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Riemann surfaces and restrictively-marked hypermaps

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Abstract

If $\mathcal{S}$ is a compact Riemann surface of genus $g > 1$, then $\mathcal{S}$ has at most $84(g-1)$ (orientation preserving) automorphisms (Hurwitz). On the other hand, if G is a group of automorphisms of $\mathcal{S}$ and $|G| > 24(g-1)$ then G is the automorphism group of a regular oriented map (of genus g) and if $|G| > 12(g-1)$ then G is the automorphism group of a regular oriented hypermap of genus g (Singerman). We generalise these results and prove that if $|G| > g-1$ then G is the automorphism group of a regular restrictedly-marked hypermap of genus g. As a special case we also show that a marked finite transitive permutation group (Singerman) is a restrictedly-marked hypermap with the same genus.

Keywords: Groups, Riemann surface, hypermaps, maps, restrictedly-marked, restrictedly regular.

Math. Subj. Class.: 05C15, 05C10, 30F10

1 Introduction

A compact Riemann surface $\mathcal{S}$ of genus $g > 1$ has at most $84(g-1)$ orientation-preserving automorphisms (Hurwitz [5]). On the other hand, if a group G of automorphisms of a compact Riemann surface $\mathcal{S}$ of genus g is sufficiently large, then it represents the automorphism group of a regular oriented map (if $|G| > 24(g-1)$) or hypermap (if $|G| > 12(g-1)$), (Singerman [8]). Hypermaps can be seen as restrictedly-marked maps, in this case restricted to the subgroup $\Upsilon^{\hat{0}} = \langle R_1, R_2, R_1^{R_0}\rangle \cong C_2 * C_2 * C_2$ of index 2 in $\Upsilon = \langle R_0, R_1, R_2 \mid R_i^2 = (R_2R_0)^2 = 1\rangle$ (Jones and Breda [2]). As such they are represented by bipartite-maps (2-coloured maps called Walsh bipartite maps) where the hypermap's automorphisms (similarly for homomorphisms) are bipartition-preserving map's automorphisms. Orientable maps (Υ^+-conservative maps) give rise to the restricted forms

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known as "oriented" maps (Υ^+-marked maps), described by triples (Ω, R, L) consisting of a finite set Ω and two Ω-permutations R and L, with $L^2 = 1$, generating a transitive group on Ω. Oriented hypermaps (Δ^+-marked hypermaps), the restricted forms of Δ^+-conservative (or orientable) hypermaps, are direct generalisations of that. More generally yet are the restrictedly-marked hypermaps, the restricted forms of "Θ-conservative" hypermaps, where Θ is a normal subgroup of finite index of $\Delta = C_2 * C_2 * C_2$. Often these result in "multi-coloured" maps.

In this paper we generalise Singerman's results and prove that if G is a group of automorphisms of a compact Riemann surface of genus $g > 1$, and $|G| > g - 1$, then G is the automorphism group of a regular restrictedly-marked hypermap of genus g.

1.1 Regular restrictedly-marked hypermaps

Algebraically, hypermaps correspond to finite transitive permutations representations $\nu : \Delta \longrightarrow G$, $R_i \mapsto r_i$, where Δ is the free product $C_2 * C_2 * C_2$ generated by the reflections in the sides of a hyperbolic triangle with zero internal angles in the hyperbolic plane. Let Θ be a normal subgroup of finite index n. A hypermap $\mathcal{H}$ is Θ-*conservative* if its fundamental subgroup H is a subgroup of Θ. In such case Θ acts on its flags uniformly dividing them into Θ-orbits of equal length. A Θ-conservative hypermap $\mathcal{H}$ is Θ-regular if and only if the group $Aut^{\Theta}(\mathcal{H})$, of the automorphisms of $\mathcal{H}$ preserving each Θ-orbit, acts transitively on each Θ-orbit; and this happens if and only if H is normal in Θ. A hypermap is *restrictedly-marked* if it is Θ-conservative, and is *restrictedly-regular* if it is Θ-regular, for some normal subgroup Θ of finite index in Δ. Not every hypermap is restrictedly-regular, see [1].

By the Kurosh Subgroup Theorem (see [7], Corollary 4.9.1 and remarks after Corollary 4.9.2), Θ freely decomposes uniquely (up to a permutation of factors) in a free product $C_2 * \cdots * C_2 * C_\infty * \cdots * C_\infty = \langle X_1, \ldots, X_m \mid X_i^2 = 1,\ i = 1 \ldots s\rangle$, for some $0 \leq s \leq m$. A Θ-conservative hypermap is combinatorially described by a $(m + 1)$-tuple

