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Relating Embedding and Coloring Properties of Snarks

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Abstract

In 1969, Grünbaum conjectured that snarks do not have polyhedral embeddings into orientable surfaces. To describe the deviation from polyhedrality, we define the defect of a graph and use it to study embeddings of superpositions of cubic graphs into orientable surfaces. Superposition was introduced in [4] to construct snarks with arbitrary large girth. It is shown that snarks constructed in [4] do not have polyhedral embeddings into orientable surfaces. For each $k \geq 2$ we construct infinitely many snarks with defect precisely k. We then relate the defect with the resistance $r(G)$ of a cubic graph G which is the size of a minimum color class of a 4-edge-coloring of G. These results are then extended to deal with some weaker versions of the Grünbaum Conjecture.

Keywords: Graph embedding, resistance, snark, superposition.

Math. Subj. Class.: 05C10, 05C15

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1 Introduction

In this paper we study 2-cell embeddings of cubic graphs into closed orientable surfaces. All graphs will be simple and without loops. We follow the notation of [5]. A (orientable) combinatorial embedding Π of a graph G is given by a rotation system and two embeddings are (combinatorially) *equivalent* if they define the same collection of facial walks. To describe an embedding it is sufficient to list all facial walks. An embedding given by a collection $\mathcal{F}$ of facial walks is orientable if we can orient walks in $\mathcal{F}$ so that each edge appears twice along walks in $\mathcal{F}$, once in each direction. Any such orientation is called a *consistent orientation*. Unless stated otherwise, all embeddings considered are orientable.

An embedding of a graph is *polyhedral* if all facial walks are cycles and any two of them are either disjoint, intersect in one vertex or intersect in one edge. Therefore, an embedding of a cubic graph is polyhedral if all facial walks are cycles and any two of them are either disjoint or they intersect in precisely one edge.

The study of polyhedral embeddings of cubic graphs is motivated by the following conjecture of Grünbaum [3].

Conjecture 1 (Grünbaum Conjecture)**.** *If a cubic graph admits a polyhedral embedding into an orientable surface, then it is 3-edge-colorable.*

Grünbaum proposed this conjecture in 1969 [3] as a generalization of the Four Color Theorem, which was not yet proved at that time. At present it is known that it is true for the plane (follows from the Four Color Theorem). Recently, Kochol announced that he found counterexamples to the conjecture on surfaces with genus at least five. The conjecture is still open for other surfaces.

For each orientable surface S we can state the Grünbaum Conjecture for S.

Conjecture 2 (Grünbaum Conjecture for S)**.** *If a cubic graph admits a polyhedral embedding into the orientable surface S, then it is 3-edge-colorable.*

Let Π be an embedding of a 3-connected cubic graph G into a surface S. Consider its dual multigraph G^*, and observe that the embedding is polyhedral if and only if G^* is a simple graph. Let w be the length of a shortest cycle in G^* which is non-contractible in S. Then w is called the *face-width* of the embedding Π. It is easy to see that Π is polyhedral if and only if the face-width is at least 3 (and G is 3-connected). See [5] for more details.

Robertson and Mohar proposed a weakening of Grünbaum's Conjecture, where the assumption on the face-width is strengthened (see [5]).

Conjecture 3 (Grünbaum, Mohar, Robertson)**.** *There exists an integer k such that the following holds. If a cubic graph admits an embedding into an orientable surface with face-width at least k, then it is 3-edge-colorable.*

In [2] it is also conjectured that 3-connected cubic graphs with embeddings of face-width at least 4 are 3-edge-colorable. However, it is not even known if any lower bound on the face-width guarantees 3-edge-colorability for any non-simply connected fixed surface.

By a theorem of Vizing, simple cubic graphs are either 3- or 4-edge-colorable. Cyclically 4-edge-connected cubic graphs with girth at least 4 which are not 3-edge-colorable are called *snarks*. Grünbaum's conjecture is equivalent to the statement that snarks do not admit polyhedral embeddings into orientable surfaces.

Grünbaum's conjecture has been verified for many families of snarks since its formulation in 1969. Szekeres [10] proved it for flower snarks J_{2k+1} and the Szekeres graph, in [6] it was proved for Goldberg snarks G_{2k+1}. However the proofs do not rely on the coloring properties of graphs. It can be shown that no graph J_n or G_n, $n \geq 3$, admits an orientable polyhedral embedding and that the graphs J_n neither have polyhedral embeddings into non-orientable surfaces [6].

Let Π be an embedding of a cubic graph G and let $\mathcal{F} = \{W_1, \ldots, W_k\}$ be the collection of facial walks of Π. For a walk $W_i \in \mathcal{F}$ we define the *defect* $d(W_i)$ of W_i to be the number of edges which appear twice along W_i. For two facial walks $W_i, W_j \in \mathcal{F}$, $i \neq j$, we define the *defect* $d(W_i, W_j) = \max\{0, |E(W_i) \cap E(W_j)| - 1\}$. The defect of the embedding Π is defined as

$$d(G, \Pi) = d(\Pi) = \sum_{i=1}^{k} d(W_i) + \sum_{1 \leq i < j \leq k} d(W_i, W_j).$$

and the defect of the graph G is defined as

$$d(G) = \min\{d(G, \Pi) \mid \Pi \text{ is an orientable embedding of } G\}.$$

In an embedding Π of G, a pair of facial walks is a *bad pair* if they have more than one edge in common. An edge e is a *bad edge* if it appears twice along a facial cycle of Π or if there is another edge f such that e and f both appear on two facial walks W_i and W_j. In the latter case we call the edges e and f a *bad pair of edges*.

It is clear from the definition of the defect that a graph G admits a polyhedral embedding into an orientable surface if and only if $d(G) = 0$. The Grünbaum Conjecture is therefore equivalent to the statement that for any snark G the defect $d(G)$ is at least 1. We give a stronger connection in Section 4.

The defect $d(G, \Pi)$ can also be expressed as follows.

