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Analitično spremljanje razvoja elastoplastičnega stanja med upogibom nosilcev pravokotnega prereza

Analytical Tracing of the Evolution of the Elasto-Plastic State during the Bending of Beams with a Rectangular Cross-Section

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Prispevek obravnava elastoplastično analizo upogibnice nosilcev pravokotnega prereza, ki so obremenjeni z določenimi vrstami obremenitev, pri čemer material ni utrjevalen. Z upoštevanjem teorije majhnih pomikov in majhnih deformacij so izpeljane analitične rešitve, s katerimi lahko analiziramo elastoplastični problem upogiba nosilcev v celoti analitično. To omogoča spremljanje razvoja elastoplastičnega odziva s širjenjem plastičnega območja po trdnini s povečevanjem obremenitve, t.j. tako širjenje vzdolž osi nosilca kot širjenje po globini prereza, od nastanka plastičnih deformacij do porušitve.

(Ključne besede: nosilci, elastoplastične analize, analitične funkcije, porušitve)

The deflection analysis of beams with rectangular cross-sections is considered under specific loading conditions and assuming elasto-plastic behaviour with no hardening. Within the framework of the small-strain and small-displacement approach, analytical solutions are derived that enable the elasto-plastic analyses of beams to be performed in a closed analytical form. As a consequence, clear tracing of the evolution of the elasto-plastic response with a propagation of the plastic zone through the solid body, i.e., its spreading along the beam's longitudinal axis as well as its penetration through the cross-section, is enabled as loads increase, from the appearance of a first plastic yielding in the structure until its collapse.

(Keywords: beams, elastioplastic analysis, analytic functions, beam collapse)

0 UVOD

Upogib nosilcev, ki se pogosto pojavlja v tehnični praksi, je bil že mnogokrat obravnavan tudi z zahtevnejšimi postopki, še posebej v primeru elastičnih problemov ([4], [7], [10] in [11]). Elastoplastične analize nosilcev, pri katerih predpostavimo nastanek plastičnih členkov, z uporabo katerih izračunamo mejne obremenitve, ki povzročijo porušitev konstrukcije, spadajo v področje teorije mejnih stanj ([2] in [5]). Ker vodilne enačbe elastoplastičnega problema upogiba nosilcev v splošnem niso rešljive analitično, v literaturi srečamo predvsem numerične in eksperimentalne rezultate ([1], [6], [8] in [9]). Analitična rešitev je navedena kvečjemu za nespremenljivo porazdelitev notranjega momenta, medtem ko je za kvadratično porazdelitev prikazan le izračun posameznega primera [3]. Ob

0 INTRODUCTION

The bending of beams, which is frequently addressed in technical practice, has been adequately and thoroughly analysed, considering even more rigorous approaches, especially for elastic problems ([4], [7], [10] and [11]). The elasto-plastic analyses of beams, which with the assumed formation of plastic hinges limit the fully plastic loads, are evaluated. Those causing a structure to collapse are treated within a framework of the limit-analysis theory ([2] and [5]). Since, in general, the governing equation of the elasto-plastic beam-bending problem is not to be solved analytically; it is numerical solutions and experimental results that are met in the literature ([1], [6], [8] and [9]). An analytical solution is described, at most, for a constant bending-moment distribution, while for a quadratic bending moment the distribution is derived only for a predpostavki elastoplastičnega materiala brez utrjevanja smo se v prispevku osredotočili na raziskavo elastoplastičnega upogiba nosilcev pravokotnega prereza. Posamezne rešitve, izpeljane ob predpostavki, da je porazdelitev notranjega momenta največ kvadratična, v celoti omogoča analitično spremljati razvoj elastoplastičnega stanja v konstrukciji ob monotonem in sorazmernem povečevanju obremenitev.

1 VODILNE ENAČBE PROBLEMA

Naj bo raven nosilec prečnega prereza A (sl. 1) izpostavljen elastoplastičnemu upogibu v ravnini (x,z), kjer je x vzdolžna os nosilca, medtem ko sta y in z glavni osi prereza. Medtem ko upoštevamo, da je obnašanje snovi v elastičnem področju skladno s Hookovim zakonom, posebnih zahtev za nepovračljivo obnašanje trenutno ne postavimo. Upoštevamo še Bernoulli-Navierjevo hipotezo o ravnosti prereza in njegovi pravokotnosti na nevtralno os. V skladu z naravo obravnavanega problema je napetostno stanje $\sigma_{ij}(x,y,z)$ popisano tako, da so naslednje rezultante enake nič:

particular problem [3]. By assuming an elasto-plastic material with no hardening, we concentrate in this paper on an investigation of the elasto-plastic bending of beams with a rectangular cross-sectional area. The particular solutions derived under the assumption of, at most, a quadratic bending moment distribution enable the fully analytical tracing of the evolution of the elasto-plastic state in structural components using the monotonic and proportional application of loads to a structure.

1 GOVERNING EQUATIONS

Let us consider a straight beam with a cross-sectional area A, (Fig. 1), subject to elasto-plastic bending in the (x,z) plane, where x is the longitudinal axis of the beam, and y and z are the principal axes of the cross-section. While the material behaviour is assumed to obey Hooke's law in the elastic region, no restrictions on the nature of the irreversible inelastic response are imposed on the moment. Also, the Bernoulli-Navier assumptions on the cross-section's planarity and perpendicularity to the neutral axis are respected. The established stress state $\sigma_{ij}(x,y,z)$ is characterized, in accordance with the nature of the considered problem, by the following resultants being zero:

$$\int_A \sigma_{xy}\mathrm{d}A = T_y(x) = 0 \ , \ \int_A \left(\sigma_{xz}y - \sigma_{xy}z\right)\mathrm{d}A = M_x(x) = 0 \ , \ \int_A \sigma_{xx} y\mathrm{d}A = -M_z(x) = 0 \qquad (1).$$

Predpostavljeno je tudi, da so zunanje sile v skladu z:

Furthermore, it is assumed, that the external loads are in accordance with:

$$\int_A \sigma_{xx}\mathrm{d}A = N(x) = 0 \qquad (2),$$

medtem ko sta preostali rezultanti:

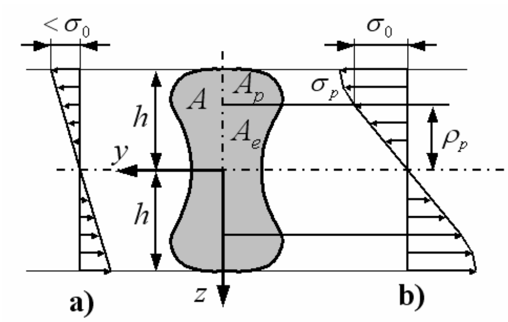
while the remaining two resultants:

$$\int_A \sigma_{xx} z\mathrm{d}A = M_y(x) \ , \ \int_A \sigma_{xz}\mathrm{d}A = T_z(x) = \frac{\mathrm{d}M_y(x)}{\mathrm{d}x} \qquad (3)$$

v splošnem od nič različni. Z upoštevanjem ničelnih napetostnih komponent ($\sigma_{yy}(x,y,z)= \sigma_{zz}(x,y,z)= \sigma_{yz}(x,y,z)=0$) in Bernoulli-Navierjeve hipoteze sledi linearna porazdelitev osnih deformacij ε_{xx} po višini prereza, kar v primeru elastičnega odziva (sl. 1a) vodi tudi do linearne porazdelitve napetosti $\sigma_{xx}(x,y,z)$. Če uvedemo poenostavljen zapis $\sigma_{xx}=\sigma$ in $M_y=M$, lahko to zvezo glede na prvo od enačb (3) zapišemo kot:

are non-zero, in general. By considering the zero stress components, $\sigma_{yy}(x,y,z)=\sigma_{zz}(x,y,z)= \sigma_{yz}(x,y,z)=0$, the Bernoulli-Navier assumptions lead to a linear distribution of the axial strain ε_{xx} across the cross-sectional height, and consequently, in the case of an elastic response (Fig. 1a), to a linear stress $\sigma_{xx}(x,y,z)$ distribution, too. By introducing simplified notations, $\sigma_{xx}=\sigma$ and $M_y=M$, this relationship can be written, in accordance with first of Equations (3), as:

$$\sigma(x,z) = \frac{M(x)}{I} z \qquad (4),$$



Sl. 1. *Porazdelitev napetosti pri problemu upogiba nosilcev: a) elastična, b) elastoplastična*
Fig. 1. *Stress distribution in a beam-bending problem. a) elastic, b) elasto-plastic*

kjer je I vztrajnostni moment prereza glede na glavno os y. Slika 1b prikazuje napetostno stanje, ko upogibne napetosti presežejo mejo tečenja σ_0. V primeru elastoplastičnega odziva je napetostno stanje odvisno od narave trenutnega odziva:

where I is the area moment of inertia with respect to the principal axis y. Fig. 1b describes the stress state when the bending stress exceeds the yield stress σ_0. In the case of an elastic-plastic response, the respective stress distribution is governed by the nature of the actual response:

$$\sigma(x,z) = \operatorname{sign}(M(x)) \begin{cases} \dfrac{z}{\rho_p(x)}\sigma_0 & \dots\ |z| \le \rho_p(x) \\ \operatorname{sign}(z)\ \sigma_p(\varepsilon_p(x,z)) & \dots\ |z| > \rho_p(x) \end{cases} \quad (5).$$

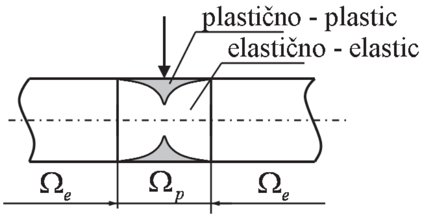
Pri tem je primerjalna plastična deformacija ε_p vzeta kot absolutna vrednost vzdolžne plastične deformacije $\varepsilon_{xx}=\varepsilon$, medtem ko $\sigma_p(\varepsilon_p)$ označuje zvezo med napetostmi in plastičnimi deformacijami, ki jo določa enoosni natezni preizkus. Parameter $\rho_p(x)$ popisuje stopnjo plastične deformacije določenega prereza in označuje ločnico med elastičnim in plastičnim območjem prereza. Ta parameter je povezan z upogibnim momentom z enačbo (3), pri čemer je upoštevana porazdelitev napetosti (5):

Here, the equivalent plastic strain ε_p is taken as the absolute value of the axial plastic strain $\varepsilon_{xx}=\varepsilon$, while $\sigma_p(\varepsilon_p)$ denotes the stress-plastic strain relationship, as determined from a uniaxial tensile test. The parameter $\rho_p(x)$ describes the degree of plastic loading of the particular section and shows the boundary between the elastic and plastic regions of the section. This parameter is related to the bending moment with Equation (3), considering the stress distribution (5):

$$\int_{A_p} \operatorname{sign}(z)\ \sigma_p(\varepsilon_p(x,z))\, z\, dA + \int_{A_e} \frac{z^2}{\rho_p(x)}\sigma_0\, dA = |M(x)| \quad (6).$$

Najprej definirajmo elastični in plastični mejni moment $M_e(x)$ in $M_p(x)$. Elastični mejni moment $M_e(x)$ je notranji upogibni moment, ki še vedno povzroča samo elastične deformacije v prerezu pri vzdolžni koordinati x in ga izračunamo z limitiranjem enačbe (6), upoštevajoč $A_e \to A$, $A_p \to 0$ in $\rho_p(x) \to h$. Plastični mejni moment je notranji upogibni moment, pri katerem postane prerez v celoti plastično deformiran in ga izračunamo z limitiranjem enačbe (6), upoštevajoč $A_e \to 0$, $A_p \to A$ in $\rho_p(x) \to 0$. Naj poudarimo, da velja enačba (6) za dani notranji upogibni moment $|M(x)|$ le, če je prerez delno plastično deformiran, tj.

Let us first define the elastic and plastic limit moments, $M_e(x)$ and $M_p(x)$. The elastic limit moment $M_e(x)$ is the bending moment, which still causes only an elastic deformation in the cross-section at a longitudinal position x, and can be computed using the limit of Equation (6), i.e., $A_e \to A$, $A_p \to 0$ and $\rho_p(x) \to h$. The plastic limit moment $M_p(x)$ is the bending moment where the cross-section is fully plastically deformed, and can be computed using the limit of Equation (6), i.e., $A_e \to 0$, $A_p \to A$ and $\rho_p(x) \to 0$. Let us emphasize that for a given bending moment $|M(x)|$ Equation (6) is valid only when the cross-section is partially plastically deformed, i.e., $0<M_e(x) \le |M(x)| \le M_p(x)$. With regard to the limit elastic



Sl. 2. *Razdelitev na elastična in elastoplastična polja vzdolž nosilca*
Fig. 2. *Elastic and elastic-plastic domain decomposition along the beam*

$0<M_e(x)\leq|M(x)|\leq M_p(x)$. Glede na velikost elastičnega mejnega momenta $M_e(x)$ lahko nosilec razdelimo na polja tako, da Ω_e zajema prereze z elastičnim odzivom, Ω_p pa prereze, na katerih je elastični mejni moment presežen (sl. 2).

Zveza med ukrivljenostjo nevtralne osi, ki je podana z $R(x)$, in njenim povesom $w(x)=w(x,z=0)$ v z smeri je enaka:

moment $M_e(x)$, the whole beam can be decomposed into domains along the beam, where Ω_e consists of cross-sections with a pure elastic response, while Ω_p consists of all the cross-sections where the elastic limit moment is exceeded (Fig. 2).

