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Realisation of groups as automorphism groups in permutational categories

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Abstract

It is shown that in various categories, including many consisting of maps or hypermaps, oriented or unoriented, of a given hyperbolic type, or of coverings of a suitable topological space, every countable group A is isomorphic to the automorphism group of uncountably many non-isomorphic objects, infinitely many of them finite if A is finite. In particular, the latter applies to dessins d'enfants, regarded as finite oriented hypermaps.

Keywords: Permutation group, centraliser, automorphism group, map, hypermap, dessin d'enfant.

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1 Introduction

In 1939 Frucht published his celebrated theorem [15] that every finite group is isomorphic to the automorphism group of a finite graph; in 1960, by allowing infinite graphs, Sabidussi [46] extended this result to all groups. Similar results have been obtained, realising all finite groups (or in some cases all groups, or all countable groups) as automorphism groups of various other mathematical structures. Examples include the following, in chronological order: distributive lattices, by Birkhoff [4] in 1946; regular graphs of a given degree, by Sabidussi [45] in 1957; Riemann surfaces, by Greenberg [18, 19] in 1960 and 1973; projective planes, by Mendelsohn [37] in 1972; Steiner triple and quadruple systems, by Mendelsohn [38] in 1978; fields, by Fried and Kollár [14] in 1978; matroids of rank 3, by Babai [1] in 1981; oriented maps and hypermaps, by Cori and Machì [10] in 1982; finite volume hyperbolic manifolds of a given dimension, by Belolipetsky and Lubotzky [3] in

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2005; abstract polytopes, by Schulte and Williams [47] in 2015 and by Doignon [12] in 2016. Babai has given comprehensive surveys of this topic in [1, 2].

In many of these cases, each group is represented as the automorphism group of not just one object, but infinitely many non-isomorphic objects. The aim of this paper is to obtain results of this nature for certain 'permutational categories', introduced and discussed in [24]. These are categories $\mathfrak{C}$ which are equivalent to the category of permutation representation of some 'parent group' $\Gamma = \Gamma_{\mathfrak{C}}$: thus each object $\mathcal{O}$ in $\mathfrak{C}$ can be identified with a permutation representation $\theta : \Gamma \to S := \mathrm{Sym}(\Omega)$ of Γ on some set Ω, and the morphisms $\mathcal{O}_1 \to \mathcal{O}_2$ can be identified with the functions $\Omega_1 \to \Omega_2$ which commute with the actions of Γ on the corresponding sets Ω_i. They include the categories of maps or hypermaps on surfaces, oriented or unoriented, and possibly of a given type. Other examples include the category of coverings of a 'suitably nice' topological space; this includes the category of dessins d'enfants, regarded as finite coverings of the thrice-punctured sphere, or equivalently as finite oriented hypermaps.

The automorphism group $\mathrm{Aut}_{\mathfrak{C}}(\mathcal{O})$ of an object $\mathcal{O}$ in a permutational category $\mathfrak{C}$ is identified with the centraliser $C := C_S(G)$ in S of the monodromy group $G := \theta(\Gamma)$ of $\mathcal{O}$. Now $\mathcal{O}$ is connected if and only if G is transitive on Ω, as we will assume throughout this paper. Such objects correspond to conjugacy classes of subgroups of Γ, the point-stabilisers. An important result is the following:

Theorem 1.1. $\mathrm{Aut}_{\mathfrak{C}}(\mathcal{O}) \cong N_G(H)/H \cong N_\Gamma(M)/M$, *where* H *and* M *are the stabilisers in* G *and* Γ *of some* $\alpha \in \Omega$, *and* $N_G(H)$ *and* $N_\Gamma(M)$ *are their normalisers.*

There are analogous results in various contexts, ranging from abstract polytopes to covering spaces, which can be regarded as special cases of Theorem 1.1. Proofs of this result for particular categories can be found in the literature: for instance, in [28] it is deduced for oriented maps from a more general result about morphisms in that category; in [29, Theorem 2.2 and Corollary 2.1] a proof for dessins is briefly outlined; similar results for covering spaces are proved in [35, Appendix] and [39, Theorem 81.2], and for abstract polytopes in [36, Propositions 2D8 and 2E23(a)]. Theorem 1.1 follows immediately from the following 'folklore' result, proved in [26, Theorem 2(1)] (see also [44, Theorem 3.2]):

Theorem 1.2. *Let* G *be a transitive permutation group on a set* Ω, *with* H *the stabiliser of some* $\alpha \in \Omega$, *and let* $C := C_S(G)$ *be the centraliser of* G *in the symmetric group* $S := \mathrm{Sym}(\Omega)$. *Then* $C \cong N_G(H)/H$.

Of course, finite objects in any category have finite automorphism groups. In most of the permutational categories we will consider, the parent groups are finitely generated, so by Theorem 1.1 the automorphism groups of connected objects are all countable. Let us define a category $\mathfrak{C}$ to be *countably* (resp. *finitely*) *abundant* if every countable (resp. finite) group A is isomorphic to $\mathrm{Aut}_{\mathfrak{C}}(\mathcal{O})$ for some connected object (resp. finite connected object) $\mathcal{O}$ in $\mathfrak{C}$. Let us define $\mathfrak{C}$ to be *countably* (resp. *finitely*) *superabundant* if there are $2^{\aleph_0}$ (resp. $\aleph_0$) isomorphism classes of such objects $\mathcal{O}$ realising each A.

In the case of a permutational category $\mathfrak{C}$, these properties follow immediately from Theorem 1.1 if the associated parent group Γ has the corresponding abundance properties, namely that every countable group A is isomorphic to $N_\Gamma(M)/M$ for the required number of conjugacy classes of subgroups M of Γ, and these can be chosen to have finite index in Γ if A is finite. If Γ is finitely generated, then the cardinalities $2^{\aleph_0}$ (resp. $\aleph_0$) are the best that can be achieved, since they are upper bounds on the number of conjugacy classes of subgroups (resp. subgroups of finite index) in Γ, and hence on the number of isomorphism classes of objects (resp. finite objects) available.

We will be mainly concerned with permutational categories consisting of maps and hypermaps of various types (p, q, r), where $p, q, r \in \mathbb{N} \cup \{\infty\}$. For these, the parent groups are either extended triangle groups $\Delta[p, q, r]$, generated by reflections in the sides of a triangle with internal angles π/p, π/q and π/r, or (for subcategories of oriented objects) their orientation-preserving subgroups, the triangle groups $\Delta(p, q, r)$. We say that a triple (p, q, r) is *spherical, euclidean* or *hyperbolic* as $p^{-1} + q^{-1} + r^{-1} > 1, = 1$ or < 1 respectively (where by convention we take $\infty^{-1} = 0$), so that these groups act on the sphere, euclidean plane, or hyperbolic plane. We will call the triple *cocompact* if these groups act cocompactly, or equivalently $p, q, r \in \mathbb{N}$. We will use Theorem 1.1 to prove:

Theorem 1.3.

(a) *If (p, q, r) is a hyperbolic triple, where $p, q, r \in \mathbb{N} \cup \{\infty\}$, then the groups $\Delta(p, q, r)$ and $\Delta[p, q, r]$, together with their associated categories of oriented hypermaps and of all hypermaps of type (p, q, r), are finitely superabundant.*

(b) *If, in addition, (p, q, r) is not cocompact then these groups and categories are all countably superabundant.*

By contrast, if we take Γ to be a Tarski monster [42], an infinite group in which every subgroup $M \neq \Gamma, 1$ has order p for some (very large) prime p, then $N_\Gamma(M) = M$ for each $M \neq 1$, and hence the only groups realised as automorphism groups in the corresponding category are 1 and Γ.

The spherical and euclidean triples must be excluded from Theorem 1.3 since the corresponding triangle groups are either finite or solvable, so the same restriction applies to the automorphism groups of connected objects in the associated categories. By taking $p = r = \infty$ and $q = 2$ or ∞ we see that the categories $\mathfrak{M}$ and $\mathfrak{H}$ of all maps and hypermaps, together with their subcategories $\mathfrak{M}^+$ and $\mathfrak{H}^+$ of oriented maps and hypermaps, satisfy Theorem 1.3. In the case of $\mathfrak{M}^+$ and $\mathfrak{H}^+$, Cori and Machì [9] showed in 1982 that every finite group arises as an automorphism group; they considered only finite groups, but their proof extends to countable groups. In fact, by Theorem 1.3(a) the category of Grothendieck's dessins d'enfants [20] of any given hyperbolic type is finitely superabundant. Of course these categories are not countably abundant. Nevertheless, in §8 we will prove a result, based on work of Conder [7, 8] on alternating and symmetric quotients of triangle groups, to support the following conjecture:

Conjecture 1.4. *The non-cocompactness condition can be omitted from Theorem 1.3(b), so that the triangle groups $\Delta(p, q, r)$ and $\Delta[p, q, r]$ of any hyperbolic type, and their associated categories, are countably superabundant.*

The proof of Theorem 1.3 is divided into several cases, depending on the particular group $\Gamma = \Gamma_{\mathfrak{C}}$ involved and whether we wish to realise countable or finite groups. In each case we construct a primitive permutation representation of Γ, of infinite or unbounded finite degree, such that a point stabiliser N has an epimorphism onto a free group of countably infinite or unbounded finite rank, and hence onto an arbitrary countable or finite group A. By arranging that the kernel M is not normal in Γ we see from the maximality of N in Γ that $N_\Gamma(M)/M = N/M \cong A$, so Theorem 1.1 gives $\mathrm{Aut}_{\mathfrak{C}}(\mathcal{O}) \cong A$ for the object $\mathcal{O} \in \mathfrak{C}$ corresponding to M. Variations in the constructions yield $2^{\aleph_0}$ or $\aleph_0$ conjugacy classes of such subgroups $M \leq \Gamma$, and hence that number of objects $\mathcal{O}$ realising A. These objects are regular coverings, with covering group A, of the object $\mathcal{N} \cong \mathcal{O}/\mathrm{Aut}_{\mathfrak{C}}(\mathcal{O})$ in $\mathfrak{C}$ corresponding to N and its conjugates.

In fact a deep result of Belolipetsky and Lubotzky [3, Theorem 2.1] implies finite superabundance for every finitely generated group which is large, that is, has a subgroup of finite index with an epimorphism onto a non-abelian free group. This applies to every non-elementary finitely generated Fuchsian group, and in particular to every hyperbolic triangle group, as in Theorem 1.3(a). However, the proof of [3, Theorem 2.1] is long, delicate and non-constructive, so here we offer a shorter, more direct argument, specific to the context of this paper in using maps and hypermaps.

One should not confuse countable abundance with the SQ-universality of a group Γ, a concept introduced by P. M. Neumann in [41], and proved there for (among others) all hyperbolic triangle groups and extended triangle groups: this requires that every countable group is isomorphic to a subgroup of a quotient of Γ, that is, to N/M where $M \leq N \leq \Gamma$ and M is normal in Γ, so that $N_\Gamma(M) = \Gamma$, while countable abundance requires that $N_\Gamma(M) = N$. In terms of permutational categories, SQ-universality of the parent group Γ means that every countable group A is embedded in the automorphism group of some regular object $\mathcal{O}$ (one with a transitive monodromy group G, so that M is normal in Γ and $\mathrm{Aut}_{\mathfrak{C}}(\mathcal{O}) \cong \Gamma/M \cong G$), whereas countable abundance means that A is isomorphic to the automorphism group of some object, not necessarily regular. Both properties mean that any phenomenon exhibited by some countable group, no matter how exotic or pathological, can be realised within Γ, and hence within $\mathfrak{C}$: see [23] for some examples where $\mathfrak{C} = \mathfrak{H}^+$.

Soon after this paper was submitted, a very interesting paper [5] by Bottinelli, Grave de Peralta and Kolpakov appeared on the arXiv. It independently introduces some of the concepts and proves some of the results presented here: for instance their concept of a ‘telescopic group’ coincides with our notion of finite abundance, and they prove this for all free products of cyclic groups (except $C_2 * C_2$). However, their methods of construction differ substantially from ours, and they obtain asymptotic estimates for the number of finite objects realising a given finite group as their automorphism group, a topic not considered here.

2 Permutational categories

Following [24], let us define a *permutational category* $\mathfrak{C}$ to be a category which is equivalent to the category of permutation representations $\theta : \Gamma \to S := \mathrm{Sym}(\Omega)$ of a *parent group* $\Gamma = \Gamma_{\mathfrak{C}}$. We then define the *automorphism group* $\mathrm{Aut}(\mathcal{O}) = \mathrm{Aut}_{\mathfrak{C}}(\mathcal{O})$ of an object $\mathcal{O}$ in $\mathfrak{C}$ to be the group of all permutations of Ω commuting with the action of Γ on Ω; thus it is the centraliser $C_S(G)$ of the *monodromy group* $G = \theta(\Gamma)$ of $\mathcal{O}$ in the symmetric group S. In this paper we will restrict our attention to the *connected* objects $\mathcal{O}$ in $\mathfrak{C}$, those corresponding to transitive representations of Γ. We will pay particular attention to those categories for which the parent group Γ is either an extended triangle group

$$\Delta[p,q,r] = \langle R_0, R_1, R_2 \mid R_i^2 = (R_1R_2)^p = (R_2R_0)^q = (R_0R_1)^r = 1\rangle,$$

or its orientation-preserving subgroup of index 2, the triangle group

$$\Delta(p,q,r) = \langle X, Y, Z \mid X^p = Y^q = Z^r = XYZ = 1\rangle,$$

where $X = R_1R_2$, $Y = R_2R_0$ and $Z = R_0R_1$. Here $p, q, r \in \mathbb{N} \cup \{\infty\}$, and we ignore any relations of the form $W^\infty = 1$. We will now give some important examples of such categories; for more details, see [24]. In what follows, C_n denotes a cyclic group of order n, F_n denotes a free group of rank n, V_4 denotes a Klein four-group $C_2 \times C_2$ and $*$ denotes a free product.

1. The category $\mathfrak{M}$ of all maps on surfaces (possibly non-orientable or with boundary) has parent group

$$\Gamma = \Gamma_{\mathfrak{M}} = \Delta[\infty, 2, \infty] \cong V_4 * C_2.$$

This group acts on the set Ω of incident vertex-edge-face flags of a map (equivalently, the faces of its barycentric subdivision), with each generator R_i $(i = 0, 1, 2)$ changing the i-dimensional component of each flag (whenever possible) while preserving the other two.

2. The subcategory $\mathfrak{M}^+$ of $\mathfrak{M}$ consists of the oriented maps, those in which the underlying surface is oriented and without boundary. This category has parent group

$$\Gamma = \Gamma_{\mathfrak{M}^+} = \Delta(\infty, 2, \infty) \cong C_\infty * C_2,$$

the orientation-preserving subgroup of index 2 in $\Delta[\infty, 2, \infty]$. This group acts on the directed edges of an oriented map: X uses the local orientation to rotate them about their target vertices, and the involution Y reverses their direction, so that Z rotates them around incident faces. Here, and in the preceding example, $\Delta(p, 2, r)$ and $\Delta[p, 2, r]$ are the parent groups for the subcategories of maps of type $\{r, p\}$ in the notation of [10], meaning that the valencies of all vertices and faces divide p and r respectively, so that $X^p = Z^r = 1$. (By convention, all positive integers divide ∞.)

3. Hypermaps are natural generalisation of maps, without the restriction that each edge is incident with at most two vertices and faces which implies that $Y^2 = 1$. There are several ways of defining or representing hypermaps. The most convenient way is via the Walsh bipartite map [54], where the black and white vertices correspond to the hypervertices and hyperedges of the hypermap, the edges correspond to incidences between them, and the faces correspond to its hyperfaces. The category $\mathfrak{H}$ of all hypermaps (possibly unoriented and with boundary) has parent group

$$\Gamma = \Gamma_{\mathfrak{H}} = \Delta[\infty, \infty, \infty] \cong C_2 * C_2 * C_2.$$

This group acts on the incident edge-face pairs of the bipartite map, with R_0 and R_1 preserving the face and the incident white and black vertex respectively, while R_2 preserves the edge. As in the case of maps, $\Delta[p, q, r]$ is the parent group for the subcategory of hypermaps of type (p, q, r).

4. For the subcategory $\mathfrak{H}^+$ of oriented hypermaps, where the underlying surface is oriented and without boundary, the parent group is the even subgroup

$$\Gamma = \Gamma_{\mathfrak{H}^+} = \Delta(\infty, \infty, \infty) \cong C_\infty * C_\infty \cong F_2$$

of index 2 in $\Delta[\infty, 2, \infty]$. This acts on the edges of the bipartite map, with X and Y using the local orientation to rotate them around their incident black and white vertices, so that Z rotates them around incident faces. Again $\Delta(p, q, r)$ is the parent group for the subcategory of oriented hypermaps of type (p, q, r). Hypermaps of type $(p, 2, r)$ can be regarded as maps of type $\{r, p\}$ by deleting their white vertices; conversely maps correspond to hypermaps with $q = 2$.

5. One can regard the category $\mathfrak{D}$ of dessins d'enfants, introduced by Grothendieck [20], as the subcategory of $\mathfrak{H}^+$ consisting of its finite objects, where the bipartite graph is finite and the surface is compact. The parent group is $\Gamma = \Delta(\infty, \infty, \infty) \cong F_2$, and its action is the same as for $\mathfrak{H}^+$.

Here we briefly mention two other classes of permutational categories where Theorem 1.1 applies.

6. Abstract polytopes [36] are higher-dimensional generalisations of maps. Those n-polytopes associated with the Schläfli symbol $\{p_1, \ldots, p_{n-1}\}$ can be regarded as transitive permutation representations of the string Coxeter group Γ with presentation

$$\langle R_0, \ldots, R_n \mid R_i^2 = (R_{i-1}R_i)^{p_i} = (R_iR_j)^2 = 1 \ (|i-j| > 1)\rangle,$$

acting on flags. For instance maps, in Example 1, correspond to the symbol $\{\infty, \infty\}$. However, in higher dimensions, not all transitive representations of Γ correspond to abstract polytopes, since they need to satisfy the intersection property [36, Proposition 2B10].