Lemma 4. *If G is a 3-connected Π-embedded cubic graph, then $d(G, \Pi)$ equals the minimum cardinality of an edge set $F \subseteq E(G)$ with the property that every non-contractible cycle C^* of length 1 or 2 in the dual graph G^* intersects at least one edge in F.*

Proof. If $F \subseteq E(G)$ has the property that every non-contractible cycle C^* in the dual graph G^* intersects at least one edge in F, then we say that F is a blocking set. It is clear that every blocking set has to contain all edges which appear twice in some facial walk. Namely, since G is 2-connected, every such edge gives rise to a non-contractible loop in the dual G^*. Similarly, 3-connectivity implies that every pair of edges shared by two facial walks W_i, W_j gives rise to a non-contractible 2-cycle in G^*. Therefore, if $|E(W_i) \cap E(W_j)| \geq 2$, every blocking set contains all but at most one of the edges in $E(W_i) \cap E(W_j)$. This proves that $|F| \geq d(\Pi)$. The converse inequality holds for the set F in which we put all the edges shown above to be necessarily included. This completes the proof. □

Embeddings of snarks and in particular snarks obtained as dot products have been studied in [11, 8, 7]. The genus of the Petersen graph is 1 and the genera of the two Blanuša snarks, which are the two possible dot products of the Petersen graph, are 1 and 2. Using a computer program which goes over all possible rotation systems we also computed the defects of some small snarks. In particular for the Petersen graph P and the Blanuša graph B_1 of genus 1 we have the following lemma.

Lemma 5. *Let P be the Petersen graph and B_1 the Blanuša snark of genus 1. Then $d(P) = 5$ and $d(B_1) = 3$.*

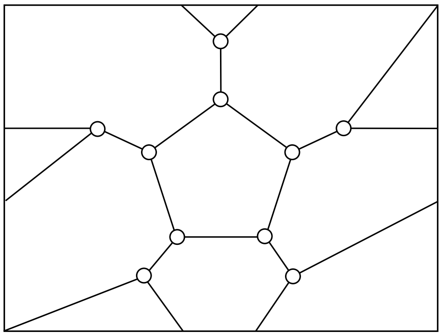

Figure 1: An embedding of the Petersen graph in the torus.

Figure 1 shows an embedding of the Petersen graph in the torus with defect 5 and Figure 2 shows the graph B_1 embedded in the torus with defect 3.

