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On the largest subsets avoiding the diameter of $(0, \pm 1)$-vectors

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Abstract

Let $L_{mkl} \subset \mathbb{R}^{m+k+l}$ be the set of vectors which have m of entries -1, k of entries 0, and l of entries 1. In this paper, we investigate the largest subset of L_{mkl} whose diameter is smaller than that of L_{mkl}. The largest subsets for $m = 1$, $l = 2$, and any k will be classified. From this result, we can classify the largest 4-distance sets containing the Euclidean representation of the Johnson scheme $J(9, 4)$. This was an open problem in Bannai, Sato, and Shigezumi (2012).

Keywords: The Erdős–Ko–Rado theorem, s-distance set, diameter graph, independent set, extremal set theory.

Math. Subj. Class.: 05D05, 05C69

1 Introduction

The famous theorem in Erdős–Ko–Rado [8] stated that for $n \geq 2k$ and a family $\mathfrak{A}$ of k-element subsets of $I_n = \{1, \ldots, n\}$, if any two distinct $A, B \in \mathfrak{A}$ satisfy $A \cap B \neq \emptyset$, then

$$|\mathfrak{A}| \leq \binom{n-1}{k-1}.$$

For $n > 2k$, the set $\{A \subset I_n \mid |A| = k, 1 \in A\}$ is the unique family achieving equality, up to permutations on I_n. For $n = 2k$, the largest set is any family which contains only one of A or $I_n \setminus A$ for any k-element $A \subset I_n$. This result plays a central role in extremal set theory, and similar or analogous theorems are proved for various objects [2, 5, 9].

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We can naturally interpret $A \subset I_n$ as $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ by the manner $x_i = 1$ if $i \in A$, $x_i = 0$ if $i \notin A$. By this identification, the Erdős–Ko–Rado Theorem can be rewritten that for $n \geq 2k$ and a subset X of $L_k = \{x \in \mathbb{R}^n \mid x_i \in \{0, 1\}, \sum x_i = k\}$ if any distinct $x, y \in X$ satisfy $d(x, y) < D(L_k) = \sqrt{2k}$, then

$$|X| \leq \binom{n-1}{k-1},$$

where $d(,)$ is the Euclidean distance, and $D(L_k)$ is the diameter of L_k. We would like to consider the following problem to generalize the Erdős–Ko–Rado Theorem.

Problem 1.1. Let $L_{mkl} \subset \mathbb{R}^{m+k+l}$ be the set of vectors which have m of entries -1, k of entries 0, and l of entries 1. Classify the largest $X \subset L_{mkl}$ with $D(X) < D(L_{mkl})$.

It is almost obvious for the cases $m = l$ (Proposition 2.1) and $m + k \leq l$ (Proposition 2.2). In this paper, we solve the first non-trivial case $m = 1$, $l = 2$ and any k (Theorem 2.5). Using the largest sets for the case $(m, k, l) = (1, 6, 2)$, we can classify the largest 4-distance sets containing the Euclidean representation of the Johnson scheme $J(9, 4)$. This was an open problem in [1].

We will give a brief survey on related results. Let $\mathfrak{L}_{nm}$ be the set of $(0, \pm 1)$-vectors in $\mathbb{R}^n$ which have m non-zero coordinates. For a fixed set D of integers, let $V(n, m, D)$ be the family of subsets $V = \{v_1, \dots, v_k\}$ of $\mathfrak{L}_{nm}$ such that $(v_i, v_j) \in D$ for any $i \neq j$. There are several results relating to the largest sets in $V(n, m, D)$ for some (n, m, D) [4, 6, 7]. Since $X \subset \mathfrak{L}_{nm}$ is on a sphere, if $|D| = s$ holds, then $|X| \leq \binom{n+s-1}{s} + \binom{d+s-2}{s-1}$ [3]. The case $D = \{d\}$ is investigated in [4]. For non-negative integers $d < m$, $t \geq 2$, and $n > n_0(m)$ (see [4] about $n_0(m)$), if $X \in V(n, m, \{d, d+1, \dots, d+t-1\})$, then $|X| \leq \binom{n-d}{t} / \binom{m-d}{t}$ [6]. This equality can be attained whenever a Steiner system $S(n-d, m-d, t)$ (equivalently t-$(n-d, m-d, 1)$ design) exists . We also have if $X \in V(n, m, \{-(t-1), -(t-2), \dots, t-1\})$, then $|X| \leq 2^{t-1}(m-t+1)\binom{n}{t} / \binom{m}{t}$ [7]. When $m = t + 1$, this equality can be attained whenever a Steiner system $S(n, m, m-1)$ exists.

2 Largest subsets avoiding the diameter of L_{mkl}

Let L_{mkl} denote the finite set in $\mathbb{R}^n = \mathbb{R}^{m+k+l}$, which consists of all vectors whose number of entries -1, 0, 1 is equal to m, k, l, respectively. For two subsets X, Y of L_{mkl}, X is *isomorphic* to Y if there exists a permutation $\sigma \in S_n$ such that $X = \{(y_{\sigma(1)}, \dots, y_{\sigma(n)}) \mid (y_1, \dots, y_n) \in Y\}$. The *diameter* $D(X)$ of $X \subset \mathbb{R}^n$ is defined to be

$$D(X) = \max\{d(x, y) \mid x, y \in X\},$$

where $d(,)$ is the Euclidean distance. Let M_{mkl} denote the largest possible number of cardinalities of $X \subset L_{mkl}$ such that $D(X) < D(L_{mkl})$. The *diameter graph* of $X \subset \mathbb{R}^n$ is defined to be the graph (X, E), where $E = \{(x, y) \mid d(x, y) = D(X)\}$. The problem of determining M_{mkl} is equivalent to determining the independence number of the diameter graph of L_{mkl}. Note that $M_{mkl} = M_{lkm}$ because we have $L_{mkl} = -L_{lkm} = \{-x \mid x \in L_{lkm}\}$. Thus we may assume $m \leq l$. In this section, we determine M_{mkl}, and classify the largest sets for several cases of m, k, l.