$$\mathcal{Q} = (\Omega; x_1, \ldots, x_m) \tag{1.1}$$

where Ω is a finite set (the set of the "Θ-slices"), $x_1, \ldots, x_m$ are permutations of Ω generating a group G acting transitively on Ω such that the function $\rho : X_i \mapsto x_i$ extends to an epimorphism from Θ to G. Such $(m + 1)$-tuple is called a Θ-*marked hypermap*. For example, if Θ is the even-word subgroup $\Delta^+ \cong F(2)$, a Δ^+-conservative (i.e. orientable) hypermap is described by a Δ^+-marked (i.e. oriented) hypermap $\mathcal{Q} = (\Omega; x_1, x_2)$, where x_1, x_2 are usually denoted by the letters R, L or by ρ, λ. On the other hand, if Q is the fundamental Θ-marked subgroup of $\mathcal{Q}$, i.e. the stabiliser of some fixed $\omega \in \Omega$ under the action of Θ on Ω, then the "Δ-*form*" hypermap ${}^{\Delta}\mathcal{Q} = (\Delta/_r Q; Q_\Delta R_0, Q_\Delta R_1, Q_\Delta R_2)$, where Q_Δ is the core of Q in Δ and $\Delta/_r Q$ is the set of the right cosets of Q in Δ, is Θ-conservative and shares with $\mathcal{Q}$ the same underlying surface, the same underlying hypergraph, the same set $\mathcal{V}$ of hypervertices, the same set $\mathcal{E}$ of hyperedges and the same set $\mathcal{F}$ of hyperfaces; only the set Ω of Θ-slices of $\mathcal{Q}$ is just an orbit of the action of Θ on the set of flags $F = \Delta/_r Q$ of ${}^{\Delta}\mathcal{Q}$. The sizes of Ω and F are $|\Theta : Q|$ and $|\Delta : Q|$ respectively. The permutations $x_1, \ldots, x_m$ are restrictions to Ω of the permutations $X_1\nu, \ldots, X_m\nu \in \mathrm{Sym}(F)$, where $\nu : \Delta \longrightarrow Mon({}^{\Delta}\mathcal{Q})$ is the canonical transitive permutation representation. A Θ-marked hypermap $\mathcal{Q}$ is regular if its Δ-form ${}^{\Delta}\mathcal{Q}$ is Θ-regular. Moreover, $\mathcal{Q}$ has boundary if ${}^{\Delta}\mathcal{Q}$ has boundary. That is, if $R_i \in Q^d$ for some $d \in \Delta$ and $i = 0, 1, 2$. If $\mathcal{Q}$ has no boundary, then the Euler characteristic of $\mathcal{Q}$ (we mean the Euler characteristic of its underlying surface, which is the Euler characteristic of ${}^{\Delta}\mathcal{Q}$) is given by $\chi(\mathcal{Q}) = |\mathcal{V}| + |\mathcal{E}| + |\mathcal{F}| - \frac{|F|}{2}$, where $|F| = n|\Omega|$.

Let $(k; l; m)$ be the type of the trivial Θ-marked hypermap $\mathcal{T}_{\Theta} = (\Theta/_r\Theta; \Theta X_1, \dots, \Theta X_m)$, with just one Θ-slice; geometrically a Θ-slice is the connected (polygonal) region obtained by the elements of a fixed Schreier transversal for Θ in Δ acting on a single flag (a triangular region). The Δ-form of $\mathcal{T}_{\Theta}$ is the hypermap ${}^{\Delta}\mathcal{T}_{\Theta} = (\Delta/_r\Theta; \Theta R_0, \Theta R_1, \Theta R_2)$ with n flags. If $\mathcal{Q}$ is a regular Θ-marked hypermap then $\mathcal{Q}$ regularly covers[1] $\mathcal{T}_{\Theta}$ and, as a consequence, its hypervertices (resp. hyperedges, hyperfaces) will be regularly partitioned (or coloured) in q_v parts, each part projecting to a hypervertex (resp. hyperedge, hyperface) of $\mathcal{T}_{\Theta}$. Hypervertices (resp. hyperedges, hyperfaces) of the same colour have the same valency. This induces a sequence

$$(k_1, \dots, k_{q_v}\,;\, l_1, \dots, l_{q_e}\,;\, m_1, \dots, m_{q_f})$$

called the Θ-*type* of $\mathcal{Q}$, where $k_1, \dots, k_{q_v}$ are the common valencies of the hypervertices in the same coloured-parts 1, ..., q_v, respectively, and similarly for the rest of the numbers. Here q_v, q_e and q_f are the numbers of hypervertices, hyperedges and hyperfaces of $\mathcal{T}_{\Theta}$, respectively. The Euler characteristic of $\mathcal{Q}$ is then expressed by (cf §9.1 of [1])

$$\chi(\mathcal{Q}) = |\Omega| \left(\sum_{i=1}^{q_v} \frac{\mu_v\, k}{2k_i} + \sum_{i=1}^{q_e} \frac{\mu_e\, l}{2l_i} + \sum_{i=1}^{q_f} \frac{\mu_f\, m}{2m_i} - \frac{n}{2} \right) \tag{1.2}$$

where $\mu_v = 1$ or 2 according as the hypervertex v of $\mathcal{T}_{\Theta}$ lies on the boundary or not, and similarly for μ_e and μ_f. Note that n is the index of Θ in Δ. For further reading on the subject, and for geometric illustrations, we refer the reader to [1].

2 Two special restrictedely-marked subgroups

Example 1. Let Θ be the subgroup Δ_n in Δ of index $2n$ generated by the $n+1$ generators

$$Z_1 = R_1R_2\,,\ \ Z_2 = (R_1R_2)^{R_0R_2}, \dots, Z_n = (R_1R_2)^{(R_0R_2)^{n-1}} \text{ and } Z_{n+1} = (R_2R_0)^n\,.$$

