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Avoidance in bowtie systems

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Abstract

There are ten configurations of two bowties that can arise in a bowtie system. The avoidance spectrum for three of these was determined in a previous paper (Aequat. Math. **85** (2013), 347–358). In this paper the avoidance spectrum for a further five configurations is determined.

Keywords: Bowtie system, configuration, avoidance, Steiner triple system.

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1 Introduction

Let $X = (V, E)$ be the graph with vertex set $V = \{x, a, b, c, d\}$ and edge set $E = \{xa, xb, xc, xd, ab, cd\}$. Such a graph is called a *bowtie* and will be represented throughout this paper by the notation $a, b - x - c, d$. The vertex x is called the *centre* of the bowtie and the other vertices are called *endpoints*. A decomposition of the complete graph K_n into subgraphs isomorphic to X is called a *bowtie system* of order n and denoted by $\mathrm{BTS}(n)$. An elementary counting argument shows that a necessary condition for the existence of a $\mathrm{BTS}(n)$ is $n \equiv 1$ or $9 \pmod{12}$. In a $\mathrm{BTS}(n)$, if every vertex of the complete graph K_n occurs the same number of times as the centre of a bowtie, then the bowtie

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system is said to be *balanced*, otherwise the system is said to be *unbalanced*. A necessary condition for the existence of a balanced $\mathrm{BTS}(n)$ is $n \equiv 1 \pmod{12}$.

It is easy to see that, given a $\mathrm{BTS}(n)$, by regarding each of the two triangles of every bowtie as separate entities, we have a Steiner triple system $\mathrm{STS}(n)$. We call this the *associated* Steiner triple system of the bowtie system. Conversely, if $n \equiv 1$ or $9 \pmod{12}$, it is also true that the triangles of every $\mathrm{STS}(n)$ can be amalgamated to form bowties. This is a consequence of the fact that the block intersection graph of every Steiner triple system is Hamiltonian, see for example [2, Section 13.6]. If $n \equiv 1 \pmod{12}$, there exists a cyclic $\mathrm{STS}(n)$, see also [2, Section 7.2], and this system will have an even number of full orbits. It is then immediate that we can amalgamate triangles from pairs of orbits to form a balanced $\mathrm{BTS}(n)$. Hence the necessary conditions for both $\mathrm{BTS}(n)$ and balanced $\mathrm{BTS}(n)$ given above are also sufficient.

A *configuration* in a bowtie system (resp. Steiner triple system) is a small collection of bowties (resp. triangles) which may occur in the system. The study of configurations in $\mathrm{STS}(n)$ is now well established and the whole of Chapter 13 of [2] is devoted to various results about them and in particular includes formulae for the number of occurrences of all possible configurations of four or fewer triangles. Those for configurations of one, two or three triangles are functions of n. Such configurations are called *constant* because the number of occurrences is independent of the structure of the $\mathrm{STS}(n)$. Other configurations are *variable*. There are 16 non-isomorphic configurations of four triangles of which 5 are constant and 11 are variable. An important concept is that of avoidance; given any particular configuration in a bowtie system (resp. Steiner triple system), to determine the spectrum of n for which there exists a $\mathrm{BTS}(n)$ (resp. $\mathrm{STS}(n)$) which does not contain that configuration. Avoidance sets for all configurations of four or fewer triangles in Steiner triple systems are known. Most, particularly those for constant configurations, are easy to determine but that for the so-called *Pasch configuration* (four triangles isomorphic to $\{a, b, c\}$, $\{a, y, z\}$, $\{x, b, z\}$, $\{x, y, c\}$) was more challenging. It is $n \equiv 1$ or $3 \pmod{6}$, $n \neq 7, 13$ and a complete solution appears in the two papers [7] and [6].

In this paper we will be concerned with the avoidance sets of configurations of two bowties in a BTS(n). There are ten such configurations which were determined in [3] and are illustrated in Figure 1. In this figure each triangle of a bowtie is represented by a path on three vertices and, in each case, one bowtie is represented by solid lines and the second by dashed lines. The intersection of two solid lines or two dashed lines is the centre of the bowtie and the other four points are the endpoints. The ten configurations are each labelled $\hat{C}_i$ for some value of i, $1 \leq i \leq 16$, to reflect the fact that the bowtie configuration with that label gives the configuration C_i in the standard listing of configurations of four triangles in Steiner triple systems as given in [5] or [2, Section 13.1]. Indeed it was by examining all 16 possible configurations of four triangles in a Steiner triple system and identifying which could be obtained from two bowties that the ten possible configurations of two bowties were obtained.