The relationship between the curvature of the neutral axis, given by the respective radius $R(x)$ and its deflection $w(x)=w(x,z=0)$, in the z direction is:

$$\varepsilon(x,z)=\frac{z}{R(x)}\approx -z\frac{d^2w}{dx^2} \quad (7).$$

Posledica Bernoulli-Navierjeve hipoteze o ravnosti prereza je, da na tistem prerezu, kjer se pojavijo tako elastične kakor plastične deformacije, porazdelitev deformacij sledi Hookovemu zakonu, ki velja v elastičnem delu prereza:

The Bernoulli-Navier assumptions about the cross-section's planarity have the consequence that at a particular cross-section where both the elastic and plastic strains are present, the strain distribution of the cross-section is followed by Hooke's law, which is valid in the elastic part of the cross-section, therefore:

$$\varepsilon(x,z)=\varepsilon_e(x,z)=\frac{\sigma_e(x,z)}{E} \quad (8).$$

Z upoštevanjem enačb (7) in (8) sledi:

Considering Equations (7) and (8) it follows that:

$$-z\frac{d^2w}{dx^2}=\frac{\sigma_e(x,z)}{E} \quad (9).$$

Če sedaj uporabimo enačbi (4) in (5), dobimo diferencialno enačbo za oba možna primera, tako za elastični odziv kakor za odziv, ko so se v prerezu A že pojavile tudi plastične deformacije:

Using Equations (4) and (5) the differential equation governing the two possible cases, i.e., that of the pure elastic response and that of the evolved plastic strains in the cross-section A, can be deduced:

$$\frac{d^2w}{dx^2}=\begin{cases}-\dfrac{M(x)}{EI} & \dots\ x\in\Omega_e\\[2ex] \operatorname{sign}(-M(x))\dfrac{\sigma_0}{E\rho_p(x)} & \dots\ x\in\Omega_p\end{cases} \quad (10).$$

Da se izognemo težavam, ki nedvomno nastanejo pri reševanju diferencialne enačbe (10) v primeru elastoplastičnega odziva in poljubnega prereza A, se v nadaljevanju osredotočimo na nosilce pravokotnega prereza, definiranega s širino b in višino $2h$, obe konstanti vzdolž osi x ($A=2bh$, $I=2/3bh^3$). Snovne lastnosti vzamemo

In order to escape from a stack of difficulties, which arise when solving the differential equation (10) in the case of an elasto-plastic response and an arbitrary cross-sectional area A, we will limit ourselves in what follows to a consideration of beams with a rectangular cross-section, defined by a width b and a height $2h$, constant along the x axis ($A=2bh$, $I=2/3bh^3$). The material properties are likewise kot nespremenljive in izotropne ($\sigma_p(\varepsilon_p)=\sigma_0$). Globina plastičnega območja je določena z velikostjo upogibnega momenta $M(x)$ in s pripadajočo stopnjo preseganja elastičnega mejnega momenta M_e ($|M(x)|>M_e$):

assumed to be constant and isotropic, assuming no plastic hardening ($\sigma_p(\varepsilon_p)=\sigma_0$). The depth of the plastic zone is determined with the magnitude of the bending moment $M(x)$ and the degree by which the limit elastic moment M_e is exceeded ($|M(x)|>M_e$):

$$\rho_p(x) = h\sqrt{3\left(1-\frac{|M(x)|}{bh^2\sigma_0}\right)} \quad (11).$$

Upoštevajoč $\rho_p=h$ in $\rho_p=0$ dobimo iz zgornje enačbe mejne vrednosti upogibnega momenta $M_e = 2bh^2\sigma_0/3$ in $M_p = bh^2\sigma_0$. Torej so z $|M(x)|=M_e$ vzpostavljeni pogoji za začetek plastičnih deformacij, medtem ko se pri plastičnem mejnem momentu $|M(x)|=M_p$ plastična cona razširi čez celoten prerez.

Vodilno enačbo upogibnega problema (10), definirano na območju Ω, ($\Omega=\Omega_e \cap \Omega_p$, $\Omega_e \cup \Omega_p=0$), lahko končno preoblikujemo v:

By considering $\rho_p=h$ and $\rho_p=0$, the limit values $M_e = 2bh^2\sigma_0/3$ and $M_p = bh^2\sigma_0$ of the bending moment are obtained from this equation. Thus, with $|M(x)|=M_e$ the conditions for the initiation of plastic deformation are established, whereas the fulfilment of $|M(x)|=M_p$, the latter being termed the fully plastic moment, causes the plastic zone to spread over the whole of the cross-section.

The governing equation (10) of the bending problem, which is defined over a domain Ω, ($\Omega=\Omega_e \cap \Omega_p$, $\Omega_e \cup \Omega_p=0$), is finally transformed to:

$$\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = \begin{cases} -\dfrac{3M(x)}{2Ebh^3} & \dots \quad x \in \Omega_e \\[2ex] \dfrac{K}{\sqrt{M_p - |M(x)|}} & \dots \quad x \in \Omega_p \end{cases} \quad (12),$$

kjer je K definiran kot:

where the constant K is defined as:

$$K = \mathrm{sign}\left(-M(x)\right)\sqrt{\frac{b\sigma_0^{\,3}}{3E^2}} \quad (13).$$

2 ANALITIČNA REŠITEV

Medtem ko je analitična rešitev razmeroma preprosta v primeru elastičnega odziva ($x\in\Omega_e$), pa to ne velja za elastoplastični primer ($x\in\Omega_p$). Toda za najpogostejše primere obremenitve, pri katerih je porazdelitev notranjega upogibnega momenta $M(x)$ dana v polinomski obliki $M(x)=P_n(x)$, lahko analitično rešitev izpeljemo. Pri obremenitvi nosilca s zunanjimi točkovnimi silami in momenti ter zvezno porazdeljeno obtežbo je odvisnost momenta $M(x)$ največ polinom drugega reda $M(x) = P_2(x) = m_2x^2 + m_1x + m_0$. Ta zveza gotovo zadovoljivo pokriva večino dejanskih obremenilnih primerov.

V primeru elastičnega odziva ($x\in\Omega_e$) se torej lahko rešitev $w(x)$ eksplicitno zapiše v obliki:

2 ANALYTICAL SOLUTION

While a closed-form explicit solution is relatively trivial in the case of the elastic response ($x\in\Omega_e$), this is not true for the elasto-plastic case ($x\in\Omega_p$). However, for the most common loading cases that result in bending-moment distributions $M(x)$ with a polynomial form, $M(x)=P_n(x)$, such analytical solutions can be readily deduced. Namely, the application of concentrated loads, moments and uniformly distributed loads on a beam structure results in moment functions $M(x)$ of, at most, a second-order polynomial $M(x) = P_2(x) = m_2x^2 + m_1x + m_0$. This functional relationship allows for a satisfactory analysis of the majority of real loading cases.

