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Gray code numbers for graphs*

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Abstract

A graph H has a Gray code of k-colourings if it is possible to list all of its k-colourings in such a way that consecutive elements in the list differ in the colour of exactly one vertex. We prove that for any graph H, there is a least integer $k_0(H)$ such that H has a Gray code of k-colourings whenever $k \geq k_0(H)$. We then determine $k_0(H)$ whenever H is a complete graph, tree, or cycle.

Keywords: Graph colouring, cyclic Gray code.

Math. Subj. Class.: 05C15, 05A05

1 Introduction

The term *combinatorial Gray code* refers to a list of combinatorial objects so that each object is listed exactly once, and consecutive objects in the list differ in some prespecified, small way. According to Savage [13], this term was introduced in 1980 as a generalization of *minimal change listings*. The precise definition of minimal change depends on what is being listed but, informally, the idea is that consecutive objects in the list differ from each other as little as possible. For example, the Binary Reflected Gray Code (BRGC) is a listing of all 2^n binary strings of length n in which consecutive strings differ from one another at exactly one position.

A combinatorial Gray code corresponds to a Hamilton path or cycle in the graph where the vertices are the objects, and two vertices are joined by an edge if the objects differ in the prespecified way. The graph has a Hamilton path if and only if a combinatorial Gray code exists. If the graph has a Hamilton cycle, then there is a listing in which the first and last objects in the list also differ in the prespecified way, and the corresponding combinatorial Gray code is said to be *cyclic*.

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A survey of results concerning combinatorial Gray codes can be found in Savage [13]. A significant amount of research described there occurred following Herb Wilf's invited address *Generalized Gray Codes* at the 1988 SIAM Conference on Discrete Mathematics in San Francisco. The results and open problems he described appear in his SIAM monograph [16].

A combinatorial family which has not yet been explored in terms of Gray codes is the set of proper k-colourings of a graph. Let H be a graph. We regard two k-colourings f_1 and f_2 of H as being different if there is a vertex $x \in V(H)$ such that $f_1(x) \neq f_2(x)$. The following is one of the most important definitions in the paper. A *(cyclic) Gray code of k-colourings for H* is a listing of all of the distinct k-colourings of H so that consecutive colourings in the list, including the last and first, differ in the colour of exactly one vertex. Unless otherwise stated, the Gray codes of k-colourings in this paper are cyclic and we will therefore drop the adjective "cyclic".

The problem of recolouring a graph so that the colour of precisely one vertex is changed at each step has been considered in the literature [2, 3, 4, 5, 7, 9, 10]. These results will be briefly surveyed in the next section.

We conclude this section with a brief summary of the remainder of the paper. Section two provides an introduction to some concepts in graph theory and combinatorial Gray codes that are directly related to this work. In section three we introduce C-graphs, a class of graphs that arise frequently in our work, and determine some sufficient conditions for C-graphs to be Hamiltonian. These graphs appear in the proof of the existence theorem: For any graph H, there is a least integer $k_0(H)$ such that there exists a Gray code for k-colourings of H for all integers $k \geq k_0(H)$. The remainder of the paper is devoted to determining $k_0(H)$ for complete graphs, trees, and cycles. This is done in sections four through six, respectively.

2 Preliminaries

This section summarises results that will be used concerning vertex colourings, Hamilton paths and cycles, and combinatorial Gray codes. Our terminology is consistent with Bondy and Murty [1]. Throughout this section, let H be a graph with vertex set $V(H) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(H) = \{e_1, e_2, \ldots, e_m\}$.

A *proper k-colouring* of H is a function $f : V(H) \to \{1, 2, \ldots, k\}$ such that whenever vertex x is adjacent to vertex y, $f(x) \neq f(y)$. Since we only consider colourings of this type, for brevity we will drop the adjective "proper" and refer to these as *k-colourings*. Note that the function f need not be onto, that is, not every colour must be assigned to some vertex of H. A graph H is *k-colourable* if it has a k-colouring. The *chromatic number* $\chi(H)$ is the minimum k for which H has a k-colouring. A graph with chromatic number equal to k is sometimes called a *k-chromatic graph*. With respect to the enumeration $v_1, v_2, \ldots, v_n$ of $V(H)$, we will sometimes write a k-colouring of H in *one-line notation*: $f(v_1)f(v_2)\ldots f(v_n)$.

A *Hamilton path* in H is a path that contains every vertex of H. Analogously, a *Hamilton cycle* in H is a cycle that contains every vertex of H. A graph with a Hamilton cycle is said to be *Hamiltonian*. Having a Hamilton cycle is a stronger condition than having a Hamilton path. A graph with a Hamilton path between any two distinct vertices is called *Hamilton connected*.

Let H be a graph and let $\pi = x_1, x_2, \ldots, x_n$ be an enumeration of the vertices of H.

Let H_i be the subgraph of H induced by $\{x_1, x_2, \ldots, x_i\}$, for $i = 1, 2, \ldots, n$. Define $D_\pi = \max_{1 \le i \le n} d_{H_i}(x_i)$, where the notation $d_F(v)$ denotes the degree of the vertex v in the graph F. The quantity $\min_\pi D_\pi + 1$ is known as the *colouring number of* H and is denoted $col(H)$ [8]. It satisfies the bound $\chi(H) \le col(H) \le \Delta(H) + 1$, where $\Delta(H)$ is the maximum degree of H. We note that other authors whose work is referenced below define the colouring number in such a way that the "+1" is not included (cf. [4, 7]).

The *k-colouring graph* of H is defined to be the graph $G_k(H)$ with vertex set the set of all k-colourings of H, with two k-colourings f_i and f_j being adjacent in $G_k(H)$ if and only if $f_i(x) = f_j(x)$ for all vertices $x \in V(H)$ except one. The graph H has a Gray code for k-colourings if and only if the graph $G_k(H)$ has a Hamilton cycle. The number of vertices of $G_k(H)$ is the value of the chromatic polynomial of H at k (see [1], Section 8.4).

Connectedness of the k-colouring graph has been fairly extensively studied. It arises in the study of efficient algorithms for almost-uniform sampling of k-colourings [7, 9, 10]. For $k \in \{2, 3\}$, the k-colouring graph of a k-chromatic graph is never connected, while for each $k \ge 4$ there are k-chromatic graphs for which G_k is connected, and others for which it is not connected [4]. The problem of deciding if the 3-colouring graph of a bipartite graph is not connected is NP-complete, but Polynomial for planar bipartite graphs [5]. On the other hand, the k-colouring graph of H is connected for all $k \ge col(H) + 1$, and this bound is best possible [3] (also see [7]).