Further, using a computer, we found that the smallest possible defect among snarks with up to 28 vertices is 2. We describe an orientable embedding of a snark G_{26} on 26 vertices with defect 2. The graph G_{26} has vertices labeled with integers from 1 to 26 and the embedding has the following facial cycles:

```
Face  1:    1  2  5  9  4
Face  2:    2  1  3  8 14 18 12  6
Face  3:    3  1  4 10  7
Face  4:    5  2  6 11 13  8  3  7
Face  5:    5  7 10 16 25 26 23 15  9
Face  6:    4  9 15 22 24 16 10
Face  7:   11  6 12 19 17
Face  8:    8 13 20 21 14
Face  9:   13 11 17 24 22 20
Face 10:   12 18 26 25 19
Face 11:   18 14 21 23 26
Face 12:   17 19 25 16 24
Face 13:   21 20 22 15 23
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The snark G_{26} has two orientable embeddings with defect 2. There are two snarks with 28 vertices with defect 2. All other snarks with at most 28 vertices have defect at least 3.

2 Superposition

We give a short description of superposition of graphs. For the details see [4]. A multipole $M = (V, E, S)$ consists of a set of vertices V, edges E and semiedges S. A semiedge s is incident with one vertex v and denoted by $s = (v)$. We assume that the degrees of vertices in a multipole are all 3 (the degree of a vertex v in a multipole is the number of edges and semiedges incident with v). A $(k_1, \ldots, k_n)$-pole is a multipole (V, E, S) with a partition of semiedges into sets $S = S_1 \cup \cdots \cup S_n$ with $|S_i| = k_i$, $i = 1, \ldots, n$. The sets $S_1, \ldots, S_n$ are called the *connectors* of the multipole. A (k_1, k_2)- pole is called a *superedge* and a

(k_1, k_2, k_3)-pole is called a *supervertex*. A $(1,1,1)$-pole consisting of a single vertex v and three semiedges incident with v is called a *trivial supervertex*.

Let G be a snark. We remove two non-adjacent vertices v and u from G and replace all edges vx_i incident with v with semiedges (x_i), $i = 1, 2, 3$, and all edges uy_i with semiedges (y_i), $i = 1, 2, 3$. We define $S_1 = \{(x_1), (x_2), (x_3)\}$ and $S_2 = \{(y_1), (y_2), (y_3)\}$ to obtain a $(3, 3)$-multipole with connectors S_1 and S_2. We say that we have obtained a *proper superedge* by removing vertices v and u from G. An empty multipole will be considered as a special $(1, 1)$-multipole and a proper superedge. An empty multipole is also called a *trivial superedge*. For a broader definition of a proper superedge see [4].

Let $G = (V, E)$ be a cubic graph. To each vertex $v \in V$ we assign a supervertex $\mathcal{S}(v)$ and additionally to each edge incident to v we assign one of the connectors of $\mathcal{S}(v)$. To each edge $xy \in E$ we assign a (proper) superedge $\mathcal{E}(x, y)$. If $\mathcal{E}(x, y)$ is not an empty multipole we assign one of the connectors of $\mathcal{E}(x, y)$ to x and the other to y.

Assume that for each edge $e = xy \in E$ the following holds. If $\mathcal{E}(x, y)$ is an empty multipole then the connectors assigned to e in supervertices $\mathcal{S}(x)$ and $\mathcal{S}(y)$ have cardinality 1. Otherwise the connector assigned to the edge e in the supervertex $\mathcal{S}(x)$ has the same cardinality as the connector assigned to x in the superedge $\mathcal{E}(x, y)$ and the same holds for y.

We can then construct a new graph as follows. If the superedge assigned to $e = xy$ is an empty multipole, then we remove the semiedge (v) in the connector of $\mathcal{S}(x)$ assigned to e and the semiedge (u) in the connector of $\mathcal{S}(y)$ assigned to e and add an edge uv. Otherwise we have semiedges $\{(u_1), (u_2), (u_3)\}$ in the connector of $\mathcal{S}(x)$ and semiedges $\{(x_1), (x_2), (x_3)\}$ in the connector of e assigned to x. We remove them, add the edges $\{u_1x_1, u_2x_2, u_3x_3\}$ and do the same for the vertex y. By repeating the procedure for all edges $e \in E$ we get a cubic graph G' called a superposition of G. If we have assigned proper superedges to all edges, then the graph G' is called a proper superposition of G.

The following result is proved in [4]:

Theorem 6. *A proper superposition of a snark is a snark.*

Superposition is a powerful tool used to construct new snarks from smaller snarks which generalizes many previously known constructions. This construction was used to obtain snarks with arbitrary large girth. In Section 5 we show that the snarks with high girth obtained by Kochol in [4] do not have orientable polyhedral embeddings.

3 Defect of a graph and the Grünbaum Conjecture

Let $M = (V, E, S)$ be a multipole. A *combinatorial embedding* of M is an assignment of rotations to vertices V. As with combinatorial embeddings of graphs, we can define the collection of facial walks $\mathcal{F}$, which consists of closed walks and walks which start and end with semiedges of a connector. Again we can describe the embedding of M by specifying $\mathcal{F}$. If in the definition of the defect we replace graphs with multipoles, we get the definition of a defect of a multipole.

Suppose we have an orientable embedding of a superedge $M = (V, E, S_1 \cup S_2)$. Let the connectors be $S_1 = \{(u_1), (u_2), (u_3)\}$ and $S_2 = \{(v_1), (v_2), (v_3)\}$. Suppose that in the consistent orientation of facial walks we have walks $W_1 = u_1P_1v_1$, $U_1 = u_2R_1u_1$, $U_2 = u_3R_2u_2$, $W_2 = v_3P_2u_3$, $V_1 = v_1Q_1v_2$ and $V_2 = v_2Q_2v_3$. Suppose further that the walks P_1 and P_2 are disjoint. An embedding as described is called a *nice embedding* of a superedge.

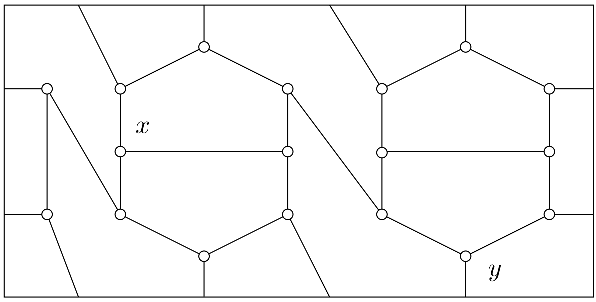