This group is normal in $\Delta = \langle R_0, R_1, R_2\rangle = \langle Z_1, R_0R_2, R_2\rangle$, for $Z_i^{R_0R_2} = Z_{i+1}$, $i = 1, 2, \dots, n-1$, $Z_n^{R_0R_2} = Z_1^{Z_{n+1}^{-1}}$, $Z_{n+1}^{R_0R_2} = Z_{n+1}$, $Z_1^{R_2} = Z_1^{-1}$, $Z_i^{R_2} = (Z_{n-(i-2)}^{Z_{n+1}})^{-1}$, $i = 2, \dots, n$ and $Z_{n+1}^{R_2} = Z_{n+1}^{-1}$. The quotient Δ/Δ_n is a dihedral group D_n of order $2n$. By the Reidmaster-Schreier Rewriting Process, Δ_n is isomorphic to a free product $C_\infty * \cdots * C_\infty$ ($n+1$ times), that is, a free group of rank $n+1$, and consequently any Δ_n-marked hypermap has the form $\mathcal{Q} = (\Omega, z_1, \dots, z_{n+1})$. As the generators lie in $\Delta_1 = \Delta^+ = \langle R_1R_2, R_2R_0\rangle$ (the subgroup of the even-length words of Δ), any Δ_n-marked hypermap is orientable. The trivial Δ_n-marked hypermap $\mathcal{T}_{\Delta_n}$ is the regular hypermap of type $(1; n; n)$ on the sphere (pictured below for the case when $n = 7$) with n hypervertices, one hyperedge and one hyperface. Any Δ_n-conservative hypermap $\mathcal{H}$ covers $\mathcal{T}_{\Delta_n}$ and so its hypervertices are n-coloured, which makes $\mathcal{H}$ a $(n+2)$-coloured map.

[1] A (hyper)map $\mathcal{M}'$ with fundamental subgroup M' regularly covers $\mathcal{M}$ with fundamental subgroup M if M' is a normal subgroup of M (Jones and Singerman [6]).

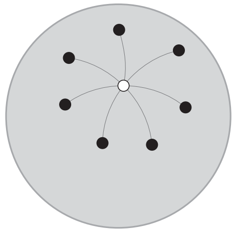

Figure 1: The trivial Δ_7-marked hypermap $\mathcal{T}_{\Delta_7}$.

The Δ_n-type of $\mathcal{Q}$ is therefore $(k_1, \ldots, k_n; l; m)$ for some positive integers $k_1, \ldots, k_n$, l and m such that n divides both l and m. By (1.2), the Euler characteristic of a regular Δ_n-marked hypermap of Δ_n-type $(k_1, \ldots, k_n; l; m)$ is given by

$$\chi(\mathcal{Q}) = |\Omega| \left(\frac{1}{k_1} + \cdots + \frac{1}{k_n} + \frac{n}{l} + \frac{n}{m} - \frac{2n}{2} \right) . \tag{2.1}$$

If $\mathcal{Q} = (\Omega, z_1, \ldots, z_{n+1})$ is a Δ_n-marked hypermap then each orbit of z_i, for $i = 1, \ldots, n$, corresponds to a i-coloured hypervertex v_i (denoting the i-coloured hypervertices by v_i, w_i, etc.) while each orbit of z_{n+1} corresponds to a hyperedge (a white vertex in the picture below) and each orbit of the product $z_1 \ldots z_{n+1}$ correspond to a hyperface.

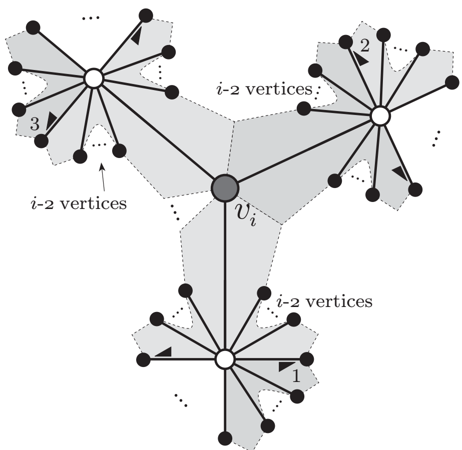

Figure 2: Part of a z_i-orbit ($i \geq 2$) labeled 1,2,3,...

If $\mathcal{Q}$ is regular (equivalently, its Δ-form ${}^{\Delta}\mathcal{Q} = (F, r_0, r_1, r_2)$ is Δ_n-regular) the automorphism group $Aut(\mathcal{Q})$, which is the Δ_n-automorphism group of its Δ-form ${}^{\Delta}\mathcal{Q}$, coincides with the monodromy group $Mon(\mathcal{Q})$ - only the actions of $Mon(\mathcal{Q})$ and $Aut(\mathcal{Q})$ on F are different. Hence $Aut(\mathcal{Q})$ is generated by $z_1, \ldots, z_{n+1}$ (considered as automorphisms). Each $Z_i \in \Delta_n$, $i = 1, \ldots, n$, is a conjugate of the product of two of the hyperbolic reflections R_0, R_1 and R_2, so $Z_1, \ldots, Z_n$ are (parabolic) limit rotations about hypervertices, Z_{n+1} is a n-step limit rotation about a hyperedge and the product $Z_1 \ldots Z_n$ is a n-step limit rotation about a hyperface. They project via ρ to automorphisms $z_i \in Aut(\mathcal{Q})$. Notice that ${}^{\Delta}\mathcal{Q}$ may be not regular (that is, $\mathcal{Q}$ may be not Δ-symmetric); if this is the case, the size of the automorphism group $Aut({}^{\triangle}\mathcal{Q})$ is smaller than $|F|$ and consequently at least one of r_0, r_1 and r_2 cannot be realised as an automorphism of ${}^{\triangle}\mathcal{Q}$. However, $Aut({}^{\triangle}\mathcal{Q})$ acts regularly on the Δ_n-orbit Ω and thus $z_1, \ldots, z_n$ can be realised as automorphisms of ${}^{\triangle}\mathcal{Q}$. Now $\mathcal{Q}$ has n hypervertex-colours $v_1, v_2, \ldots, v_n$, only one hyperedge-colour e and only one hyperface-colour f. These n hypervertex-colours appear around any hyperedge as well as around any hyperface as shown on Figure 3. The generators $z_1, \ldots, z_n$, being the projections of $Z_1, \ldots, Z_n$, are clearly rotations about the hypervertices $v_1, \ldots, v_n$ of orders $k_1, \ldots, k_n$ respectively. The generator z_{n+1} is a rotation about e of order $\frac{l}{n}$ and the product $z_{n+2} = z_1 z_2 \ldots z_{n+1} = (r_1 r_0)^n$ is a rotation about f of order $\frac{m}{n}$.

