

Classification of cubic vertex-transitive tricirculants*

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Abstract

A finite graph is called a tricirculant if admits a cyclic group of automorphism which has precisely three orbits on the vertex-set of the graph, all of equal size. We classify all finite connected cubic vertex-transitive tricirculants. We show that except for some small exceptions of order less than 54, each of these graphs is either a prism of order $6k$ with k odd, a Möbius ladder, or it falls into one of two infinite families, each family containing one graph for every order of the form $6k$ with k odd.

Keywords: Graph, cubic, semiregular automorphism, tricirculant, vertex-transitive.

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1 Introduction

All graphs in this paper are finite. A connected graph Γ admitting a cyclic group of automorphisms H having k orbits of vertices of equal size larger than 1 is called a k -multicirculant and a generator of H is then called a k -multicirculant automorphism of Γ . In particular, 1-, 2- and 3-multicirculants are generally called circulants, bicirculants and tricirculants, respectively. A graph is called cubic if it is connected and regular of valence 3.

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A famous and longstanding *polycirculant conjecture* claims that every vertex-transitive graph and digraph is a k -multicirculant for some k (see [3, 13, 14, 15]). It is particularly interesting to study conditions under which vertex-transitive graphs admit k -multicirculant automorphisms of large order (and thus relatively small k); see, for example, [2, 20]. On the other hand, existence of a k -multicirculant automorphism with small k often restricts the structure of a vertex-transitive graph to the extent that allows a complete classification. There is a series of classification results about cubic arc-transitive k -multicirculant for $k \leq 5$ (see [7, 10]). In particular, cubic arc-transitive trirculants were completely classified in [10], where it was shown that only 4 such graphs exist: $K_{3,3}$, the Pappus graph, Tutte's 8-cage and a graph on 54 vertices. This work culminated in a beautiful paper [9], where it was shown that for all square-free values of k coprime to 6 there exist only finitely many cubic arc-transitive k -multiculants.

In this paper we widen the focus to a much wider and structurally richer class of cubic vertex-transitive graphs that are not necessarily arc-transitive. It is known that every cubic vertex-transitive graph has a k -multicirculant automorphism [16] and that no fixed k exists such that every cubic vertex-transitive graph is a k -multicirculant [20]. As for the classification results, it is clear that a cubic graph is a vertex-transitive 1-circulant if and only if it is a cubic Cayley graph on a cyclic group. Furthermore, vertex-transitive cubic bicirculants were classified in [17]. The goal of this paper is to provide a complete classification of cubic vertex-transitive trirculants.

The following theorem and remarks are a brief summary of the contents of this paper.

Theorem 1.1. *A cubic graph Γ is a vertex-transitive trirculant if and only if the order of Γ is $6k$ for some positive integer k and one of the following holds:*

1. $k > 1$, k is odd and Γ is isomorphic to one of the graphs $X(k)$ or $Y(k)$, described in Section 4 (Definition 4.1) and Section 5 (Definition 5.1), respectively.
2. Γ is a prism and k is odd, or Γ is a Möbius ladder (defined in Section 6).
3. Γ is the truncated tetrahedron or Tutte's 8-cage.

Remark 1.2. All Möbius ladders are circulants. A prism P_k is a circulant whenever k is odd. Meanwhile $X(k)$ and $Y(k)$ are bicirculants if and only if k is odd and $3 \nmid k$. In this case, $X(k)$ is isomorphic to the generalized Petersen graph $GP(3k, k + (-1)^\alpha)$ while $Y(k)$ is isomorphic to the graph $H(3k, 1, \alpha k + 2)$, where $\alpha \in \{1, 2\}$ and $\alpha \equiv k \pmod{3}$ (see [17] for definitions pertaining to bicirculant graphs).

Remark 1.3. $Y(k)$ can be embedded on the torus yielding the toroidal map $\{6, 3\}_{\alpha, 3}$ where $\alpha = \frac{1}{2}(k - 1)$ (see [5] for the definitions pertaining to the maps on the torus).

Remark 1.4. For $k \geq 7$, the graphs $X(k)$ and $Y(k)$ have girth 8 and 6, respectively. Meanwhile, prisms and Möbius ladders have girth 4. The graph $X(k)$ is not bipartite for any integer k , while $Y(k)$ is bipartite for all odd $k > 1$.

Remark 1.5. There are exactly four cubic arc-transitive trirculants: $K_{3,3}$, the Pappus graph, Tutte's 8-cage and the graph with 54 vertices denoted F054A in [1]. All other cubic vertex-transitive trirculants have two edge orbits and their arc-type is $2 + 1$ (see [4] for details). There are no semi-symmetric cubic trirculants.

2 Diagrams

Even though this paper is about simple graphs, it will be convenient in the course of the analysis to have a slightly more general definition, which allows the graphs to have loops, parallel edges and semi-edges. To distinguish between a simple graph (which we will refer to simply as a graph) and these more general objects, that we shall call *pregraphs*. In what follows, we briefly introduce the concept of a pregraph and refer the reader to [11, 12] for more detailed explanation.

A *pregraph* is an ordered 4-tuple $(V, D; \text{beg}, \text{inv})$ where D and $V \neq \emptyset$ are disjoint finite sets of *darts* and *vertices*, respectively, $\text{beg}: D \rightarrow V$ is a mapping which assigns to each dart x its *initial vertex* $\text{beg } x$, and $\text{inv}: D \rightarrow D$ is an involution which interchanges every dart x with its *inverse dart*, also denoted by x^{-1} . The *neighbourhood* of a vertex v is defined as the set of darts that have v for its initial vertex and the *valence* of v is the cardinality of the neighbourhood. Note that a simple graph Γ can be represented as a pregraph by letting $V = V(\Gamma)$ be the vertex-set of Γ , letting $D = \{(u, v) : u, v \in V(\Gamma), u \sim_{\Gamma} v\}$ be the arc-set of Γ , and by letting $\text{beg}(u, v) = u$ and $\text{inv}(u, v) = (v, u)$. Notions such as morphism, isomorphism and automorphism of pregraphs are obvious generalisations of those for (simple) graphs and precise definitions can be found in [11, 12].

The orbits of inv are called *edges*. The edge containing a dart x is called a *semi-edge* if $\text{inv } x = x$, a *loop* if $\text{inv } x \neq x$ while $\text{beg}(x^{-1}) = \text{beg } x$, and is called a *link* otherwise. The *endvertices of an edge* are the initial vertices of the darts contained in the edge. Two links are *parallel* if they have the same endvertices. When we present a pregraph as a drawing, the links are drawn in the usual way as a line between the points representing its endvertices, a loop is drawn as a closed curve at its unique endvertex and a semi-edge is drawn as a segment attached to its unique endvertex. The following lemma, which will serve as the starting point of our analysis of cubic tricirculants, is an easy exercise illustrating the definition of a pregraph.

Lemma 2.1. *Up to isomorphism there are precisely four non-isomorphic cubic pregraphs on three vertices such that no vertex is the initial vertex of more than one semi-edge, namely the pregraphs $\Delta_1, \Delta_2, \Delta_3$ and Δ_4 depicted in Figure 1.*

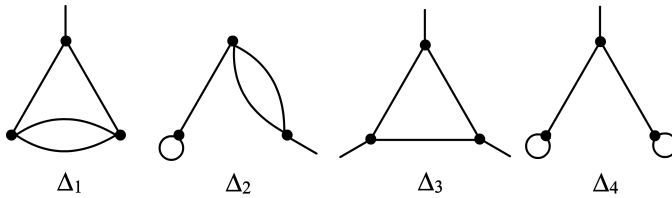


Figure 1: Cubic pregraphs on three vertices without parallel semi-edges.

Given a pregraph Γ and a group $H \leq \text{Aut}(\Gamma)$ we define the quotient Γ/H to be the pregraph $(V', D'; \text{beg}', \text{inv}')$ where V' and D' are the sets $V(\Gamma)/H$ and $D(\Gamma)/H$ of orbits of H on V and D , respectively, and beg' and inv' are defined by $\text{beg}'(x^H) = (\text{beg } x)^H$ and $\text{inv}'(x^H) = (\text{inv } x)^H$ for every $x \in D(\Gamma)$. The mapping $\wp_H: V(\Gamma) \cup D(\Gamma) \rightarrow V' \cup D'$ that maps a vertex or a dart x to its H -orbit x^H is then an epimorphism of pregraphs, called *the quotient projection with respect to H* . If H acts semiregularly on $V(\Gamma)$ (that is, if the vertex-stabiliser H_v is trivial for every vertex $v \in V(\Gamma)$), then the quotient projection \wp_H

is a *regular covering projection* and in particular, it preserves the valence of the vertices. Moreover, in this case the graph Γ is isomorphic to the *derived covering graph* of Γ/H , a notion which we now define.

Let $\Gamma' = (V', D', \text{beg}', \text{inv}')$ be an arbitrary connected pregraph, let N be a group and let $\zeta: D' \rightarrow N$ be a mapping (called a *voltage assignment*) satisfying the condition $\zeta(x) = \zeta(\text{inv}' x)^{-1}$ for every $x \in D'$. Then $\text{Cov}(\Gamma', \zeta)$ is the graph with $D' \times N$ and $V' \times N$ as the sets of darts and vertices, respectively, and the functions beg and inv defined by $\text{beg}(x, a) = (\text{beg}' x, a)$ and $\text{inv}(x, a) = (\text{inv}' x, a\zeta(x))$. If, for a voltage assignment $\zeta: D(\Gamma') \rightarrow N$, there exists a spanning tree T in Γ' such that $\zeta(x)$ is the trivial element of N for every dart x in T , then we say that ζ is *normalised*; note that in this case $\text{Cov}(\Gamma', \zeta)$ is connected if and only if the images of ζ generate N . The following lemma is a well-known fact in the theory of covers:

Lemma 2.2. *Let H be a group of automorphisms acting semiregularly on the vertex-set of a connected graph Γ , let $\Gamma' = \Gamma/H$ and let T be a spanning tree in Γ' . Then there exists a voltage assignment $\zeta: D(\Gamma') \rightarrow N$ which is normalised with respect to T and such that $\Gamma \cong \text{Cov}(\Gamma', \zeta)$.*

Let us now move our attention back to cubic trirculants. Consider the voltage assignments $\zeta_i: \Delta_i \rightarrow \mathbb{Z}_{2k}$ given in Figure 2 where the value $\zeta_i(x)$ is written next to the drawing of each dart $x \in D(\Delta_i)$.

Lemma 2.3. *Let Γ be a cubic trirculant, let ρ be a corresponding trirculant automorphism and let n be the order of ρ . Then $n = 2k$ for some positive integer k and there exist elements $r, s \in \mathbb{Z}_n$ and $i \in \{1, 2, 3, 4\}$ such that $\Gamma \cong \text{Cov}(\Delta_i, \zeta_i)$ where $\zeta_i: D(\Delta_i) \rightarrow \mathbb{Z}_n$ is the voltage assignment defined by Figure 2.*

Proof. Note first that the quotient $\Gamma/\langle\rho\rangle$ is a pregraph with three vertices of valency 3, one for each orbit under the action of $\langle\rho\rangle$. Since ρ is a semiregular automorphism of Γ of order n , the group $\langle\rho\rangle$ is isomorphic to \mathbb{Z}_n and acts semiregularly on $V(\Gamma)$. By Lemma 2.2, $\Gamma \cong \text{Cov}(\Gamma/\langle\rho\rangle, \zeta)$ for some voltage assignment $\zeta: D(\Gamma/\langle\rho\rangle, \mathbb{Z}_n)$. Note that by definition of voltage assignments, it follows that $\zeta(x) = -\zeta(x^{-1})$ for every $x \in D(\Gamma/\langle\rho\rangle)$, implying that if x is a semi-edge, then $\zeta(x)$ is an element of order at most 2 in \mathbb{Z}_n , and since Γ has no semi-edges, $\zeta(x)$ must in fact have order 2 in \mathbb{Z}_n . Since every pregraph with three vertices in which every vertex has valence 3 contains at least one semi-edge, it follows that $n = 2k$ for some positive integer k , and moreover, $\zeta(x) = k$ for every semi-edge x of $\Gamma/\langle\rho\rangle$. Since Γ has no parallel edges, this also implies that every vertex of $\Gamma/\langle\rho\rangle$ is the initial vertex of at most one semi-edge. By Lemma 2.1, $\Gamma/\langle\rho\rangle \cong \Delta_i$ for some $i \in \{1, 2, 3, 4\}$ and we may thus assume that $\Gamma \cong \text{Cov}(\Delta_i, \zeta_i)$ for some $i \in \{1, 2, 3, 4\}$ and some voltage assignment $\zeta_i: \Delta_i \rightarrow \mathbb{Z}_n$. Finally, in view of Lemma 2.2, we may assume that $\zeta_i(x) = 0$ for every edge x belonging to a chosen spanning tree of Δ_i . In particular, ζ_i can be chosen as shown in Figure 2. \square

A cubic trirculant isomorphic to $\text{Cov}(\Delta_i, \zeta_i)$, $i \in \{1, 2, 3, 4\}$, is said to be of *Type i* and is denoted by $T_i(k, r, s)$, if $i \neq 3$, or $T_3(k, r)$. From Lemma 2.3 we get the following characterization of connected cubic trirculants.