First we determine M_{mkl} for the cases $m = l$ and $m + k \leq l$.

Proposition 2.1. *Assume* $m = l$*. Then we have*

$$M_{mkl} = \frac{1}{2}\binom{n}{m}\binom{k+m}{m} = \frac{1}{2}|L_{mkl}|,$$

and the largest sets contain only one of x *or* $-x$ *for any* $x \in L_{mkl}$*.*

Proof. For any $x \in L_{mkl}$, we have $\{y \mid d(x, y) = D(L_{mkl})\} = \{-x\}$. Therefore the diameter graph of L_{mkl} is the set of independent edges. The proposition can be easily proved from this fact. □

For $X \subset L_{mkl}$, we use the notation

$$N_i(X, j) = \{(x_1, \ldots, x_n) \in X \mid x_i = j\}, \qquad \text{and} \qquad n_i(X, j) = |N_i(X, j)|.$$

Proposition 2.2. *Assume* $m + k \leq l$*. Then we have*

$$M_{mkl} = \binom{n-1}{m+k-1}\binom{m+k}{m}.$$

For $m + k > l$*, the largest set is* $N_1(L_{mkl}, -1) \cup N_1(L_{mkl}, 0)$*, up to isomorphism. For* $m + k = l$*, then the largest sets contain only one of* $\{(x_1, \ldots, x_n) \in L_{mkl} \mid x_i = 1, \forall i \in J\}$ *or* $\{(x_1, \ldots, x_n) \in L_{mkl} \mid x_i = 1, \forall i \in I_n \setminus J\}$ *for any* $J \subset I_n$ *of order* l*.*

Proof. A finite subset X of L_{mkl} satisfies $D(X) < D(L_{mkl})$ if and only if $\{i \mid x_i = -1, 0\} \cup \{i \mid y_i = -1, 0\}$ is not empty for any distinct $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in X$. We can therefore apply the Erdős–Ko–Rado Theorem [8] to determine the positions of entries -1 or 0. The number of possible positions of -1, 0 is $\binom{n-1}{m+k-1}$. After fixing the position, -1, 0 can be placed in $\binom{m+k}{k}$ ways. This determines M_{mkl}. The largest sets are classified from the optimal sets of the Erdős–Ko–Rado Theorem. □

The remaining part of this section is devoted to proving

$$M_{1k2} = \mathfrak{M}_k = \binom{k+3}{3} + 2,$$

and determining the classification of the largest sets. Note that $D(L_{1k2}) = \sqrt{10}$ and if $X \subset L_{1k2}$ satisfies $D(X) < D(L_{1k2})$, then $D(X) \leq \sqrt{8}$. The following two lemmas are used later.

Lemma 2.3. *Let* $X \subset L_{1k2}$ *with* $D(X) < D(L_{1k2})$*. Suppose* $k \geq 4$*, and* $|X| \geq \mathfrak{M}_k$*. Then there exists* $i \in \{1, \ldots, n\}$ *such that* $n_i(X, 0) \geq \mathfrak{M}_{k-1}$*.*

Proof. This lemma is immediate because the average of $n_i(X, 0)$ is

$$\frac{1}{n}\sum_{i=1}^{n} n_i(X, 0) = \frac{k|X|}{k+3} \geq \frac{k\mathfrak{M}_k}{k+3} = \mathfrak{M}_{k-1} - \frac{6}{k+3} > \mathfrak{M}_{k-1} - 1. \qquad \square$$

Lemma 2.4. *Let* $G = (V, E)$ *be a connected simple graph, and* E' *a matching in* G*. Assume that* G *has an independent set* I *of size* $|V| - |E'|$*. Then for* $z \in I$ *if* $x \in V$ *satisfies* $(x, y) \in E'$ *for some* y *adjacent to* z*, then* $x \in I$*.*

Proof. Since the cardinality of I is $|V| - |E'|$, only one of x or y is an element of I for any $(x, y) \in E'$. By assumption, $y \notin I$, and hence $x \in I$. □

The subsets $S_k(i)$, $T_k(i)$, $U_k(i)$ of L_{1k2} are defined by

$$\begin{aligned}
S_k(i) &= \{(x_1, \ldots, x_n) \in L_{1k2} \mid x_1 = \cdots = x_{i-1} = 0, x_i = -1\},\\
T_k(i) &= \{(x_1, \ldots, x_n) \in L_{1k2} \mid x_1 = \cdots = x_{i-1} = 0, x_i = 1\},\\
U_k(i) &= \left\{(x_1, \ldots, x_n) \in L_{1k2} \mid \begin{array}{l} x_1 = 1, x_l = -1, x_j = 1, \\ \exists l \in \{2, \ldots, i\}, \exists j \in \{l+1, \ldots, n\} \end{array}\right\}
\end{aligned}$$

for $i = 2, \ldots, k+2$. We define $S_k(1) = N_1(L_{1k2}, -1)$, and $T_k(1) = N_1(L_{1k2}, 1)$. The following are candidates of the largest subsets avoiding the largest distance $\sqrt{10}$.

