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The modified Wiener index of some graph operations

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Abstract

Graovac and Pisanski [On the Wiener index of a graph, J. Math. Chem. **8** (1991) 53 – 62] applied an algebraic approach to generalize the Wiener index by symmetry group of the molecular graph under consideration. In this paper, exact formulas for this graph invariant under some graph operations are presented.

Keywords: Modified Wiener index, graph operation, automorphism group.

Math. Subj. Class.: 20C15, 20D15

1 Introduction

Throughout this paper graph means simple connected graphs. The distance between the vertices u, v of a graph G, $d_G(u, v)$ (or $d(u, v)$ for short), is defined as the number of edges in a shortest path connecting them. The sum of all distances between vertices in G is called the *Wiener index* of G [9]. The first study of this number were made by Harold Wiener in 1947 who realized that there are correlations between this graph invariant and the boiling points of paraffin. We encourage the reader to consult [1, 2] and references therein for information about the effect of this graph invariant on trees and hexagonal systems and [4] for some applications in chemistry.

Let $G = (V, E)$ be a simple graph with the vertex set V and the edge set E. Graovac and Pisanski [3] in a seminal paper applied the symmetry group of the graph under consideration to generalize the Wiener index. To explain, we assume that Γ is the automorphism group of G. Then the *distance number* of an automorphism g, $\delta(g)$, is defined as the average of $d(u, g(u))$ over all vertices $u \in V(G)$ and

$$\delta(G) = \frac{1}{|\Gamma|}\sum_{g\in\Gamma}\delta(g) = \frac{1}{|\Gamma||V(G)|}\sum_{u\in V(G)}\sum_{g\in\Gamma}d(u, g(u)).$$

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Define:

$$\widehat{W}(G) = \frac{|V(G)|}{2|\Gamma|} \sum_{u \in V(G)} \sum_{g \in \Gamma} d(u, g(u)).$$

It can be easily shown that

$$\widehat{W}(G) = \frac{1}{2}|V(G)|^2 \delta(G).$$

The authors of [3], in their pioneering work used the name "*modified Wiener index*" for this graph invariant. Suppose $e = uv \in E(G)$ and $V(e) = \{u, v\}$. The line graph $L(G)$ is a graph with $E(G)$ as vertex set. Two different vertices of $V(L(G))$ are adjacent if and only if they have a common vertex in G. The subdivision graph $S(G)$ is the graph obtained by inserting an additional vertex in each edge of G. In other words, each edge of the subdivision graph is replaced by a path of length 2.

Suppose G is a graph. Following Yan et al. [8], we set $EE(G) = \{\{e, f\} \mid e, f \in E(G) \ \& \ |V(e) \cap V(f)| = 1\}$ and $EV(G) = \{\{e, v\} \mid v \in V(G) \ \& \ v \in V(e) \ \& \ e \in E(G)\}$, where $V(e)$ is the set of all end vertices of the edge e. Define the line graph $L(G)$, the subdivision graph $S(G)$, the total graph $T(G)$ and the graphs $R(G)$ and $Q(G)$ as follows:

$$\begin{aligned}
V(L(G)) &= E(G), & E(L(G)) &= EE(G), \\
V(S(G)) &= V(G) \cup E(G), & E(S(G)) &= EV(G), \\
V(T(G)) &= V(G) \cup E(G), & E(T(G)) &= E(G) \cup EV(G) \cup EE(G), \\
V(R(G)) &= V(G) \cup E(G), & E(R(G)) &= E(G) \cup EV(G), \\
V(Q(G)) &= V(G) \cup E(G), & E(Q(G)) &= EV(G) \cup EE(G).
\end{aligned}$$

Throughout this paper we use the standard notations of group theory and graph theory. We refer to [5], for the main properties of product graphs. If G is a connected graph then the diameter $d = \mathrm{diam}(G)$ is defied as the length of the largest distance between two vertices in G. Moreover, define

$$\overline{D}(g, i) = |\{\{u, g(u)\} \mid u \in V(G) \ \& \ d_G(u, g(u)) = i\}|; 1 \le i \le d,$$

and $\overline{D}(\Gamma, i) = \sum_{g \in \Gamma} \overline{D}(g, i)$; $1 \le i \le d$. The number of $\{u, v\}$ such that $d(u, v) = i$ in $\widehat{W}(G)$ is equal to $\overline{D}(G, i) = \overline{D}(\Gamma, i)$. Suppose x and y are vertices of G. We write $x \sim_G y$ to show that x, y are adjacent in G. They are called equivalent, $x \approx_G y$, if there exists an automorphism α such that $\alpha(x) = y$. The path, cycle and complete graphs with n vertices are denoted by P_n, C_n and K_n, respectively. The number of edges in a path P is denoted by $l(P)$ and named the length of P. Our other notations are standard and taken mainly from the standard books on these topics.

Main Theorem. *Suppose G is a tree of diameter d. Then the following relations hold:*

1. $\widehat{W}(L(G)) = \frac{n-1}{n}\widehat{W}(G) - \frac{n-1}{2|\Gamma|}\sum_{i=1}^{d} \overline{D}(G, i)$,
2. $\widehat{W}(S(G)) = \frac{4n-2}{n}\widehat{W}(G) + \frac{4n-2}{n-1}\widehat{W}(L(G))$,
3. $\widehat{W}(T(G)) = \frac{2n-1}{n}\widehat{W}(G) + \frac{2n-1}{n-1}\widehat{W}(L(G))$,

4. $\widehat{W}(Q(G)) = \frac{4n-2}{n}\widehat{W}(G)$,
5. $\widehat{W}(R(G)) > \frac{4n-2}{n}\frac{|\Gamma|}{|\mathrm{Aut}(R(G))|}\widehat{W}(G)$.

2 Proof of the Main Theorem

In [6], a character theoretical method for computing the modified Wiener index of graphs is presented and in [8], the authors computed exact formulas for the Wiener index under five graph operations. The aim of this paper is to continue these papers by computing the modified Wiener index of trees under the graph operations $L(-), S(-), T(-), Q(-)$ and $R(-)$. For simplicity of our argument, we assume that $\Gamma = \mathrm{Aut}(G)$ and $\overline{W}(G) = \frac{2|\Gamma|}{|G|}\widehat{W}(G)$ then $\overline{W}(G) = \sum_{i=1}^{d} i.\overline{D}(G,i)$. We will start by stating a well-known result in algebraic graph theory.

Lemma 2.1. *Suppose G is a tree with at least three vertices. Then* $\mathrm{Aut}(L(G)) \cong \mathrm{Aut}(G)$.

Proof. It is an immediate consequence of [7, Corollary 1.4]. □

Theorem 2.2. *Let G be a tree with $n \geq 3$ vertices and $\Gamma = \mathrm{Aut}(G)$. Then, $\widehat{W}(L(G)) = \frac{n-1}{n}\widehat{W}(G) - \frac{n-1}{2|\Gamma|}\sum_{i=1}^{d}\overline{D}(G,i)$.*

Proof. Suppose $e = uv$ and $f = xy$ are vertices of $L(G)$ such that $e \approx_{L(G)} f$ and $d_{L(G)}(e,f) = i$. So, there are $\sigma \in \mathrm{Aut}(G)$ and $\bar{\sigma} \in \mathrm{Aut}(L(G))$ such that $\bar{\sigma}(e) = f$, $\sigma(u) = x$ and $\sigma(v) = y$. Choose a shortest path $e = e_0, e_1, \ldots, e_i = f$ in $L(G)$. Set $e = e_0 = u_0u_1, e_1 = u_1u_2, \ldots, e_{i-1} = u_{i-1}u_i$, $f = e_i = u_iu_{i+1}$, $u = u_0$, $v = u_1$, $y = u_i$ and $x = u_{i+1}$. Since G is a tree, $u = u_0, u_1, \ldots, u_i, u_{i+1} = x$ is a shortest path in G connecting $u = u_0$ and $x = u_{i+1}$. Thus $d_G(u,x) = i+1$. Therefore, for any vertices e and f of $L(G)$ at distance i, we fined two vertices u_e and u_f of G at distance $i+1$, corresponding to e and f, respectively.

We now assume that r and s are vertices in G at distance i and $r = v_0, v_1, \ldots, v_{i-1}$, $v_i = s$ is the unique shortest path connecting r and s. Then the edges v_0v_1 and $v_{i-1}v_i$ are at distance $i-1$ in $L(G)$. Hence, $\overline{D}(L(G),i) = \overline{D}(G,i+1)$. Therefore,

$$\begin{aligned}\overline{W}(L(G)) &= \sum_i i\overline{D}(L(G),i)\\ &= \sum_i (i-1)\overline{D}(G,i)\\ &= \sum_i i\overline{D}(G,i) - \sum_i \overline{D}(G,i)\\ &= \overline{W}(G) - \sum_i \overline{D}(G,i).\end{aligned}$$

Therefore, $\frac{2|\mathrm{Aut}(L(G))|}{|V(L(G))|}\widehat{W}(L(G)) = \frac{2|\Gamma|}{|V(G)|}\widehat{W}(G) - \sum_i \overline{D}(G,i)$, which completes our argument. □

Lemma 2.3. *Suppose G is a tree. Then* $\mathrm{Aut}(S(G)) \cong \mathrm{Aut}(G)$.