Theorem 2.4. *A cubic graph Γ is a trirculant if and only if $n = 6k$ and it is isomorphic to $T_i(k, r, s)$ or $T_3(k, r)$, for some $i \in \{1, 2, 4\}$ and $r, s \in \mathbb{Z}_{2k}$. Furthermore, $T_i(k, r, s)$*

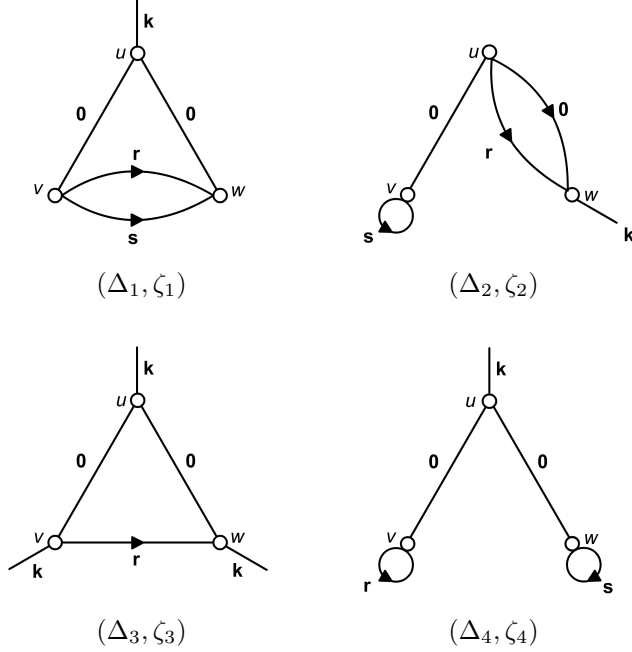


Figure 2: The voltage assignments giving rise to cubic tricirculants.

is connected if and only if $\gcd(2k, k, r, s) = 1$ while $T_3(k, r)$ is connected if and only if $\gcd(2k, k, r) = 1$.

Note that in principle, a cubic tricirculant could be of more than one type. However, each vertex-transitive cubic tricirculant is of Type i for exactly one $i \in \{1, 2, 3, 4\}$, with the exception of the triangular prism P_3 which is both of Type 1 as well as of Type 3.

The following sections are devoted to the analysis of the tricirculants graphs arising from the voltage assignments ζ_i , and in particular, to determining sufficient and necessary conditions for vertex-transitivity. We will consider the graphs of order at most 48 separately in Section 3 and as for the graphs of larger order, we will show that no graph of Type 4 is vertex-transitive, that Type 3 yields prisms and Möbius ladders, and that Types 1 and 2 each yield one infinite family of vertex-transitive graphs, namely the graphs $X(k)$ defined in Definition 4.1 and the graphs $Y(k)$ defined in Definition 5.1.

Finally, to facilitate discussion in the forthcoming sections, we specify the notion of a walk. A walk ω in a (pre)graph is a sequence of darts (x_1, x_2, \dots, x_n) such that $\text{beg}(x_{i+1}) = \text{beg}(x_i^{-1})$ for all $i \in \{1, \dots, n-1\}$; that is, the initial vertex of x_{i+1} is the end vertex of x_i . We say that ω is closed if the initial vertex of x_1 is the endvertex of x_n . If additionally $\text{beg}(x_i) \neq \text{beg}(x_j)$ for all $i \neq j$, then we say that ω is a cycle. We define the inverse of ω as the walk $\omega^{-1} = (x_n^{-1}, x_{n-1}^{-1}, \dots, x_1^{-1})$. We say that a walk ω is a *reduced walk* if no two consecutive arcs are inverse to one another (if ω is closed, we will consider the first arc and the last arc of ω to be consecutive). It is pertinent to point out that, from this definition, all walks and cycles are directed and have an initial vertex. Therefore, no reduced walk is equal to its inverse.

3 Graphs of small order

To make this classification as general and as neat as possible, we will treat cubic tricirculant graphs of small order separately, as if we allow graphs that are “too small”, special cases and exceptions will inevitably occur. Since a cubic tricirculant must have order $6k$, for some positive k , we present, in Table 1, the complete list of all cubic vertex-transitive tricirculants of order at most 48 obtained from the census [18] of cubic vertex-transitive graphs. Notice that with the exception of the truncated tetrahedron, $\text{Tr}(K_4)$, of order 12 and Tutte’s 8-cage, T8, of order 30 (see Figure 3), all vertex-transitive cubic tricirculants of order at most 48 are isomorphic to either $X(k)$, $Y(k)$, P_k (the prism of order $2k$) or M_k (the Möbius ladder of order $2k$) for some integer k .

Table 1: Graphs of small order.

Order	Type	Graph	Order	Type	Graph	Order	Type	Graph
6	1, 3	P_3	18	3	M_9	36	3	M_{18}
6	3	M_3	24	3	M_{12}	42	1	$X(7)$
12	1	$\text{Tr}(K_4)$	30	1	$X(5)$	42	2	$Y(7)$
12	3	M_6	30	2	$Y(5)$	42	3	P_{21}
18	1	$X(3)$	30	3	P_{15}	42	3	M_{21}
18	2	$Y(3)$	30	3	M_{15}	48	3	M_{24}
18	3	P_9	30	4	T8			

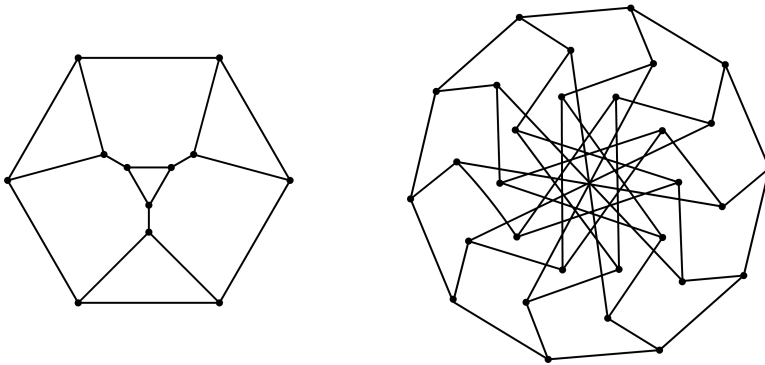


Figure 3: The truncated tetrahedron $\text{Tr}(K_4)$ and Tutte’s 8-cage.

The graphs $K_{3,3}$, $Y(3)$ (the Pappus graph) and Tutte’s 8-cage are arc-transitive. The fourth arc-transitive cubic tricirculant, denoted F054A in [1], does not appear in Table 1, as it has 54 vertices; it corresponds to the graph $Y(9)$ defined in Section 5. The four aforementioned graphs are the only edge-transitive cubic tricirculants. Indeed, a cubic graph that is both vertex- and edge-transitive is automatically arc-transitive [22]. A graph that is edge-transitive but not vertex-transitive is called a semi-symmetric graph, and must have two orbits of vertices under its full automorphism group [6]. It follows that no tricirculant can be semi-symmetric, as its tricirculant automorphism would mix the two vertex orbits, making the graph vertex-transitive.

For the remainder of this paper, all cubic tricirculants will have order at least 54.

4 Type 1

Let $k \geq 9$ be an integer, and let r and s be two distinct elements of \mathbb{Z}_{2k} . Recall that $T_1(k, r, s)$ is the covering graph arising from (Δ_1, ζ_1) shown in Figure 2; for convenience, we repeat the drawing here (see Figure 4).

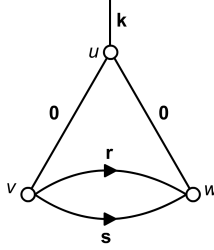


Figure 4: The voltage assignment giving rise to the graph $T_1(k, r, s)$.

By the definition of a derived cover, the vertices of $T_1(k, r, s)$ are then the pairs (x, i) where $x \in \{u, v, w\}$ is a vertex of Δ_1 and $i \in \mathbb{Z}_{2k}$. Note that $T_1(k, r, s)$ is connected if and only if k, r and s generate \mathbb{Z}_{2k} , that is, if and only if $\gcd(k, r, s) = 1$.

To simplify notation, we write u_i instead of (u, i) and similarly for v_i and w_i . Then $U = \{u_0, u_1, \dots, u_{2k-1}\}$, $V = \{v_0, v_1, \dots, v_{2k-1}\}$ and $W = \{w_0, w_1, \dots, w_{2k-1}\}$ are the respective fibres of vertices u, v and w , and the edge-set of $T_1(k, r, s)$ is then the union of the sets

$$\begin{aligned} E_K &= \{u_i u_{i+k} : i \in \mathbb{Z}_{2k}\}, \\ E_0 &= \{u_i v_i : i \in \mathbb{Z}_{2k}\} \cup \{u_i w_i : i \in \mathbb{Z}_{2k}\}, \\ E_R &= \{v_i w_{i+r} : i \in \mathbb{Z}_{2k}\}, \\ E_S &= \{v_i w_{i+s} : i \in \mathbb{Z}_{2k}\}. \end{aligned}$$

Definition 4.1. For an odd integer $k > 1$, let

$$r^* = \begin{cases} \frac{k+3}{2}, & \text{if } k \equiv 1 \pmod{4} \\ \frac{k+3}{2} + k, & \text{if } k \equiv 3 \pmod{4} \end{cases}$$

and let $X(k) = T_1(k, r^*, 1)$.

We can now state the main theorem of this section.

Theorem 4.2. *Let $k \geq 9$ and let Γ be a connected cubic tricirculant of Type 1 with $6k$ vertices. Then the following holds:*

- (1) Γ is vertex-transitive if and only if it is isomorphic to $X(k)$ with k odd.
- (2) If Γ is not vertex-transitive, then it has two vertex orbits under its full automorphism group, one twice the size of the other.

The next theorem gives more information about the graph $X(k)$.

Theorem 4.3. *Let k be an odd integer; $k > 1$. Then the following holds:*

- (1) $X(k)$ is a bicirculant if and only if $3 \nmid k$, in which case it is isomorphic to the generalized Petersen graph $GP(3k, k + (-1)^\alpha)$ where $\alpha \in \{1, 2\}$ and $\alpha \equiv k \pmod{3}$.
- (2) $X(k)$ has arc-type $2 + 1$.

The rest of the section is devoted to the proof of Theorems 4.2 and 4.3. Clearly, as a tricirculant of Type 1, Γ is isomorphic to $T_1(k, r, s)$ for some integer k and elements $r, s \in \mathbb{Z}_{2k}$ (see Lemma 2.3). We shall henceforth assume that $\Gamma = T_1(k, r, s)$ for some $k \geq 9$ and that Γ is connected. For a symbol X from the set of symbols $\{0, R, S, K\}$, edges in E_X will be called edges of type X , or simply X -edges.

Define ρ as the permutation given by $\rho(u_i) = u_{i+1}$, $\rho(v_i) = v_{i+1}$ and $\rho(w_i) = w_{i+1}$, and note that ρ is a tricirculant automorphism of $T_1(k, r, s)$. Further, observe that

$$T_1(k, r, s) \cong T_1(k, s, r), \quad (4.1)$$

and if $a \in \mathbb{Z}$ is such that $\gcd(2k, a) = 1$, then

$$T_1(k, ar, as) \cong T_1(k, r, s).$$

Lemma 4.4. *Let k be an odd integer, $k > 1$, and let r^* and $X(k)$ be as in Definition 4.1. Then the graph $X(k)$ is vertex-transitive.*

Proof. Recall that $X(k) \cong T_1(k, r^*, 1)$. Define ϕ as the mapping given by:

$$\begin{array}{lll} u_i \mapsto w_{i-r^*+2}, & w_i \mapsto v_{i-2r^*+2}, & v_i \mapsto u_{i-r^*+2} \quad \text{if } i \text{ is even;} \\ u_i \mapsto v_{i+r^*-2}, & v_i \mapsto w_{i+2r^*-2}, & w_i \mapsto u_{i+r^*-2} \quad \text{if } i \text{ is odd.} \end{array}$$

That ϕ is indeed an automorphism follows from the fact that k is odd, r^* is even and the congruence $k + 3 - 2r^* \equiv 0$ holds. Since ϕ transitively permutes the three $\langle \rho \rangle$ -orbits U , V and W , the group $\langle \rho, \phi \rangle$ acts transitively on the vertices of $T_1(k, r^*, 1)$. \square

Lemma 4.5. *For $k > 1$, $X(k)$ has arc-type $2 + 1$.*

Proof. Let G be the full automorphism group of $X(k)$ and consider the automorphism φ interchanging u_i with u_{-i} and v_i with w_{-i} , $i \in \mathbb{Z}_{2k}$. Note that φ is an element of G_{u_0} , the stabiliser of u_0 . Moreover, φ fixes u_k but interchanges v_0 with w_0 . This is, φ fixes one of the neighbours of u_0 while interchanging the remaining two. This implies that the arc-type of $X(k)$ is either 3, when $X(k)$ is arc-transitive, or $2 + 1$. However, there are no arc-transitive graphs of Type 1. The result follows. \square

Remark 4.6. The automorphism given in the proof above is an automorphism of $\Gamma \cong T_1(k, r, s)$, from which we see that if Γ is not vertex-transitive, then $V \cup W$ is a vertex orbit under its full automorphism group $\text{Aut}(\Gamma)$. Then Γ must have two vertex orbits: U and $V \cup W$. This shows that item (2) of Theorem 4.2 holds.