In fact the corresponding solution $w(x)$ in the case of an elastic response ($x\in\Omega_e$) results explicitly in the form of:

$$w(x) = -\frac{x^2}{8Ebh^3}\left(m_2x^2 + 2m_1x + 6m_0\right) + C_1x + C_2 \quad , \quad x \in \Omega_e \quad (14),$$

medtem ko je v primeru elasto-plastičnega odziva ($x \in \Omega_p$) rešitev $w(x)$ izpeljana za vsako stopnjo n=0,1,2 predpostavljenega polinoma $M(x)=P_n(x)$ posebej. Eksplicitno izpeljane rešitve so torej:

while in the case of an elasto-plastic response ($x \in \Omega_p$) the respective particular solutions $w_p(x)$ are deduced separately for each of the possible degrees n=0, 1 and 2 of the assumed polynomial function $M(x)=P_n(x)$. The explicit results of these deductions are as follows:

n=0: n=0:

$$w(x) = \frac{K}{2\sqrt{M_p - |m_0|}} x^2 + C_1 x + C_2 \quad , \quad x \in \Omega_p \tag{15}$$

n=1: n=1:

$$w(x) = \frac{4K}{3m_1^2} \sqrt{\left(M_p - |M(x)|\right)^3} + C_1 x + C_2 \quad , \quad x \in \Omega_p \tag{16}$$

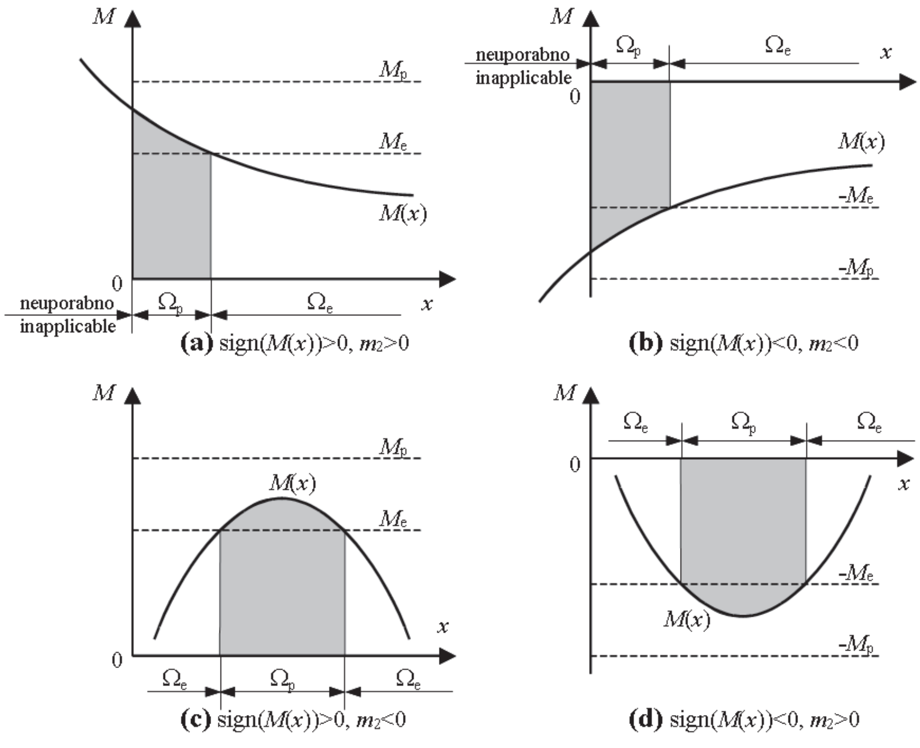
n=2: n=2:

$$w(x) = \begin{cases} \dfrac{K}{2\sqrt{|m_2|^3}} \left[T(x) \arcsin \dfrac{T(x)}{|D|} + \sqrt{D^2 - T^2(x)} \right] + C_1 x + C_2 & ; \quad \operatorname{sign}(M(x)) m_2 > 0 \\ \dfrac{K}{2\sqrt{|m_2|^3}} \left[T(x) \operatorname{arsh} \dfrac{T(x)}{|D|} - \sqrt{D^2 + T^2(x)} \right] + C_1 x + C_2 & ; \quad \operatorname{sign}(M(x)) m_2 < 0 \end{cases} \tag{17}$$

$$, \quad x \in \Omega_p$$

V zgornji enačbi je $T(x)=dM(x)/dx=2m_2x+m_1$ notranja prečna sila, D pa konstanta, izračunana iz koeficientov polinoma drugega reda kot

From the above, $T(x)=dM(x)/dx=2m_2x+m_1$ is the shear force and D is a constant obtained from the second-order polynomial coefficients as



Sl. 3. *Mogoče kvadratične porazdelitve momenta, ki povzročijo plastično tečenje*
Fig. 3. *Possible parabolic moment distributions causing plastic yielding*

$D^2 = B^2 - 4AC$, pri čemer je ta polinom definiran z enačbo $f(x)=M_p-|M(x)|=Ax^2+Bx+C$.

Pravilna izbira rešitve za upogibnico $w(x)$ v primeru kvadratične porazdelitve momenta $M(x)$ je odvisna od oblike uvedene odvisnosti $f(x)$ v elastoplastičnem območju Ω_p. Oblika odvisnosti, ki jo prikazuje slika 3, predstavlja dve fizikalno različni možnosti, kar posledično vodi k dvojnosti rešitve za $w(x)$. Matematično vzeto sta ti dve možnosti za dano porazdelitev momenta $M(x)$ enolično definirani z zmnožkom vodilnega koeficienta m_2 in predznaka momenta $M(x)$. Iz prikazanih grafov za vse mogoče porazdelitve $M(x)$ izhaja, da sta porazdelitvi (a) in (b), ki ju lahko popišemo tudi s $\mathrm{sign}(M(x))\cdot m_2>0$, enakovredni glede na potek odvisnosti $f(x)$, predstavljeno z ukrivljenostjo $d^2w/dx^2<0$. Podobna enakovrednost, opredeljena z ukrivljenostjo $d^2w/dx^2>0$, velja tudi za porazdelitvi (c) in (d), ki pa ju lahko popišemo s $\mathrm{sign}(M(x))\cdot m_2<0$.

Skratka, analitično lahko izračunamo upogibnico nosilcev pravokotnega prereza z uporabo enačb (14) in (17), pri čemer uporabimo enačbo (13) za K.

3 SPLOŠNI OPIS POSTOPKA REŠEVANJA

Ko določimo porazdelitev notranjega upogibnega momenta $M(x)$ za celotni nosilec, ki je za statično določene probleme odvisna samo od zunanjih sil in ne od razvoja plastičnega stanja, območje konstrukcije Ω razdelimo na $N_{ED}+N_{PD}$ polj glede na trenutno mehansko stanje: N_{ED} elastičnih polj $\Omega_e^{(i)}$, $(i=1,...,N_{ED})$ in N_{PD} elastoplastičnih polj $\Omega_p^{(j)}$, $(j=1,...,N_{PD})$. Število polj, na katero razdelimo konstrukcijo, je odvisno od števila različnih predpisov odvisnosti porazdelitve momenta $M(x)$. Takšna razdelitev, ki v celoti pokriva reševanje elastičnega odziva, zahteva dodatne razdelitve, ko se v kateremkoli polju pojavijo plastične deformacije. Takšna razdelitev na koncu prinese N_{PD} polj $\Omega_p^{(j)}$, kjer absolutna vrednost upogibnega momenta $M(x)$ preseže elastični mejni moment M_e, ter N_{ED} polj $\Omega_e^{(i)}$, kjer elastični mejni moment ni presežen. V vsakem od tako dobljenih polj je upogibnica $w(x)$ popisana z eno od enačb (14) do (17), kar privede do problema

$D^2 = B^2 - 4AC$, the considered polynomial being defined by the relation $f(x)=M_p-|M(x)|=Ax^2+Bx+C$.