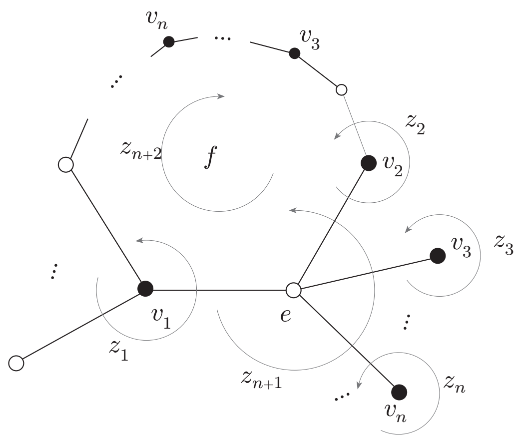

Figure 3: $z_1, \ldots, z_{n+2}$ seen as automorphisms of $\mathcal{Q}$.

Formula (2.1) can then be rewritten as follows.

Lemma 2.1. *If $\mathcal{Q} = (\Omega, z_1, \ldots, z_{n+1})$ is a regular Δ_n-marked hypermap then the Euler characteristic of $\mathcal{Q}$ is given by*

$$\chi(\mathcal{Q}) = |\Omega| \left(\frac{1}{|z_1|} + \cdots + \frac{1}{|z_n|} + \frac{1}{|z_{n+1}|} + \frac{1}{|z_1 \ldots z_{n+1}|} - n \right) .$$

If G is a group generated by $g_1, \ldots, g_m$ such that the function $x_i \mapsto g_i$ extends to an epimorphism from Θ to G, then $\mathcal{Q} = (G, g_1, \ldots, g_m)$ is a regular Θ-marked hypermap (Theorem 22 of [1]). A slightly more general statement is obtained by taking the free group $\Theta = \Delta_{m-1} < \Delta^+$ of rank m:

Lemma 2.2. *If G is a group generated by $g_1, \ldots, g_m$, then $\mathcal{Q} = (G, g_1, \ldots, g_m)$ is a regular Δ_{m-1}-marked hypermap, and so, a regular restrictedely-marked hypermap.*

Example 2. The subgroup $\Theta = K_3$ of Δ of index 6 generated by $A = R_0 {R_1}^{R_2}$, $B = R_0 {R_2}^{R_1}$, $C = R_1 {R_0}^{R_2}$ and $D = R_1 {R_2}^{R_0}$ is normal in Δ, for $A^{R_0} = A^{-1}$, $A^{R_1} = DB$, $A^{R_2} = C^{-1}$, $B^{R_0} = B^{-1}$, $B^{R_1} = DA$, $B^{R_2} = C^{-1}B^{-1}A$, $C^{R_0} = BD$, $C^{R_1} = C^{-1}$, $C^{R_2} = A^{-1}$, $D^{R_0} = BC$, $D^{R_1} = D^{-1}$ and $D^{R_2} = A^{-1}D^{-1}C$. This group factors Δ into a dihedral group D_3 with 6 elements. By the Reidmaster-Schreier Rewriting Process,

K_3 is isomorphic to a free product $C_\infty * C_\infty * C_\infty * C_\infty$ of rank 4. Therefore a regular K_3-marked hypermap has representative form $\mathcal{Q} = (G; a, b, c, d)$ where G is a group generated by a, b, c, d, without further restrictions. Moreover, as K_3 is a subgroup of Δ^+, any K_3-marked hypermap is orientable. The trivial K_3-hypermap $\mathcal{T}_{K_3}$ is the reflexible hypermap $(3, 3, 3)_{1,0}$ of type $(3, 3, 3)$ on the torus whose Walsh map is the regular map $\{6, 3\}_{1,0}$.

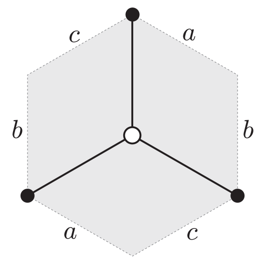

Figure 4: The trivial K_3-hypermap $\mathcal{T}_{K_3}$.

This has 1 hypervertex, 1 hyperedge and 1 hyperface. Therefore the K_3-type of any regular K_3-marked hypermap coincides with its topological type (or Δ-type) $(k; l; m)$ for some $k = 0 \mod 3$, $l = 0 \mod 3$ and $m = 0 \mod 3$. By (1.2), $\mathcal{Q}$ has Euler characteristic given by

$$\chi(\mathcal{Q}) = |G| \left(\frac{3}{k} + \frac{3}{l} + \frac{3}{m} - 3 \right) .$$

Since

$$k = |r_1 r_2| = 3|(r_1 r_2)^3| = 3|a^{-1}b|,$$

and similarly $l = 3|c^{-1}d|$ and $m = 3|bca^{-1}d^{-1}|$, we have:

Lemma 2.3. *The Euler characteristic of a regular K_3-marked hypermap $\mathcal{Q} = (G; a, b, c, d)$ is given by*

$$\chi(\mathcal{Q}) = |G| \left(\frac{1}{|a^{-1}b|} + \frac{1}{|c^{-1}d|} + \frac{1}{|bca^{-1}d^{-1}|} - 3 \right) .$$

3 Riemann surfaces

Let $\mathcal{S}$ be a Riemann surface of genus $g \geq 2$. By the uniformization theorem S is a quotient $\mathbb{U}/\Gamma$, where Γ is a cocompact torsion-free discrete subgroup of $Isom^+(\mathbb{U}) \cong PSL(2, \mathbb{R})$, the group of orientation-preserving isometries of the hyperbolic plane $\mathbb{U}$ (modelled on the complex upper half-plane). The group Γ, the *surface-group* corresponding to S, is a Fuchsian group with signature $(g; -)$; it is unique up to a conjugacy in $PSL(2, \mathbb{R})$. Automorphisms of $\mathcal{S}$ lift to isometries of $\mathbb{U}$ normalising Γ, so if G is a group of automorphisms of $\mathcal{S}$ then $G = \Lambda/\Gamma$, where Λ is a Fuchsian group containing Γ as a normal subgroup. Let $(\lambda; m_1, \ldots, m_r)$ be the signature of Λ. This means that Λ has presentation

$$\langle x_1, ..., x_r, a_1, b_1, ..., a_\lambda, b_\lambda \mid x_1^{m_1} = ... = x_r^{m_r} = \prod_{i=1}^{r} x_i \prod_{i=1}^{\lambda} [a_i, b_i] = 1 \rangle .$$

This has measure $\mu(\Lambda) = 2\pi\left(2\lambda - 2 + \sum_{i=1}^{r}(1-\frac{1}{m_i})\right)$. The Riemann-Hurwitz formula $\mu(\Gamma) = |\Lambda/\Gamma|\,\mu(\Lambda)$ can be written as

$$2g-2 = |G|\left(2\lambda - 2 + \sum_{i=1}^{r}(1-\frac{1}{m_i})\right) \Leftrightarrow \chi(\mathcal{S}) = |G|\left(\sum_{i=1}^{r}\frac{1}{m_i} - (r+2\lambda-2)\right), \tag{3.1}$$

where $\chi(\mathcal{S}) = 2-2g$ and the sum is considered empty in the case $r=0$. Since the order of the periods in the signature is irrelevant, we assume that $m_1 \leq m_1 \leq \cdots \leq m_r$. Using the same terminology as in [3], we say that a group G acts on genus g with signature $(\lambda; m_1,\ldots,m_r)$ if $G = \Lambda/\Gamma$ where Λ and Γ are Fuchsian groups such that (1) Λ has signature $(\lambda; m_1,\ldots,m_r)$ and (2) Γ has signature $(g;-)$ and is normal in Λ. In this case G acts on the Riemann surface $\mathcal{S} = \mathbb{U}/\Gamma$ as a group of automorphisms. The canonical epimorphism from Λ to G with kernel Γ is called a *surface-kernel epimorphism*. Surface-kernel epimorphisms are order-preserving, so the images of the generators $x_1,\ldots,x_r$ in G have orders $m_1,\ldots,m_r$ respectively.

Theorem 3.1. *Let $\mathcal{S}$ be a Riemann surface of genus $g>1$ and let G be a group of automorphism of $\mathcal{S}$. If $|G|>g-1$ then G is the automorphism group of a regular restrictedly-marked hypermap $\mathcal{Q}$ of genus g. Moreover, G is the automorphism group of a regular K_3-marked hypermap if G acts with signature $(1; m_1,\ldots,m_r)$, and of a regular Δ_{r-2}-marked hypermap if G acts with signature $(0; m_1,\ldots,m_r)$.*

Proof. Let $G = \Lambda/\Gamma$ for some Fuchsian group Λ normalising Γ (the surface-group) and let $(\lambda; m_1,\ldots,m_r)$ be the signature of Λ. By the Riemann-Hurwitz formula,

$$|G| = \frac{2(g-1)}{2\lambda - 2 + \sum_{i=1}^{r}\left(1-\frac{1}{m_i}\right)}.$$

Now $|G|>g-1$ implies

$$0 < 2\lambda - 2 + \sum_{i=1}^{r}\left(1-\frac{1}{m_i}\right) < 2,$$

and this inequality implies $\lambda \leq 1$.

I : $\lambda = 1$. In this case $0<r<4$.

(a) $\mathbf{r=3}$ Then Λ has signature $(1; m_1, m_2, m_3)$ and thus

$$G = \langle x_1, x_2, x_3, a, b \mid x_1^{m_1} = x_2^{m_2} = x_3^{m_3} = x_1x_2x_3[a,b] = 1, \ldots\rangle.$$

Fourth relation gives $x_2 = x_1^{-1}[a,b]^{-1}x_3^{-1}$. Replacing x_2 by its inverse we may take $x_2 = x_3[a,b]x_1$ to get

$$G = \langle x_1, x_3, a, b \mid x_1^{m_1} = x_3^{m_3} = (x_3[a,b]x_1)^{m_2} = 1, \ldots\rangle.$$

Equation (3.1) can then be written as

$$\chi(\mathcal{S}) = |G|\left(\frac{1}{m_1}+\frac{1}{m_2}+\frac{1}{m_3}-3\right).$$

Now we will try to see G as the automorphism group of a regular K_3-marked hypermap $\mathcal{Q} = (G; A, B, C, D)$. Taking into account Lemma 2.3, we will try to solve

$$|A^{-1}B| = m_3\,,\quad |C^{-1}D| = m_1 \text{ and } |BCA^{-1}D^{-1}| = m_2\,,$$

knowing that $|x_1| = m_1$, $|x_3| = m_3$ and $|x_3[a,b]x_1| = m_2$.