Now, recall that Γ is connected and equals $T_1(k, r, s)$, for some $k \geq 9$, $r, s \in \mathbb{Z}_{2k}$. We may also assume that Γ is vertex-transitive, however, for some of the results in this section vertex-transitivity is not needed. When possible, we will not assume the graph to be vertex-transitive, but rather only to have some weaker form of symmetry, that we will define in the following paragraphs.

A simple graph Γ is said to be *c-vertex-regular*, for some $c \geq 3$, if there are the same number of c -cycles through every vertex of Γ .

For an edge e of Γ and a positive integer c denote by $\epsilon_c(e)$ the number of c -cycles that pass through e . For a vertex v of Γ , let $\{e_1, e_2, e_3\}$ be the set of edges incident to v ordered in such a way that $\epsilon_c(e_1) \leq \epsilon_c(e_2) \leq \epsilon_c(e_3)$. The triplet $(\epsilon_c(e_1), \epsilon_c(e_2), \epsilon_c(e_3))$ is then called the *c-signature* of v . If for $c \geq 3$ all vertices in Γ have the same c -signature, then we say Γ is *c-cycle-regular* and the signature of Γ is the signature of any of its vertices. Note that for every $c \geq 3$, a vertex-transitive graph is necessarily c -cycle-regular, and every c -cycle-regular graph is c -vertex-regular.

If c equals the girth of the graph, then following [19], a c -cycle-regular graph will be called *girth-regular*.

Lemma 4.7. *If Γ is 4-vertex-regular, then neither r nor s equals k .*

Proof. Suppose $r = k$. Since Γ has no parallel edges, we see that $s \neq k$. Observe that (u_0, v_0, w_r, u_r) and $(u_0, w_0, v_{-r}, u_{-r})$ are the only 4 cycles of Γ through u_0 .

Meanwhile, there exists a unique 4-cycle through v_0 , namely (v_0, w_r, u_r, u_0) , which contradicts Γ being 4-vertex-regular. Hence $r \neq k$, and since $T_1(k, s, r) \cong T_1(k, r, s)$ (see Equation (4.1)), this also shows that $s \neq k$. \square

For the sake of simplicity denote a dart in Δ_1 starting at vertex a , pointing to vertex b and having voltage x by $(ab)_x$. Recall that a walk in Δ_1 is a sequence of darts (x_1, x_2, \dots, x_n) , for instance $((vw)_r, (wu)_0, (uw)_k)$ is a walk of length 3. However, since Γ is a simple graph a walk in Γ will be denoted as a sequence of vertices, as it is normally done.

Lemma 4.8. *If Γ is 8-cycle-regular, then neither r nor s equals 0.*

Proof. Suppose $r = 0$. Since Γ has no parallel edges, we see that $s \neq 0$. Suppose there is an 8-cycle C in Γ . Such a cycle, when projected to Δ_1 , yields a reduced closed walk ω in Δ_1 whose ζ_1 -voltage is 0. Note that ω cannot trace three darts with voltage 0 consecutively, as this would lift into a 3-cycle contained in C . This implies ω necessarily visits the dart $(vw)_s$ or its inverse at least once. Furthermore, since $\gcd(k, s) = 1$ and $8 < 9 \leq k$, ω must trace $(wv)_{-s}$ as many times as it does $(vw)_s$. By observing Figure 4, the reader can see that if ω traces the dart $(vw)_s$, then it must also trace the semi-edge $(uu)_k$ before tracing $(wv)_{-s}$, as ω cannot trace three consecutive darts with voltage 0. Since ω has net voltage 0, it necessarily traces $(uu)_k$ an even number of times. Moreover, ω must trace a dart in $\{(uv)_0, (vu)_0, (uw)_0, (wu)_0\}$ immediately before and immediately after tracing $(uu)_k$. Hence, ω must visit the set $\{(vw)_s, (wv)_{-s}\}$ at least twice; the semi-edge $(uu)_k$ at least twice; and the set $\{(uv)_0, (vu)_0, (uw)_0, (wu)_0\}$ at least 4 times. This already amounts to 8 darts, none of which is $(vw)_r$ or its inverse. Therefore, no 8-cycle in Γ visits an R -edge. However, for $X \neq R$ there is at least one 8-cycle through every X -edge, as $(u_0, v_0, w_s, u_s, u_{s+k}, w_{s+k}, v_k, u_k)$ and $(u_0, w_0, v_{-s}, u_{-s}, u_{k-s}, v_{k-s}, w_k, u_k)$ are 8-cycles in Γ . This contradicts our hypothesis of Γ being 8-cycle-regular. The proof when $s = 0$ is analogous. \square

Now, observe that $(u_0, u_k, v_k, w_{k+r}, u_{k+r}, u_r, w_r, v_0, u_0)$ is a cycle of length 8 in Γ starting at u_0 . In what follows we will study the 8-cycle structure of Γ to determine conditions for vertex-transitivity. Recall that each 8-cycle in Γ quotients down into a closed walk

of length 8 having net voltage 0 in Δ_1 . We can thus, provided we are careful, determine how many 8-cycles pass through any given vertex of Γ by counting closed walks of length 8 and net voltage 0 in the quotient Δ_1 . Note that only closed walks that are reduced will lift into cycles of Γ . Therefore, we may safely ignore non-reduced walks and focus our attention exclusively on reduced ones. Define \mathcal{W}_8 as the set of all reduced closed walks of length 8 in Δ_1 .

Every element of \mathcal{W}_8 having net voltage 0 lifts into a closed walk of length 8 in Γ . In principle, the latter walk might not be a cycle of Γ . However, we will show that this never happens in our particular case. For this, it suffices to show that Γ does not admit cycles of length 4 or smaller, as any closed walk of length 8 that is not a cycle is a union of smaller cycles, one of which will have length smaller or equal to 4.

Lemma 4.9. Γ has girth at least 5.

Proof. Suppose Γ has a 3-cycle. Since Γ is vertex-transitive, this would mean that there is a reduced closed walk with net voltage 0 of length 3 in Δ_1 that visits u . From Figure 4 we get that such a closed walk must be either $((uv)_0, (vw)_r, (wu)_0)$, $((uv)_0, (vw)_s, (wu)_0)$ or one of their inverses. This would imply that either $r = 0$ or $s = 0$, a contradiction. Similarly, the existence of 4-cycles in Γ would imply there is a closed walk with net voltage 0 of length 4 visiting u . The only such walks are

$$\begin{aligned} &((uv)_0, (vw)_r, (wu)_0, (uu)_k), & ((uu)_k, (uv)_0, (vw)_r, (wu)_0), \\ &((uv)_0, (vw)_s, (wu)_0, (uu)_k), & ((uv)_0, (vw)_s, (wu)_0, (uu)_k) \end{aligned}$$

and their inverses. In all eight cases, we have that $r = k$ or $s = k$, contradicting Lemma 4.7. \square

Now, let N be the set of the net voltages of walks in \mathcal{W}_8 , expressed as linear combinations of k , r and s , where we view r and s as indeterminants over \mathbb{Z}_{2k} . For instance, the closed walk $((uv)_0, (vw)_r, (vw)_{-s}, (vw)_r, (wu)_0, (uv)_0, (vw)_r, (wu)_0)$ has net voltage $3r - s$. For an element $\nu \in N$, let $\mathcal{W}(\nu)$ be the set of walks in \mathcal{W} with net voltage ν and for a vertex x of Δ_1 , let $\mathcal{W}_8^x(\nu)$ be the set of walks in $\mathcal{W}(\nu)$ starting at x .

For each $\nu \in N$, we have computed the number of elements in $\mathcal{W}_8^u(\nu)$ and in $\mathcal{W}_8^v(\nu)$. The result is displayed in Table 2. Notice that, if $\omega \in \mathcal{W}_8(\nu)$, then $\omega^{-1} \in \mathcal{W}_8^x(-\nu)$. This means ω has voltage 0 if and only if ω^{-1} does too. For this reason, and since we are interested in walks with net voltage 0, we have grouped walks with net voltage ν along with their inverses having net voltage $-\nu$. This computation is straightforward but somewhat lengthy. For this reason and to avoid human error, we have done it with the help of a computer programme written in SageMath [21].

From the first Row of Table 2 we know that there are always 12 walks starting at u in \mathcal{W}_8 that have net voltage 0, regardless of the values of r and s . Similarly, there are always 10 such walks starting at v . Since Γ is vertex-transitive, the number of walks in \mathcal{W}_8 having net voltage 0 (after evaluating r and s in \mathbb{Z}_{2k}) starting at u and those starting at v must be the same. It follows that, for some $\nu \in N$ with $|\mathcal{W}_8^v(\nu)| > |\mathcal{W}_8^u(\nu)|$, the equation $\nu \equiv 0 \pmod{2k}$ holds when we evaluate r and s in \mathbb{Z}_{2k} . We thus see that at least one expression in VIII–XIII is congruent to 0 modulo $2k$. In fact, if Γ is vertex-transitive, then at most one of these expressions can be congruent to 0.

Table 2: Net voltages of closed walks of length 8 in Δ_1 .

Label	Net voltage	Starting at u	Starting at v
I	0	12	10
II	$\pm(2r)$	8	8
III	$\pm(2s)$	8	8
IV	$\pm(r + s)$	8	4
V	$\pm(r - s)$	8	4
VI	$\pm(k + 2r - s)$	12	10
VII	$\pm(k + 2s - r)$	12	10
VIII	$\pm(k + 3r - 2s)$	4	6
IX	$\pm(k + 3s - 2r)$	4	6
X	$\pm(3r - s)$	4	6
XI	$\pm(3s - r)$	4	6
XII	$\pm(2r - 2s)$	4	6
XIII	$\pm(4r - 4s)$	0	2

Lemma 4.10. *If Γ is 4-vertex-regular and 8-cycle-regular, then exactly one of the following equations modulo $2k$ holds:*

$$3s - 2r + k \equiv 0, \quad (4.2)$$

$$3r - 2s + k \equiv 0, \quad (4.3)$$

$$3r - s \equiv 0, \quad (4.4)$$

$$3s - r \equiv 0, \quad (4.5)$$

$$4r - 4s \equiv 0. \quad (4.6)$$

Proof. We will slightly abuse notation and, for a label $x \in \{\text{I, II, } \dots, \text{XIII}\}$, we will refer by x to the congruence equation modulo $2k$ obtained by making the expression labelled x in Table 2 congruent to 0. For instance, the equation $2r \equiv 0$ will be referred to as II.

First note that if II or III holds then $r = k$ or $s = k$, contradicting Lemma 4.7. If V holds then (v_0, w_r, v_0) is a 2-cycle in Γ , which is not possible. Similarly, XII implies the existence of a 4-cycle in Γ , contradicting Lemma 4.9. Now, if XIII holds, then either $2r - 2s \equiv 0$, which is not possible, or $2r - 2s + k \equiv 0$, which gives Equation (4.6).

If VI holds, then neither X nor XI can hold as this would imply $2r - 2s \equiv 0$ (subtracting IV from X or from XI, respectively). Therefore, IV excludes X and XI. If IV and I are the only equations to hold, we would have 6 more elements of \mathcal{W}_8 through u than through v . This means that if IV holds then necessarily VII, IX and XII also hold. However, XII implies $2r - 2s + k \equiv 0$ and subtracting this from VIII we get $r = 0$, in contradiction with Lemma 4.8. Hence, VII, IX and XII cannot all hold at the same time and so IV can never hold.

Suppose VI holds, then one of the equations in $\{\text{VIII, IX, X, XI, XIII}\}$ must also hold. Note that VI and VIII imply III; VI and IX imply V; VI and X imply $r = k$; VI and XIII imply $s = k$. Therefore, by Lemma 4.7, only XI can hold. But VI and XI imply VII, and so we would have 40 elements of \mathcal{W}_8 starting at u but only 36 starting at v . This would contradict Γ being 8-cycle-regular. Thus VI cannot hold. An analogous reasoning, where the roles of r and s are interchanged, shows that VII cannot hold. We have shown that

the only equations that can hold are in $\{\text{VIII}, \text{IX}, \text{X}, \text{XI}, \text{XIII}\}$. However, if two or more of these equations hold, then we would have more elements of \mathcal{W}_8 through v than we would have through u . It follows that exactly one equation in $\{\text{VIII}, \text{IX}, \text{X}, \text{XI}, \text{XIII}\}$ holds. \square

Lemma 4.10 tells us, for each of the 5 possible equations, exactly which walks in \mathcal{W}_8 will lift to 8-cycles and so we can count exactly how many 8-cycles pass through a given vertex or edge of Γ . For instance, if $3s - 2r + k \equiv 0$, then there are 16 walks in \mathcal{W}_8 through u that will lift to an 8-cycle (see Table 2). Since a closed walk in \mathcal{W}_8 and its inverse lift to the same cycle in Γ , we see that there are 8 cycles of length 8 through every vertex in Γ . Moreover, there are exactly 5 cycles of length 8 through every edge of type R or θ while there are 6 such cycles through each edge of type K or S . It follows that the 8-signature of a vertex x of Γ is $(5, 5, 6)$ whenever $3s - 2r + k \equiv 0$ and thus, in this case, Γ is 8-cycle-regular with 8-signature $(5, 5, 6)$. Table 3 shows the number of 8-cycles through each vertex and through each edge, depending on its type, for each of the 5 cases described in Lemma 4.10.