Proof. Define $\Phi : \mathrm{Aut}(G) \longrightarrow \mathrm{Aut}(S(G))$ given by $\Phi(\alpha)|_{V(G)} = \alpha$ and if $e = xy \in E(G)$ then $\Phi(\alpha)(e) = \alpha(x)\alpha(y) \in E(G)$. Notice that if $x \sim_G y$ and t, t' are vertices in $S(G)$ such that $x \sim_{S(G)} t \sim_{S(G)} y$ and $\Phi(\alpha)(x) \sim_{S(G)} t' \sim_{S(G)} \Phi(\alpha)(y)$ then $\Phi(\alpha)(t) = \Phi(\alpha)(t')$. It can easily be proved that $\Phi(\alpha)$ is a permutation of $S(G)$. We show that $ab \in E(S(G))$ if and only if $\Phi(\alpha)(a)\Phi(\alpha)(b) \in E(S(G))$. Suppose $a \sim_{S(G)} b$ and $a \in G$. Then there exists $c \in G$ such that $a \sim_G c$ and $a \sim_{S(G)} b \sim_{S(G)} c$. Since $\alpha \in \mathrm{Aut}(G)$, $ac \in E(G)$ if and only if $\alpha(a)\alpha(c) \in E(G)$. Hence, there exists $l \in S(G)$ such that $\Phi(\alpha)(a) \sim_{S(G)} l \sim_{S(G)} \Phi(\alpha)(c)$. This implies that $\Phi(\alpha)(b) = l$. So, $ab \in E(S(G))$ if and only if $\Phi(\alpha)(a)\Phi(\alpha)(b) \in E(S(G))$. Similarly, if $\overline{\sigma} \in \mathrm{Aut}(S(G))$ then $\sigma = \overline{\sigma}|_G \in \mathrm{Aut}(G)$. Thus, Φ is invertible. □

Theorem 2.4. *Let G be a tree with $n \geq 3$ vertices. Then*

$$\widehat{W}(S(G)) = \frac{4n-2}{n}\widehat{W}(G) + \frac{4n-2}{n-1}\widehat{W}(L(G)).$$

Proof. Suppose $x, y \in V(S(G))$ are in the same orbit of $\mathrm{Aut}(S(G))$, $d_{S(G)}(x,y) = k$ and $\tau(x) = y$, where $\tau \in \mathrm{Aut}(S(G))$. It is obvious that both of x and y must be together in $V(G)$ or $E(G)$. We first assume that $x, y \in V(G)$. Choose the shortest path $P_1 : x = u_0, u_1, \ldots, u_k = y$ in $S(G)$. Obviously, if i is even then $u_i \in G$ and so $k = 2k'$. Since G is tree, $P_2 : x = u_0, u_2, \ldots, u_{k'} = y$ is the unique path connecting x and y in G. Hence, $d_{S(G)}(x,y) = 2d_G(x,y)$.

Next we assume that $x, y \notin V(G)$, $P_3 : x = u_0, u_1, \ldots, u_k = y$ is a shortest path in $S(G)$. Choose edges $ab, cd \in E(G)$ such that $a \sim_{S(G)} x \sim_{S(G)} b$ and $c \sim_{S(G)} y \sim_{S(G)} d$. Suppose that $\tilde{x}$ and $\tilde{y}$ are corresponding vertices of x and y in $L(G)$, respectively. Since $S(G)$ is tree, the path P_3 is unique and so the vertices u_i, i is even, are corresponding to vertices $\tilde{u}_i$ in $L(G)$. This proves that k is even, say $k = 2k'$. In a similar way, there exists a path $P_2 : \tilde{x} = \tilde{u}_0, \tilde{u}_1, \ldots, \tilde{u}_{k'} = \tilde{y}$ in $L(G)$. So, we have again $d_{S(G)}(x,y) = 2d_{L(G)}(\tilde{x},\tilde{y})$. Thus,

$$\begin{aligned}
\overline{W}(S(G)) &= \frac{1}{2}\sum_i i\overline{D}(S(G),i) \\
&= \frac{1}{2}\sum_i 2i\overline{D}(G,i) + \frac{1}{2}\sum_i 2i\overline{D}(L(G),i) \\
&= 2\overline{W}(G) + 2\overline{W}(L(G)).
\end{aligned}$$

Therefore, $\frac{|\mathrm{Aut}(S(G))|}{|V(S(G))|}\widehat{W}(S(G)) = \frac{2|\mathrm{Aut}(G)|}{|V(G)|}\widehat{W}(G) + \frac{2|\mathrm{Aut}(L(G))|}{|V(L(G))|}\widehat{W}(L(G))$, which completes our proof. □

Lemma 2.5. *Suppose G is a tree with at least three vertices. Then* $\mathrm{Aut}(T(G)) \cong \mathrm{Aut}(G)$.

Proof. The map $\Phi : \mathrm{Aut}(G) \longrightarrow \mathrm{Aut}(T(G))$ defined in a similar way as Lemma 2.3, is an isomorphism. Since if $\alpha \in \mathrm{Aut}(G)$ then $\Phi(\alpha) \in \mathrm{Aut}(T(G))$ and for $\beta \in \mathrm{Aut}(T(G))$ we have $\alpha = \beta|_G \in \mathrm{Aut}(G)$, as desired. □

Theorem 2.6. *With hypothesis of Lemma 2.5,* $\widehat{W}(T(G)) = \frac{2n-1}{n}\widehat{W}(G) + \frac{2n-1}{n-1}\widehat{W}(L(G))$.

Proof. Suppose x and y are vertices of $T(G)$ such that $x \approx_{T(G)} y$ and $d_{T(G)}(x,y) = k$. If $x, y \in G$ then we claim that $d_G(x,y) = k$. To prove, we first notice that G is a subgraph of $T(G)$. Next we assume that $P : x = u_0, u_1, \ldots, u_h = y$ is the unique path in G

and $P' : x = v_0, v_1, \ldots, v_k = y$ is a shortest path in $L(G)$ connecting x and y. If v_1 is a vertex in $L(G)$ and v_2 is a vertex in G then by interchanging v_0, v_1, v_2 by v_0, v_2 we obtain another path P'' in $L(G)$ such that $l(P'') < l(P')$, a contradiction. Thus, if $v_1 \in V(L(G)))$ then $v_2, v_3, \ldots, v_{k-1} \in V(L(G))$ and $l(P) < l(P')$. This shows that $v_1 \notin V(L(G))$. By continuing this method, one can see that all vertices of P' are in vertices of G. Therefore, $P = P'$. A similar argument shows that in other case that x and y are corresponding to vertices in $L(G)$, a shortest path in $L(G)$ and $T(G)$ will be the same and so $\overline{W}(T(G)) = \overline{W}(G) + \overline{W}(L(G))$. Therefore, $\frac{|\mathrm{Aut}(T(G))|}{|V(T(G))|}\widehat{W}(T(G)) = \frac{|\Gamma|}{|V(G)|}\widehat{W}(G) + \frac{|\mathrm{Aut}(L(G))|}{|V(L(G))|}\widehat{W}(L(G))$, which completes the proof. □

Lemma 2.7. *Suppose G is a tree with at least three vertices. Then* $\mathrm{Aut}(Q(G)) \cong \mathrm{Aut}(G)$.

Proof. Since $Q(G) = L(G) \cup S(G)$, Lemmas 2.1, 2.3 and a similar argument as Lemma 2.3 completes the proof. □

Theorem 2.8. *With hypothesis of Lemma 2.7,* $\widehat{W}(Q(G)) = \frac{4n-2}{n}\widehat{W}(G)$.

Proof. Suppose x and y are vertices of $Q(G)$ such that $x \approx_{Q(G)} y$. If x, y are corresponding to the vertices of $L(G)$ then $d_{Q(G)}(x, y) = d_{L(G)}(x, y)$. Suppose x, y are vertices of G with distance k and $P : x = u_0, u_1, \ldots, u_{k-1}, u_k = y$ is a shortest path in G connecting x and y. If x and y are adjacent in G then distance between them in $Q(G)$ will be 2. In other case, the path $P' : x = u_0, e_1, \ldots, e_k, u_k = y$ has length $k + 1$, where $e_i = u_{i-1}u_i$, $1 \leq i \leq k$. In the case that $x, y \in L(G)$, the sum of distances is $\sum_i i\overline{D}(L(G), i)$ and in the second case the summation will be $\sum_i \overline{D}(G, i) + \sum_i \widehat{D}(G, i)$. Then we have $\overline{W}(Q(G)) = \overline{W}(G) + \frac{1}{2}\sum_i \widehat{D}(G, i) + \overline{W}(L(G))$. By applying Theorem 1, the result is obtained. □

Lemma 2.9. *Suppose G is a tree with at least three vertices. Then* $\mathrm{Aut}(G)$ *is isomorphic to a proper subgroup of* $\mathrm{Aut}(R(G))$.

Proof. It is easy to see that the mapping $\Phi : \mathrm{Aut}(G) \longrightarrow \mathrm{Aut}(R(G))$ given by the same definition as Lemma 2.3 is a one-to-one homomorphism, as desired. Since G is a tree, it has at least a pendant vertex and so $R(G)$ has an automorphism of order 2 in $\mathrm{Aut}(R(G)) \setminus \Phi(\mathrm{Aut}(G))$, proving the lemma. □

Theorem 2.10. *With hypothesis of Lemma 2.9,* $\widehat{W}(R(G)) > \frac{4n-2}{n}\frac{|\mathrm{Aut}(G)|}{|\mathrm{Aut}(R(G))|}\widehat{W}(G)$.

Proof. Suppose x and y are vertices in $R(G)$. Similar to Theorem 2.8, if $x, y \in G$ then $d_{R(G)}(x, y) = d_G(x, y)$. In this case, the sum of distances is at least $\sum_i i\overline{D}(G, i)$. If x and y are corresponding to vertices w and z of $L(G)$ such that $d_{L(G)}(w, z) = k$ then $d_{R(G)}(x, y) = k + 1$ and so the sum of distances is at least $\sum_i i\overline{D}(L(G), i) + \sum_i \widehat{D}(G, i)$.

Now, we assume that u and v are two pendants in G and in the same orbits under the action of $\mathrm{Aut}(G)$ on vertices and e and f are edges such that u is incident to e and f is incident to v. Then e and f are in the same orbits under the action of $\mathrm{Aut}(G)$ on edges. Hence, in $R(G)$, all elements of $\{u, v, e, f\}$ are in the same orbits under the action of $\mathrm{Aut}(R(G))$. Since, corresponding to $d_G(u, v)$ in $\overline{W}(G)$ there exists at least a quantity in the form of $d_{R(G)}(u, v) + d_{R(G)}(e, f) + d_{R(G)}(u, f) + d_{R(G)}(v, e) + d_{R(G)}(u, e) +$

$d_{R(G)}(v, f)$ in $\overline{W}(R(G))$, by Theorem 1, $\overline{W}(R(G)) > 2\overline{W}(G)$, which completes our argument. $\square$

3 Examples

In this section, we apply our results in the pervious section. We denote the cyclic group of order n by $\mathbb{Z}_n$ and the symmetric group on n symbols by $\mathrm{Sym}(n)$. We notice that by Lemmas 2.1–2.7, if G is a tree with at least three vertices then

$$\mathrm{Aut}(G) \cong \mathrm{Aut}(L(G)) \cong \mathrm{Aut}(S(G)) \cong \mathrm{Aut}(T(G)) \cong \mathrm{Aut}(Q(G)),$$

and also by Lemma 2.9, $\mathrm{Aut}(G) \leq \mathrm{Aut}(R(G))$.

