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Reflexible complete regular dessins and antibalanced skew morphisms of cyclic groups

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Abstract

A skew morphism of a finite group A is a bijection φ on A fixing the identity element of A and for which there exists an integer-valued function π on A such that $\varphi(ab) = \varphi(a)\varphi^{\pi(a)}(b)$, for all $a, b \in A$. In addition, if $\varphi^{-1}(a) = \varphi(a^{-1})^{-1}$, for all $a \in A$, then φ is called antibalanced. In this paper we develop a general theory of antibalanced skew morphisms and establish a one-to-one correspondence between reciprocal pairs of antibalanced skew morphisms of the cyclic additive groups and isomorphism classes of reflexible regular dessins with complete bipartite underlying graphs. As an application, reflexible complete regular dessins are classified.

Keywords: Graph embedding, antibalanced skew morphism, reciprocal pair.

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1 Introduction

A *map* $\mathcal{M}$ is a 2-cell embedding $i : \Gamma \hookrightarrow \mathcal{S}$ of a connected graph Γ, possibly with loops or multiple edges, into a closed surface $\mathcal{S}$ such that each component of $\mathcal{S} \setminus i(\Gamma)$ is homeomorphic to an open disc. A map is *orientable* if its supporting surface $\mathcal{S}$ is orientable,

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otherwise it is *non-orientable*. Throughout the paper, maps considered are all orientable. A map with a 2-coloured bipartite underlying graph is called a *dessin*. An automorphism of a dessin $\mathcal{D}$ is a permutation of the edges of the underlying bipartite graph which preserves the graph structure and vertex colouring, and extends to an orientation-preserving self-homeomorphism of the supporting surface. It is well known that the automorphism group of a dessin acts semi-regularly on its edges. In the case where this action is transitive, and hence regular, the dessin is called *regular* as well.

A regular dessin is *reflexible* if it is isomorphic to its mirror image, otherwise it is called *chiral*. Moreover, a regular dessin is *symmetric* if it has an external symmetry transposing the vertex colors. Thus, a symmetric regular dessin may be viewed as a regular map, that is, a map whose orientation-preserving automorphism group acts transitively on the arcs.

A regular dessin is *complete* if its underlying graph is the complete bipartite graph $K_{m,n}$. Due to its important connection to generalized Fermat curves, the classification problem of complete regular dessins has attracted much attention. A full classification of the symmetric complete regular dessins was obtained in a series of papers [8, 9, 17, 18, 19, 22]. For the general case, complete bipartite graphs which underly a unique regular dessin were determined by Fan and Li [10], and complete regular dessins of odd prime power order have been recently classified by Hu, Nedela and Wang [13]. These results were proved by group-theoretic methods through a translation of complete regular dessins to exact bicyclic groups with two distinguished generators.

Recently, Feng et al discovered an alternative approach to this problem by establishing a surprising correspondence between complete regular dessins and reciprocal pairs of skew morphisms of cyclic groups [12]. A skew morphism of a finite group A is a bijection φ on A fixing the identity element of A and for which there exists an integer function π on A such that $\varphi(ab) = \varphi(a)\varphi^{\pi(a)}(b)$, for all $a, b \in A$. Suppose that φ and $\tilde{\varphi}$ are a pair of skew morphisms of the cyclic additive groups $\mathbb{Z}_n$ and $\mathbb{Z}_m$, and π and $\tilde{\pi}$ are associated power functions, respectively. The skew morphism pair $(\varphi, \tilde{\varphi})$ is called *reciprocal* if they satisfy the following conditions:

(a) the order of φ divides m and the order of $\tilde{\varphi}$ divides n,

(b) $\pi(x) \equiv \tilde{\varphi}^x(1) \pmod{|\varphi|}$ and $\tilde{\pi}(y) \equiv \varphi^y(1) \pmod{|\tilde{\varphi}|}$ for all $x \in \mathbb{Z}_n$ and $y \in \mathbb{Z}_m$.

In [12, Theorem 5] the authors proved that the isomorphism classes of complete regular dessins with underlying graphs $K_{m,n}$ are in one-to-one correspondence with the reciprocal pairs of skew morphisms of the cyclic groups $\mathbb{Z}_n$ and $\mathbb{Z}_m$.

The aim of this paper is to classify the reflexible complete regular dessins. Employing methods used in [12] we are led naturally to introduce a new concept of antibalanced skew morphism. More precisely, a skew morphism of a finite group A is *antibalanced* if $\varphi^{-1}(a) = \varphi(a^{-1})^{-1}$, for all $a \in A$. We note that there is a big difference between antibalanced skew morphisms and skew morphisms arising from antibanlanced regular Cayley maps, because the latter have a generating orbit which is closed under taking inverses [4, 16], while the former do not have such a restriction.

In Section 3 we develop a general theory of antibalanced skew morphisms, and present a classification and enumeration of antibalanced skew morphisms of cyclic groups, extending the results obtained by Conder, Jajcay and Tucker [4, Theorem 7.1]. In Section 4, we establish a one-to-one correspondence between reflexible complete regular dessins and reciprocal pairs of antibalanced skew morphisms of cyclic groups. In Section 5 all reciprocal pairs of antibalanced skew morphisms of cyclic groups are completely determined.

2 Preliminaries

The theory of skew morphisms has been developed and expanded by various authors. In this section we summarize and prove some preliminary results for future reference.

Let φ be a skew morphism of a finite group A, and let π be a power function of φ. In general, the function π is not uniquely determined by φ. However, if φ has order k, then π may be regarded as a function $\pi : A \to \mathbb{Z}_k$, which is unique. In this case we will refer to π as *the power function of* φ. A subgroup N of A is φ*-invariant* if $\varphi(N) = N$, in which case the restriction of φ to N is a skew morphism of N. Moreover, it is well known [16] that $\operatorname{Ker}\varphi$ and $\operatorname{Fix}\varphi$ defined by

$$\operatorname{Ker}\varphi = \{a \in A \mid \pi(a) = 1\} \quad \text{and} \quad \operatorname{Fix}\varphi = \{a \in A \mid \varphi(a) = a\}$$

are subgroups of A, and in particular, $\operatorname{Fix}\varphi$ is φ-invariant. Note that, for any two elements $a, b \in A$, $\pi(a) = \pi(b)$ if and only if $ab^{-1} \in \operatorname{Ker}\varphi$, so the power function π of φ takes exactly $|A : \operatorname{Ker}\varphi|$ distinct values in $\mathbb{Z}_k$. The index $|A : \operatorname{Ker}\varphi|$ will be called the *skew type* of φ. It follows that a skew morphism of A is an automorphism if and only if it has skew type 1. A skew morphism which is not an automorphism will be called a *proper* skew morphism.

Moreover, define

$$\operatorname{Core}\varphi = \bigcap_{i=1}^{k} \varphi^i(\operatorname{Ker}\varphi).$$

Then $\operatorname{Core}\varphi$ is a φ-invariant normal subgroup of A, and it it is the largest φ-invariant subgroup of A contained in $\operatorname{Ker}\varphi$. In particular, if A is abelian, then $\operatorname{Core}\varphi = \operatorname{Ker}\varphi$ [4, Lemma 5.1].

The following properties of skew morphisms are fundamental.