7. Under suitable connectedness conditions (see [35, 39] for example), the connected, unbranched coverings $Y \to X$ of a topological space X can be identified with the transitive permutation representations $\theta : \Gamma \to S = \mathrm{Sym}(\Omega)$ of its fundamental group $\Gamma = \pi_1 X$, acting by unique path-lifting on the fibre Ω over a base-point in X. The automorphism group of an object $Y \to X$ in this category is its group of covering transformations, the centraliser in S of the monodromy group $\theta(\Gamma)$ of the covering. For instance, dessins (see Example 5 above) correspond to finite unbranched coverings of the thrice-punctured sphere $X = \mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\} = \mathbb{C} \setminus \{0, 1\}$, and hence to transitive finite permutation representations of its fundamental group $\Gamma = \pi_1 X \cong F_2 \cong \Delta(\infty, \infty, \infty)$. If we compactify surfaces by filling in punctures, then the unit interval $[0, 1] \subset \mathbb{P}^1(\mathbb{C})$ lifts to a bipartite map on the covering surface Y, with black and white vertices over 0 and 1, and face-centres over ∞. See [17, 29, 31] for further details of these and other properties of dessins.

3 Preliminary results

In this section we will prove some general results which ensure that certain groups have various automorphism realisation properties.

Lemma 3.1. *Let $\theta : \Gamma \to \Gamma'$ be an epimorphism of groups. If Γ' is finitely or countably abundant or superabundant, then so is Γ.*

Proof. If $A \cong N'/M'$ where $M' \le N' \le \Gamma'$ and $N' = N_{\Gamma'}(M')$, then $A \cong N/M$ where $M = \theta^{-1}(M')$ and $N = \theta^{-1}(N') = N_\Gamma(M)$, with $|\Gamma : M| = |\Gamma' : M'|$, so Γ inherits finite or countable abundance from Γ'. Moreover, non-conjugate subgroups M' lift to non-conjugate subgroups M, so the superabundance properties are also inherited. □

Our basic tool for proving finite superabundance will be the following:

Proposition 3.2. *Let Γ be a group with a sequence $\{N_n \mid n \ge n_0\}$ of maximal subgroups N_n of finite index such that for each $a, d \in \mathbb{N}$ there is some n with $|N_n : K_n| > a$, where K_n is the core of N_n in Γ, and there is an epimorphism $N_n \to F_d$. Then Γ is finitely superabundant.*

Proof. Any finite group A is an d-generator group for some $d \in \mathbb{N}$, so there is an epimorphism $F_d \to A$. By hypothesis, for some maximal subgroup $N = N_n$ of Γ there is an epimorphism $N \to F_d$, and the core K of N satisfies $|N : K| > |A|$. Composition gives an epimorphism $N \to A$, and hence a normal subgroup M of N with $N/M \cong A$. Then $N_\Gamma(M) \geq N$, so the maximality of N implies that either $N = N_\Gamma(M)$ or M is a normal subgroup of Γ. If M is normal in Γ then M must be contained in the core K of N, so that $|N : M| \geq |N : K|$. But this is impossible, since $|N : M| = |A|$ and we chose $N = N_n$ so that $|N : K| > |A|$. Hence $N = N_\Gamma(M)$, as required. Moreover, given A we can find such subgroups N with $|N : K|$ arbitrarily large, so infinitely many of them are mutually non-conjugate, and hence so are their corresponding subgroups M, since conjugate subgroups have conjugate normalisers. □

In order to deal with countable abundance or superabundance we need an analogue of Proposition 3.2 for countable groups A. Here we have the advantage that, instead of an infinite sequence of maximal subgroups, which are finitely generated if Γ is, a single infinitely generated maximal subgroup is sufficient. However, when A is infinite we cannot ensure that M is not normal in Γ simply by comparing indices of subgroups, since these are not finite; a new idea is therefore needed.

Proposition 3.3. *Let Γ be a group with a non-normal maximal subgroup N and an epimorphism $\phi : N \to F_\infty$. Then Γ is countably abundant. Moreover, each countable group $A \neq 1$ is realised as $N_\Gamma(M)/M$ by $2^{\aleph_0}$ conjugacy classes of subgroups M in Γ with $N_\Gamma(M) = N$.*

Proof. Given any countable group A there exist epimorphisms $\alpha : F_\infty \to A$; composing any of these with the epimorphism $\phi : N \to F_\infty$ gives an epimorphism $\phi \circ \alpha : N \to A$, and hence a normal subgroup $M = \ker(\phi \circ \alpha)$ of N with $N/M \cong A$. As before, the maximality of N implies that either $N = N_\Gamma(M)$, as required, or M is a normal subgroup of Γ. In the latter case M is contained in the core K of N in Γ, so to prove the result we need to show that we can choose α so that $M \not\leq K$. Since N is not normal in Γ we have $N \setminus K \neq \emptyset$, so choose any element $g \in N \setminus K$, and define $f := g\phi \in F_\infty$. Then we can choose $\alpha : F_\infty \to A$ so that all of the (finitely many) free generators of F_∞ appearing in f are in $\ker(\alpha)$, and hence $g \in M$. Thus $M \not\leq K$, so Γ is countably abundant.

If $A \neq 1$ we can choose such epimorphisms α with $2^{\aleph_0}$ different kernels, lifting back to distinct subgroups M of Γ; these all have normaliser N, which is its own normaliser in Γ, so they are mutually non-conjugate in Γ, giving us $2^{\aleph_0}$ conjugacy classes of subgroups M realising A. □

Remark 3.4. Unfortunately, if $A = 1$ then α is unique, so that $M = N$, and the subgroup N yields only one conjugacy class of subgroups realising A. In this case, in order to prove that Γ is countably superabundant by this construction we would need to find not one but $2^{\aleph_0}$ conjugacy classes of non-normal maximal subgroups N. For certain specific groups Γ we will be able to do this.

4 Finite superabundance of hyperbolic triangle and extended triangle groups

In this section we will use Proposition 3.2 to prove Theorem 1.3(a).

Case 1: $\Gamma = \Delta(p,q,r)$*, cocompact.* First assume that $\Gamma = \Delta(p,q,r)$, acting cocompactly on the hyperbolic plane, that is, with finite periods p,q and r. By Dirichlet's Theorem on primes in an arithmetic progression, there are infinitely many primes $n \equiv -1 \bmod (l)$, where $l := \mathrm{lcm}\{2p,2q,2r\}$. For each such n there is an epimorphism $\theta_n : \Gamma \to \mathrm{PSL}_2(n)$ sending the standard generators X, Y and Z of Γ to elements x, y and z of $\mathrm{PSL}_2(n)$ of orders p,q and r (see [16, Corollary C], for example). This gives an action of Γ on the projective line $\mathbb{P}^1(\mathbb{F}_n)$, which is doubly transitive and hence primitive, so the subgroup N_n of Γ fixing ∞ is a non-normal maximal subgroup of index $n+1$. Since p,q and r all divide $(n+1)/2$, the elements x,y and z are semi-regular permutations on $\mathbb{P}^1(\mathbb{F}_n)$, with all their cycles of length p,q or r. Thus no non-identity powers of X, Y or Z have fixed points, so by a theorem of Singerman [48] N_n is a surface group

$$N_n = \langle A_i, B_i \ (i = 1,\ldots,g) \mid \prod_{i=1}^{g}[A_i,B_i] = 1\rangle$$

of genus g given by the Riemann–Hurwitz formula:

$$2(g-1) = (n+1)\left(1 - \frac{1}{p} - \frac{1}{q} - \frac{1}{r}\right). \tag{4.1}$$

This shows that $g \to \infty$ as $n \to \infty$. Now we can map N_n onto the free group F_g by sending the generators A_i to a free basis, and the generators B_i to 1. The core $K_n = \ker(\theta_n)$ of N_n in Γ satisfies $|N_n : K_n| = |\mathrm{PSL}_2(n)|/(n+1) = n(n-1)/2$, so Proposition 3.2 gives the result.

Case 2: $\Gamma = \Delta(p,q,r)$*, not cocompact.* Now assume that Γ has k infinite periods p,q,r for some $k = 1,2$ or 3. We can adapt the above argument by first choosing an infinite set of primes $n \geq 13$ such that any finite periods of Γ divide $(n+1)/2$, as before. For each such n we can map Γ onto a cocompact triangle group Γ_n, where each infinite period of Γ is replaced with $(n+1)/2$. Since $(n+1)/2 \geq 7$, the triangle group Γ_n is also hyperbolic, so as before there is an epimorphism $\Gamma_n \to \mathrm{PSL}_2(n)$, giving (by composition) a primitive action of Γ on $\mathbb{P}^1(\mathbb{F}_n)$. Again, no non-identity powers of any elliptic generators among X, Y and Z have fixed points, but any parabolic generator (one of infinite order) now induces two cycles of length $(n+1)/2$ on $\mathbb{P}^1(\mathbb{F}_n)$, so by [48] it introduces two parabolic generators P_i into the standard presentation of the point-stabiliser N_n in Γ. We therefore have

$$N_n = \langle A_i, B_i \ (i = 1,\ldots,g), P_i \ (i = 1,\ldots,2k) \mid \prod_{i=1}^{g}[A_i,B_i].\prod_{i=1}^{2k}P_i = 1\rangle,$$

a free group of rank $2g+2k-1$, where the Riemann–Hurwitz formula now gives

$$2(g-1) + 2k = (n+1)\left(1 - \frac{1}{p} - \frac{1}{q} - \frac{1}{r}\right) \tag{4.2}$$

with $1/\infty = 0$. Since $k \leq 3$ we have $g \to \infty$ as $n \to \infty$, so Proposition 3.2 again gives the result.

Case 3: $\Gamma = \Delta[p,q,r]$. The proof when Γ is an extended triangle group $\Delta[p,q,r]$ of hyperbolic type is similar to that for $\Delta(p,q,r)$. If Γ is cocompact then, as before, we consider epimorphisms $\theta_n : \Gamma^+ = \Delta(p,q,r) \to \mathrm{PSL}_2(n)$ for primes $n \equiv -1 \bmod (l)$, where now $l = \mathrm{lcm}\{2p, 2q, 2r, 4\}$; the stabilisers of ∞ form a series of maximal subgroups N_n of index $n+1$ in Γ^+. By an observation of Singerman [50] the core K_n of N_n in Γ^+ is normal in Γ, with quotient Γ/K_n isomorphic to $\mathrm{PSL}_2(n) \times C_2$ or $\mathrm{PGL}_2(n)$ as the automorphism of $\mathrm{PSL}_2(n)$ inverting x and y is inner or not. Thus θ_n extends to a homomorphism $\theta_n^* : \Gamma \to \mathrm{PGL}_2(n)$; in the first case its image is $\mathrm{PSL}_2(n)$ and its kernel K_n^* contains K_n with index 2, and in the second case it is an epimorphism with kernel $K_n^* = K_n$. In either case the action of Γ^+ on $\mathbb{P}^1(\mathbb{F}_n)$ extends to an action of Γ, and the stabiliser N_n^* of ∞ is a maximal subgroup of index $n+1$ in Γ, containing N_n with index 2. In order to apply Proposition 3.2 to these subgroups N_n^* it is sufficient to show that they map onto free groups of unbounded rank.

Now N_n^* is a non-euclidean crystallographic (NEC) group, and N_n is its canonical Fuchsian subgroup of index 2. We can obtain the signature of N_n^* by using Hoare's extension to NEC groups [22] of Singerman's results [48] on subgroups of Fuchsian groups. As before, N_n is a surface group of genus g given by (4.1). There are no elliptic or parabolic elements in N_n, and hence none in N_n^*. The reflections R_i $(i = 0,1,2)$ generating Γ induce involutions on $\mathbb{P}^1(\mathbb{F}_n)$, each with at most two fixed points. If $\Gamma/K_n \cong \mathrm{PSL}_2(n) \times C_2$ these involutions are elements of $\mathrm{PSL}_2(n)$, so they are even permutations by the simplicity of this group, and hence they have no fixed points since $n+1 \equiv 0 \bmod (4)$. Thus N_n^* contains no reflections; however, it is not a subgroup of Γ^+, so it is a non-orientable surface group

$$N_n^* = \langle G_1, \ldots, G_{g^*} \mid G_1^2 \ldots G_{g^*}^2 = 1 \rangle$$

generated by glide-reflections G_i, with its genus g^* given by the Riemann–Hurwitz formula

$$2 - 2g = 2(2 - g^*)$$

for the inclusion $N_n \leq N_n^*$, so $g^* = g + 1$. Thus there is an epimorphism $N_n^* \to F_d = \langle X_1, \ldots, X_d \mid - \rangle$ where $d = \lfloor g^*/2 \rfloor$, given by $G_{2i-1} \mapsto X_i$ and $G_{2i} \mapsto X_i^{-1}$ for $i = 1, \ldots, d$ and $G_{g^*} \mapsto 1$ if g^* is odd. Since $g \to \infty$ as $n \to \infty$, we have $d \sim g/2 \to \infty$ also, so Proposition 3.2 gives the result.

Similar arguments also deal with the case where $\Gamma/K_n \cong \mathrm{PGL}_2(n)$. Since $n \equiv -1 \bmod (4)$, each generating reflection R_i of Γ induces an odd permutation of $\mathbb{P}^1(\mathbb{F}_n)$ with two fixed points, contributing two reflections to the standard presentation of the NEC group N_n^*. The Riemann–Hurwitz formula for the inclusion $N_n \leq N_n^*$ then takes the form

$$2 - 2g = 2(2 - h^* + s),$$

where $h^* = 2g^*$ or g^* as N_n^* has an orientable or non-orientable quotient surface of genus g^* with s boundary components for some $s \leq 6$. Thus $h^* \sim g \to \infty$ as $n \to \infty$. We obtain an epimorphism $N_n^* \to F_d$ with $d \sim h^*/2$ as in the orientable or non-orientable cases above, this time by mapping the additional standard generators of N_n^*, associated with the boundary components, to 1, so Proposition 3.2 again gives the result. Finally, in the non-cocompact case, any periods $p, q, r = \infty$ can be dealt with as above for $\Delta(p,q,r)$. This completes the proof of Theorem 1.3(a) .

Remark 4.1. It seems plausible that an argument based on the Čebotarev Density Theorem would show that, given $\Gamma = \Delta[p,q,r]$, the cases $\Gamma/K_n \cong \mathrm{PSL}_2(n) \times C_2$ and $\mathrm{PGL}_2(n)$ each occur for infinitely many primes $n \equiv -1 \bmod (l)$, so that only one case would need to be considered; however, the resulting shortening of the proof would not justify the effort. Nevertheless this dichotomy, for general prime powers n, is interesting in its own right and deserves further study.

Remark 4.2. The restrictions on the prime n in the above proof are partly for convenience of exposition, rather than necessity. Relaxing them would allow X, Y and Z to have one or two fixed points on $\mathbb{P}^1(\mathbb{F}_n)$, thus adding extra standard generators to N_n and N_n^* and extra summands to the Riemann–Hurwitz formulae used. However, these extra terms are bounded as $n \to \infty$, so asymptotically they make no significant difference. One advantage of these restrictions is that since x, y and z are semi-regular permutations, the hypermaps realising A in Case 1 are uniform, that is, their hypervertices, hyperedges and hyperfaces all have valencies p, q and r. If we choose these periods so that $\Delta(p, q, r)$ is cocompact, maximal (see [49]) and non-arithmetic (see [52]), then by a result of Singerman and Syddall [51, Theorem 12.1] each hypermap (regarded as a dessin) has the same automorphism group as its underlying Riemann surface. By [49, 52] these conditions apply to 'most' hyperbolic triples, such as $(2, 3, 13)$, so we have the following:

Corollary 4.3. *The category of compact Riemann surfaces is finitely superabundant.*

In fact, Greenberg [19, Theorem 6′] showed in 1973 that, given a compact Riemann surface S and a finite group $A \neq 1$, there is a normal covering $T \to S$ with covering group and $\mathrm{Aut}(T)$ both isomorphic to A, while Teichmüller theory yields uncountably many compact Riemann surfaces realising A. Since the Riemann surfaces realising a finite group A in Corollary 4.3 are uniformised by subgroups of finite index in triangle groups, by Grothendieck's reinterpretation [20] of Belyĭ's Theorem they are all defined (as algebraic curves with automorphism group A) over number fields.

5 Countable abundance of non-cocompact hyperbolic triangle groups

We now turn to Theorem 1.3(b) and consider countable abundance, starting with the hyperbolic triangle groups $\Gamma = \Delta(p, q, r)$. We would like to show that Γ satisfies the hypotheses of Proposition 3.3, that is, it has a non-normal maximal subgroup which has an epimorphism onto F_∞. Given Γ, it is easy to find maximal subgroups of finite index by mapping Γ onto primitive permutation groups of finite degree; however, such subgroups are finitely generated, so they do not map onto F_∞; a maximal subgroup of infinite index is needed, and these seem to be harder to find. They certainly exist: by a result of Ol'shanskiĭ [43], Γ has a quotient $Q \neq 1$ with no proper subgroups of finite index; by Zorn's Lemma, Q has maximal subgroups, which must have infinite index, and these lift back to maximal subgroups of infinite index in Γ. These are not normal (otherwise they would have prime index), but does one of them map onto F_∞? Conceivably, they could be generated by elliptic elements, which have finite order, in which case they would not map onto a free group of any rank. As a first step we consider the case where Γ is not cocompact, that is, it has an infinite period, so it is a free product of two cyclic groups. For simplicity of exposition we first consider countable abundance, postponing superabundance until the next section.

Theorem 5.1. *If Γ is a non-cocompact hyperbolic triangle group $\Delta(p, q, r)$, then Γ and the corresponding category of oriented hypermaps are countably abundant.*

Proof. By Proposition 3.3 it is sufficient to show that Γ has a non-normal maximal subgroup N with an epimorphism $N \to F_\infty$. Using the usual isomorphisms between triangle groups, we may assume that $r=\infty$, and that $p\geq 3$ and $q\geq 2$, so that $\Gamma\cong C_p*C_q$ with $p,q\in\mathbb{N}\cup\{\infty\}$.

Case 1: $p=3, q=2$. First we consider the case where $p=3$ and $q=2$, so that Γ is isomorphic to the modular group $\mathrm{PSL}_2(\mathbb{Z})\cong C_3*C_2$. We can construct a maximal subgroup N of infinite index in Γ as the point stabiliser in a primitive permutation representation of Γ of infinite degree. Since Γ is the parent group

$$\Delta(3,2,\infty)=\langle X,Y,Z\mid X^3=Y^2=XYZ=1\rangle$$

for the category $\mathfrak{C}=\mathfrak{M}_3^+$ of oriented trivalent maps, we can take N to be the subgroup of Γ corresponding to an infinite map $\mathcal{N}_3$ in $\mathfrak{C}$.