There are four equations which connect the number of occurrences of the various configurations of two bowties and these were proved in [3]. Denoting the number of occurrences of the configuration $\hat{C}_i$ by c_i, the equations are the following.

$$4c_7 + c_8 + c_{11} + c_{15} = n(n-1)(n-5)/24. \tag{1.1}$$

$$c_{11} + c_{12} + 2c_{14} + 3c_{15} + 4c_{16} = n(n-1)/3. \tag{1.2}$$

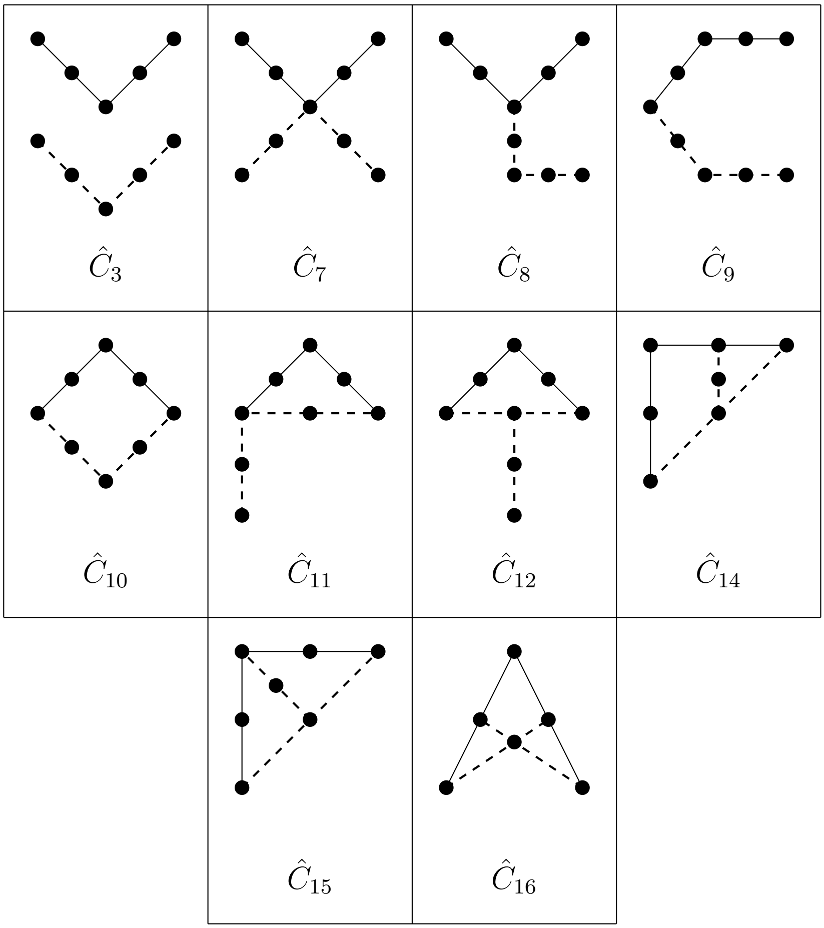

Figure 1: Configurations of two bowties.

$$c_8 + c_9 + 2c_{10} + c_{11} + c_{12} + c_{14} = n(n-1)(n-7)/12. \tag{1.3}$$
$$4c_3 + c_8 + 2c_9 + c_{12} = n(n-1)(n-7)(n-9)/72. \tag{1.4}$$

If the bowtie system is balanced, there is a further equation.

$$c_7 = n(n-1)(n-13)/288. \tag{1.5}$$

All configurations are variable except that $\hat{C}_7$ is constant in balanced bowtie systems.

Avoidance sets for the three most compact configurations, $\hat{C}_{14}, \hat{C}_{15}$ and $\hat{C}_{16}$ have already been determined in [3]. The following theorem was proved.

Theorem 1.1. *For each $n \equiv 1 \pmod{12}$ there exists both a balanced and an unbalanced $\mathrm{BTS}(n)$ simultaneously avoiding $\hat{C}_{14}, \hat{C}_{15}$ and $\hat{C}_{16}$. For each $n \equiv 9 \pmod{12}$, $n \neq 9$ there exists a (necessarily unbalanced) $\mathrm{BTS}(n)$ simultaneously avoiding $\hat{C}_{14}, \hat{C}_{15}$ and $\hat{C}_{16}$.*

Thus not only can each of these three configurations be avoided for all values of n for which both balanced $\mathrm{BTS}(n)$ and unbalanced $\mathrm{BTS}(n)$ exist except for $n = 9$, they can all be avoided simultaneously. There are precisely 12 non-isomorphic $\mathrm{BTS}(9)$s which were enumerated in [4]. All avoid $\hat{C}_{16}$, none avoid $\hat{C}_{15}$ and just one avoids $\hat{C}_{14}$. The details are in [3].