$$X_k = T_k(k+1) \cup (\bigcup_{i=1}^{k+1} S_k(i)) \text{ for } k \geq 1,$$

$$Y_1 = T_1(1), \qquad Y_k = T_k(k) \cup (\bigcup_{i=1}^{k-1} S_k(i)) \text{ for } k \geq 2,$$

$$Z_2 = T_2(1), \qquad Z_k = T_k(k-1) \cup (\bigcup_{i=1}^{k-2} S_k(i)) \text{ for } k \geq 3.$$

Note that $|X_k| = |Y_k| = |Z_k| = \mathfrak{M}_k$, and they can be inductively constructed by

$$\begin{aligned}
X_k &= \{(0, x) \mid x \in X_{k-1}\} \cup N_1(L_{1k2}, -1),\\
Y_k &= \{(0, x) \mid x \in Y_{k-1}\} \cup N_1(L_{1k2}, -1),\\
Z_k &= \{(0, x) \mid x \in Z_{k-1}\} \cup N_1(L_{1k2}, -1).
\end{aligned}$$

We also use the following notation.

$$\begin{aligned}
X'_k &= X_k \setminus S_k(1) = \{(0, x) \mid x \in X_{k-1}\} \quad (k \geq 2),\\
Y'_k &= Y_k \setminus S_k(1) = \{(0, x) \mid x \in Y_{k-1}\} \quad (k \geq 2),\\
Z'_k &= Z_k \setminus S_k(1) = \{(0, x) \mid x \in Z_{k-1}\} \quad (k \geq 3).
\end{aligned}$$

Theorem 2.5. *Let $X \subset L_{1k2}$ with $D(X) < D(L_{1k2})$. Then we have*

$$|X| \leq \mathfrak{M}_k.$$

If equality holds, then

(1) *for $k = 1$, $X = X_1$, or Y_1,*

(2) *for $k \geq 2$, $X = X_k$, Y_k, or Z_k,*

up to isomorphism.

This theorem will be proved by induction. We first prove the inductive step.

Lemma 2.6. *Let $k \geq 2$. Assume that the statement in Theorem 2.5 holds for some $k-1$. Let $X \subset L_{1k2}$ with $D(X) < D(L_{1k2})$, such that $n_i(X, 0) = \mathfrak{M}_{k-1}$ for some i. Then we have $|X| \leq \mathfrak{M}_k$. If equality holds, then $X = X_k$, Y_k, or Z_k, up to isomorphism.*

Proof. Without loss of generality, $n_1(X, 0) = \mathfrak{M}_{k-1}$, and hence X contains X'_k, Y'_k, or Z'_k for $k \geq 3$, and X'_1, or Y'_1 for $k = 2$.

(i) Suppose $X'_k \subset X$ for $k \geq 2$. The set of other candidates of elements of X is $S_k(1) \cup U_k(k)$. The diameter graph G of $S_k(1) \cup U_k(k)$ is a bipartite graph of the partite sets $S_k(1)$ and $U_k(k)$. Since the three elements

$$(-1, 0, \ldots, 0, 0, 1, 1), (-1, 0, \ldots, 0, 1, 0, 1), (-1, 0, \ldots, 0, 1, 1, 0) \in S_k(1)$$

are isolated vertices in G, they may be contained in X. Let G' be the subgraph of G formed by removing the three isolated vertices. A perfect matching of G' is given as follows.

Matching (i)

$S_k(1)$ $(-1, x_2, \ldots, x_{k+3})$	$U_k(k)$ $(1, y_2, \ldots, y_{k+3})$
$x_i = 1, x_j = 1$ $(2 \leq i \leq k, i < j < n)$	$y_i = -1, y_{j+1} = 1$
$x_i = 1, x_n = 1$ $(2 \leq i \leq k)$	$y_i = -1, y_{i+1} = 1$

By this matching, we can show

$$|X| \leq \mathfrak{M}_{k-1} + |S_k(1)| = \mathfrak{M}_k.$$

We will classify the sets attaining this bound. First assume that $x \in X$ for some $x \in S_k(1)$ with $x_2 = 1$. By Lemma 2.4, X must contain any $x \in S_k(1)$ with $x_2 = 1$. In particular, $(-1, 1, 1, 0, \ldots, 0) \in X$. Using Lemma 2.4 again, X must contain $x \in S_k(1)$ with $x_3 = 1$. By a similar manner, X must contain any $x \in S_k(1)$. Therefore $X = X_k$.

Assume X does not contain any $x \in S_k(1)$ with $x_2 = 1$, namely $n_2(X, 1) = 0$. By assumption, we have

$$|X| = n_2(X, -1) + n_2(X, 0) \leq \binom{k+2}{2} + \mathfrak{M}_{k-1} = \mathfrak{M}_k.$$

If $|X| = \mathfrak{M}_k$, then we have $n_2(X, -1) = \binom{k+2}{2}$ and $n_2(X, 0) = \mathfrak{M}_{k-1}$. This implies that X is isomorphic to X_k, Y_k, or Z_k.