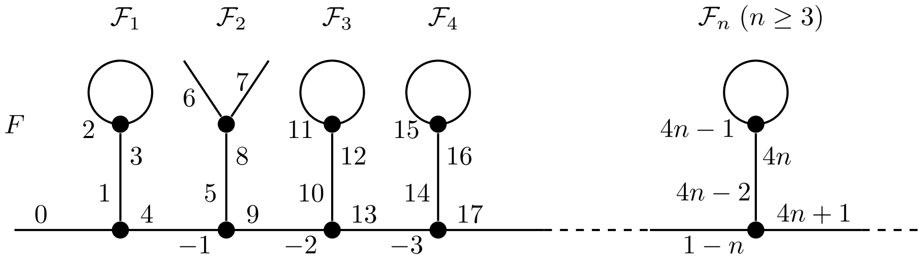

Figure 1: The trivalent map $\mathcal{N}_3$.

We will take $\mathcal{N}_3$ to be the infinite planar trivalent map shown in Figure 1, oriented with the positive (anticlockwise) orientation of the plane. The monodromy group $G=\langle x,y\rangle$ of this map gives a transitive permutation representation $\theta:\Gamma\to G, X\mapsto x, Y\mapsto y, Z\mapsto z$ of Γ on the set Ω of directed edges of $\mathcal{N}_3$, with x rotating them anticlockwise around their target vertices, and y reversing their direction. The vertices, all of valency 3, correspond to the 3-cycles of x (it has no fixed points). The edges correspond to the cycles of y, with three free edges corresponding to its fixed points and the other edges corresponding to its 2-cycles. The faces correspond to the cycles of $z=yx^{-1}$, and in particular, the directed edge α labelled 0 and fixed by y is in an infinite cycle $C=(\ldots,\alpha z^{-1},\alpha,\alpha z,\ldots)$ of z, corresponding to the unbounded face F of $\mathcal{N}_3$; the directed edges αz^i in C are indicated by integers i in Figure 1. The unlabelled directed edges are fixed points of z, one incident with each 1-valent face. The pattern seen in Figure 1 repeats to the right in the obvious way. The 'flowers' $\mathcal{F}_n$ $(n\geq 1)$ above the horizontal axis continue indefinitely to the right, with $\mathcal{F}_n$ an identical copy of $\mathcal{F}_1$ for each $n\geq 3$; we will later need the fact that for each $n\geq 2$ the 'stem' of $\mathcal{F}_n$ (the vertical edge connecting it to the horizontal axis) carries two directed edges in C, with only one of their two labels divisible by n.

Lemma 5.2. *The group* Γ *acts primitively on* Ω.

Proof. Suppose that $\sim$ is a Γ-invariant equivalence relation on Ω; we need to show that it is either the identity or the universal relation. Since $\alpha y=\alpha$, the equivalence class $E=[\alpha]$ containing α satisfies $Ey=E$. Since $\langle Z\rangle$ acts regularly on C we can identify C with $\mathbb{Z}$ by identifying each $\alpha z^i\in C$ with the integer i, so that Z acts by $i\mapsto i+1$. Then $\sim$ restricts to a translation-invariant equivalence relation on $\mathbb{Z}$, which must be congruence mod (n) for some $n \in \mathbb{N} \cup \{\infty\}$, where we include $n = 1$ and ∞ for the universal and identity relations on $\mathbb{Z}$.

Suppose first that $n \in \mathbb{N}$, so $E \cap C$ is the subgroup (n) of $\mathbb{Z}$. If $n = 1$ then $C \subseteq E$. Now Ex^{-1} is an equivalence class, and it contains $\alpha x^{-1} = 1$; this is in C, and hence in E, so $Ex^{-1} = E$. We have seen that $Ey = E$, so $E = \Omega$ since $G = \langle x^{-1}, y\rangle$, and hence $\sim$ is the universal relation on Ω.

We may therefore assume that $n > 1$. The vertical stem of the flower $\mathcal{F}_n$ is an edge carrying two directed edges in C, with only one of its two labels divisible by n, so one is in E whereas the other is not. However, these two directed edges are transposed by y, contradicting the fact that $Ey = E$.

Finally suppose that $n = \infty$, so that all elements of C are in distinct equivalence classes, and hence the same applies to Cy. In particular, since $\alpha \in C \cap Cy$ we have $E \cap C = \{\alpha\} = E \cap Cy$. By inspection of Figure 1, $\Omega = C \cup Cy$ and hence $E = \{\alpha\}$. It follows that all equivalence classes for $\sim$ are singletons, so $\sim$ is the identity relation, as required. □

We now return to the proof of Theorem 5.1. It follows from Lemma 5.2 that the subgroup $N = \Gamma_\alpha$ of Γ fixing α is maximal. Clearly N is not normal in Γ, since G is not a regular permutation group, so it sufficient to find an epimorphism $N \to F_\infty$. One could use the Reidemeister–Schreier algorithm to find a presentation for N: truncation converts $\mathcal{N}_3$ into a coset diagram for N in Γ, and then deleting edges to form a spanning tree yields a Schreier transversal. In fact a glance at Figure 1 shows that N is a free product of cyclic groups: three of these, corresponding to the fixed points of y and generated by conjugates of Y, have order 2, and there are infinitely many of infinite order, generated by conjugates of Z and corresponding to the fixed points of z, that is, the 1-valent faces of $\mathcal{N}_3$, one in each flower $\mathcal{F}_n$ for $n \neq 2$. By mapping the generators of finite order to the identity we obtain the required epimorphism $N \to F_\infty$.

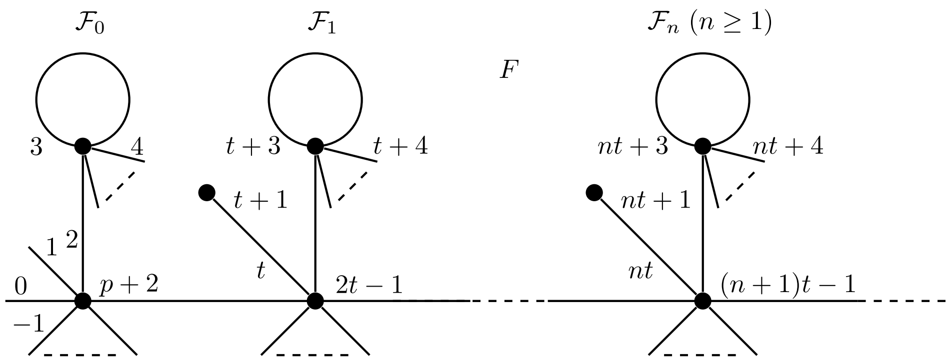

Figure 2: The p-valent map $\mathcal{N}_p$, with $t := p + 3$.

Case 2: Finite $p \geq 4, q = 2$. We now assume that Γ is a Hecke group $C_p * C_2$ for some finite $p \geq 4$. Let $\mathcal{N}_p$ be the infinite p-valent planar map in Figure 2. Apart from $\mathcal{F}_0$, the flowers are all identical copies of $\mathcal{F}_1$, with a 'leaf' growing out of its base and leading to a vertex of valency 1, representing a fixed point of x. The 'fans' indicated by short dashed lines represent however many free edges are needed in order that the incident vertex should have valency p, that is, $p - 3$ free edges for vertices at the top of a stem, and $p - 4$ for those at the base. As before, the elements αz^i of the cycle C of z containing the directed edge $\alpha = 0$ are labelled with integers i; to save space in the diagram only a few labels are shown. We define $t = p + 3$, since a translation from a flower $\mathcal{F}_n$ $(n \geq 1)$ to the next flower $\mathcal{F}_{n+1}$ adds that number to all labels.

The proof that the monodromy group $G = \langle x, y \rangle$ of $\mathcal{N}_p$ is primitive is very similar to that in Lemma 5.2 for $\mathcal{N}_3$. Any Γ-invariant equivalence relation $\sim$ on Ω restricts to C as congruence mod (n) for some $n \in \mathbb{N} \cup \{\infty\}$. The equivalence class $E = [\alpha]$ satisfies $Ey = E$, so if $n \neq 1, \infty$ then the fact that y transposes the directed edges labelled nt and $nt + 1$, with the first but not the second in $E = (n)$, gives a contradiction. If $n = 1$ then $C \subseteq E$, so both x and y preserve E and hence $\sim$ is the universal relation. If $n = \infty$ then $E \cap C = \{\alpha\}$, and hence also $E \cap Cy = \{\alpha\}$; but $\Omega = C \cup Cy$ and hence $E = \{\alpha\}$ and $\sim$ is the identity relation.

This shows that the subgroup N of Γ fixing α is maximal. As before, it is not normal, and it is a free product of cyclic groups, now of order p, 2 or ∞, corresponding to the fixed points of x, y and z (infinitely many in each case). Sending the generators of finite order to the identity gives the required epimorphism $N \to F_\infty$.

Case 3: Finite $p, q \geq 3$. We modify the map $\mathcal{N}_p$ in the proof of Case 2 by removing the leaf attached to the base of each flower $\mathcal{F}_n$ $(n \geq 1)$, adding a white vertex to every remaining edge (including one at the free end of each free edge), and finally adding edges incident with 1-valent black vertices where necessary to ensure that all white vertices have valency q or 1. The resulting map $\mathcal{N}_{p,q}$ is shown in Figure 3.

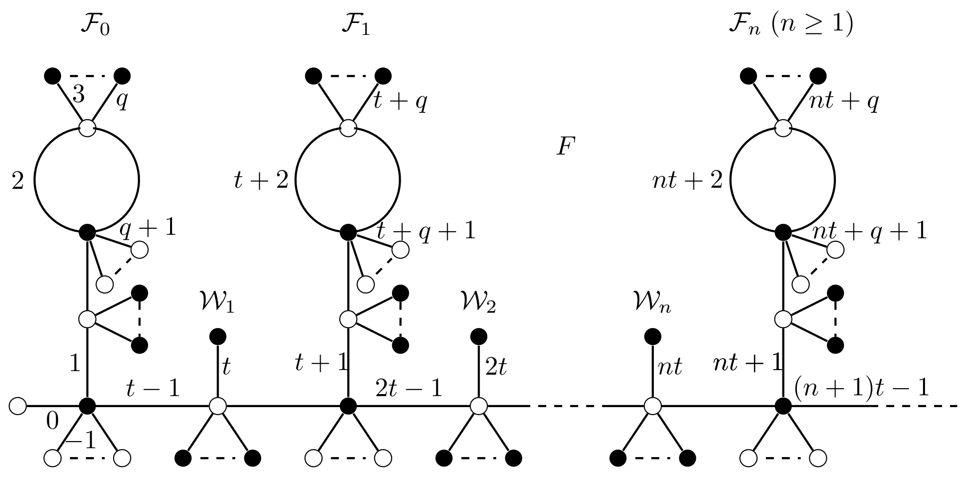

Figure 3: The bipartite map $\mathcal{N}_{p,q}$, with $t := p + 2q - 2$.

Note that while the flowers $\mathcal{F}_n$ have grown since Case 2 was proved, small 'weeds' $\mathcal{W}_n$ $(n \geq 1)$ have grown between them. This bipartite map is the Walsh map for an oriented hypermap of type (p, q, ∞). Its monodromy group G is generated by permutations x and y, of order p and q, which rotate edges around their incident black and white vertices. It is sufficient to show that G acts primitively on the set Ω of edges, and that in the induced action of Γ on Ω, the subgroup N fixing an edge has an epimorphism onto F_∞.

The proof is similar to that for Case 2. The elements αz^i of the cycle C of z containing the edge $\alpha = 0$ are labelled with integers i. (To save space in Figure 3, only a few significant labels are shown.) Any Γ-invariant equivalence relation $\sim$ restricts to C as congruence mod (n) for some $n \in \mathbb{N} \cup \{\infty\}$. If $E = [\alpha]$ then since $\alpha y = \alpha$ we have $Ey = E$. If $n \neq \infty$ then $Ex = E$ since the edge $\beta \in E$ labelled nt is fixed by x, so that $E\Gamma = E$ and hence $E = \Omega$. Thus we may assume that $n = \infty$, so all elements of C are in distinct conjugacy classes and hence $E \cap C = \{\alpha\}$. Similarly $E \cap Cy = \{\alpha\}$. But $\Omega = C \cup Cy$, so $E = \{\alpha\}$ and $\sim$ is the identity relation. Thus G is primitive, so the subgroup N of Γ fixing α is maximal. It is a free product of cyclic groups, of orders p, q and ∞, corresponding to the fixed points of x, y and z. There are infinitely many of each, and mapping those of finite order to the identity gives an epimorphism $N \to F_\infty$.

Case 4: p or $q = \infty$. If $p = \infty$ or $q = \infty$ we can use the natural epimorphism from $\Gamma = \Delta(p, q, \infty)$ to a hyperbolic triangle group $\Gamma' = \Delta(p', q', \infty)$ with p' and q' both finite, use Case 1, 2 or 3 to establish countable abundance for Γ', and finally use Lemma 3.1 to deduce it for Γ. □

Remark 5.3. Constructions similar to those in the proof of Theorem 5.1 have been used in [25] to prove that if Γ is a non-cocompact hyperbolic triangle group then Γ has uncountably many conjugacy classes of maximal subgroups of infinite index. This strengthens and generalises results of B. H. Neumann [40], Magnus [33, 34], Tretkoff [53], and Brenner and Lyndon [6] on maximal nonparabolic subgroups of the modular group, and has some overlap with work of Kulkarni [30] on maximal subgroups of free products of cyclic groups.

6 Countable superabundance of non-cocompact hyperbolic triangle groups

In order to prove countable superabundance for non-cocompact hyperbolic triangle groups $\Gamma = \Delta(p, q, r)$, we need $2^{\aleph_0}$ objects realising each countable group A. The proofs of countable abundance for the various cases in Theorem 5.1 all used Proposition 3.3, and by Remark 3.4 this yields the required number of objects in all cases except when $A \neq 1$.

In fact, for any countable group A these proofs can be adapted (as in Remark 5.3) to produce not just one but $2^{\aleph_0}$ conjugacy classes of subgroups N satisfying the conditions of Proposition 3.3. We thus obtain $2^{\aleph_0}$ non-isomorphic objects $\mathcal{N}$, each with $2^{\aleph_0}$ coverings $\mathcal{O}$ realising any countable group $A \neq 1$, and with one covering (namely $\mathcal{O} = \mathcal{N}$) realising $A = 1$; each $\mathcal{O}$ has the property that $\mathcal{O}/\mathrm{Aut}(\mathcal{O}) \cong \mathcal{N}$, so A is realised by $2^{\aleph_0}$ non-isomorphic objects.

We will give the required details for Case 1 of Theorem 5.1, where $p = 3$, $q = 2$ and Γ is the modular group; the argument is similar in the other cases. We can modify the map $\mathcal{N}_3$ in Figure 1 by adding 'stalks' between the flowers $\mathcal{F}_n$, each consisting of a new vertex on the horizontal axis, and a new free edge pointing upwards. Adding a stalk between $\mathcal{F}_m$ and $\mathcal{F}_{m+1}$ adds 2 to the value of all labels on flowers $\mathcal{F}_n$ for $n > m$. For the proof of Theorem 5.1 to work we need to preserve the property that only one of the two labels on the stem of each flower $\mathcal{F}_n$ $(n > 1)$ is divisible by n. This can be done, in $2^{\aleph_0}$ different ways, by ensuring that for each $n > 1$ the total number of stalks added between $\mathcal{F}_1$ and $\mathcal{F}_n$ is a multiple of n. The proof for Case 1 then proceeds as before, except that it now yields $2^{\aleph_0}$ conjugacy classes of maximal subgroups N.

In the remaining cases of Theorem 5.1 we could use similar modifications to the maps

$\mathcal{N}_p$ and $\mathcal{N}_{p,q}$ in Figures 2 and 3, or alternatively add extra vertices and edges to those below the horizontal axis, so that the non-negative labels above the axis are unaltered.

7 Countable superabundance of non-cocompact extended triangle groups

We now consider countable superabundance for extended triangle groups $\Gamma = \Delta[p,q,r]$ and their associated categories of unoriented hypermaps, again restricting attention to non-cocompact groups. Earlier we realised countable groups A as automorphism groups in various categories $\mathfrak{C}^+$ of oriented hypermaps of a given type by constructing specific objects $\mathcal{N} = \mathcal{N}_p$ $(p \geq 3)$ or $\mathcal{N}_{p,q}$ $(p,q \geq 3)$ in those categories, and then forming regular coverings $\mathcal{M}$ of $\mathcal{N}$, with covering group A, constructed so that $\mathcal{M}$ has only those automorphisms induced by A. These objects $\mathcal{M}$ and $\mathcal{N}$ correspond to subgroups M and N of the parent group $\Gamma^+ = \Delta(p,q,r)$ for $\mathfrak{C}^+$ with $N = N_{\Gamma^+}(M)$. We can also regard $\mathcal{M}$ and $\mathcal{N}$ as objects in the corresponding category $\mathfrak{C}$ of unoriented maps or hypermaps of type (p,q,r), for which the parent group is the extended triangle group Γ.

Lemma 7.1. *For these objects $\mathcal{M}$ we have $\mathrm{Aut}_{\mathfrak{C}}(\mathcal{M}) = \mathrm{Aut}_{\mathfrak{C}^+}(\mathcal{M}) \cong A$.*

Proof. We have $\mathrm{Aut}_{\mathfrak{C}}(\mathcal{M}) \cong N_\Gamma(M)/M$ and $\mathrm{Aut}_{\mathfrak{C}^+}(\mathcal{M}) \cong N_{\Gamma^+}(M)/M \cong A$ by Theorem 1.1, so it is sufficient to show that $N_\Gamma(M) = N_{\Gamma^+}(M)$. Clearly $N_\Gamma(M) \geq N_{\Gamma^+}(M)$. If this inclusion is proper then since $N_{\Gamma^+}(M) = N_\Gamma(M) \cap \Gamma^+$ with $|\Gamma : \Gamma^+| = 2$ we have $|N_\Gamma(M) : N_{\Gamma^+}(M)| = 2$, so the subgroup $N = N_{\Gamma^+}(M)$ is normalised by some elements of $\Gamma \setminus \Gamma^+$. This is impossible, since in all cases the map or hypermap $\mathcal{N}$ corresponding to N is chiral (without orientation-reversing automorphisms), by the proof of Theorem 5.1 and by inspection of Figures 1, 2 and 3. The same applies to the modified maps required to produce $2^{\aleph_0}$ such objects $\mathcal{O}$. □

Corollary 7.2. *Each non-cocompact hyperbolic extended triangle group $\Delta[p,q,r]$ and its associated category of all hypermaps of type (p,q,r) are countably superabundant.*

This completes the proof of Theorem 1.3(b).