In this paper, we consider five further configurations. In particular we show that $\mathrm{BTS}(n)$ avoiding three of the least compact configurations $\hat{C}_3$, $\hat{C}_7$ and $\hat{C}_8$ do not exist if $n > 13$. Our main results are that for each of the configurations $\hat{C}_{11}$ and $\hat{C}_{12}$, and for all admissible values of n, there exists a $\mathrm{BTS}(n)$ avoiding that configuration, with the single exception of $\hat{C}_{11}$ when $n = 13$. The situation for the two configurations $\hat{C}_9$ and $\hat{C}_{10}$ remains unresolved.

2 Avoiding $\hat{C}_3$, $\hat{C}_7$ and $\hat{C}_8$

We begin with $\hat{C}_7$. The number of bowties in a $\mathrm{BTS}(n)$ is $n(n-1)/12$. Hence if $n > 13$, there will be two bowties with a common centre. So the only possible systems which may avoid $\hat{C}_7$ are balanced $\mathrm{BTS}(13)$s, and indeed all such systems do avoid $\hat{C}_7$, and $\mathrm{BTS}(9)$s. Checking the data of the 12 non-isomorphic $\mathrm{BTS}(9)$s from [3] shows that six of these do avoid $\hat{C}_7$ and the other six do not. We state this formally as a theorem.

Theorem 2.1. *The only bowtie systems to avoid $\hat{C}_7$ are six of the twelve non-isomorphic $\mathrm{BTS}(9)$s and all balanced $\mathrm{BTS}(13)$s.*

Next we consider $\hat{C}_8$ and begin with some observations. First, if $a, b - x - c, d$ is a bowtie in a $\mathrm{BTS}(n)$ which has no $\hat{C}_8$ configurations, then there are at most two bowties whose centre is a. This is because any such bowtie must intersect the bowtie $a, b - x - c, d$ in a further point which can only be c or d. Similarly, there are at most two bowties whose centre is b, c or d.

Secondly, in any $\mathrm{BTS}(n)$, a point x can be the centre of at most $(n-1)/4$ bowties. Thus if the $\mathrm{BTS}(n)$ has no $\hat{C}_8$ configurations and x is the centre of less then $(n-1)/4$ bowties, then it is an endpoint of at least one other bowtie and so, by the above, there are at most two bowties whose centre is x. As a consequence, in a $\mathrm{BTS}(n)$ which has no $\hat{C}_8$ configurations, each point x is the centre of 0, 1, 2 or $(n-1)/4$ bowties. Furthermore, if a point is the centre of $(n-1)/4$ bowties, then all remaining points are the centre of at most two bowties. We can now prove the following theorem.

Theorem 2.2. *A* $\mathrm{BTS}(n)$ *avoiding* $\hat{C}_8$ *can only exist if* $n \le 13$.

Proof. Subtracting equation 1.2 from equation 1.1 and re-arranging terms gives

$$c_8 = n(n-1)(n-13)/24 - 4c_7 + c_{12} + 2c_{14} + 2c_{15} + 4c_{16}.$$

Hence $c_8 \ge n(n-1)(n-13)/24 - 4c_7$.

Now let a_x be the number of bowties in a $\mathrm{BTS}(n)$ whose centre is x. Then $c_7 = \sum_{x \in V} \binom{a_x}{2}$ where V denotes the set of n points in the design. Suppose that $n > 13$ and that the $\mathrm{BTS}(n)$ has no $\hat{C}_8$ configurations. Let m be the maximum number of bowties centred on any point in the $\mathrm{BTS}(n)$. Then from the argument above either $m = 2$ or $m = (n-1)/4$ and all but one point is the centre of at most two bowties.
In either case

$$c_7 = \sum_{x \in V} \binom{a_x}{2} \le \binom{(n-1)/4}{2} + (n-1) = (n-1)(n+27)/32.$$

Hence

$$c_8 \ge n(n-1)(n-13)/24 - (n-1)(n+27)/8 = (n-1)(n^2 - 16n - 81)/24.$$

The right hand side of this expression is strictly positive for $n \ge 21$, and the result follows. □

In order to complete the avoidance spectrum for the configuration $\hat{C}_8$, we have the following result.

Theorem 2.3. *All* $\mathrm{BTS}(9)$*s avoid* $\hat{C}_8$ *but no balanced* $\mathrm{BTS}(13)$ *avoids* $\hat{C}_8$. *There exist unbalanced* $\mathrm{BTS}(13)$*s which avoid* $\hat{C}_8$.