Table 3: Number of 8-cycles through each edge-type of $T_1(k, r, s)$.

Congruence	θ -edge	R -edge	S -edge	K -edge	8-signature
$3s - 2r + k$	5	5	6	6	$(5, 5, 6)$
$3r - rs + k$	5	6	5	6	$(5, 5, 6)$
$3r - s$	6	6	4	4	$(4, 6, 6)$
$3s - r$	6	4	6	4	$(4, 6, 6)$
$4r - 4s$	4	4	4	4	$(4, 4, 4)$

We will now show that Equations (4.4), (4.5) and (4.6) cannot hold when Γ is vertex-transitive. This will be proved in Lemmas 4.12, 4.13 and 4.14. But first we need to show a result about the subgraph induced by the θ - and R -edges.

Recall that $\Gamma = T_1(k, r, s)$ with $k \geq 9$ and that it is connected. Henceforth we also assume that Γ is vertex-transitive. Denote by $\Gamma_{\theta, R}$ the subgraph that results from deleting the edges of type S and K from Γ . Equivalently, $\Gamma_{\theta, R}$ is the subgraph induced by θ - and R -edges. We see that $\Gamma_{\theta, R}$ is 2-valent and that it has $\gcd(2k, r)$ connected components, each of which is a cycle of length $6k/\gcd(2k, r)$.

Since $\Gamma_{\theta, R}$ only has edges of type θ and R , any two vertices in the same connected components must have indices that differ in a multiple of r . It is not hard to see that, indeed, each connected component consist precisely of all those vertices whose indices are congruent modulo $\gcd(2k, r)$. We now prove an auxiliary result that will be used both in the case where Equation (4.4) as well as in the case where Equation (4.2) or Equation (4.3) holds.

Suppose that no automorphism of Γ maps an S -edge to a θ -edge. Since Γ is vertex-transitive, there is an automorphism that maps a vertex from V to a vertex in U . Such an automorphism then maps an S -edge to a K -edge. Since all S -edges as well as all the K -edges are in the same orbit, this then implies that $E_S \cup E_K$ forms a single edge-orbit of $\text{Aut}(\Gamma)$.

Lemma 4.11. *Suppose that no automorphism of Γ maps an S -edge to a θ -edge, then $\Gamma_{\theta, R}$ is disconnected, and the set of connected components induce a block system for the vertex-set of Γ .*

Proof. By the discussion above the lemma, $E_S \cup E_K$ forms a single edge-orbit in Γ . Now note that $\text{Aut}(\Gamma)$ also acts as a group of automorphism on $\Gamma_{\theta,R}$ and that this action is transitive, since the edges removed consist of an edge-orbit of Γ . It follows that connected components of $\Gamma_{\theta,R}$ form a block system for the vertex set of Γ . Now, suppose $\Gamma_{\theta,R}$ consists of a single cycle of length $6k$, $(w_0, u_0, v_0, w_r, u_r, v_r, \dots, v_{(2k-1)r})$. Notice that the vertex antipodal to u_0 in this cycle is u_{kr} which is the same as u_k , as r is odd. We see that edges of type K in Γ join vertices that are antipodal in $\Gamma_{\theta,R}$, while S -edges do not. Since this $6k$ -cycle is a block of imprimitivity, K -edges can only be mapped into K -edges and thus U is an orbit of $\text{Aut}(\Gamma)$, contradicting the assumption that Γ is vertex-transitive. Hence, $\Gamma_{\theta,R}$ is disconnected. \square

Lemma 4.12. *If $3r - s \equiv 0 \pmod{2k}$, $k \geq 9$, then $T_1(k, r, s)$ is not vertex-transitive.*

Proof. Let $\Gamma = T_1(k, r, s)$ where $3r - s \equiv 0 \pmod{2k}$. A computer assisted counting shows that S -edges and K -edge have exactly 4 distinct 8-cycles passing through them, while θ - and R -edges have 6 such cycles (see Table 3). It follows that any automorphism sending a vertex $u \in U$ to a vertex $v \in V$ must necessarily map the K -edge incident to u into the S -edge incident to v . Thus $E_S \cup E_K$ is an orbit of edges under the action of $\text{Aut}(\Gamma)$.

Since we are assuming that $3r - s \equiv 0$ we get that any number dividing k and r must also divide s , but since $\gcd(k, r, s) = 1$, we have that $\gcd(k, r) = 1$ and then $\gcd(2k, r) \in \{1, 2\}$. In light of Lemma 4.11, $\Gamma_{\theta,R}$ is disconnected, implying $\gcd(2k, r) \neq 1$ and therefore $\gcd(2k, r) = 2$. From the congruence $3r - s \equiv 0$ we also get that r and s must have the same parity thus making s even and k odd. Recall that each connected component of $\Gamma_{\theta,R}$ consists of all the vertices whose indices are congruent modulo $\gcd(2k, r)$, so $\Gamma_{\theta,R}$ has two connected components: one containing all the vertices with even index, and the other containing those with odd index. Since s is even and k is odd, S -edges in Γ join vertices with same parity, while K -edges join vertices with distinct parity. We know that each of the two connected components of $\Gamma_{\theta,R}$ is a block of imprimitivity of Γ , and so any automorphism of Γ must either preserve the parity of all indices, or of none at all. It follows that no automorphism can send S -edges into K -edges, and therefore no automorphism can send a vertex in V to a vertex in U . We conclude Γ cannot be vertex-transitive. \square

Lemma 4.13. *If $3s - r \equiv 0 \pmod{2k}$, $k \geq 9$, then $T_1(k, r, s)$ is not vertex-transitive.*

Proof. Set $\Gamma = T_1(k, r, s)$ and $\Gamma' = T_1(k, r', s')$, where $r' = s$ and $s' = r$. Notice that the triplet (k, r', s') satisfies Equation (4.4) and so, by Lemma 4.12, Γ' cannot be vertex-transitive. By observation (4.1), $\Gamma \cong \Gamma'$. Therefore Γ is not vertex-transitive. \square

Lemma 4.14. *If $4r - 4s \equiv 0 \pmod{2k}$, $k \geq 9$, then $T_1(k, r, s)$ is not vertex-transitive.*

Proof. Let $\Gamma = T_1(k, r, s)$, with $4r - 4s \equiv 0 \pmod{2k}$ and $k \geq 9$. From the congruence $4r - 4s \equiv 0$, we see that either $2r - 2s \equiv 0$ or $2r - 2s + k \equiv 0$, but the former contradicts Lemma 4.9. Hence $2r - 2s + k \equiv 0$ and the closed walk $((uv)_0, (vw)_r, (wv)_{-s}, (vw)_r, (wv)_{-s}, (vu)_0, (wu)_k)$ in Δ_1 lifts to a 7-cycle in Γ . We will show that Γ cannot be 7-vertex-regular. Following a similar procedure as the one used to obtain Table 2, we have counted all closed walks of length 7 in Δ_1 starting at u and those starting at v , and we have computed their net voltage. The result is displayed in Table 4.

It is plain to see that if, in addition to I, any one of the voltages in Rows III–VI is congruent to 0, then either $r = 0$, $r = k$, $s = 0$ or $s = k$, contradicting Lemmas 4.7 or 4.8.

Table 4: Net voltages of closed walks of length 7 in Δ_1 .

Label	Net voltage	Starting at u	Starting at v
I	$\pm(k + 2r - 2s)$	8	10
II	$\pm(k + r + s)$	12	8
III	$\pm(k + 2r)$	6	4
IV	$\pm(k + 2s)$	6	4
V	$\pm(3r - 2s)$	2	6
VI	$\pm(3s - 2r)$	2	6

We see that only the voltages in Row I or II can hold, and thus Γ is not 7-vertex-regular and hence cannot be vertex-transitive. \square

In what follow we deal with the case where Equation (4.2) holds, that is, when $3s - 2r + k \equiv 0 \pmod{2k}$ and $k \geq 9$. From Table 3 we see that S -edges and K -edge have exactly 6 distinct 8-cycles passing through them, while θ - and R -edges have only 5. In particular, no automorphism of Γ maps an S -edge to a θ -edge, implying that $E_S \cup E_K$ is an edge-orbit of Γ (see the discussion above Lemma 4.11).

Lemma 4.15. *Suppose that $3s - 2r + k \equiv 0 \pmod{2k}$, $k \geq 9$. If e is an edge in $E_K \cup E_S$, then the endpoints of e belong to different connected components of $\Gamma_{\theta,R}$.*

Proof. Suppose a K -edge e has both of its endpoints in the same connected component, C , of $\Gamma_{\theta,R}$. Then, because Γ is vertex-transitive and C is a block of imprimitivity, both of the endpoints of any K -edge must belong to the same connected component of $\Gamma_{\theta,R}$. This means that the subgraph induced by θ -edges, R -edges and K -edges is disconnected. Recall that K -edges and S -edges belong to a single edge-orbit in Γ and thus, for every S -edge, e' , there exists $\phi \in \text{Aut}(\Gamma)$ such $\phi(e) = e'$. Since ϕ also acts as an automorphism on $\Gamma_{\theta,R}$, it follows that the endpoints of $\phi(e)$ are contained in the component $\phi(C)$, thus making Γ a disconnected graph, which is a contradiction. \square

Lemma 4.16. *Suppose that $3s - 2r + k \equiv 0 \pmod{2k}$, $k \geq 9$. Then the subgraph $\Gamma_{\theta,R}$ has an even number of connected components.*

Proof. Let e be a K -edge whose endpoints are in two different connected components C_1 and C_2 . Let $\rho^r \in \text{Aut}(\Gamma)$ be an “ r -fold” rotation. That is, ρ^r adds r to the index of every vertex. It is plain to see that ρ^r fixes the connected components of $\Gamma_{\theta,R}$ set-wise and that it send K -edges into K -edges. Moreover, since it is transitive on $U \cap C_1$, we have that all K -edges having an endpoint in C_1 will have the other endpoint in C_2 . Hence K -edges “pair up” connected components of $\Gamma_{\theta,R}$. The result follows. \square

Proposition 4.17. *If $T_1(k, r, s)$ is connected and vertex-transitive, then the following statements (or the analogous three statements obtained by interchanging r and s) hold:*

- (1) $3s - 2r + k \equiv 0 \pmod{2k}$
- (2) k and s are odd and $\gcd(k, s) = 1$
- (3) r is even and $\gcd(k, r) = \{1, 3\}$.

Proof. First, note that item (1) follows from the combination of Lemma 4.10 and Lemmas 4.12, 4.13 and 4.14.

We will continue by proving (3). Set $d = \gcd(k, r)$, so that $r = dr'$ and $k = dk'$ for some integers k' and r' . Since $3s - 2r + k \equiv 0 \pmod{2k}$, we see that, for some odd integer a :

$$3s = d(ak' + 2r').$$

By connectedness, $\gcd(k, r, s) = 1$, and hence $\gcd(d, s) = 1$ implying that d divides 3. Hence $d = \gcd(k, r) \in \{1, 3\}$. Moreover, since $\Gamma_{\theta, R}$ has $\gcd(2k, r)$ connected components and this number must be even by Lemma 4.16, we see that $\gcd(2k, r) \in \{2, 6\}$, and therefore r is even.

Now, from the congruence $3s - 2r + k \equiv 0 \pmod{2k}$ we see that s and k are either both even, or both odd. Since $\gcd(2k, r, s) = 1$ and r is even, s must necessarily be odd and then k is also odd. Set $d' := \gcd(k, s)$ so that $s = d's'$ and $k = d'k'$ for some integers s' and k' . Again, from Equation (4.2) we obtain

$$2r = d'(ak' + 3s')$$

for some odd integer a . Thus $d' \in \{1, 2\}$, but since s is odd, we see that $\gcd(k, s) = 1$. This proves item (2) and completes the proof. \square

Remark 4.18. Conditions (1)–(3) (or their analogues) in Proposition 4.17 are also sufficient to prove vertex-transitivity of a connected graph $T_1(k, r, s)$. This will be shown after the proof on Lemma 4.20. Moreover, observe that if k and s satisfy condition (2) of Lemma 4.17, then $\gcd(2k, 1) = 1$ and $T_1(k, r, s)$ is isomorphic to $T_1(k, rs^{-1}, 1)$ where s^{-1} is the multiplicative inverse of s in \mathbb{Z}_{2k} . Hence, in what follows we may assume that $s = 1$.

Lemma 4.19. *Let k be an odd number and set $s = 1$. There exists a unique element $r \in \mathbb{Z}_{2k}$ such that conditions (1) and (3) of Proposition 4.17 are satisfied. This unique r equals r^* from Definition 4.1.*

Proof. Suppose that such r exists. From condition (1) of Lemma 4.19, we get $r = (3 - ak)/2$ for some odd integer a . Since r must be a positive integer smaller than $2k$, it follows that $a \in \{-1, -3\}$. Hence $(3 + k)/2$ and $(3 + 3k)/2 = (3 + k)/2 + k$ are the only possible candidates for r . Since k is odd, the integers $(3 + k)/2$ and $(3 + k)/2 + k$ have different parity. Set r to be whichever one of these two expressions is even. Note that any odd number that divides both r and k must divide 3 and that this is true whether r equals $(3 + k)/2$ or $(3 + k)/2 + k$. Thus condition (3) of Lemma 4.19 is satisfied and r equals r^* from Definition 4.1. \square

We have just proved that once an odd $k \geq 9$ is prescribed, then there is at most one graph $T_1(k, r, 1)$ that is connected and vertex-transitive. The following lemma generalizes this to an arbitrary value of s .