Remark 7.3. It would not have been possible to use Lemma 7.1 also in the proof of Theorem 1.3(a) in §4, since the maximal subgroups N_n of $\Gamma^+ = \Delta(p,q,r)$ constructed there are normalised by orientation-reversing elements of $\Gamma = \Delta[p,q,r]$. Instead of the natural representation of $\mathrm{PSL}_2(n)$, we could have used its representation on the cosets of a maximal subgroup $H \cong A_5$ for $n \equiv \pm 1 \bmod (5)$, or $H \cong S_4$ for $n \equiv \pm 1 \bmod (8)$: in both of these cases there are two conjugacy classes of subgroups H, transposed by conjugation in $\mathrm{PGL}_2(n)$ (see [11, Ch. XII]) and corresponding to a chiral pair of hypermaps. However, in either case the point stabilisers H have constant order as $n \to \infty$, whereas Proposition 3.2 requires $|N_n : K_n| = |H|$ to be unbounded, so we would need an alternative argument to show that M is not normal in Γ^+, as in the proof of Proposition 3.3.

8 Countable superabundance of some cocompact triangle groups

Theorem 1.3(b) proves countable superabundance only for non-cocompact hyperbolic triangle groups $\Delta(p,q,r)$ and $\Delta[p,q,r]$. We would like to extend to this property to the cocompact case. The arguments we used to prove Theorem 1.3(b) depend on a standard generator of $\Delta(p,q,r)$ (Z, without loss of generality) having infinite order, so that it can have a cycle C of infinite length in some permutation representation, which is then proved to be primitive by identifying C with $\mathbb{Z}$. Clearly this is impossible in the cocompact case, so a different approach is needed. The following is a first step in this direction.

Proposition 8.1. *If one of p, q and r is even, another is divisible by 3, and the third is at least 7, then the cocompact triangle groups $\Delta(p, q, r)$ and $\Delta[p, q, r]$ and their associated categories are countably superabundant.*

Proof. By permuting periods and applying Lemma 3.1 we may assume that $p = 3$, $q = 2$ and $r \geq 7$. First suppose that $r = 7$. We will construct an infinite transitive permutation representation of the group

$$\Gamma = \Delta[3, 2, 7] = \langle X, Y, T \mid X^3 = Y^2 = T^2 = (XY)^7 = (XT)^2 = (YT)^2 = 1\rangle$$

(where $T = R_2$) in which the subgroup $\Gamma^+ = \Delta(3, 2, 7) = \langle X, Y\rangle$ acts primitively, and we will then apply Proposition 3.3 to a point-stabiliser in Γ^+. This representation is constructed by adapting the Higman–Conder technique of 'sewing coset diagrams together', used in [7] to realise finite alternating and symmetric groups as quotients of Γ^+ and Γ. We refer the reader to [7] for full technical details of this method. (Note that we have changed Conder's notation, which has $X^2 = Y^3 = 1$, by transposing the symbols X and Y; this has no significant effect on the following proof.)

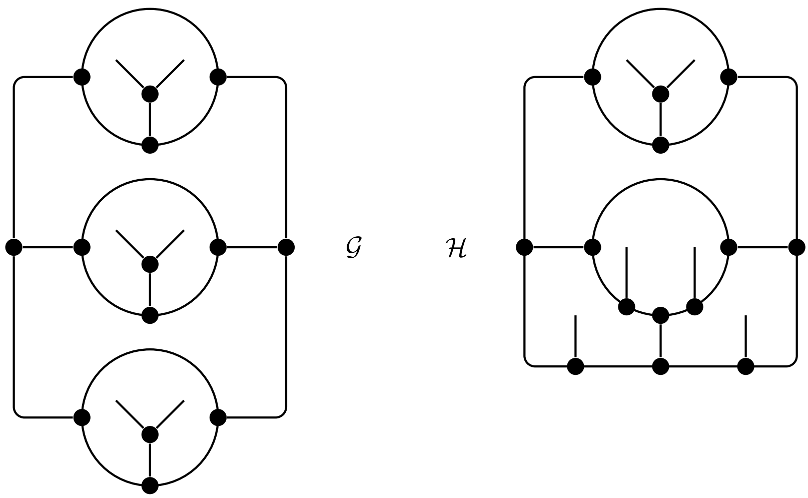

Figure 4: The maps $\mathcal{G}$ and $\mathcal{H}$.

Conder gives 14 coset diagrams $A, \ldots, N$ for subgroups of index $n = 14, \ldots, 108$ in Γ^+, with respect to the generators X and Y; these can be interpreted as describing transitive representations of Γ^+ of degree n. Each diagram is bilaterally symmetric, so this action of Γ^+ extends to a transitive representation of Γ of degree n, with T fixing vertices on the vertical axis of symmetry, and transposing pairs of vertices on opposite sides of it. Although Conder does not do this, in the spirit of the proof of Theorem 5.1 we can convert each of his diagrams into a planar map of type $\{7, 3\}$ (equivalently a hypermap of type $(3, 2, 7)$) by contracting the small triangles representing 3-cycles of X to trivalent vertices, so that the cycles of X, Y and Z on directed edges correspond to its vertices, edges and faces. (Warning: although Γ^+ acts as the monodromy group of this oriented map, permuting directed edges as described in Example 2 of §2, Γ does *not* act as the monodromy group of the unoriented map, as in Example 1: the latter permutes flags, whereas T uses the symmetry of the map to extend the action of Γ^+ on directed edges to an action of Γ on directed edges.)

We will construct an infinite coset diagram from Conder's diagrams G and H of degree $n = 42$; the corresponding maps $\mathcal{G}$ and $\mathcal{H}$ are shown in Figure 4. Conder defines a (1)-handle in a diagram to be a pair α, β of fixed points of Y with $\beta = \alpha X = \alpha T$, represented in the corresponding map by two free edges incident with the same vertex on the axis of symmetry. Thus $\mathcal{G}$ has three (1)-handles, while $\mathcal{H}$ has one. If diagrams D_i $(i = 1, 2)$ of degree n_i have (1)-handles α_i, β_i then one can form a new diagram, called a (1)-join $D_1(1)D_2$, by replacing these four fixed points of Y with transpositions (α_1, α_2) and (β_1, β_2), and leaving the permutations representing X, Y and T in D_1 and D_2 otherwise unchanged; the result is a new coset diagram giving a transitive representation of Γ of degree $n_1 + n_2$. In terms of the corresponding maps $\mathcal{D}_i$, this is a connected sum operation, in which the two surfaces are joined across cuts between the free ends of the free edges representing the fixed points α_i and β_i; in particular, if $\mathcal{D}_i$ has genus g_i then $\mathcal{D}_1(1)\mathcal{D}_2$ has genus $g_1 + g_2$. This is illustrated in Figure 5, where the (1)-handle at the bottom of $\mathcal{G}$ is joined to that at the top of $\mathcal{H}$ by two dashed edges to form $\mathcal{G}(1)\mathcal{H}$; these edges can be carried by a tube connecting the two surfaces, showing the additivity of the genera (both equal to 0 here). (Further details about this and more general joining operations on dessins can be found in [27].)

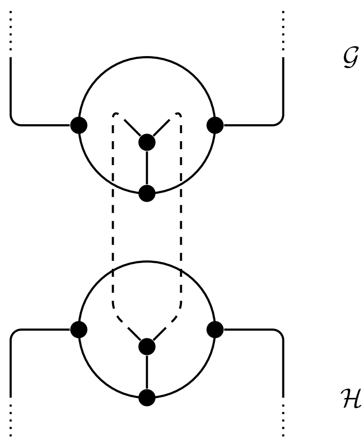

Figure 5: Joining $\mathcal{G}$ and $\mathcal{H}$ to form $\mathcal{G}(1)\mathcal{H}$.

Using (1)-handles in G and H, we first form an infinite diagram

$$H(1)G(1)G(1)G(1)G\cdots.$$

corresponding to an infinite planar map $\mathcal{H}(1)\mathcal{G}(1)\mathcal{G}(1)\mathcal{G}(1)\mathcal{G}\cdots$ of type $\{7, 3\}$: the (1)-handle at the top of each map $\mathcal{H}$ or $\mathcal{G}$ is joined, as in Figure 5, to that at the bottom of the next map $\mathcal{G}$. In this chain, each copy of $\mathcal{G}$ has an unused (1)-handle; we join these arbitrarily in pairs, using (1)-compositions. Each such join adds a bridge to the underlying surface, increasing the genus by 1, so the result is an oriented trivalent map $\mathcal{N}$ of type $\{7, 3\}$ and of infinite genus. This gives an infinite transitive permutation representation $X \mapsto x$, $Y \mapsto y$, $T \mapsto t$ of Γ on the directed edges of $\mathcal{N}$, with Γ^+ again acting as its monodromy group, and T acting as a reflection.

We need to prove that Γ and Γ^+ act primitively. As shown by Conder [7] the permutation $w = yxt$ $(= xyt$ in his notation) induced by YXT has cycle structures $1^3 13^3$ and $1^1 3^1 10^1 11^1 17^1$ in G and H. In each of the (1)-compositions we have used, two fixed points of w are paired to form a cycle of length 2 of w, and a cycle of w of length 13 in G is merged with one of length 13 or 10 in G or H to form a cycle of length 26 or 23. All other cycles of w are unchanged, so in particular its cycle C of length 17 in H remains a cycle in the final diagram. Since all other cycles of w have finite length coprime to 17, some power of w acts on C as w and fixes all other points. Since 17 is prime, it follows that if Γ^+ acts imprimitively, then all points in C must lie in the same equivalence class E. Now C is what Conder calls a 'useful cycle', since it contains a fixed point of y not in a (1)-handle (the right-hand free edge β in the central circle in $\mathcal{H}$ in Figure 4) and a pair of points from a 3-cycle of x (namely β and $\beta x = \beta w^8$). It follows that X and Y leave E invariant, which is impossible since they generate the transitive group Γ^+. Thus Γ^+ acts primitively (as therefore does Γ), so the point-stabilisers $N = \Gamma_\alpha$ and $N^+ = \Gamma^+_\alpha$ of a directed edge α are maximal subgroups of Γ and Γ^+. By the Reidemeister–Schreier algorithm, N^+ is a free product of four cyclic groups of order 2 (arising from fixed points of y in H not in the (1)-handle), and infinitely many of infinite order (two arising from each bridge between a pair of copies of G). Thus N^+ admits an epimorphism onto F_∞, so Proposition 3.3 shows that Γ^+ is countably abundant.

We can choose α to be fixed by t, so that $T \in N$, and hence N is a semidirect product of N^+ by $\langle T \rangle$. The action of t is to reflect H and all the copies of G in the diagram, together with the bridges added between pairs of them. Acting by conjugation on N^+, T therefore induces two transpositions on the elliptic generators of order 2. Each bridge contributes a pair of free generators to N^+, one of them (represented by a loop crossing the bridge and returning 'at ground level'), centralised by T, the other (represented by a loop transverse to the first, following a cross-section of the bridge) inverted by T; by sending T, together with the inverted generators and the four elliptic generators of N^+, to the identity, we can map N onto the free group of countably infinite rank generated by the centralised generators, so Proposition 3.3 shows that Γ is countably abundant. In fact, there are $2^{\aleph_0}$ ways of pairing the copies of G to produce bridges, giving mutually inequivalent permutation representations and hence mutually non-conjugate subgroups N and N^+ of Γ and Γ^+, so this argument establishes countable superabundance.

The extension to the case $r \geq 7$ is essentially the same, but based on the coset diagrams in Conder's later paper [8] on alternating and symmetric quotients of $\Delta(3, 2, r)$ and $\Delta[3, 2, r]$. In this case his coset diagrams $S(h, d)$ and $U(h, d)$ play the roles of G and H, where $r = h + 6d$ with $d \in \mathbb{N}$ and $h = 7, \ldots, 12$. $\square$

Proposition 8.1 accounts for a proportion $121/216$ of all hyperbolic triples. It seems plausible that coset diagrams of Everitt [13] and others, constructed to extend Conder's results on alternating group quotients to all finitely generated non-elementary Fuchsian groups, could be used to prove that all cocompact hyperbolic triangle groups $\Delta(p, q, r)$ and $\Delta[p, q, r]$, together with their associated categories, are countably superabundant, thus proving Conjecture 1.4.

9 Realisation of other groups

Finally, although this paper is mainly about triangle groups and their associated categories of maps and hypermaps, we can deduce realisation properties for many other groups and categories.

Theorem 9.1. *If a group Γ has an epimorphism onto a non-abelian free group then it is finitely and countably superabundant.*

Proof. By Theorem 1.3 the free group $F_2 = \Delta(\infty, \infty, \infty)$ has these properties. Since F_2 is an epimorphic image of every other non-abelian free group, the result follows from Lemma 3.1. □

For example, Theorem 9.1 applies to the fundamental groups Γ of many topological spaces, so their categories of coverings are finitely and countably superabundant.. Examples include compact orientable surfaces of genus g with k punctures, where $2g + k \geq 3$. Taking $g = 0$, $k = 3$ shows that the category $\mathfrak{D}$ of all dessins is finitely superabundant (see Theorem 1.3(a) for a more specific result); in fact, Cori and Machì [9] proved that every finite group is the automorphism group of a finite oriented hypermap, two years before Grothendieck introduced dessins in [20].

Hidalgo [21] has proved the stronger result that every action of a finite group A by orientation-preserving self-homeomorphisms of a compact oriented surface S is topologically equivalent to the automorphism group of a dessin. One way to see this is to triangulate S/A, with all critical values of the projection $\pi : S \to S/A$ among the vertices, none of which has valency 1, and then to add an edge to an additional 1-valent vertex v in the interior of a face. This gives a map (that is, a dessin with $q = 2$) on S/A which lifts via π to a dessin $\mathcal{D}$ on S with $A \leq \mathrm{Aut}(\mathcal{D})$. The only 1-valent vertices in $\mathcal{D}$ are the $|A|$ vertices in $\pi^{-1}(v)$. These are permuted by $\mathrm{Aut}(\mathcal{D})$, with A and hence $\mathrm{Aut}(\mathcal{D})$ acting transitively; however, the stabiliser of a 1-valent vertex (in any dessin) must be the identity, so $\mathrm{Aut}(\mathcal{D}) = A$, as required. By starting with inequivalent triangulations of S/A one can obtain infinitely many non-isomorphic dessins realising this action of A.

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Complex uniformly resolvable decompositions of K_v

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Dedicated to the good friend and colleague Lorenzo Milazzo
who passed away in March 2019.

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Abstract

In this paper we consider complex uniformly resolvable decompositions of the complete graph K_v into subgraphs such that each resolution class contains only blocks isomorphic to the same graph from a given set $\mathcal{H}$ and at least one parallel class is present from each graph of $\mathcal{H}$. We completely determine the spectrum for the cases $\mathcal{H} = \{K_2, P_3, K_3\}$, $\mathcal{H} = \{P_4, C_4\}$, and $\mathcal{H} = \{K_2, P_4, C_4\}$.

Keywords: Resolvable decomposition, complex uniformly resolvable decomposition, path, cycle.

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1 Introduction and definitions

Given a set $\mathcal{H}$ of pairwise non-isomorphic graphs, an *$\mathcal{H}$-decomposition* (or *$\mathcal{H}$-design*) of a graph G is a decomposition of the edge set of G into subgraphs (called *blocks*) isomorphic to some element of $\mathcal{H}$. We say that an $\mathcal{H}$-decomposition is *complex*[1] if all $H \in \mathcal{H}$ are present in it.

An *$\mathcal{H}$-factor* of G is a spanning subgraph of G which is a vertex-disjoint union of some copies of graphs belonging to $\mathcal{H}$. If $\mathcal{H} = \{H\}$, we will briefly speak of an H-factor. An $\mathcal{H}$-decomposition of G is *resolvable* if its blocks can be partitioned into $\mathcal{H}$-factors (*$\mathcal{H}$-factorization* or resolution of G). An $\mathcal{H}$-factor in an $\mathcal{H}$-factorization is referred to as a *parallel class*. Note that the parallel classes are mutually edge-disjoint, by definition.

An $\mathcal{H}$-factorization $\mathcal{F}$ of G is called *uniform* if each factor of $\mathcal{F}$ is an H-factor for some graph $H \in \mathcal{H}$. A K_2-factorization of G is known as a *1-factorization* and its factors are called *1-factors*; it is well known that a 1-factorization of K_v exists if and only if v is even ([10]). If $\mathcal{H} = \{F_1, \dots, F_k\}$ and $r_i \geq 0$ for $i = 1, \dots, k$, we denote by $(F_1, \dots, F_k)$-URD$(v; r_1, \dots, r_k)$ a uniformly resolvable decomposition of the complete graph K_v having exactly r_i F_i-factors. A *complex* $(F_1, \dots, F_k)$-URD$(v; r_1, \dots, r_k)$ is a uniformly resolvable decomposition of the complete graph K_v into $r_1 + \cdots + r_k$ parallel classes with the requirement that at least one parallel class is present for each $F_i \in \mathcal{H}$, i.e., $r_i > 0$ for $i = 1, \dots, k$.