Proposition 2.1 ([16]). *Let φ be a skew morphism of a finite group A, let π be the power function of φ, and let k be the order of φ. Then the following hold:*

(a) *for any positive integer ℓ and for any $a, b \in A$, $\varphi^\ell(ab) = \varphi^\ell(a)\varphi^{\sigma(a,\ell)}(b)$, where $\sigma(a, \ell) = \sum_{i=1}^{\ell} \pi(\varphi^{i-1}(a))$;*

(b) *for all $a, b \in A$, $\pi(ab) \equiv \sum_{i=1}^{\pi(a)} \pi(\varphi^{i-1}(b)) \pmod{k}$.*

Proposition 2.2 ([1]). *Let φ be a skew morphism of a finite group A, let π be the power function of φ, and let k be the order of φ. Then $\mu = \varphi^\ell$ is a skew morphism of A if and only if the congruence $\ell x \equiv \sigma(a, \ell) \pmod{k}$ is soluble for every $a \in A$, in which case $\pi_\mu(a)$ is the solution*

Proposition 2.3 ([15]). *If φ is a skew morphism of a finite group A, then $O_a^{-1} = O_{a^{-1}}$ for any $a \in A$, where O_a and $O_{a^{-1}}$ denote the orbits of φ containing a and a^{-1}, respectively.*

Proposition 2.4 ([1, 25]). *Let φ be a skew morphism of a finite group A, and let π be the power function of φ. Then for any automorphism θ of A, $\mu = \theta^{-1}\varphi\theta$ is a skew morphism of A with power function $\pi_\mu = \pi\theta$.*

Proposition 2.5 ([26, 25])**.** *Let φ be a skew morphism of a finite group A, and let π be the power function of φ. If $A = \langle a_1, a_2, \dots, a_r \rangle$, then $|\varphi| = \mathrm{lcm}(|O_{a_1}|, |O_{a_2}|, \dots, |O_{a_r}|)$. Moreover, the skew morphism φ and its power function π are completely determined by the action of φ and the values of π on the generating orbits $O_{a_1}, O_{a_2}, \dots, O_{a_r}$, respectively.*

Proposition 2.6 ([26])**.** *Let φ be a skew morphism of a finite group A, and let π be the power function of φ. If N is a φ-invariant normal subgroup of A, then $\bar{\varphi}$ defined by $\bar{\varphi}(\bar{x}) = \overline{\varphi(x)}$ is a skew morphism of $\bar{A} := A/N$ and the power function $\bar{\pi}$ of $\bar{\varphi}$ is determined by $\bar{\pi}(\bar{x}) \equiv \pi(x) \pmod{|\bar{\varphi}|}$.*

Since $\mathrm{Core}\,\varphi$ is a φ-invariant normal subgroup of A, by Proposition 2.6, φ induces a skew morphism $\bar{\varphi}$ of $\bar{A} = A/\mathrm{Core}\,\varphi$. It is shown in [25] that the $\bar{\varphi}$-invariant subgroup $\mathrm{Fix}\,\bar{\varphi}$ of $\bar{A}$ lifts to a φ-invariant subgroup $\mathrm{Smooth}\,\varphi$ of A, namely,

$$\mathrm{Smooth}\,\varphi = \{a \in A \mid \bar{\varphi}(\bar{a}) = \bar{a}\}.$$

In particular, if $\mathrm{Smooth}\,\varphi = A$, then the skew morphism φ is called a *smooth* skew morphism.

Proposition 2.7 ([25])**.** *Let φ be a skew morphism of a finite group A, and let π be the power function of φ. Then φ is smooth if and only if $\pi(\varphi(a)) = \pi(a)$ for all $a \in A$.*

The most important properties of smooth skew morphisms are summarized as follows.

Proposition 2.8 ([25])**.** *Let φ be a smooth skew morphism of A, $|\varphi| = k$, and let π be the power function of φ. Then the following hold:*

(a) *$\varphi(\mathrm{Ker}\,\varphi) = \mathrm{Ker}\,\varphi$;*

(b) *$\pi : A \to \mathbb{Z}_k$ is a group homomorphism of A into the multiplicative group $\mathbb{Z}_k^*$ with $\mathrm{Ker}\,\pi = \mathrm{Ker}\,\varphi$;*

(c) *for any φ-invariant normal subgroup N of A, the induced skew morphism $\bar{\varphi}$ on A/N is also smooth, and in particular, if $N = \mathrm{Ker}\,\varphi$ then $\bar{\varphi}$ is the identity permutation;*

(d) *for any positive integer ℓ, $\mu = \varphi^\ell$ is a smooth skew morphism of A;*

(e) *for any automorphism θ of A, $\mu = \theta^{-1}\varphi\theta$ is a smooth skew morphism of A.*

Lemma 2.9. *Let φ be a skew morphism of a finite group A. Then φ is smooth if and only if there exists a φ-invariant normal subgroup N of A contained in $\mathrm{Ker}\,\varphi$ such that the induced skew morphism $\bar{\varphi}$ of $\bar{A} = A/N$ is the identity permutation.*

Proof. If φ is smooth, then by Proposition 2.8(a), φ is kernel-preserving, and so $\mathrm{Ker}\,\varphi = \mathrm{Core}\,\varphi$. Take $N = \mathrm{Ker}\,\varphi$, then by Proposition 2.8(c) the induced skew morphism $\bar{\varphi}$ of A/N is the identity permutation.

Conversely, suppose that there exists a φ-invariant normal subgroup N of A contained in $\mathrm{Ker}\,\varphi$ such that the induced skew morphism $\bar{\varphi}$ of $\bar{A} = A/N$ is the identity permutation. Then, for each $a \in A$, there is an element $u \in N \le \mathrm{Ker}\,\varphi$ such that $\varphi(a) = ua$. Thus, $\pi(\varphi(a)) = \pi(a)$, and therefore φ is smooth by Proposition 2.7. $\square$

There is a fundamental correspondence between skew morphisms and groups with cyclic complements.

Proposition 2.10 ([5]). *If $G = AC$ is a factorisation of a finite group G into a product of a subgroup A and a cyclic subgroup $C = \langle c\rangle$ with $A \cap C = 1$, then c induces a skew morphism φ of A via the commuting rule $ca = \varphi(a)c^{\pi(a)}$, for all $a \in A$; in particular $|\varphi| = |C : C_G|$, where $C_G = \cap_{g\in G} C^g$.*

Conversely, if φ is a skew morphism of a finite group A, then $G = L_A\langle\varphi\rangle$ is a transitive permutation group on A with $L_A \cap \langle\varphi\rangle = 1$ and $\langle\varphi\rangle$ core-free in G, where L_A is the left regular representation of A.

3 Antibalanced skew morphisms

In this section we develop a theory of antibalanced skew morphisms and classify all antibalanced skew morphisms of cyclic groups.

A skew morphism φ of a finite group A will be called *antibalanced* if

$$\varphi^{-1}(a) = \varphi(a^{-1})^{-1}, \quad \text{for all } a \in A.$$

Since $1 = \varphi(aa^{-1}) = \varphi(a)\varphi^{\pi(a)}(a^{-1})$, we have $\varphi(a)^{-1} = \varphi^{\pi(a)}(a^{-1})$. Thus, φ is antibalanced if and only if $\varphi^{\pi(a)}(a^{-1}) = \varphi^{-1}(a^{-1})$, or equivalently, $\pi(a) \equiv -1 \pmod{|O_{a^{-1}}|}$, for all $a \in A$. By Proposition 2.3, $|O_a| = |O_{a^{-1}}|$. It follows that φ is antiblanced if and only if $\pi(a) \equiv -1 \pmod{|O_a|}$ for all $a \in A$. Note that for any $a \in \operatorname{Ker}\varphi$, $|O_a|$ is 1 or 2.

Remark 3.1. It was proved in [16, Theorem 1] that a Cayley map $\mathrm{CM}(A, X, p)$ is regular (on the arcs) if and only if there is a skew morphism φ of A such that the restriction of φ to X is equal to p. Since X is a generating set of A and is closed under taking inverses, the associated skew morphism φ has a generating orbit which is closed under taking inverses. For brevity, such a skew morphism will be called a *Cayley* skew morphism.

Moreover, a regular Cayley map $\mathrm{CM}(A, X, p)$ was termed antibalanced if $p^{-1}(x) = p(x^{-1})^{-1}$ for all $x \in X$ [24]. It follows that a regular Cayley map is antibalanced if and only if the associated Cayley skew morphism is antibalanced. However, neither generating orbit, nor inverse-closed orbit are assumed in the preceding definition of antibalanced skew morphisms. Therefore, antibalanced skew morphisms may be regarded as a natural generalization of the skew morphisms arising from antibalanced regular Cayley maps.

We give an example to clarify the concept.