Lemma 4.20. *Let $k \geq 9$ be an odd integer and let $T_1(k, r, s)$ be vertex-transitive. Then $T_1(k, r, s) \cong X(k)$.*

Proof. Recall that $X(k) \cong T_1(k, r^*, 1)$, where r^* is as in Definition 4.1. Since $T_1(k, r, s)$ is connected and vertex-transitive, conditions (1)–(3) of Proposition 4.17 hold. In particular,

s is relatively prime to $2k$. Then we have $T_1(k, r, s) \cong T_1(k, s^{-1}r, 1)$, where s^{-1} is the multiplicative inverse of s in \mathbb{Z}_{2k} . It follows by Lemma 4.19 that $T_1(k, s^{-1}r, 1) = T_1(k, r^*, 1)$. \square

Notice that in the previous proof vertex-transitivity is only used to ensure that the graph $T_1(k, r, s)$ satisfies conditions (1)–(3) of Proposition 4.17, and from there $T_1(k, r, s)$ is shown to be isomorphic to $T_1(k, r^*, 1)$. Since we know from Lemma 4.4 that $T_1(k, r^*, 1)$ is vertex-transitive, we have that conditions (1)–(3) of Proposition 4.17 are not only necessary, but also sufficient for vertex-transitivity, and thus Proposition 4.17 may be regarded a characterization theorem.

Remark 4.21. For $k \geq 9$, $X(k)$ has girth 8. The proof of this fact is straightforward but lengthy and for this reason we decide to omit it. Note as well that $X(k)$ is not a bipartite graph as every connected component of $X(k)_{0,R}$ is a cycle of odd length.

We have now completed the proof of Theorem 4.2. Item (1) follows from Lemmas 4.4 and 4.20 while item (2) follows from Remark 4.6. Now that we have a characterization for vertex-transitivity, we would like to know when a cubic vertex-transitive trirculant of Type 1 is also a bicirculant. Note that the only cubic vertex-transitive circulants are prisms and Möbius ladders, which have girth 4. It follows from Lemma 4.9 that no cubic vertex-transitive trirculant of Type 1 is a circulant.

Lemma 4.22. *Let $\Gamma = T_1(k, r, s)$ be connected and vertex-transitive. If $3 \nmid k$, then Γ is a bicirculant.*

Proof. Let $\varphi: V(\Gamma) \rightarrow V(\Gamma)$ be the mapping given by:

$$\begin{array}{llll} u_i \mapsto v_i, & v_i \mapsto w_{i+r}, & w_i \mapsto u_i & \text{if } i \text{ is even;} \\ u_i \mapsto w_{i+k+s}, & v_i \mapsto u_{i+k+s}, & w_i \mapsto v_{i+k+s-r} & \text{if } i \text{ is odd.} \end{array}$$

It can be readily seen that φ is indeed a graph automorphism. Moreover, since r is even and both k and s are odd, φ preserves the parity of the index of all vertices. In fact, it is plain to see that φ is the product of the following two disjoint permutation cycles ϕ_1 and ϕ_2 of length $3k$:

$$\begin{aligned} \phi_0 &= (u_0, v_0, w_r, u_r, v_r, w_{2r}, u_{2r}, v_{2r}, \dots, u_{(k-1)r}, v_{(k-1)r}, w_0); \\ \phi_1 &= (u_k, w_s, v_{k+2s-r}, u_{k+r}, w_{r+s}, v_{k+2s}, u_{k+2r} \dots v_{k+2s+(k-2)r). \end{aligned}$$

Hence, Γ is a bicirculant. \square

Lemma 4.23. *If $3 \mid k$, then $X(k)$ is not a bicirculant.*

Proof. Suppose $X(k)$ admits a semiregular automorphism φ having two orbits, O_1 and O_2 of size $3k$. Following the notation in [17], $X(k)$ must be isomorphic to $H(3k, i, j)$, $I(3k, i, j)$ or $F(3k, i)$ for some $i, j \in \mathbb{N}$. Since $X(k)$ is not bipartite, it cannot be isomorphic to $H(3k, i, j)$. Further, if $X(k) \cong F(3k, i)$ then each of its orbits under φ would admit a matching, which is not possible because $3k$ is odd. Suppose $X(k) \cong I(3k, i, j)$ for some $i, j \in \mathbb{N}$. It is known that vertex-transitive I -graphs are generalized Petersen graphs and as such, one of the two orbits, say O_1 , consists of a single cycle having, in this case, length $3k$. Recall that the connected components of $X(k)_{0,R}$ are blocks of imprimitivity. Since

$\gcd(2k, r) = 6$, there are 6 such blocks, each of size k . It follows that no two adjacent vertices of O_1 are in the same block. On the other hand, it is plain to see that any 2-path in $X(k)$ has at least two vertices in the same block (see Figure 5). We conclude that $X(k)$ cannot be a bicirculant. \square

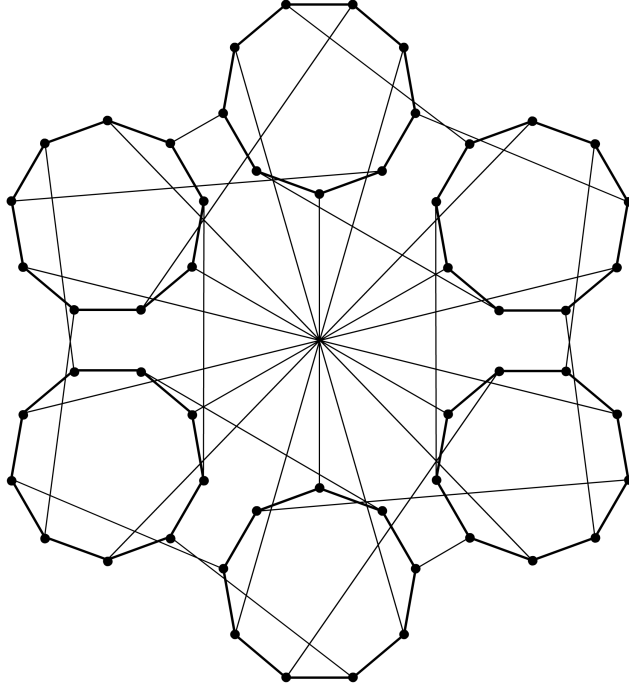


Figure 5: $T_1(9, 6, 1)$. Bold edges correspond to θ - and R -edges.

Corollary 4.24. *If $T_1(k, r, s)$ is vertex-transitive with $3 \nmid k$, then $T_1(k, r, s) \cong \text{GP}(3k, k + (-1)^\alpha)$ where $\alpha \in \{1, 2\}$ and $\alpha \equiv k \pmod{3}$.*

Proof. Let $T_1(k, r, s)$ be vertex-transitive and suppose $k \equiv 1 \pmod{3}$. Let $\varphi = \phi_0\phi_1$ be the automorphism described in the proof of Lemma 4.22. We will show that u_k is adjacent to $\varphi^{(k+1)}(u_k)$. Notice that $\varphi^{(k+1)}(u_k) = v_{k+2s-r+\frac{k-1}{3}r}$, but $k+2s-r \equiv r-s$ so we may write $\varphi^{(k+1)}(u_k) = v_{r-s+\frac{k-1}{3}r}$. Recall that $\gcd(2k, 3) = 1$ and r is even. From the congruence $k+2r-3s \equiv 0$ we see that:

$$\begin{aligned} 2r - 3s &\equiv 3k, \\ 3r - 3s + r(k-1) &\equiv 3k, \\ r - s + r(k-1)/3 &\equiv k. \end{aligned}$$

So that $\varphi^{(k+1)}(u_k) = v_k$, which is adjacent to u_k . The case when $k \equiv 2 \pmod{3}$ can be proved with the use of a similar argument. \square

This completes the proof of Theorem 4.3. Item (1) follows from Lemma 4.22 and Corollary 4.24, while item (2) is proved in Lemma 4.5.

5 Type 2

Let $k \geq 9$ be an integer, and let $r, s \in \mathbb{Z}_{2k}$. Recall that $T_2(k, r, s)$ is the derived graph of Δ_2 with the normalized voltage assignment for \mathbb{Z}_{2k} shown in Figure 6.

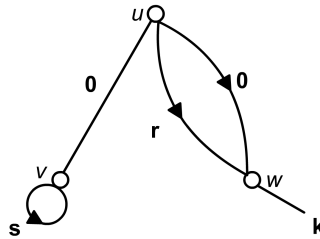


Figure 6: Type 2.

Then, with the same notation as in Section 4,

$$U = \{u_0, u_1, \dots, u_{2k-1}\}, \quad V = \{v_0, v_1, \dots, v_{2k-1}\} \text{ and } W = \{w_0, w_1, \dots, w_{2k-1}\}$$

are the respective fibres of vertices u, v and w in Δ_2 . The set of edges of $T_2(k, r, s)$ can be expressed as the union $E_K \cup E_R \cup E_S \cup E_0$ where:

$$\begin{aligned} E_K &= \{w_i u_{i+k} : i \in \mathbb{Z}_{2k}\}, \\ E_0 &= \{u_0 v_0 : i \in \mathbb{Z}_{2k}\} \cup \{u_0 w_0 : i \in \mathbb{Z}_{2k}\}, \\ E_R &= \{u_i w_{i+r} : i \in \mathbb{Z}_{2k}\}, \\ E_S &= \{v_i v_{i+s} : i \in \mathbb{Z}_{2k}\}. \end{aligned}$$

Similarly as with Type 1 cubic tricirculants, we see that every cubic tricirculant of Type 2 is isomorphic to $T_2(k, r, s)$ for an appropriate choice of k and $r, s \in \mathbb{Z}_{2k}$.

Definition 5.1. For an odd integer $k > 1$, let $Y(k) = T_2(k, 2, 1)$.

Theorem 5.2. Let $k \geq 9$ and let Γ be a connected cubic tricirculant of Type 2 with $6k$ vertices. Then the following holds:

1. Γ is vertex-transitive if and only if it is isomorphic to $Y(k)$ for some odd integer k .
2. If Γ is not vertex-transitive, then it has three distinct vertex orbits under its full automorphism group.

The next theorem gives more information about the graph $Y(k)$.

Theorem 5.3. For an odd integer $k > 1$ the following holds:

- (1) $Y(k)$ is a bicirculant if and only if $3 \nmid k$, in which case it is isomorphic to the graph $H(6k, 1, \alpha k + 2)$ where $\alpha \in \{1, 2\}$ and $\alpha \equiv k \pmod{3}$ (see [17]).
- (2) $Y(k)$ is the underlying graph of the toroidal map $\{6, 3\}_{\alpha, 3}$ where $\alpha = \frac{1}{2}(k - 3)$.
- (3) $Y(k)$ has arc-type $2 + 1$.

We devote the remainder of this section to proving Theorems 5.2 and 5.3.

Remark 5.4. Note that $T_2(k, r, s)$ and $T_2(k, r, -s)$ are in fact the exact same graph. Then, we can safely assume $s < k$.

Lemma 5.5. For an odd integer $k > 1$, $Y(k)$ is vertex-transitive.

Proof. Since $Y(k) = T_2(k, r, s)$, it suffices to give an automorphism of $T_2(k, 2, 1)$ that mixes the sets U , V and W . Let φ be the mapping given by:

$$\begin{array}{llll} u_i \mapsto v_{i+1}, & v_i \mapsto u_{i+1}, & w_i \mapsto v_i & \text{if } i \text{ is even;} \\ u_i \mapsto w_{i+2+k}, & v_i \mapsto w_{i+2}, & w_i \mapsto u_{i+k} & \text{if } i \text{ is odd.} \end{array}$$

Now consider a vertex $u_i \in U$ with i even. Observe that φ maps u_i to v_{i+1} and that the neighbourhood $N(u_i) := \{v_i, w_i, w_{i+2}\}$ is mapped to $\{u_{i+1}, v_i, v_{i+2}\}$, which is precisely the neighbourhood of v_{i+1} . That is, φ maps the neighbourhood of any vertex u_i into the neighbourhood of its image, when i is even. The reader can verify the remaining cases and see that φ is indeed a graph automorphism. \square

Lemma 5.6. For an odd integer $k > 9$, $Y(k)$ has arc-type $2 + 1$.

Proof. Let G be the full automorphism group of $Y(k)$, $k > 9$, and consider the automorphism φ interchanging w_i with w_{-i} , u_i with u_{-i-2} and v_i with v_{-i-2} , $i \in \mathbb{Z}_{2k}$. Note that $\varphi \in G_{w_0}$ and that it fixes w_k while interchanging u_0 with u_{-2} . Since no $Y(k)$ with $k > 9$ is arc-transitive, it follows that its arc-type is $2 + 1$. \square

For the rest of this section, let $k \geq 9$ be an integer, let $r, s \in \mathbb{Z}_{2k}$ and suppose $\Gamma = T_2(k, r, s)$ is vertex-transitive. In what follows we will show that Γ is isomorphic to $Y(k)$. As with cubic trirculants of Type 1, the strategy will be to count reduced closed walks in the quotient Δ_2 . Observe that the walk $((wu)_0, (uw)_r, (ww)_k, (wu)_{-r}, (uw)_0, (ww)_k)$ starting at w is a closed walk of length 6 having net voltage 0, regardless of what r and s evaluate to in \mathbb{Z}_{2k} . As was done with graphs of Type 1, we computed all closed walks in Δ_2 along with their net voltages. Again, it would be convenient if closed walks of length 6 with net voltage 0 always lift into 6-cycles. For this it suffices to show that the girth of Γ is at least 4.