(ii) Suppose $Y'_k \subset X$ for $k \geq 2$. The set of other candidates of elements of X is the union of $S_k(1)$, $U_k(k-1)$, and

$$\mathcal{S}_1 = \{(x_1, \ldots, x_{k+3}) \in L_{1k2} \mid x_1 = 1, x_k = 1, x_j = -1, k < j\}$$

for $k \geq 3$, and $S_2(1) \cup \mathcal{S}_1$ for $k = 2$. The diameter graph G of $S_k(1) \cup U_k(k-1) \cup \mathcal{S}_1$ is a bipartite graph of the partite sets $S_k(1)$ and $U_k(k-1) \cup \mathcal{S}_1$. Since the three elements

$$(-1, 0, \ldots, 0, 1, 1, 0, 0), (-1, 0, \ldots, 0, 1, 0, 1, 0), (-1, 0, \ldots, 0, 1, 0, 0, 1) \in S_k(1)$$

are isolated vertices in G, they may be contained in X. Let G' be the subgraph of G formed by removing the three isolated vertices. A perfect matching of G' is given as follows.

Matching (ii)

$S_k(1)$ $(-1, x_2, \ldots, x_{k+3})$	$U_k(k-1)$ $(1, y_2, \ldots, y_{k+3})$
$x_i = 1, x_j = 1$ $(2 \leq i \leq k-1, i < j < n)$	$y_i = -1, y_{j+1} = 1$
$x_i = 1, x_n = 1$ $(2 \leq i \leq k-1)$	$y_i = -1, y_{i+1} = 1$

$S_k(1)$	$\mathcal{S}_1$
$(-1,0,\dots,0,1,1,0)$	$(1,0,\dots,0,1,-1,0,0)$
$(-1,0,\dots,0,0,1,1)$	$(1,0,\dots,0,1,0,-1,0)$
$(-1,0,\dots,0,1,0,1)$	$(1,0,\dots,0,1,0,0,-1)$

By this maching, we can show $|X| \le \mathfrak{M}_k$.

We will classify the sets attaining this bound. For $k = 2$, the maximum indepdent sets of G' is $\{(-1,0,0,1,1),(-1,0,1,0,1),(-1,0,1,1,0)\} \subset S_2(1)$ or $\mathcal{S}_1$. This implies that $X = Y_2$ or Z_2. For $k \ge 3$, we assume that $x \in X$ for some $x \in S_k(1)$ with $x_2 = 1$. By Lemma 2.4, X must contain any $x \in S_k(1)$. Therefore $X = Y_k$. If X does not contain any $x \in S_k(1)$ with $x_2 = 1$, namely $n_2(X,1) = 0$. It can be proved that X is isomorphic to X_k, Y_k, or Z_k.

(iii) Suppose $k \ge 3$, and $Z'_k \subset X$. The set of other candidates of elements of X is the union of $S_k(1)$, $U_k(k-2)$, and

$$\mathcal{S}_2 = \{(x_1,\dots,x_{k+3}) \in L_{1k2} \mid x_1 = 1, x_{k-1} = 1, x_j = -1, k < j\}$$

for $k \ge 4$, and $S_3(1) \cup \mathcal{S}_2$ for $k = 3$. The diameter graph G of $S_k(1) \cup U_k(k-2) \cup \mathcal{S}_2$ is a bipartite graph of the partite sets $S_k(1)$ and $U_k(k-2) \cup \mathcal{S}_2$. Since the four vectors

$$(-1,0,\dots,0,1,1,0,0,0),(-1,0,\dots,0,1,0,1,0,0),$$
$$(-1,0,\dots,0,1,0,0,1,0),(-1,0,\dots,0,1,0,0,0,1) \in S_k(1)$$

are isolated vertices in G, they may be contained in X. Let G' be the subgraph of G formed by removing the four isolated vertices. A maximum matching of G' is given as follows.

Matching (iii)

$S_k(1)$ $(-1,x_2,\dots,x_{k+3})$	$U_k(k-2)$ $(1,y_2,\dots,y_{k+3})$
$x_i = 1, x_j = 1$ $(2 \le i \le k-2, i < j < n)$	$y_i = -1, y_{j+1} = 1$
$x_i = 1, x_n = 1$ $(2 \le i \le k-2)$	$y_i = -1, y_{i+1} = 1$

$S_k(1)$	$\mathcal{S}_2$
$(-1,0,\dots,0,1,1,0,0)$	$(1,0,\dots,0,1,-1,0,0,0)$
$(-1,0,\dots,0,0,1,1,0)$	$(1,0,\dots,0,1,0,-1,0,0)$
$(-1,0,\dots,0,0,0,1,1)$	$(1,0,\dots,0,1,0,0,-1,0)$
$(-1,0,\dots,0,1,0,0,1)$	$(1,0,\dots,0,1,0,0,0,-1)$

Note that the two vectors

$$(-1,0,\dots,0,1,0,1,0),(-1,0,\dots,0,0,1,0,1) \in S_k(1) \tag{2.1}$$

are unmatched in this matching. By this matching, we can show $|X| \le \mathfrak{M}_k$.

We will classify the sets attaining this bound. If $|X| = \mathfrak{M}_k$, then the two vectors in (2.1) must be contained in X. Therefore X does not contain any element of $\mathcal{S}_2$, and contains an element of $S_k(1)$ which matches some element of $\mathcal{S}_2$. For $k = 3$, X therefore contains $S_k(1)$, and $X = Z_3$. For $k \ge 4$, we assume that $x \in X$ for some $x \in S_k(1)$ with $x_2 = 1$. By Lemma 2.4, X must contain any $x \in S_k(1)$. Therefore $X = Z_k$. If X does not contain any $x \in S_k(1)$ with $x_2 = 1$, namely $n_2(X,1) = 0$. Therefore X is isomorphic to X_k, Y_k, or Z_k. □

Matchings (i)–(iii) and the notation $\mathcal{S}_1, \mathcal{S}_2$ defined in the proof of Lemma 2.6 are used again later. The base case in the induction is the case $k = 3$. We will prove the cases $k = 1, 2, 3$ in order.