Example 3.1. In this example, $\widehat{W}(P_n)$, $\widehat{W}(L(P_n))$, $\widehat{W}(S(P_n))$, $\widehat{W}(T(P_n))$, $\widehat{W}(Q(P_n))$ and $\widehat{W}(R(P_n))$ are calculated where P_n is a path with n vertices. To do this, we assume that $V(P_n) = \{v_i\}_{i=1}^{n}$ and $E(P_n) = \{e_i = v_i v_{i+1}\}_{i=1}^{n-1}$. We first notice that the automorphism group of P_n is generated by an element α of order 2, where

$$\alpha = \begin{cases} (v_1, v_n)(v_2, v_{n-1}) \ldots (v_{\frac{n}{2}}, v_{\frac{n+2}{2}}) & n \text{ is even} \\ (v_1, v_n)(v_2, v_{n-1}) \ldots (v_{\frac{n-1}{2}}, v_{\frac{n+1}{2}}) & n \text{ is odd.} \end{cases}$$

Therefore, $\mathrm{Aut}(P_n) \cong \mathbb{Z}_2$. The modified Wiener index of P_n was computed in [3, Example 5.6] as

$$\widehat{W}(P_n) = \begin{cases} \frac{n^3}{8} & n \text{ is even} \\ \frac{n^3-n}{8} & n \text{ is odd.} \end{cases}$$

On the other hand, if n is even, then

$$\overline{D}(P_n, i) = \begin{cases} 0 & i \text{ is even} \\ 2 & i \text{ is odd,} \end{cases}$$

and if n is odd, then

$$\overline{D}(P_n, i) = \begin{cases} 2 & i \text{ is even} \\ 0 & i \text{ is odd.} \end{cases}$$

Therefore,

$$\sum_{i=1}^{n-1} \overline{D}(P_n, i) = \begin{cases} n & n \text{ is even} \\ n-1 & n \text{ is odd.} \end{cases}$$

By applying Theorems 2.2–2.10, we obtained the following equations,

$$\widehat{W}(L(P_n)) = \begin{cases} \frac{n^3-3n^2+2n}{8} & n \text{ is even} \\ \frac{(n-1)^3}{8} & n \text{ is odd,} \end{cases}$$

$$\widehat{W}(S(P_n)) = \frac{2n^3-3n^2+n}{4},$$

$$\widehat{W}(T(P_n)) = \frac{2n^3-3n^2+n}{4},$$

$$\widehat{W}(Q(P_n)) = \begin{cases} \frac{2n^3-n^2}{4} & n \text{ is even} \\ \frac{2n^3-n^2-2n+1}{4} & n \text{ is odd,} \end{cases}$$

$$\widehat{W}(R(P_n)) = \begin{cases} \frac{2n^3-n^2}{4} & n \text{ is even} \\ \frac{2n^3-n^2+2n-1}{4} & n \text{ is odd.} \end{cases}$$

For the last equality, we notice that the automorphism group of $R(P_n)$ can be generated by three elements α, β and γ as follows:

$$\alpha = (v_1, e_1),$$
$$\beta = (v_n, e_n),$$
$$\gamma = \begin{cases} (v_1, v_n)(v_2, v_{n-1})\dots\left(v_{\frac{n}{2}}, v_{\frac{n}{2}+1}\right)(e_1, e_{n-1})(e_2, e_{n-2})\dots\left(e_{\frac{n}{2}-1}, e_{\frac{n}{2}+1}\right) & n \text{ is even} \\ (v_1, v_n)(v_2, v_{n-1})\dots\left(v_{\frac{n-1}{2}}, v_{\frac{n+3}{2}}\right)(e_1, e_{n-1})(e_2, e_{n-2})\dots\left(e_{\frac{n+1}{2}}, e_{\frac{n+3}{2}}\right) & n \text{ is odd.} \end{cases}$$

It is easy to see that this group is isomorphic to the dihedral group D_8.

In the following example to compute the modified Wiener index, we apply the concept of semidirect product and wreath product of groups together with our results in last section. Let G be a group with a subgroup H and a normal subgroup N such that $G = HN$ and $H \cap N = 1$. Then G is called the semidirect product of N by H. To define the notion of wreath product, we assume that A and H are groups, X is a set and H acts on X. Define $K = \prod_{w\in X} A_w$, where $A_w \cong A$. If we consider the elements of K as arbitrary sequences of elements of A with componentwise multiplication then the action of H on X can be extended in a natural way to an action of H on the group K by $h(a_\omega) = (a_{h^{-1}\omega})$. Then the wreath product $A \wr H$ of A by H is the semidirect product H by K.

Example 3.2. Suppose S_n is an star with a vertex of degree n and n pendant vertices. Since S_n has exactly one vertex of degree n, this vertex will be fixed under each automorphism of S_n. On the other hand, all pendants can be imaged under permutations to each other. Thus Aut$(S_n) \cong \mathrm{Sym}(n)$. According to [3, Example 5.8], the modified Wiener index of S_n with $n+1$ vertices is equal to

$$\widehat{W}(S_n) = n^2 - 1.$$

Also,

$$\overline{D}(S_n, i) = \begin{cases} 0 & i = 1 \\ (n-1)n! & i = 2. \end{cases}$$

Therefore, by applying Theorems 2.2–2.10,

$$\widehat{W}(L(S_n)) = \frac{n^2 - n}{2},$$
$$\widehat{W}(S(S_n)) = 5n^2 - 3n - 3,$$
$$\widehat{W}(T(S_n)) = \frac{6n^2 - 3n - 3}{2},$$
$$\widehat{W}(Q(S_n)) = 4n^2 - 2n - 2,$$
$$\widehat{W}(R(S_n)) = \frac{8n^2 - 2n - 3}{2}.$$

For the proof of last equality, we assume that

$$V(S_n) = \{v_0, v_1, v_2, \ldots, v_n\} \text{ and } E(S_n) = \{e_i = v_0v_i \mid 1 \leq i \leq n\}.$$

The automorphism group of $R(S_n)$ has automorphisms τ_i such that $\tau_i : v_i \mapsto e_i$ and τ_i fixes other vertices of the graph. Suppose $A_i = \langle \tau_i \rangle$. Then $A_i \cong \mathbb{Z}_2$ and

$$K = \underbrace{\mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2}_{n \text{ times}}$$

is isomorphic to a subgroup of $\mathrm{Aut}(R(S_n))$. On the other hand, $\mathrm{Sym}(n)$ acts on K by $\alpha(a_i) = (a_{\alpha^{-1}i})$. Hence $\mathrm{Aut}(R(S_n)) \cong \mathbb{Z}_2 \wr \mathrm{Sym}(n)$.

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On the inertia of weighted $(k-1)$-cyclic graphs*

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Abstract

Let G_w be a weighted graph. The inertia of G_w is the triple $\mathrm{In}(G_w) = (i_+(G_w), i_-(G_w), i_0(G_w))$, where $i_+(G_w), i_-(G_w), i_0(G_w)$ are, respectively, the number of the positive, negative and zero eigenvalues of the adjacency matrix $A(G_w)$ of G_w including their multiplicities. A simple n-vertex connected graph is called a $(k-1)$-cyclic graph if its number of edges equals $n+k-2$. Let $\theta(r_1, r_2, \dots, r_k)_w$ be an n-vertex simple weighted graph obtained from k weighted paths $(P_{r_1})_w, (P_{r_2})_w, \dots, (P_{r_k})_w$ by identifying their initial vertices and terminal vertices, respectively. Set $\Theta_k := \{\theta(r_1, r_2, \dots, r_k)_w : r_1 + r_2 + \cdots + r_k = n + 2k - 2\}$. The inertia of the weighted graph $\theta(r_1, r_2, \dots, r_k)_w$ is studied. Also, the weighted $(k-1)$-cyclic graphs that contain $\theta(r_1, r_2, \dots, r_k)_w$ as an induced subgraph are studied. We characterize those graphs among Θ_k that have extreme inertia. The results generalize the corresponding results obtained by Tan and Liu in 2013 and Yu et al., 2014.

Keywords: Weighted k-cyclic graph, adjacency matrix, inertia.

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1 The first section

In this paper, we only consider simple weighted graphs on positive weight set. Let G_w be a weighted graph with vertex set $\{v_1, v_2, \dots, v_n\}$, edge set $E(G_w) \neq \emptyset$ and weight set $W(G_w) = \{w(e) > 0, e \in E(G)\}$. The function $w : E(G_w) \to W(G_w)$ is called a weight function of G_w. It is obvious that each weighted graph corresponds to a weight function. The *adjacency matrix* of G_w is defined as the matrix $A(G_w) = (a_{ij})$ such that $a_{ij} = w(v_iv_j)$ if $v_iv_j \in E(G_w)$ and 0 otherwise. The eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ of

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$A(G_w)$ are said to be the eigenvalues of the weighted graph G_w. The *inertia* of G_w is defined to be the triple $\mathrm{In}(G_w) = (i_+(G_w), i_-(G_w), i_0(G_w))$, where $i_+(G_w), i_-(G_w)$ and $i_0(G_w)$ are the numbers of the positive, negative and zero eigenvalues of $A(G_w)$ including multiplicities, respectively. $i_+(G_w)$ and $i_-(G_w)$ are called the *positive, negative index of inertia* (for short, *positive, negative index*) of G_w, respectively. The number $i_0(G_w)$ is called the *nullity* of $A(G_w)$. The nullity and the rank of $A(G_w)$ are also called the nullity and the rank of G_w, and denoted by $\eta(G)$ and $\mathrm{R}(G)$, respectively. Obviously, $\mathrm{R}(G_w) = i_+(G_w) + i_-(G_w)$ and $i_+(G_w) + i_-(G_w) + i_0(G_w) = n$. For convenience, in the whole context, we let G denote the unweighted graph with respect to the weighted graph G_w; G can be also viewed as a trivial weighted graph in which the weight for each edge is 1.

An *induced subgraph* of G_w is an induced subgraph of G having the same weights with those of G_w. For an induced weighted subgraph H_w of G_w, let $G_w - H_w$ be the subgraph obtained from G_w by deleting all vertices of H_w and all incident edges. A m*-cyclic graph* is a simple connected graph in which the number of edges equals the number of vertices plus $m-1$. A weighted path and a weighted cycle of order n are denoted by $(P_n)_w$, $(C_n)_w$, respectively. An isolated vertex is denoted by K_1.