Regarding the correct selection of the deflection line solution $w(x)$ in the case of a parabolic moment distribution $M(x)$, attention is to be paid to the established functional behaviour of the function $f(x)$ in the elasto-plastic domain Ω_p. This behaviour is characterized, as shown in Fig. 3, by two physically different situations which appear alternatively, and lead, as a consequence, to the duality of the solution $w(x)$. Mathematically, the moment distribution $M(x)$ given these situations is uniquely defined by the product of the leading polynomial coefficient m_2 and the sign of the moment $M(x)$. From the displayed graphs of all possible moment distributions $M(x)$ it follows that the distributions (a) and (b), which are otherwise characterized by the sign of $(M(x))\cdot m_2>0$ are equivalent with respect to the functional behaviour of the function $f(x)$, represented by the curvature $d^2w/dx^2<0$. Similar equivalence, yielding the curvature $d^2w/dx^2>0$, can be attributed to the distributions (c) and (d) that are characterized by the sign of $(M(x))\cdot m_2<0$.

To summarize, the explicit deflection curve for beams with a rectangular cross-section can be calculated with Equations (14) to (17), using Equation (13) for the constant K.

3 GENERAL DESCRIPTION OF THE SOLVING PROCEDURE

After the determination of the bending-moment distribution $M(x)$ over the whole beam, which is for statically determinate problems dependent only on the external loads and not on the plastic state evolution, the structure domain Ω should be partitioned in accordance with the actual mechanical state into $N_D=N_{ED}+N_{PD}$ sub-domains: N_{ED} elastic sub-domains $\Omega_e^{(i)}$, $(i=1,...,N_{ED})$ and N_{PD} elasto-plastic sub-domains $\Omega_p^{(j)}$, $(j=1,...,N_{PD})$. The number of sub-domains that the structure is divided into is first dictated by the number of different functions defining the moment distribution $M(x)$. This partitioning, which completely covers the solution of the problem under the presumed elastic response, needs, however, a further subdivision if in any of these sub-domains the plastic yielding occurs. The latter subdivision finally yields N_{PD} sub-domains $\Omega_p^{(j)}$, i.e., regions where the absolute value of the bending moment $M(x)$ exceeds the elastic limit moment M_e, and N_{ED} sub-domains $\Omega_e^{(i)}$, where the elastic limit moment is not exceeded. In each of the thus obtained sub-domains the deflection $w(x)$ is governed by a corresponding iskanja vrednosti pripadajočih neznanih integracijskih konstant $\{C_1,C_2\}_j$; (j=1,...,N_D), saj je preostanek odvisnosti $w(x)$ eksplicitno znan. Sistem enačb, ki ga je treba rešiti, če želimo izračunati skupno $2{\cdot}N_D$ neznanih konstant, je sestavljen z upoštevanjem robnih pogojev in pogojev skladnega prehoda. Naj na tem mestu poudarimo, da je za statično določene konstrukcije sistem vedno linearen in razmeroma majhen, saj je število polj razmeroma majhno celo za zahtevne konstrukcije. Pri statično nedoločenih konstrukcijah pa se pojavijo dodatne neznanke, toda celotni sistem enačb se da v splošnem prevesti na majhen sistem nelinearnih enačb, ki ga je pa treba rešiti numerično. Opisana metoda je zelo učinkovita in preprosta za uporabo, kar bo prikazano v računskem primeru.

function given by (14) to (17), which results in the problem of finding the respective values of a set of unknown integration constants $\{C_1,C_2\}_j$; (j=1,...,N_D), because the rest of the function $w(x)$ is explicitly known. The system of equations needed to solve the unknown constants, their number being $2{\cdot}N_D$ in total, is obtained upon the implementation of the corresponding boundary and continuity conditions. Here, let us emphasize that for an arbitrary statically determinate structure this system is always linear and rather small, as the number of sub-domains is relatively low, even in the case of complex structures. For statically indeterminate structures additional unknowns appear, but the whole system of equations can be generally reduced to a small system of nonlinear equations that should be solved numerically. The described method is highly efficient and simple to use, which will be demonstrated through a numerical example.

4 RAČUNSKI PRIMER

V nadaljevanju bo obravnavan mehanski odziv dvakrat členkasto podprtega nosilca s previsnim poljem, ki je obremenjen s točkovno silo F ($F \geq 0$) na prostem koncu in z zvezno obtežbo q ($q \geq 0$) med podporama (sl. 4). Naj parameter λ ($0<\lambda\leq 1$) pomeni obremenitveni parameter, ki določa stopnjo vnesene obremenitve glede na mejno obremenitev ($\lambda=1$), pri kateri se pojavi izguba funkcionalnosti konstrukcije, ki preide v mehanizem. Nadalje naj λ_e pomeni stopnjo obremenitve, ki ustreza tisti mejni elastični obremenitvi, pri kateri se plastične deformacije še niso pojavile. Za obremenjevanje, katerega posledica je elastični odziv ($\lambda \leq \lambda_e$), je značilna sorazmernost, torej je:

4 NUMERICAL EXAMPLE

The mechanical response of a simply supported overhanging beam that is subject to a concentrated force F ($F \geq 0$) at its free end and to a uniformly distributed load q ($q \geq 0$) between the supports (Fig. 4), will be considered in the following. Let the parameter λ ($0<\lambda 1$) be the role of the loading parameter that defines the level of the applied loading measured with respect to the limit loading ($\lambda=1$), by which the loss of the structural functionality with the transition to a mechanism occurs. Furthermore, let λ_e denote the loading level corresponding to the elastic limit loading, which is still characterized by the absence of plastic strain in the structure. The application of the loads that result in a linear elastic response ($\lambda \leq \lambda_e$) is characterised by proportionality, therefore:

$$M^{\lambda}(x) = \frac{\lambda}{\lambda_e} M^{e}(x) = \lambda M^{p}\left(x\right) \quad (18),$$