Proof. Checking the data of the 12 non-isomorphic $\mathrm{BTS}(9)$s from [3] shows that all avoid $\hat{C}_8$. The fact that no balanced $\mathrm{BTS}(13)$ avoids $\hat{C}_8$ follows from an exhaustive computer search of all $1,411,422$ non-isomorphic systems identified in [4]. Two unbalanced $\mathrm{BTS}(13)$s on the point set $\{0, 1, 2, \ldots, 12\}$ which avoid $\hat{C}_8$ are given below. In the first case the associated $\mathrm{STS}(13)$ is cyclic and in the second case it is non-cyclic.

(1) $0,4-1-2,5$; $0,7-2-3,6$; $2,9-4-3,7$;
$0,6-8-1,3$; $4,5-8-9,12$; $1,7-9-5,6$;
$0,9-10-6,7$; $2,8-10-3,5$; $0,5-11-1,10$;
$2,12-11-4,6$; $3,9-11-7,8$; $0,3-12-4,10$;
$1,6-12-5,7$.

(2) $1,4-0-2,7$; $6,8-0-9,10$; $0,12-3-1,8$;
$2,6-3-4,7$; $2,9-4-5,8$; $1,2-5-3,10$;
$1,7-9-5,6$; $2,8-10-6,7$; $0,5-11-1,6$;
$2,12-11-4,10$; $3,9-11-7,8$; $1,10-12-5,7$;
$4,6-12-8,9$. □

Finally in this section we consider $\hat{C}_3$. We have a parallel result to Theorem 2.2 for the configuration $\hat{C}_8$.

Theorem 2.4. *A* $\mathrm{BTS}(n)$ *avoiding* $\hat{C}_3$ *can only exist if* $n \leq 13$.

Proof. Assume that $n > 13$, so that from Theorem 2.2, $c_8 > 0$. From equation 1.3, $c_9 < n(n-1)(n-7)/12$ and $c_8 + c_9 + c_{12} \leq n(n-1)(n-7)/12$. So by addition $c_8 + 2c_9 + c_{12} < n(n-1)(n-7)/6$. From equation 1.4,

$$4c_3 = n(n-1)(n-7)(n-9)/72 - (c_8 + 2c_9 + c_{12}).$$

Therefore $4c_3 > n(n-1)(n-7)(n-21)/72$. Throughout this proof all inequalities are strict and since $n > 13$, i.e. $n \geq 21$, we have that $c_3 > 0$. □

Again, to complete the avoidance spectrum for the configuration $\hat{C}_3$, we have the following result.

Theorem 2.5. *The avoidance spectrum of the configuration* $\hat{C}_3$ *is the set* $\{9, 13\}$.

Proof. The configuration $\hat{C}_3$ has ten points so all $\mathrm{BTS}(9)$s avoid $\hat{C}_3$. A balanced $\mathrm{BTS}(13)$ on the set Z_{13} which avoid $\hat{C}_3$ is the set of bowties $(i+1),(i+4) - i - (i+2),(i+7)$, $0 \leq i \leq 12$, with arithmetic modulo 13. An unbalanced $\mathrm{BTS}(13)$ can be obtained by replacing the bowties $1,4-0-2,7$ and $7,10-6-8,0$ with the bowties $1,4-0-6,8$ and $6,10-7-0,2$. □

3 Avoiding $\hat{C}_{11}$ and $\hat{C}_{12}$

The method we use to construct bowtie systems which avoid the configurations $\hat{C}_{11}$ and $\hat{C}_{12}$ is similar to how we proved Theorem 1.1 on avoiding $\hat{C}_{14}, \hat{C}_{15}$ and $\hat{C}_{16}$ and uses standard techniques involving group divisible designs. It is however more intricate. We note that all GDDs used in this paper exist (see [1, Section IV 4.1]). An essential component of the construction is the following $\mathrm{BTS}(9)$ which is System (a)(I) in [4] and avoids both $\hat{C}_{11}$ and $\hat{C}_{12}$.

$$\begin{array}{lll} 1,2-0-3,6; & 4,8-0-5,7; & 3,5-4-1,7; \\ 6,7-8-2,5; & 5,6-1-3,8; & 3,7-2-4,6. \end{array}$$

We begin by proving the following result.

Theorem 3.1. *For each* $n \equiv 1, 9 \pmod{24}$*, there exists a BTS*(n) *avoiding* $\hat{C}_{12}$.