Example 3.2. The cyclic group $\mathbb{Z}_{12}$ has exactly eight skew morphisms, four of which are proper:

$$\begin{aligned}
\varphi &= (0)(2)(4)(6)(8)(10)(1,3,5,7,9,11), & \pi_\varphi &= [1][1][1][1][1][1][5,5,5,5,5,5];\\
\psi &= (0)(2)(4)(6)(8)(10)(1,11,9,7,5,3), & \pi_\psi &= [1][1][1][1][1][1][5,5,5,5,5,5];\\
\mu &= (0)(2)(4)(6)(8)(10)(1,5,9)(3,7,11), & \pi_\mu &= [1][1][1][1][1][1][2,2,2][2,2,2];\\
\gamma &= (0)(2)(4)(6)(8)(10)(1,9,5)(3,11,7), & \pi_\gamma &= [1][1][1][1][1][1][2,2,2][2,2,2].
\end{aligned}$$

It is easily seen that all the above skew morphisms are antibalanced. Note that the first two skew morphisms contain a generating orbit closed under taking inverses, but the last two skew morphisms do not contain such an orbit. Therefore, φ and ψ are antibalanced Cayley skew morphism, and μ and γ are antibalanced non-Cayley skew morphisms.

We summarise some properties of antibalanced skew morphisms as follows.

Lemma 3.3. *Let φ be an antibalanced skew morphism of a finite group A, and let π be the associated power function. Then the following hold:*

(a) *for any positive integer ℓ, $\varphi^{-\ell}(a) = \varphi^{\ell}(a^{-1})^{-1}$ for all $a \in A$;*

(b) *for any automorphism θ of A, the skew morphism $\mu = \theta^{-1}\varphi\theta$ is antibalanced;*

(c) *for any φ-invariant normal subgroup N of A, the induced skew morphism $\bar{\varphi}$ of A/N is antibalanced;*

(d) *for any $c \in \operatorname{Ker}\varphi$ and $a \in A$, $\pi(a) \equiv 1 \pmod{|O_c|}$.*

Proof. (a) The case $\ell = 1$ is the definition. Assume the result for ℓ, i.e. $\varphi^{-\ell}(a) = \varphi^{\ell}(a^{-1})^{-1}$ for all $a \in A$. Then

$$\varphi^{-(\ell+1)}(a) = \varphi^{-1}(\varphi^{-\ell}(a)) = \varphi^{-1}(\varphi^{\ell}(a^{-1})^{-1}) = \varphi(\varphi^{\ell}(a^{-1}))^{-1} = \varphi^{\ell+1}(a^{-1})^{-1},$$

and the result follows by induction.

(b) For any $a \in A$, we have

$$\mu^{-1}(a) = \theta^{-1}\varphi^{-1}\theta(a) = \theta^{-1}(\varphi(\theta(a)^{-1}))^{-1}) = \big(\theta^{-1}(\varphi(\theta(a^{-1})))\big)^{-1} = \mu(a^{-1})^{-1},$$

so μ is antibalanced.

(c) Since $\varphi^{-1}(a) = \varphi(a^{-1})^{-1}$, we have $\bar{\varphi}^{-1}(\bar{a}) = \bar{\varphi}(\bar{a}^{-1})^{-1}$, and so $\bar{\varphi}$ is antibalanced.

(d) For any $c \in \operatorname{Ker}\varphi$ and any $a \in A$, we have

$$\begin{aligned}
\varphi(c)a^{-1} &= \varphi(c)[\varphi^{-1}(\varphi(a))]^{-1} = \varphi(c)\varphi(\varphi(a)^{-1}) = \varphi(c\varphi(a)^{-1}) \\
&= \varphi^{-1}(\varphi(a)c^{-1})^{-1} = \big(\varphi^{-1}(\varphi(a))\varphi^{-\pi\varphi^{-1}(\varphi(a))}(c^{-1})\big)^{-1} \\
&= (a\varphi^{-\pi(a)}(c^{-1}))^{-1} = (a\varphi^{\pi(a)}(c)^{-1})^{-1} = \varphi^{\pi(a)}(c)a^{-1},
\end{aligned}$$

so $\varphi^{\pi(a)}(c) = \varphi(c)$, and hence $\pi(a) \equiv 1 \pmod{|O_c|}$. □

Lemma 3.4. *Let φ be an automorphism of a finite group A. Then φ is antibalanced if and only if $\varphi^2 = 1$, that is, φ is an involutory automorphism.*

Proof. By definition, φ is antibalanced if and only if for all $a \in A$, $\varphi^{-1}(a) = \varphi(a^{-1})^{-1}$. Since φ is an automorphism, $\varphi(a^{-1})^{-1} = \varphi(a)$, and hence φ is antibalanced if and only if for all $a \in A$, $\varphi^{-1}(a) = \varphi(a)$, that is, $\varphi^2(a) = a$. □

Corollary 3.5. *Every antibalanced automorphism of the cyclic additive group $\mathbb{Z}_n$ is of the form $\varphi(x) = sx$, $x \in \mathbb{Z}_n$, where $s^2 \equiv 1 \pmod{n}$.*

Let φ be a skew morphism of a finite group A. Suppose that φ has an orbit X generating A. The words of even length in the generators from X form a subgroup of A, which will be called the *even word subgroup* of A with respect to X and denoted by A_X^+. Note that the index of A_X^+ in A is 1 or 2.

The following results generalize the properties of antibalanced Cayley skew morphisms (or more precisely, antibalanced regular Cayley maps) obtained in [4]

Lemma 3.6. *Let φ be a skew morphism of a finite group A containing an orbit X which generates A, and let π be the associated power function. Then φ is antibalanced if and only if $\pi(x) \equiv -1 \pmod{|X|}$ for all $x \in X$ and φ restricted to A_X^+ is an involutory automorphism. Furthermore, if φ is antibalanced, then φ is a smooth skew morphism of skew type 1 or 2.*

Proof. Since X is a generating orbit of φ, we have $|\varphi| = |X|$ by Proposition 2.5, and the value of π on A is completely determined by the value of π on X. Suppose that $\pi(x) \equiv -1 \pmod{|X|}$ for all $x \in X$. Then by Proposition 2.1, for any $x, y \in X$,

$$\pi(xy) = \sum_{i=1}^{\pi(x)} \pi(\varphi^{i-1}(y)) \equiv \pi(x)\pi(y) \equiv 1 \pmod{|X|}.$$

Since $A = \langle X \rangle$, every element of A is expressible as a word of finite length in the elements of X. By induction, $\pi(a) \equiv 1 \pmod{|X|}$ if a is an even word, and $\pi(a) \equiv -1 \pmod{|X|}$ if a is an odd word. Note that if a is an even word (resp. an odd word), then both $\varphi(a)$ and a^{-1} are also even words (resp. odd words). Thus, φ is a smooth skew morphism of skew type 1 or 2 and φ restricted to A_X^+ is an automorphism of A_X^+.

If φ is antibalanced, then it is evident that $\pi(x) \equiv -1 \pmod{|X|}$ for all $x \in X$. This implies that φ restricted to A_X^+ is an automorphism of A_X^+, and hence, for any $a \in A_X^+$, $\varphi(a^{-1}) = \varphi(a)^{-1}$. Since φ is antibalanced, we have $\varphi(a^{-1})^{-1} = \varphi^{-1}(a)$, and hence, for any $a \in A_X^+$, $\varphi^{-1}(a) = \varphi(a)$. Therefore, φ restricted to A_X^+ is an involutory automorphism.

Conversely, assume that $\pi(x) \equiv -1 \pmod{|X|}$ for all $x \in X$ and φ restricted to A_X^+ is an involutory automorphism. For any even word $a \in A_X^+$, $\varphi^{-1}(a) = \varphi(a) = \varphi(a^{-1})^{-1}$. For any odd word b,

$$1 = \varphi(b^{-1}b) = \varphi(b^{-1})\varphi^{\pi(b^{-1})}(b) = \varphi(b^{-1})\varphi^{-1}(b),$$

and so $\varphi^{-1}(b) = \varphi(b^{-1})^{-1}$. Therefore, φ is antibalanced. □

Remark 3.7. Let φ be a skew morphism of a finite group A containing an orbit X which generates A, and let π be the associated power function. Lemma 3.6 implies that

(a) if $|A : A_X^+| = 1$, then the skew morphism φ is antibalanced if and only if φ is an involutory automorphism of A;

(b) if $|A : A_X^+| = 2$, then φ is antibalanced if and only if $\pi(a) \equiv 1 \pmod{|\varphi|}$ for all $a \in A_X^+$, $\pi(a) \equiv -1 \pmod{|\varphi|}$ for all $a \in A \setminus A_X^+$ and φ restricted to A_X^+ is an involutory automorphism.