We do this in two steps.

First, since $m_1 = |C^{-1}D| = |DC^{-1}| = |CD^{-1}|$, we set $CD^{-1} = x_1$, from which we get $D = x_1^{-1}C$. Then putting $x_3[a,b]x_1 = BCA^{-1}D^{-1} = BCA^{-1}C^{-1}x_1$, we get $x_3[a,b] = BCA^{-1}C^{-1}$.

Second, since $|A^{-1}B| = |BA^{-1}|$, set $BA^{-1} = x_3$. This gives $B = x_3A$. Replacing B in the above equation we get $[a,b] = ACA^{-1}C^{-1} = [A^{-1}, C^{-1}]$.

Hence $A = a^{-1}$, $C = b^{-1}$, $B = x_3A = x_3a^{-1}$ and $D = x_1^{-1}C = x_1^{-1}b^{-1}$ is a solution. Since they generate G,

$$\mathcal{Q} = (G; a^{-1}, x_3a^{-1}, b^{-1}, x_1^{-1}b^{-1})$$

is a regular K_3-marked hypermap with automorphism group G and Euler characteristic

$$\begin{aligned}\chi(\mathcal{Q}) &= |G|\left(\tfrac{1}{|A^{-1}B|} + \tfrac{1}{|C^{-1}D|} + \tfrac{1}{|BCA^{-1}D^{-1}|} - 3\right)\\ &= |G|\left(\tfrac{1}{|ax_3a^{-1}|} + \tfrac{1}{|bx_1^{-1}b^{-1}|} + \tfrac{1}{|x_3[a,b]x_1|} - 3\right)\\ &= |G|\left(\tfrac{1}{m_3} + \tfrac{1}{m_1} + \tfrac{1}{m_2} - 3\right)\,,\end{aligned}$$

which gives $genus(\mathcal{Q}) = g = genus(\mathcal{S})$.

(b) r = 2 In this case $G = \langle x_1, x_2, a, b \mid x_1^{m_1} = x_2^{m_2} = x_1x_2[a,b] = 1, \dots\rangle = \langle x, a, b \mid x^{m_1} = ([a,b]x)^{m_2} = 1, \dots\rangle$. Equation (3.1) takes the form

$$\chi = |G|\left(\frac{1}{m_1} + \frac{1}{m_2} - 2\right)\,.$$

To see G as the automorphism group of a regular K_3-marked hypermap $\mathcal{Q} = (G; A, B, C, D)$ with $\chi = |G|(\frac{1}{|A^{-1}B|} + \frac{1}{|C^{-1}D|} + \frac{1}{BCA^{-1}D^{-1}|} - 3)$ we start by putting $A^{-1}B = 1$, that is, $B = A$. Comparing formulas we must now have $|C^{-1}D| = m_1$ and $|ACA^{-1}D^{-1}| = m_2$. Following a similar procedure as above, we easily get a solution $A = a^{-1}$, $B = A = a^{-1}$, $C = b^{-1}$, $D = x^{-1}b^{-1}$ and so $\mathcal{Q} = (G; a^{-1}, a^{-1}, b^{-1}, x^{-1}b^{-1})$ (or $(G; a, a, b, x^{-1}b)$) is a regular K_3-marked hypermap with automorphism group G and Euler characteristic

$$\chi(\mathcal{Q}) = |G|\left(\tfrac{1}{m_1} + \tfrac{1}{m_2} - 2\right) = \chi(\mathcal{S}).$$

(c) r = 1 If G acts with signature $(1; m)$ then G has presentation $\langle a, b \mid [a,b]^m = \dots = 1\rangle$. Expression (3.1) becomes

$$|G|\left(\frac{1}{m} - 1\right) = \chi(\mathcal{S})\,. \tag{3.2}$$

In this case the regular K_3-marked hypermap $\mathcal{Q} = (G; a, a, b, b)$ has Euler characteristic

$$\chi(\mathcal{Q}) = |G| \left(\frac{1}{m} - 1 \right) \;=\; \chi(\mathcal{S}) \,. \tag{3.3}$$

II : $\lambda = \mathbf{0}$. In this case $2 < \sum_{i=1}^{r} (1 - \frac{1}{m_i}) < 4$ implies $2 < r < 8$.

If G acts with signature $(0; m_1, \ldots, m_r)$ then G has presentation $\langle x_1, \ldots, x_{r-1} \mid x_1^{m_1} = \cdots = x_{r-1}^{m_{r-1}} = (x_1..x_{r-1})^{m_r} = ... = 1 \rangle$ with $r-1$ elliptic generators and no parabolic generators. In this case the Riemann-Hurwitz formula yields

$$\chi(\mathcal{S}) = |G| \left(\frac{1}{m_1} + \cdots + \frac{1}{m_{r-1}} + \frac{1}{m_r} - (r-2) \right) .$$