Proof. Take a 3-GDD of type 4^t, where $t = 3s$ or $3s+1$ and $s \geq 1$. Denote the points of the i^{th} group, $1 \leq i \leq t$, by $(i,1),(i,2),(i,3)$ and $(i,4)$. Inflate each point to two points, i.e. a point (i,j) becomes two points (i,j) and (i,j'). Add a single new point ∞. On each inflated group of 8 points augmented with the ∞ point place a copy of the $\mathrm{BTS}(9)$ above, identifying the points as follows.

$$\begin{array}{lllll} \infty = 0, & (i,1) = 1, & (i,1') = 3, & (i,2) = 2, & (i,2') = 6, \\ & (i,3) = 4, & (i,3') = 5, & (i,4) = 8, & (i,4') = 7. \end{array}$$

On each of the original blocks of the GDD, say $\{(i_1,j_1),(i_2,j_2),(i_3,j_3)\}$, where $i_1 \neq i_2 \neq i_3 \neq i_1$, place the two bowties $(i_2,j_2),(i_3,j_3) - (i_1,j_1) - (i_2,j_2'),(i_3,j_3')$ and $(i_2,j_2),(i_3,j_3') - (i_1,j_1') - (i_2,j_2'),(i_3,j_3)$. The bowties in the resulting $\mathrm{BTS}(8t+1)$ can be thought of as being of two types; (i) those resulting from a $\mathrm{BTS}(9)$ which we will call

BTS *bowties* and (ii) those resulting from the blocks of the GDD which we will call GDD *bowties*. We need to consider pairs of bowties which arise from all possibilities. There are five cases to consider.

(1) Two GDD bowties which come from the same block of the GDD. By the construction these form a configuration $\hat{C}_{16}$.

(2) Two GDD bowties which come from different blocks of the GDD. There are four possible scenarios.

  (a) If the two bowties are disjoint then they form a configuration $\hat{C}_3$.

  (b) If the centres of the two bowties are the same, then they have no further points in common and we have a configuration $\hat{C}_7$.

  (c) If the centre of one of the bowties is an endpoint of the other bowtie, then again they have no further points in common and we have a configuration $\hat{C}_8$.

  (d) If the two bowties have an endpoint in common, then they also have a further endpoint in common and they form a configuration $\hat{C}_{10}$.

(3) Two BTS bowties which come from the same $\mathrm{BTS}(9)$. The configuration they form is completely determined by the structure of the $\mathrm{BTS}(9)$ and so avoids $\hat{C}_{12}$ (and $\hat{C}_{11}$).

(4) Two BTS bowties which come from different $\mathrm{BTS}(9)$s. If the two bowties are disjoint then they form a configuration $\hat{C}_3$. Otherwise they can only intersect in the point ∞ which will be the centre of both bowties and we have a configuration $\hat{C}_7$.

(5) A BTS bowtie and a GDD bowtie. If the two bowties are disjoint then they form a configuration $\hat{C}_3$. If they have just one point in common then they also avoid $\hat{C}_{12}$. Otherwise they have two points in common and these points will both be endpoints of the GDD bowtie. Further, the two points will be (i, j) and (i, j') for some i, j such that $1 \leq i \leq t$ and $1 \leq j \leq 4$. If either of these points is the centre of the BTS bowtie, then the other point is an endpoint and we have a configuration $\hat{C}_{11}$. Otherwise both points are endpoints of the BTS bowtie and, because of the way in which the points of the $\mathrm{BTS}(9)$ were assigned to the points ∞, (i, j) and (i, j'), they are in different triangles. Hence we have a configuration $\hat{C}_{10}$. □

We now prove a parallel result for the configuration $\hat{C}_{11}$.

Theorem 3.2. *For each* $n \equiv 1, 9 \pmod{24}$, *there exists a* $\mathrm{BTS}(n)$ *avoiding* $\hat{C}_{11}$.

Proof. This follows the same steps as the previous theorem. However the way in which each inflated group of 8 points augmented with the ∞ point is identified with the points of the $\mathrm{BTS}(9)$ is different. In this case it is as follows.

$$\begin{array}{lllll} \infty = 0, & (i,1) = 1, & (i,1') = 2, & (i,2) = 3, & (i,2') = 6, \\ & (i,3) = 4, & (i,3') = 8, & (i,4) = 5, & (i,4') = 7. \end{array}$$

The construction of the GDD bowties is the same. Also, in the analysis of pairs of bowties, the first four cases are the same. So we only need to consider case (5) of a BTS bowtie and a GDD bowtie. Again, if the two bowties are disjoint then they form a configuration $\hat{C}_3$.

