks.

Keywords: General position set, edge geodesic cover problem, edge k-general position problem, torus network, hypercube, Benes network.

Math. Subj. Class.: 05C12, 05C76

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1 Introduction

Dual min-max invariant combinatorial problems are central topics in graph theory and more generally in combinatorics, cf. [2]. Here we consider an instance of such dual problems, the edge geodesic cover problem and the edge general position problem, where we will use the first as a tool to study the second.

An *edge geodesic cover* of G is a set S of geodesics such that each edge of G belongs to at least one geodesic of S. The *edge geodesic cover number* of G, $\mathrm{gcover_e}(G)$, is the minimum cardinality of an edge geodesic cover of G. The *edge geodesic cover problem* is to find a minimum cardinality edge geodesic cover of G, cf. [19]. An *edge geodesic partition* of G is a set S of geodesics such that each edge of G belongs to exactly one geodesic of S. The *edge geodesic partition number* of G, $\mathrm{gpart_e}(G)$, is the minimum cardinality of an edge geodesic partition of G. The *edge geodesic partition problem* is to find a minimum cardinality edge geodesic partition of G. A survey on edge geodesic cover and partition problems up to 2018 can be found in [19].

For $k \geq 3$, we introduce the edge k-general position sets as follows. An *edge k-general position set* is a set S of edges of G such that no k edges of S lie on a common geodesic, that is, $|S \cap E(P)| \leq k-1$ for any geodesic P of G. An edge k-general position set of maximum cardinality in G is called an *edge k-gp set*. Its cardinality is denoted by $k\text{-}\mathrm{gp_e}(G)$ and called the *edge k-gp number*. The *edge k-general position problem* is to find an edge k-gp set. The edge 3-general position problem is known as the *edge general position problem* and was studied for the first time in [23] and then continued in [13, 14]. The corresponding invariant is called the *gp_e-number* of G and denoted by $\mathrm{gp_e}(G)$. The related (vertex) general position problem, which was independently introduced in [3, 22], has already been extensively studied, see [1, 5, 10–12, 16, 28, 30, 31, 33].

The main objective of this paper is to demonstrate that the edge geodesic cover problem and the edge k-general position problem form a pair of dual min-max combinatorial problems. To do so, we first establish some basic results in Section 2. The advantage of dual min-max invariant combinatorial problems is that one problem can be solved by means of the other problem. In this paper we apply this approach to the above-mentioned problems. In Section 3 we determine the edge k-gp number for torus graphs $C_{2^r} \,\square\, C_{2^r}$ and $k = 2^t + 1$. Then, in Section 4, we demonstrate that partial cubes contain large edge k-gp sets and prove that the edge k-gp number of a hypercube Q_d is $(k-1)2^{d-1}$. In Section 5 we solve the edge k-gp problem for Benes networks for $k \in \{3, 5\}$. In the rest of the introduction, we give some further definitions needed.

Let P_n denote the path on n vertices and C_n the cycle on n vertices. The distance $d_G(u, v)$ between vertices u and v of G is the number of edges on a shortest u, v-path. Shortest paths are also known as *isometric paths* or *geodesics*. The *diameter* $diam(G)$ of G is the maximum distance between vertices of G. A *diametral path* is an isometric path whose length is equal to the diameter of G. If $X, Y \subseteq V(G)$, then $d_G(X, Y) = \min_{x\in X, y\in Y} d_G(x, y)$. If H and H' are subgraphs of G, then $d_G(H, H') = d_G(V(H), V(H'))$. In this manner, if $e, f \in E(G)$, then $d_G(e, f)$ is the minimum distance between a vertex of e and a vertex of f.

2 A few preliminary results

In this section we present some preliminary results on edge k-general position sets. We first show that if the diameter of a graph is at most $2k - 2$, then the matching of cardinality

k of the graph are edge k-general position sets. (Recall that a *matching* of G is a set of independent edges of G.)

Proposition 2.1. *Let G be a graph and $k \geq 3$. Then $\operatorname{diam}(G) \leq 2k-2$ if and only if every matching of size k is an edge k-general position set.*

Proof. Suppose that $\operatorname{diam}(G) \leq 2k-2$. Let M be an arbitrary matching of G of order k and assume that the edges from M lie on a common geodesic P. Since M is a matching, the length of P is at least $2k-1$ and so $\operatorname{diam}(G) \geq 2k-1$ would hold. As this contradicts our assumption we get that M is an edge k-general position set.