A variation on the problem of deciding whether the k-colouring graph is not connected is the problem of deciding whether two given k-colourings of the graph H lie in the same component of $G_k(H)$. The diameter of any component of the 3-colouring graph of a graph with n vertices is $O(n^2)$ [3]. By contrast, for each fixed $k \ge 4$, the problem of deciding if two k-colourings of H lie in the same component of $G_k(H)$ is PSPACE-complete, and the diameter of a component can be superpolynomial in the number of vertices of H [2].

In order to prove the existence of Gray codes for k-colourings, we sometimes rely on the existence of Gray codes for other combinatorial families. We next review some results about the Binary Reflected Gray Code (BRGC) and a particular Gray code of permutations.

The BRGC is a listing of the 2^n binary strings of length n so that consecutive strings in the list, including the last and first, differ in exactly one position. It is defined recursively by $L_1 = 0, 1$; and for $n > 1$, $L_n = 0 \cdot L_{n-1}, 1 \cdot \overline{L_{n-1}}$, where '$\cdot$' denotes concatenation and $\overline{L_{n-1}}$ lists the elements of L_{n-1} from last to first [13]. Thus, $L_2 = 00, 01, 11, 10$ and $L_3 = 000, 001, 011, 010, 110, 111, 101, 100$.

The *n-cube* is the graph Q_n whose 2^n vertices are the binary strings of length n, with two binary strings being adjacent if they differ in exactly one position. Since the BRGC is cyclic, it describes a Hamilton cycle in Q_n for all integers $n \ge 2$.

In our work on trees (see Theorem 5.2), we will be interested in when Q_n has a Hamilton path starting at $00 \cdots 0$ and ending at $11 \cdots 1$. The circumstances under which it does are characterised in the following lemma, which appears as an exercise in [12] (page 185, #20) and also follows from a result in [14]. We shall include a short proof for completeness.

Lemma 2.1. *There exists a Hamilton path from* $00 \cdots 0$ *to* $11 \cdots 1$ *in* Q_n *if and only if* $n \ge 1$ *and* n *is odd.*

Proof. ($\Rightarrow$) The graph Q_n is bipartite with bipartition (X, Y), where X is the set of binary sequences of length n with an even number of ones, and Y is the set of binary sequences of length n with an odd number of ones. Since $|X| = |Y|$ and $00 \cdots 0 \in X$, a Hamilton path ending at $11\cdots1$ is possible only when $11\cdots1 \in Y$. Thus, if a Hamilton path with the specified ends exists, $n \geq 1$ and n is odd.

($\Leftarrow$) Any odd integer n equals $2k+1$ for some integer $k \geq 0$. The proof is by induction on k. The statement is clearly true when $k = 0$. Assume, for some $t \geq 0$, that the graph Q_{2t+1} has a Hamilton path P from $00\cdots0$ to $11\cdots1$. Without loss of generality, the second vertex on P is $100\cdots0$. By symmetry, Q_{2t+1} has a Hamilton cycle that uses the edge $(00\cdots0)(100\cdots0)$, that is, Q_{2t+1} has a Hamilton path S from $00\cdots0$ to $100\cdots0$. The desired Hamilton path in Q_{2t+3} is obtained from $R = P\cdot00, \overline{P}\cdot01, P\cdot11$ by inserting $S\cdot01$ between the first two vertices of R (as before, '$\cdot$' denotes concatenation and $\overline{P}$ is denotes the reverse of the sequence P). Thus, by induction, if $n \geq 1$ is odd, then Q_n has a Hamilton path from $00\cdots0$ to $11\cdots1$. □

We now look at a particular Gray code for permutations. Recall that the *symmetric group* S_n is the group of all permutations of $\{1, 2, \ldots, n\}$ under the operation of function composition. When listing all the permutations of an n-set, an obvious choice for a minimal change condition is that consecutive permutations in the list differ by a *transposition* (a permutation that exchanges two elements and leaves all others fixed). Let X be a subset of the permutations in S_n such that the identity element $e \notin X$ and X is closed with respect to inverses ($x^{-1} \in X$ whenever $x \in X$). The *Cayley graph with cymbal X on the symmetric group S_n*, denoted $Cay(X : S_n)$, is defined to have vertex set $V(Cay(X : S_n)) = \{g : g \in S_n\}$ and an edge joining g to g' if and only if $g' = gx$ for some $x \in X$. If every element $g \in S_n$ can be written $g = g_1g_2\ldots g_j$ for some elements $g_1, g_2, \ldots, g_j \in X$ then X is a *generating set* of S_n. It is known that $Cay(X : S_n)$ is connected if and only if X is a generating set. It is an open question as to whether every connected Cayley graph on S_n is Hamiltonian [6]. This question has been settled by Kompel'makher and Liskovets in the case where the generating set X is a set of transpositions [11]. Corollary 2.2 below, due to Slater [15], is a special case of this result. It implies that it is possible to list all permutations in S_n so that consecutive permutations in the list differ by a transposition of the number in the first position and a number in some other position.

Corollary 2.2. [15] *The Cayley graph $Cay(\{(1,2),(1,3),\ldots,(1,n)\} : S_n)$ is Hamiltonian.*

3 C-graphs and the existence theorem

The main result of this section states that for every graph H, there exists a constant $k_0(H)$ such that H has a Gray code of k-colourings for every integer $k \geq k_0(H)$. The proof of this theorem, and others in this paper, involves a particular family of nicely structured graphs which we call *C-graphs*. These are introduced below, and some conditions under which such graphs are Hamiltonian are developed. Throughout this section subscripts are to be interpreted modulo N.

A *C-graph with vertex partition* $F_0, F_1, \ldots, F_{N-1}$ is a graph G together with a partition $F_0, F_1, \ldots, F_{N-1}$ of $V(G)$ so that, for $i = 0, 1, \ldots, N-1$, the subgraph induced by F_i is a Hamilton connected graph with at least three vertices. The basic structure of a C-graph, as it is used in Lemma 3.1 and its corollary, is shown in Figure 1.