Recently, the existence problem for $\mathcal{H}$-factorizations of K_v has been studied and a lot of results have been obtained, especially on the following types of uniformly resolvable $\mathcal{H}$-decompositions: for a set $\mathcal{H}$ consisting of two complete graphs of orders at most five in [3, 13, 14, 15]; for a set $\mathcal{H}$ of two or three paths on two, three, or four vertices in [5, 6, 9]; for $\mathcal{H} = \{P_3, K_3 + e\}$ in [4]; for $\mathcal{H} = \{K_3, K_{1,3}\}$ in [8]; for $\mathcal{H} = \{C_4, P_3\}$ in [11]; for $\mathcal{H} = \{K_3, P_3\}$ in [12]; for 1-factors and n-stars in [7]; and for $\mathcal{H} = \{P_2, P_3, P_4\}$ in [9]. In connection with our current studies the following cases are most relevant:

- perfect matchings and parallel classes of triangles or 4-cycles (that is, $\{K_2, K_3\}$ or $\{K_2, C_4\}$, Rees [13]);
- perfect matchings and parallel classes of 3-paths ($\{K_2, P_3\}$, Bermond et al. [1], Gionfriddo and Milici [5]);
- parallel classes of 3-paths and triangles ($\{K_3, P_3\}$, Milici and Tuza [12]).

In this paper we give a complete characterization of the spectrum (the set of all admissible combinations of the parameters) for the following two triplets of graphs and for the pair contained in one of them which is not covered by the cases known so far:

- complex $\{K_2, P_3, K_3\}$-decompositions of K_v (Section 3, Theorem 3.1);
- complex $\{K_2, P_4, C_4\}$-decompositions of K_v (Section 4, Theorem 4.1);
- complex $\{P_4, C_4\}$-decompositions of K_v (Section 5, Theorem 5.1).

We summarize the formulation of those results in the concluding section, where a conjecture related to the method of "metamorphosis" of parallel classes is also raised. We provide the basis for this approach by applying linear algebra in Section 2.

[1] In the terminology of [2] one may say that in a complex decomposition all $H \in \mathcal{H}$ are *essential* (page 131) or *mandatory* (page 232). In Chapter II.7.9 on uniformly resolvable designs the default is that all block sizes are essential, while in Chapter IV.1 on partially balanced designs no block size is required to be mandatory.

2 Local metamorphosis

In this section we prove three relations between uniform parallel classes of 4-cycles and 4-paths that will be used in the proofs of our main theorems. Before presenting the new statements, we recall the Milici–Tuza–Wilson Lemma from [12].

It will be assumed throughout, without further mention, that all parallel classes are meant on the same vertex set.

Theorem 2.1 ([12]). *The union of two parallel classes of 3-cycles can be decomposed into three parallel classes of P_3.*

The next two results, Theorems 2.2 and 2.3, will directly imply Theorem 2.4 which states a pure metamorphosis from 4-cycles to 4-paths.

Theorem 2.2. *The union of two parallel classes of C_4 is decomposable into two parallel classes of P_4 and one perfect matching.*

Proof. Let the vertices be $v_1, \dots, v_n$ where n is a multiple of 4. The union of two parallel classes of C_4 forms a 4-regular graph G with $2n$ edges, say $e_1, \dots, e_{2n}$. We associate a Boolean variable x_i with each edge e_i ($1 \le i \le 2n$) and construct a system of linear equations over $GF(2)$, which has $\frac{3n}{2} - 1$ equations over the $2n$ variables. Let us set

$$x_{i_1} + x_{i_2} + x_{i_3} + x_{i_4} = 1 \qquad (\text{mod } 2)$$

for each 4-tuple of indices such that $e_{i_1}, e_{i_2}, e_{i_3}, e_{i_4}$ are either the edges of a C_4 in a parallel class (call this a C-equation) or are the four edges incident with a vertex v_i (a V-equation). This gives $\frac{3}{2}n$ equations, but the V-equation for v_n can be omitted since the n V-equations sum up to 0 (as each edge is counted twice in the total sum) and therefore the one for v_n follows from the others.

We claim that this system of equations is contradiction-free over $GF(2)$. To show this, we need to prove that if the left sides of a subcollection $\mathcal{E}$ of the equations sum up to 0, then also the right sides have zero sum; that is, the number $|\mathcal{E}|$ of its equations is even.

Observe that each variable is present in precisely three equations: in one C-equation and two V-equations. Hence, to have zero sum on the left side, any x_i should either not appear in any equations of $\mathcal{E}$ or be present in precisely two. This means one of the following two situations.

(T1) If $(e_{i_1}, e_{i_2}, e_{i_3}, e_{i_4})$ is a 4-cycle (in this cyclic order of edges) and its C-equation belongs to $\mathcal{E}$, then precisely two related V-equations must be present in $\mathcal{E}$, namely either those for the vertices $e_{i_1} \cap e_{i_2}$ and $e_{i_3} \cap e_{i_4}$ or those for $e_{i_2} \cap e_{i_3}$ and $e_{i_1} \cap e_{i_4}$.

(T2) If $(e_{i_1}, e_{i_2}, e_{i_3}, e_{i_4})$ is a 4-cycle such that its C-equation does not belong to $\mathcal{E}$ but some x_{i_j} ($1 \le j \le 4$) is involved in $\mathcal{E}$, then all the four V-equations for $v_{i_1}, v_{i_2}, v_{i_3}, v_{i_4}$ must be present in $\mathcal{E}$.

In the first and second parallel class of 4-cycles, respectively, let us denote the number of cycles of type (T1) by a_1 and b_1, and that of type (T2) by a_2 and b_2. Then the number of V-equations in $\mathcal{E}$ is equal to both $2a_1 + 4a_2$ and $2b_1 + 4b_2$, which is the same as the average of these two numbers. Thus, the number $|\mathcal{E}|$ of equations is equal to

$$a_1 + b_1 + \frac{1}{2}((2a_1 + 4a_2) + (2b_1 + 4b_2)) = 2(a_1 + a_2 + b_1 + b_2)$$

that is even, as needed.

Since the system of equations is non-contradictory, it has a solution $\xi \in \{0,1\}^{2n}$ over $GF(2)$. We observe further that in any C-equation $x_{i_1} + x_{i_2} + x_{i_3} + x_{i_4} = 1$ we may switch the values from $\xi(x_{i_j})$ to $1 - \xi(x_{i_j})$ simultaneously for all $1 \le j \le 4$, and doing so the modified values remain a solution because the parities of sums in the V-equations do not change either. In this way, we can transform ξ to a basic solution ξ_0 in which every C-equation contains precisely one 1 and three 0s. Since each V-equation contains precisely two or zero variables from each C-equation, it follows that in the basic solution ξ_0 each V-equation, too, contains precisely one 1 and three 0s.

As a consequence, the variables which have $x_i = 1$ in the basic solution define a perfect matching in G (since at most two 1s may occur at each vertex, and then the corresponding V-equation implies that there is precisely one). Moreover, removing those edges from G, each cycle of each parallel class becomes a P_4. In this way we obtain two parallel classes of P_4, and one further class which is a perfect matching. □

Theorem 2.3. *The edge-disjoint union of a perfect matching and a parallel class of C_4 is decomposable into two parallel classes of P_4.*

Proof. We apply several ideas from the previous proof, but in a somewhat different way. We now introduce Boolean variables $x_1, \dots, x_n$ for the edges $e_1, \dots, e_n$ of the 4-cycles only; but still there will be two kinds of linear equations, namely $n/4$ of them for 4-cycles (called C-equations) and $n/2$ of them for the edges of the perfect matching (M-equations). They are of the same form as before:

$$x_{i_1} + x_{i_2} + x_{i_3} + x_{i_4} = 1 \qquad (\text{mod } 2)$$

The C-equations require $e_{i_1}, e_{i_2}, e_{i_3}, e_{i_4}$ to be the edges of a C_4 in the parallel class. The M-equations take $e_{i_1}, e_{i_2}, e_{i_3}, e_{i_4}$ as the four edges incident with a matching edge. If a matching edge is the diagonal of a 4-cycle, then their equations coincide; and if a matching edge shares just one vertex with a 4-cycle then the M-equation and the C-equation share two variables which correspond to consecutive edges on the cycle. Further, we recall that the matching is edge-disjoint from the cycles, therefore each variable associated with a cycle-edge occurs in precisely two distinct M-equations.

These facts imply that only two types of C-equations can occur in a subcollection $\mathcal{E}$ of equations whose left sides sum up to 0 over $GF(2)$.

(T1) If $(e_{i_1}, e_{i_2}, e_{i_3}, e_{i_4})$ is a 4-cycle and its C-equation belongs to $\mathcal{E}$, then the corresponding C_4 has precisely two (antipodal) vertices for which the M-equations of the incident matching edges are present in $\mathcal{E}$. (At the moment it is unimportant whether those two vertices form a matching edge or not.)

(T2) If $(e_{i_1}, e_{i_2}, e_{i_3}, e_{i_4})$ is a 4-cycle whose C-equation does not belong to $\mathcal{E}$ but some x_{i_j} $(1 \le j \le 4)$ is involved in $\mathcal{E}$, then each M-equation belonging to a matching edge incident with some of the four vertices $v_{i_1}, v_{i_2}, v_{i_3}, v_{i_4}$ is present in $\mathcal{E}$. (It is again unimportant whether one or both or none of the diagonals of the C_4 in question is a matching edge.)

Let now a_1 and a_2 denote the number of cycles with type (T1) and type (T2), respectively. By what has been said, the number of vertices requiring an M-equation is equal to $2a_1 + 4a_2$. Since each of those equations is now counted at both ends of the corresponding matching edge, we obtain that $\mathcal{E}$ contains exactly $a_1 + 2a_2$ M-equations; moreover it has a_1 C-equations, by definition. Thus, the number $|\mathcal{E}|$ of equations is equal to $a_1 + (a_1 + 2a_2) = 2(a_1 + a_2)$ which is even. Thus, if the left sides in $\mathcal{E}$ sum up to zero, then also the right sides have sum 0 in $GF(2)$. It proves that the system of the $3n/4$ equations is contradiction-free and has a solution $\xi \in \{0,1\}^n$ over $GF(2)$.

Now, we observe that in any C-equation $x_{i_1} + x_{i_2} + x_{i_3} + x_{i_4} = 1$ we may switch the values from $\xi(x_{i_j})$ to $1 - \xi(x_{i_j})$ simultaneously for all $1 \le j \le 4$. Doing so, the modified values remain a solution as the parities of sums in the M-equations do not change either. In this way we can transform ξ to a basic solution ξ_0 in which every C-equation contains precisely one 1 and three 0s. Since each M-equation has precisely two or four or zero variables from any C-equation, it follows that in the basic solution ξ_0 each M-equation, too, contains precisely one 1 and three 0s.

As a consequence, the variables (cycle-edges) which have $x_i = 1$ in the basic solution establish a pairing between the edges of the original matching. Hence the set $I = \{e_i : \xi_0(x_i) = 1\}$ together with the edges of the given matching factor determines a P_4-factor. Moreover, removing the edges of I from the 4-cycles, we obtain another parallel class of paths P_4. □

These two types of metamorphosis can be combined to obtain the following third one.

Theorem 2.4. *The union of three parallel classes of C_4 is decomposable into four parallel classes of P_4.*

Proof. Applying Theorem 2.2 we transform the union of the first and the second C_4-class into two P_4-classes and a perfect matching. After that we combine the third C_4-class with the perfect matching just obtained into two further P_4-classes, by Theorem 2.3. □

3 The spectrum for $\mathcal{H} = \{K_2, P_3, K_3\}$

In this section we consider complex uniformly resolvable decompositions of the complete graph K_v into m classes containing only copies of 1-factors (perfect matchings), p classes containing only copies of paths P_3 and t classes containing only copies of triangles K_3. The current problem is to determine the set of feasible triples (m, p, t) such that $m \cdot p \cdot t \neq 0$, for which there exists a complex (K_2, P_3, K_3)-URD$(v; m, p, t)$. A little more than that, for $v = 6$ and $v = 12$ we shall list all the possible (m, p, t) with $m, p, t \ge 0$ such that there exists a uniformly resolvable decomposition (K_2, P_3, K_3)-URD$(v; m, p, t)$.

Theorem 3.1. *The necessary and sufficient conditions for the existence of a complex (K_2, P_3, K_3)-URD$(v; m, p, t)$ are:*

(i) $v \ge 12$ *and* v *is a multiple of* 6*;*

(ii) $3m + 4p + 6t = 3v - 3$.

Moreover, the parameters m, p, t are in the following ranges:

(iii) $1 \le m \le v - 7$ *and* m *is odd,*
$3 \le p \le 3 \cdot \lfloor \frac{v}{4} - 1 \rfloor$,
$1 \le t \le \lfloor \frac{v}{2} - 3 \rfloor$.

Proof. We first prove that the conditions are necessary. Divisibility of v by 6 is immediately seen, due to the presence of K_3-classes and 1-factors. We observe further that the number of edges in a parallel class is v for a triangle-class, $2v/3$ for a P_3-class and $v/2$ for a matching. Thus, in any (K_2, P_3, K_3)-URD$(v; m, p, t)$ we must have

$$\frac{mv}{2} + \frac{2pv}{3} + tv = \binom{v}{2}.$$

Dividing it by $v/6$, the assertion of (ii) follows.

As (ii) implies, p is a multiple of 3, say $p = 3x$. Then we obtain

$$m + 4x + 2t = v - 1$$

and also conclude that m is odd. Since $m \geq 1$, $t \geq 1$, and $x \geq 1$, this equation yields

$$m \leq v - 7, \qquad x \leq v/4 - 1, \qquad t \leq v/2 - 3,$$

implying the conditions listed in (iii), and the first one also excludes $v = 6$. This completes the proof that the conditions (i) – (iii) are necessary.

To prove the sufficiency of (i) – (ii), we consider $v \geq 18$ first. Since v is a multiple of 6 according to (i), there exists a Nearly Kirkman Triple System of order v, which means $m = 1$ perfect matching and $t = \frac{v}{2} - 1$ parallel classes of triangles. More generally, for every odd m in the range $1 \leq m \leq v - 7$, there exists a collection of m perfect matchings and $t = \frac{v-1-m}{2}$ parallel classes of triangles, which together decompose K_v; this was proved in [13]. From such a system, for every $0 < x < \frac{v-1-m}{4}$, we can take $2x$ parallel classes of triangles. Applying Theorem 2.1 [12], also proved independently by Wilson (unpublished), we obtain $3x$ parallel classes of paths P_3. This gives a complex (K_2, P_3, K_3)-URD$(v; m, 3x, \frac{v-1-m}{2} - 2x)$. For $v = 12$, the statement follows by Proposition 3.3 below. □

3.1 Small cases

Obviously, the proofs of the necessary conditions that v is a multiple of 6, and that the equality $3m + 4p + 6t = 3v - 3$ must be satisfied by every (K_2, P_3, K_3)-URD$(v; m, p, t)$, do not use the assumption $m \cdot p \cdot t \neq 0$.

Proposition 3.2. *There exists a* (K_2, P_3, K_3)*-URD*$(6; m, p, t)$ *if and only if* $(m, p, t) \in \{(5, 0, 0), (3, 0, 1), (1, 3, 0)\}$.

Proof. Putting $v = 6$, the equation $3m + 4p + 6t = 15$ has exactly four solutions (m, p, t) over the nonnegative integers. The case $(1, 0, 2)$ would correspond to an NKTS(6) which is known not to exist [13]. The case $(5, 0, 0)$ corresponds to a 1-factorization of the complete graph K_6 which is known to exist [10]. The case of $(1, 3, 0)$ is just the same as a (K_2, P_3)-URD$(6; 1, 3)$ that is known to exist [6]. To see the existence for $(3, 0, 1)$, consider $V(K_6) = \mathbb{Z}_6$ and the following classes: $\{\{1, 4\}, \{2, 5\}, \{3, 6\}\}$, $\{\{1, 5\}, \{2, 6\}, \{3, 4\}\}$, $\{\{1, 6\}, \{2, 4\}, \{3, 5\}\}$, $\{(1, 2, 3), (4, 5, 6)\}$. □

Proposition 3.3. *There exists a* (K_2, P_3, K_3)*-URD*$(12; m, p, t)$ *if and only if* $(m, p, t) \in \{(11, 0, 0),\ (9, 0, 1),\ (7, 3, 0),\ (7, 0, 2),\ (5, 0, 3),\ (5, 3, 1),\ (3, 6, 0),\ (3, 3, 2),\ (3, 0, 2),\ (1, 6, 1),\ (1, 3, 3)\}$.

Proof. Checking the nonnegative integer solutions of $3m + 4p + 6t = 33$, the case of $(1, 0, 5)$ would correspond to an NKTS(12) which is known not to exist [13]. The case of $(11, 0, 0)$ corresponds to a 1-factorization of the complete graph K_{12} that is known to exist [2]. The result for the cases $(9, 0, 1)$, $(7, 0, 2)$, $(5, 0, 3)$, $(3, 0, 4)$ follows by [13]. Applying Theorem 2.1 to $(7, 0, 2)$, $(5, 0, 3)$, $(3, 0, 4)$, we obtain the existence for $(7, 3, 0)$, $(5, 3, 1)$, $(3, 3, 2)$, and $(3, 6, 0)$. The existence for the case $(1, 3, 3)$ is shown by the following construction. Let $V(K_{12}) = \mathbb{Z}_{12}$, and consider the following parallel classes:

- matching: $\{\{1, 6\}, \{2, 4\}, \{3, 0\}, \{5, 11\}, \{7, 9\}, \{8, 10\}\}$;
- paths: $\{\{5, 0, 11\}, \{8, 1, 7\}, \{9, 2, 3\}, \{6, 4, 10\}\}$, $\{\{1, 3, 8\}, \{7, 5, 10\}, \{4, 9, 0\}, \{2, 11, 6\}\}$, $\{\{0, 6, 5\}, \{2, 7, 4\}, \{9, 8, 11\}, \{3, 10, 1\}\}$;
- triangles: $\{\{0, 1, 2), \{3, 4, 5\}, \{6, 7, 8\}, \{9, 10, 11\}\}$, $\{\{0, 4, 8\}, \{3, 7, 11\}, \{2, 6, 10\}, \{1, 5, 9\}\}$, $\{\{0, 7, 10\}, \{3, 6, 9\}, \{2, 5, 8\}, \{1, 4, 11\}\}$.

Finally, we apply Theorem 2.1 to the case $(1, 3, 3)$ and infer that a (K_2, P_3, K_3)-URD$(12; 1, 6, 1)$ exists, too. □

4 The spectrum for $\mathcal{F} = \{K_2, P_4, C_4\}$

In this section we consider complex uniformly resolvable decompositions of the complete graph K_v into m parallel 1-factors, p parallel classes of 4-paths, and c parallel classes of 4-cycles. The current problem is to determine the set of feasible triples (m, p, c) such that $m \cdot p \cdot c \neq 0$, for which there exists a complex (K_2, P_3, C_4)-URD$(v; m, p, c)$. The case $\{P_4, C_4\}$ will be discussed in Section 5.