If they have just one point in common then they also avoid $\hat{C}_{11}$. Otherwise they have two points in common and they are (i, j) and (i, j') as before. Because of the way in which the points of the $\mathrm{BTS}(9)$ were assigned to the points ∞, (i, j) and (i, j'), no BTS bowtie has its centre at a point (i, j) (resp. (i, j')) and an endpoint at the point (i, j') (resp. (i, j)). So both points are endpoints of the BTS bowtie. If they are in the same triangle then we have a configuration $\hat{C}_{12}$. If they are in different triangles then we have a configuration $\hat{C}_{10}$. □

We next consider the cases $n \equiv 13, 21 \pmod{24}$. In order to deal with bowtie systems in these residue classes avoiding $\hat{C}_{12}$, the following further $\mathrm{BTS}(13)$ is used.

$$\begin{array}{lll}
1,4-0-9,10; & 2,7-0-6,8; & 3,12-0-5,11; \\
1,5-2-3,6; & 1,8-3-5,10; & 2,10-8-4,5; \\
7,9-1-10,11; & 1,12-6-7,10; & 2,4-9-5,6; \\
4,7-3-9,11; & 2,12-11-4,6; & 4,10-12-8,9; \\
5,12-7-8,11. & &
\end{array}$$

This system avoids the configuration $\hat{C}_{12}$ and has the property that one point, namely 0, is at the centre of three bowties and never appears as an endpoint. We can now prove the following result.

Theorem 3.3. *For each $n \equiv 13, 21 \pmod{24}$, there exists a $\mathrm{BTS}(n)$ avoiding $\hat{C}_{12}$.*

Proof. Take a 3-GDD of type $4^t 6^1$, where $t = 3s$ or $3s + 1$ and $s \geq 1$. Proceed as in Theorem 3.1 where in addition the points of the long group are denoted by $(t + 1, j)$, $1 \leq j \leq 6$. On this inflated group of 12 points augmented with the ∞ point place a copy of the $\mathrm{BTS}(13)$ above, identifying the points as follows.

$$\begin{array}{llll}
\infty = 0, & & & \\
(t+1,1) = 1, & (t+1,1') = 10, & (t+1,2) = 4, & (t+1,2') = 9, \\
(t+1,3) = 2, & (t+1,3') = 6, & (t+1,4) = 7, & (t+1,4') = 8, \\
(t+1,5) = 3, & (t+1,5') = 5, & (t+1,6) = 12, & (t+1,6') = 11.
\end{array}$$

The proof now follows that of Theorem 3.1 . This proves the result for all stated values of n except $n = 21$. A solution for this value is the following.

$$\begin{array}{lll}
15,9-3-11,17; & 17,9-5-11,15; & 18,10-3-14,19; \\
19,10-5-14,18; & 16,12-3-13,20; & 20,12-5-13,16; \\
18,9-4-11,19; & 19,9-8-11,18; & 16,10-4-14,20; \\
20,10-8-14,16; & 15,12-4-13,17; & 17,12-8-13,15; \\
16,9-6-11,20; & 20,9-7-11,16; & 15,10-6-14,17; \\
17,10-7-14,15; & 18,12-6-13,19; & 19,12-7-13,18; \\
0,7-3-1,5; & 2,3-6-5,7; & 2,7-8-3,4; \\
6,8-1-4,7; & 0,6-4-2,5; & 2,9-12-11,13; \\
2,13-14-9,10; & 12,14-1-10,13; & 0,14-11-1,9; \\
0,12-10-2,11; & 0,19-15-1,17; & 2,15-18-17,19; \\
2,19-20-15,16; & 18,20-1-16,19; & 0,18-16-2,17; \\
1,2-0-9,13; & 8,5-0-20,17. &
\end{array}$$

□

Turning our attention to avoiding $\hat{C}_{11}$, we have shown by an exhaustive computer search that there is no $\mathrm{BTS}(13)$ that avoids this configuration. So for the residue classes 13 and 21 $\pmod{24}$ we use the modified constructions given in Theorems 3.4 and 3.5. For balanced $\mathrm{BTS}(13)$s the minimum number of $\hat{C}_{11}$ configurations is 10 for both associated cyclic and non-cyclic $\mathrm{STS}(13)$s. For unbalanced systems with the associated cyclic $\mathrm{STS}(13)$, we find that the minimum is 5, but for unbalanced systems with the associated non-cyclic $\mathrm{STS}(13)$, we find that the minimum is 4 and an example is given below.

$$\begin{array}{lll}
0,12-3-1,8; & 2,6-3-4,7; & 3,5-10-6,7;\\
2,4-9-3,11; & 2,7-0-5,11; & 0,10-9-8,12;\\
0,8-6-4,12; & 1,7-9-5,6; & 0,4-1-10,12;\\
2,5-1-6,11; & 2,11-12-5,7; & 2,8-10-4,11;\\
4,5-8-7,11. & &
\end{array}$$

Theorem 3.4. *For each $n \equiv 21 \pmod{24}$, there exists a BTS(n) avoiding $\hat{C}_{11}$.*