Figure 2: The Blanuša snark embedded in the torus with defect 3.

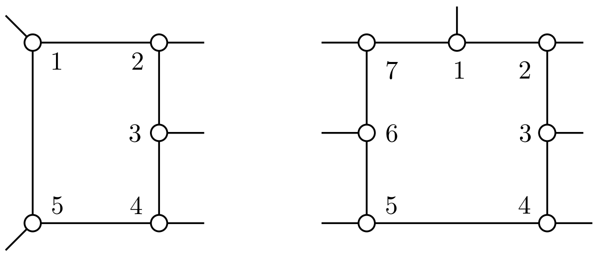

Figure 3: Supervertices used for replacing edges.

Take the Blanuša snark B_1 embedded in the torus and remove vertices x and y (see Figure 2) to obtain the proper *Blanuša superedge* B_1'. Note that the embedding of B_1 in the torus induces a nice embedding of B_1' with defect 1. Using a computer program, which goes over all possible rotations of vertices, we have proved the following result.

Lemma 7. *The Blanuša superedge* B_1' *has defect 1.*

We now describe what we mean by replacing an edge in an embedded graph with a nicely embedded superedge. Suppose Π is an embedding of G and $e = xy \in E(G)$ is an edge. Denote the neighbors of x with $\{y, x_1, x_2\}$ and the neighbors of y with $\{x, y_1, y_2\}$ so that in the embedding Π there are facial walks $C_1 = *x_1xyy_1*$, $C_2 = *y_1yy_2*$, $C_3 = *y_2yxx_2*$ and $C_4 = *x_2xx_1*$.

We will use the (1,1,3)-supervertex $\mathcal{V}$ from the left hand side of Figure 3 where the connectors are $\{(1)\}, \{(5)\}$ and $\{(2),(3),(4)\}$. To vertices x and y we assign $\mathcal{V}(x)$ and $\mathcal{V}(y)$, both copies of $\mathcal{V}$, to e we assign the nicely embedded superedge (with the notation defined at the beginning of this section) and to all other vertices and edges we assign trivial supervertices and superedges. We denote the vertices in $\mathcal{V}(y)$ with $1', 2', \ldots$ to distinguish them from vertices in $\mathcal{V}(x)$. In $\mathcal{V}(x)$ we assign connectors $\{(1)\}, \{(5)\}, \{(2),(3),(4)\}$ to xx_1, xx_2, e and in $\mathcal{V}(x)$ we assign connectors $\{(1')\}, \{(5')\}, \{(2'),(3'),(4')\}$ to yy_1, yy_2, e. In the superposition we add the edges $2u_1, 3u_2, 4u_3$ and v_12', v_23', v_34'.

This superposition has an induced embedding defined by facial walks $\mathcal{F}$ defined as follows. Take all facial walks of Π which do not contain vertices x and y and modify the facial walks C_i, $i = 1, 2, 3, 4$, to get walks C_i', $i = 1, 2, 3, 4$, as follows: $C_1' = *x_1 12 u_1 P_1 v_1 2' 1' y_1 *$, $C_2' = *y_1 1' 5' y_2 *$, $C_3' = *y_2 5' 4' v_3 P_2 u_3 45 x_2 *$ and $C_4' = *x_2 51 x_1 *$. Add walks 543215 and $1'2'3'4'5'1'$. Add all closed walks in the embedding of the superedge M. Add walks $23u_2R_1u_1 2$, $34u_3R_2u_2 3$, $3'2'v_1Q_1v_2 3'$ and $4'3'v_2Q_2v_3 4'$. We have described an orientable embedding of a graph G'. If in the embedding Π the cycles C_1 and C_3 are distinct then the bad edges in the induced embedding of G' are the bad edges of Π minus possibly e and the bad edges in the embedding of the superedge M.

Using the $(3, 1, 3)$-supervertex from Figure 3 we can similarly replace all edges on a facial cycle C in G. Again the bad edges in the induced embedding of the superposition are bad edges in the original graph minus possibly the edges of C and the bad edges in superedges.

Lemma 8. *The following statements are equivalent:*

(1) *The Grünbaum Conjecture is true.*

(2) *All snarks have defect at least 2.*

(3) *All nicely embedded proper superedges have defect at least 1.*

Proof. First we prove that (1) is equivalent to (3).

If the Grünbaum Conjecture is false, then there exists an embedding of a snark with defect 0. If we remove two non-adjacent vertices from one facial cycle in the embedding we get a nicely embedded superedge with an induced embedding of defect 0.

Suppose we have a nicely embedded superedge with defect 0. Take the embedding of P in the torus from Figure 1 and replace each edge along the unique 9-cycle with the nicely embedded proper superedge to get a snark with defect 0.

It is clear that (2) implies (1). The Grünbaum Conjecture implies that snarks have defect at least 1. We show that (3) implies that there is no snark with defect precisely 1, which completes the proof.

Suppose Π is an embedding of a snark G with defect 1. First we show that all facial walks are cycles and that there are two facial cycles C and D which have two edges $e = xy$ and $f = uv$ in common and that e and f are at distance at least 2 along C and D.

If there is a vertex v in G which appears twice along a facial walk W, then there is an edge incident with v which appears twice along W and contributes 1 to the defect of Π. There is another facial walk which contains v and it intersects W in at least two edges incident with v. So the defect of Π is at least 2, which shows that all facial walks are cycles.

There are two facial cycles C and D which intersect at two edges e and f. Suppose that e and f are at distance at most 2 on C. Edges e and f can not be adjacent since in this case C and D could not be facial cycles in an embedding of G. If they are at distance 1 on C, assume y and u are adjacent and there are vertices $x_1 \neq x, u$ and $v_1 \neq y, v$ such that x_1 is adjacent to y and v_1 is adjacent to u. The cycle C contains the path $xyuv$ and the cycle D contains paths x_1yx and vuv_1. There is another facial cycle which contains the path v_1uyx_1 and we get that the defect of the embedding is more than 1.

Now we can choose two vertices w and z on C which are not incident with e or f and w and z separate e and f on C. Since the defect is 1, the vertices w and z are not on the cycle D. Remove the vertices w and z from G to obtain a superedge. This is a nicely embedded superedge with defect 0. □

Theorem 9. *For each $k \geq 2$ there exist infinitely many snarks with defect precisely k. For each $k \geq 1$ there exist infinitely many nicely embedded superedges with defect precisely k.*