Lemma 3.1. *Let G be a C-graph with vertex partition $F_0, F_1, \ldots, F_{N-1}$. If, for $i = 0, 1, \ldots, N-1$, there are vertex disjoint edges x_iy_{i+1} with $x_i \in F_i$ and $y_{i+1} \in F_{i+1}$, then G is Hamiltonian.*

Proof. For $0 \le i \le N-1$ and $u, w \in F_i$, let $P_i(u,w)$ denote a Hamilton path from u to w in the subgraph of G induced by F_i. Then the concatenation of $x_0, P_1(y_1, x_1)$, $P_2(y_2, x_2), \ldots, P_{N-1}(y_{N-1}, x_{N-1})$, $P_0(y_0, x_0)$ is a Hamilton cycle in G. □

We now develop some conditions under which Lemma 3.1 applies. These will be useful in our subsequent work.

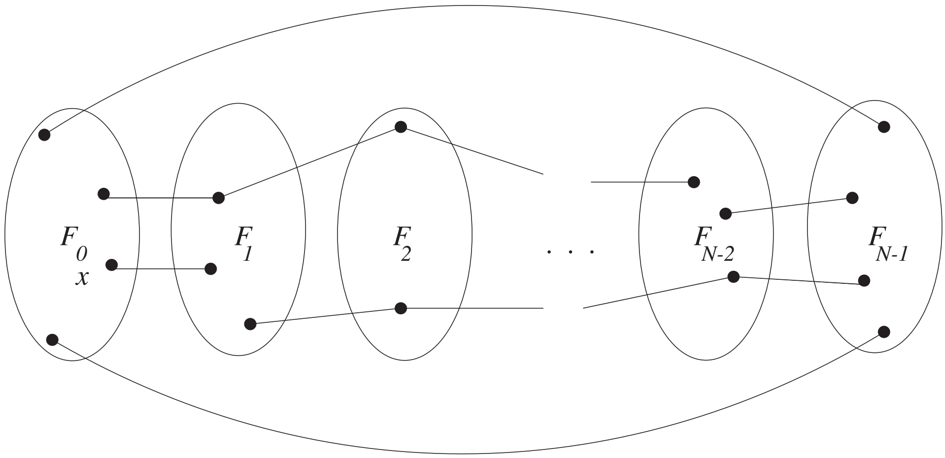


Figure 1: The basic structure of a C-graph in Lemma 3.1 and Corollary 3.2.

Let G be a graph, and X and Y be disjoint subsets of $V(G)$. We use $[X, Y]$ to denote the set of edges with one end in X and the other end in Y.

Corollary 3.2. *Let G be a C-graph with vertex partition $F_0, F_1, \ldots, F_{N-1}$. Suppose that, for $j = 0, 1, \ldots, N-1$, the set $[F_j, F_{j+1}]$ contains at least two vertex disjoint edges. If there exists i, $0 \le i \le N-1$, such that some vertex $x \in F_i$ has a neighbour in F_{i+1} and $[F_{i-1}, F_i - \{x\}]$ (still) contains at least two vertex disjoint edges, then G is Hamiltonian.*

Proof. By the symmetry in the definition of C-graphs we may, without loss of generality, assume that $i = 0$. For $j = 0, 1, \ldots, N-2$, define the edges $x_j y_{j+1} \in [F_j, F_{j+1}]$ inductively as follows. Let $x_0 = x$ and y_1 be any neighbour of x in F_1. The vertex y_1 exists by hypothesis. Suppose $x_0y_1, x_1y_2, \ldots, x_{j-1}y_j$, $(j \ge 1)$, have been chosen. It follows from the hypothesis that the set of edges $[F_j - \{y_j\}, F_{j+1}]$ is not empty. Let x_jy_{j+1} be any edge in this set, where $x_j \in F_j - \{y_j\}$ and $y_{j+1} \in F_{j+1}$. Finally, it follows from our hypothesis that the set of edges $[F_{N-1} - \{y_{N-1}\}, F_0 - \{x_0\}]$ is not empty. (Remember that $x_0 = x$.) Let $x_{N-1}y_0$ be any edge in this set, where $x_{N-1} \in F_{N-1} - \{y_{N-1}\}$ and $y_0 \in F_0 - \{x_0\}$. The result now follows from Lemma 3.1. □

Corollary 3.3. *Let G be a C-graph with vertex partition $F_0, F_1, \ldots, F_{N-1}$. Suppose that, for $j = 0, 1, \ldots, N-1$, the set $[F_j, F_{j+1}]$ contains at least two vertex disjoint edges. If there exists i, $0 \le i \le N-1$, such that $[F_i, F_{i+1}]$ contains at least three vertex disjoint edges, then G is Hamiltonian.*

Proof. By the symmetry in the definition of C-graphs we may, without loss of generality, assume that $i = N-1$. Let $x \in F_0$ be any vertex such that there are two vertex disjoint edges in $[F_0, F_1]$, one of which is incident with x. Since at most one of the three disjoint edges in $[F_{N-1}, F_0]$ is incident with x, the result follows from Corollary 3.2. □

We now prove that every graph has a Gray code of k-colourings when k is sufficiently large. As a consequence, it becomes of interest to determine the minimum number of colours necessary for certain graphs to have a Gray code of k-colourings. This is the topic of the remaining sections of the paper.

Theorem 3.4. *Let H be a graph. If $k \geq 2 + col(H)$, then $G_k(H)$ is Hamiltonian.*

Proof. Consider an enumeration $\sigma = v_1, v_2, \ldots, v_n$ of the vertices of H such that $D_\sigma = \min_\pi D_\pi$. Let $k \geq 3 + D_\sigma = 2 + col(H)$. We show by induction on i that each graph $G_k(H_i), 1 \leq i \leq n$, is Hamiltonian.

Since $k \geq 3$, the sequence $1, 2, \ldots, k, 1$ is a Hamilton cycle in $G_k(H_1)$. Suppose that the sequence $f_0, f_1, \ldots, f_{N-1}, f_0$ is a Hamilton cycle in $G_k(H_{i-1})$, for some integer i with $2 \leq i \leq n$.

Consider $G_k(H_i)$. For $j = 0, 1, \ldots, N-1$, let F_j be the set of k-colourings of H_i that agree with f_j on $V(H_{i-1})$. Since $k \geq 3 + D_\sigma \geq 3 + d_{H_i}(v_i)$, for each k-colouring of H_{i-1} there are at least three colours that can be assigned to v_i in order to extend it to a k-colouring of H_i. Therefore, each set F_j contains at least three k-colourings of H_i. Furthermore, by its definition, each set F_j induces a complete subgraph of $G_k(H_i)$. Since complete graphs with at least three vertices are Hamilton connected and $F_0, F_1, \ldots, F_{N-1}$ is a partition of the vertex set of $G_k(H_i)$, we have that $G_k(H_i)$ is a C-graph.