The study of eigenvalues of graph has been received a lot of attention due to its applications in chemistry (see [2, 7, 10, 15] for details). Gregory et al. [8] studied the subadditivity of the positive, negative indices of inertia and developed certain properties of Hermitian rank which were used to characterize the biclique decomposition number. Gregory et al. [9] investigated the inertia of a partial join of two graphs and established a few relations between the inertia and biclique decompositions of partial joins of graphs. Daugherty [3] characterized the inertia of unicyclic graphs in terms of matching number and obtained a linear-time algorithm for computing it. Yu et al. [19] investigated the minimal positive index of inertia among all unweighted bicyclic graphs of order n with pendants, and characterized the bicyclic graphs with positive index 1 or 2. Very recently, it is interesting to see that Marina et al. [1] studied the inertia set of a signed graph in algebraic approach.

The nullity of unweighted graphs has been studied extensively in the literature. Tan and Liu [18] gave the nullity set of unicyclic graphs and characterized the unicyclic graphs with maximum nullity. In addition, Nath and Sarma [17] presented another version of characterization of an acyclic or unicyclic graph to be singular. One of the present authors [13] studied the nullity of graphs with pendant vertices. Fan and Qian [6] characterized the bipartite graphs with the second largest nullity and the regular bipartite graphs with the third largest nullity. Fan and Wang [5] characterized the unicyclic signed graphs of order n with nullity $n-2, n-3, n-4, n-5$, respectively.

Our paper is motivated directly by [4, 11, 13, 19, 20, 21]. On the one hand, Fan et al. [4] studied the nullity of signed bicyclic graph (which is, in fact, the bicyclic graph with edge weight 1 or -1); Li [13] and Hu [11] studied the nullity of unweighted bicyclic graph. On the other hand, Yu et al. [20] characterized all n-vertex weighted uicyclic graphs with positive index 1 or 2; Tan and Liu [21] studied the nullity of unweighted $(k-1)$-cyclic graphs. It is natural and interesting for us to consider the extremal problems on the inertia of weighted $(k-1)$-cyclic graphs, which may generalize the corresponding results obtained in [20, 21].

This paper is organized as follows. In Section 2, some preliminaries are presented. In Section 3, we define two classes of weighted $(k-1)$-cyclic graph, denoted by Θ_k and $\Gamma_{n,k-1}$. Moreover, we give a method to determine the inertia of a weighted graph in Θ_k.

In Section 4, we characterize all weighted $(k-1)$-cyclic graphs in $\Gamma_{n,k-1}$ having just one or two positive (resp. negative) eigenvalues. In Section 5, we characterize all weighted $(k-1)$-cyclic graphs in $\Gamma_{n,k-1}$ of rank $2, 3, 4$, respectively.

2 Preliminaries

In this section, we list some lemmas which will be used to prove our main results. Suppose M, N are two Hermitian matrices of order n, if there exists an invertible matrix Q of order n such that $QMQ^T = N$, where Q^T denotes the conjugate transpose of Q, then we say that M is *congruent to* N, denoted by $M \cong N$.

Lemma 2.1 ([12]). *Let M, N be two Hermitian matrices of order n satisfying $M \cong N$. Then $i_+(M) = i_+(N)$, $i_-(M) = i_-(N)$ and $i_0(M) = i_0(N)$.*

Let M be a Hermitian matrix. We denote three types of elementary congruence matrix operations (ECMOs) on M as follows:

(1) interchanging i-th and j-th rows of M, while interchanging i-th and j-th columns of M;

(2) multiplying i-th row of M by a non-zero number k, while multiplying i-th column of M by k;

(3) adding i-th row of M multiplied by a non-zero number k to j-th row, while adding i-th column of M multiplied by k to j-th column.

By Lemma 2.1, the ECMOs do not change the inertia of a Hermitian matrix.

Lemma 2.2 ([14]). *Let H_w be an induced subgraph of G_w. Then $i_+(H_w) \leqslant i_+(G_w)$ and $i_-(H_w) \leqslant i_-(G_w)$.*

Lemma 2.3 ([14]). *Let G_w be a weighted graph containing a pendant vertex v with its unique neighbor u. Then $i_+(G_w) = i_+(G_w - u - v) + 1$ and $i_-(G_w) = i_-(G_w - u - v) + 1$.*

The following result is a direct consequence of Lemma 2.3.

Lemma 2.4. *Let $(P_n)_w$ be a weighted path of order n. Then $\mathrm{In}((P_n)_w) = (\frac{n}{2}, \frac{n}{2}, 0)$ if n is even and $(\frac{n-1}{2}, \frac{n-1}{2}, 1)$ otherwise.*

By Lemma 2.4, we can show that the adjacency matrix of $(P_{2k})_w$ is invertible. In fact, let $\{v_1, v_2, \ldots, v_{2k}\}$ be the vertex set of the weighted path $(P_{2k})_w$ such that $v_iv_{i+1} \in E((P_{2k})_w)\,(i = 1, \ldots, 2k-1)$ and $w_{ii} = w(v_{2i-1}v_{2i})\,(i = 1, \ldots, k)$, $w_{i,i+1} = w(v_{2i}v_{2i+1})\,(i = 1, \ldots, k-1)$. Then the adjacency matrix of $(P_{2k})_w$ has the following block form:

$$A = \begin{pmatrix} A_{11} & A_{12} & \ldots & \mathbf{0} & \mathbf{0} \\ A_{21} & A_{22} & \ldots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \ldots & A_{k-1,k-1} & A_{k-1,k} \\ \mathbf{0} & \mathbf{0} & \ldots & A_{k,k-1} & A_{k,k} \end{pmatrix}$$

where $A_{ii} = \begin{pmatrix} 0 & w_{ii} \\ w_{ii} & 0 \end{pmatrix}, (i = 1, \ldots, k)$ and

$$A^T_{i+1,i} = A_{i,i+1} = \begin{pmatrix} 0 & 0 \\ w_{i,i+1} & 0 \end{pmatrix}, (i = 1, \ldots, k-1).$$

Let $B = (B_{ij})_{i,j=1}^k$, where

$$B_{ij} = \begin{cases} \begin{pmatrix} 0 & \dfrac{1}{w_{ii}} \\ \dfrac{1}{w_{ii}} & 0 \end{pmatrix} & \text{if } i = j; \\ \begin{pmatrix} 0 & \dfrac{w_{i,i+1} \cdots w_{j-1,j}}{w_{i,i} \cdots w_{j,j}} \\ 0 & 0 \end{pmatrix} & \text{if } i < j \text{ and } j - i \equiv 0 \pmod 2; \\ \begin{pmatrix} 0 & -\dfrac{w_{i,i+1} \cdots w_{j-1,j}}{w_{i,i} \cdots w_{j,j}} \\ 0 & 0 \end{pmatrix} & \text{if } i < j \text{ and } j - i \equiv 1 \pmod 2; \\ B_{ji}^T, & \text{if } i > j. \end{cases}$$

Lemma 2.5. *Let A and B be the matrices defined as above. Then $AB = I$.*

Proof. Let $C = (C_{ij})_{i,j=1}^k = AB$. It suffices to show that $C_{ii} = I_2$ for $i = 1, \ldots, k$, where I_2 is the identity matrix of order 2, and $C_{ij} = \mathbf{0}$ if $i \neq j$. Note that the first (resp. last) row of A contains just two non-zero blocks, whereas each of the rest rows of A contains just three non-zero blocks, the proofs are a little different between them. First we consider the cases that $i \neq 1, k$.

If $1 < i = j < k$, then

$$\begin{aligned} C_{ii} &= \sum_{s=1}^k A_{is} B_{si} = A_{i,i-1} B_{i-1,i} + A_{ii} B_{ii} + A_{i,i+1} B_{i+1,i} \\ &= \begin{pmatrix} 0 & w_{i-1,i} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -\dfrac{w_{i-1}}{w_{i-1,i-1} w_{i,i}} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & w_{ii} \\ w_{ii} & 0 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{w_{ii}} \\ \frac{1}{w_{ii}} & 0 \end{pmatrix} \\ &\quad + \begin{pmatrix} 0 & 0 \\ w_{i,i+1} & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ -\dfrac{w_{i,i+1}}{w_{ii} w_{i+1,i+1}} & 0 \end{pmatrix} \\ &= I_2. \end{aligned}$$

If $1 < i < j < k$, we distinguish the following three possible cases to prove our result.

Case 1: $j - i \equiv 0 \pmod 2$. In this case, we have

$$\begin{aligned} C_{ij} &= \sum_{s=1}^k A_{is} B_{sj} = A_{i,i-1} B_{i-1,j} + A_{i,i} B_{i,j} + A_{i,i+1} B_{i+1,j} \\ &= \begin{pmatrix} 0 & w_{i-1,i} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -\frac{w_{i-1,i} \cdots w_{j-1,j}}{w_{i-1,i-1} \cdots w_{jj}} \\ 0 & 0 \end{pmatrix} \\ &\quad + \begin{pmatrix} 0 & w_{ii} \\ w_{ii} & 0 \end{pmatrix} \begin{pmatrix} 0 & \frac{w_{i,i+1} \cdots w_{j-1,j}}{w_{i,i} \cdots w_{jj}} \\ 0 & 0 \end{pmatrix} \\ &\quad + \begin{pmatrix} 0 & 0 \\ w_{i,i+1} & 0 \end{pmatrix} \begin{pmatrix} 0 & -\frac{w_{i+1,i+2} \cdots w_{j-1,j}}{w_{i+1,i+1} \cdots w_{jj}} \\ 0 & 0 \end{pmatrix} \\ &= \mathbf{0}. \end{aligned}$$

Case 2: $j-i=1$. In this case, we have

$$
\begin{aligned}
C_{ij} &= \sum_{s=1}^{k} A_{is}B_{sj} = A_{i,i-1}B_{i-1,j} + A_{i,i}B_{i,j} + A_{i,i+1}B_{i+1,j} \\
&= \begin{pmatrix} 0 & w_{i-1,i} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & \dfrac{w_{i-1,i}w_{i,j}}{w_{i-1,i-1}w_{ii}w_{jj}} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & w_{ii} \\ w_{ii} & 0 \end{pmatrix} \begin{pmatrix} 0 & -\dfrac{w_{ij}}{w_{i,i}w_{jj}} \\ 0 & 0 \end{pmatrix} \\
&\quad + \begin{pmatrix} 0 & 0 \\ w_{i,i+1} & 0 \end{pmatrix} \begin{pmatrix} 0 & \dfrac{1}{w_{jj}} \\ \dfrac{1}{w_{jj}} & 0 \end{pmatrix} \\
&= \mathbf{0}.
\end{aligned}
$$