Proof. Suppose we have an embedding Π of a snark G with defect k in which all facial walks are cycles and there are k bad edges which form an independent set. Let B_1' be the nicely embedded superedge obtained from the Blanuša snark by removing vertices x and y. Replace each bad edge in G by B_1' to obtain an embedded snark G'. By the construction we see that the defect of G' is at most k. By Lemma 7 each superedge contributed at least 1 to the defect of G', so we get that the defect of G' is precisely k.

Suppose that in G we can choose $k+1$ edges such that k of them are bad and one of them is good and they form an independent set of edges. If we replace each edge with B_1' we get a snark with the defect precisely $k+1$.

Note that if we take the snark G_{26} with the embedding described in the Introduction we can perform both operations. Also it is easy to see that after we have performed one operation, the embedding of the superposition is such that allows us to perform both operations again. Thus for any $k \geq 3$ we can generate infinitely many snarks with defect precisely k.

Let M be a nicely embedded superedge such that all semiedges are good. Then we can perform the above operations on M to obtain a nicely embedded superedge M' such that all semiedges of M' are good. Thus starting with the nice embedding of B_1' we can for each $k \geq 1$ construct infinitely many nicely embedded proper superedges with defect precisely k. ☐

Suppose G is a cubic graph and let $c : E(G) \to \{1, 2, 3, 4\}$ be an edge-coloring of G. A coloring c is minimum if the number of edges colored with color 4 is minimum possible among all edge-colorings of G with at most four colors. The number of edges colored with color 4 in a minimum coloring is called the *resistance*, $r(G)$, of G. It is easy to see that a cubic graph is not 3-edge-colorable if and only if the resistance of G is at least 2, cf. [9].

Suppose Π is an orientable embedding of a cubic graph G. A vertex is called a *bad vertex* if in the embedding Π it appears three times along a facial walk. Denote the number of bad vertices in the embedding Π with $d_V(\Pi)$. We define the *modified defect* $d'(\Pi)$ of the embedding Π with

$$d'(\Pi) = d(\Pi) + 2d_V(\Pi).$$

and the modified defect of the graph G with

$$d'(G) = \min\{d'(\Pi) \mid \Pi \text{ an orientable embedding of } G\}.$$

Note that $d(G) = 0$ if and only if $d'(G) = 0$.

Assume that in the embedding Π of G the vertex v appears three times along a facial walk F. This implies that the edges, incident with v, each appear twice along a facial walk F, which contributes 3 to the defect of Π. Suppose that v' is adjacent to v. The facial walk which contains edges incident with v and distinct from vv' intersects F twice at edges incident with v'. For each vertex adjacent to v we get a contribution 1 to the defect of Π. All together we have $d(\Pi) \geq 6d_V(\Pi)$.

Obviously for each graph G we have $d'(G) \geq d(G)$ and the Grünbaum Conjecture is equivalent to the statement that $d'(G) > 0$ for every snark G. Stated with resistance, the Grünbaum Conjecture is equivalent to the statement that for every graph G, $d'(G) > 0$ if $r(G) > 0$. The following theorem gives a stronger relation.

Theorem 10. *If there exists a snark G with $d'(G) < \frac{1}{2}r(G) + 1$, then there exists a snark G' with $d'(G) = 0$.*

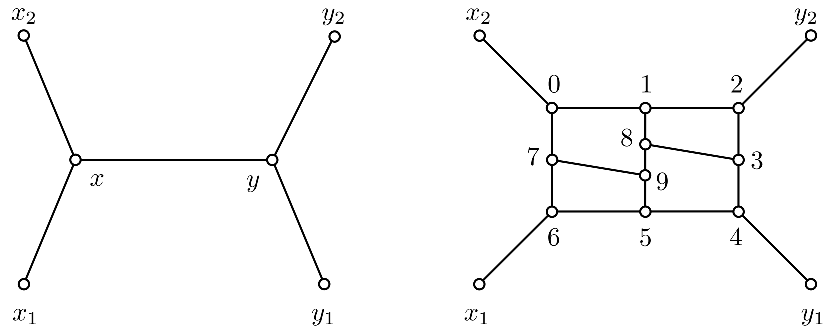

Figure 4: Thickening an edge.

Proof. Suppose there exists a snark G which has an embedding into an orientable surface with $2d'(G) < r(G) + 2$.

We will construct a sequence of graphs $G_1 = G, G_1, G_2, \ldots, G_k$ such that $d'(G_i) > 1$ for $i < k$, $d'(G_k) \leq 1$, $d'(G_i) \leq d'(G_{i-1}) - 1$ for $i = 2, \ldots, k$ and $r(G_i) \geq r(G_{i-1}) - 2$ for $i = 2, \ldots, k$. Since $k < d'(G)$ and $r(G) > 2k - 2$ we see that $r(G_k) > 0$. So G_k is a non 3-edge-colorable cubic graph which has an embedding of defect at most 1 into an orientable surface. By Lemma 8 the Grünbaum Conjecture is false and therefore there exists a snark G' with $d'(G') = 0$.

Suppose we have an embedding $\Pi(G_i)$ of G_i for which $d'(\Pi) = d'(G_i)$. We replace a bad edge $e = xy$ in the embedding of G_i with a graph on 10 vertices (see Figure 4) to get a graph G_{i+1} with an induced embedding of a smaller modified defect (see Figure 4). In the embedding of G_i we can assume we have facial walks W_1, W_2, W_3, W_4 which contain the paths x_1xyy_1, y_1yy_2, y_2yxx_2 and x_2xx_1 respectively, where some of W_1, W_2, W_3, W_4 may be equal. To define an embedding of G_{i+1} we take facial walks of the embedding of G_i, replace the paths x_1xyy_1, y_1yy_2, y_2yxx_2 and x_2xx_1 on the walks W_1, W_2, W_3, W_4 with the paths x_1654y_1, y_1432y_2, y_2210x_2 and x_2076x_1 and add facial cycles 018970, 12381, 34583 and 56785. By appropriately choosing the bad edge e we can guarantee that the modified defect decreases by at least one.

We distinguish four choices for the bad edge e. At each step we can make Choice 3 only if we can not make Choices 1 or 2 and can make Choice 4 if we can not make Choices 1, 2, or 3. As long as the defect of the embedding is more that 0 we can make one of the choices.

Choice 1: bad edge $e = xy$ where x and y are bad vertices. In this case $W_1 = W_2 = W_3 = W_4$.