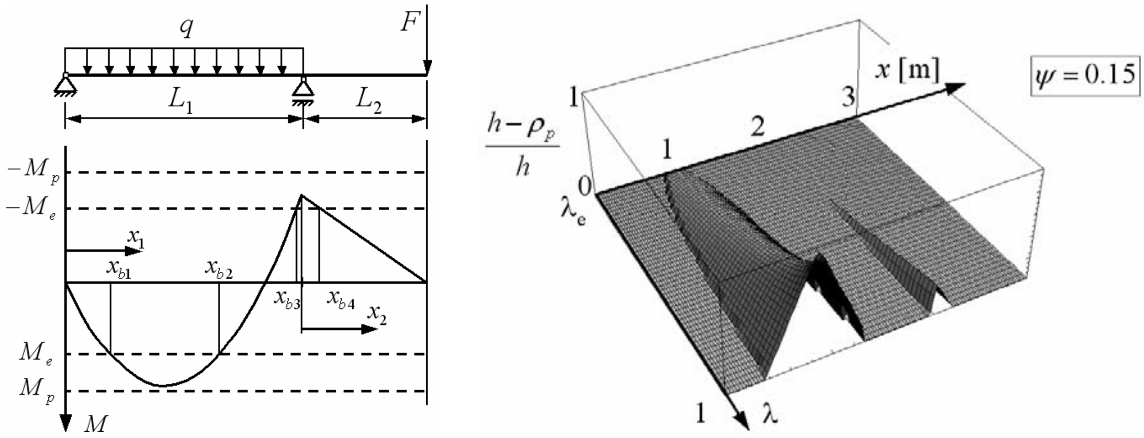
kjer $M^{e}(x)$ in $M^{p}(x)$ označujeta odvisnost upogibnega momenta pri stopnji obremenitve $\lambda=\lambda_e$ in $\lambda=1$, medtem ko je $M^{\lambda}(x)$, ki je shematično prikazan na sliki 4, porazdelitev momenta glede na stopnjo obremenitve λ.

Pri razvoju plastičnih deformacij je poleg obremenitvenega parametra λ pomembno tudi razmerje obeh obremenitev. To razmerje označimo s koeficientom ψ:

where $M^{e}(x)$ and $M^{p}(x)$ denote the functional dependencies of the bending moment at the load levels $\lambda=\lambda_e$ and $\lambda=1$, respectively, and $M^{\lambda}(x)$, shown schematically in Fig. 4, is the respective moment dependence corresponding to the load level λ.

Of importance for the evolution of the plastic strains there is, along with the increase of the loading parameter λ, the ratio established between the two loads. Let us denote the considered load ratio with the coefficient ψ:

$$\psi = \frac{F}{qL_1} \quad (19).$$



Sl. 4. *Računski primer ter širjenje plastičnega območja s povečevanjem stopnje obremenitve* λ ($\lambda_e < \lambda \leq 1$)
Fig. 4. *Numerical example and plastic-zone propagation with increasing load level* λ ($\lambda_e < \lambda \leq 1$)

Ker pri predpostavljenem sorazmernem obremenjevanju razmerje med obremenitvama q in F ostaja nespremenjeno, oznaki obremenitve zaradi celovitosti informacije razširimo z indeksom ψ, torej $q \to q_\psi$ in $F \to F_\psi$. Obremenjevanje ($d\lambda > 0$) je torej podano z:

Since by assumed proportional loading the ratio between the loads q and F remains fixed for any load level λ, the load notations may be enlarged in order to complete the information by adding the index ψ, i.e., $q \to q_\psi$ and $F \to F_\psi$. The application of the loads ($d\lambda > 0$) is thus characterised by:

$$F_\psi = \lambda F_\psi^{\mathrm{p}} \;\wedge\; q_\psi = \lambda q_\psi^{\mathrm{p}} \quad ; \quad 0 < \lambda \leq 1 \tag{20}$$

kjer q_ψ^{p} in F_ψ^{p} označujeta velikost obremenitev, ki povzročijo porušitev konstrukcije, kar se dejansko zgodi s pojavom plastičnega členka. Odvisnost upogibnega momenta $M(x)$ ($0 \leq x = x_1 \leq L_1 \;\wedge\; 0 \leq x = x_2 \leq L_2$) je izražena z brezrazsežnima koordinatama $\xi(\xi L_1 = x_1)$ in $\eta(\eta L_2 = x_2)$, kjer je $0 \leq \xi, \eta \leq 1$,

where q_ψ^{p} and F_ψ^{p} denote the magnitudes of the loads causing the collapse of the structure, which actually happens during the occurrence of a plastic hinge. The bending-moment function $M(x)$ ($0 \leq x = x_1 \leq L_1 \;\wedge\; 0 \leq x = x_2 \leq L_2$) is expressed in terms of the non-dimensional coordinates $\xi(\xi L_1 = x_1)$ and $\eta(\eta L_2 = x_2)$, where obviously $0 \leq \xi, \eta \leq 1$,

$$\begin{aligned} M_1(\xi) &= F_\psi L_2 \left[\frac{m}{2\psi}(1-\xi) - 1\right]\xi = q_\psi L_1^2 \left[\frac{1}{2}(1-\xi) - \frac{\psi}{m}\right]\xi \\ M_2(\eta) &= F_\psi L_2 (\eta - 1) = q_\psi L_1^2 \frac{\psi}{m}(\eta - 1) \qquad ; \quad m = \frac{L_1}{L_2} \end{aligned} \tag{21}$$

Glede na razvoj elastoplastičnega stanja se v odvisnosti od obremenilnega razmerja ψ pojavi več mogočih primerov. Če prevladuje obremenitev q, tj. $0 \leq \psi \leq \psi_1 = \left(2 - \sqrt{15}/2\right)m$, se plastifikacija nosilca pojavi med podporama, če pa prevladuje sila F, tj. $\psi_3 = \left(7 - 2\sqrt{15}\right)m/6 \leq \psi < \infty$, se nosilec plastificira pod desno podporo, v obeh primerih ne glede na stopnjo obremenitve ($\lambda_e < \lambda \leq 1$). Značilnost primerov, kjer je obremenilno razmerje $\psi_1 < \psi < \psi_3$, je obstoj dveh plastičnih področij pri velikosti momenta ob porušitvi. V teh primerih, razen za obremenilno razmerje

In relation to the load ratio ψ, several possible cases can occur ψ, regarding the elasto-plastic state evolution. While the prevailing of the load q, i.e., $0 \leq \psi \leq \psi_1 = \left(2 - \sqrt{15}/2\right)m$ results in the plastification of the beam between the supports, the prevailing of the load F, i.e., $\psi_3 = \left(7 - 2\sqrt{15}\right)m/6 \leq \psi < \infty$, leads to the beam plastification at the right support, both regardless of the load level ($\lambda_e < \lambda \leq 1$). The load cases corresponding to the load ratio $\psi_1 < \psi < \psi_3$ are characterized at the moment of collapse by the existence of two separate plastic regions. Typical for the elasto-plastic evolution in the latter cases, except for $\psi = \psi_2 = \left(3/2 - \sqrt{2}\right)m$,

$\psi = \psi_2 = \left(3/2 - \sqrt{2}\right)m$, pri katerem nastaneta dva plastična členka hkrati, je za razvoj elasto-plastičnega stanja značilno, da se najprej pojavi eno plastično območje ($\lambda_e<\lambda\leq\lambda^*<1$), šele nato pa tudi drugo ($\lambda^*<\lambda\leq1$).