The following lemma deals with antibalanced skew morphisms of abelian groups.

Lemma 3.8. *Let φ be a skew morphism of a finite abelian group A containing an orbit X which generates A, and let π be the associated power function. Then φ is antibalanced if and only if $\pi(x) \equiv -1 \pmod{|X|}$ for all $x \in X$.*

Proof. By Lemma 3.6, it suffices to prove the sufficient part. Assume that $\pi(x) \equiv -1 \pmod{|X|}$ for all $x \in X$. Then, $\pi(a) \equiv 1 \pmod{|X|}$ if a is an even word, and $\pi(a) \equiv -1$

$\pmod{|X|}$ if a is an odd word. Furthermore φ restricted to A_X^+ is an automorphism of A_X^+. For any $a \in A_X^+$ and for any odd word b,

$$\varphi(b)\varphi^{-1}(a) = \varphi(ba) = \varphi(ab) = \varphi(a)\varphi(b) = \varphi(b)\varphi(a),$$

and hence $\varphi^2(a) = a$. Thus, by Lemma 3.6, φ is antibalanced. $\square$

Now we are ready to determine antibalanced skew morphisms of cyclic groups.

Theorem 3.9. *Let φ be an antibalanced skew morphism of the cyclic additive group $\mathbb{Z}_n$.*

(a) *If n is odd, then φ is an involutory automorphism of the form $\varphi(x) = sx$, $x \in \mathbb{Z}_n$, where $s^2 \equiv 1 \pmod{n}$.*

(b) *If n is even, then φ and the associated power function π are of the form*

$$\varphi(x) = \begin{cases} xs, & x \text{ is even}, \\ (x-1)s + 2r + 1, & x \text{ is odd}, \end{cases} \quad \textit{and} \quad \pi(x) = \begin{cases} 1, & x \text{ is even}, \\ -1, & x \text{ is odd}, \end{cases} \tag{3.1}$$

where r, s are integers in $\mathbb{Z}_{n/2}$ such that

$$s^2 \equiv 1 \pmod{n/2} \quad \textit{and} \quad (r+1)(s-1) \equiv 0 \pmod{n/2}. \tag{3.2}$$

In this case, the order of φ is equal to $n/\gcd(n, r(s+1))$, and in particular φ is an automorphism if and only if $s \equiv 2r+1 \pmod{n/2}$ and $(2r+1)^2 \equiv 1 \pmod{n}$.

Proof. First suppose that φ is an antibalanced skew morphism of $\mathbb{Z}_n$ with the associated power function π. Note that the orbit X of φ containing 1 generates $\mathbb{Z}_n$. Let $\mathbb{Z}_n^+$ be the even word subgroup of $\mathbb{Z}_n$ with respect to X. Then $|\mathbb{Z}_n : \mathbb{Z}_n^+| = 1$ or 2. By Lemma 3.6, φ restricted to $\mathbb{Z}_n^+$ is an involutory automorphism.

If n is odd, then $\mathbb{Z}_n^+ = \mathbb{Z}_n$, so φ is an involutory automorphism of $\mathbb{Z}_n$, and the result follows from Corollary 3.5.

Now assume that n is an even number. By Lemma 3.6, φ is a smooth skew morphism of skew type 1 or 2, so $\langle 2\rangle$ is a φ-invariant normal subgroup of $\mathbb{Z}_n$ contained in $\operatorname{Ker}\varphi$, and the induced skew morphism $\bar{\varphi}$ of $\mathbb{Z}_n/\langle 2\rangle$ is the identity permutation. Thus, there are integers $r, s \in \mathbb{Z}_{n/2}$ such that

$$\varphi(1) \equiv 2r+1 \pmod{n} \quad \text{and} \quad \varphi(2) \equiv 2s \pmod{n},$$

where $\gcd(s, n/2) = 1$. By Lemma 3.6,

$$\pi(x) = 1 \text{ if } x \text{ is even}, \quad \pi(x) = -1 \text{ if } x \text{ is odd}.$$

It follows that

$$\varphi(x) = \begin{cases} xs, & x \text{ is even}, \\ \varphi(x-1) + \varphi(1) = (x-1)s + 2r + 1, & x \text{ is odd}. \end{cases} \tag{3.3}$$

Since φ restricted to $\mathbb{Z}_n^+$ is an involutory automorphism, we have $s^2 \equiv 1 \pmod{n/2}$. Furthermore, we have

$$2s = \varphi(2) = \varphi(1) + \varphi^{-1}(1) = 2r + 1 - 2rs + 1 \pmod{n},$$

and hence $(r+1)(s-1) \equiv 0 \pmod{n/2}$. Moreover, by induction we have

$$\varphi^j(1) \equiv 1 + 2r \sum_{i=1}^{j} s^{i-1} \pmod{n}.$$

Let k be the smallest positive integer such that $\varphi^k(1) = 1$. By Proposition 2.5, $k = |\varphi|$. If k is odd, then $s \equiv 1 \pmod{n/2}$, since the length of the orbit of φ containing 2 divides k. Upon substitution we get $k = n/\gcd(n, 2r)$. If k is even, then the congruence $2r\sum_{i=1}^{k} s^{i-1} \equiv 0 \pmod{n}$ reduces to $rk(s+1) \equiv 0 \pmod{n}$, so $k = n/\gcd(n, r(s+1))$. Note that in either cases $k = n/\gcd(n, r(s+1))$. In particular, if φ is an automorphism, then for any $x \in \mathbb{Z}_n$, $\varphi(x) = x\varphi(1) = x(2r+1)$ and $(2r+1)^2 \equiv 1 \pmod{n}$.

Conversely, we need to verify that φ given by (3.1) is an antibalanced skew morphism of $\mathbb{Z}_n$, provided that the stated numerical conditions in (3.2) are fulfilled. It is easily seen that $\varphi(0) = 0$ and φ is a bijection on $\mathbb{Z}_n$. Now for any $x \in \mathbb{Z}_n$ and for any $y \in \mathbb{Z}_n$, if x is even, then one can easily show that

$$\varphi(x) + \varphi(y) = \varphi(x+y).$$

If x is odd and y is even, then

$$\varphi(x) + \varphi^{-1}(y) = (x-1)s + 2r + 1 + ys = (x+y-1)s + 2r + 1 = \varphi(x+y).$$

Finally, if both x and y are odd, then $\varphi(-2rs + (y-1)s + 1) = y$, and so $\varphi^{-1}(y) = -2rs + (y-1)s + 1$. From the condition $(r+1)(s-1) \equiv 0 \pmod{n/2}$ we deduce that $-2rs \equiv 2s - 2r - 2 \pmod{n}$ and hence $\varphi^{-1}(y) = (y+1)s - 2r - 1$. Consequently,

$$\varphi(x) + \varphi^{-1}(y) = (x-1)s + 2r + 1 + (y+1)s - 2r - 1 = (x+y)s = \varphi(x+y).$$

Therefore, φ is a skew morphism of $\mathbb{Z}_n$. By Lemma 3.8, it is antibalanced. □

From the proof of Theorem 3.9 we obtain the following corollary.

Corollary 3.10. *Let φ be an antibalanced skew morphism of $\mathbb{Z}_n$. If φ is of odd order, then the restriction of φ to* Ker φ *is the identity automorphism of* Ker φ.