Proof. Take a 3-GDD of type 3^t, where $t = 4s+3$ and $s \geq 0$. Denote the points of the i^{th} group, $1 \leq i \leq t$, by $(i,1),(i,2)$ and $(i,3)$. As before inflate each point to two points, i.e. a point (i,j) becomes two points (i,j) and (i,j'). Add three new points ∞_0, ∞_1 and ∞_2. On each inflated group of 6 points augmented with the three ∞ points first place a copy of the $\mathrm{BTS}(9)$ at the beginning of this Section, identifying the points as follows.

$$\begin{array}{llll}
\infty_0 = 0, & \infty_1 = 1, & \infty_2 = 2, & \\
(i,1) = 3, & (i,1') = 6, & (i,2) = 4, & (i,2') = 8,\\
(i,3) = 5, & (i,3') = 7. & &
\end{array}$$

The triangle $\{\infty_0,\infty_1,\infty_2\}$ now occurs $4s+3$ times. Remove the bowties

$$\infty_1,\infty_2 - \infty_0 - (i,1),(i,1')$$

for all i such that $2 \leq i \leq 4s+3$ and replace them by the bowties

$$(2i,1),(2i,1') - \infty_0 - (2i+1,1),(2i+1,1'), \quad 1 \leq i \leq 2s+1.$$

We call these $\mathrm{BTS}^\star$ *bowties*. The construction of the GDD bowties is as in the previous three theorems.

We need to prove that a bowtie system constructed in this way avoids configuration $\hat{C}_{11}$. The proof for the five cases involving just BTS bowties and GDD bowties is as in Theorem 3.2. So any putative configuration $\hat{C}_{11}$ must contain a $\mathrm{BTS}^\star$ bowtie. We show that this is impossible. A configuration $\hat{C}_{11}$ consists of two bowties isomorphic to $c,y-x-b,z$ and $a,z-y-d,e$. The centre of every $\mathrm{BTS}^\star$ bowtie is ∞_0; however this point never occurs as the endpoint of any bowtie. So $y \neq \infty_0$. Now suppose that $x = \infty_0$ and that $c,y-x-b,z$ is a $\mathrm{BTS}^\star$ bowtie. Then without loss of generality $y = (2i,1)$ and $z = (2i+1,1)$ for some i such that $1 \leq i \leq 2s+1$, say $i = q$. Therefore the bowtie $a,z-y-d,e$ is a GDD bowtie and either d or $e = (2q+1,1') = b$ which means that we do not have a configuration $\hat{C}_{11}$. $\square$

We note that, by using a 3-GDD of type 3^t where $t = 4s+1$, $s \geq 1$, the above theorem can also be used to provide an alternative proof of the existence of a $\mathrm{BTS}(n)$ avoiding $\hat{C}_{11}$ for the residue class $9 \pmod{24}$.

A $\mathrm{BTS}(21)$ avoiding $\hat{C}_{11}$ from the above theorem is given below. This will be needed in the proof of the final theorem. It has the crucial property that one point, again namely 0, is at the centre of five bowties and never appears as an endpoint.

$$
\begin{array}{lll}
1,2-0-3,6; & 4,8-0-5,7; & 10,14-0-11,13; \\
16,20-0-17,19; & 9,12-0-15,18; & \\
3,5-4-1,7; & 9,11-10-1,13; & 15,17-16-1,19; \\
6,7-8-2,5; & 12,13-14-2,11; & 18,19-20-2,17; \\
5,6-1-3,8; & 11,12-1-9,14; & 17,18-1-15,20; \\
3,7-2-4,6; & 9,13-2-10,12; & 15,19-2-16,18; \\
9,15-3-12,18; & 10,17-3-14,19; & 11,16-3-13,20; \\
9,18-6-12,15; & 10,19-6-14,17; & 11,20-6-13,16; \\
10,16-4-14,20; & 11,15-4-13,18; & 9,17-4-12,19; \\
10,20-8-14,16; & 11,18-8-13,15; & 9,19-8-12,17; \\
11,17-5-13,19; & 9,16-5-12,20; & 10,15-5-14,18; \\
11,19-7-13,17; & 9,20-7-12,16; & 10,18-7-14,15.
\end{array}
$$

Theorem 3.5. *For each* $n \equiv 13 \pmod{24}$*, except for* $n = 13$*, there exists a* $\mathrm{BTS}(n)$ *avoiding* $\hat{C}_{11}$.