Conversely, suppose that every matching of order k is an edge k-general position set. Assume on the contrary that $\operatorname{diam}(G) \geq 2k-1$ and let P be a geodesic of length $\operatorname{diam}(G)$. Selecting every second edge of P we construct a matching M of order at least k. Let M' be a subset of M with $|M'| = k$. As M' is a matching, by our assumption we have that M' is an edge k-general position set, but this is not the case as all the edges from M' lie on P. $\square$

A j-geodesic is a geodesic of length j. The following proposition is a useful tool to prove that a given set of edges is an edge general position set.

Proposition 2.2. *Let S be a set of edge-disjoint geodesics in G each of length j and let $\ell = \min_{P,Q \in S} d_G(P,Q)$. If $k \geq 2$ and $d_G(P,Q) < \ell(k-1) + j(k-2)$ holds for every $P, Q \in S$ such that P and Q lie in a common geodesic, then no k paths from S lie on a common geodesic.*

Proof. Suppose on the contrary that the paths $P_1, \ldots, P_k$ from S lie (in this order) on a common geodesic. Then

$$\begin{aligned} d_G(P_1, P_k) &= \sum_{i=1}^{k-2} (d_G(P_i, P_{i+1}) + j) + d_G(P_{k-1}, P_k) \\ &= \sum_{i=1}^{k-1} d_G(P_i, P_{i+1}) + j(k-2) \\ &\geq \ell(k-1) + j(k-2), \end{aligned}$$

a contradiction since P_1 and P_k lie on a common geodesic and we have assumed that then $d_G(P_1, P_k) < \ell(k-1) + j(k-2)$ holds. $\square$

Setting $j = 1$ in Proposition 2.2 we have the following consequence.

Corollary 2.3. *Let $S \subseteq E(G)$, $\ell = \min_{e,f \in S} d_G(e,f)$, and $L = \max_{e,f \in S} d_G(e,f)$. If $L < \ell(k-1) + (k-2)$, then S is an edge k-general position set.*

We conclude the section by the following simple, yet fundamental inequalities comparing $\operatorname{gp_e}(G)$, $\operatorname{gcover_e}(G)$, and $\operatorname{gpart_e}(G)$. The result establishes how the edge geodesic cover problem and the edge general position problem constitute dual min-max combinatorial problems.

Lemma 2.4. *If G is a graph and $k \geq 3$, then*

$$k\text{-}\operatorname{gp_e}(G) \leq (k-1) \cdot \operatorname{gcover_e}(G) \leq (k-1) \cdot \operatorname{gpart_e}(G)\,.$$

Proof. Each geodesic from a geodesic cover can contain at most $k-1$ edges from an edge k-general position set. Hence the left inequality. The right inequality follows because a geodesic partition is a geodesic cover, cf. [19]. □

3 The edge k-gp problem for torus

The *Cartesian product* $G \,\square\, H$ of graphs G and H is defined on the vertex set $V(G \,\square\, H) = V(G) \times V(H)$, vertices (g,h) and (g',h') are adjacent if either $gg' \in E(G)$ and $h = h'$, or $g = g'$ and $hh' \in E(H)$. See the book [8] for more information on this graph operation. Cartesian products of cycles $C_n \,\square\, C_m$ are known as *torus graphs*. As in this paper we will consider only these products, we simplify the general terminology for products as follows. Two edges $x = (x_1, x_2)$ and $y = (y_1, y_2)$ of a torus are said to be *parallel* if $d(x_1, y_2) = d(x_2, y_1) = d(x_1, y_1) + 1 = d(x_2, y_2) + 1$. The edges of $C_n \,\square\, C_m$ that project on the edges of the first factor will be called *horizontal edges*, while the edges that project on C_m are *vertical edges*. Analogously we will speak of *horizontal cycles* (copies of C_n in the product) and of *vertical cycles*.