Theorem 3.11. *Let $n = 2^{\alpha} p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_\ell^{\alpha_\ell}$ be the prime power factorization of a positive integer n. Then the number $\nu(n)$ of antibalanced skew morphisms of the cyclic additive group $\mathbb{Z}_n$ is determined by the following formula:*

$$\nu(n) = \begin{cases} 2^\ell, & \alpha = 0, \\ \prod_{i=1}^{\ell} (p_i^{\alpha_i} + 1), & \alpha = 1, \\ 2\prod_{i=1}^{\ell} (p_i^{\alpha_i} + 1), & \alpha = 2, \\ 6\prod_{i=1}^{\ell} (p_i^{\alpha_i} + 1), & \alpha = 3, \\ (4 + 2^{\alpha-2} + 2^{\alpha-1})\prod_{i=1}^{\ell} (p_i^{\alpha_i} + 1), & \alpha \geq 4. \end{cases}$$

Proof. If $\alpha = 0$, then n is odd. By Theorem 3.9(a), every antibalanced skew morphism of $\mathbb{Z}_n$ is an automorphism of the form $\varphi(x) = xs$, $x \in \mathbb{Z}_n$, where $s^2 \equiv 1 \pmod{n}$. It follows that the number $\nu(n)$ is equal to the number of solutions of the quadratic congruence $s^2 \equiv 1 \pmod{n}$, which is equal to 2^ℓ.

Now assume $\alpha \geq 1$, so n is an even number. By Theorem 3.9(b), the number of antibalanced skew morphisms of $\mathbb{Z}_n$ is equal to the number of integer solutions (r, s) in $\mathbb{Z}_{n/2}$ of the system

$$\begin{cases} s^2 \equiv 1 \pmod{n/2}, \\ (r+1)(s-1) \equiv 0 \pmod{n/2}. \end{cases}$$

By the Chinese Remainder Theorem, (r, s) is a solution of the system if and only if it is a solution of each of the following $\ell + 1$ systems

$$\begin{cases} s^2 \equiv 1 \pmod{2^{\alpha-1}}, \\ (r+1)(s-1) \equiv 0 \pmod{2^{\alpha-1}} \end{cases} \tag{3.4}$$

and

$$\begin{cases} s^2 \equiv 1 \pmod{p_i^{\alpha_i}}, \\ (r+1)(s-1) \equiv 0 \pmod{p_i^{\alpha_i}}, \end{cases} \quad i = 1, 2, \dots, \ell. \tag{3.5}$$

We first determine the solutions of (3.5). By assumption, for each i, $i = 1, 2, \dots, \ell$, p_i is an odd prime. It follows from the congruence $s^2 \equiv 1 \pmod{p_i^{\alpha_i}}$ that either $s \equiv 1 \pmod{p_i^{\alpha_i}}$ or $s \equiv -1 \pmod{p_i^{\alpha_i}}$. If $s \equiv 1 \pmod{p_i^{\alpha_i}}$, then upon substitution the congruence $(r+1)(s-1) \equiv 0 \pmod{p_i^{\alpha_i}}$ holds for every $r \in \mathbb{Z}_{p_i^{\alpha_i}}$. On the other hand, if $s \equiv -1 \pmod{p_i^{\alpha_i}}$, then upon substitution the congruence $(r+1)(s-1) \equiv 0 \pmod{p_i^{\alpha_i}}$ reduces to $r \equiv -1 \pmod{p_i^{\alpha_i}}$. Therefore, for each odd prime p_i, the system (3.5) has precisely $(p_i^{\alpha_i} + 1)$ solutions in $\mathbb{Z}_{p_i^{\alpha_i}}$.

Now we turn to solutions of (3.4). If $\alpha = 1$, then it only has the trivial solution $(r, s) = (1, 1)$. If $\alpha = 2$, then $(r, s) = (0, 1), (1, 1)$ in $\mathbb{Z}_2$. If $\alpha = 3$, then $(r, s) = (0, 1), (1, 1), (2, 1), (3, 1), (1, 3), (3, 3)$ in $\mathbb{Z}_4$. If $\alpha \geq 4$, then by the congruence $s^2 \equiv 1 \pmod{2^{\alpha-1}}$ we have $s \equiv \pm 1, 2^{\alpha-2} \pm 1 \pmod{2^{\alpha-1}}$. Combining this with the congruence $(r+1)(s-1) \equiv 0 \pmod{2^{\alpha-1}}$ we obtain the following solutions (r, s) in $\mathbb{Z}_{2^{\alpha-1}}$: (a) $r \in \mathbb{Z}_{2^{\alpha-1}}$ and $s = 1$; (b) $r = 2^{\alpha-1} - 1, 2^{\alpha-2} - 1$ and $s = -1$; (c) $r = 2^{\alpha-1} - 1, 2^{\alpha-2} - 1$ and $s = 2^{\alpha-2} - 1$; (d) $r \equiv 1 \pmod{2}$ and $s = 2^{\alpha-2} + 1$.

Finally, multiplying the numbers of solutions for the prime power cases we obtain the number $\nu(n)$, as required. □

4 Correspondence

A correspondence between complete regular dessins and pairs of certain skew morphisms of cyclic groups has been established in [12, Theorem 5]. In this section we extend the correspondence to reflexible complete regular dessins.

Definition 4.1. Let $\varphi : \mathbb{Z}_n \to \mathbb{Z}_n$ and $\tilde{\varphi} : \mathbb{Z}_m \to \mathbb{Z}_m$ be a pair of skew morphisms of the cyclic additive groups $\mathbb{Z}_n$ and $\mathbb{Z}_m$, associated with power functions $\pi : \mathbb{Z}_n \to \mathbb{Z}_{|\varphi|}$ and $\tilde{\pi} : \mathbb{Z}_m \to \mathbb{Z}_{|\tilde{\varphi}|}$, respectively. The pair $(\varphi, \tilde{\varphi})$ will be called *reciprocal* if they satisfy the following conditions:

(a) $|\varphi|$ divides m and $|\tilde{\varphi}|$ divides n,

(b) $\pi(x) \equiv \tilde{\varphi}^x(1) \pmod{|\varphi|}$ and $\tilde{\pi}(y) \equiv \varphi^y(1) \pmod{|\tilde{\varphi}|}$ for all $x \in \mathbb{Z}_n$ and $y \in \mathbb{Z}_m$.

Suppose that $\mathcal{D}$ is a complete regular dessin with underling graph $K_{m,n}$. Take an arbitrary pair of vertices u and v of valency m and n, respectively. Then the stabilizers G_u and G_v of $G = \mathrm{Aut}(\mathcal{D})$ are cyclic of orders m and n, respectively. Assume $G_u = \langle a \rangle$ and $G_v = \langle b \rangle$. Then by the regularity we have $G = \langle a, b \rangle$ and $|G| = mn$. Since the underlying graph $K_{m,n}$ is simple, $\langle a \rangle \cap \langle b \rangle = 1$. Consequently, from the product formula we deduce that $G = \langle a \rangle \langle b \rangle$. Thus each complete regular dessin determines a triple (G, a, b) such that $G = \langle a \rangle \langle b \rangle$ and $\langle a \rangle \cap \langle b \rangle = 1$.