Proposition 2.7. *Let $X \subset L_{112}$ with $D(X) < D(L_{112})$. Then we have*

$$|X| \leq \mathfrak{M}_1 = 6.$$

If equality holds, then $X = X_1$, or Y_1, up to isomorphism.

Proof. Since the diameter graph G of L_{112} is isomorphic to $C_4 \cup C_4 \cup C_4$, where C_4 is the 4-cycle, the bound $|X| \leq 6$ clearly holds. Considering the permutation of coordinates, G has the automorphism group S_4. Since the stabilizer of X_1 in S_4 is of order 6, the orbit of X_4 has length 4. Similarly the orbit of Y_1 has length 4. Since the number of maximum independent sets of G is $2^3 = 8$, this proposition follows. □

For $k = 2$, we also classify $(\mathfrak{M}_2 - 1)$-point sets X with $D(X) < D(L_{122})$ in order to prove the case $k = 3$.

Proposition 2.8. *Let $X \subset L_{122}$ with $D(X) < D(L_{122})$. Then we have*

$$|X| \leq \mathfrak{M}_2 = 12.$$

If $|X| = 12$, then $X = X_2$, Y_2, or Z_2, up to isomorphism. If $|X| = 11$, then X is

$$V_2 = X_2' \cup \{(-1,0,0,1,1), (-1,0,1,0,1), (-1,0,1,1,0), (-1,1,1,0,0), (1,-1,1,0,0)\},$$

$$W_2 = Y_2' \cup \{(-1,1,1,0,0), (-1,1,0,1,0), (-1,1,0,0,1), (-1,0,0,1,1), (1,1,-1,0,0)\},$$

or the set obtained by removing a point from X_2, Y_2, or Z_2, up to isomorphism.

Proof. First suppose $n_i(X,0) = 6$ for some i. Then we have $|X| \leq 12$, and X with $|X| = 12$ is X_2, Y_2, or Z_2 by Lemma 2.6. In order to find X with $|X| = 11$, we consider 5-point independent sets in the diameter graph of $S_2(1) \cup U_2(2)$ or $S_2(1) \cup U_2(1) \cup \mathcal{S}_1$. If X is not isomorphic to a subset of X_2, Y_2, or Z_2, then $X = V_2$ from $S_2(1) \cup U_2(2)$, and $X = W_2$ from $S_2(1) \cup U_2(1) \cup \mathcal{S}_1$.

Suppose $n_i(X,0) \leq 5$ for any i. If $|X| \geq 11$, then the average of $n_i(X,0)$ is greater than 4. Without loss of generality, we may assume $n_1(X,0) = 5$. Since the diameter graph of L_{112} is $C_4 \cup C_4 \cup C_4$, we can show that X contains a 5-point subset of X_2' or Y_2'.

(i) Suppose X contains a 5-point subset of X_2'. By considering the automorphism group of X_2', we may assume X contains the 5-point subset obtained by removing $(0,-1,0,1,1)$ or $(0,0,-1,1,1)$. First assume that X contains the 5-point subset obtained by removing $(0,-1,0,1,1)$. Since other candidates of elements of X are still in $S_2(1) \cup U_2(2)$, we have $|X| \leq 11$, and if $|X| = 11$, then X is isomorphic to a subset of X_2, Y_2, or Z_2. Assume that X contains the 5-point subset obtained by removing $(0,0,-1,1,1)$. The set of other candidates of elements of X is $S_2(1) \cup U_2(2) \cup \{(1,0,1,-1,0), (1,0,1,0,-1)\}$. If X does not contain both $(1,0,1,-1,0)$ and $(1,0,1,0,-1)$, then $|X| \leq 11$, and X

attaining this bound is isomorphic to a subset of X_2, Y_2, or Z_2. To make a new set, X may contain $(1,0,1,-1,0)$. The two vectors $(-1,1,0,1,0), (-1,0,0,1,1) \in S_2(1)$, which are at distance $\sqrt{10}$ from $(1,0,1,-1,0)$, are not contained in X. The set P_1 consisting of the two isolated vertices

$$(-1,0,1,0,1), (-1,0,1,1,0) \in S_2(1)$$

and 6 points

$$(-1,1,1,0,0), (-1,1,0,0,1), (1,-1,1,0,0),$$
$$(1,-1,0,1,0), (1,-1,0,0,1), (1,0,1,0,-1)$$

has the unique maximum 6-point independent set

$$\left\{ \begin{array}{l} (-1,0,1,0,1), (-1,0,1,1,0), (1,-1,1,0,0), \\ (1,-1,0,1,0), (1,-1,0,0,1), (1,0,1,-1,0) \end{array} \right\},$$

which gives X isomorphic to Y_2, and $n_2(X,0) = 6$. If X contains a 5-point independent set in P_1 and is not isomorphic to a subset of Y_2, then X contains the 5-point independent set

$$\{(-1,0,1,0,1), (-1,0,1,1,0), (-1,1,1,0,0), (1,-1,1,0,0), (1,0,1,0,-1)\}.$$

Then X is isomorphic to W_2 and $n_2(X,0) = 6$.