Theorem 4.1. *The necessary and sufficient conditions for the existence of a complex* (K_2, P_4, C_4)*-URD*$(v; m, p, c)$ *are:*

(i) $v \geq 8$ *and* v *is a multiple of* 4*;*

(ii) $2m + 3p + 4c = 2v - 2$.

Moreover, the parameters m, p, c *are in the following ranges:*

(iii) $1 \leq c \leq \frac{v}{2} - 3$,
$1 \leq m \leq v - 6$,
$2 \leq p \leq 2\lfloor \frac{v-4}{3} \rfloor$;

(iv) p *is even; and if* $p \equiv 2 \pmod 4$*, then also* m *is even.*

Proof. We first show that the conditions are necessary. Since K_v has a C_4-factor, v must be a multiple of 4. Further, as a C_4-, K_2-, and P_4-factor respectively cover exactly v, $v/2$, and $3v/4$ edges, in a (K_2, P_4, C_4)-URD$(v; m, p, c)$ we have

$$\frac{mv}{2} + \frac{3pv}{4} + cv = \binom{v}{2}.$$

This equality directly implies (ii) and we may also conclude that p is even and, further, if $p \equiv 2 \pmod 4$, then m must be even as well. Putting $p = 2x$ we obtain

$$m + 3x + 2c = v - 1.$$

By our condition, all the three types of parallel classes are present in the decomposition, i.e. we have $c \geq 1$, $m \geq 1$, and $x \geq 1$. These, together with the equality above, imply the necessity of (iii).

Next we prove the sufficiency of (i) – (ii). We first take a 1-factorization of $K_{v/2}$ into $v/2 - 1$ perfect matchings, which exists because v is a multiple of 4. Now, replace each vertex of $K_{v/2}$ with two non-adjacent vertices, and each edge with a copy of the complete bipartite graph $K_{2,2}$, i.e. a 4-cycle whose missing diagonals are the non-adjacent vertex pairs.[2] This blow-up results in $v/2 - 1$ parallel classes of C_4 inside K_v, and the missing edges can be taken as a perfect matching. Let C be the set of the parallel classes of C_4 and x be a nonnegative integer such that $0 < x \leq \lfloor \frac{v-4}{3} \rfloor$. The construction splits into two cases depending on the parity of x.

If x is even, take $\frac{3x}{2}$ parallel classes from C. Applying Theorem 2.4, we transform the $\frac{3x}{2}$ parallel classes of C_4 into $2x = p$ parallel classes of paths P_4. For any given $y = c$ in the range $0 < y \leq \lfloor \frac{v-2}{2} - \frac{3x}{2} \rfloor$, keep y classes of C_4 and transform the remaining $\frac{v-2}{2} - \frac{3x}{2} - y$ classes of C_4 into $2(\frac{v-2}{2} - \frac{3x}{2} - y) = m - 1$ classes of 1-factors. In this way we obtain a complex (K_2, P_4, C_4)-URD$(v; v - 3x - 2y - 1, 2x, y)$.

If x is odd, take $\frac{3(x-1)}{2} + 2$ parallel classes from C. By Theorems 2.2 and 2.4, we can transform the $\frac{3(x-1)}{2} + 2$ parallel classes of C_4 into $2x = p$ parallel classes of paths P_4 and a 1-factor. For any given $y = c$ in the range $0 < y \leq \frac{v-2}{2} - \frac{3(x-1)}{2} - 2$, keep y classes of C_4 and transform the remaining $\frac{v-2}{2} - \frac{3(x-1)}{2} - 2 - y$ classes of C_4 into $2(\frac{v-2}{2} - \frac{3(x-1)}{2} - 2 - y) = m - 2$ classes of 1-factors. In this way, we obtain a complex (K_2, P_4, C_4)-URD$(v; v - 3x - 2y - 1, 2x, y)$.

The result, for every $v \equiv 0 \pmod 4$, $0 < x \leq \lfloor \frac{v-4}{3} \rfloor$, and $0 < y \leq \lfloor \frac{v-2}{2} - \frac{3x}{2} \rfloor$, is a uniformly resolvable decomposition of K_v into into $v - 1 - 3x - 2y = m$ classes containing only copies of 1-factors, $2x = p$ classes containing only copies of paths P_4, and $y = c$ classes containing only copies of 4-cycles C_4. This finishes the proof of the theorem. □

5 The spectrum for $\mathcal{H} = \{P_4, C_4\}$

Finally, we consider complex uniformly resolvable decompositions of the complete graph K_v into p classes containing only copies of paths P_4 and c classes containing only copies of 4-cycles C_4.

Theorem 5.1. *The necessary and sufficient conditions for the existence of a complex (P_4, C_4)-URD$(v; p, c)$ are:*

(i) $v \geq 8$ *and* v *is a multiple of* 4*;*

(ii) $3p + 4c = 2v - 2$.

Proof. Necessity is a consequence of Theorem 4.1, since we did not need to assume $m > 0$ in that part of its proof. Turning to sufficiency, the condition $3p + 4c = 2v - 2$ implies that $3p \equiv 2v - 2 \pmod 4$. This gives $p = 2 + 4x$ and $c = \frac{v-4}{2} - 3x$. For a construction, we start with a decomposition of K_v into a perfect matching F and $\frac{v-2}{2}$ parallel classes of C_4 as in the proof of Theorem 4.1. By Theorem 2.3, we can transform one class of C_4 and F

[2]With another terminology, this "blow-up" is the lexicographic product $K_{v/2}[2K_1]$.

into two classes of paths P_4. Then, by Theorem 2.4, we transform $3x$ parallel classes of C_4 into $4x$ parallel classes of P_4. The result, for every x such that $0 \leq x \leq \lfloor \frac{v-6}{6} \rfloor$, is a uniformly resolvable decomposition of K_v into $2 + 4x$ classes containing only copies of paths P_4 and $\frac{v-4}{2} - 3x$ classes containing only copies of 4-cycles C_4. This completes the proof. □

6 Conclusion

Combining Theorems 3.1, 4.1, and 5.1, we obtain the main result of this paper.

Theorem 6.1.

(i) A complex (K_2, P_3, K_3)-URD$(v; m, p, t)$ exists if and only if $v \geq 12$, v is a multiple of 6, and $3m + 4p + 6t = 3v - 3$.

(ii) A complex (K_2, P_4, C_4)-URD$(v; m, p, t)$ exists if and only if $v \geq 8$, v is a multiple of 4, and $2m + 3p + 4t = 2v - 2$.

(iii) A complex (P_4, C_4)-URD$(v; p, t)$ exists if and only if $v \geq 8$, v is a multiple of 4, and $3p + 4t = 2v - 2$.

Concerning the local metamorphosis studied in Section 2, we pose the following conjecture as a common generalization of Theorems 2.1 and 2.4.

Conjecture 6.2. *The union of $k-1$ parallel classes of C_k is decomposable into k parallel classes of P_k.*

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General d-position sets*

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Abstract

The general d-position number $\mathrm{gp}_d(G)$ of a graph G is the cardinality of a largest set S for which no three distinct vertices from S lie on a common geodesic of length at most d. This new graph parameter generalizes the well studied general position number. We first give some results concerning the monotonic behavior of $\mathrm{gp}_d(G)$ with respect to the suitable values of d. We show that the decision problem concerning finding $\mathrm{gp}_d(G)$ is NP-complete for any value of d. The value of $\mathrm{gp}_d(G)$ when G is a path or a cycle is computed and a structural characterization of general d-position sets is shown. Moreover, we present some relationships with other topics including strong resolving graphs and dissociation sets. We finish our exposition by proving that $\mathrm{gp}_d(G)$ is infinite whenever G is an infinite graph and d is a finite integer.

Keywords: General d-position sets, dissociation sets, strong resolving graphs, computational complexity, infinite graphs.

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1 Introduction

A *general position set* of a graph G is a set of vertices $S \subseteq V(G)$ such that no three vertices from S lie on a common shortest path of G. The order of a largest general position set, shortly called a *gp-set*, is the *general position number* $\mathrm{gp}(G)$ of G (also written gp-number). This concept was recently and independently introduced in [6, 16]. We should mention though that the same concept was studied on hypercubes already in 1995 by Körner [14]. Following [16] and its notation and terminology, the concept received a lot of attention, see the series of papers [8, 13, 11, 12, 17, 18, 21, 22]. In particular, in [18] the general position problem was studied on complementary prisms. In order to characterize an extremal case for the general position number of these graphs, the concept of general 3-position was introduced as an essential ingredient of the characterization. In this paper we extend this idea as follows.

Let $d \in \mathbb{N}$ and let G be a (connected) graph. Then $S \subseteq V(G)$ is a *general d-position set* if the following holds:

$$\{u, v, w\} \in \binom{S}{3}, v \in I_G(u, w) \Rightarrow d_G(u, w) > d\,, \tag{1.1}$$

where $d_G(u, w)$ denotes the shortest-path distance in G between u and w, and $I_G(u, w) = \{x \in V(G) :\ d_G(u, w) = d_G(u, x) + d_G(x, w)\}$ is the *interval* between u and w. In words, S is a general d-position set if no three different vertices from S lie on a common geodesic of length at most d. We will say that vertices u, v, w that fulfill condition (1.1) lie in *general d-position*. The cardinality of a largest general d-position set in a graph G is the *general d-position number* of G and is denoted by $\mathrm{gp}_d(G)$.

We proceed as follows. In the rest of this section we recall needed definitions and state some basic facts and results on the general d-position number. Then, in Section 2, we demonstrate that in the inequality chain $\mathrm{gp}_{\mathrm{diam}(G)}(G) \le \mathrm{gp}_{\mathrm{diam}(G)-1}(G) \le \cdots \le \mathrm{gp}_2(G)$ all kinds of equality and strict inequality cases are possible. Using one of the corresponding constructions we also prove that the problem of determining the gp_d number is NP-complete. In Section 3 we determine the gp_d number of paths and cycles and give a general upper bound on the gp_d number in term of the diameter of a given graph. In the subsequent section we prove a structural characterization of general d-position sets. In Section 5 we report on the connections between general d-position sets and two well-established concepts, the dissociation number and strong resolving graphs. In the concluding section we consider the gp_d number of infinite graphs and pose several open questions.

1.1 Preliminaries

For a positive integer k we will use the notation $[k] = \{1, \ldots, k\}$. The *clique number* and the *independence number* of G are denoted by $\omega(G)$ and $\alpha(G)$. If $S \subseteq V(G)$, then the subgraph of G induced by S is denoted by $\langle S \rangle$ and $\binom{S}{k}$ denotes the set of all subsets of S having cardinality k. A subgraph H of a graph G is *isometric* if $d_H(u, v) = d_G(u, v)$ holds for all $u, v \in V(H)$. If H_1 and H_2 are subgraphs of G, then the distance $d_G(H_1, H_2)$ between H_1 and H_2 is defined as $\min\{d_G(h_1, h_2) :\ h_1 \in V(H_1), h_2 \in V(H_2)\}$. In particular, if H_1 is the one vertex graph with u being its unique vertex, then we will write $d_G(u, H_2)$ for $d_G(H_1, H_2)$. We say that the subgraphs H_1 and H_2 are *parallel*, denoted by $H_1 \parallel H_2$, if for every pair of vertices $h_1 \in V(H_1)$ and $h_2 \in V(H_2)$ we have $d_G(h_1, h_2) = d_G(H_1, H_2)$. If H_1 and H_2 are not parallel, we will write $H_1 \nparallel H_2$. The *open neighborhood* and the *closed neighborhood* of a vertex v of G will be denoted by $N_G(x)$ and $N_G[x]$, respectively. Vertices x and y of G are *true twins* if $N_G[u] = N_G[v]$. We may omit the subscript G in the above definitions if the graph G is clear from the context.

Clearly, if $d = 1$, then every subset of vertices of G is a general 1-position set, and if $d \geq \mathrm{diam}(G)$, then S is a general d-position set if and only if S is a general position set. Moreover, note that

$$\mathrm{gp}_{\mathrm{diam}(G)}(G) \leq \mathrm{gp}_{\mathrm{diam}(G)-1}(G) \leq \cdots \leq \mathrm{gp}_2(G)\,. \tag{1.2}$$

We conclude the preliminaries with the following useful property.

Proposition 1.1. *Let G be a graph and let $2 \leq d \leq \mathrm{diam}(G) - 1$ be a positive integer. If $H_1, \ldots, H_r$ are isometric subgraphs of G such that $d_G(H_i, H_j) \geq d$ for $i \neq j$, then $\mathrm{gp}_d(G) \geq \sum_{i=1}^{r} \mathrm{gp}_d(H_i)$.*

Proof. For each $i \in [r]$, let S_i be a general d-position set of H_i such that $|S_i| = \mathrm{gp}_d(H_i)$. We claim that $S = \bigcup_{i=1}^{r} S_i$ is a general d-position set of G. Suppose $\{x, y, z\} \in \binom{S}{3}$ such that $y \in I_G(x, z)$ and $d_G(x, z) \leq d$. That is, there exists a shortest x,z-path of length at most d in G that contains y. Since $d_G(u, v) \geq d$ for any two vertices $u \in V(H_i)$ and $v \in V(H_j)$ with $i \neq j$, there exists $k \in [r]$ such that $\{x, y, z\} \subseteq V(H_k)$. Now, since H_k is an isometric subgraph of G, it follows that $d_{H_k}(x, y) = d_G(x, y)$, $d_{H_k}(y, z) = d_G(y, z)$ and $d_{H_k}(x, z) = d_G(x, z)$. This implies that there is a x,z-geodesic in H_k that contains y. Hence, $y \in I_{H_k}(x, z)$, and since S_k is a general d-position set of H_k, we infer that $d_G(x, z) = d_{H_k}(x, z) > d$, which is a contradiction. Therefore, S is a general d-position set of G, and it follows that $\mathrm{gp}_d(G) \geq \sum_{i=1}^{r} \mathrm{gp}_d(H_i)$. □

2 On the inequality chain (1.2) and computational complexity

In this section we investigate the inequality chain (1.2) by constructing different classes of graphs which demonstrate that all kinds of equality and strict inequality cases can happen. We conclude the section by applying one of these constructions to prove that the GENERAL d-POSITION PROBLEM is NP-complete.

Equality in (1.2) simultaneously.
For $n \geq 2$, let S be a star with center x and leaves $u_1, \ldots, u_n, v_1, \ldots, v_n$. Construct a graph G_n of order $2n + 3$ by taking the disjoint union of S and an independent set of vertices $\{u, v\}$ together with the set of edges $\{uu_i, vv_i : i \in [n]\}$. The diameter of G_n is 4, and we have $\mathrm{gp}_4(G_n) = \mathrm{gp}_3(G_n) = \mathrm{gp}_2(G_n) = 2n$.

Equality in (1.2) simultaneously again.
Let T_r, $r \geq 2$, be the tree obtained from the path P_{r+1} on $r + 1$ vertices, by attaching two leaves to each of its internal vertices. Then we claim that

$$\mathrm{gp}_r(T_r) = \mathrm{gp}_{r-1}(T_r) = \cdots = \mathrm{gp}_2(T_r)\,.$$

Indeed, first note that $\mathrm{diam}(T_r) = r$. Since the gp-number of a tree is the number of its leaves (cf. [16, Corollary 3.7]), we have $\mathrm{gp}_r(T_r) = 2r$. Let next S be a general 2-position set. If u is a vertex of T_r adjacent to exactly two leaves, say v and w, then $|S \cap \{u, v, w\}| \leq 2$. Moreover, if u is a vertex of T_r adjacent to exactly three leaves, say v, w, and z, then $|S \cap \{u, v, w, z\}| \leq 3$. It follows that $\mathrm{gp}_2(T_r) \leq 2(r - 3) + 2 \cdot 3 = 2r$. In conclusion, $2r = \mathrm{gp}_r(T_r) \leq \mathrm{gp}_{r-1}(T_r) \leq \cdots \leq \mathrm{gp}_2(T_r) \leq 2r$, hence equality holds throughout.

Strict inequality in (1.2) **in exactly one case**.
Let $k, \ell \geq 4$ and let $G_{k,\ell}$ be a graph defined as follows. Its vertex set is

$$V(G_{k,\ell}) = \bigcup_{j=1}^{k} \{u_j, w_j, x_{j,1}, \ldots, x_{j,\ell}\} \cup \{x_{k+1,1}\}.$$

For $j \in [k]$, each of the vertices u_j and w_j is adjacent to $x_{j,1}, \ldots, x_{j,\ell}$ and to $x_{j+1,1}$. There are no other edges in $G_{k,\ell}$. Note that $|V(G_{k,\ell})| = k(\ell+2)+1$ and that $\mathrm{diam}(G_{k,\ell}) = 2k$.

It is straightforward to see that the set $X = \bigcup_{j=1}^{k} \{x_{j,1}, \ldots, x_{j,\ell}\} \cup \{x_{k+1,1}\}$ is a largest independent set of $G_{k,\ell}$. Moreover, X is also a largest general 2-position set and a largest general 3-position set. Furthermore, it is not difficult to infer that the set $X \setminus \{x_{2,1}, \ldots, x_{k,1}\}$ is a largest general d-position set for each $d \in \{4, \ldots, 2k\}$. In conclusion,

$$\mathrm{gp}_{2k}(G_{k,\ell}) = \mathrm{gp}_{2k-1}(G_{k,\ell}) = \cdots = \mathrm{gp}_4(G_{k,\ell}) < \mathrm{gp}_3(G_{k,\ell}) = \mathrm{gp}_2(G_{k,\ell}) = \alpha(G).$$

Strict inequality in (1.2) **in every case**.
Given a positive integer t, construct the graph H_t as follows. Begin with a complete graph K_{4t} with vertex set $V(K_{4t}) = A \cup B$ where $|A| = |B| = 2t$. Next, add a path $P_{t-1} = v_1 \ldots v_{t-1}$, and join with an edge every vertex of B with the leaf v_1 of P_{t-1}. Then, add a pendant vertex u_i to every vertex $v_i \in \{v_2, \ldots, v_{t-1}\}$, and finally, for every $i \in \{2, \ldots, t-1\}$, add the edge $u_i v_{i-1}$. As an example, the graph H_8 is represented in Figure 1.