Suppose f_j and f_{j+1} differ in the colour assigned to vertex w_j. If w_j is not adjacent to v_i, then every colour assigned to v_i by some colouring in F_j is also assigned to v_i by some colouring in F_{j+1}. Thus, every vertex in F_j has a neighbour in F_{j+1} corresponding to the k-colouring that agrees with it on v_i. Since these edges are vertex disjoint, in this case $[F_j, F_{j+1}]$ contains at least three vertex disjoint edges. If w_j is adjacent to v_i, the vertex in F_j corresponding to the k-colouring of H_i that assigns colour $f_{j+1}(w_j)$ to v_i has no neighbour in F_{j+1}, but every other vertex in F_j has a neighbour in F_{j+1}. In this case $[F_j, F_{j+1}]$ contains at least two vertex disjoint edges. Thus, if there exists j such that w_j is not adjacent to v_i, the graph $G_k(H_i)$ is Hamiltonian by 3.3.

Otherwise, for all j, $0 \leq j \leq N-1$, w_j is adjacent to v_i. From the above argument, we have that $[F_j, F_{j+1}]$ always contains at least two vertex disjoint edges. Choose a colouring $c_{N-1} \in F_{N-1}$ which has a neighbour $c_0 \in F_0$. There exists a largest integer $r \leq N-1$ such that f_{r-1} uses colour $c_{N-1}(v_i)$ but f_r does not. By definition of r, no colouring in F_{r-1} assigns colour $c_{N-1}(v_i)$ to v_i but, for $t = r, r+1, \ldots, N-1$, there exists a colouring $c_t \in F_t$ such that $c_t(v_i) = c_{N-1}(v_i)$. Hence, there exists $x = c_r \in F_r$ which has a neighbour in F_{r+1} but no neighbour in F_{r-1}. Since $[F_{r-1}, F_r]$ contains two vertex disjoint edges, and no edge in $[F_{r-1}, F_r]$ is incident with x, we have that $[F_{r-1}, F_r - \{x\}]$ (still) contains two vertex disjoint edges. The induction step that $G_k(H_i)$ is Hamiltonian now follows from 3.2 □

Corollary 3.5. *For any graph H, there exists a least integer $k_0(H)$ such that H has a Gray code of k-colourings for any integer $k \geq k_0(H)$.*

We call the integer $k_0(H)$ in Corollary 3.5 the *Gray code number of H*.

4 The Gray code number of complete graphs

For all integers $n \geq 1$, the Gray code number $k_0(K_n)$ is determined by the following theorem.

Theorem 4.1. *$k_0(K_1) = 3$, and $k_0(K_n) = n + 1$ for $n \geq 2$.*

Proof. The first statement is easy to see. We prove the second statement.

For $n \geq 2$, any two n-colourings of K_n differ in the colour of at least two vertices. Hence $G_n(K_n)$ has $n! > 1$ vertices and no edges. Therefore, if $G_k(K_n)$ is Hamiltonian, then $k \geq n + 1$.

Since $col(K_n) = n$, by Theorem 3.4 we have $k_0(K_n) \leq n + 2$. Hence we need only consider the case $k = n + 1$. We claim that $G_{n+1}(K_n)$ is isomorphic to *Cay($X : S_{n+1}$)*, where X is the generating set of transpositions $X = \{(1, 2), (1, 3), \ldots, (1, n + 1)\}$.

Let $\phi : V(\text{Cay}(X : S_{n+1})) \to V(G_{n+1}(K_n))$ be the isomorphism defined by $\phi(\pi) = \pi(2)\pi(3)\cdots\pi(n + 1)$, where the right hand side is an $(n + 1)$-colouring of K_n written in one-line notation. Then, $\pi_1\pi_2 \in E(\text{Cay}(X : S_{n+1}))$ if and only if π_1 and π_2 differ by a transposition in X. The permutations π_1 and π_2 differ by a transposition involving 1 if and only if $\phi(\pi_1)$ and $\phi(\pi_2)$ differ in the colour of exactly one vertex, that is, if and only if $\phi(\pi_1)\phi(\pi_2) \in E(G_{n+1}(K_n))$. Hence, by Corollary 2.2, the graph $G_{n+1}(K_n)$ is Hamiltonian. □

5 Gray code numbers for trees

In this section we determine the Gray code number of any tree. Since, for a tree T, $col(T) \leq 2$, we have by Theorem 3.4 that $k_0(T) \leq 4$.

Proposition 5.1. *If H is a connected bipartite graph, then $k_0(H) \geq 3$.*

Proof. A connected bipartite graph has exactly two 2-colourings, so $G_2(H)$ has exactly two vertices and cannot be Hamiltonian. □

Thus, $3 \leq k_0(T) \leq 4$ for any tree T. It turns out that equality can occur in both the upper and lower bound.

A complete bipartite graph $K_{1,n}$ is also called a *star*. For $n \geq 2$, the unique vertex v adjacent to all vertices of the star is called its *centre*. When $n = 1$, either vertex can be chosen to be the centre of the star. A star is called *odd* or *even* when its number of vertices, $(n + 1)$, is odd or even, respectively.

Theorem 5.2. *Let T be a star with $n + 1 \geq 2$ vertices. Then $G_3(T)$ is Hamiltonian if and only if T is even.*

Proof. Let the vertex set of the star $K_{1,n}$ be $\{v_0, v_1, v_2, \ldots, v_n\}$, where v_0 is the centre.

($\Leftarrow$) When $n = 1$, $K_{1,1}$ is just K_2 and $G_3(K_2)$ is Hamiltonian by Theorem 4.1.

Suppose $n \geq 3$ is odd. By Lemma 2.1, the set of all 2^n sequences of length n with entries from $\{a, b\}$ can be listed starting with $aa\cdots a$ and ending with $bb\cdots b$. Let $P(a, b)$ denote any such listing (an acyclic Gray code). Then, a Hamilton cycle in $G_3(T)$ is $1 \cdot P(2, 3), 2 \cdot P(3, 1), 3 \cdot P(1, 2)$, where '$\cdot$' denotes concatenation.