Proof. Take a 3-GDD of type $4^t 10^1$, where $t = 3s+2$, $s \geq 1$. Proceed as in Theorem 3.2 where the points of the long group are denoted by $(t+1, j)$, $1 \leq j \leq 10$. On this inflated group of 20 points augmented with the ∞ point place a copy of the $\mathrm{BTS}(21)$ above, identifying the points as follows.

$$
\begin{array}{llll}
\infty = 0, & & & \\
(t+1,1) = 1, & (t+1,1') = 2, & (t+1,2) = 3, & (t+1,2') = 6, \\
(t+1,3) = 4, & (t+1,3') = 8, & (t+1,4) = 5, & (t+1,4') = 7, \\
(t+1,5) = 10, & (t+1,5') = 14, & (t+1,6) = 11, & (t+1,6') = 13, \\
(t+1,7) = 16, & (t+1,7') = 20, & (t+1,8) = 17, & (t+1,8') = 19, \\
(t+1,9) = 9, & (t+1,9') = 12, & (t+1,10) = 15, & (t+1,10') = 18.
\end{array}
$$

The proof now follows that of Theorem 3.2. This proves the result for all stated values of n except $n = 37$. A solution for this value is given in Table 1 below. □

Finally, we again note that, by using a 3-GDD of type $4^t 10^1$ where $t = 3s$, $s \geq 1$, the above theorem can also be used to provide an alternative proof of the existence of a $\mathrm{BTS}(n)$ avoiding $\hat{C}_{11}$ for the residue class $21 \pmod{24}$.

Table 1: A $\mathrm{BTS}(37)$ avoiding $\hat{C}_{11}$.

$16,34-0-17,35;$	$18,36-0-1,19;$	$8,26-0-9,27;$
$10,28-0-11,29;$	$12,30-0-13,31;$	$14,32-0-15,33;$
$2,20-0-7,25;$	$3,21-0-4,22;$	$5,23-0-6,24;$
$7,18-6-36,1;$	$25,36-24-18,19;$	$24,1-7-19,36;$
$6,19-25-1,18;$	$9,8-1-26,27;$	$27,8-19-26,9;$
$7,5-2-23,25;$	$25,5-20-23,7;$	$14,13-3-31,32;$
$32,13-21-31,14;$	$15,11-4-29,33;$	$33,11-22-29,15;$
$12,10-6-28,30;$	$30,10-24-28,12;$	$11,10-1-28,29;$
$29,10-19-28,11;$	$15,13-2-31,33;$	$33,13-20-31,15;$
$6,5-3-23,24;$	$24,5-21-23,6;$	$12,8-4-26,30;$
$30,8-22-26,12;$	$14,9-7-27,32;$	$32,9-25-27,14;$
$13,12-1-30,31;$	$31,12-19-30,13;$	$11,9-2-27,29;$
$29,9-20-27,11;$	$7,4-3-22,25;$	$25,4-21-22,7;$
$15,10-5-28,33;$	$33,10-23-28,15;$	$14,8-6-26,32;$
$32,8-24-26,14;$	$15,14-1-32,33;$	$33,14-19-32,15;$
$6,4-2-22,24;$	$24,4-20-22,6;$	$11,8-3-26,29;$
$29,8-21-26,11;$	$12,9-5-27,30;$	$30,9-23-27,12;$
$13,10-7-28,31;$	$31,10-25-28,13;$	$18,17-16-35,36;$
$36,17-34-35,18;$	$2,3-1-21,16;$	$20,21-19-3,34;$
$19,16-2-34,21;$	$1,34-20-16,3;$	$10,14-4-32,16;$
$28,32-22-14,34;$	$22,16-10-34,32;$	$4,34-28-16,14;$
$8,13-5-31,16;$	$26,31-23-13,34;$	$23,16-8-34,31;$
$5,34-26-16,13;$	$9,15-6-33,16;$	$27,33-24-15,34;$
$24,16-9-34,33;$	$6,34-27-16,15;$	$11,12-7-30,16;$
$29,30-25-12,34;$	$25,16-11-34,30;$	$7,34-29-16,12;$
$4,5-1-23,17;$	$22,23-19-5,35;$	$19,17-4-35,23;$
$1,35-22-17,5;$	$12,14-2-32,17;$	$30,32-20-14,35;$
$20,17-12-35,32;$	$2,35-30-17,14;$	$9,10-3-28,17;$
$27,28-21-10,35;$	$21,17-9-35,28;$	$3,35-27-17,10;$
$11,13-6-31,17;$	$29,31-24-13,35;$	$24,17-11-35,31;$
$6,35-29-17,13;$	$8,15-7-33,17;$	$26,33-25-15,35;$
$25,17-8-35,33;$	$7,35-26-17,15;$	$8,10-2-28,18;$
$26,28-20-10,36;$	$20,18-8-36,28;$	$2,36-26-18,10;$
$12,15-3-33,18;$	$30,33-21-15,36;$	$21,18-12-36,33;$
$3,36-30-18,15;$	$9,13-4-31,18;$	$27,31-22-13,36;$
$22,18-9-36,31;$	$4,36-27-18,13;$	$11,14-5-32,18;$
$29,32-23-14,36;$	$23,18-11-36,32;$	$5,36-29-18,14.$

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