Lemma 3.1. *If $r \geq 3$ and $2^t \leq 2^{r-1} - 2$, then*

$$(2^t + 1)\text{-gp}_{\rm e}(C_{2^r}) = 2^{t+1}\,.$$

Proof. Set $k = 2^t + 1$. Since $\text{gcover}_{\rm e}(C_{2^r}) = \text{gpart}_{\rm e}(C_{2^r}) = 2$, Lemma 2.4 implies that it is enough to show that $k\text{-gp}_{\rm e}(C_{2^r}) \geq 2(k-1)$. That is, we need to construct an edge k-general position set S of C_{2^r} with $|S| = 2k - 2 = 2^{t+1}$. We proceed to construct such a set S.

Let $V(C_{2^r}) = \{v_1, v_2, \ldots, v_{2^r}\}$. Define S to be the set containing edges $e_i = u_i v_i$, $i \in [2^{t+1}]$, where the edges are equidistant on C_{2^r} , that is, we select them such that

$$d_{C_{2^r}}(e_i, e_{i+1}) = 2^{r-t-1} - 1$$

holds for all $i \in [2^{t+1} - 1]$, cf. Figure 1.

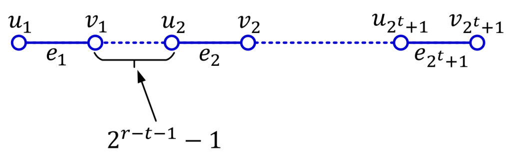

Figure 1: The edges $e_1, e_2 \cdots e_{2^t+1}$ of S' lie on geodesic P. The end-vertices of the edge e_i are u_i and v_i, such that $u_1, v_1, u_2, v_2, \ldots, u_{2^t+1}, v_{2^t+1}$ are in increasing order. We have $d(e_i, e_{i+1}) = 2^{r-t-1} - 1$.

We claim that S is an edge k-general position set. If not, then there exists a subset S' of S such that $|S'| = k = 2^t + 1$ and the edges of S' lie on a common geodesic P. By the equidistant distribution of the edges from S we may without loss of generality assume that $S' = \{e_1, e_2, \ldots, e_{2^t+1}\}$. As P is a geodesic which contains $2^t + 1$ edges with the distance $2^{r-t-1} - 1$ between the consecutive edges, we have

$$d_{C_{2^r}}(u_1, v_{2^t+1}) = (2^t + 1) + 2^t(2^{r-t-1} - 1) = 2^{r-1} + 1\,,$$

a contradiction because $\text{diam}(C_{2^{r-1}}) = 2^{r-1}$. □

Note that Lemma 3.1 provides an example where both inequalities of Lemma 2.4 are attained.

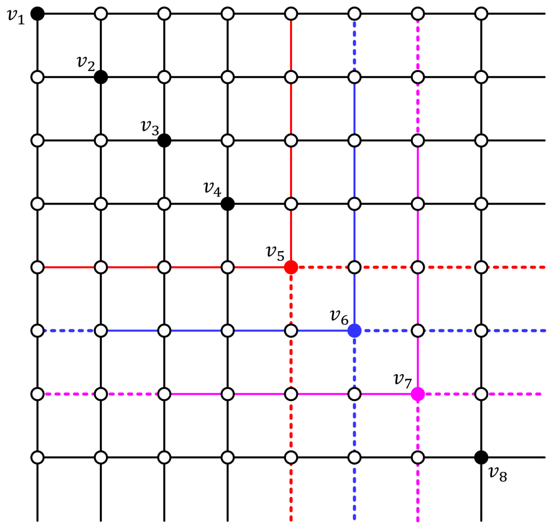

Figure 2: $v_1, \ldots, v_8$ are diagonal vertices of $C_8 \,\square\, C_8$. The vertex v_5 is the red bullet. There are two red lines. One is red solid line and the other is red dotted line. Both geodesics (the red solid line and the red dotted line) are diametral paths such that v_5 is the mid point of both geodesics. The pairs of diametral paths at $v_1, \ldots, v_8$ partition $E(G)$.

Proposition 3.2. *If $r \geq 2$, then* $\mathrm{gcover_e}(C_{2r} \,\square\, C_{2r}) = \mathrm{gpart_e}(C_{2r} \,\square\, C_{2r}) = 4r$.

Proof. Set $G = C_{2r} \,\square\, C_{2r}$.