($\Rightarrow$) Now suppose $n \geq 2$ and $f_0, f_1, \ldots f_{N-1}, f_0$ is a Hamilton cycle in $G_3(T)$. Note that two colourings can differ in the colour assigned to v_0 only if the remaining vertices are all coloured with the third colour. Thus all 2^n colourings of T in which v_0 is coloured

1 appear consecutively on the cycle, say with $f_0 = 1222\cdots 2$ at one end of the segment and $f_{2^n-1} = 1333\cdots 3$ at the other. Consider each of the colourings in the sequence $f_0, f_1, \ldots, f_{2^n-1}$ restricted to $V(T) - \{v_0\} = \{v_1, v_2, \ldots, v_n\}$. If all 2's are replaced by 0's and all 3's are replaced by 1's, then the resulting sequence describes a Hamilton path in Q_n starting at $00\cdots 0$ and ending at $11\cdots 1$. Hence, by Lemma 2.1, n is odd. □

We now show in several steps that $G_3(T)$ is Hamiltonian in all remaining cases. An *odd flare* is a tree obtained from an even star with at least four vertices and centre v by a single subdivision of an edge. That is, it is obtained by deleting an edge vw, adding a new vertex z, and adding the edges vz and zw. The vertex v is (still) called the *centre* of the odd flare.

Lemma 5.3. *Let T be a tree with $n \geq 6$ vertices. If T is neither a star nor an odd flare, then there exist leaves v and w such that $d(v,w) \geq 3$ and $T - \{v,w\}$ is not an odd star.*

Proof. Let $P = x_0x_1\ldots x_p$ be a longest path in T. Since P is a longest path, x_0 and x_p are leaves of T.

Suppose first that n is even. Since T is not a star, $p \geq 3$. Let $v = x_0$ and $w = x_p$. Then the distance $d(v,w) \geq 3$ and $T - \{v,w\}$ is not an odd star (since n is even).

Now suppose that n is odd. If $p \geq 5$, the result follows on letting $v = x_0$ and $w = x_p$. Suppose that $p = 4$, so that $P = x_0x_1x_2x_3x_4$. If the degree $d(x_1) > 2$ or $d(x_3) > 2$ then the result follows as above with $v = x_0$ and $w = x_4$. Otherwise, since $n \geq 6$, it must be that $d(x_2) > 2$. Let $Q = x_2y_1y_2\ldots y_q$ be a longest path starting at x_2 that contains no other vertex of P. As above, since Q is a longest path, the vertex y_q is a leaf of T. The result follows by letting $v = x_0$ and $w = y_q$.

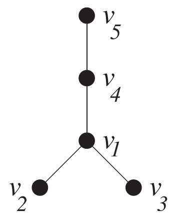
Finally, suppose that n is odd and $p = 3$, so that $P = x_0x_1x_2x_3$. Since T is not an odd flare and $n \geq 6$ is odd, there exist at least three other vertices, of which at least one is adjacent to x_1 and at least one is adjacent to x_2. Let $v = x_0$ and $w = x_3$. Then $d(v,w) \geq 3$ and $T - \{v,w\}$ is not an odd star. □

Theorem 5.4. *If T is an odd flare, then $G_3(T)$ is Hamiltonian.*

Proof. The proof is by induction on t, where $|V(T)| = 2t+1,\ t \geq 2$. We prove the stronger statement that there is a Hamilton cycle in which any colour assigned to the centre of the flare remains constant for an even number of consecutive 3-colourings (vertices of $G_3(T)$). This stronger statement is true for the Gray code of 3-colourings for the unique odd flare with five vertices shown in Figure 2. Suppose it holds for all odd flares on $2t-1$ vertices for some $t \geq 3$, and let T be an odd flare on $2t+1$ vertices.

Let c be the centre of the odd flare T, and v and w be leaves adjacent to c. Then the tree $T' = T - \{v,w\}$ is an odd flare with $2t-1$ vertices and the same centre. By the induction hypothesis, $G_3(T')$ has a Hamilton cycle $C = f_0, f_1, \ldots, f_{N-1}, f_0$ in which any colour assigned to c remains constant for an even number of consecutive 3-colourings. For the remainder of this proof, subscripts are to be interpreted modulo N.

For $i = 0, 1, \ldots, N-1$, let F_i be the set of 3-colourings of T that agree with f_i on $V(T')$. Since v and w are leaves, each set F_i contains four 3-colourings of T. Two different vertices $c_1, c_2 \in V(G_3(T))$ belonging to F_i are adjacent if and only if $c_1(v) = c_2(v)$ or $c_1(w) = c_2(w)$. Thus, the subgraph of $G_3(T)$ induced by F_i is a 4-cycle. Denote this 4-cycle by $x_{i,1}, x_{i,2}, x_{i,3}, x_{i,4}, x_{i,1}$ where the colourings $x_{i,1}$ and $x_{i,3}$ are such that $x_{i,1}(v) = x_{i,1}(w)$ and $x_{i,3}(v) = x_{i,3}(w)$.



12223	23332	21331	13331	32221	31212
12323	23312	21131	13321	31221	31112
12321	23313	21132	13323	31223	31113
12331	23113	21112	13223	31213	31123
12332	21113	23112	13221	32213	31121
12232	21313	23132	13231	32113	32121
13232	21312	23131	12231	32112	32123
13332	21332	23331	12221	32212	32223

Figure 2: A Gray code of 3-colourings for a flare with five vertices.

Suppose $f_i(c) = f_{i+1}(c)$. Then every colouring in F_i has a neighbour in F_{i+1}, namely the colouring that agrees with it on v and w. Thus, in this case, $[F_i, F_{i+1}]$ consists of four vertex disjoint edges.

Suppose $f_i(c) \neq f_{i+1}(c)$. Let $\alpha = \{1, 2, 3\} - \{f_i(c), f_{i+1}(c)\}$. Then the colouring in F_i that assigns α to both v and w has a neighbour in F_{i+1}, as above, and no other colouring in F_i has a neighbour in F_{i+1}.

The colourings f_j, for which $f_j(c) \neq f_{j+1}(c)$, partition C into paths, each having an even number of 3-colourings of T'. For each such path $f_i, f_{i+1}, \ldots, f_{i+2q-1}$, let $H_{i,i+2q-1}$ be the subgraph of $G_3(T)$ induced by $F_i \cup F_{i+1} \cup \cdots \cup F_{i+2q-1}$, and call $H_{i,i+2q-1}$ a *segment*. Note that the positive integer q may differ between segments.