Lemma 5.7. Γ has no cycles of length 3.

Proof. Suppose to the contrary, that Γ contains a triangle T . By observing Figure 6 we can see that all three vertices of T belong to V or they belong to $U \cup W$. Since Γ is vertex-transitive, there is at least one triangle through every vertex in W . We can thus assume without loss of generality that all three vertices of T are in $U \cup W$. This implies that $r = k$ and further, that there are 2 distinct triangles through every vertex in W : the lifts of $((wu)_0, (uw)_r, (ww)_k)$ and $((wu)_{-r}, (uw)_0, (ww)_k)$. However, since the subgraph induced by V is 2-valent, there is at most one triangle through every vertex in V , contradicting the vertex-transitivity of Γ . \square

Note that since Γ is simple, $s \neq k$ and if $r = k$, Γ would contain a triangle. Thus the following corollary holds.

Corollary 5.8. *Neither r nor s equals k .*

Now, let \mathcal{W}_6 be the set of all walks of length 6 in Δ_2 and let N be the set of net voltages of elements of \mathcal{W}_6 , expressed in terms of k , r and s . It follows from Lemma 5.7 that walks in \mathcal{W}_6 with net voltage 0 will lift into 6-cycles, and not just closed walks. Table 5 shows, for each element n of N , how many closed walks with net voltage n start at each vertex of Δ_2 .

Table 5: Net voltages of closed walks of length 6 in Δ_2 .

Label	Net voltage	Starting at u	Starting at v	Starting at w
I	0	2	0	4
II	$\pm(k - s)$	8	8	8
III	$\pm(k - r - s)$	4	4	4
IV	$\pm(k + r - s)$	4	4	4
V	$\pm(3r)$	2	0	2
VI	$\pm(2r)$	2	0	4
VII	$\pm(6s)$	0	2	0
VII	$\pm(r - 2s)$	4	6	2
IX	$\pm(r + 2s)$	4	6	2

From Row I of Table 5 we see that there are at least 4 walks $\omega \in \mathcal{W}_6$ starting at w . Therefore, at least one expression from Rows II–IX must be congruent to 0 (mod $2k$). However, a careful inspection of Table 5 shows that if neither of the expressions in Rows VIII and IX are congruent to 0 (mod $2k$), then there are at least 2 more walks in \mathcal{W}_6 with net voltage 0 starting at w than there are those starting at v . This means that if Γ is vertex-transitive, then one of the following two equations modulo $2k$ hold:

$$\begin{aligned} r &\equiv 2s, \\ r &\equiv -2s \end{aligned} \tag{5.1}$$

Suppose Equation (5.1) holds. By Remark 5.4 we can in fact write $r = 2s$. Then the mapping $\phi: T_2(k, 2s, s) \rightarrow T_2(k, -2s, -s)$ given by $x_i \mapsto x_{-i}$, for all $x \in U \cup V \cup W$, is a graph isomorphism. But, again by Remark 5.4, $T_2(k, -2s, -s) = T_2(k, -2s, s)$ so that any graph $T_2(k, r, s)$ with $r \equiv 2s$ is isomorphic to a graph $T_2(k, r', s)$ satisfying $r' \equiv -2s$. We can therefore limit our analysis to the case when Equation (5.1) holds.

Let $\Gamma = T_2(k, r, s)$ be connected and vertex-transitive, and suppose $r \equiv 2s$. Note that any number dividing both k and s must also divide r , and since by connectedness of Γ $\gcd(k, r, s) = 1$ we see that $\gcd(k, s) = 1$ and $\gcd(2k, s) \in \{1, 2\}$. We will show that $\gcd(2k, s)$ cannot be 2.

Lemma 5.9. *If Γ is vertex-transitive, $\gcd(2k, s) = 1$ and $\gcd(2k, r) = 2$.*

Proof. In order to get a contradiction, suppose $\gcd(2k, s) = 2$. Then s is even and k is odd, as $\gcd(k, s) = 1$. Furthermore, the 2-valent subgraph of Γ induced by the set V is the union of two k -cycles. We will show that these are in fact the only two k -cycles in

Γ which will imply that Γ cannot be vertex-transitive. Suppose there is a k -cycle C' that visits a vertex not in V . Define $\Gamma[U \cup W]$ as the subgraph of Γ induced by the set $U \cup W$. Since r is even and k is odd, it is not difficult to see that $\Gamma[U \cup W]$ is bipartite, with sets $\{w_i : i \text{ is even}\} \cup \{u_i : i \text{ is odd}\}$ and $\{w_i : i \text{ is odd}\} \cup \{u_i : i \text{ is even}\}$. Therefore no k -cycle of Γ is contained in $\Gamma[U \cup W]$. This means that C' visits at least one vertex in each of sets U , V and W .

Now, C' projects onto a closed walk C in Δ_2 that has net voltage 0. Define x_S as the number of times C traces the dart $(vv)_s$ minus the number of times it traces $(vv)_{-s}$, so that x_{SS} is the total voltage contributed to C by darts in $\{(vv)_{\pm s}\}$. Define x_R similarly, and define x_K as the number of times C traces the semi-edge $(ww)_k$. We obtain the following congruence modulo $2k$:

$$x_R r + x_{SS} + x_K k \equiv 0.$$

Recall that both r and s are even, making both $x_R r$ and x_{SS} even. It follows that $x_K k$ is even, but since k is odd, x_K must be even, and hence $x_K k \equiv 0 \pmod{2k}$. We thus have

$$x_R r + x_{SS} \equiv 0$$

and since $r \equiv 2s \pmod{2k}$,

$$\begin{aligned} x_R 2s + x_{SS} &\equiv 0, \\ s(2x_R + x_S) &\equiv 0. \end{aligned}$$

From this, and because $\gcd(k, s) = 1$, we see that $2x_R + x_S = 0$ or $2x_R + x_S \geq k$. To see that $2x_R + x_S$ cannot equal 0, define A as the set of darts in Δ_2 with voltage 0 or r and observe that C must visit A an even number of times. Since C has odd length, C visits $\{(vv)_{\pm s}\} \cup \{(ww)_k\}$ an odd number of times. Then, since x_K is even, x_S must be odd. But $2x_R$ is even so $2x_R + x_S \neq 0$. We thus have

$$2x_R + x_S \geq k.$$

Now, notice that if C traces the dart $(ww)_r$, then it must immediately trace a dart in $\{(wu)_0, (ww)_k\}$. This means that C traces a dart in the subgraph of Δ_2 induced by $U \cup W$ at least $2x_R$ times. Then $2x_R + x_S < k$, since C has length k and it must visit $(vu)_0$ at least once. We thus have $k > 2x_R + x_S \geq k$, a contradiction. The result follows. \square

Lemma 5.10. *If $T_2(k, r, s)$ is vertex-transitive, k is odd.*

Proof. Suppose to the contrary that k is even. Since s is odd, we have $sk \equiv k \pmod{2k}$, and thus $s(2 \cdot \frac{1}{2})k \equiv k$. But k is even, so we can rewrite that expression as $2s \cdot \frac{k}{2} \equiv k$. Recall that $r \equiv 2s$ and then $r \frac{k}{2} \equiv k$.

Now, consider the walk of length $k + 1$ in Δ_2 that starts in w , traces the semi-edge $(ww)_k$ once and then traces the 2-path $((wu)_r, (uw)_0)$ exactly $\frac{k}{2}$ times. Note that this walk is closed and has net voltage $\frac{k}{2}r + k \equiv 0$. Furthermore, it is easy to see that it lifts into a $(k + 1)$ -cycle in Γ and that it does not visit any vertex in V .

Since Γ is vertex-transitive, there must be a $(k + 1)$ -cycle C' through each vertex in V . Note that the subgraph of Γ induced by V is a single $2k$ -cycle, and thus C' must visit all three set U , V and W . It follows that C' projects into a walk C of length $k + 1$ having net voltage 0 and that it visits all three vertices of Δ_2 . Furthermore, since $k + 1$ is odd, the

number of times C visits $\{(vv)_{\pm s}\}$ must be of different parity than the number of times it traces $(ww)_k$. Define x_S , x_R and x_K as in the proof of Lemma 4.16. Hence

$$x_R r + x_S s + x_K k \equiv 0.$$

Now, r and k are even, and so $x_R r$ and $x_K k$ are also even. It follows that $x_S s$ is even, but since s is odd, s_S is necessarily even. This means C visits $\{(vv)_{\pm s}\}$ an even number of times. It follows that C traces $(ww)_k$ and odd number of times, that is, x_K is odd. We thus have

$$\begin{aligned} x_R r + x_S s + k &\equiv 0, \\ x_R r + x_S s &\equiv k, \end{aligned}$$

and since $r \equiv 2s$,

$$s(2x_R + x_S) \equiv k.$$

This implies $2x_R + x_S \geq k$, as s and k are relatively prime. However, $2x_R + x_S \leq k - 1$ since C has length $k + 1$ and C visits the set $\{(vu)_0, (uv)_0\}$ at least twice. This contradiction arises from the assumption that k is even. We conclude that k is odd. \square

Lemma 5.11. *For an odd $k > 9$, if $T_2(k, r, s)$ is vertex-transitive then we have*

$$T_2(k, r, s) \cong Y(k).$$

Proof. Let $T_2(k, r, s)$ be vertex-transitive. Then, by Lemmas 5.10 and 5.9 k is an odd integer, $\gcd(2k, s) = 1$ and $r \equiv 2s$. Denote by s^{-1} the multiplicative inverse of s in \mathbb{Z}_{2k} . Now, define φ as the mapping between $T_2(k, 2, 1)$ and $T_2(k, r, s)$ given by $x_i \mapsto x_{s^{-1}i}$, for $x \in U \cup V \cup W$. It is plain to see that φ is the desired isomorphism. Therefore $T_2(k, r, s) \cong T_2(k, 2, 1) = Y(k)$. \square

This, together with Lemma 5.5, proves the first claim of Theorem 5.2. We now proceed to show that a tricirculant of Type 2 that is not vertex-transitive must have 3 distinct orbits on vertices.

Lemma 5.12. *Let Γ be a tricirculant of Type 2. If Γ is not vertex-transitive, then it has 3 distinct vertex orbits.*

Proof. We will slightly abuse notation and say $\Gamma = T_2(k, r, s)$ for some $k \in \mathbb{Z}$ and $r, s \in \mathbb{Z}_{2k}$. Let $\varphi \in \text{Aut}(\Gamma)$ and let $u_i \in U$. Suppose that φ maps u_i to some vertex $v_j \in V$. Then φ must map the neighbourhood of u_i , $\{v_i, w_i, w_{i+r}\}$ to $\{u_j, v_{j-s}, v_{j+s}\}$, the neighbourhood of v_j . It is straightforward to see that φ must map a vertex in W to a vertex in V . This is, φ “mixes” the set V with the sets U and W implying that Γ is vertex-transitive, a contradiction. Similarly, if φ maps u_i to some $w_j \in W$ then it must map $\{v_i, w_i, w_{i+r}\}$ to $\{w_{j+k}, u_j, u_{j-r}\}$. Thus φ maps v_i into either U or W contradicting again Γ not being vertex-transitive. This shows that the set U is a single vertex orbit of Γ under $\text{Aut}(\Gamma)$. An analogous argument shows that V is a single vertex orbit and therefore Γ has three distinct orbits on vertices: U , V and W . \square

The proof of Theorem 5.2 is now complete. In the following paragraphs we will give further details about the structure of a vertex-transitive tricirculant of Type 2. In particular, we will show that $Y(k)$ can be seen as a map on the torus and we will give sufficient and necessary conditions for $Y(k)$ to be a bicirculant. This will prove items (1) and (2) of Theorem 5.3.

Proposition 5.13. *Let $k \geq 3$ be an odd integer, then $Y(k)$ admits an embedding on the torus with hexagonal faces, yielding a map of type $\{6, 3\}_{\alpha, 3}$, where $\alpha = \frac{1}{2}(k - 3)$ (see [5] for details about the notation).*

Proof. First, notice that vertices in $U \cup W$ with even index form a cycle of length $2k$, $C_1 = (w_0, u_0, w_2, u_2, \dots, w_{2k-2}, u_{2k-2})$. Likewise, vertices in $U \cup W$ with odd index form a $2k$ -cycle, C_2 . The vertices in V form a third cycle of length $2k$, C_3 .

Now, $w_i w_{i+k} \in E$ for all $i \in \mathbb{Z}_{2k}$, and $i+k$ has different parity than i . This means each vertex of C_1 of the form w_i is adjacent, through a K -edge, to a vertex of C_2 . Further, each vertex of the form u_i in C_1 is adjacent to a vertex in C_3 , namely v_i . Note that the vertices of C_2 that are not adjacent to a vertex of C_1 are precisely those of the form u_i (with i odd), and that $u_i v_i \in E$ for all odd i . This is, every other vertex in C_2 has a neighbour in C_3 .