Oglejmo si primer razvoja elasto-plastičnega stanja za primer, ko je m=2 in ψ=0,15. Ker je $\psi_1<\psi<\psi_2$, se plastične deformacije najprej pojavijo med podporama pri stopnji obremenitve λ_e in šele nato se pojavijo plastične deformacije pri stopnji obremenitve λ^* pri desni podpori. Rešitev problema, ki naj vsebuje tudi izračun upogibnice konstrukcije $w(x)$, je dobljena z upoštevanjem pripadajočih splošnih rešitev (14) do (17) glede na izkazan mehanski odziv pri danem obremenilnem razmerju ψ in pri vsiljeni stopnji obremenitve λ, pri čemer so izpolnjeni robni pogoji in pogoji skladnega prehoda na mejah med elastičnimi in elastoplastičnimi polji. Če si ogledamo odvisnost elasto-plastičnega dela rešitve za $\lambda_e<\lambda\leq\lambda^*$, opazimo, da rešitev sestavljajo štiri različne funkcije, ki popisujejo obnašanje v treh elastičnih in enem elastoplastičnem polju. Upoštevaje naravo snovnega odziva v teh poljih, je enačba (14) kot delna rešitev uporabljena trikrat, enačba (17) pa enkrat. Toda povečevanje obremenitve, izpolnjujoč odvisnost $\lambda^*<\lambda\leq1$, je povezano s plastifikacijo novega polja, kar privede do šestih različnih odvisnosti za popis končne rešitve, pri čemer je v treh poljih obnašanje elastično, v treh pa elastoplastično, kar je jasno razvidno iz porazdelitve momentov, prikazane na sliki 4. Tu je znova glede na prikazano porazdelitev momenta enačba (14) uporabljena trikrat, enačba (16) enkrat in (17) dvakrat. Da bi določili rešitev upogibnice $w(x)$ v celoti, preostane še izračun sistema 8 (ali 12) linearnih enačb z 8 (ali 12) neznanimi konstantami C_i. Z rešitvijo sistema, ki je dobljen z upoštevanjem robnih pogojev in pogojev skladnega prehoda, na koncu dobimo eksplicitni izraz za upogibnico $w(x)$.

Širjenje plastičnega območja vzdolž nosilca enolično določa enačba $M^\lambda(x)=M_e$, ko se vsiljene obremenitve povečujejo ($d\lambda>0 \wedge 0<\lambda\leq1$), medtem ko je širjenje plastičnega območja v globino nosilca dano z enačbo (11). Ker je sorazmernost notranjih veličin ohranjena med celotnim postopkom obremenjevanja za katerokoli obremenilno razmerje ψ, je velikost stopnje

i.e., the load ratio, which leads to the appearance of two plastic hinges at the same time, is that the first plastic region is created ($\lambda_e<\lambda\leq\lambda^*<1$), and only afterwards ($\lambda^*<\lambda\leq1$) is it followed by the appearance of the second plastic region.

Let us now present the elasto-plastic state evolution for the case where m=2 and ψ=0.15. Because $\psi_1<\psi<\psi_2$, the plastic strains first occur between the supports at load level λ_e and then at load level λ^* the plastic strains also occur at the right support. The solution of the problem, when the deflection of the structure $w(x)$ is to be evaluated as well, is obtained by considering the respective general solutions (14) to (17), according to the proven deformation response behaviour at a given load ratio ψ and load level λ applied and fulfilling the associated boundary conditions and continuity conditions at the interfaces between the elastic and the elasto-plastic sub-domains. Observing the functional dependence of the elasto-plastic part of the solution for $\lambda_e<\lambda\leq\lambda^*$, we find that it consists of four distinct functions that describe the respective response behaviour in three elastic sub-domains and one elasto-plastic sub-domain. Considering the evidenced nature of the material response in these sub-domains, Equation (14) is applied as the corresponding particular solution three times, and Equation (17) once. But a further increase in the loads, fulfilling the relation $\lambda^*<\lambda\leq1$, is accompanied by the plastification of new sub-domains, which demands six distinct functions for the corresponding solution description, three covering the elastic response and three covering the elasto-plastic response in the respective sub-domains, which is clearly seen from the moment distribution in Fig. 4. Again with regard to the shown bending-moment distribution, Equation (14) is applied three times, while Equation (16) is used once and (17) twice. There remains, in order to determine the solution of the deflection line functions $w(x)$ entirely, a derivation of a corresponding system of 8 (or 12) linear equations with 8 (or 12) constants C_i as unknowns. By solving this system, which is obtained by the imposition of the corresponding boundary and continuity conditions, an explicit expression for the solution $w(x)$ is finally determined.

The propagation of the plastic domain along the beam structure is uniquely defined by the equation $M^\lambda(x)=M_e$ as the applied loads are increased ($d\lambda>0 \wedge 0<\lambda\leq1$), while the penetration of the plastic region inside the beam is given by Equation (11). Also, as the proportionality of the internal forces is preserved through the whole loading, the magnitude of the elastic limit load level obremenitve, ki ustreza mejni elastični obremenitvi, vedno enaka λ_e=2/3. V nadaljevanju prikazani rezultati so dobljeni z upoštevanjem naslednjih geometrijskih in snovnih podatkov: dolžini nosilca L_1=2m in L_2=1m, širina prereza b=20mm ter njegova polovična višina h=20mm, modul elastičnosti E=210 GPa in meja tečenja σ_0=300MPa. Za dano geometrijsko obliko nosilca je na sliki 4 prikazan razvoj plastičnih deformacij, ko se stopnja obremenitve zveča in je v koraku $\lambda_e<\lambda\leq 1$. S te slike, ki predstavlja širino in globino plastičnih območij, lahko opazimo, da se je plastifikacija najprej pojavila med podporama in šele nato ob desni podpori. Ker je razvoj plastičnih deformacij sorazmeren obremenitvi, se edini plastični členek prav tako pojavi med podporama.

Vpliv upoštevanja snovne nelinearnosti v mehanskem odzivu vsekakor velja ovrednotiti kvantitativno, s čimer ocenimo smiselnost uporabe elastoplastičnih enačb (14) do (17) v primerjavi z izključno uporabo enačbe (14), pri katerih je neelastični vpliv zanemarjen. Ker so statične veličine, kakor so napetostne rezultante in reakcijske sile, neodvisne od narave odziva snovi v primeru statično določenega problema, analizo omejimo samo na izračun deformacijskih veličin, kot so poves $w(x)$, relativni zasuki $\varphi_\varepsilon(x)$ v okolici x ter mesto največjega pomika x_w med podporama. Kot merilo odstopanja od linearnega odziva, označenega z indeksom "e", uporabimo naslednje veličine: $r_w(x)$ za odstopanje pomikov ter $r_\varphi(x)$ za odstopanje relativnih zasukov okrog točke (v ε-okolici x), kjer se pojavi plastični členek. Opazovane veličine so definirane z:

is always, irrespective of the load ratio ψ, equal to λ_e=2/3. The results were calculated using the following geometrical and material data: lengths of the beam L_1=2m and L_2=1m, cross-sectional width b=20mm and half-height h=20mm, the modulus of elasticity E=210 GPa and the yield stress σ_0=300MPa. For this particular geometry of the structure there is, plotted in Fig. 4, a plastic strain evolution, as the load level is increased within the interval $\lambda_e<\lambda\leq 1$. From this figure, which represents the depth and the width of the plastic zone, it can be seen that the first plastic yielding occurs between the supports, but after that it appears also at the right support. Since the plastic strain evolution is proportional to the loading, the only plastic hinge occurs between the supports.