Since $\mathrm{gcover_e}(G) \geq \lceil |E(G)|/\mathrm{diam}(G) \rceil$ (cf. [20]), $\mathrm{diam}(G) = 2r$, and $|E(G)| = 2 \cdot 2r \cdot 2r$, we infer that $\mathrm{gcover_e}(G) \geq 4r$.

To prove that $\mathrm{gcover_e}(G) \leq 4r$ we proceed by construction. Let $v_1, \ldots, v_{2r}$ be the diagonal vertices of G as demonstrated in Figure 2. For each diagonal vertex v_i let P'_{v_i} and P''_{v_i} be two edge disjoint diametral paths as shown in Figure 2 for the vertices v_5, v_6, and v_7. Then v_i is the midpoint of P'_{v_i} and P''_{v_i} which is possible because both factors of G are even paths. The set of paths $\{P'_{v_i}, P''_{v_i} : i \in [2r]\}$ then partitions $E(G)$. Thus, $\mathrm{gcover_e}(G) \leq \mathrm{gpart_e}(G) \leq 4r$. □

Theorem 3.3. *If $r \geq 3$ and $2^t \leq 2^{r-1} - 2$, then*

$$(2^t + 1)\text{-}\mathrm{gp_e}(C_{2^r} \,\square\, C_{2^r}) = 2^{r+t+1}\,.$$

Proof. The technique of the proof is parallel to the one from the proof of Lemma 3.1. More precisely, we set $k = 2^t + 1$ and are going to construct an edge k-general position set S of G with $|S| = (k-1) \cdot \mathrm{gpart_e}(G) = 2^t \cdot 2^{r+1} = 2^{r+t+1}$.

Consider a horizontal cycle of $C_{2^r} \,\square\, C_{2^r}$, say C^h. Using Lemma 3.1, construct a set S^h of edges from C^h of cardinality 2^t which contains equidistant edges $e^h_1, e^h_2, \ldots, e^h_{2^t}$ of the cycle, that is, $d(e^h_i, e^h_{i+1}) = d(e^h_j, e^h_{j+1}) = 2^{r-t+1} - 1$ for $1 \leq i, j \leq 2^t$. Now add all

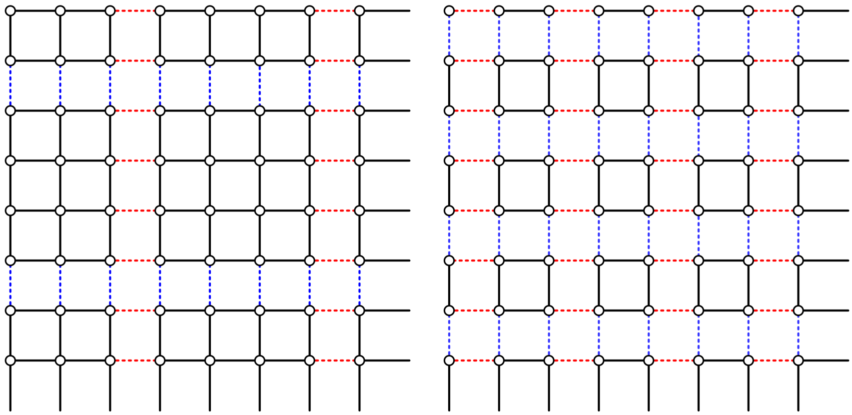

Figure 3: There are two type of edges: red dotted edges and blue dotted edges. The blue dotted edges are vertical and the red dotted edges are horizontal. (a) The red dotted edges and the blue dotted edges form an edge 3-general position set of the 8×8 torus. (b) The red dotted edges and the blue dotted edges form an edge 5-general position set of the 8×8 torus.

the edges of S^h to S. Further, add to S all the edges parallel to the edges from S^h. See Figure 3 where the edges from S are the red dotted edges. At this stage we have added to $|S|$ precisely $2^t \cdot 2^r = 2^{r+t}$ edges because there are 2^r parallel edges for every given edge in $C_{2^r} \,\square\, C_{2^r}$. We proceed analogously for the vertical cycles of $C_{2^r} \,\square\, C_{2^r}$. That is, we select one such cycle, select in it 2^t equidistant edges at the distance $2^{r-t+1} - 1$, and add to S all these edges as well as all the edges parallel to them. At this stage, the cardinality of S is doubled and thus we have ended up with $|S| = 2^{r+t+1}$.