It is clear that the (canonical) regular Δ_{r-2}-marked hypermap $\mathcal{Q} = (G; x_1, \ldots, x_{r-1})$ will do. The product $x_r = x_1 \ldots x_{r-1}$ has order m_r and so by Lemma 2.1,

$$\chi(\mathcal{Q}) = |G| \left(\sum_{i=1}^{r} \frac{1}{m_i} - (r-2) \right) = \chi(\mathcal{S})$$

which shows that $genus(\mathcal{Q}) = g$. □

4 A note on finite marked permutations groups

Marked finite transitive permutation groups (MFTPG) are triples $\mathfrak{M} = (G, \Omega, D)$ consisting of a finite set Ω and a set $D = \{z_1, \ldots, z_{n-1}\}$ of permutations of Ω generating G and acting transitively on Ω (Singerman [9]). The genus of a marked finite transitive permutation group $\mathfrak{M}$ was defined by Singerman as being the genus of a certain Riemann surface $\mathcal{S} := \mathbb{X}/N$ "naturally" associated to $\mathfrak{M}$. Although not originally aimed to give an insight on higher dimensional combinatorial structures associated to higher dimensional manifolds, MFTPGs are related with Vince's combinatorial maps [10], or the more general Ferri's connected $(n+1)$-coloured graphs [4]. In fact, any finite combinatorial map $\mathcal{G}$ of rank I is a marked finite transitive permutation group $(G, V(G), D)$ where $V(G)$ is the set of vertices of $\mathcal{G}$ and D consists of $|I|$ fixed-point free involutory permutations of $V(G)$ induced by the $|I|$ edge-colours of $\mathcal{G}$. The transitivity of G on $V(G)$ is a consequence of the connectivity of $\mathcal{G}$ and the fixed-point free action of the involutory permutations a consequence of the non-existence of free edges in $\mathcal{G}$ (when realised as cell decompositions this corresponds to boundary-free manifolds). These were all designed to describe cell decompositions of n-dimensional manifolds, n-polytopes and tessellations ($n = |I| - 1$)), though in general not all such combinatorial constructions of rank > 3 are realised in such a way. Despite the name, a combinatorial map is just a graph and when its degree (rank) is 3 it actually describes a 2-dimensional simplicial complex (best known as a hypermap).

In the special case when $D = \{z_1, z_2\}$ and z_1 is a fixed-point free involution, which corresponds to an oriented map, Singerman showed that the associated "Riemann" genus coincide with the genus of the map. In general Singerman did not associate MFTPGs to embeddings of graphs on surfaces (of genus g), except in the above mentioned case. We can now show that actually any marked finite transitive permutation group $\mathfrak{M}$ of genus g represents a restrictedly-marked hypermap $\mathcal{Q}$ of genus g. These "coloured" maps representing restrictedly-marked hypermaps are not related with Vince's and Ferri's edge-coloured graphs.

Let $\mathfrak{M} = (G, \Omega, D)$ be a marked finite transitive permutation group with $D = \{z_1, \ldots, z_{n-1}\}$. If $rank(\mathfrak{M}) = |D| = 1$ then G is cyclic generated by one element and by the transitivity of G on Ω we must have $|\Omega| = |G|$. So $\mathfrak{M}$ is a regular oriented (Δ^+-restricted) map $\mathcal{M} = (G; z_1, 1)$.

For the rest of the paper let $rank(\mathfrak{M}) > 1$ ($n > 2$). Let $k_1, \ldots, k_{n-2}$ be the orders of $z_1, \ldots, z_{n-2}$ respectively, $l := n|z_{n-1}|$ and $m := n|z_n|$, where $z_n = z_1 \ldots z_{n-1}$. Let Γ be a F-group with signature $(0; k_1, \ldots, k_{n-2}, \frac{l}{n}, \frac{m}{n})$; this means that Γ has presentation

$$\langle x_1, \ldots, x_{n-2}, x_{n-1}, x_n \mid x_1^{k_1} = \cdots = x_{n-2}^{k_{n-2}} = x_{n-1}^{\frac{l}{n}} = x_n^{\frac{m}{n}} = \prod_{i=1}^{n} x_i = 1\rangle .$$

This has measure given by $\mu(\Gamma) = 2\pi\left(n - 2 - (\Sigma_{i=1}^{n-2} \frac{1}{k_i} + \frac{n}{l} + \frac{n}{m})\right)$. We have an obvious epimorphism $\varrho : \Gamma \to G$ defined by $x_i \mapsto z_i$ for $i = 1, \ldots, n-1$. Let $N = Stab_G(w)\varrho^{-1} \lhd_q \Gamma$, where $q = |\Theta : Q| = |\Omega|$. This group N is also a F-group and $\mathcal{S} := \mathbb{X}/N$ is a Riemann surface, where $\mathbb{X}$ is the sphere $\mathbb{S}$ if $\mu(\Gamma) < 0$, the Euclidean plane $\mathbb{C}$ if $\mu(\Gamma) = 0$ or the hyperbolic plane $\mathbb{H}$ if $\mu(\Gamma) > 0$ (Singerman [9]). Singerman defined the genus of $\mathfrak{M}$ to be the genus of the Riemann surface $\mathcal{S}$. On the other hand, $\mathfrak{M}$ corresponds to an orientable Δ_{n-2}-marked hypermap $\mathcal{Q} = (\Omega, z_1, \ldots, z_{n-1})$, where $\Delta_{n-2} = \langle Z_1, \ldots, Z_{n-1}\rangle \lhd_{2(n-2)} \Delta$ is the group generated by $Z_1 = R_1R_2$, $Z_2 = (R_1R_2)^{R_0R_2}$, ..., $Z_{n-2} = (R_1R_2)^{(R_0R_2)^{n-3}}$ and $Z_{n-1} = (R_2R_0)^{n-2}$. Let Z_n be the product $Z_1 \ldots Z_{n-1} = (R_1R_0)^{n-2}$ and Q its Δ_{n-2}-marked fundamental subgroup . The orientable restricted marked hypermap $\mathcal{Q}$ has genus $g(\mathcal{Q}) = \frac{2-\chi}{2}$ where $\chi = |\mathcal{V}| + |\mathcal{E}| + |\mathcal{F}| - \frac{|F|}{2}$ is the Euler characteristic of the underlying surface (F is the set of flags of its Δ-form ${}^{\Delta}\mathcal{Q}$). Although natural, it is not yet clear whether $\mathcal{Q}$ has genus g (see theorem 6), since different Θ-marked hypermaps can be associated to a given group, with different genus.