The impact of considering the material non-linearity in the mechanical response is certainly appropriate for being analysed quantitatively, thus estimating the reasonableness of using the elasto-plastic Equations (14) to (17) in comparison to Equation (14), where the inelasticity of the response is neglected. With the static quantities, i.e., the stress resultants and support reaction forces, independent of the nature of the material response in a statically determinate problem, such an estimation analysis reduces to an evaluation of the deformation quantities, such as the displacement $w(x)$, the relative rotation $\varphi_\varepsilon(x)$ in the ε-neighbourhood of x and the location of the maximum displacement x_w between the supports. As a measure for the deviation from the linear response, denoted by a suffix "e", the following quantities can be used: $r_w(x)$ for the displacement deviations and $r_\varphi(x)$ for the relative rotation deviations around the point (in the neighbourhood of x), where the plastic hinge occurs. The inspected quantities' definitions are as follows:

$$r_w(x)=\left(\frac{w(x)}{w^e(x)}-1\right)\cdot 100\% \,,\quad r_\varphi(x)=\left(\frac{\varphi(x+\Delta)-\varphi(x-\Delta)}{\varphi^e(x+\Delta)-\varphi^e(x-\Delta)}-1\right)\cdot 100\% \qquad (22).$$

V obravnavanem primeru je smiselno opazovati naslednje veličine: $r_w(x_C)$ na prostem koncu, $r_w(x_M)$ in $r_\varphi(x_M)$ pa na mestu največjega notranjega upogibnega momenta med podporama, pri čemer je Δ=10mm. Opazovane veličine so predstavljene v preglednici 1.

Vrednosti veličin v preglednici kažejo, da se pri sorazmernem večanju obremenitve od $\lambda=\lambda_e$ do $\lambda\to 1$ vpliv nelinearnosti povečuje, pri čemer je tabeliranje narejeno za boljši prikaz ob uporabi enakomernega obremenilnega koraka ($\Delta\lambda$=konst.). Konec koncev je to tudi

It is reasonable in the investigated example to observe the following quantities: $r_w(x_C)$ at the free end, $r_w(x_M)$ and $r_\varphi(x_M)$ at the position of the maximum bending moment between the supports, taking Δ=10mm. The observed quantities are presented in Table 1.

The tabulated quantities show that the proportional increase of the considered loading from $\lambda=\lambda_e$ to $\lambda\to 1$, the tabulation being made for greater evidence on the basis of an equidistant load-step incrementation ($\Delta\lambda$=const.), leads to the growing influence of the non-linearity. As a matter of fact, this

Preglednica 1. *Odstopanje elastoplastičnega od elastičnega odziva*
Table 1. *Deviations of elastic-plastic from the linear elastic response*

λ_i	$r_\varphi(x_M)$ [%]	$r_w(x_M)$ [%]	$r_w(x_C)$ [%]
$8/12 = \lambda_e$	0,0	0,0	0,0
9/12	2,6	0,9	3,7
10/12	13,1	5,7	22,3
11/12	45,4	20,4	79,0
$1 - 1{\cdot}10^{-9}$	13972,5	582,8	2027,9

pričakovano. S širjenjem plastičnega območja in s posledičnim slabljenjem elastične odpornosti konstrukcije je odstopanje od elastične rešitve vedno izrazitejše s približevanjem $\lambda \rightarrow 1$. V bližini singularne točke, kjer je dosežena končna vrednost λ=1, vse opazovane veličine izkazujejo naglo povečanje, pri čemer se asimptotično približujejo neskončnosti. Pri λ=1 je najbolj obremenjen prerez v celoti plastificiran in konstrukcija, v tem primeru statično določena, se spremeni v mehanizem s pojavom plastičnega členka. Ko se to zgodi, je izvirna namembnost in nosilnost konstrukcije izgubljena, kar je enako porušitvi.

is to be expected. Namely, with the intensification of the plastic propagation and consecutive weakening of the structural elastic resistance the departure from linearity becomes more and more apparent as the loads approach $\lambda \rightarrow 1$. Near the singularity point, where the ultimate load level λ=1 is reached, all the observed quantities exhibit a rapid increase, with their values asymptotically approaching infinity. At λ=1 the most stressed cross-section becomes fully plastified, and the structure, the problem being a statically determinate one, transforms into a mechanism due to the occurrence of a plastic hinge. When this happen, the originally designed functionality and load-carrying capacity of the structure are lost, which is equivalent to a collapse.

5 SKLEP

V prispevku je predstavljena zelo preprosta in učinkovita metoda za obravnavanje elastoplastičnega obnašanja ravnih nosilcev. Postopek temelji na eksplicitnih odvisnostih upogibnice za delno plastificirana polja nosilca pravokotnega prereza, obremenjenega z značilnimi obremenitvami, vključujoč točkovne sile in momente ter zvezno porazdeljeno obtežbo. Eksplicitne oblike odvisnosti upogibnic, prikazane z enačbami (14) do (17), so bile uspešno izpeljane analitično na osnovi klasične teorije upogiba nosilcev, kar pomeni trden matematični okvir za elastoplastično analizo tovrstnih črtnih konstrukcij.

Z uporabo eksplicitno izpeljanih rešitev postane postopek reševanja elastoplastičnih problemov upogiba nosilcev zelo preprost, ne glede na to, da snov izkazuje nelinearno obnašanje. Ker je končna rešitev dobljena z razrešitvijo razmeroma majhnega sistema enačb, je tak postopek tudi učinkovit. Predstavljena metoda je torej uporabna za spremljanje razvoja plastičnega deformiranja med sorazmernim obremenjevanjem nosilcev.

5 CONCLUSIONS

In the paper a very simple and efficient method for the elasto-plastic behaviour of straight beams is presented. The approach is based on the explicit deflection line functions for partially plastic domains of a beam with a rectangular cross-section under characteristic loading cases, including the application of concentrated forces and moments as well as the application of uniformly distributed loads. Explicit forms of the deflection lines, listed in Equations (14) to (17), were successfully analytically derived, following the classical beam-bending formulation, thus forming a firm mathematical framework for the elastic-plastic analyses of the considered kind of beam structures.

By taking the derived explicit solutions into account the procedure needed to solve an elasto-plastic bending-beam problem becomes very simple, irrespective of the exhibited non-linear material behaviour. Since the solution is obtained by solving a relatively small system of equations, this procedure is also efficient. The presented method can be used to trace the plastic-yielding evolution during the proportional loading of beams.

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