Case 3: $j-i \equiv 1 \pmod 2$ and $j-i>1$. In this case, we have

$$
\begin{aligned}
C_{ij} &= \sum_{s=1}^{k} A_{is}B_{sj} = A_{i,i-1}B_{i-1,j} + A_{i,i}B_{i,j} + A_{i,i+1}B_{i+1,j} \\
&= \begin{pmatrix} 0 & w_{i-1,i} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & \dfrac{w_{i-1,i}\cdots w_{j-1,j}}{w_{i-1,i-1}\cdots w_{jj}} \\ 0 & 0 \end{pmatrix} \\
&\quad + \begin{pmatrix} 0 & w_{ii} \\ w_{ii} & 0 \end{pmatrix} \begin{pmatrix} 0 & -\dfrac{w_{i,i+1}\cdots w_{j-1,j}}{w_{i,i}\cdots w_{jj}} \\ 0 & 0 \end{pmatrix} \\
&\quad + \begin{pmatrix} 0 & 0 \\ w_{i,i+1} & 0 \end{pmatrix} \begin{pmatrix} 0 & \dfrac{w_{i+1,i+2}\cdots w_{j-1,j}}{w_{i+1,i+1}\cdots w_{jj}} \\ 0 & 0 \end{pmatrix} \\
&= \mathbf{0}.
\end{aligned}
$$

For $i=1$ or $i=k$, all the proofs above are still correct if we set the corresponding blocks to be **0** whenever one of its subscripts equals 0 or $k+1$, such as $A_{10}=A_{k,k+1}=\mathbf{0}$.

If $1 \leqslant j < i \leqslant k$, the proof is similar to the case $1 \leqslant i < j \leqslant k$. We omit the procedure here. □

3 The inertia of weighted graphs in Θ_k

For $m \geqslant 1$, a *m-cyclic graph* is a simple connected graph in which the number of edges equals the number of vertices plus $m-1$. Let P_{r_i} be a path of order $r_i\,(r_i \geqslant 2)$ and $\{P_{r_i} | 1 \leqslant i \leqslant k\}$ be the set of $k\,(k \geqslant 2)$ vertex-disjoint paths, where there exists at most one path of order 2. Identify the k initial vertices as u_0 and terminal vertices as v_0, respectively. The resultant graph, denoted by $\theta(r_1, r_2, \ldots, r_k)$, is called a *$\Theta$-graph*. Denote by Θ_k the set of all n-vertex weighted Θ-graphs having form $\theta(r_1, r_2, \ldots, r_k)_w$. Note that any weighted Θ-graph is also a weighted $(k-1)$-cyclic graph. Denote the set of all weighted $(k-1)$-cyclic graphs of order n, which contain a weighted Θ-graph as an induced subgraph, by $\Gamma_{n,k-1}$. In this section, we'll give a method to determine the inertia of weighted graphs in Θ_k.

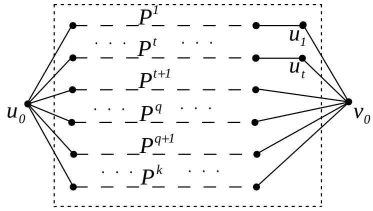

Figure 1: The structure of $\theta(r_1, r_2, \ldots, r_k)$

Let $G_w := \theta(r_1, r_2, \ldots, r_k)_w$ be a graph of order n. Let n_i be the number of r_j's which satisfy $r_j - 2 \equiv i \pmod 4$, $1 \leqslant j \leqslant k$, $0 \leqslant i \leqslant 3$ and set $t := n_1 + n_3$ and $q := t + n_2$. It is easy to see that $G_w \in \Theta_k$, we arrange the structure of G_w as follows: First come the paths $P_{r_1}, \ldots, P_{r_{n_1}}$ with $r_1 \leqslant r_2 \leqslant \ldots \leqslant r_{n_1}$ and $r_i \equiv 3 \pmod 4$, $i = 1, 2, \ldots, n_1$; next $P_{r_{n_1+1}}, \ldots, P_{r_t}$ with $r_{n_1+1} \leqslant r_{n_1+2} \leqslant \ldots \leqslant r_t$ and $r_i \equiv 1 \pmod 4$, $i = n_1+1, n_1+2, \ldots, t$; then $P_{r_{t+1}}, \ldots, P_{r_q}$ with $r_{t+1} \leqslant r_{t+2} \leqslant \ldots \leqslant r_q$ and $r_i \equiv 2 \pmod 4$, $i = t+1, t+2, \ldots, q$; finally $P_{r_{q+1}}, \ldots, P_{r_k}$ with $r_{q+1} \leqslant r_{q+2} \leqslant \ldots \leqslant r_k$ and $r_i \equiv 0 \pmod 4$, $i = q+1, q+2, \ldots, k$. Let u_i be the neighbor of v_0 in the odd path P_{r_i}, $i = 1, 2, \ldots, t$. Let $P^i = u_1^i u_2^i \ldots u_{2s_i}^i$ $(1 \leqslant i \leqslant k)$ be the path in P_{r_i} $(1 \leqslant i \leqslant k)$ obtained by deleting u_0, v_0 and u_i if r_i is odd; see Fig. 1. Further on we will label the weight for each edge of G_w according to the following possible cases.

Case 1: $\min\{r_1, r_2, \ldots, r_k\} = 4$**.** In this case, partition the vertex set of G_w as follows: $\{u_0\}, V(P^1), \ldots, V(P^k), \{u_1, \ldots, u_t\}, \{v_0\}$. Let $a_i = w(u_0 u_1^i)$ $(i = 1, \ldots, k)$, $b_i = w(u_i u_{2s_i}^i)$ $(i = 1, \ldots, t)$, $b_j = w(v_0 u_{2s_j}^j)$ $(j = t+1, \ldots, k)$, $d_i = w(v_0 u_i)$ $(i = 1, \ldots, t)$, $w_{jj}^i = w(u_{2j-1}^i u_{2j}^i)$ $(i = 1, \ldots, k; j = 1, \ldots, \frac{1}{2}|V(P^i)|)$ and $w_{j,j+1}^i = w(u_{2j}^i u_{2j+1}^i)$ $(i = 1, \ldots, k; j = 1, \ldots, \frac{1}{2}|V(P^i)| - 1)$. Then the adjacency matrix of G_w has the following form:

$$A(G_w) = \left(\begin{array}{c|c|c|c|c}
0 & \alpha_1^T \ldots \alpha_t^T & \alpha_{t+1}^T \ldots \alpha_k^T & \mathbf{0} & 0 \\
\hline
\begin{matrix}\alpha_1^T \\ \vdots \\ \alpha_t^T\end{matrix} & \begin{matrix}A_1 & & \\ & \ddots & \\ & & A_t\end{matrix} & \mathbf{0} & \begin{matrix}\beta_1 & & \\ & \ddots & \\ & & \beta_t\end{matrix} & \mathbf{0} \\
\hline
\begin{matrix}\alpha_{t+1}^T \\ \vdots \\ \alpha_k^T\end{matrix} & \mathbf{0} & \begin{matrix}A_{t+1} & & \\ & \ddots & \\ & & A_k\end{matrix} & \mathbf{0} & \begin{matrix}\beta_{t+1}^T \\ \vdots \\ \beta_k^T\end{matrix} \\
\hline
\mathbf{0} & \begin{matrix}\beta_1^T & & \\ & \ddots & \\ & & \beta_t^T\end{matrix} & \mathbf{0} & \mathbf{0} & \begin{matrix}d_1 \\ \vdots \\ d_t\end{matrix} \\
\hline
0 & \mathbf{0} & \beta_{t+1}^T \ldots \beta_k^T & d_1 \ldots d_t & 0
\end{array}\right),$$

where $\alpha_i^T = (a_i, 0, \ldots, 0)$ and $\beta_i^T = (0, \ldots, 0, b_i)$.

We apply the ECMOs on $A(G_w)$: using $-\alpha_i^T A_i^{-1}$ to multiply the $(i+1)$-th row, then adding it to the first row, we can cancel α_i^T $(i = 1, \ldots, k)$ in the first row. Similarly, using $-\beta_i^T A_i^{-1}$ to multiply the $(i+1)$-th row, then adding it to $(k+i+1)$-th row if $i \leqslant t$, and adding it to the last row if $t+1 \leqslant i \leqslant k$, we can cancel $\beta_i^T (i = 1, \dots, k)$. After that, column operations are applied so that each α_i and β_i are reduced to **0**s. By Lemma 2.5, $-\alpha_i^T A_i^{-1} \alpha_i = -\beta_i^T A_i^{-1} \beta_i = 0$ and $c_i = -\alpha_i^T A_i^{-1} \beta_i = -\beta_i^T A_i^{-1} \alpha_i$, where

$$c_i = \begin{cases} -\dfrac{a_i b_i w_{12}^i w_{23}^i \dots w_{s_i-1,s_i}^i}{w_{11}^i w_{22}^i \dots w_{s_i,s_i}^i}, & \text{if } |A_i| = 2s_i \equiv 2 \pmod 4; \\[2ex] \dfrac{a_i b_i w_{12}^i w_{23}^i \dots w_{s_i-1,s_i}^i}{w_{11}^i w_{22}^i \dots w_{s_i,s_i}^i}, & \text{if } |A_i| = 2s_i \equiv 0 \pmod 4. \end{cases}$$

So $A(G_w)$ can be reduced to the following matrix:

$$B = \left(\begin{array}{c|c|c|c|c} 0 & \mathbf{0} & \mathbf{0} & c_1 \dots c_t & s \\ \hline \mathbf{0} & \begin{matrix} A_1 & & \\ & \ddots & \\ & & A_t \end{matrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \begin{matrix} A_{t+1} & & \\ & \ddots & \\ & & A_k \end{matrix} & \mathbf{0} & \mathbf{0} \\ \hline \begin{matrix} c_1 \\ \vdots \\ c_t \end{matrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \begin{matrix} d_1 \\ \vdots \\ d_t \end{matrix} \\ \hline s & \mathbf{0} & \mathbf{0} & d_1 \dots d_t & 0 \end{array}\right),$$

where $s = \sum_{i=t+1}^{k} c_i$.