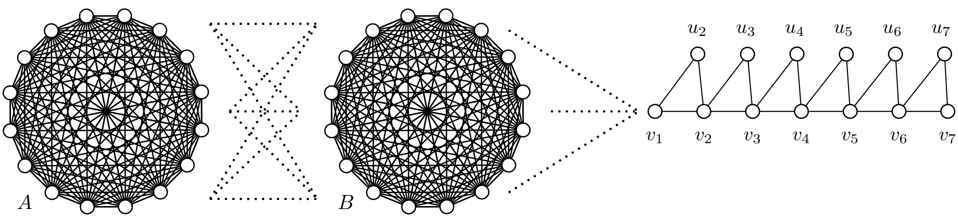

Figure 1: The graph H_8. Edges joining the sets A and B, as well as joining B with the vertex v_1 are indicated with dotted lines.

Notice that the graph H_t has diameter t. The general d-position number of H_t for all possible d is given in the following result.

Proposition 2.1. *If* $2 \leq d \leq t$, *then*

$$\mathrm{gp}_d(H_t) = \begin{cases} 4t; & d = t, \\ 4t+2; & d = t-1, \\ 5t-d+1; & \textit{otherwise}. \end{cases}$$

Proof. We first note that the set $A \cup B$ is a general position set of H_t, or equivalently a general t-position set. Thus, $\mathrm{gp}(H_t) = \mathrm{gp}_t(H_t) \geq 4t$. Suppose $\mathrm{gp}(H_t) = \mathrm{gp}_t(H_t) > 4t$ and let S be a general t-position set. Hence, there exists at least one vertex not in $A \cup B$ which is in S. Since every shortest path joining a vertex of A with a vertex not in $A \cup B$ passes through a vertex in B, it follows that $S \cap A = \varnothing$ or $S \cap B = \varnothing$. This implies that $|S| \leq 2t + 2t - 3 = 4t - 3$, and this is not possible. Therefore $\mathrm{gp}(H_t) = \mathrm{gp}_t(H_t) = 4t = 5t - d$, when $d = t$.

We next consider the case $d = t - 1$. The set $A \cup B \cup \{v_{t-1}, u_{t-1}\}$ is a general $(t-1)$-position set of H_t, and so, $\mathrm{gp}_{t-1}(H_t) \geq 4t+2$. If we suppose that $\mathrm{gp}_{t-1}(H_t) > 4t+2$, then a similar argument to that above for $d = t$ leads to a contradiction. Therefore, $\mathrm{gp}_{t-1}(H_t) = 4t+2$.

We finally consider $d = t - k$ with $2 \leq k \leq t-2$. Notice that the set $A \cup B \cup \{v_{t-1}, u_{t-1}, u_{t-2}, \ldots, u_{t-k}\}$ is a general d-position set of H_t of cardinality $4t + k + 1 = 4t + (t - d + 1) = 5t - d + 1$, and so, $\mathrm{gp}_d(H_t) \geq 5t - d + 1$. Again, an argument similar to the two cases above leads to $\mathrm{gp}_d(H_t) = 5t - d + 1$. □

Proposition 2.1 yields strict inequalities in the chain (1.2), that is, for any graph H_t with $t \geq 3$, we have

$$\mathrm{gp}_t(H_t) < \mathrm{gp}_{t-1}(H_t) < \cdots < \mathrm{gp}_2(H_t)\,. \tag{2.1}$$

We shall finish this section by considering the computational complexity of the decision problem related to finding the general d-position number of graphs, in which we also show the usefulness of the above graphs H_t.

GENERAL d-POSITION PROBLEM
Input: A graph G, an integer $d \geq 2$, and a positive integer r.
Question: Is $\mathrm{gp}_d(G)$ larger than r?

We first remark that the GENERAL d-POSITION PROBLEM is known to be NP-complete for every $d \geq \mathrm{diam}(G)$ (see [16]). Hence, we may center our attention on the cases $d \in \{2, \ldots, \mathrm{diam}(G) - 1\}$, although our reduction also works for the case $d = \mathrm{diam}(G)$.

Theorem 2.2. *If $d \geq 2$, then the* GENERAL d-POSITION PROBLEM *is NP-complete.*

Proof. First, we can readily observe that the problem belongs to the class NP, since checking that a given set is indeed a general d-position set can be done in polynomial time. From now on, we make a reduction from the MAXIMUM CLIQUE PROBLEM to the GENERAL d-POSITION PROBLEM.

In order to present the reduction, for a given graph G of order $t \geq 3$, we shall construct a graph G' by using the above graphs H_t. We construct G' from the disjoint union of G and H_t, by adding all possible edges between $A \cup B \cup \{v_1\}$ and $V(G)$. It is then easily observed that $\omega(G') = |A| + |B| + \omega(G)$. Moreover, using similar arguments as in the proof of Proposition 2.1, we deduce that $\mathrm{gp}_d(G') = \mathrm{gp}_d(H_t) + \omega(G)$. From this fact, since the value $\mathrm{gp}_d(H_t)$ is known from Proposition 2.1, the reduction is completed, and the theorem is proved. □

3 Paths and cycles

In this section we determine the general d-position number of paths and cycles. The first result in turn implies a general upper bound on the general d-position number in term of the diameter of a given graph.

Proposition 3.1. *If $n \geq 3$ and $2 \leq d \leq n-1$, then*

$$\mathrm{gp}_d(P_n) = \begin{cases} 2\left\lceil \frac{n}{d+1} \right\rceil - 1; & n \equiv 1 \pmod{d+1}, \\ 2\left\lceil \frac{n}{d+1} \right\rceil; & \textit{otherwise}. \end{cases}$$

Proof. Let $d \in \{2, \dots, n-1\}$ and $P_n = v_1 v_2 \dots v_n$. If $n \equiv 1 \pmod{d+1}$, then let

$$S = \{v_{(d+1)i+1}, v_{(d+1)i+2} : 0 \le i \le \lfloor n/(d+1) \rfloor - 1\} \cup \{v_n\},$$

and if $n \not\equiv 1 \pmod{d+1}$, then let

$$S = \{v_{(d+1)i+1}, v_{(d+1)i+2} : 0 \le i \le \lceil n/(d+1) \rceil - 1\}.$$

It can be readily seen that S is a general d-position set of P_n, which gives the lower bound

$$\mathrm{gp}_d(P_n) \ge \begin{cases} 2\left\lceil \frac{n}{d+1} \right\rceil - 1; & n \equiv 1 \pmod{d+1}, \\ 2\left\lceil \frac{n}{d+1} \right\rceil; & \text{otherwise.} \end{cases}$$

On the other hand, suppose

$$\mathrm{gp}_d(P_n) > \begin{cases} 2\left\lceil \frac{n}{d+1} \right\rceil - 1; & n \equiv 1 \pmod{d+1}, \\ 2\left\lceil \frac{n}{d+1} \right\rceil; & \text{otherwise,} \end{cases}$$

and let S' be a general d-position set of cardinality $\mathrm{gp}_d(P_n)$. By the pigeonhole principle, we deduce that there exists a subpath in P_n of length d that contains at least three elements of S', but this is not possible. Therefore, the desired equality follows. □

Specializing to $n = 14$ in Proposition 3.1, we next show a table with the values of $\mathrm{gp}_d(P_n)$ for every possible value of d. Notice that, equalities and inequalities occur in distinct positions with respect to the chain (1.2).

d	2	3	4	5	6	7	8	9	10	11	12	13
$\mathrm{gp}_d(P_{14})$	10	8	6	6	4	4	4	4	4	4	3	2

Table 1: The values of $\mathrm{gp}_d(P_{14})$ for every $2 \le d \le 13$.

The result for paths gives the following general lower bound.

Corollary 3.2. *Let G be a connected graph of diameter d. If $2 \le k \le d$, then*

$$\mathrm{gp}_k(G) \ge \begin{cases} 2\left\lceil \frac{d+1}{k+1} \right\rceil - 1; & d \equiv 0 \pmod{k+1}, \\ 2\left\lceil \frac{d+1}{k+1} \right\rceil; & \text{otherwise.} \end{cases}$$

Proof. Shortest paths are isometric subgraphs; in particular, this holds for diametrical paths. Hence G contains an isometric P_{d+1}, and therefore $\mathrm{gp}_k(G) \ge \mathrm{gp}_k(P_{d+1})$ by Proposition 1.1 with $r = 1$. Applying Proposition 3.1 yields the result. □

In a similar manner as done for paths, we can compute the general d-position number for cycles. It is easy to show that $\mathrm{gp}_d(C_3) = 3$ for any d, $\mathrm{gp}_1(C_4) = 4$, and $\mathrm{gp}_d(C_4) = 2$ for $d \ge 2$.

Proposition 3.3. *If $n \ge 5$ and $2 \le d < \left\lfloor \frac{n}{2} \right\rfloor$, then*

$$\mathrm{gp}_d(C_n) = \begin{cases} 2\left\lfloor \frac{n}{d+1} \right\rfloor + 1; & n \equiv d \pmod{d+1}, \\ 2\left\lfloor \frac{n}{d+1} \right\rfloor; & \textit{otherwise.} \end{cases}$$

If $d \ge \left\lfloor \frac{n}{2} \right\rfloor$, then $\mathrm{gp}_d(C_n) = 3$.

Proof. Let $C_n = v_1 v_2 \dots v_n v_1$. Note that $\mathrm{diam}(C_n) = \lfloor \frac{n}{2} \rfloor$, and the argument naturally splits into two cases.

First assume that $2 \le d < \lfloor \frac{n}{2} \rfloor$. Let $m = \lfloor \frac{n}{d+1} \rfloor$ and for each $k \in [m]$ we define X_k by $X_k = \{v_i : (k-1)(d+1)+1 \le i \le k(d+1)\}$. Let $X = V(C_n) \setminus \bigcup_{k=1}^{m} X_k$. Note that $|X| = x$ where $n \equiv x \pmod{d+1}$ and x is the unique integer such that $0 \le x \le d$. If $x \ne d$, then let $S = \{v_{(k-1)(d+1)+1}, v_{(k-1)(d+1)+2} : 1 \le k \le m\}$. If $x = d$, then let $S = \{v_{(k-1)(d+1)+1}, v_{(k-1)(d+1)+2} : 1 \le k \le m\} \cup \{v_{m(d+1)+1}\}$. It is straightforward to check that in both cases S is a general d-position set, which shows that the claimed value is a lower bound for $\mathrm{gp}_d(C_n)$. As in the proof of Proposition 3.1, an application of the pigeonhole principle establishes the upper bound.

Since $\mathrm{diam}(C_n) = \lfloor \frac{n}{2} \rfloor$, to prove the second statement it is sufficient to show that $\mathrm{gp}_d(C_n) = 3$ for $d = \lfloor \frac{n}{2} \rfloor$. For this purpose, let $S = \{v_1, v_3, v_{\lceil \frac{n}{2} \rceil + 2}\}$. For $n = 2r$, we see that $S = \{v_1, v_3, v_{r+2}\}$ and $d = r$. On the other hand, for $n = 2r+1$, we have $S = \{v_1, v_3, v_{r+3}\}$ and $d = r$. In both cases an easy computation shows that none of the three vertices lies on a shortest path in C_n between the other two vertices. Therefore, S is a general d-position set, and it follows that $\mathrm{gp}_d(C_n) \ge 3$. Suppose T is an arbitrary general d-position set of C_n. We may assume without loss of generality that $v_1 \in T$. It follows that $|T \cap \{v_2, \dots, v_{r+1}\}| \le 1$ and $|T \cap \{v_{r+2}, \dots v_n\}| \le 1$, for otherwise T contains three vertices that lie on a path of length at most d. Therefore, $\mathrm{gp}_d(C_n) \le |T| \le 3$. □

4 A characterization of general d-position sets

In [1, Theorem 3.1] a structural characterization of general position sets of a given graph was proved. In this section we give such a characterization for general d-position sets and as a consequence deduce the characterization from [1].

Theorem 4.1. *Let G be a connected graph and let $d \ge 2$ be an integer. Then $S \subseteq V(G)$ is a general d-position set if and only if the following conditions hold:*

(i) *$\langle S \rangle$ is a disjoint union of complete graphs $Q_1, \dots, Q_\ell$.*

(ii) *If $Q_i \nparallel Q_j$, $i \ne j$, then $d_G(Q_i, Q_j) \ge d$.*

(iii) *If $d_G(Q_i, Q_j) + d_G(Q_j, Q_k) = d_G(Q_i, Q_k)$ for $\{i,j,k\} \in \binom{[\ell]}{3}$, then $d_G(Q_i, Q_k) > d$.*

Proof. Let S be a general d-position set of G and let H be a connected component of $\langle S \rangle$. If H is not complete, then it contains an induced P_3. The vertices of this P_3 are on a geodesic of length 2 which is not possible since they belong to S and $d \ge 2$. Hence H must be complete.

Consider next two cliques Q_i and Q_j that are not parallel. Let $d_G(Q_i, Q_j) = p$ and let $u \in Q_i$ and $v \in Q_j$ be vertices with $d_G(u,v) = p$. Since $Q_i \nparallel Q_j$, we may assume without loss of generality that there is a vertex $w \in Q_i$ such that $d_G(w, Q_j) = p+1$. Then u lies on a w,v-geodesic of length $p+1$ which implies that $p+1 \ge d+1$ and so, $d_G(Q_i, Q_j) = p \ge d$.

Assume next that $d_G(Q_i, Q_j) + d_G(Q_j, Q_k) = d_G(Q_i, Q_k)$ for some $\{i,j,k\} \in \binom{[\ell]}{3}$. If $Q_i \nparallel Q_j$, then by the already proved condition (ii) we immediately get that $d_G(Q_i, Q_j) \ge d$ and thus $d_G(Q_i, Q_k) > d$. The same holds if $Q_j \nparallel Q_k$. Hence assume next that $Q_i \parallel Q_j$ and $Q_j \parallel Q_k$. Let $u \in Q_i$ and $w \in Q_k$ be vertices with $d_G(u,w) = d_G(Q_i, Q_k)$. Since $d_G(Q_i, Q_j) + d_G(Q_j, Q_k) = d_G(Q_i, Q_k)$, $Q_i \parallel Q_j$, and $Q_j \parallel Q_k$, it follows that

$d_G(u,w) = d_G(u,v) + d_G(v,w)$ for every vertex v of Q_j. We conclude that $d_G(Q_i, Q_k) > d$.

To prove the converse, assume that conditions (i), (ii), and (iii) are fulfilled for a given set S and let $\{u,v,w\} \in \binom{S}{3}$. We need to show that u,v,w lie in general d-position.

If u,v,w lie in the same connected component of $\langle S\rangle$, then by (i), this component is complete and the assertion is clear. Suppose next that u,v,w lie in the union of cliques Q_i and Q_j. If $Q_i \parallel Q_j$, then u,v,w are clearly in general d-position. And if $Q_i \nparallel Q_j$, then u,v,w lie in general d-position by (ii).

In the last case to consider the three vertices lie in different cliques, say $u \in Q_i$, $v \in Q_j$, and $w \in Q_k$. If the assertion does not hold, then the three vertices lie on a common geodesic and we may assume without loss of generality that $d_G(u,w) = d_G(u,v) + d_G(v,w)$. If $Q_i \nparallel Q_j$, then by (ii), we get $d_G(Q_i,Q_j) \geq d$ and hence $d_G(u,w) = d_G(u,v) + d_G(v,w) \geq d_G(Q_i,Q_j) + d_G(Q_j,Q_k) \geq d+1 > d$. Analogously, if $Q_j \nparallel Q_k$, we also get $d_G(u,w) > d$. Suppose then that $Q_i \parallel Q_j$ and $Q_j \parallel Q_k$. If also $Q_i \parallel Q_k$, then $d_G(u,w) = d_G(u,v) + d_G(v,w)$ implies that $d_G(Q_i,Q_j) + d_G(Q_j,Q_k) = d_G(Q_i,Q_k)$ and so $d_G(Q_i,Q_k) > d$ by (iii). Again using the fact that $Q_i \parallel Q_k$, it follows that $d_G(u,w) > d$. We are left with the case that $Q_i \parallel Q_j$, $Q_j \parallel Q_k$, and $Q_i \nparallel Q_k$. If $d_G(u,w) = d_G(Q_i,Q_k)$, then by (iii), we get that $d_G(u,w) > d$. Otherwise we may assume without loss of generality that there exists a vertex $u' \in Q_i$, $u' \neq u$, such that $d_G(Q_i,Q_k) = d_G(u',Q_k) < d_G(u,w)$. Since $Q_i \nparallel Q_k$, (ii) implies that $d_G(u',Q_k) \geq d$. But then $d_G(u,w) > d_G(Q_i,Q_k) \geq d$. □

Corollary 4.2 ([1, Theorem 3.1]). *Let G be a connected graph. Then $S \subseteq V(G)$ is a general position set if and only if the following conditions hold:*

(i) $\langle S\rangle$ *is a disjoint union of complete graphs* $Q_1,\dots,Q_\ell$.

(ii) $Q_i \parallel Q_j$ *for every* $i \neq j$.

(iii) $d_G(Q_i,Q_j) + d_G(Q_j,Q_k) \neq d_G(Q_i,Q_k)$ *for every* $\{i,j,k\} \in \binom{[\ell]}{3}$.

Proof. Set $d = \operatorname{diam}(G)$, so that general d-position sets are precisely general position sets. Condition (ii) of Theorem 4.1 implies that in cliques Q_i and Q_j, which are not parallel, we can find a pair of vertices at distance larger than $\operatorname{diam}(G)$. Since this is not possible, every two cliques must be parallel. Similarly, if the assumption of condition (iii) would be fulfilled for some cliques Q_i, Q_j, and Q_k, then we would again have vertices at distance larger than $\operatorname{diam}(G)$. Therefore, $d_G(Q_i,Q_j) + d_G(Q_j,Q_k) \neq d_G(Q_i,Q_k)$ must hold for every $\{i,j,k\} \in \binom{[\ell]}{3}$. □

5 Connections with other topics

In this section we connect general d-position sets with the dissociation number and with strong resolving graphs.

