BLED WORKSHOPS
IN PHYSICS
VOL. 8, NO. 1
p. 66

Proceedings of the Mini-Workshop
Hadron Structure and Lattice QCD
Bled, Slovenia, July 9-16, 2007

Sigma meson in a two-level Nambu – Jona-Lasinio model ⋆

Mitja Rosina[a,b] and Borut Tone Oblak[c]

[a] Faculty of Mathematics and Physics, University of Ljubljana, Jadranska 19, P.O. Box 2964, 1001 Ljubljana, Slovenia
[b] J. Stefan Institute, 1000 Ljubljana, Slovenia
[c] National Institute of Chemistry, Hajdrihova 19, 1000 Ljubljana, Slovenia

Abstract. We continue the study of a schematic quasispin model similar to the Nambu – Jona-Lasinio model. The model is characterized by a finite number of quarks occupying a finite number of states in the Dirac sea as well as in the valence space (due to a sharp momentum cutoff and a periodic boundary condition). This allows the use of first quantization and an explicit wavefunction. Most low-lying states in the excitation spectrum can be interpreted as multi-pion states and one can deduce the effective pion-pion interaction and scattering length. However, the intruder states can be recognized as sigma-meson excitations or their admixtures to multi-pion states.

1 Introduction

In the Mini-Workshop Bled 2006 [1] we presented a soluble two-level quasispin model of spontaneous chiral symmetry breaking, inspired by the Nambu – Jona Lasinio model. It is the hadronic analogue of the Lipkin model in nuclear physics [2].

In our schematic model we enclose $N = \mathcal{N}$ quarks in a periodic box $\mathcal{V}$ and use a sharp momentum cutoff Λ, leading to a finite number $\mathcal{N} = N_c N_f \mathcal{V}\Lambda^3/3\pi^2$ of states in the Dirac sea and the same number of states in the valence "shell". We further simplify the one-flavour Nambu – Jona-Lasinio Hamiltonian by taking all quark kinetic energies equal to $\frac{3}{4}\Lambda$ and by neglecting the interaction terms which change the individual quark momenta:

$$H = \sum_{k=1}^{N}\left(\gamma_5(k)h(k)\,\tfrac{3}{4}\Lambda + m_0\beta(k)\right) - \frac{2G}{\mathcal{V}}\left(\sum_{k=1}^{N}\beta(k)\sum_{l=1}^{N}\beta(l) + \sum_{k=1}^{N}i\beta(k)\gamma_5(k)\sum_{l=1}^{N}i\beta(l)\gamma_5(l)\right)$$

Here $h = \boldsymbol{\sigma}\cdot\mathbf{p}/p$ is helicity and γ_5 and β are Dirac matrices. We use the popular model parameters close to [3,4], $\Lambda = 648$ MeV, $G = 40.6$ MeV fm, $m_0 = 4.58$

⋆ Talk delivered by M. Rosina

MeV, which yield the phenomenological values of quark constituent mass, quark condensate and pion mass both in full Nambu – Jona-Lasinio model as well as in our quasispin model (using in both cases the Hartree-Fock + RPA approximations). It has been shown in [1] that in the large N limit the exact results of our quasispin model tend in fact to the Hartree-Fock + RPA values.

2 What can we learn from the excitation spectrum?

It is very convenient to introduce the quasispin formalism using the fact that the following operators obey (quasi)spin commutation relations

$$j_x = \frac{1}{2}\beta\,, \quad j_y = \frac{1}{2}i\beta\gamma_5\,, \quad j_z = \frac{1}{2}\gamma_5\,,$$

The (quasi)spin commutation relations are also obeyed by separate sums over quarks with right and left helicity as well as by the total sum ($\alpha = x,\, y,\, z$)

$$R_\alpha = \sum_{k=1}^{N} \frac{1+h(k)}{2} j_\alpha(k)\,, \quad L_\alpha = \sum_{k=1}^{N} \frac{1-h(k)}{2} j_\alpha(k)\,, \quad J_\alpha = R_\alpha + L_\alpha = \sum_{k=1}^{N} j_\alpha(k)\,.$$

The model Hamiltonian can then be written as

$$H = 2P(R_z - L_z) + 2m_0 J_x - 2g(J_x^2 + J_y^2) \quad . \tag{1}$$

It commutes with R^2 and L^2 but not with R_z and L_z. Nevertheless, it is convenient to work in the basis $|\,R, L, R_z, L_z\,\rangle$.The Hamiltonian matrix elements can be easily calculated using the angular momentum algebra. By diagonalisation we then obtain the energy spectrum of the system.

Table 1. The spectrum of the quasispin model with $N = 144$, quantum numbers $R+L = 36$ and model parameters listed in the Introduction

Parity	$(E - E_0)$[MeV]	n	$\bar{V}$[MeV]	A	B	C
+	932	10	-9.5	-0.9	-0.0	0.3
−	803	9	-11.7			
+	771	8	-11.3	-0.0	-0.0	-0.0
−	767	7	-8.8			
+	646	6	-11.4	4.8	0.9	-2.2
+	634	6	-12.2	0.3	0.1	-0.1
−	580	5	-10.0			
+	482	4	-10.5	-0.3	-0.2	-0.0
−	378	3	-10.1			
+	261	2	-10.3	3.5	2.3	-0.2
−	136	1				
+	0	0		-18.4	-18.4	-30.0

The ground state is the vacuum. Most excited states can be interpreted as multi-pion states while the intruder state is suggestive of the sigma mesons. Here n is the guessed number of pions while other columns will be explained in the following subsections where we discuss the harvest of the excitation spectrum.