(ii) Suppose X contains a 5-point subset of Y_2'. By considering the automorphism group of Y_2', we may assume X contains the 5-point subset obtained by removing $(0,1,-1,0,1)$. The set of other candidates of elements of X is $S_2(1) \cup \mathcal{S}_1 \cup \{(1,0,1,0,-1)\}$. To make a new set, X may contain $(1,0,1,0,-1)$. The two vectors $(-1,1,0,0,1), (-1,0,0,1,1) \in S_2(1)$, which are at distance $\sqrt{10}$ from $(1,0,1,0,-1)$, are not contained in X. The set consisting of the two isolated vertices

$$(-1,1,1,0,0), (-1,1,0,1,0) \in S_2(1)$$

and 5 points

$$(-1,0,1,1,0), (-1,0,1,0,1), (1,1,-1,0,0), (1,1,0,-1,0), (1,1,0,0,-1)$$

has the unique maximum 5-point independent set

$$\{(-1,1,1,0,0), (-1,1,0,1,0), (1,1,-1,0,0), (1,1,0,-1,0), (1,1,0,0,-1)\},$$

which gives X is isomorphic to a subset of Z_2. □

Proposition 2.9. *Let $X \subset L_{132}$ with $D(X) < D(L_{132})$. Then we have*

$$|X| \leq \mathfrak{M}_3 = 22.$$

If equality holds, then $X = X_3$, Y_3, or Z_3, up to isomorphism.

Proof. If $n_i(X,0) = 12$ for some i, then we have $|X| \leq 22$, and the set attaining this bound is X_3, Y_3, or Z_3 by Lemma 2.6.

Suppose $n_i(X, 0) \leq 11$ for any i. If $|X| > 22$, then the average of $n_i(X, 0)$ is greater than 11, which gives a contradiction. Therefore $|X| \leq 22$, and if $|X| = 22$, then the average of $n_i(X, 0)$ is 11, and $n_i(X, 0) = 11$ for any i. By Proposition 2.8, X may contain

$$V'_3 = \{(0, v) \in L_{132} \mid v \in V_2\},$$

$$W'_3 = \{(0, w) \in L_{132} \mid w \in W_2\},$$

or an 11-point set obtained by removing a point from X'_3, Y'_3, or Z'_3.

(i) Suppose X contains an 11-point subset of X'_3. By considering the automorphism group of X'_3, X may contain the set in X'_3 obtained by removing $(0, -1, 0, 0, 1, 1)$, $(0, -1, 1, 1, 0, 0)$, $(0, 0, -1, 0, 1, 1)$, or $(0, 0, 0, -1, 1, 1)$. If X contains the set X'_3 with $(0, -1, 0, 0, 1, 1)$, $(0, -1, 1, 1, 0, 0)$, or $(0, 0, -1, 0, 1, 1)$ removed, then the set of other candidates of X is still $S_3(1) \cup U_3(3)$, and $|X| < 22$. Suppose X contains the set X'_3 with $(0, 0, 0, -1, 1, 1)$ removed. Then new candidates of vectors of X are only $(1, 0, 0, 1, -1, 0)$ and $(1, 0, 0, 1, 0, -1)$, and X may contain $(1, 0, 0, 1, -1, 0)$. The three vectors $(-1, 1, 0, 0, 1, 0)$, $(-1, 0, 1, 0, 1, 0)$, and $(-1, 0, 0, 0, 1, 1)$, which are at distance $\sqrt{10}$ from $(1, 0, 0, 1, -1, 0)$, are not contained in X. Therefore by $|X| = 22$, the other new candidate $(1, 0, 0, 1, 0, -1)$, and two isolated vectors $(-1, 0, 0, 1, 0, 1)$, and $(-1, 0, 0, 1, 1, 0)$ must be contained in X. Moreover a 7-point independent set must be obtained from Matching (i). Since $(-1, 1, 0, 0, 1, 0)$ and $(-1, 0, 1, 0, 1, 0)$ are not contained in X, by Lemma 2.4, $(1, -1, 0, 0, 0, 1)$ and $(1, 0, -1, 0, 0, 1)$ must be contained in X, and consequently any element of $U_2(2)$ is contained in X. This implies $n_2(X, 1) = 0$, and X is isomorphic to X_3, Y_3, or Z_3.

(ii) Suppose X contains an 11-point subset of Y'_3. By considering the automorphism group of Y'_3, X may contain the set in Y'_3 obtained by removing $(0, -1, 0, 0, 1, 1)$, $(0, -1, 1, 1, 0, 0)$, or $(0, 0, 1, -1, 0, 1)$. If X contains the set Y'_3 with $(0, -1, 0, 0, 1, 1)$, or $(0, -1, 1, 1, 0, 0)$ removed, then the set of other candidates of X is still $S_3(1) \cup U_3(2) \cup \mathcal{S}_1$, and $|X| < 22$. Suppose X contains the set Y'_3 with $(0, 0, 1, -1, 0, 1)$ removed. Then a new candidate of an element of X is only $(1, 0, 0, 1, 0, -1)$, and X may contain $(1, 0, 0, 1, 0, -1)$. The three vectors $(-1, 1, 0, 0, 0, 1)$, $(-1, 0, 1, 0, 0, 1)$, and $(-1, 0, 0, 0, 1, 1)$, which are at distance $\sqrt{10}$ from $(1, 0, 0, 1, 0, -1)$, are not contained in X. By considering Matching (ii), we can show $|X| < 22$.

(iii) Suppose X contains an 11-point subset of Z'_3. By considering the automorphism group of Z'_3, X may contain the set in Z'_3 obtained by removing $(0, 1, -1, 0, 0, 1)$. Then a new candidate of an element of X is only $(1, 0, 1, 0, 0, -1)$, and X may contain $(1, 0, 1, 0, 0, -1)$. The three vectors $(-1, 1, 0, 0, 0, 1)$, $(-1, 0, 0, 1, 0, 1)$, and $(-1, 0, 0, 0, 1, 1)$, which are at distance $\sqrt{10}$ from $(1, 0, 1, 0, 0, -1)$, are not contained in X. By considering Matching (iii), we can show $|X| < 22$.