To calculate the modified defect of the embedding of G_{i+1} observe that bad edges in the embedding of G_{i+1} are bad edges of the embedding of G_i minus e plus bad pairs $\{70, 01\}$, $\{12, 23\}$, $\{34, 45\}$ and $\{56, 67\}$. So $d(\Pi(G_{i+1})) = d(\Pi(G_i)) - 1 + 4 = d(G_i) + 3$. Since we removed two bad vertices x and y and created no new bad vertices we have $d_V(\Pi(G_{i+1})) = d_V(\Pi(G_i)) - 2$ and therefore the modified defect is $d'(\Pi(G_{i+1})) \leq d'(\Pi(G_i)) - 1$. Since $d(\Pi(G_i)) = d'(G)$ we conclude that $d'(G_{i+1}) \leq d'(G_i) - 1$.

Choice 2: bad edge with $e = xy$ where x is a bad vertex and y is not. In this case $W_1 = W_3 = W_4$ and $W_2 \neq W_1$.

The defect of the induced embedding of G_{i+1} is

$$d(\Pi(G_{i+1})) = d(\Pi(G_i)) - 1 + 2 = d(G_i) + 1$$

and

$$d_V(\Pi(G_{i+1})) = d_V(\Pi(G_i)) - 1.$$

Therefore the modified defect is $d'(\Pi(G_{i+1})) = d'(\Pi(G_i)) - 1$. We conclude that $d'(G_{i+1}) \leq d'(G_i) - 1$.

Choice 3: bad edge $e = xy$ which appears twice along one facial walk. Since we can not make choices 1 or 2 we can assume that $W_1 = W_3$ and $W_2 \neq W_1$ and $W_4 \neq W_1$ (but it is possible that $W_2 = W_4$).

In the embeddings of G_i and G_{i+1} there are no bad vertices. The defect of the embedding of G_{i+1} is $d(\Pi(G_{i+1})) = d(\Pi(G_i)) - 1$ and therefore $d'(G_{i+1}) \leq d'(G_i) - 1$.

Choice 4: $e = xy$ which does not appear twice along one facial walk. Since we can not make choices 1, 2, or 3 it is only possible that maybe $W_2 = W_4$.

In the embeddings of G_i and G_{i+1} there are no bad vertices. The defect of the embedding of G_{i+1} is $d(\Pi(G_{i+1})) = d(\Pi(G_i)) - 1$ and therefore $d'(G_{i+1}) \leq d'(G_i) - 1$.

It remains to show that $r(G_{i+1}) \geq r(G_i) - 2$. Suppose we have a minimum coloring c of the graph G_{i+1}. We define a coloring c' of G_i as follows: $c'(f) = c(f)$ if f is not incident with x or y and $c'(x_1x) = c(x_16)$, $c'(yy_2) = c(2y_2)$. We can color the edge f with one of the colors 1, 2, 3 and color edges x_20 and y_14 with color 4. So $r(G_i) \leq r(G_{i+1}) + 2$. □

Theorem 10 implies that if Grünbaum's conjecture is true, we can bound $d'(G)$ from below with $r(G)$, which would be a very strong connection between the defect, which is a topological property, and resistance, which is a coloring property. A similar bound holds for the unmodified defect as shown below.

Corollary 11. *The following statements are equivalent:*
(1) *The Grünbaum Conjecture is true.*
(2) *For all snarks G:* $d'(G) \geq \frac{1}{2}r(G) + 1$.
(3) *For all snarks G:* $d(G) \geq \frac{3}{8}r(G) + \frac{3}{4}$.

Proof. It is obvious that either of (2) and (3) imply (1). For (2), the reverse holds by Theorem 10.

To see that (3) follows from (1), observe that $d(\Pi) \geq 6d_V(\Pi)$. From this fact we get

$$d'(\Pi) = d(\Pi) + 2d_V(\Pi) \leq \frac{4}{3}d(\Pi)$$

and so $d'(G) \leq \frac{4}{3}d(\Pi)$.

So, assuming (1) we obtain (2), and consequently

$$r(G) \leq 2d'(G) - 2 \leq \frac{8}{3}d(G) - 2$$

or equivalently $\frac{3}{8}r(G) + \frac{3}{4} \leq d(G)$. □

A similar result follows for Conjecture 2. If a cubic graph G is embeddable into an orientable surface S we define the *defect of* G *in* S as

$$d_S(G) = \min\{d(\Pi) \mid \Pi \text{ is an embedding of } G \text{ into } S\}.$$

and similarly the *modified defect of G in S* as

$$d'_S(G) = \min\{d'(\Pi) \mid \Pi \text{ is an embedding of } G \text{ into } S\}.$$

Theorem 12. *If there exists a snark G embeddable into a surface S with $d_S(G) < \frac{r(G)}{2}$ then there exists a snark G' embeddable into S with $d_S(G') = 0$.*

Proof. We follow the proof of Theorem 10. Note that when we thicken an edge in G_i, the graph G_{i+1} is embedded into the same surface. □

Corollary 13. *For every orientable surface S the following statements are equivalent:*
(1) *Conjecture 2 is true for S.*
(2) *For all snarks G embeddable into S: $d'_S(G) \geq \frac{r(G)}{2}$.*
(3) *For all snarks G embeddable into S: $d_S(G) \geq \frac{3}{8}r(g)$.*

The proof is similar to the proof of Corollary 11 and is omitted. Let us observe that the bounds in Corollary 11 can be made slightly better since we can use Lemma 8(2) to strengthen the base case, but here we can not since the proof of that lemma involves snarks which may not be embeddable in S.

4 Larger face-width and blocking weight

We now give similar results for Conjecture 3. Let Π be an embedding of a cubic graph G into a surface S and let $k > 0$ a fixed integer. Let G^* be the dual of G in Π. A function $w : E(G) \to \mathbb{Z}^+$ is a *k-blocking weight* if for each cycle C in G^*, which is not contractible in the surface S, we have

$$w^*(C) := \sum_{e \in E(C)} w(e) \geq k.$$

Obviously, an embedding Π has face-width at least k if and only if the trivial function with $w(e) = 1$ for each $e \in E(G)$, is a k-blocking weight.

Let $\phi : \mathbb{Z}^+ \to \mathbb{Z}^+$ be the function defined recursively as follows. We set $\phi(1) = 0$, and for $x \geq 2$ define recursively

$$\phi(x) = 5\phi\left(\left\lceil \frac{x}{2} \right\rceil\right) + 5\phi\left(\left\lfloor \frac{x}{2} \right\rfloor\right) + 1. \tag{1}$$

For a k-blocking weight w of an embedding Π, we define

$$d^k(G, \Pi, w) = \sum_{e \in E(G)} \phi(w(e))$$

and set

$$d^k(G, \Pi) = \min\{d^k(G, \Pi, w) \mid w \text{ a } k\text{-blocking weight for } \Pi\}.$$

Finally, the *k-defect of G in S* is

$$d^k_S(G) = \min\{d^k(G, \Pi) \mid \Pi \text{ an embedding of } G \text{ into } S\}.$$

If G cannot be embedded in S, we set $d^k_S(G) = \infty$.

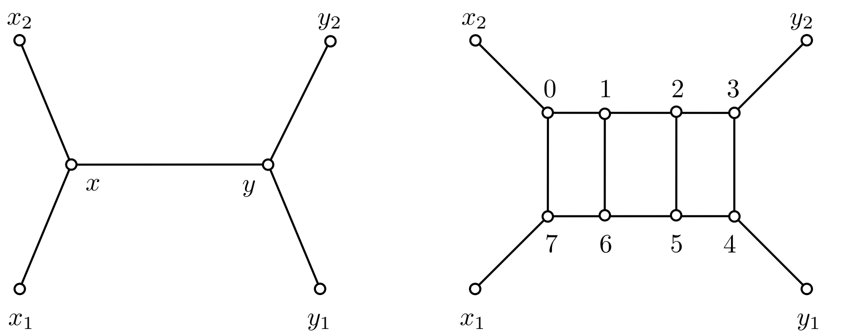

Figure 5: Thickening an edge.

Theorem 14. *Let k be an integer and S a fixed surface. If there exists a snark G embeddable into S with $d^k_S(G) < \frac{r(G)}{2}$, then there exists a snark G' embeddable into S with face-width at least k.*