It is enough to prove that for any vertices $a \in \{x_{i,1}, x_{i,3}\}$, $b \in \{x_{i+2q-1,1}, x_{i+2q-1,3}\}$ the graph $H_{i,i+2q-1}$ has a Hamilton path starting at a and ending at b. By choosing a and b to have neighbours in F_{i-1} and F_{i+2q}, respectively, we can obtain a Hamilton cycle in $G_3(T)$ by concatenating the Hamilton paths through the segments that arise from the partition of C. Without loss of generality assume $i = 1$, so that $H_{1,2q}$ is the subgraph of $G_3(T)$ induced by $F_1 \cup F_2 \cup \cdots \cup F_{2q}$. By the argument above, for $r = 1, 2, \ldots, 2q$ we have $[F_r, F_{r+1}] = \{x_{r,1}x_{r+1,1}, x_{r,2}x_{r+1,2}, x_{r,3}x_{r+1,3}, x_{r,4}x_{r+1,4}\}$. Let $P_r(w, z)$ denote a Hamilton path from w to z in the 4-cycle induced by F_r. A Hamilton path in $H_{1,2q}$ starting at a and ending at b is $P_1(a, x_{1,2})$, $P_2(x_{2,2}, x_{2,1})$, $P_3(x_{3,1}, x_{3,2}), \ldots, P_{2q}(x_{2q,2}, b)$.

Since the number of vertices in each segment is a multiple of four in the Hamilton cycle constructed above, the number of consecutive 3-colourings in which any colour assigned to the centre of the flare remains constant is a multiple of four. The result now follows by induction. $\square$

Theorem 5.5. *Let T be a tree which is not an odd star. Then $G_3(T)$ is Hamiltonian.*

Proof. The proof is by induction on $n = |V(T)|$. Suppose $n \leq 5$. If T is a star with an even number of vertices, then $G_3(T)$ is Hamiltonian by Theorem 5.2. If T is an odd flare, then $G_3(T)$ is Hamiltonian by Theorem 5.4. Otherwise, T is isomorphic to P_4 or P_5. Gray codes for 3-colourings of these trees are shown in Figure 3.

Suppose for some $n \geq 6$ that if T' is a tree on $n-1$ or fewer vertices which is not an odd star, then $G_3(T')$ is Hamiltonian. Let T be a tree on n vertices which is not an odd star. If T is an odd flare or a star with an even number of vertices, then the result follows from Theorem 5.4 or Theorem 5.2, respectively. Otherwise, by Lemma 5.3, there exist leaves v and w such that $d(v,w) \geq 3$ and $T' = T - \{v,w\}$ is not an odd star. By the induction hypothesis, $G_3(T')$ has a Hamilton cycle $f_0, f_1, \ldots, f_{N-1}, f_0$. For the remainder of the proof, subscripts are to be interpreted modulo N.

For $i = 0, 1, \ldots, N-1$, let F_i be the set of 3-colourings of T that agree with f_i on $V(T')$. Since v and w are leaves, each set F_i contains four 3-colourings of T. Two different vertices $c_1, c_2 \in V(G_3(T))$ belonging to F_i are adjacent if and only if $c_1(v) = c_2(v)$ or $c_1(w) = c_2(w)$. Thus, the subgraph of $G_3(T)$ induced by F_i is a 4-cycle.

Let a and b be the unique neighbours of v and w in T, respectively. Then $a \neq b$ since $d(v,w) \geq 3$. If $f_i(a) = f_{i+1}(a)$ and $f_i(b) = f_{i+1}(b)$, then every colouring in F_i has a neighbour in F_{i+1}, namely the colouring that agrees with it on v and w. Thus, in this case, $[F_i, F_{i+1}]$ consists of four vertex disjoint edges. Suppose $f_i(a) = f_{i+1}(a)$ and $f_i(b) \neq f_{i+1}(b)$. Let $\alpha = \{1,2,3\} - \{f_i(b), f_{i+1}(b)\}$. Then the two colourings in F_i that assign α to w each have a neighbour in F_{i+1}, as above. Futhermore, the subgraph induced by these four vertices is a 4-cycle. The case where $f_i(a) \neq f_{i+1}(a)$ and $f_i(b) = f_{i+1}(b)$ is similar and also leads to two vertex disjoint edges that join adjacent vertices in F_i to adjacent vertices of F_{i+1}.

We claim that if $[F_{i-1}, F_i]$ and $[F_i, F_{i+1}]$ each consist of two vertex disjoint edges, then these edges are incident with at least three vertices of F_i. By the argument above, $\{f_{i-1}(a), f_{i-1}(b)\} \neq \{f_i(a), f_i(b)\}$, and $\{f_i(a), f_i(b)\} \neq \{f_{i+1}(a), f_{i+1}(b)\}$. The colourings f_{i-1}, f_i and f_{i+1} can not assign three different colours to a, otherwise one of these colourings assigns the same colour to a as to one of its neighbours. Similarly, these colourings can not assign three different colours to b. Since $f_{i-1} \neq f_{i+1}$, it follows that $f_{i-1}(a) \neq f_{i+1}(a)$ and $f_{i-1}(b) \neq f_{i+1}(b)$. Without loss of generality $f_{i-1}(a) \neq f_i(a)$ and $f_i(b) \neq f_{i+1}(b)$. The two edges in $[F_{i-1}, F_i]$ are incident with colourings in F_i that assign the unique colour $\alpha \in \{1,2,3\} - \{f_{i-1}(a), f_i(a)\}$ to v. Similarly, the two edges in $[F_i, F_{i+1}]$ are incident with vertices in F_i that assign the unique colour $\beta \in \{1,2,3\} - \{f_i(b), f_{i+1}(b)\}$ to w. Since only one colouring in F_i assigns α to v and β to w, the proof of the claim is complete.

It may be assumed without loss of generality that $|[F_0, F_1]| = 2$. We now define vertices $s_i, s'_i, t_i, t'_i \in F_i$ which will be used to construct a Hamilton cycle in $G_3(T)$. The vertices $s_0, s'_0, t_0, t'_0, s_1, t_1$ are defined first.

- The vertices s'_0, t'_0, s_1 and t_1 are chosen so that $[F_0, F_1] = \{s'_0 s_1, t'_0 t_1\}$. (The vertex s_1 is adjacent to t_1 because $G_3(T)$ contains the 4-cycle $s'_0 s_1 t_1 t'_0 s'_0$.)
- The vertices s_0 and t_0 are chosen so that the 4-cycle $s_0, s'_0, t'_0, t_0, s_0$ is the subgraph of $G_3(T)$ induced by F_0.

Note that $s'_0 t'_0, s_0 t_0 \in E(G_3(T))$.