Define

$$D = \left(\begin{array}{cc|ccc} 0 & s & c_1 & \dots & c_t \\ s & 0 & d_1 & \dots & d_t \\ \hline c_1 & d_1 & & & \\ \vdots & \vdots & & \mathbf{0} & \\ c_t & d_t & & & \end{array}\right). \tag{3.1}$$

After interchanging rows and columns, we get the equivalent matrix of B:

$$\begin{pmatrix} D & & & \\ & A_1 & & \\ & & \ddots & \\ & & & A_k \end{pmatrix}. \tag{3.2}$$

It follows that

$$\begin{aligned}
i_+(G_w) &= i_+(D) + \sum_{j=1}^{k} i_+(A_k) = i_+(D) + \frac{1}{2}\sum_{j=1}^{k} |A_i| \\
&= i_+(D) + \frac{1}{2}\left(\sum_{j=1}^{t}(r_i - 3) + \sum_{j=t+1}^{k}(r_i - 2)\right) \\
&= i_+(D) + \frac{1}{2}\left(\sum_{j=1}^{k}(r_i - 2) - t\right) \\
&= i_+(D) + \frac{1}{2}(n - 2 - t).
\end{aligned}$$

Similarly, $i_-(G_w) = i_-(D) + \frac{1}{2}(n-2-t), i_0(G_w) = t + 2 - \mathrm{R}(D)$.

Case 2.: $\min\{r_1, r_2, \ldots, r_k\} = 3$**.** We suppose, without loss of generality, that the first ℓ paths $P_i = u_0u_iv_0$ $(i = 1, \ldots, \ell)$ are of length 3. Partition the vertex of G_w as follows: $\{u_0\}, V(P^{\ell+1}), \ldots, V(P^k), \{u_1, \ldots, u_\ell\}$,$\{u_{\ell+1}, \ldots, u_t\}, \{v_0\}$. Then we label the weight for each edge of G_w as follows: $c_i = w(u_0u_i)\,(i = 1, \ldots, \ell)$, $d_i = w(v_0u_i)\,(i = 1, \ldots, t)$, $a_i = w(u_0u_1^i)\,(i = \ell+1, \ldots, k)$, $b_i = w(u_iu_{2s_i}^i)\,(i = \ell+1, \ldots, t)$, $b_j = w(v_0u_{2s_j}^j)\,(j = t+1, \ldots, k)$ and $w_{jj}^i = w(u_{2j-1}^iu_{2j}^i)\,(i = \ell+1, \ldots, k; j = 1, \ldots, \frac{1}{2}|V(P^i)|)$, $w_{j,j+1}^i = w(u_{2j}^iu_{2j+1}^i)\,(i = \ell+1, \ldots, k; j = 1, \ldots, \frac{1}{2}|V(P^i)| - 1)$. Then the adjacency matrix of G_w has the following form:

$$A(G_w) = \left(\begin{array}{c|c|c|c|c|c}
0 & \alpha_{\ell+1}^T \ldots \alpha_t^T & \alpha_{t+1}^T \ldots \alpha_k^T & c_1 \ldots c_\ell & \mathbf{0} & 0 \\
\hline
\alpha_{\ell+1}^T & A_{\ell+1} & & & \beta_{\ell+1} & \\
\vdots & \ddots & \mathbf{0} & \mathbf{0} & \ddots & \mathbf{0} \\
\alpha_t^T & A_t & & & \beta_t & \\
\hline
\alpha_{t+1}^T & & A_{t+1} & & & \beta_{t+1}^T \\
\vdots & \mathbf{0} & \ddots & \mathbf{0} & \mathbf{0} & \vdots \\
\alpha_k^T & & A_k & & & \beta_k^T \\
\hline
c_1 & & & & & d_1 \\
\vdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \vdots \\
c_\ell & & & & & d_\ell \\
\hline
 & \beta_{\ell+1}^T & & & & d_{\ell+1} \\
\mathbf{0} & \ddots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \vdots \\
 & \beta_t^T & & & & d_t \\
\hline
0 & \mathbf{0} & \beta_{t+1}^T \ldots \beta_k^T & d_1 \ldots d_\ell & d_{\ell+1} \ldots d_t & 0
\end{array}\right).$$

After applying ECMOs on the above matrix, we can get a diagonal matrix similar to (3.2), hence the result is still holds in this case.

Case 3: $\min\{r_1, r_2, \ldots, r_k\} = 2$. Let $c_{t+1} = w(u_0v_0)$, then we only need to delete the row and the column corresponding to A_{t+1} and replace the upper right and the lower left elements of $A(G_w)$ with c_{t+1}, and the rest arguments are similar.

Theorem 3.1. *Let $G_w = \theta(r_1, r_2, \ldots, r_k)_w$ be a weighted graph of order n. Denote by n_i the number of r_j's which satisfy $r_j - 2 \equiv i \pmod 4$ $(1 \leqslant j \leqslant k, 0 \leqslant i \leqslant 3)$ and let $t = n_1 + n_3$. The matrix D is defined as in* (3.1). *Then*

$$(i_+(G_w), i_-(G_w), i_0(G_w)) = \Bigg(i_+(D) + \frac{1}{2}(n-2-t), i_-(D) + \frac{1}{2}(n-2-t), t+2-\mathrm{R}(D)\Bigg). \quad (3.3)$$

In particular,

(i) *if $n_1 + n_3 = 0,\ s = 0$, then $(i_+(G_w), i_-(G_w), i_0(G_w)) = \left(\frac{1}{2}n - 1, \frac{1}{2}n - 1, 2\right)$.*

(ii) *if $n_1 + n_3 = 0,\ s \neq 0$, then $(i_+(G_w), i_-(G_w), i_0(G_w)) = \left(\frac{1}{2}n, \frac{1}{2}n, 0\right)$.*

(iii) *if $n_1n_3 > 0$, then*

$$(i_+(G_w), i_-(G_w), i_0(G_w)) = \left(\frac{1}{2}(n-t)+1, \frac{1}{2}(n-t)+1, t-2\right).$$

(iv) *if $n_1 + n_3 \neq 0,\ n_1n_3 = 0$ and $d_ic_t \neq c_id_t$ holds for some $i \in \{1, 2, \ldots, t-1\}$, then*

$$(i_+(G_w), i_-(G_w), i_0(G_w)) = \left(\frac{1}{2}(n-t)+1, \frac{1}{2}(n-t)+1, t-2\right).$$

(v) *if $n_1 + n_3 \neq 0,\ n_1n_3 = 0,\ s > 0$ and $d_ic_t = c_id_t$ holds for $i = 1, 2, \ldots, t$, then*

$$(i_+(G_w), i_-(G_w), i_0(G_w)) = \begin{cases} \left(\frac{1}{2}(n-t), \frac{1}{2}(n-t)+1, t-1\right), & \text{if } n_1 > 0, n_3 = 0; \\ \left(\frac{1}{2}(n-t)+1, \frac{1}{2}(n-t), t-1\right), & \text{if } n_3 > 0, n_1 = 0. \end{cases}$$

(vi) *if $n_1 + n_3 \neq 0,\ n_1n_3 = 0,\ s = 0$ and $d_ic_t = c_id_t$ holds for $i = 1, 2, \ldots, t$, then*

$$(i_+(G_w), i_-(G_w), i_0(G_w)) = \left(\frac{1}{2}(n-t), \frac{1}{2}(n-t), t\right).$$

(vii) *if $n_1 + n_3 \neq 0,\ n_1n_3 = 0,\ s < 0$ and $d_ic_t = c_id_t$ holds for $i = 1, 2, \ldots, t$, then*

$$(i_+(G_w), i_-(G_w), i_0(G_w)) = \begin{cases} \left(\frac{1}{2}(n-t)+1, \frac{1}{2}(n-t), t-1\right), & \text{if } n_1 > 0, n_3 = 0; \\ \left(\frac{1}{2}(n-t), \frac{1}{2}(n-t)+1, t-1\right), & \text{if } n_3 > 0, n_1 = 0. \end{cases}$$

Proof. By the discussion of Cases 1-3 above, the first part of Theorem 3.1 follows directly. Furthermore, by the first part of Theorem 3.1 it is routine to check that (i) and (ii) hold.

(iii) If $n_1 n_3 > 0$, applying ECMOs on D yields the following matrix:

$$\left(\begin{array}{cc|ccc} 0 & s & 0 & \dots & c_t \\ s & 0 & \alpha_1 & \dots & d_t \\ \hline 0 & \alpha_1 & & & \\ \vdots & \vdots & & \mathbf{0} & \\ c_t & d_t & & & \end{array}\right),$$

where $\alpha_i = d_i - \frac{d_t}{c_t} c_i$. Noted that $c_1 > 0$ and $c_t < 0$, hence $\alpha_1 \neq 0$, which implies that $i_+(D) = i_-(D) = 2$ and $\mathrm{R}(D) = 4$. By (3.3), we have $(i_+(G_w), i_-(G_w), i_0(G_w)) = (\frac{1}{2}(n-t), \frac{1}{2}(n-t)+1, t-1)$. By a similar discussion as in the proof of (iii), we can show that (iv) also holds.

(v) In this case, applying ECMOs to D yields the following matrix:

$$\left(\begin{array}{cc|cccc} 0 & s & 0 & \dots & 0 \\ s & 0 & 0 & \dots & 0 \\ \hline 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \dots & -\frac{2c_t d_t}{s} \end{array}\right).$$

If $n_1 > 0, n_3 = 0$, then $-\frac{2c_t d_t}{s} < 0$ for $c_t > 0$, hence $i_+(D) = 1, i_-(D) = 2$ and $\mathrm{R}(D) = 3$. In view of (3.3), we have $(i_+(G_w), i_-(G_w), i_0(G_w)) = (\frac{1}{2}(n-t), \frac{1}{2}(n-t)+1, t-2)$. If $n_1 = 0, n_3 > 0$, then $-\frac{2c_t d_t}{s} > 0$ for $c_t < 0$, hence $i_+(D) = 2, i_-(D) = 1$ and $\mathrm{R}(D) = 3$. In view of (3.3), we have $(i_+(G_w), i_-(G_w), i_0(G_w)) = (\frac{1}{2}(n-t)+1, \frac{1}{2}(n-t), t-2)$. By a similar discussion, we can also show that (vi) and (vii) hold.