Therefore, we can think of $Y(k)$ as three stacked $2k$ -cycles C_1 , C_2 and C_3 where every other vertex of C_i has a neighbour in C_{i-1} while each of the remaining vertices has a neighbour in C_{i+1} , where $i \pm 1$ is computed modulo 3 (see Figure 7 for a detailed example). From here it is clear that $T_2(k, 2, 1)$ can be embedded on a torus, tessellating it with $3k$ hexagons. It can be readily verified that this yields the map $\{6, 3\}_{\alpha, 3}$, where $\alpha = \frac{1}{2}(k - 3)$, following the notation in [5]. \square

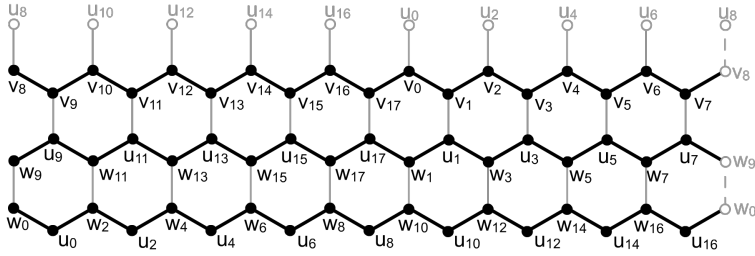


Figure 7: The graph $Y(9)$. The subgraph induced by bold edges has three connected components that correspond, from bottom to top, to the cycles C_1 , C_2 and C_3 described in the proof of Proposition 5.13.

Corollary 5.14. *For $k \geq 3$, $Y(k)$ has girth 6 and is bipartite.*

Observe that $Y(9)$ corresponds to the map $\{6, 3\}_{3, 3}$, which is a regular map (see Chapter 8.4 of [5]). It follows that $T_2(9, 2, 1)$ is arc-transitive, making it one of the four possible cubic arc-transitive tricirculant graphs (called F054A in [10], following Foster's notation).

Lemma 5.15. *If $k > 1$ is an odd integer such that $3 \nmid k$, then $Y(k)$ is a bicirculant.*

Proof. Consider the mapping φ defined by:

$$\begin{array}{llll} u_i \mapsto w_{i+2+k}, & v_i \mapsto w_{i+2}, & w_i \mapsto u_{i+k} & \text{if } i \text{ is even;} \\ u_i \mapsto v_{i+1}, & v_i \mapsto u_{i+1}, & w_i \mapsto v_i & \text{if } i \text{ is odd.} \end{array}$$

For a vertex $u_i \in U$, we have $\varphi^l(u_i) \in U$ if and only if $3 \mid l$. This is, if we start at U , every third iteration of φ lands us in U again. Moreover, we have $\varphi^3(u_i) = u_{i+3+k}$. In general $\varphi^{3l}(u_i) = u_{i+3l+lk}$. Hence $\varphi^{3l}(u_i) = u_i$ if and only if $3l + lk \equiv 0$. If $3 \nmid k$, then

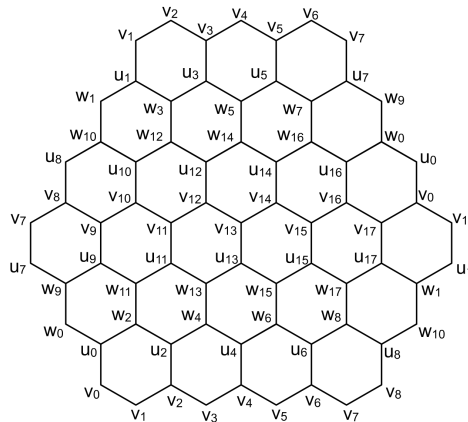


Figure 8: A drawing of $Y(9)$ as the map on the torus $\{6, 3\}_{3,3}$.

$l = k$ is the smallest value for l that satisfies $3l + lk \equiv 0$. It follows that the orbit of u_i has size $3k$. It is plain to see that the vertices not in this orbit form an orbit of size $3k$ on their own, namely, the orbit of u_{i+1} . Hence $Y(k)$ is a bicirculant and is isomorphic to the graph $H(3k, 1, \alpha k + 2)$, where $\alpha \in \{1, 2\}$ and $\alpha \equiv k \pmod{3}$ (see [17] for details). It is worthwhile to mention that if 3 does divide k , then φ has 6 orbits of size k . \square

Lemma 5.16. *If $k > 1$ is an odd integer such that $3 \mid k$, then $Y(k)$ is not a bicirculant.*

Proof. Let k be an odd integer divisible by 3. To make this proof more succinct, we will assume that $Y(k)$ is not arc-transitive, this is, that $k > 9$. The graphs $Y(3)$ and $Y(9)$ can be verified to not be bicirculants individually. By Lemma 5.6, $Y(k)$ has arc-type $2 + 1$ and the set $E' = \{u_i v_i \mid i \in \mathbb{Z}_{2k}\} \cup \{w_i w_{k+1} \mid i \in \mathbb{Z}_{2k}\}$ is an orbit of edges under the full automorphism group of $Y(k)$. Then each connected component of $Y(k) - E'$ is a block of imprimitivity for $\text{Aut}(Y(k))$. Furthermore $Y(k) - E'$ has three connected components that correspond precisely to the three $2k$ -cycles C_i , $i \in \{1, 2, 3\}$, in the proof of Proposition 5.13. For convenience, we will relabel the vertices of $Y(k)$ as follows: for each $i \in \{1, 2, 3\}$, let $V(C_i) = \{v_{i,j} \mid j \in \mathbb{Z}_k\}$, where $v_{i,j}$ is adjacent to $v_{i,j-1}$, $v_{i,j+1}$ and $v_{i+1,j-1}$. It is straightforward to see that this labelling is always possible.

Now, suppose $Y(k)$ admits a bicirculant automorphism ρ . Since a ρ -orbit has size $3k$, and every $2k$ -cycle C_i is a block of imprimitivity, ρ must cyclically permute them. Without loss of generality $\rho(C_i) = C_{i+1}$. Then, ρ is determined, up to two possibilities, by its action on a single vertex. Indeed, if $\rho(v_{i,0}) = v_{i+1,a}$ then the two neighbours of $v_{i,0}$ in C_i must be mapped to the neighbours of $\rho(v_{i,0})$ in C_{i+1} . In other words, the set $\{v_{i,-1}, v_{i,1}\}$ is bijectively mapped to $\{v_{i+1,a-1}, v_{i+1,a+1}\}$ in one of two possible ways, and this completely determines the action of ρ . In short, for all $i \in \{1, 2, 3\}$ and $j \in \mathbb{Z}_{2k}$ one of the following holds:

$$(1) \quad \rho(v_{i,j}) = v_{i+1,j+a},$$

$$(2) \quad \rho(v_{i,j}) = v_{i+1,-j+a}.$$

If (2) holds, then $\rho^3(v_{i,j}) = v_{i,-j+a}$ and so $\rho^6(v_{i,j}) = v_{i,j}$. This is, ρ has order 6, which implies that $Y(k)$ has order 12 and hence $k = 2$, which contradicts k being odd and

divisible by 3. If (1) holds, then $\rho^3(v_{i,j}) = v_{i,j+3a}$ and since $3 \mid k$, a vertex $v_{1,j}$ can only be mapped to a vertex $v_{1,b}$ by an element of $\langle \rho \rangle$ if $j \equiv b \pmod{3}$. This implies that ρ has at least 3 orbits on vertices, contradicting that it is a bicirculant automorphism. The result follows. \square

Theorem 5.3 now follows from Proposition 5.13 and Lemmas 5.6, 5.15 and 5.16.

6 Type 3

Let us begin this section by giving the definition of two well-known families of cubic graphs. For a positive integer k , the *prism* P_k is the graph of order $2k$ with vertex set $\{u_0, u_1, \dots, u_{k-1}\} \cup \{v_0, v_1, \dots, v_{k-1}\}$ and edges of the form $u_i u_{i+1}$, $v_i v_{i+1}$ and $u_i v_i$, where the indices are taken modulo k . The *Möbius ladder* M_k is the graph with vertex set $\{u_0, u_1, \dots, u_{2k-1}\}$ and edges of the form $u_i u_{i+1}$ and $u_i u_{i+k}$, where indices are taken modulo $2k$.

Let $k \geq 9$ be an integer, and let $r \in \mathbb{Z}_{2k}$. Recall that $T_3(k, r)$ is the derived graph of Δ_3 with the normalized voltage assignment for \mathbb{Z}_{2k} shown in Figure 9.

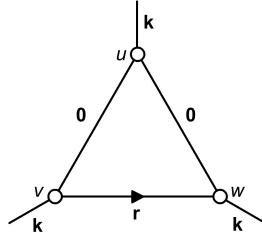


Figure 9: Type 3.

Let $U = \{u_0, u_1, \dots, u_{2k-1}\}$, $V = \{v_0, v_1, \dots, v_{2k-1}\}$ and $W = \{w_0, w_1, \dots, w_{2k-1}\}$ be the respective fibers of vertices u , v and w in Δ_3 . It is clear that any cubic trirculant of Type 3 is isomorphic to $T_3(k, r)$ for some k and r .

Theorem 6.1. *If $T_3(k, r)$ is connected, then it is isomorphic to either a prism or a Möbius ladder.*

Proof. First, recall that $T_3(k, r)$ is connected if and only if $\gcd(k, r) = 1$. Then $\gcd(2k, r) \in \{1, 2\}$, depending on whether r is even or odd.

If r is odd, then the subgraph induced by 0 - and R -edges is a single cycle of length $6k$, $(w_0, u_0, v_0, w_r, u_r, v_r, \dots, w_{2k-r}, u_{2k-r}, v_{2k-r}, w_0)$. It is straightforward to see that K -edges join antipodal vertices in this cycle. Hence, in this case $T_3(k, r)$ is a Möbius ladder.

If r is even, then the graph induced by 0 - and R -edges is the union of two disjoint cycles of length $3k$: one consisting of all the vertices with even index, and the other consisting on those with odd index. Observe that K -edges connect these two cycles creating a prism. \square

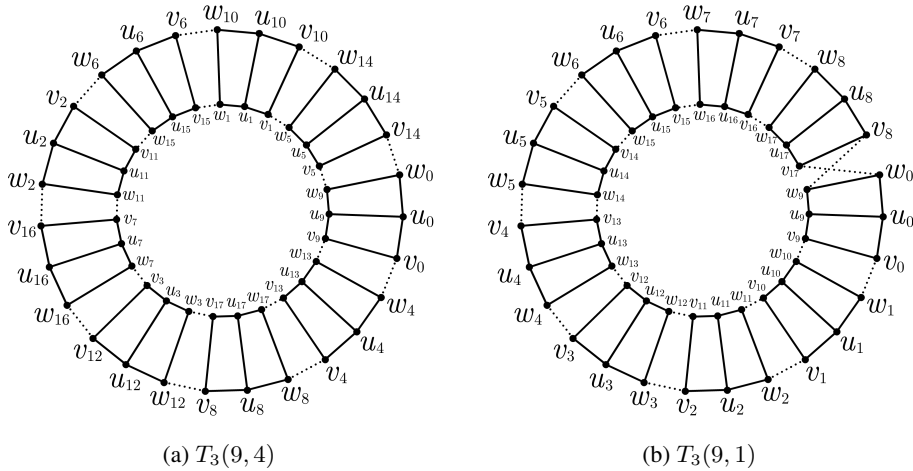


Figure 10: A prism and a Möbius ladder on 54 vertices.

7 Type 4

Let k be a positive integer, and let r and s be two distinct integers in \mathbb{Z}_{2k} . Recall that $T_4(k, r, s)$ is the derived graph of Δ_4 with the normalized voltage assignment for \mathbb{Z}_{2k} shown in Figure 11. Let $U = \{u_0, u_1, \dots, u_{2k-1}\}$, $V = \{v_0, v_1, \dots, v_{2k-1}\}$ and

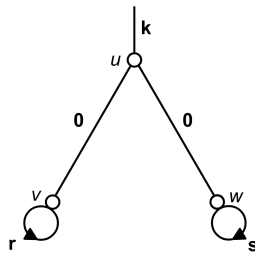


Figure 11: Type 4.

$W = \{w_0, w_1, \dots, w_{2k-1}\}$ be the respective fibers of vertices u , v and w in Δ_4 . Then, the set of edges of $T_4(k, r, s)$ can be expressed as the union $E_K \cup E_R \cup E_S \cup E_0$ where:

$$\begin{aligned}
 E_K &= \{u_i u_{i+k} : i \in \mathbb{Z}_{2k}\}, \\
 E_0 &= \{u_i v_i : i \in \mathbb{Z}_{2k}\} \cup \{u_i w_i : i \in \mathbb{Z}_{2k}\}, \\
 E_R &= \{v_i v_{i+r} : i \in \mathbb{Z}_{2k}\}, \\
 E_S &= \{w_i w_{i+s} : i \in \mathbb{Z}_{2k}\}.
 \end{aligned}$$

For $X \in \{0, R, S, K\}$, edges in E_X will be called edges of type X , or simply X -edges. Similarly as with trirculants of Types 1, 2 and 3, every cubic trirculant of Type 4 with $6k$ vertices is isomorphic to $T_4(k, r, s)$ for an appropriate choice of r and s .