Proof. Suppose that there exists a snark G embeddable into S with $d^k_S(G) < \frac{r(G)}{2}$. We will construct a sequence of graphs $G_0 = G, G_1, G_2, \ldots, G_n$ such that $d^k_S(G_i) > 0$ for $i < n$, $d^k_S(G_n) = 0$, and $d^k_S(G_{i+1}) \leq d^k_S(G_i) - 1$ and $r(G_{i+1}) \geq r(G_i) - 2$ for $i = 0, \ldots, n-1$. These properties imply that $n \leq d^k_S(G)$. Since $r(G) > 2d^k_S(G) \geq 2n$, the stated recursive property for the resistances of consecutive terms also implies that $r(G_n) > 0$. So G_n is a non-3-edge-colorable graph embedded into S with face-width at least k.

Again G_{i+1} is constructed from G_i by thickening an edge. Suppose G_i is Π_i-embedded into S and w_i is a k-blocking weight of G_i such that $d^k_S(G_i) = d^k(G, \Pi_i, w_i)$. Let e be an edge of G_i whose weight $w(e)$ is maximum. Suppose $w_i(e) = w$. We replace the edge e in G_i as shown in Figure 5 to get a graph G_{i+1} embedded in the same surface S. We define the weight function $w_{i+1} : E(G_{i+1}) \to \mathbb{Z}^+$ as follows. If $f \in E(G_i)$, then $w_{i+1}(f) = w_i(f)$. For the new edges we set: $w_{i+1}(01) = w_{i+1}(12) = w_{i+1}(23) = w_{i+1}(34) = w_{i+1}(07) = \lceil \frac{w}{2} \rceil$ and $w_{i+1}(45) = w_{i+1}(52) = w_{i+1}(56) = w_{i+1}(61) = w_{i+1}(67) = \lfloor \frac{w}{2} \rfloor$.

It is easy to check that w_{i+1} is a k-blocking weight for G_{i+1} and that $r(G_i) \leq r(G_{i+1}) + 2$. Since we have replaced one edge of weight w with five edges of weight $\lceil \frac{w}{2} \rceil$ and five of weight $\lfloor \frac{w}{2} \rfloor$, the definition (1) of the valuation function ϕ implies that

$$d^k_S(G_{i+1}) \leq d^k(G_{i+1}, \Pi_{i+1}, w_{i+1}) = d^k(G_i, \Pi_i, w_i) - 1 = d^k_S(G_i) - 1.$$

This proves our claims and completes the proof. □

Corollary 15. *For a fixed surface S and an integer k, the following statements are equivalent:*
(1) *Conjecture 3 is true for S and k.*
(2) *If a snark G is embeddable into S, then $d^k_S(G) \geq \frac{r(G)}{2}$.*

5 Snarks with large girth

Let G be a superposition of the Petersen graph. A vertex of G that arises from a vertex v of P by replacing it with a trivial supervertex $\mathcal{S}(v)$ will be called an *original vertex*. Each edge

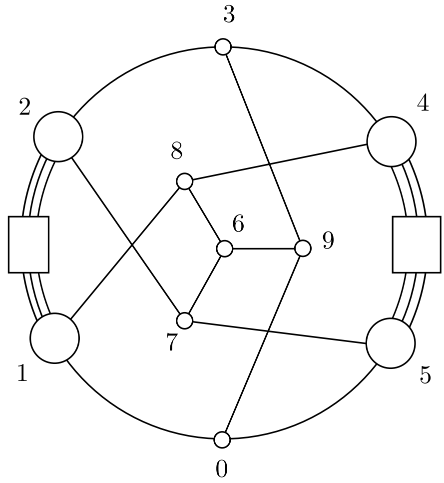

Figure 6: Cyclically 4-edge-connected graphs $\mathcal{G}^4$.

incident with an original vertex will be called an *original edge*. A connected subgraph of G which is induced by nontrivial supervertices and superedges between them is called a *block*.

We will describe cycles in G. If a cycle C contains a path $x_1 \ldots x_k$ this will be denoted by $C = *x_1 \ldots x_k*$. If a cycle enters a block in a supervertex $\mathcal{S}(x_2)$ from an original vertex x_1 and leaves this block from a supervertex $\mathcal{S}(y_1)$ to an original vertex y_2, this will be denoted by $C = *x_1x_2.y_1y_2*$. It is possible that $x_2 = y_1$ in which case we will sometimes write $C = *x_1x_2y_2*$. There are no original vertices on C between x_1 and y_2.

In [4] two families of snarks of large girth were constructed. The first family has cyclic edge-connectivity four. All these graphs can be expresses as a proper superposition of the Petersen graph where we assign trivial supervertices to vertices 0, 3, 6, 7, 8, 9 of P (see also Figure 6). Let $\mathcal{G}^4$ be the set of all cubic graphs that can be obtained in this way.

Theorem 16. *If $G \in \mathcal{G}^4$ then G has no polyhedral embeddings into orientable surfaces.*

Proof. Let $G \in \mathcal{G}^4$ and suppose that it is polyhedrally embedded into an orientable surface. We use the notation from Figure 6.

Look at the facial cycles on the edges 01 and 81. There are at least three distinct facial cycles on these two edges, otherwise the embedding would not be polyhedral.

We now show that there are exactly three. Suppose we have four facial cycles $A = *01.23*$, $B = *01.27*$, $C = *81.27*$ and $D = *81.23*$ Since the embedding is polyhedral, the cycle C must be $C = 81.2768$ (otherwise it would intersect the facial cycle which contains the path 867 twice) and the cycle A must be $A = 01.2390$ (otherwise it would intersect the facial cycle which contains the path 390 twice). Since B already intersects cycles A and C it can not use the edge 43 or 48, therefore it must be $B = 01.2750$ and similarly $D = 81.2348$. There is another facial cycle which contains the vertex 3. It must be $F = 439675.4$ since the embedding is polyhedral. Since the embedding is orientable, we can consistently orient the facial cycles. Suppose that F is oriented so that the edges 43 and 67 are in the direction of the orientation. Then the cycle D is oriented so that the edges 34 and

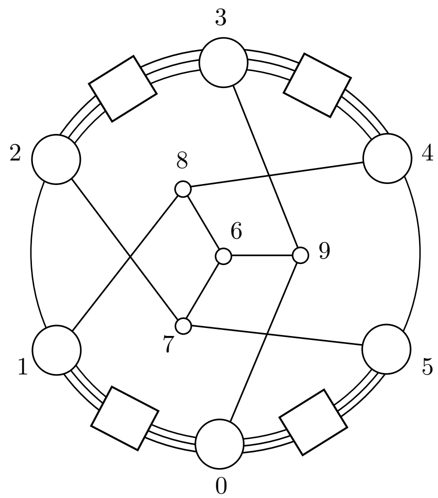

Figure 7: A cyclically 5-edge-connected graph.

81 are in the direction of the orientation. Finally the cycle C is directed so that the edges 18 and 67 are in the direction of the orientation. This is a contradiction since the facial cycles C and F are oriented in the same direction on the edge 67.

By symmetry we have exactly three facial cycles at the edges from the other supervertices. The facial cycles which contain original edges therefore induce an embedding of the underlying Petersen graph. Since the embedding of G is orientable we have a consistent orientation of cycles. We use this orientation in the induced embedding of P. Since facial walks are oriented consistently on original edges of G, this orientation is consistent on all edges of P and so the embedding is orientable.