Theorem 4.1. *A marked finite permutation group* $\mathfrak{M} = (G, \Omega, D)$*, where* $D = \{z_1, \ldots, z_{n-1}\}$*, is an orientable* Δ_{n-2}*-marked hypermap* $\mathcal{Q} = (\Omega, z_1, \ldots, z_{n-1})$ *with* $genus(\mathcal{Q}) = genus(\mathfrak{M})$.

Proof. Let $g = genus(\mathfrak{M})$. Decompose $z_1, \ldots, z_n$ in a product of cycles:

$$\begin{aligned} z_1 &= \sigma_{1,1} \cdots \sigma_{1,t_1} \\ &\cdots \\ z_{n-2} &= \sigma_{n-2,1} \cdots \sigma_{n-2,t_{n-2}} \\ z_{n-1} &= \alpha_1 \cdots \alpha_s \\ z_n &= \beta_1 \cdots \beta_r . \end{aligned}$$

Then the number of hypervertices $|\mathcal{V}|$, the number of hyperedges $|\mathcal{E}|$ and the number of hyperfaces $|\mathcal{F}|$ are given by $|\mathcal{V}| = \sum_{i-1}^{n-2} t_i$, $|\mathcal{E}| = s$ and $|\mathcal{F}| = r$, respectively. The number of flags $|\Omega| = |\Theta : Q| = q$ is related to the above cycle decomposition as

$$|\Omega| = \sum_{j=1}^{t_i} |\sigma_{i,j}| = \sum_{i=1}^{s} |\alpha_i| = \sum_{i=1}^{r} |\beta_i| .$$

Now the F-group N has signature (Singerman [9])

$$\left(g; \frac{k_1}{\delta_{1,1}}, \ldots, \frac{k_1}{\delta_{1,t_1}}, \ldots, \frac{k_{n-2}}{\delta_{n-2,1}}, \ldots, \frac{k_{n-2}}{\delta_{n-2,t_{n-2}}}, \frac{\frac{l}{n}}{a_1}, \ldots, \frac{\frac{l}{n}}{a_s}, \frac{\frac{m}{n}}{b_1}, \ldots, \frac{\frac{m}{n}}{b_r}\right),$$

where $\delta_{i,j} = |\sigma_{i,j}|$, $a_i = |\alpha_i|$ and $b_i = |\beta_i|$. The measures of the F-groups N and Γ are related by $\mu(N) = |\Omega|\,\mu(\Gamma)$. This formula translates as

$$2g-2+\sum_{i=1}^{t_1}(1-\frac{\delta_{1,i}}{k_1})+\cdots+\sum_{i=1}^{t_{n-2}}(1-\frac{\delta_{n-2,i}}{k_{n-2}})+\sum_{i=1}^{s}(1-\frac{a_i}{\frac{l}{n}})+\sum_{i=1}^{r}(1-\frac{b_i}{\frac{m}{n}}) = |\Omega|(n-2-\sum_{i=1}^{n-2}\frac{1}{k_i}-\frac{n}{l}-\frac{n}{m}).$$

Since $\sum_{j=1}^{t_i}\delta_{i,j} = |\Omega|$, $\sum_{i=1}^{s} a_i = |\Omega|$ and $\sum_{i=1}^{r} b_i = |\Omega|$, we get

$$2g-2+\sum_{i=1}^{n-2}t_i+s+r-\frac{|\Omega|}{k_1}-\cdots-\frac{|\Omega|}{k_{n-2}}-\frac{|\Omega|}{\frac{l}{n}}-\frac{|\Omega|}{\frac{m}{n}} = |\Omega|(n-2)-\sum_{i=1}^{n-2}\frac{|\Omega|}{k_i}-|\Omega|(\frac{n}{l}+\frac{n}{m}).$$

Now replacing $\sum_{i=1}^{n-2}t_i$ by $|\mathcal{V}|$, s by $|\mathcal{E}|$, r by $|\mathcal{F}|$ and taking into account that $|\Delta:\Delta_n| = 2(n-2)$, and thus $|\Omega|2(n-2) = |\Theta:Q||\Delta:\Theta| = |\Delta:Q| = |F|$, we get

$$|\mathcal{V}|+|\mathcal{E}|+|\mathcal{F}|-\frac{|F|}{2} = 2-2g\,.$$

Thus $\chi(\mathcal{Q}) = 2-2g$ and hence $genus(\mathcal{Q}) = g$. □

Any hypermap $\mathcal{H}$ is a Δ-marked hypermap. As $\mathcal{H}$ may also be represented by some Θ-marked hypermap $\mathcal{Q}$, with flags in $\mathcal{H}$ being represented by Θ-slices in $\mathcal{Q}$, the genus of the Δ-marked hypermap $\mathcal{H}$ may be different from the genus of the Θ-marked hypermap $\mathcal{Q}$. Theorem 4.1 just emphasises how natural is the choice of Δ_n in the restrictedly-marked subgroups considered earlier.

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