Remark 7.1. Note that for any k, r and s the following isomorphism holds:

$$T_4(k, r, s) \cong T_4(k, -r, s) \cong T_4(k, r, -s) \cong T_4(k, s, r).$$

Theorem 7.2. *There are no cubic vertex-transitive tricirculants of Type 4 of order greater or equal to 54.*

The rest of this section is devoted to prove Theorem 7.2 as well as to characterise Type 4 tricirculants with 2 and 3 vertex orbits. We will assume henceforth that $k \geq 9$ and thus the order of $T_4(k, r, s)$ is at least 54.

Lemma 7.3. *If $T_4(k, r, s)$ is vertex-transitive, then $r \neq s$ and $r \neq -s$.*

Proof. Suppose that $r = s$ and consider the graph $\Gamma := T_4(k, r, r)$, with $k \geq 9$ and $1 \leq r \leq k - 1$. Observe that $(v_0, v_r, u_r, w_r, w_0, u_0, v_0)$ is a 6-cycle of Γ that does not contain any K -edge but does contain edges of all other types. For Γ to be vertex-transitive, there must be at least one 6-cycle through a K -edge; otherwise K -edges conform a single edge-orbit and thus U is a single vertex-orbit. However, such a cycle would quotient down to a closed walk of length 6 in Δ_4 having voltage 0 and tracing the semi-edge $(uu)_k$. By observing Figure 11 we see that any closed walk of length 6 visiting $(uu)_k$ has net voltage $k \pm 3r$. That is, if Γ is vertex-transitive, then $3r \equiv k \pmod{2k}$. Observe that under these conditions, Γ is connected if and only if $r = 1$ and thus making $k = 3$, contradicting that $k \geq 9$. This shows that $r \neq s$ and in view of Remark 7.1, $r \neq -s$. \square

Lemma 7.4. *Let e be a K -edge of $T_4(k, r, s)$ and let a be an integer greater than 4. If C is a cycle of length a containing e then there exists another cycle of length a , C' , such that the intersection $C \cap C'$ contains a 3-path whose middle edge is e .*

Proof. Without loss of generality, we can assume $e = u_0u_k$. Define H as the subgraph of Γ induced by vertices at distance at most one from e , and let C be a k -cycle through e (see Figure 12). Observe that C intersects H in a 3-path, P , whose middle edge is e . Now, let ϕ be the mapping that acts by multiplying the index of each vertex by -1 ; that is ϕ maps u_i into u_{-i} , v_i into v_{-i} , and w_i to w_{-i} . Observe that ϕ is in fact an automorphism of Γ . Moreover, ϕ fixes each vertex and each edge of H . In particular, it fixes P . Therefore, $\phi(C)$ is a k -cycle through e and P lies in the intersection of C and $\phi(C)$. To see that $\phi(C)$ is different from C , observe that ϕ interchanges the two vertices in each of the following sets: $\{v_{k+r}, v_{k-r}\}$, $\{v_r, v_{-r}\}$, $\{w_{k+s}, w_{k-s}\}$ and $\{w_s, w_{-s}\}$ (white vertices in Figure 12). It is clear that C can visit at most one vertex in each of these four sets, and if C visits one vertex in one of these sets, then $\phi(C)$ must visit the other. \square

Lemma 7.5. *If $T_4(k, r, s)$ is vertex-transitive, then in \mathbb{Z}_{2k} one of the equalities (A1)–(A4) below and one of the equalities (B1)–(B4) below hold:*

(A1)	$k + 2r + s = 0,$	(B1)	$k + 2s + r = 0,$
(A2)	$k - 2r + s = 0,$	(B2)	$k - 2s + r = 0,$
(A3)	$r + 3s = 0,$	(B3)	$s + 3r = 0,$
(A4)	$r - 3s = 0,$	(B4)	$s - 3r = 0.$

Proof. Suppose $T_4(k, r, s)$ is vertex-transitive. We will first show that one of the equalities (A1)–(A4) holds in \mathbb{Z}_{2k} . The rest of the lemma will follow from the fact that $T_4(k, r, s) \cong$

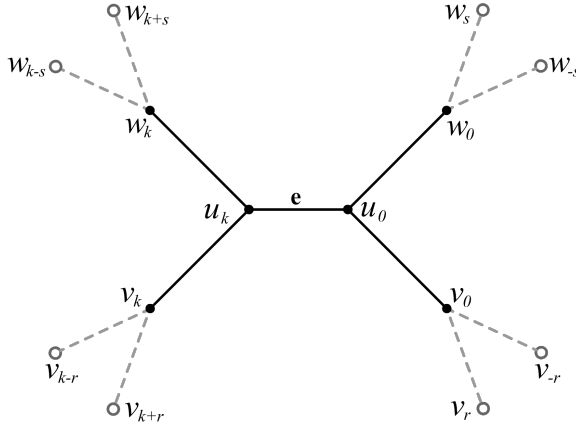


Figure 12: Neighbourhood of $u_0 u_k$. The subgraph H shown in solid edges and vertices.

$T_4(k, s, r)$. Since $T_4(k, r, s)$ is vertex-transitive, the edge-neighbourhood of any vertex in U can be mapped by an automorphism to the edge-neighbourhood of any vertex in V . In particular, either there exists an automorphism mapping the K -edge $u_0 u_k$ to the θ -edge $u_0 v_0$ or there is an automorphism mapping $u_0 u_k$ to the R -edge $v_0 v_r$. Therefore, the property described in Lemma 7.4 should also hold for $u_0 v_0$ or for $v_0 v_r$ (or possibly both). We will see what this means in terms of k, r and s .

Suppose that the property described in Lemma 7.4 holds for $u_0 v_0$. Observe that there is an 8-cycle $C = (u_0, v_0, v_{-r}, u_{-r}, u_{k-r}, v_{k-r}, v_k, u_k, u_0)$ through $u_0 v_0$ (see Figure 13). Then there must be another 8-cycle C' whose intersection with C contains the 3-path (v_{-r}, v_0, u_0, u_k) , but no 4-path containing this 3-path. This in turns implies that some vertex in $\{v_{-3r}, u_{-2r}\}$ is adjacent to a vertex in $\{w_{k-s}, w_{k+s}\}$. Since no vertex in V is adjacent to a vertex in W , we have that either $u_{-2r} w_{k-s} \in E$ or $u_{-2r} w_{k+s} \in E$. This implies $2r \equiv k + s$ or $2r \equiv k - s$, and so (A1) or (A2) holds.

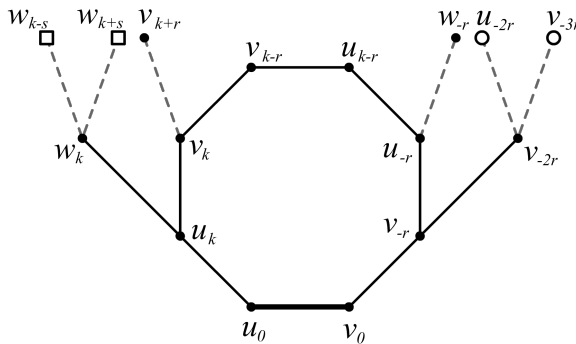


Figure 13: A part of the graph $T_4(k, r, s)$ containing $u_0 v_0$.

If, on the other hand, the property described in Lemma 7.4 holds for $v_0 v_r$, then one vertex in $\{w_{r+s}, w_{r-s}\}$ must be adjacent to a vertex in $\{w_s, w_{-s}\}$ (see Figure 14), implying

that one of the following expressions must be equal to 0 in \mathbb{Z}_{2k} : $r + 3s$, $r + s$, $r - 3s$, $r - s$. However, in view of Lemma 7.3, $r + s \neq 0$ and $r - s \neq 0$. Hence (A3) or (A4) holds. \square

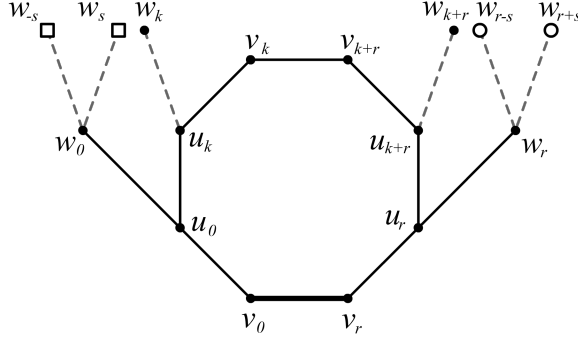


Figure 14: A subgraph containing $v_0 v_r$.

We are now ready to prove Theorem 7.2. Let $k \geq 9$ be an integer and suppose $T_4(k, r, s)$ is vertex-transitive. Then, by Lemma 7.5, one of the equalities (A1)–(A4) and one of the equalities (B1)–(B4) must hold. If, for instance both (A1) and (B1) hold, then by adding them we see that $r = -s$, which contradicts Lemma 7.3. Similarly, we get that $r \pm s$ if both equalities in any of the following pair hold: (A1) and (B2), (A2) and (B2), (A3) and (B3). If (A1) and (B4), or (A2) and (B4) hold, then $k = r$, which contradicts the simplicity of Γ . If (A1) and (B3), (A2) and (B3), or (A3) and (B4) hold, then either $k = 5$ or $\gcd(k, r, s) \neq 1$. It is readily seen that we get a contradiction in each of the 16 possible cases that arise from Lemma 7.5. We conclude that a connected $T_4(k, r, s)$ cannot be vertex-transitive if $k \geq 9$.

Lemma 7.6. *Let $k \geq 9$ and $T_4(k, r, s)$ be connected. Let $\alpha = \gcd(2k, r)$, $R = r/\alpha$, $S = s/\alpha$. Then $T_4(k, r, s)$ has two orbits on vertices if and only if $\alpha \in \{1, 2\}$ and $(SR^{-1})^2 \equiv \pm 1 \pmod{2k/\alpha}$, where R^{-1} is the multiplicative inverse of R modulo $2k/\alpha$ (or the equivalent statement obtained from interchanging r and s).*

Proof. For a graph Γ of Type 4, let Γ^* be the graph obtained from Γ by deleting all vertices in U (along with the edges incident to them) and making v_i adjacent to w_i , $i \in \mathbb{Z}_{2k}$. Then Γ^* is isomorphic to the bicirculant I -graph $I(2k, r, s)$ of order $4k$. Moreover, each automorphism $\varphi \in \text{Aut}(\Gamma)$ induces an automorphism $\varphi^* \in \text{Aut}(\Gamma^*)$ in a natural way. Denote by $\text{Aut}^*(\Gamma^*)$ the group of all automorphism of Γ^* induced by an automorphism of Γ .

Now suppose Γ has two orbits on vertices. Since K -edges can only be mapped to K -edges, these two orbits must necessarily be U and $V \cup W$. This is, $\text{Aut}(\Gamma)$ acts transitively on $V \cup W$ and so $\text{Aut}^*(\Gamma^*)$ is transitive on the vertices of Γ^* .

If $\Gamma^* = I(2k, r, s)$ is connected then it must be a vertex-transitive generalised Petersen graph [17]. This is, without loss of generality, the graph induced by V is a $2k$ -cycle, and thus $\gcd(2k, r) = 1$. Let r^{-1} be the multiplicative inverse of r modulo $2k$ and let $\Lambda = T_4(k, 1, sr^{-1})$. Then $\Lambda \cong \Gamma = T_4(k, r, s)$. Moreover, Λ^* is isomorphic to the generalised Petersen graph $\text{GP}(2k, sr^{-1})$, and since it is vertex-transitive we have $(sr^{-1})^2 \equiv \pm 1 \pmod{2k}$ [8].

If $\Gamma^* = I(2k, r, s)$ is disconnected, then it must have two connected components. Indeed, since Γ is connected we have $\gcd(2k, k, r, s) = 1$, but $I(2k, r, s)$ is disconnected if and only if $\gcd(2k, r, s) \neq 1$. Then necessarily $\gcd(2k, r, s) = 2$ and both r and s are even while k is odd. Each connected component is isomorphic to $I(k, r/2, s/2)$. An analogous argument as the one used in the connected case shows that $\gcd(k, r/2) = 1$, (and since k is odd, $\gcd(k, r) = 1$ and $\gcd(2k, r) = 2$) and $(SR^{-1})^2 = ((s/2)(r/2)^{-1})^2 \equiv \pm 1 \pmod{k}$.

For the reverse implication let $k \geq 9$ and $r, s \in \mathbb{Z}_{2k}$. Suppose that $\alpha = \gcd(2k, r) \in \{1, 2\}$ and $(SR^{-1})^2 \equiv \pm 1 \pmod{2k/\alpha}$, where R^{-1} is the multiplicative inverse of R modulo $2k/\alpha$. Let $\Gamma = T_4(k, r, s)$. Define $\varphi: \Gamma \rightarrow \Gamma$ as follows:

$$u_i \mapsto u_{(SR^{-1})i}, \quad v_i \mapsto w_{(SR^{-1})i}, \quad w_i \mapsto v_{(SR^{-1})i} \quad \text{for all } i \in \mathbb{Z}_{2k}.$$

Observe that $SR^{-1}r \equiv s \pmod{2k}$ and $sSR^{-1} \equiv \pm r \pmod{2k}$. From here, it is routine to check that φ is an automorphism of Γ . We conclude that $V \cup W$ is a vertex orbit of Γ . \square

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