Suppose that the vertices $s_i, s'_i, t_i, t'_i \in F_i$ have been defined for all $0 \leq i < k$, that s_k and t_k have also been defined, and further, that

- $\{s'_{k-1} s_k, t'_{k-1} t_k\} \subseteq [F_{k-1} F_k]$;

P_4				P_5					
2121	3232	1313		12123	31212	21213	32121	32132	31232
2123	3231	1312		12323	31312	21313	32321	12132	21232
3123	1231	2312		12313	21312	21323	12321	13132	23232
3121	1232	2313		12312	21212	21321	12121	13232	23132
3131	1212	2323		32312	23212	31321	13121	13231	23131
2131	3212	1323		32313	13212	31323	13131	23231	23121
2132	3213	1321		31313	13213	32323	12131	21231	23123
3132	1213	2321		31213	23213	32123	32131	31231	13123

Figure 3: Gray codes of 3-colourings for P_4 and P_5.

- s_k is adjacent to t_k;
- the vertices s'_k and t'_k are distinct;
- the vertices s_k, s'_k, t'_k and t_k occur in this order along the path of length three from s_k to t_k (both $s_k = s'_k$ and $t_k = t'_k$ are possible); and
- if $|[F_k, F_{k+1}]| = 2$, then s_k or t_k has no neighbours in F_{k+1} (or neither do).

There are then three cases to consider.

- $k = N - 1$.
  Define s'_{N-1} and t'_{N-1} so that the 4-cycle $s_{N-1}, s'_{N-1}, t'_{N-1}, t_{N-1}, s_{N-1}$ is the subgraph of $G_3(T)$ induced by F_{N-1}.
- $k < N - 1$ *and* $|[F_k, F_{k+1}]| = 2$.
  Let R be the unique (s_k, t_k)-path of length three in the subgraph of $G_3(T)$ induced by F_k. Define s'_k and t'_k such that $[F_k, F_{k+1}] = \{s'_k s_{k+1}, t'_k t_{k+1}\}$ and s'_k precedes t'_k on R.
- $k < N - 1$ *and* $|[F_k, F_{k+1}]| = 4$.
  Let R be the unique (s_k, t_k)-path of length three in the subgraph of $G_3(T)$ induced by F_k. Define s'_k and t'_k to be adjacent vertices on R such that s'_k precedes t'_k and, if $k < N - 2$ and $|[F_{k+1}, F_{k+2}]| = 2$, then the neighbours of s'_k and t'_k in F_{k+1} are not both incident with an edge in $[F_{k+1}, F_{k+2}]$. (This is always possible as there are three choices for the pair of vertices s'_k and t'_k.) Then, define s_{k+1} and t_{k+1} so that $\{s'_k s_{k+1}, t'_k t_{k+1}\} \subseteq [F_k, F_{k+1}]$.

For $i = 0, 1, \ldots, N - 1$, we now define paths P_i and Q_i in the subgraph of $G_3(T)$ induced by F_i, and then use them to construct a Hamilton cycle in $G_3(T)$. The path P_i is the unique (s_i, s'_i)-path (perhaps of length zero) that does not include t_i. The path Q_i is the unique (t'_i, t_i)-path (perhaps of length zero) that does not include s_i. Then, the sequence $P_0, P_1, \ldots, P_{N-1}, Q_{N-1}, Q_{N-2}, \ldots, Q_0, s_0$ is a Hamilton cycle in $G_3(T)$.

The result now follows by induction. □

Corollary 5.6. *Let T be a tree. If T is an odd star with at least three vertices, then $k_0(T) = 4$. Otherwise, $k_0(T) = 3$.*

6 The Gray code number of cycles

In this section we prove that $k_0(C_n) = 4$ for all integers $n \geq 3$. We assume throughout this section that the n-cycle C_n has vertex set $V(C_n) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(C_n) = \{v_i v_{i+1} : 1 \leq i < n\} \cup \{v_n v_1\}$.

Theorem 6.1. *For any integer $n \geq 3$, $k_0(C_n) \geq 4$.*

Proof. For $n \geq 3$, the graph $G_3(C_n)$ is connected only if $n = 4$ [4]. It remains to show that $G_3(C_4)$ is not Hamiltonian. The 3-colourings 1213 and 1312 both have degree two in $G_3(C_4)$. Both of them are adjacent to 1212 and 1313. Since 1212, 1213, 1313, 1312, 1212 is not a Hamilton cycle, the graph $G_3(C_4)$ is not Hamiltonian. □

X_1: 21231 31231 31232 31212 31312 32312 12312 12313 12323 12123
13123 23123 23121 23131 23231

X_2: 12132 13132 13232 13212 13213 23213 21213 21313 21323 21321
31321 32321 32121 32131 32132

Figure 4: The two disjoint 15-cycles X_1 and X_2 that comprise $G_3(C_5)$.

For any $n \geq 3$, the colouring number $col(C_n) = 3$. Thus, by Theorems 3.4 and 6.1, we have that $4 \leq k_0(C_n) \leq 5$. Since $C_3 = K_3$, we know from Theorem 4.1 that $k_0(C_3) = 4$. We now show that $k_0(C_n) = 4$ for all $n \geq 3$.

Theorem 6.2. *For any integer $n \geq 3$, the graph $G_4(C_n)$ is Hamiltonian.*

Proof. By Theorem 4.1, we may assume $n \geq 4$. In what follows, we use the four colours a, b, c, d. Let P_{n-3} be the path on $n - 3$ vertices induced by $\{v_1, v_2, \ldots, v_{n-3}\}$. By Corollary 5.6 the graph $G_4(P_{n-3})$ is Hamiltonian. Let $f_0, f_1, \ldots, f_{N-1}, f_0$ be a Hamilton cycle in $G_4(P_{n-3})$. For $i = 0, 1, 2, \ldots, N - 1$, let F_i be the set of 4-colourings of C_n that agree with f_i on $V(P_{n-3})$.

Suppose $f_i(v_1) = f_i(v_{n-3})$. Then f_i can be extended to a colouring of C_n in 21 ways, so $|F_i| = 21$. The subgraph of $G_4(C_n)$ induced by F_i does not depend on the colour assigned to v_1 and v_{n-3}, and is isomorphic to the graph H_1 shown in Figure 5. The vertex labels are the possible colourings of $v_{n-2}v_{n-1}v_n$ (in one-line notation) in the case that $f_i(v_1) = f_i(v_{n-3}) = c$.