This completes the proof. □

4 Characterization of weighted graphs in $\Gamma_{n,k-1}$ with small positive (negative) indices

In this section, we'll characterize all the weighted graphs in $\Gamma_{n,k-1}$ with 1 or 2 positive (negative) indices.

Theorem 4.1. *Let $G_w \in \Gamma_{n,k-1}$. Then $i_+(G_w) = 1$ if and only if G_w is one of the following graphs: the weighted graph $\theta(3,\dots,3)_w$ with $c_k d_i = c_i d_k$, $i = 1, 2, \dots, k$; the weighted graph $\theta(3,\dots,3,2)_w$ with $c_{k-1} d_i = c_i d_{k-1}$, $i = 1, 2, \dots, k-1$.*

Proof. The sufficiency follows directly from Theorem 3.1. Here we only show the necessity in what follows.

Note that if $G_w \in \Gamma_{n,k-1}$ with pendants, then assume, without loss of generality, that x is a pendent vertex of G_w. Let $N(x) = \{y\}$ and $G'_w = G_w - \{x, y\}$. It's routine to check that G'_w is not a weighted empty graph, which contradicts to the fact that $i_+(G_w) = 1$.

Now we consider the case that G_w contains no pedants and $i_+(G_w) = 1$. In view of Theorem 3.1,

• $t = 0$ and $s = 0$. In this subcase, we have $i_+(G_w) = \frac{1}{2}n - 1 = 1$ holds for $n = 4$. Then $G_w = \theta(2,4)_w$ with weighted condition $c_1 w_{11}^2 = a_2 b_2$ for $s = 0$. Note that the weighted graph $\theta(2,4)_w$ with $c_1 w_{11}^2 = a_2 b_2$ is, in fact, the weighted graph $\theta(3,3)_w$ with $c_2 d_i = c_i d_2$, $i = 1,2$.

• $t = 0$ and $s \neq 0$. In this subcase, we have $n \geqslant 4$, hence $i_+(G_w) = \frac{n}{2} \geqslant 2$.

• $n_1 > 0$ and $n_3 > 0$. In this subcase, we have $n-t \geqslant 4$, hence $i_+(G_w) = \frac{1}{2}(n-t) + 1 \geqslant 3$.

• Just one of n_1 and n_3 is 0, and $d_i c_t \neq c_i d_t$ holds for some $i \in \{1,2,\ldots,t\}$. In this subcase, we have $n-t \geqslant 2$ if $n_3 = 0$ and $n-t \geqslant 6$ if $n_1 = 0$. Hence $i_+(G_w) = \frac{1}{2}(n-t)+1 \geqslant 2$.

• Just one of n_1 and n_3 is 0, $s = 0$ and $d_i c_t = c_i d_t$ holds for $i = 1,2,\ldots,t$. In this subcase, we have $n-t \geqslant 2$ if $n_3 = 0$ and $n-t \geqslant 6$ if $n_1 = 0$. Hence, $i_+(G_w) = 1$ if and only if $n-t = 2$ and $n_3 = 0$. This gives that G_w must be the weighted graph $\theta(3,\ldots,3)_w$ with $c_k d_i = c_i d_k$ holding for $i = 1,2,\ldots,k$.

• Just one of n_1 and n_3 is 0, $s > 0$ and $d_i c_t = c_i d_t$ holds for $i = 1,2,\ldots,t$. In this subcase, we have $n-t \geqslant 2$ if $n_3 = 0$ and $n-t \geqslant 4$ if $n_1 = 0$. Hence, $i_+(G_w) = 1$ if and only if $n-t = 2$ and $n_3 = 0$. This gives that G_w must be the weighted graph $\theta(3,\ldots,3,2)_w$ with $c_{k-1} d_i = c_i d_{k-1}$ holding for $i = 1,2,\ldots,k-1$.

• Just one of n_1 and n_3 is 0, $s < 0$ and $d_i c_t = c_i d_t$ holds for $i = 1,2,\ldots,t$. In this subcase, we have $n-t \geqslant 4$ if $n_3 = 0$ and $n-t \geqslant 6$ if $n_1 = 0$, which implies that $i_+(G_w) = \frac{1}{2}(n-t)+1 > 1$.

Hence, we conclude that $i_+(G_w) = 1$ if and only if G_w is the weighted graph $\theta(3,\ldots,3)_w$ with $c_k d_i = c_i d_k$ holding for $i = 1,2,\ldots,k$ or, G_w is the weighted graph $\theta(3,\ldots,3,2)_w$ with $c_{k-1} d_i = c_i d_{k-1}$ holding for $i = 1,2,\ldots,k-1$. □

Theorem 4.2. *Let $G_w \in \Theta_k$. Then $i_+(G_w) = 2$ if and only if G_w is one of the following graphs: the weighted graph $\theta(2,4,4)_w$ with $c_1 = \frac{a_2 b_2}{w_{11}^2} + \frac{a_3 b_3}{w_{11}^3}$; the weighted graph $\theta(3,\ldots,3)_w$ with $d_i c_t \neq c_i d_k$ for some $i \in \{1,2,\ldots,k\}$; the weighted graph $\theta(3,\ldots,3,2)_w$ with $d_i c_{k-1} \neq c_i d_{k-1}$ for some $i \in \{1,2,\ldots,k-1\}$; the weighted graph $\theta(3,\ldots,3,2,4)_w$ with $c_{k-2} d_i = c_i d_{k-2}$, $i = 1,2,\ldots,k-2$ and $c_{k-1} w_{11}^k \geqslant a_k b_k$.*

Proof. The sufficiency is clear by Theorem 3.1. To prove the necessity, suppose that $G_w \in \Theta_k$ with $i_+(G_w) = 2$. We proceed by distinguishing the following subcases.

• $t = 0$ and $s = 0$. In this subcase, $i_+(G_w) = \frac{1}{2}n - 1 = 2$, hence we have $n = 6$. Then G_w may be $\theta(2,4,4)_w$, $\theta(2,6)_w$ or $\theta(4,4)_w$. If G_w is the weighted graph $\theta(2,4,4)_w$, then $c_1 w_{11}^2 = a_2 b_2$ for $s = 0$, whereas the s of $\theta(2,6)_w$ is positive and the s of $\theta(4,4)_w$ is negative, which contradicts the assumption that $s = 0$.

• $t = 0$ and $s \neq 0$. In this subcase, $i_+(G_w) = \frac{1}{2}n = 2$, hence we have $n = 4$. Then G_w is just the weighted graph $\theta(2,4)_w$ with $c_1 w_{11}^2 \neq a_2 b_2$. In fact, the weighted graph $\theta(2,4)_w$ with $c_1 w_{11}^2 \neq a_2 b_2$ is also the weighted graph $\theta(3,3)_w$ with $c_k d_i \neq c_i d_k$ for $i = 1,2$.

• $n_1 > 0, n_3 > 0$. In this subcase, we have $n-t \geqslant 4$. Hence, $i_+(G_w) = \frac{1}{2}(n-t)+1 \geqslant 3$, which implies that there does not exist such weighted graph G_w.

• Just one of n_1 and n_3 is 0, and $d_i c_t \neq c_i d_t$ holds for some $i \in \{1,2,\ldots,t\}$. In this subcase, by a similar discussion in the proof of Theorem 4.1, $i_+(G_w) = 2$ holds only if

$n_3 = 0$ in which $i_+(G_w) = \frac{1}{2}(n-t)+1$. So we have $n - t = 2$. Hence G_w must be the weighted graph $\theta(3,\ldots,3)_w$ with $d_i c_t \neq c_i d_k$ for some $i \in \{1,2,\ldots,k\}$, or the weighted graph $\theta(3,\ldots,3,2)_w$ with $d_i c_{k-1} \neq c_i d_{k-1}$ for some $i \in \{1,2,\ldots,k-1\}$.

• Just one of n_1 and n_3 is 0, $s = 0$ and $d_i c_t = c_i d_t$ holds for $i = 1,2,\ldots,t$. In this subcase, $i_+(G_w) = \frac{1}{2}(n-t)$. Hence, by a similar discussion in the proof of Theorem 4.1, $i_+(G_w) = 2$ if and only if $n - t = 4$ and $n_3 = 0$, which implies that G_w must be the weighted graph $\theta(3,\ldots,2,4)_w$ with $c_{k-2}d_i = c_i d_{k-2}$ $i = 1,2,\ldots,k-2$ and $c_{k-1}w_{11}^k = a_k b_k$.

• Just one of n_1 and n_3 is 0, $s > 0$ and $d_i c_t = c_i d_t$ holds for $i \in \{1,2,\ldots,t\}$. In this subcase, $i_+(G_w) = \frac{1}{2}(n-t)$. Hence, by a similar discussion in the proof of Theorem 4.1, $i_+(G_w) = 2$ if and only if $n - t = 4$ and $n_3 = 0$, which implies that G_w must be the weighted graph $\theta(3,\ldots,2,4)_w$ with $c_{k-2}d_i = c_i d_{k-2}$ for $i \in \{1,2,\ldots,k-2\}$ and $c_{k-1}w_{11}^k > a_k b_k$.

• Just one of n_1 and n_3 is 0, $s < 0$ and $d_i c_t = c_i d_t$ holds for $i \in \{1,2,\ldots,t\}$. In this subcase, by a similar discussion in the proof of Theorem 4.1, we have $n - t \geqslant 4$ if $n_3 = 0$ and $n - t \geqslant 6$ if $n_1 = 0$. Hence, we have $i_+(G_w) = \frac{1}{2}(n-t)+1 > 2$.