2.1 Pion-pion scattering

Since we are working in a finite volume $\mathcal{V}$ with periodic boundary conditions we cannot impose scattering boundary conditions. Instead of a continuous spectrum of scattering states we obtain a discrete spectrum. Energy levels of n-pion states can be interpreted to contain the average effective pion-pion potential $\bar{V}$ given in Table 1:

$$E_{n\pi} = n\, m_\pi + \frac{n(n-1)}{2}\bar{V}.$$

Let us repeat our results presented last year [1]. We calculate the s-state scattering length in the first-order Born approximation

$$a = \frac{m_\pi/2}{2\pi}\int V(\mathbf{r})\, d^3r = \frac{m_\pi}{4\pi}\bar{V}\mathcal{V}. \tag{2}$$

This formula was first quoted by M.Lüscher [5] in 1986 and 1991 and later by many authors. It was derived in a much more sophisticated way, but in our context it is just the first-order Born approximation.

In our example for $N = 144$ we have $\bar{V} = -10.3\,\text{MeV}$ and $\mathcal{V} = \pi^2 N/\Lambda^3 = 40\,\text{fm}^3$ This gives

$$a\, m_\pi = \frac{m_\pi^2}{4\pi}\bar{V}\mathcal{V} = -0.0836. \tag{3}$$

Since there are no experiments with one-flavour pions we compare with the two-flavour value ($I = 2$). The chiral perturbation theory (soft pions) suggests in leading order $a_0^{I=2}\, m_\pi = -m_\pi^2/16\pi f_\pi^2 = -0.0445$. The old analysis of Gasser and Leutwyler gave -0.019 and the more recent analysis by Lesniak gave -0.034 ("non-uniform fit") or -0.044 ("uniform fit"). We get about twice larger value in our one-flavour model due to the artifact that we made up for the second flavour by replacing $G \to 2G$.

2.2 The sigma meson

In the spectrum in Table 1 one can clearly distinguish the presence of the sigma meson by noticing the doubling of the positive parity states at 634 and 646 MeV. Moreover, the state at 646 MeV has strong transition matrix elements from the ground state for positive parity one-body operators (see Table 2):

$$2\hat{A} = R_+ + L_- = J_x + i(R_y - L_y)$$
$$2\hat{B} = R_- + L_+ = J_x - i(R_y - L_y)$$
$$2\hat{C} = R_z - L_z$$

On the other hand, the state at 634 MeV has much smaller transition matrix elements. This is s a good argument that the state at 646 MeV is a rather pure sigma meson. To conclude, we are still devising a method how to extract from the spectrum the width of the sigma meson for the $\sigma \to \pi\pi$ decay

2.3 Comparison with different particle-hole methods

In *particle-hole methods* (=approximations) sigma meson is introduced as

$$|\sigma\rangle = (a\,\hat{A} + b\,\hat{B} + c\,\hat{C})|g\rangle\,.$$

In Table 2. we present excitation energies as well as transition matrix elements $A = \langle\sigma|\,\hat{A}\,|g\rangle$ and similar for B and C.

Table 2. Excitation energies and transition matrix elements for various approximations

	$(E - E_0)$[MeV]	A	B	C
Exact	646	4.8	0.9	-2.2
TD	555	5.8	1.8	-2.4
HOM	530	5.8	1.8	-2.4
RPA	668	5.6	1.0	-2.2

In the Tamm-Dancoff approximation (TD) the coefficients a, b, c are determined by diagonalizing the 3×3 Hamiltonian matrix in the corresponding particle-hole space. Similarly, in the Hermitian Operator Method (HOM) [6] $a = b$ and one diagonalizes a 2×2 Hamiltonian matrix. IN The Random Phase Approximation (RPA) one solves the RPA equations assuming for $\mathcal{O}^\dagger = (a\,\hat{A} + b\,\hat{B} + c\,\hat{C})$:

$$|\,\sigma\,\rangle = \mathcal{O}^\dagger|\,\Phi_0\,\rangle\ ,\ \text{with}\ \mathcal{O}|\,\Phi_0\,\rangle = 0\ ,\ \text{and}\ |\,\Phi_0\,\rangle = |\,HF\,\rangle.$$

References

1. M. Rosina and B. T. Oblak, Bled Workshops in Physics **7**, No.1, 92 (2006); also available at http://www-f1.ijs.si/BledPub.
2. H. J. Lipkin, N. Meshkov, A. J. Glick, Nucl. Phys. **62**, 188 (1965).
   N. Meshkov, A. J. Glick, H. J. Lipkin, Nucl. Phys. **62**, 199 (1965).
   A. J. Glick, H. J. Lipkin, N. Meshkov, Nucl. Phys. **62**, 211 (1965).
   D. Agassi, H. J. Lipkin, N. Meshkov, Nucl. Phys. **86**, 321 (1966).
3. M. Fiolhais, J. da Providência, M. Rosina and C. A. de Sousa, Phys. Rev. C **56**, 3311 (1997).
4. M. Buballa, Phys. Reports **407**, 205 (2005).
5. M. Lüscher, Commun. Math. Phys. **104**, 177 (1986); **105**, 153 (1986); Nucl. Phys. **B354**, 531 (1991).
6. M. Bouten, P. van Leuven, M. V. Mihailović and M. Rosina, Nucl. Phys. **A202**, 127 (1973).