By checking all possibilities, it can be seen that the graph H_1 is Hamilton connected. However, we sketch a proof of this fact. Referring to Figure 5, let A, B, and D be the subgraphs of H_1 induced by the vertices whose labels start with a, b, and d, respectively. Then A, B and D are isomorphic. Further, there is an automorphism of H_1 that send $V(A)$ to $V(B)$, $V(B)$ to $V(D)$, and $V(D)$ to $V(A)$. By inspection, the graph A has a Hamilton path joining any pair of different vertices except those labelled $\{aba, acd\}$ and $\{ada, acb\}$. Thus, the graph B has a Hamilton path joining any pair of different vertices except those labelled $\{bdb, bca\}$ and $\{bab, bcd\}$, and the graph D has a Hamilton path joining any pair of different vertices except those labelled $\{dad, dcb\}$ and $\{dbd, dca\}$. Hamilton paths in A, B and D can be concatenated to form Hamilton paths between any given pair of vertices in H_1.

Otherwise, $f_i(v_1) \neq f_i(v_{n-3})$. In this case, f_i can be extended to a colouring of C_n in 20 ways, so $|F_i| = 20$. As above, the subgraph of $G_4(C_n)$ induced by F_i does not depend on the colours assigned to v_1 and v_{n-3}, and is isomorphic to the graph H_2 shown in Figure 6. The vertex labels are the possible colourings of $v_{n-2}v_{n-1}v_n$ (in one-line notation) in the case that $f_i(v_1) = f_i(v_{n-3}) = c$.

By checking all possibilities, it can be seen that the graph H_2 is Hamilton connected. However, we sketch a proof of this fact. Referring to Figure 6, let X and Y be the subgraphs of H_2 induced by the vertices labelled $\{bcb, bcd, bca, acb, acd, aca\}$ and $\{bdb, cdb, adb, bda, cda, ada\}$, respectively. Then the graphs X and Y are isomorphic to the Cartesian product of K_2 and C_3, and hence are Hamilton connected. Further, there is an automorphism of H_2 that sends $V(X)$ to $V(Y)$, and $V(Y)$ to $V(X)$. One can use appropriate Hamilton paths in X and Y as the basic building blocks towards Hamilton paths between any given two vertices of H_2.

Thus, $G_4(C_n)$ is a C-graph. We complete the proof by showing that the hypotheses of Corollary 3.3 are satisfied.

Suppose $|F_i| = |F_{i+1}|$ (here, and for the remainder of the proof subscripts are to be interpreted modulo N). If $f_i(v_1) = f_{i+1}(v_1)$ and $f_i(v_{n-3}) = f_{i+1}(v_{n-3})$, then each vertex in F_i has a neighbour in F_{i+1} corresponding to the 4-colouring in F_{i+1} that assigns the same colours to v_{n-2}, v_{n-1}, and v_n. Hence, in this case $[F_i, F_{i+1}]$ contains at least 20 vertex disjoint edges. If f_i and f_{i+1} differ in colour at one of v_1 or v_{n-3}, without loss of generality say v_1, then the 4-colourings in F_i which assign $f_{i+1}(v_1)$ to v_n have no neighbours in F_{i+1}. In this case $[F_i, F_{i+1}]$ contains 13 vertex disjoint edges.

Suppose $|F_i| \neq |F_{i+1}|$. By symmetry we can assume $|F_i| = 20$ and $|F_{i+1}| = 21$. We may also assume without loss of generality that $f_i(v_1) = f_{i+1}(v_1)$ and $f_i(v_{n-3}) \neq f_{i+1}(v_{n-3})$. The colouring f_{i+1} must assign colour $f_{i+1}(v_1)$ to v_{n-3} in order that it assign the same colour to v_1 and v_{n-3}. Since $f_{i+1}(v_1) = f_i(v_1)$, this colour is not assigned to v_n by any 4-colouring in $F_i \cup F_{i+1}$. In addition, the 4-colourings in F_i which assign the colour of $f_{i+1}(v_{n-3})$ to v_{n-2} have no neighbours in F_{i+1}. For each of the remaining 4-colourings in F_i there is a 4-colouring in F_{i+1} that assigns the same colours to v_{n-2}, v_{n-1}, and v_n. That is, $[F_i, F_{i+1}]$ contains 14 vertex disjoint edges.

By Corollary 3.3, the graph $G_4(C_n)$ is Hamiltonian. □

Corollary 6.3. *For all $n \geq 3$, we have $k_0(C_n) = 4$.*

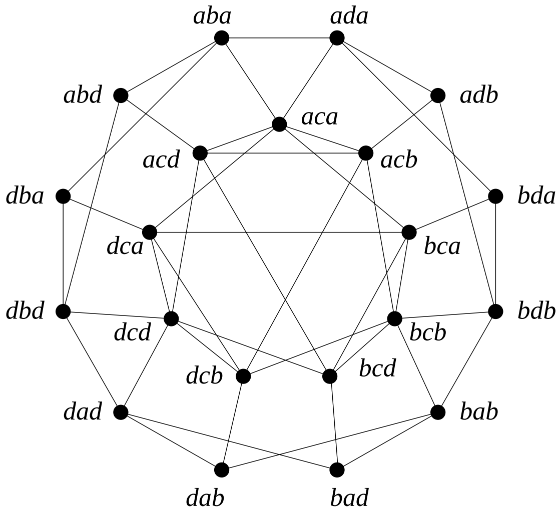


Figure 5: H_1.

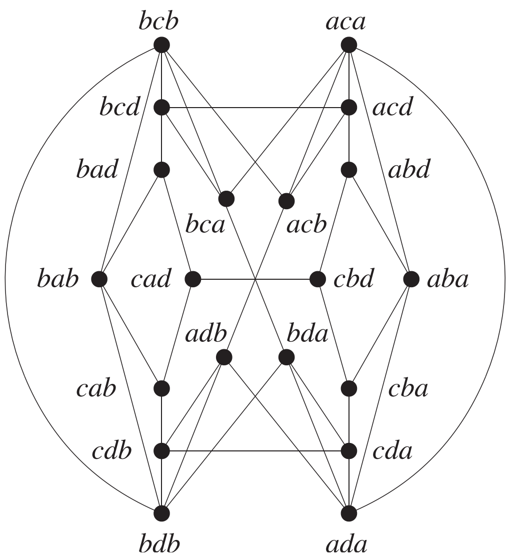


Figure 6: H_2.

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