This completes the proof. □

Theorem 4.3. *Let $G_w \in \Gamma_{n,k}$ with pedants. Then $i_+(G_w) = 2$ if and only if $G \cong G^1, G^2, \ldots, G^9$ or G^{10} (see Fig. 2) and the corresponding weighted conditions are as shown in Table 1, where the empty cell means that there is no correlation between the inertia index of G_w and its weight set.*

Table 1: The weighted condition for each $G_w \in \Gamma(n,k)$ with pedants satisfying $i_+(G_w) = 2$.

weighted graph G_w	weighted conditions of G_w
$G_w^1, G_w^2, G_w^3, G_w^4$	
G_w^5	$c_{k-1}d_i = c_i d_{k-1}$ $(1 \leqslant i \leqslant k-1)$
G_w^6, G_w^7	$c_k d_i = c_i d_k$ $(1 \leqslant i \leqslant k)$
G_w^8	$c_{k-1}d_i = c_i d_{k-1}$ $(2 \leqslant i \leqslant k-1)$
G_w^9, G_w^{10}	$c_{k-1}d_i = c_i d_{k-1}$ $(1 \leqslant i \leqslant k-1)$

Proof. It is routine to check that $i_+(G_w^i) = 2$ holds for $i = 1,2,\ldots,10$. To show the converse, suppose that $i_+(G_w) = 2$. Since G_w has at least one pendent x, let $N(x) = \{y\}$ and $G'_w = G_w - \{x,y\} = H_w + pK_1$, where H_w is obtained from G'_w by deleting all the isolated vertices. By Lemma 2.3 we have $2 = i_+(G_w) = i_+(G'_w) + 1 = i_+(H_w) + 1$. Hence, $i_+(H_w) = 1$. Recall that G_w contains a Θ-graph as an induced subgraph, we conclude that H_w is either isomorphic to a weighted star or one of the weighted graphs described in Theorem 4.1. If H_w is a star, then G must be isomorphic to $G^i, i = 1,2,3,4$. If H_w is the weighted graph $\theta(3,\ldots,3)_w$, then G must be isomorphic to G^i, $i = 5,6,7$ and if H_w is the weighted graph $\theta(3,\ldots,3,2)_w$, then G must be isomorphic to G^i, $i = 8,9,10$.

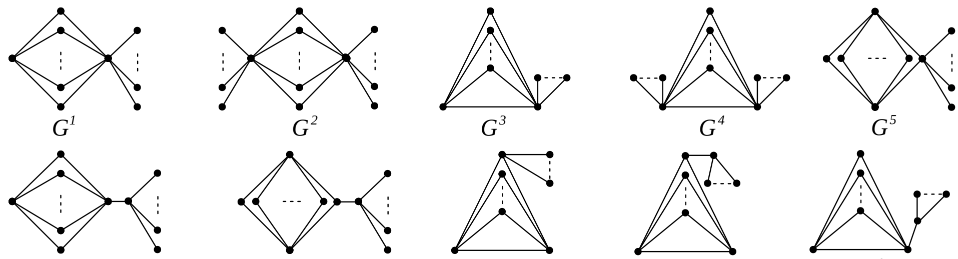

Figure 2: Graphs $G^1, G^2, \ldots, G^9$ and G^{10}.

If G is isomorphic to G^5, without loss of generality, assume that x is adjacent to the internal vertex of the k-th path P_3 (see Fig. 2), so the weighted condition is that $c_{k-1}d_i = c_i d_{k-1}$ holds for $i = 1, 2, \ldots, k-1$. If G is isomorphic to G^6 or G^7, the weighted condition is $c_k d_i = c_i d_k$ for $i = 1, 2, \ldots, k$.

If G is isomorphic to G^8, without loss of generality, assume that x is adjacent to the internal vertex of the first path P_3 (see Fig. 2), so the weighted condition is that $c_{k-1}d_i = c_i d_{k-1}$ holds for $i = 2, 3, \ldots, k-1$. If G is isomorphic to G^9 or G^{10}, the weighted condition is $c_{k-1}d_i = c_i d_{k-1}$ for $i = 1, 2, \ldots, k-1$. $\square$

Similarly, we can have the following theorems:

Theorem 4.4. *Let $G_w \in \Gamma_{n,k-1}$. Then $i_-(G_w) = 1$ if and only if G_w is the weighted $\theta(3, \ldots, 3)_w$ with the weighted condition that $c_k d_i = c_i d_k$ holds for $i = 1, 2, \ldots, k$.*

Theorem 4.5. *Let $G_w \in \Theta_k$. Then $i_-(G_w) = 2$ if and only if G_w is one of the following graphs: the weighted graph $\theta(3, \ldots, 3, 2)_w$ with an arbitrary weighted condition; the weighted graph $\theta(2, 4, 4)_w$ with weighted condition $c_1 = \frac{a_2 b_2}{w_{11}^2} + \frac{a_3 b_3}{w_{11}^3}$; the weighted graph $\theta(3, \ldots, 3)_w$ with the weighted condition that $d_i c_k \neq c_i d_k$ holds for some $i \in \{1, 2, \ldots, k\}$; the weighted graph $\theta(3, \ldots, 3, 2, 4)_w$ with the weighted condition that $c_{k-2}d_i = c_i d_{k-2}$ holds for $i = 1, 2, \ldots, k-2$ and $c_{k-1}w_{11}^k \leqslant a_k b_k$; the weighted graph $\theta(3, \ldots, 3, 4)_w$ with the weighted condition that $c_{k-1}d_i = c_i d_{k-1}$ holds for $i = 1, 2, \ldots, k-1$.*

Theorem 4.6. *Let $G_w \in \Gamma_{n,k-1}$ with pedants. Then $i_-(G_w) = 2$ if and only if G_w is one of the following graphs: the weighted graph G_w has G^1 (resp. G^2, G^3, G^4) as its unweighted graph and its weight set is arbitrary; the weighted graph G_w has G^5 as its unweighted graph satisfying the weighted condition $c_{k-1}d_i = c_i d_{k-1}$, $i = 1, 2, \ldots, k-1$; the weighted graph G_w has G^6 (resp. G^7) as its unweighted graph satisfying the weighted condition $c_k d_i = c_i d_k$, $i = 1, 2, \ldots, k$.*

5 Weighted graphs in $\Gamma_{n,k-1}$ with rank $2, 3,$ or 4

In this section, we characterize all the weighted $(k-1)$-cyclic graphs in $\Gamma_{n,k-1}$ with rank 2, 3, 4, respectively.

Theorem 5.1. *Let $G_w \in \Gamma_{n,k-1}$. Then $\mathrm{R}(G_w) = 2$ if and only if G_w is the weighted $\theta(3,\dots,3)_w$ with the weighted condition $c_k d_i = c_i d_k$ holding for $i = 1, 2, \dots, k$.*

Proof. Let $G_w \in \Gamma_{n,k-1}$, $i_+(G_w) \geqslant 1$ and $i_-(G_w) \geqslant 1$ since it contains P_2 as an induced subgraph. Then $r(G_w) = 2$ if and only if $i_+(G_w) = i_-(G_w) = 1$. By Theorems 4.1–4.6, we know G_w must be the weighted $\theta(3,\dots,3)_w$ satisfying the weighted condition that $c_k d_i = c_i d_k$ for any $1 \leqslant i \leqslant k$. □

Theorem 5.2. *Let $G_w \in \Gamma_{n,k-1}$. Then $\mathrm{R}(G_w) = 3$ if and only if G_w is the weighted $\theta(3,\dots,3,2)_w$ with the weighted condition that $c_{k-1} d_i = c_i d_{k-1}$ holds for $i = 1, 2, \dots, k-1$.*

Proof. Let $G_w \in \Gamma_{n,k-1}$, $i_+(G_w) \geqslant 1$ and $i_-(G_w) \geqslant 1$ since it contains P_{2w} as an induced subgraph. Then $\mathrm{R}(G_w) = 3$ if and only if $i_+(G_w) = 1, i_-(G_w) = 2$ or $i_+(G_w) = 2, i_-(G_w) = 1$. Note that either $i_+(G_w)$ or $i_-(G_w)$ equals 1, hence by Theorems 4.1 and 4.4 we know G_w must be the weighted graph $\theta(3,\dots,3)_w$ satisfying $c_k d_i = c_i d_k$ for $1 \leqslant i \leqslant k$. □

Theorem 5.3. *Let $G_w \in \Theta_k$. Then $\mathrm{R}(G_w) = 4$ if and only if G_w is one of the following graphs: the weighted graph $\theta(2,4,4)_w$ with weighted condition $c_1 = \frac{a_2 b_2}{w_{11}^2} + \frac{a_3 b_3}{w_{11}^3}$; the weighted graph $\theta(3,\dots,3)$ with the weighted condition that $d_i c_k \neq c_i d_k$ holds for some $i \in \{1, 2, \dots, k\}$; the weighted graph $\theta(3,\dots,3,2)_w$ with the weighted condition that $d_i c_{k-1} \neq c_i d_{k-1}$ holds for some $i \in \{1, 2, \dots, k-1\}$; the weighted graph $\theta(3,\dots,3,2,4)_w$ with the weighted condition that $c_{k-2} d_i = c_i d_{k-2}$ holds for $i = 1, 2, \dots, k-2$ and $c_{k-1} w_{11}^k = a_k b_k$.*

Proof. Let G_w be a weighted $(k-1)$-cyclic graph, it is routine to check that $i_+(G_w) \geqslant 1$ and $i_-(G_w) \geqslant 1$. Then $\mathrm{R}(G_w) = 4$ if and only if $(i_+(G_w), i_-(G_w)) = (1,3)$ or $(i_+(G_w), i_-(G_w)) = (3,1)$ or $(i_+(G_w), i_-(G_w)) = (2,2)$. If one of $i_+(G_w)$ and $i_-(G_w)$ equals 1, by Theorems 4.1 and 4.4, G_w must be the weighted graph $\theta(3,\dots,3)_w$ or $\theta(3,\dots,3,2)_w$. In this case, by Theorems 4.1, 4.2, 4.4 and 4.5 we know the rank of such graph G_w is no less than 3. Hence, it should only consider that $(i_+(G_w), i_-(G_w)) = (2,2)$. In this case, based on Theorems 4.2 and 4.5, $(i_+(G_w), i_-(G_w)) = (2,2)$ if and only if G_w is one of the weighted graphs characterized in the above result. □

Similarly, we can have the following theorem:

Theorem 5.4. *Let $G_w \in \Gamma_{n,k-1}$ with pedants. Then $\mathrm{R}(G_w) = 4$ if and only if $G \cong G^1, \dots, G^7$, what's more, the weighted condition of G_w^1 (resp. G_w^2, G_w^3, G_w^4) is arbitrary; G_w^5 satisfies the weighted condition that $c_{k-1} d_i = c_i d_{k-1}$ holds for $i = 1, 2, \dots, k-1$; while G_w^6 (resp. G_w^7) satisfies the weighted condition that $c_k d_i = c_i d_k$ holds for $i = 1, 2, \dots, k$.*

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