Now each of the cyclic factors $\langle a \rangle$ and $\langle b \rangle$ of G can be taken as the complement of the other, so in the spirit of Proposition 2.10, there are a pair of skew morphisms φ and $\tilde{\varphi}$ of the cyclic additive groups $\mathbb{Z}_n$ and $\mathbb{Z}_m$ such that

$$a^{-1}b^x = b^{\varphi(x)}a^{-\pi(x)} \quad \text{and} \quad b^{-1}a^y = a^{\tilde{\varphi}(y)}b^{-\tilde{\pi}(y)} \tag{4.1}$$

where $x \in \mathbb{Z}_n$ and $y \in \mathbb{Z}_m$. By induction we deduce that

$$a^{-k}b^x = b^{\varphi^k(x)}a^{-\sigma(x,k)} \quad \text{and} \quad b^{-l}a^y = a^{\tilde{\varphi}^l(y)}b^{-\tilde{\sigma}(y,l)}, \tag{4.2}$$

where

$$\sigma(x, k) = \sum_{i=1}^{k} \pi(\varphi^{i-1}(x)) \quad \text{and} \quad \tilde{\sigma}(y, l) = \sum_{i=1}^{l} \tilde{\pi}(\tilde{\varphi}^{i-1}(y)).$$

Inverting the above identities yields

$$b^{-x}a^k = a^{\sigma(x,k)}b^{-\varphi^k(x)} \quad \text{and} \quad a^{-y}b^l = b^{\tilde{\sigma}(y,l)}a^{-\tilde{\varphi}^l(y)}. \tag{4.3}$$

Substituting for $x = 1$ and $k = y$ we obtain $b^{-1}a^y = a^{\sigma(1,y)}b^{-\varphi^y(1)}$. By comparing this with the second identity in (4.1) we obtain

$$\tilde{\pi}(y) \equiv \varphi^y(1) \pmod{n} \quad \text{and} \quad \tilde{\varphi}(y) \equiv \sigma(1, y) \pmod{m}.$$

Similarly, inserting $y = 1$ and $l = x$ into the second identity in (4.3) we have $a^{-1}b^x = b^{\tilde{\sigma}(1,x)}a^{-\tilde{\varphi}^x(1)}$. A similar comparison with the first identity in (4.1) yields

$$\pi(x) \equiv \tilde{\varphi}^x(1) \pmod{m} \quad \text{and} \quad \varphi(x) \equiv \tilde{\sigma}(1, x) \pmod{n}.$$

By Proposition 2.10, $|\varphi| = |\langle a \rangle : \langle a \rangle_G|$ and $|\tilde{\varphi}| = |\langle b \rangle : \langle b \rangle_G|$. Thus $|\varphi|$ divides m and $|\tilde{\varphi}|$ divides n. In particular, the above four congruences are reduced to

$$\pi(x) \equiv \tilde{\varphi}^x(1) \pmod{|\varphi|}, \quad \tilde{\pi}(y) \equiv \varphi^y(1) \pmod{|\tilde{\varphi}|}$$

and

$$\varphi(x) \equiv \tilde{\sigma}(1, x) \pmod{|\tilde{\varphi}|}, \quad \tilde{\varphi}(y) \equiv \sigma(1, y) \pmod{|\varphi|}. \tag{4.4}$$

It follows that every complete regular dessin with underlying graph $K_{m,n}$ determines a pair of reciprocal skew morphisms $(\varphi, \tilde{\varphi})$ of the cyclic additive groups $\mathbb{Z}_n$ and $\mathbb{Z}_m$. Conversely, it is shown in [12, Proposition 4] that given a pair of reciprocal skew morphisms of the cyclic groups $\mathbb{Z}_n$ and $\mathbb{Z}_m$, a complete regular dessin with underlying graph $K_{m,n}$ may be reconstructed in a canonical way. Therefore, we obtain a correspondence between complete regular dessins and pairs of reciprocal skew morphisms of cyclic groups. See [12, Theorem 5] for details.

The following theorem extends this correspondence to reflexible complete regular dessins.

Theorem 4.2. *The isomorphism classes of reflexible regular dessins with complete bipartite underlying graphs $K_{m,n}$ are in one-to-one correspondence with pairs of reciprocal antibalanced skew morphisms $(\varphi, \tilde{\varphi})$ of the cyclic groups $\mathbb{Z}_n$ and $\mathbb{Z}_m$.*

Proof. It is proved in [12, Theorem 5] that the isomorphism classes of regular dessins $\mathcal{D} = (G, a, b)$ with complete bipartite underlying graphs $K_{m,n}$ are in one-to-one correspondence with the pairs $(\varphi, \tilde{\varphi})$ of reciprocal skew morphisms of the cyclic groups $\mathbb{Z}_n$ and $\mathbb{Z}_m$. It remains to show that $\mathcal{D}$ is reflexible if and only if the corresponding pair of skew morphisms $(\varphi, \tilde{\varphi})$ are both antibalanced.

First suppose that $\mathcal{D} = (G, a, b)$ is reflexible, then the identities in (4.1) determine a pair of reciprocal skew morphisms $(\varphi, \tilde{\varphi})$ of the cyclic groups $\mathbb{Z}_n$ and $\mathbb{Z}_m$. Since $\mathcal{D}$ is reflexible, the assignment $\theta : a \mapsto a^{-1}, b \mapsto b^{-1}$ extends to an automorphism of G. By the identities in (4.2) derived from (4.1) we have

$$ab^{-x} = b^{\varphi^{-1}(-x)}a^{-\sigma(-x,-1)} \quad \text{and} \quad ba^{-y} = a^{\tilde{\varphi}^{-1}(-y)}b^{-\tilde{\sigma}(-y,-1)}.$$

Applying the automorphism θ of G to the above identities we obtain

$$a^{-1}b^{x} = \theta(ab^{-x}) = \theta(b^{\varphi^{-1}(-x)}a^{-\sigma(-x,-1)}) = b^{-\varphi^{-1}(-x)}a^{\sigma(-x,-1)}$$

and

$$b^{-1}a^{y} = \theta(ba^{-y}) = \theta(a^{\tilde{\varphi}^{-1}(-y)}b^{-\tilde{\sigma}(-y,-1)}) = a^{-\tilde{\varphi}^{-1}(-y)}b^{\tilde{\sigma}(-y,-1)}.$$

By comparing these with the identities in (4.1) we get $\varphi(x) = -\varphi^{-1}(-x)$ and $\tilde{\varphi}(y) = -\tilde{\varphi}^{-1}(-y)$. Thus both φ and $\tilde{\varphi}$ are antibalanced.

Conversely, suppose that $\varphi : \mathbb{Z}_n \to \mathbb{Z}_n$ and $\tilde{\varphi} : \mathbb{Z}_m \to \mathbb{Z}_m$ form a pair of antibalanced reciprocal skew morphisms. Denote

$$\mathbb{Z}_n = \{0, 1, \ldots, (n-1)\} \quad \text{and} \quad \mathbb{Z}_m = \{0', 1', \ldots, (m-1)'\},$$

so that $\mathbb{Z}_n$ and $\mathbb{Z}_m$ are disjoint sets. Define two cyclic permutations ρ and $\tilde{\rho}$ on the sets $\mathbb{Z}_n$ and $\mathbb{Z}_m$ by setting

$$\rho = (0, 1, \ldots, (n-1)) \quad \text{and} \quad \tilde{\rho} = (0', 1', \ldots, (m-1)').$$

We extend the permutations $\varphi, \tilde{\varphi}, \rho$ and $\tilde{\rho}$ to permutations on $\Omega = \mathbb{Z}_n \cup \mathbb{Z}_m$ in a natural way, still denoted by $\varphi, \tilde{\varphi}, \rho$ and $\tilde{\rho}$. Set $a = \varphi\tilde{\rho}$, $b = \tilde{\varphi}\rho$ and $G = \langle a, b\rangle$. It is proved in [12, Proposition 4] that $|a| = m$, $|b| = n$, $\langle a\rangle \cap \langle b\rangle = 1$ and $G = \langle a\rangle\langle b\rangle$, so $\mathcal{D} = (G, a, b)$ is a complete regular dessin with underlying graph $K_{m,n}$. Now define a bijection $\gamma : \Omega \to \Omega$ on Ω to be $\gamma(x) = -x$ and $\gamma(y') = -y'$ for all $x \in \mathbb{Z}_n$ and $y' \in \mathbb{Z}_m$. Since both φ and $\tilde{\varphi}$ are antibalanced, we have

$$\begin{aligned}\gamma a(x) &= \gamma\varphi\tilde{\rho}(x) = \gamma\varphi(x) = -\varphi(x) = \varphi^{-1}(-x)\\ &= \varphi^{-1}\gamma(x) = \varphi^{-1}\tilde{\rho}^{-1}(\gamma(x)) = a^{-1}\gamma(x)\end{aligned}$$

and

$$\begin{aligned}\gamma a(y') &= \gamma\varphi\tilde{\rho}(y') = \gamma\varphi((y+1)') = \gamma((y+1)') = -(y+1)'\\ &= (-y-1)' = \varphi^{-1}\tilde{\rho}^{-1}(-y') = a^{-1}\gamma(y').\end{aligned}$$

Thus $\gamma a = a^{-1}\gamma$. Similarly, $\gamma b = b^{-1}\gamma$. Hence, $(G, a, b) \cong (G, a^{-1}, b^{-1})$, where (G, a^{-1}, b^{-1}) denotes the mirror image of $\mathcal{D}$. Therefore, (G, a, b) is reflexible, as required. □

We summarize two properties of reciprocal skew morphisms.