(iv) Suppose X contains V'_3. The set of other candidates of X is $S_3(1) \cup U_3(3) \setminus \{(1, -1, 1, 0, 0, 0)\}$, and the maximum independent set is of order at most 10 by Matching (i). Thus $|X| < 22$.

(v) Suppose X contains W'_3. The set of other candidates of X is $S_3(1) \cup U_3(2) \cup \mathcal{S}_1 \setminus \{(1, -1, 0, 1, 0, 0)\}$, and the maximum independent set is of order at most 10 by Matching (ii). Thus $|X| < 22$.

Therefore this proposition follows. □

Finally we prove Theorem 2.5.

Proof of Theorem 2.5. By Propositions 2.7–2.9, the statement holds for $k = 1, 2, 3$. By the inductive hypothesis and Lemma 2.3, if $|X| \geq \mathfrak{M}_k$, then there exists $i \in \{1, \dots, n\}$ such that $n_i(X, 0) = \mathfrak{M}_{k-1}$ for $k \geq 4$. By Lemma 2.6, this theorem holds for any k. $\square$

3 Classification of the largest 4-distance sets which contain $\tilde{J}(n, 4)$

A finite set X in $\mathbb{R}^d$ is called an s-distance set if the set of Euclidean distances of two distinct vectors in X has size s. The Johnson graph $J(n, m) = (V, E)$, where

$$V = \{\{i_1, \dots, i_m\} \mid 1 \leq i_1 < \cdots < i_m \leq n, i_j \in \mathbb{Z}\},$$
$$E = \{(v, u) \mid |v \cap u| = m - 1, v, u \in V\},$$

is represented into $\mathbb{R}^{n-1}$ as the m-distance set $\tilde{J}(n, m) = L_{0,n-m,m}$. Indeed $\tilde{J}(n, m) \subset \mathbb{R}^n$, but the summation of all entries of any $x \in \tilde{J}(n, m)$ is m, and $\tilde{J}(n, m)$ is on a hyperplane isometric to $\mathbb{R}^{n-1}$. Bannai, Sato, and Shigezumi [1] investigated m-distance sets containing $\tilde{J}(n, m)$. In their paper, for $m \leq 5$ and any n, the largest m-distance sets containing $\tilde{J}(n, m)$ are classified except for $(n, m) = (9, 4)$. In this section, the case $(n, m) = (9, 4)$ will be classified.

The set of Euclidean distances of two distinct points of $\tilde{J}(9, 4)$ is $\{\sqrt{2}, \sqrt{4}, \sqrt{6}, \sqrt{8}\}$. The set of vectors which can be added to $\tilde{J}(9, 4)$ while maintaining 4-distance is the union of the following sets [1].

$$X^{(i)} = \left(\left(\frac{2}{3}\right)^7, \left(-\frac{1}{3}\right)^2\right)^P, \qquad X^{(ii)} = \left(\left(\frac{2}{3}\right)^8, -\frac{4}{3}\right)^P,$$
$$X^{(iii)} = \left(\frac{4}{3}, \left(\frac{1}{3}\right)^8\right)^P, \qquad X^{(iv)} = \left(\left(\frac{4}{3}\right)^2, \left(\frac{1}{3}\right)^6, -\frac{2}{3}\right)^P,$$

where the exponents inside indicate the number of occurrences of the corresponding numbers, and the exponent P outside indicates that we should take every permutation. They conjectured that $\tilde{J}(9, 4) \cup X^{(i)} \cup X^{(iii)} \cup \{(-4/3, (2/3)^8)\} \cup X^{(iv)'}$ is largest, where $(-4/3, (2/3)^8) \in X^{(ii)}$, and

$$X^{(iv)'} = \left\{(x_1, \dots, x_9) \in X^{(iv)} \mid x_i = -\frac{2}{3}, x_{j_1} = \frac{4}{3}, x_{j_2} = \frac{4}{3}, i < j_1, j_2\right\}$$
$$\cup \left\{\left(\left(\frac{1}{3}\right)^6, \frac{4}{3}, -\frac{2}{3}, \frac{4}{3}\right), \left(\left(\frac{1}{3}\right)^6, \left(\frac{4}{3}\right)^2, -\frac{2}{3}\right)\right\}.$$

Actually $X^{(iv)'}$ is isometric to X_6 in Section 2 by replacing $-2/3$, $1/3$, $4/3$ to -1, 0, 1, respectively. Let $X^{(iv)''}$ (*resp.* $X^{(iv)'''}$) be the set obtained from Y_6 (*resp.* Z_6) by the same manner. Using Theorem 2.5, we can classify the largest 4-distance sets containing $\tilde{J}(9, 4)$.

Theorem 3.1. *Let $X \subset \{(x_1, \dots, x_9) \in \mathbb{R}^9 \mid x_1 + \cdots + x_9 = 1\}$ be a 4-distance set which contains $\tilde{J}(9, 4)$. Then we have*

$$|X| \leq 258.$$

If equality holds, then X is one of the following, up to permutations of coordinates.

(1) $\tilde{J}(9,4) \cup X^{(i)} \cup X^{(iii)} \cup \{(-4/3, (2/3)^8)\} \cup X^{(iv)'}$,
(2) $\tilde{J}(9,4) \cup X^{(i)} \cup X^{(iii)} \cup \{(-4/3, (2/3)^8)\} \cup X^{(iv)''}$,
(3) $\tilde{J}(9,4) \cup X^{(i)} \cup X^{(iii)} \cup \{(-4/3, (2/3)^8)\} \cup X^{(iv)'''}$.