We claim that the above constructed S is an edge k-general position set of $C_{2^r} \,\square\, C_{2^r}$. If this is not the case, then there exists a subset R of S with $|R| = 2^t + 1$ such that all the edges of R lie on a common geodesic P. Since no two parallel edges of a torus lie on a geodesic, we infer that R does not contain any parallel edges. Without loss of generality we may assume that R has at least as many horizontal edges as vertical edges. Since $|R| = 2^t + 1$, this means that R contains at least $2^{t-1} + 1$ horizontal edges. Let R^h denote the set of all those horizontal edges, $R^h = \{e_1^h, e_2^h, \dots, e_s^h\}$, where we know that $s \geq 2^{t-1} + 1$. Set $e_i^h = u_i v_i$ and assume without loss of generality that $u_1, v_1, u_2, v_2, \dots, u_s, v_s$ are in a non-decreasing order (The meaning of non-decreasing order is explained in Figure 1). Such an order is possible as no two edges from R^h are parallel. The situation is illustrated in Figure 4.

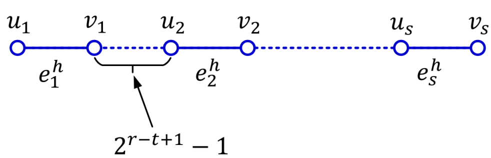

Figure 4: $R^h = \{e_1^h, e_2^h, \dots, e_s^h\}$. The vertices $u_1, v_1, u_2, v_2, \dots, u_s, v_s$ are in increasing order. Also, $d(e_i^h, e_{i+1}^h) \geq 2^{r-t+1} - 1$.

Consider now the distance $d(u_1, v_s)$ (along the geodesic P). We have

$$\begin{aligned} d(u_1, v_s) &= (1 + d(e_1^h, e_2^h)) + (1 + d(e_2^h, e_3^h)) + \cdots + (1 + d(e_{s-1}^h, e_s^h)) + 1 \\ &\geq (1 + 2^{r-t+1} - 1) + (1 + 2^{r-t+1} - 1) + \cdots + (1 + 2^{r-t+1} - 1) + 1 \\ &= (s-1) \cdot 2^{r-t+1} + 1 \\ &\geq 2^{t-1} \cdot 2^{r-t+1} + 1 \\ &\geq 2^r + 1. \end{aligned}$$

This leads to a contradiction because the length of the geodesic P in $C_{2r} \,\square\, C_{2r}$ is greater as $\mathrm{diam}(C_{2r} \,\square\, C_{2r}) = 2^r$. This proves the claim and hence the theorem. $\square$

In addition to the Cartesian product of two cycles, it would certainly be interesting to consider the Cartesian product of a cycle by a path and the Cartesian product of a path by a path. However, it seems that this would require a different approach to the one we have used for the torus graphs.

4 The edge k-gp problem for partial cubes

In this section we extend results on $\mathrm{gp_e}(G)$ of a partial cube G from [23] to k-$\mathrm{gp_e}(G)$. For this sake, we first recall the concept needed.

A graph G is a *partial cube* if G is a subgraph of some hypercube Q_d such that if $x, y \in V(G)$, then $d_G(x, y) = d_{Q_d}(x, y)$. Papers [4, 25, 26, 29] present a selection of recent developments on partial cubes. Edges xy and uv of a graph G are in relation Θ if $d_G(x, u) + d_G(y, v) \neq d_G(x, v) + d_G(y, u)$. A connected graph G is a partial cube if and only if G is bipartite and Θ is transitive [32]. As Θ is reflexive and symmetric on an arbitrary graph, it partitions the edge set of a partial cube into Θ-*classes*.