Lemma 4.3 ([12, 14]). *Let $\varphi : \mathbb{Z}_n \to \mathbb{Z}_n$ and $\tilde{\varphi} : \mathbb{Z}_m \to \mathbb{Z}_m$ be a pair of reciprocal skew morphisms of the cyclic additive groups $\mathbb{Z}_n$ and $\mathbb{Z}_m$. Then*

(a) $\varphi(x) \equiv \sum_{i=1}^{x} \tilde{\pi}(\tilde{\varphi}^{i-1}(1)) \pmod{|\tilde{\varphi}|}$ *and* $\tilde{\varphi}(y) \equiv \sum_{i=1}^{y} \pi(\varphi^{i-1}(1)) \pmod{|\varphi|}$,

(b) $|\mathbb{Z}_m : \operatorname{Ker} \tilde{\varphi}|$ *divides* $|\varphi|$ *and* $|\mathbb{Z}_n : \operatorname{Ker} \varphi|$ *divides* $|\tilde{\varphi}|$.

Lemma 4.4 ([14]). *Let $\varphi : \mathbb{Z}_n \to \mathbb{Z}_n$ and $\tilde{\varphi} : \mathbb{Z}_m \to \mathbb{Z}_m$ be a pair of reciprocal skew morphisms of the cyclic additive groups $\mathbb{Z}_n$ and $\mathbb{Z}_m$. If one of the skew morphisms is an automorphism, then the other is smooth. In particular, if one of the skew morphism is the identity automorphism, then the other is an automorphism.*

5 Classification

By Theorem 4.2, the classification of reflexible complete regular dessins is reduced to the classification of reciprocal pairs of antibalanced skew morphisms of cyclic groups. The aim of this section is to present a classification of the latter.

Proposition 5.1. *Let $\varphi : \mathbb{Z}_n \to \mathbb{Z}_n$ and $\tilde{\varphi} : \mathbb{Z}_m \to \mathbb{Z}_m$ be a reciprocal pair of antibalanced skew morphisms of the cyclic additive groups $\mathbb{Z}_n$ and $\mathbb{Z}_m$, respectively. If both n and m are odd, then $(\varphi, \tilde{\varphi}) = (\mathrm{id}_n, \mathrm{id}_m)$ where id_k denotes the identity automorphism of $\mathbb{Z}_k$, $k = n, m$.*

Proof. By Theorem 3.9(a), both φ and $\tilde{\varphi}$ are involutory automorphisms. The divisibility condition on reciprocality implies that both φ and $\tilde{\varphi}$ are the identity automorphisms. □

Theorem 5.2. *Let $\varphi : \mathbb{Z}_n \to \mathbb{Z}_n$ and $\tilde{\varphi} : \mathbb{Z}_m \to \mathbb{Z}_m$ be a reciprocal pair of antibalanced skew morphisms of the cyclic additive groups $\mathbb{Z}_n$ and $\mathbb{Z}_m$, respectively. If n is odd and m is even, then φ is an automorphism of the form $\varphi(x) \equiv sx \pmod{n}$, and $\tilde{\varphi}$ is a skew morphism of the form*

$$\tilde{\varphi}(y) = \begin{cases} y, & y \text{ is even}, \\ y + 2u, & y \text{ is odd} \end{cases} \quad \text{and} \quad \tilde{\pi}(y) = \begin{cases} 1, & y \text{ is even}, \\ -1, & y \text{ is odd} \end{cases}$$

where $s \in \mathbb{Z}_n$ and $u \in \mathbb{Z}_{m/2}$ are integers such that

$$\gcd(n, s+1) \gcd(m/2, u) \equiv 0 \pmod{m/2} \quad \text{and} \quad s^2 \equiv 1 \pmod{n}. \tag{5.1}$$

Proof. By assumption, both φ and $\tilde{\varphi}$ are antibalanced. Since n is odd and m is even, by Theorem 3.9, φ is an automorphism of the form $\varphi(x) = sx$, where $s^2 \equiv 1 \pmod{n}$ and $\tilde{\varphi}$ is a skew morphism of the form

$$\tilde{\varphi}(y) = \begin{cases} ty, & y \text{ is even}, \\ t(y-1) + 2u + 1, & y \text{ is odd}, \end{cases}$$

for some $t, u \in \mathbb{Z}_{m/2}$ satisfying the following conditions:

$$t^2 \equiv 1 \pmod{m/2} \quad \text{and} \quad (u+1)(t-1) \equiv 0 \pmod{m/2}.$$

Note that the order of φ is equal to the multiplicative order of s in $\mathbb{Z}_n$, which is a divisor of 2, and the order of $\tilde{\varphi}$ is the smallest positive integer ℓ such that

$$2u\sum_{i=1}^{\ell} t^{i-1} \equiv 0 \pmod{m}.$$

Now we employ the reciprocality to simplify these numerical conditions. By Definition 4.1(a), $|\tilde{\varphi}|$ divides n. Since n is odd, $|\tilde{\varphi}|$ is also odd, so by Corollary 3.10, $t = 1$, and consequently, $\tilde{\varphi}$ reduces to the stated form and $|\tilde{\varphi}| = m/\gcd(m, 2u)$. By Definition 4.1(b),

$$-1 \equiv \tilde{\pi}(1) \equiv \varphi(1) = s \pmod{\frac{m}{\gcd(m, 2u)}}.$$

Thus, $|\tilde{\varphi}| = m/\gcd(m, 2u)$ is a common divisor of $(s + 1)$ and n. Since m is even, $\gcd(m, 2u) = 2\gcd(m/2, u)$, and we obtain the first condition in (5.1), as required. □

By exchanging the roles of φ and $\tilde{\varphi}$, and the associated parameters, we obtain all reciprocal pairs of antibalanced skew morphisms of $\mathbb{Z}_n$ and $\mathbb{Z}_m$ where n is even and m is odd. The details are left to the reader.

Theorem 5.3. *Let $\varphi : \mathbb{Z}_n \to \mathbb{Z}_n$ and $\tilde{\varphi} : \mathbb{Z}_m \to \mathbb{Z}_m$ be a reciprocal pair of antibalanced skew morphisms of the cyclic additive groups $\mathbb{Z}_n$ and $\mathbb{Z}_m$, respectively. If both n and m are even, then*

$$\varphi(x) = \begin{cases} sx, & x \text{ is even}, \\ s(x-1) + 2r + 1, & x \text{ is odd}, \end{cases} \qquad \pi(x) = \begin{cases} 1, & x \text{ is even}, \\ -1, & x \text{ is odd} \end{cases}$$

and

$$\tilde{\varphi}(y) = \begin{cases} ty, & y \text{ is even}, \\ t(y-1) + 2u + 1, & y \text{ is odd}, \end{cases} \qquad \tilde{\pi}(y) = \begin{cases} 1, & y \text{ is even}, \\ -1, & y \text{ is odd} \end{cases}$$

where $r, s \in \mathbb{Z}_{n/2}$ and $u, t \in \mathbb{Z}_{m/2}$ are integers such that

$$\begin{cases} s^2 \equiv 1 \pmod{n/2}, \\ (r+1)(s-1) \equiv 0 \pmod{n/2}, \\ \gcd(m/2, u+1)\gcd(n, r(s+1)) \equiv 0 \pmod{n/2} \end{cases} \tag{5.2}$$

and

$$\begin{cases} t^2 \equiv 1 \pmod{m/2}, \\ (u+1)(t-1) \equiv 0 \pmod{m/2}, \\ \gcd(n/2, r+1)\gcd(m, u(t+1)) \equiv 0 \pmod{m/2}. \end{cases} \tag{5.3}$$