Proof. For any $x \in X^{(i)} \cup X^{(iii)}$, $y \in \cup_{j=1}^4 X^{(j)}$, the Euclidean distance of x, y is in $\{\sqrt{2}, \sqrt{4}, \sqrt{6}, \sqrt{8}\}$, and hence X may contain $X^{(i)} \cup X^{(iii)}$. The set $X^{(iv)}$ is isometric to L_{162} by replacing $-2/3$, $1/3$, $4/3$ to -1, 0, 1, respectively. Therefore the largest subsets of $X^{(iv)}$ with distances $\{\sqrt{2}, \sqrt{4}, \sqrt{6}, \sqrt{8}\}$ are $X^{(iv)'}$, $X^{(iv)''}$, and $X^{(iv)'''}$, up to permutations of coordinates. If X does not contain any element of $X^{(ii)}$, then

$$|X| \leq |\tilde{J}(9,4) \cup X^{(i)} \cup X^{(iii)}| + |X^{(iv)'}| = 257.$$

If X contains $x \in X^{(ii)}$ with $x_i = -4/3$, then X cannot contain $y \in X^{(iv)}$ with $y_i = 4/3$. By re-ordering the vectors, we may assume that the set

$$X^{(ii)}(t) = \{x \in X^{(ii)} \mid x_i = -4/3, \exists i \in \{1, \ldots, t\}\}$$

is in X for some t. Clearly, from the definition of $X^{(ii)}(t)$, this set must have size t. For $t = 7, 8, 9$, X contains at most one element of $X^{(iv)}$, and hence

$$|X| \leq |\tilde{J}(9,4) \cup X^{(i)} \cup X^{(iii)}| + t + 1 \leq 181.$$

If the set $X^{(ii)}(t)$ is in X for $1 \leq t \leq 6$, then consider the set of vectors in $X \cap X^{(iv)}$ in which the entry $1/3$ occurs in all of the first t positions. The final $9 - t$ entries of one of these vectors forms a vector from $L_{1,6-t,2}$; no two vectors in this set can be at the maximum distance. Thus the size of

$$|\{x \in X \cap X^{(iv)} \mid x_i = 1/3, \forall i \in \{1, \ldots, t\}\}|$$

is bounded by $\mathfrak{M}_{6-t}$. It is clear that

$$|\{x \in X \cap X^{(iv)} \mid x_i = -2/3, x_{j_1} = 4/3, x_{j_2} = 4/3, \\ \exists i \in \{1, \ldots, t\}, \exists j_1, j_2 \in \{t+1, \ldots, 9\}\}|$$

is bounded by $t\binom{9-t}{2}$. Thus, for $1 \leq t \leq 6$, we have

$$\begin{aligned} |X| &\leq |\tilde{J}(9,4) \cup X^{(i)} \cup X^{(iii)}| + t + \mathfrak{M}_{6-t} + t\binom{9-t}{2} \\ &= \frac{t^3}{3} - \frac{9t^2}{2} + \frac{31t}{6} + 257 \leq 258, \end{aligned}$$

and equality holds only if $t = 1$. The sets attaining this bound are only the three sets in the statement. □

4 Remarks on other M_{mkl}

Actually it is hard to determine M_{mkl} for other (m, k, l) by a similar manner in Section 2. Fix m, l, where $m < l$. By Proposition 2.2, if $k \leq l - m$, then $M_{mkl} = \binom{n-1}{m+k-1}\binom{m+k}{m}$.

In general there are many largest sets for $k = l - m$. For $k > l - m$, we can inductively construct a large set $X_k \subset L_{mkl}$ satisfying $D(X_k) < D(L_{mkl})$ as follows

$$X_k = \{(0, x') \mid x' \in X_{k-1}\} \cup \{(x_1, \ldots, x_n) \in L_{mkl} \mid x_1 = -1\},$$

where X_{l-m} is a largest set for $k = l - m$. Therefore we have

$$M_{mkl} \geq \mathfrak{M}_{mkl} := \binom{m+l-1}{m-1}\binom{k+m+l}{m+l} + \binom{m+l-1}{m}.$$

We can generalize Lemma 2.3 as follows.

Lemma 4.1. *Let $X \subset L_{mkl}$ with $D(X) \leq D(L_{mkl})$. Suppose $k \geq m\binom{m+l}{m} - m - l + 1$, and $|X| \geq \mathfrak{M}_{mkl}$. Then there exists $i \in \{1, \ldots, n\}$ such that $n_i(X, 0) \geq \mathfrak{M}_{m,k-1,l}$.*

Proof. This lemma is immediate because the average of $n_i(X, 0)$ is

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n} n_i(X,0) &= \frac{k|X|}{m+k+l} \geq \frac{k\mathfrak{M}_{mkl}}{m+k+l} \\ &= \mathfrak{M}_{m,k-1,l} - \frac{m+l}{m+k+l}\binom{m+k+l}{l} > \mathfrak{M}_{k-1} - 1. \quad \square\end{aligned}$$

In the manner of Section 2, it is hard to classify M_{mkl} for $m - l + 1 \leq k \leq m\binom{m+l}{m} - m - l$. Moreover it seems to be difficult to give matchings, like Matching (i) or (ii), of many possibilities of X_k. We need another idea to determine other M_{mkl}.

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