Lemma 4.1. *Let G be a partial cube, $k \geq 3$, and $F_1, \ldots, F_{k-1}$ be Θ-classes of G. Then $\bigcup_{i=1}^{k-1} F_i$ is an edge k-gp set of G.*

Proof. Consider an arbitrary set X of k edges from $\bigcup_{i=1}^{k-1} F_i$. Then by the pigeonhole principle at least two of the edges from X lie in a common Θ-*class* F_i. As no two edges from F_i lie on a common geodesic, cf. [6, Lemma 11.1] we get that the edges from X do not lie on a common geodesic. We conclude that X is an edge k-gp set of G. $\square$

Lemma 4.1 enables us to find large edge k-gp sets in partial cubes. We demonstrate this claim by the following result for hypercubes. Recall that the *d-dimensional hypercube* Q_d, $d \geq 1$, is a graph with $V(Q_d) = \{0, 1\}^r$, and there is an edge between two vertices if and only if they differ in exactly one coordinate.

Theorem 4.2. *If $3 \leq k \leq d + 1$, then*

$$k\text{-}\mathrm{gp_e}(Q_d) = (k-1)2^{d-1} = (k-1)\mathrm{gcover_e}(Q_d) = (k-1)\mathrm{gpart_e}(Q_d)\,.$$

Proof. It is well-known that Q_d is a partial cube and that it has Θ-classes F_i, $i \in [d]$, where F_i is formed by the edges whose endpoints differ in coordinate i, cf. [6–8]. Note that $|F_i| = 2^{d-1}$. Then Lemma 4.1 implies that k-$\mathrm{gp_e}(Q_r) \geq (k-1)2^{d-1}$. As we have assumed that $k \leq d + 1$ and Q_d contains $d2^{d-1}$ edges, this is indeed possible.

To prove the reverse inequality we recall from the proof of [23, Theorem 3.2] that Q_d admits an isometric path edge partition consisting of 2^{d-1} paths. Lemma 2.4 thus implies that

$$(k-1)\cdot 2^{d-1} \leq k\text{-gp}_{\rm e}(Q_d) \leq (k-1)\cdot \text{gpart}_{\rm e}(Q_d) \leq (k-1)\cdot 2^{d-1}\,,$$

hence applying Lemma 2.4 again we have equality everywhere in it for Q_d. □

5 The edge k-gp problem for Benes networks

Benes networks were presented in [21] and have been studied subsequently in different contexts, see for instance [7, 9, 15, 17, 27]. For $r \geq 3$, the r-dimensional *Benes network* $\text{BN}(r)$ is defined as follows. The vertex set consists of all ordered pairs $[s,i]$, where s runs over all r-bit binary strings and $i \in \{0,1,\ldots,2r\}$. In the *normal representation* of $BF(r)$, the first coordinate of the vertex is interpreted as the row of the vertex and its second coordinate is a column called the *level* of the vertex. The vertices $[s,i]$ and $[s',i']$, where $i,i' \leq r$, are adjacent if $|i-i'| = 1$, and either $s = s'$ or s and s' differ precisely in the $i^{\rm th}$ bit. When $i,i' \geq r$ the edges are vertically reflected (in the normal representation). The edges between level $i-1$ and level i are called *level i edges* for $1 \leq i \leq 2r$. The formal definition should be clear from Figure 5, where $\text{BN}(3)$ is shown in its normal representation. Clearly, $\text{BN}(r)$ has $(2r+1)2^r$ vertices.

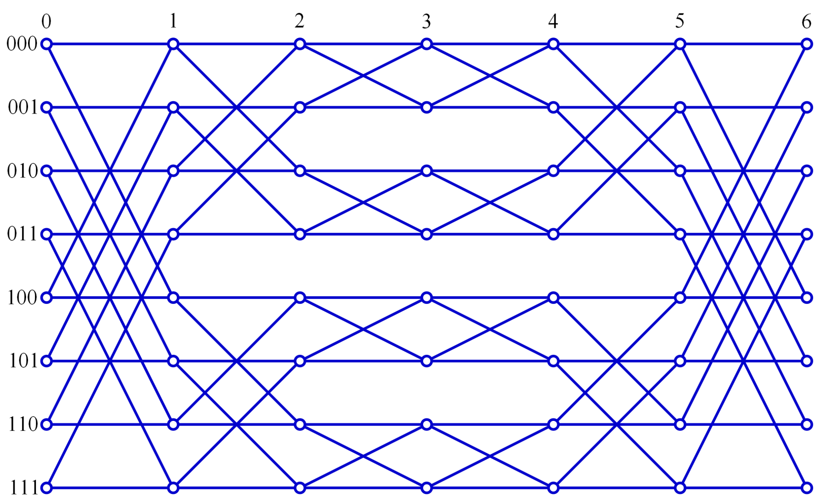

Figure 5: The Benes network $\text{BN}(3)$ consists of back-to-back butterflies and is an edge disjoint union of two butterflies.