Suppose that in the induced embedding of the Petersen graph we have two facial cycles A and B which have $k + 1$ edges in common. This implies that at least k of these edges correspond to superedges in G. It follows that the induced embedding of the Petersen graph has defect at most 2, since in G we have two superedges. This contradicts Lemma 5. □

A graph G is in $\mathcal{G}^5$ if it is a proper superposition of the Petersen graph where we assign trivial supervertices to the vertices 6, 7, 8, 9 and additionally trivial superedges to the edges 54 and 12 (see also Figure 7).

If a cycle C enters a block on a supervertex x_2 from an original vertex x_1, then it uses some vertices from a supervertex x_3 and then leaves the block from a supervertex x_3 to an original supervertex x_4, this will be denoted by $C = *x_1.x_2.x_3.x_4*$.

Theorem 17. *If $G \in \mathcal{G}^5$ then G has no polyhedral embeddings into orientable surfaces.*

Proof. Let $G \in \mathcal{G}^5$ and suppose it has a polyhedral embedding into an orientable surface. Similarly as in the proof of the previous theorem we first show that this embedding induces an embedding of the underlying Petersen graph. We use the notation of Figure 7. Call the supervertices $0, 1, 2$ with superedges between them the *lower block* and the supervertices $3, 4, 5$ with superedges between then the *upper block*.

Assume that on the edges 75 and 45 we have four distinct facial cycles, $A = *75.0*$, $B = *75.0*$, $C = *45.0*$ and $D = *45.0*$. Since the embedding is polyhedral, there must be two distinct facial cycles which enter the lower block on the edge 90. This implies that not all four of A, B, C, D can leave the lower block on the edges 12 and 18.

CASE 1: Assume that only a facial cycle, which contains the edge 75, say A, leaves the lower block on the edge 09. Since the embedding is polyhedral, we have $A = 75.0967$ and $B = *275.0.1*$ and we can assume $C = *45.0.12*$ and $D = *45.0.1.8*$. The cycle B can not leave the lower block on the edge 28 since then there would be a facial cycle at vertex 6 which would intersect it twice. So we have $B = 275.0.12$. The cycle C can not leave the upper block on the edge 48 since it already intersects the cycle D and also not on the edge 39 since it would have to continue on the path 3968. Similarly it can not leave on the edge 27, so it must be $C = 45.0.12.3.4$. We have another cycle F which enters the lower block on the edge 81, $F = *81.093*$. This cycle will intersect with the cycle which contains the path 869 twice, a contradiction to the assumption that the embedding is polyhedral.

CASE 2: Assume that only a facial cycle, which contains the edge 45, say C, leaves the lower block on the edge 09. So $C = *45.09*$, $D = *45.0.1*$, $A = *75.0.18*$ and $B = *75.0.12*$. Since the embedding is polyhedral we have $A = 75.0.1867$ and $B = 75.0.127$. If we had $D = *45.0.12*$ then there would be another facial cycle $F = *90.184*$ which would intersect the facial cycle which contains the path 869 twice, a contradiction. So we have $D = 45.0.184$ and $C = *45.093*$. There is a facial cycle $F = *21.096*$. If we had $F = *21.0967*$, then F and A would intersect twice, and if we had $F = *21.0968*$ then the cycles A, D and F could not be consistently oriented.

CASE 3: Assume there that two cycles, say A and C, which leave the lower block on the edge 09. Again we have $A = 75.0967$, $B = *275.0.1*$, $C = *45.0.93*$ and $D = *45.0.1*$. If the cycle B would leave the lower block on the edge 18, then it would be $B = *275.0.184*$ and it would intersect the cycle which contains the path 867 twice. So we have $B = *275.0.184*$ and $C = 45.0.184$. Now we have a facial cycle $F = *218693*$ and we get a contradiction since the cycles C, D and F can not be consistently oriented.

So we have that there are exactly three facial cycles on the edges 45 and 75. By symmetry the same holds for the edges at supervertices 1, 2 and 4. Since the embedding of G is polyhedral and orientable we get that the facial cycles which contain the original edges of G induce an orientable embedding of P, which has defect at most 4. This again contradicts Lemma 5. □

References

[1] D. Blanuša, Problem četiriju boja, *Glasnik Mat.-Fiz. Astr.* **2** (1946), 31–42.

[2] M. DeVos, L. Goddyn, B. Mohar, D. Vertigan and X. Zhu, Coloring-flow duality of embedded graphs, *Trans. Amer. Math. Soc.* **357** (2005), 3993–4016.

[3] B. Grünbaum, Conjecture 6, in W. T. Tutte (ed.) *Recent Progress in Combinatorics*, Academic Press, New York, 1969, p. 343.

[4] M. Kochol, Snarks without small cycles, *J. Comb. Theory Ser. B* **67** (1996), 34–47.

[5] B. Mohar and C. Thomassen, Graphs on Surfaces, The Johns Hopkins University Press, Baltimore and London, 2001.

[6] B. Mohar and A. Vodopivec, On polyhedral embeddings of cubic graphs, *Combin. Probab. Comput.* **15** (2006), 877–893.

[7] B. Mohar, A. Vodopivec, The genus of Petersen powers, preprint.

[8] A. Orbanić, T. Pisanski, M. Randić, B. Servatius, Blanuša double, *Math. Commun.* **9** (2004), 91–103.

[9] E. Steffen, Measurements of edge-uncolorability, *Discrete Math.* **280** (2004), 191–214.

[10] G. Szekeres, Polyhedral decompositions of cubic graphs, *Bull. Austral. Math. Soc.* **8** (1973), 367–387.

[11] F. C. Tinsley, J. J. Watkins, A study of snark embeddings, in *Graphs and Applications* (Boulder, Colo., 1982), Wiley-Intersci. Publ., Wiley, New York, 1985, 317–332.