Proof. By Theorem 3.9(b), the skew morphisms φ and $\tilde{\varphi}$ may be represented by the stated form, where the parameters $r, s \in \mathbb{Z}_{n/2}$ and $u, t \in \mathbb{Z}_{m/2}$ are integers such that

$$s^2 \equiv 1 \pmod{n/2}, \quad (r+1)(s-1) \equiv 0 \pmod{n/2}$$

and

$$t^2 \equiv 1 \pmod{m/2}, \quad (u+1)(t-1) \equiv 0 \pmod{m/2}.$$

In particular,

$$|\varphi| = n/\gcd(n, r(s+1)) \quad \text{and} \quad |\tilde{\varphi}| = m/\gcd(m, u(t+1)).$$

We now employ the reciprocality to simplify the numerical conditions. By Definition 4.1, we have $|\varphi| = n/\gcd(n, r(s+1))$ divides m, $|\tilde{\varphi}| = m/\gcd(m, u(t+1))$ divides n,

$$-1 \equiv \pi(1) \equiv \tilde{\varphi}(1) \equiv 2u+1 \pmod{n/\gcd(n, r(s+1))}$$

and

$$-1 \equiv \tilde{\pi}(1) \equiv \varphi(1) \equiv 2r+1 \pmod{m/\gcd(m, u(t+1))}.$$

Thus, $n/\gcd(n, r(s+1))$ divides $\gcd(m, 2(u+1))$ and $m/\gcd(m, u(t+1))$ divides $\gcd(n, 2(r+1))$, or equivalently,

$$\gcd(n, r(s+1))\gcd(m/2, u+1) \equiv 0 \pmod{n/2}$$

and

$$\gcd(m, u(t+1))\gcd(n/2, r+1) \equiv 0 \pmod{m/2},$$

as required. □

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References

[1] M. Bachratý and R. Jajcay, Powers of skew-morphisms, in: *Symmetries in graphs, maps, and polytopes*, Springer, [Cham], volume 159 of *Springer Proc. Math. Stat.*, pp. 1–25, 2016, doi: 10.1007/978-3-319-30451-9_1.

[2] M. Bachratý and R. Jajcay, Classification of coset-preserving skew-morphisms of finite cyclic groups, *Australas. J. Combin.* **67** (2017), 259–280, `https://ajc.maths.uq.edu.au/?page=get_volumes&volume=67`.

[3] M. Conder, R. Jajcay and T. Tucker, Regular Cayley maps for finite abelian groups, *J. Algebraic Combin.* **25** (2007), 259–283, doi:10.1007/s10801-006-0037-0.

[4] M. Conder, R. Jajcay and T. Tucker, Regular t-balanced Cayley maps, *J. Combin. Theory Ser. B* **97** (2007), 453–473, doi:10.1016/j.jctb.2006.07.008.

[5] M. D. E. Conder, R. Jajcay and T. W. Tucker, Cyclic complements and skew morphisms of groups, *J. Algebra* **453** (2016), 68–100, doi:10.1016/j.jalgebra.2015.12.024.

[6] M. D. E. Conder, Y. S. Kwon and J. Širáň, Reflexibility of regular Cayley maps for abelian groups, *J. Combin. Theory Ser. B* **99** (2009), 254–260, doi:10.1016/j.jctb.2008.07.002.

[7] M. D. E. Conder and T. W. Tucker, Regular Cayley maps for cyclic groups, *Trans. Amer. Math. Soc.* **366** (2014), 3585–3609, doi:10.1090/S0002-9947-2014-05933-3.

[8] S.-F. Du, G. Jones, J. H. Kwak, R. Nedela and M. Skoviera, Regular embeddings of $K_{n,n}$ where n is a power of 2. II: the non-metacyclic case, *European J. Combin.* **31** (2010), 1946–1956, doi:10.1016/j.ejc.2010.01.009.

[9] S.-F. Du, G. Jones, J. H. Kwak, R. Nedela and M. Škoviera, Regular embeddings of $K_{n,n}$ where n is a power of 2. I. Metacyclic case, *European J. Combin.* **28** (2007), 1595–1609, doi:10.1016/j.ejc.2006.08.012.

[10] W. Fan and C. H. Li, The complete bipartite graphs with a unique edge-transitive embedding, *J. Graph Theory* **87** (2018), 581–586, doi:10.1002/jgt.22176.

[11] W. Fan, C. H. Li and H. P. Qu, A classification of orientably edge-transitive circular embeddings of K_{p^e,p^f}, *Ann. Comb.* **22** (2018), 135–146, doi:10.1007/s00026-018-0373-5.

[12] Y.-Q. Feng, K. Hu, R. Nedela, M. Škoviera and N.-E. Wang, Complete regular dessins and skew-morphisms of cyclic groups, *Ars Math. Contemp.* (2020), doi:doi.org/10.26493/1855-3974.1748.ebd.

[13] K. Hu, R. Nedela and N.-E. Wang, Complete regular dessins of odd prime power order, *Discrete Math.* **342** (2019), 314–325, doi:10.1016/j.disc.2018.09.028.

[14] K. Hu, R. Nedela, N.-E. Wang and K. Yuan, Reciprocal skew morphisms of cyclic groups, *Acta Math. Univ. Comenian. (N.S.)* **88** (2019), 305–318, http://www.iam.fmph.uniba.sk/amuc/ojs/index.php/amuc/article/view/1006.

[15] R. Jajcay and R. Nedela, Half-regular Cayley maps, *Graphs Combin.* **31** (2015), 1003–1018, doi:10.1007/s00373-014-1428-y.

[16] R. Jajcay and J. Širáň, Skew-morphisms of regular Cayley maps, volume 244, pp. 167–179, 2002, doi:10.1016/S0012-365X(01)00081-4, algebraic and topological methods in graph theory (Lake Bled, 1999).

[17] G. Jones, R. Nedela and M. Škoviera, Complete bipartite graphs with a unique regular embedding, *J. Combin. Theory Ser. B* **98** (2008), 241–248, doi:10.1016/j.jctb.2006.07.004.

[18] G. A. Jones, Regular embeddings of complete bipartite graphs: classification and enumeration, *Proc. Lond. Math. Soc. (3)* **101** (2010), 427–453, doi:10.1112/plms/pdp061.

[19] G. A. Jones, R. Nedela and M. Škoviera, Regular embeddings of $K_{n,n}$ where n is an odd prime power, *European J. Combin.* **28** (2007), 1863–1875, doi:10.1016/j.ejc.2005.07.021.

[20] I. Kovács and R. Nedela, Decomposition of skew-morphisms of cyclic groups, *Ars Math. Contemp.* **4** (2011), 329–349, doi:10.26493/1855-3974.157.fc1.

[21] I. Kovács and R. Nedela, Skew-morphisms of cyclic p-groups, *J. Group Theory* **20** (2017), 1135–1154, doi:10.1515/jgth-2017-0015.

[22] J. H. Kwak and Y. S. Kwon, Regular orientable embeddings of complete bipartite graphs, *J. Graph Theory* **50** (2005), 105–122, doi:10.1002/jgt.20097.

[23] Y. S. Kwon, A classification of regular t-balanced Cayley maps for cyclic groups, *Discrete Math.* **313** (2013), 656–664, doi:10.1016/j.disc.2012.12.012.

[24] J. Širáň and M. Škoviera, Groups with sign structure and their antiautomorphisms, volume 108, pp. 189–202, 1992, doi:10.1016/0012-365X(92)90674-5, topological, algebraical and combinatorial structures. Frolík's memorial volume.

[25] N.-E. Wang, K. Hu, K. Yuan and J.-Y. Zhang, Smooth skew morphisms of dihedral groups, *Ars Math. Contemp.* **16** (2019), 527–547, doi:10.26493/1855-3974.1475.3d3.

[26] J.-Y. Zhang and S. Du, On the skew-morphisms of dihedral groups, *J. Group Theory* **19** (2016), 993–1016, doi:10.1515/jgth-2016-0027.