Theorem 5.1. *If $r \geq 3$, then*

$$\text{gcover}_{\rm e}(\text{BN}(r)) = \text{gpart}_{\rm e}(\text{BN}(r)) = 2^{r+1}\,.$$

Proof. An alternative way to represent Benes networks is that $\mathrm{BN}(r)$ consists of two back-to-back butterflies $\mathrm{BF}(r)$, cf. [18, 21, 24], that is, of two copies of $\mathrm{BF}(r)$ sharing level r vertices. It is known that $\mathrm{diam}(\mathrm{BN}(r)) = \mathrm{diam}(\mathrm{BF}(r)) = 2r$, cf. [18, 21, 24], and that the edge set of $\mathrm{BF}(r)$ can be partitioned with respect to 2^r diametral paths [21]. It follows that the edge set of $\mathrm{BN}(r)$ can be partitioned with respect to 2^{r+1} diametral paths of $\mathrm{BN}(r)$. Consequently, Lemma 2.4 implies $\mathrm{gcover_e}(\mathrm{BN}(r)) \le \mathrm{gpart_e}(\mathrm{BN}(r)) \le 2^{r+1}$.

To prove the reverse inequality, consider the set S of edges which are incident to the vertices of level 0, level r, and level $2r$. Then $|S| = 2 \cdot 2^r + 4 \cdot 2^r + 2 \cdot 2^r = 2^{r+3}$. Since a geodesic can cover a maximum of four edges of S [21], at least $2^{r+3}/4 = 2^{r+1}$ geodesics are required to cover all the edges of S. Hence, $\mathrm{gcover_e}(\mathrm{BN}(r)) \ge 2^{r+3}/4 = 2^{r+1}$. Thus, $\mathrm{gpart_e}(\mathrm{BN}(r)) \ge \mathrm{gcover_e}(\mathrm{BN}(r)) \ge 2^{r+1}$. □

Theorem 5.2. *If $k \in \{3, 5\}$, then*

$$k\text{-}\mathrm{gp_e}(BN(r)) = (k-1) \cdot 2^{r+1}\,.$$

Proof. By combining Lemma 2.4 with Theorem 5.1, it is enough to identify an edge general position set S of cardinality 2^{r+2} for $k = 3$, and an edge 5-general position set S' of cardinality 2^{r+3} for $k = 5$.

For $k = 3$, consider the set S of edges in $\mathrm{BN}(r)$ which are incident to the degree two vertices (in levels 0 and $2r$). Since each level consists of 2^r vertices and each vertex at level 0 and level $2r$ is of degree two, $|S| = 2^{r+2}$. An arbitrary geodesic contains at most two edges of S, cf. [21]. Thus, S is an edge general position set of $\mathrm{BN}(r)$ of the required cardinality.

For $k = 5$, consider the set of edges S' which are incident to the vertices of level 0, level r, and level $2r$. Then, as already noticed in the proof of Theorem 5.1, $|S'| = 2^{r+3}$. As a geodesic contains at most four edges of S', we conclude that S is an edge general position set of $\mathrm{BN}(r)$ of the required cardinality. □

6 Conclusion

In this paper, we have demonstrated that the edge geodesic cover problem and the edge k-general position problem form a pair of dual min-max invariant combinatorial problems. We have solved the edge k-general position problem completely for hypercubes and for certain cases of torus. In addition, we have solved the edge k-general position problem for Benes networks $\mathrm{BN}(3)$ and $\mathrm{BN}(5)$. The edge geodesic cover problem and the edge geodesic partition problem are completely solved for hypercubes, torus and Benes networks. Studying the interplay between these two concepts seems to be an interesting topic. Among other things, it would be interesting to explore these dual min-max invariant combinatorial problems for intersection graphs, subclasses of perfect graphs, and different Cayley graphs.

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