Strong resolving graphs

A vertex u of a connected graph G is *maximally distant* from a vertex v if every $w \in N(u)$ satisfies $d_G(v,w) \leq d_G(u,v)$. If u is maximally distant from v, and v is maximally distant from u, then u and v are *mutually maximally distant* (MMD for short). Given an integer $d \geq 2$, the *strong d-resolving graph* $G^d_{\rm SR}$ of G has vertex set $V(G)$, and two vertices u,v

are adjacent in G^d_{SR} if either u, v are MMD in G, or $d_G(u, v) \geq d$. The terminology used in this construction comes from the notion of the strong resolving graph introduced in [19] as a tool to study the strong metric dimension of graphs. See also [15].

The following observation will be useful in the proof of Theorem 5.1.

Observation 1. If G is connected and a vertex u of G is maximally distant from a vertex v of G, then $u \notin I(v, w)$ for every $w \in V(G) \setminus \{u\}$.

Proof. For the sake of contradiction suppose there exists such a vertex $w \in V(G) \setminus \{u\}$ such that $u \in I(v, w)$. Suppose that $v = v_0 \ldots v_{i-1} u = v_i v_{i+1} \ldots v_k = w$ is a v, w-geodesic. Since this is a geodesic, it follows that $d(v, u) = i$. But u is maximally distant from v, and thus $d(v, v_{i+1}) \leq d(v, u) = i$. Now, by following a shortest v, v_{i+1}-path with the path $v_{i+2} \ldots v_k = w$ we arrive at a v, w-path of length less than k, which is a contradiction. □

From Observation 1 it follows immediately that if three vertices x, y, z are pairwise MMD, then $x \notin I(y, z)$, $y \notin I(x, z)$, and $z \notin I(x, y)$. From this we infer that x, y, z lie in general d-position.

Theorem 5.1. *If G is a connected graph and $d \geq 2$ is an integer, then* $\mathrm{gp}_d(G) \geq \omega(G^d_{\mathrm{SR}})$.

Proof. We consider a set $S \subseteq V(G^d_{\mathrm{SR}})$ that induces a (largest) complete subgraph of G^d_{SR}. Then every two vertices $x, y \in S$ are MMD in G, or $d_G(x, y) \geq d$. We now consider three vertices x, y, z of S in the graph G. If they are pairwise MMD in G, then as above, x, y, z lie in general d-position. Suppose then that two of them, say x and y, are not MMD in G. Since x, y are adjacent in G^d_{SR}, it follows that $d_G(x, y) \geq d$. Suppose for instance that x, z are MMD in G. By Observation 1, it follows that $x \notin I(z, y)$ and $z \notin I(x, y)$. If $y \in I(x, z)$, then $d_G(x, z) = d_G(x, y) + d_G(y, z) \geq d + 1$, and hence x, y, z lie in general d-position. On the other hand, if $y \notin I(x, z)$, then by definition, x, y, z lie in general d-position.

It remains only to consider the case in which no pair of x, y, z is MMD in G. This means that the distance between any two of them is at least d, and this clearly means that x, y, z are in general d-position. □

Note that if $d = \mathrm{diam}(G)$, then G^d_{SR} is the standard strong resolving graph G_{SR} as defined in [19]. In this case Theorem 5.1 reduces to $\mathrm{gp}(G) \geq \omega(G_{\mathrm{SR}})$, a result earlier obtained in [12, Theorem 3.1].

Dissociation number and independence number

If G is a graph and $S \subseteq V(G)$, then S is a *dissociation set* if $\langle S \rangle$ has maximum degree at most 1. The *dissociation number* $\mathrm{diss}(G)$ of G is the cardinality of a largest dissociation set in G. This concept was introduced by Yanakkakis [23]; see also [3, 4, 10]. Further, a *k-path vertex cover* of G is a subset S of vertices of G such that every path of order k in G contains at least one vertex from S. The minimum cardinality of a k-path vertex cover in G is denoted by $\psi_k(G)$. The minimum 3-path vertex cover is a dual problem to the dissociation number because $\mathrm{diss}(G) = |V(G)| - \psi_3(G)$; see [10]. For the algorithmic state of the art on the 3-path vertex cover problem see [2].

Proposition 5.2. *If G is a triangle-free graph, $d \geq 2$, and $S \subseteq V(G)$ is a general d-position set, then S is a dissociation set. Moreover, if $d = 2$, then S is a general 2-position set if and only if S is a dissociation set.*

Proof. Let d be a positive integer such that $d \geq 2$. Suppose that S is a general d-position set in a triangle-free graph G. By Theorem 4.1 every component of the subgraph $\langle S \rangle$ of G induced by S is a complete graph. Since G is triangle-free, we conclude that each of these components has order 1 or 2. Therefore, S is a dissociation set. Now assume that $d = 2$ and S is a dissociation set in G. The components $C_1, \dots, C_k$ of $\langle S \rangle$ each have order 1 or 2 and are thus complete graphs. For every pair of distinct indices i, j in $[k]$, the fact that C_i and C_j are distinct components of the induced subgraph $\langle S \rangle$ implies that $d_G(C_i, C_j) \geq 2$. Therefore, conditions (ii) and (iii) of Theorem 4.1 follow immediately, and hence S is a general 2-position set. □

Proposition 5.2 immediately gives the following result for triangle-free graphs.

Corollary 5.3. *If G is a triangle-free graph and $d \geq 2$, then $\mathrm{gp}_d(G) \leq \mathrm{diss}(G)$. Moreover, $\mathrm{gp}_2(G) = \mathrm{diss}(G)$.*

We next relate the particular case of general 2-position number with the independence number of graphs.

Proposition 5.4. *If G is a connected graph without true twins, then $\mathrm{gp}_2(G) \geq \alpha(G)$.*

Proof. Let $x, y \in V(G)$. Suppose first that $xy \in E(G)$. Since x and y are not true twins, it follows that x and y are not MMD. By definition, we infer that $xy \notin E(G^2_{\mathrm{SR}})$. On the other hand, if $xy \notin E(G)$, then $d_G(x, y) \geq 2$ and by definition $xy \in E(G^2_{\mathrm{SR}})$. Consequently, G^2_{SR} is the complement $\overline{G}$ of G. Then by using Theorem 5.1, we have $\mathrm{gp}_2(G) \geq \omega(\overline{G}) = \alpha(G)$. □

It is straighforward to see that if $2 \leq m \leq n$, then $\mathrm{gp}_2(K_{m,n}) = n = \alpha(K_{m,n})$. Hence the bound of Proposition 5.4 is sharp. For another such family consider the grid graphs $P_{2r} \,\square\, P_{2s}$. (For the definition of the Cartesian product operation $\square$ see, for instance, [9].) As already mentioned, $\psi_3(G) = n - \mathrm{diss}(G)$ holds for any graph G of order n. Also, from [5] it is known that $\psi_3(P_{2r} \,\square\, P_{2s}) = 2rs$. Moreover, from Corollary 5.3, we have that $\mathrm{gp}_2(P_{2r} \,\square\, P_{2s}) = \mathrm{diss}(P_{2r} \,\square\, P_{2s})$. Thus,

$$\mathrm{gp}_2(P_{2r} \,\square\, P_{2s}) = \mathrm{diss}(P_{2r} \,\square\, P_{2s}) = 4rs - \psi_3(P_{2r} \,\square\, P_{2s}) = 2rs = \alpha(P_{2r} \,\square\, P_{2s}) .$$

6 Infinite graphs and some open problems

The general position problem has been partially studied also on infinite graphs. In [17] it was proved that $\mathrm{gp}(P^2_\infty) = 4$, where P^2_∞ is the 2-dimensional grid graph (alias the Cartesian product of two copies of the two way infinite path). The general position number of the 2-dimensional strong grid graph was also determined, and it was shown that $10 \leq \mathrm{gp}(P^3_\infty) \leq 16$. In [13] the latter lower bound was improved to 14. All these efforts were recently rounded off in [11] where it is proved that if $n \in \mathbb{N}$, then $\mathrm{gp}(P^n_\infty) = 2^{2^{n-1}}$. On the other hand, the following result reduces the study of the general d-position number of infinite graphs to the case $d = \infty$.

Proposition 6.1. *If G is an infinite graph and $d < \infty$, then $\mathrm{gp}_d(G) = \infty$.*

Proof. Let $d < \infty$ be a fixed positive integer. There is nothing to be proved if $d = 1$, hence assume that $d \geq 2$.

Suppose first that $\mathrm{diam}(G) = \infty$. In this case G contains an infinite isometric path $P = v_1v_2\ldots$. It is clear that $\{v_{di} : i \in \mathbb{N}\}$ is a general d-position set, and hence $\mathrm{gp}_d(G) = \infty$.

Suppose second that $\mathrm{diam}(G) < \infty$. Considering an arbitrary vertex of G and its distance levels we infer that G contains a vertex x with $\deg(x) = \infty$. Let $H = \langle N[x] \rangle$. Since H is an infinite graph, Erdős-Dushnik-Miller theorem [7] implies that H contains a (countably) infinite independent set I or an infinite clique Q (of the same cardinality as H). If H contains Q, then Q is also a clique of G, and hence G contains an infinite general d-position set. On the other hand, if H contains I, then I is also an independent set of G. Moreover, having in mind that $H = \langle N[x] \rangle$, we infer that each pair of vertices of I is at distance 2 in G. This fact in turn implies that I is an infinite general d-position set of G. We conclude that $\mathrm{gp}_d(G) = \infty$. □

6.1 Open questions

In this section we point out several questions that, in our opinion, are worthy of consideration.

- In [20, Lemma 5.1] there is a polynomial algorithm for the dissociation number of trees T and hence for $\mathrm{gp}_2(T)$. On the other hand, $\mathrm{gp}_{\mathrm{diam}(T)}(T)$ can also be efficiently computed. Hence, is it possible to compute in polynomial time $\mathrm{gp}_d(T)$ for any $2 < d < \mathrm{diam}(T)$? More generally, what can be done for the case of block graphs? We know that the simplicial vertices of a block graph form a gp-set. Can the algorithm of Papadimitriou and Yannakakis be modified for block graphs?

- Compare $\mathrm{diss}(G)$ with $\mathrm{gp}_2(G)$ for graphs G with $\omega(G) \geq 3$. Our guess is that these invariants are incomparable in such graphs. Is there some relationship when G is a block graph?

- What is $\mathrm{gp}_d(G)$ whenever G is a grid-like graph? Note that by applying Corollary 5.3 together with Theorem 4.1 in [5], one can find the value of $\mathrm{gp}_2(P_n \,\square\, P_m)$ for any n and m. Find $\mathrm{gp}_d(P_n \,\square\, P_m)$ for $d \geq 3$. Find the general d-position number of a partial grid graph for $d \geq 2$.

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On Hermitian varieties in $\mathrm{PG}(6, q^2)$

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Abstract

In this paper we characterize the non-singular Hermitian variety $\mathcal{H}(6, q^2)$ of $\mathrm{PG}(6, q^2)$, $q \neq 2$ among the irreducible hypersurfaces of degree $q + 1$ in $\mathrm{PG}(6, q^2)$ not containing solids by the number of its points and the existence of a solid S meeting it in $q^4 + q^2 + 1$ points.

Keywords: Unital, Hermitian variety, algebraic hypersurface.

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1 Introduction

The set of all absolute points of a non-degenerate unitary polarity in $\mathrm{PG}(r, q^2)$ determines the Hermitian variety $\mathcal{H}(r, q^2)$. This is a non-singular algebraic hypersurface of degree $q + 1$ in $\mathrm{PG}(r, q^2)$ with a number of remarkable properties, both from the geometrical and the combinatorial point of view; see [6, 16]. In particular, $\mathcal{H}(r, q^2)$ is a 2-character set with respect to the hyperplanes of $\mathrm{PG}(r, q^2)$ and 3-character blocking set with respect to the

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lines of $\mathrm{PG}(r, q^2)$ for $r > 2$. An interesting and widely investigated problem is to provide combinatorial descriptions of $\mathcal{H}(r, q^2)$.

First, we observe that a condition on the number of points and the intersection numbers with hyperplanes is not in general sufficient to characterize Hermitian varieties; see [1, 2]. On the other hand, it is enough to consider in addition the intersection numbers with codimension 2 subspaces in order to get a complete description; see [7].

In general, a hypersurface $\mathcal{H}$ of $\mathrm{PG}(r, q)$ is viewed as a hypersurface over the algebraic closure of $\mathrm{GF}(q)$ and a point of $\mathrm{PG}(r, q^i)$ in $\mathcal{H}$ is called a $\mathrm{GF}(q^i)$-point. A $\mathrm{GF}(q)$-point of $\mathcal{H}$ is also said to be a rational point of $\mathcal{H}$. Throughout this paper, the number of $\mathrm{GF}(q^i)$-points of $\mathcal{H}$ will be denoted by $N_{q^i}(\mathcal{H})$. For simplicity, we shall also use the convention $|\mathcal{H}| = N_q(\mathcal{H})$.

In the present paper, we shall investigate a combinatorial characterization of the Hermitian hypersurface $\mathcal{H}(6, q^2)$ in $\mathrm{PG}(6, q^2)$ among all hypersurfaces of the same degree having also the same number of $\mathrm{GF}(q^2)$-rational points.

More in detail, in [12, 13] it has been proved that if $\mathcal{X}$ is a hypersurface of degree $q+1$ in $\mathrm{PG}(r, q^2)$, $r \geq 3$ odd, with $|\mathcal{X}| = |\mathcal{H}(r, q^2)| = (q^{r+1} + (-1)^r)(q^r - (-1)^r)/(q^2 - 1)$ $\mathrm{GF}(q^2)$-rational points, not containing linear subspaces of dimension greater than $\frac{r-1}{2}$, then $\mathcal{X}$ is a non-singular Hermitian variety of $\mathrm{PG}(r, q^2)$. This result generalizes the characterization of [8] for the Hermitian curve of $\mathrm{PG}(2, q^2)$, $q \neq 2$.

The case where $r > 4$ is even is, in general, currently open. A starting point for a characterization in arbitrary even dimension can be found in [3] where the case of a hypersurface $\mathcal{X}$ of degree $q+1$ in $\mathrm{PG}(4, q^2)$, $q > 3$ is considered. There, it is shown that when $\mathcal{X}$ has the same number of rational points as $\mathcal{H}(4, q^2)$, does not contain any subspaces of dimension greater than 1 and meets at least one plane π in q^2+1 $\mathrm{GF}(q^2)$-rational points, then $\mathcal{X}$ is a Hermitian variety.

In this article we deal with hypersurfaces of degree $q+1$ in $\mathrm{PG}(6, q^2)$ and we prove that a characterization similar to that of [3] holds also in dimension 6. We conjecture that this can be extended to arbitrary even dimension.

Theorem 1.1. *Let S be a hypersurface of $\mathrm{PG}(6, q^2)$, $q > 2$, defined over $\mathrm{GF}(q^2)$, not containing solids. If the degree of S is $q+1$ and the number of its rational points is $q^{11} + q^9 + q^7 + q^4 + q^2 + 1$, then every solid of $\mathrm{PG}(6, q^2)$ meets S in at least $q^4 + q^2 + 1$ rational points. If there is at least a solid Σ_3 such that $|\Sigma_3 \cap S| = q^4 + q^2 + 1$, then S is a non-singular Hermitian variety of $\mathrm{PG}(6, q^2)$.*

Furthermore, we also extend the result of [3] to the case $q = 3$.

2 Preliminaries and notation

In this section we collect some useful information and results that will be crucial to our proof.

A Hermitian variety in $\mathrm{PG}(r, q^2)$ is the algebraic variety of $\mathrm{PG}(r, q^2)$ whose points $\langle v\rangle$ satisfy the equation $\eta(v, v) = 0$ where η is a sesquilinear form $\mathrm{GF}(q^2)^{r+1} \times \mathrm{GF}(q^2)^{r+1} \to \mathrm{GF}(q^2)$. The *radical* of the form η is the vector subspace of $\mathrm{GF}(q^2)^{r+1}$ given by

$$\mathrm{Rad}(\eta) := \{w \in \mathrm{GF}(q^2)^{r+1} : \forall v \in \mathrm{GF}(q^2)^{r+1}, \eta(v, w) = 0\}.$$

The form η is non-degenerate if $\mathrm{Rad}(\eta) = \{\mathbf{0}\}$. If the form η is non-degenerate, then the corresponding Hermitian variety is denoted by $\mathcal{H}(r, q^2)$ and it is a non-singular algebraic variety, of degree $q + 1$ containing

$$(q^{r+1} + (-1)^r)(q^r - (-1)^r)/(q^2 - 1)$$

$\mathrm{GF}(q^2)$-rational points. When η is degenerate we shall call *vertex* R_t of the degenerate Hermitian variety associated to η the projective subspace $R_t := \mathrm{PG}(\mathrm{Rad}(\eta)) := \{\langle w \rangle : w \in \mathrm{Rad}(\eta)\}$ of $\mathrm{PG}(r, q^2)$. A degenerate Hermitian variety can always be described as a cone of vertex R_t and basis a non-degenerate Hermitian variety $\mathcal{H}(r - t, q^2)$ disjoint from R_t where $t = \dim(\mathrm{Rad}(\eta))$ is the vector dimension of the radical of η. In this case we shall write the corresponding variety as $R_t\mathcal{H}(r - t, q^2)$. Indeed,

$$R_t\mathcal{H}(r - t, q^2) := \{X \in \langle P, Q \rangle : P \in R_t, Q \in \mathcal{H}(r - t, q^2)\}.$$

Any line of $\mathrm{PG}(r, q^2)$ meets a Hermitian variety (either degenerate or not) in either $1, q + 1$ or $q^2 + 1$ points (the latter value only for $r > 2$). The maximal dimension of projective subspaces contained in the non-degenerate Hermitian variety $\mathcal{H}(r, q^2)$ is $(r - 2)/2$, if r is even, or $(r - 1)/2$, if r is odd. These subspaces of maximal dimension are called *generators* of $\mathcal{H}(r, q^2)$ and the generators of $\mathcal{H}(r, q^2)$ through a point P of $\mathcal{H}(r, q^2)$ span a hyperplane $P^\perp$ of $\mathrm{PG}(r, q^2)$, the *tangent hyperplane* at P.

It is well known that this hyperplane meets $\mathcal{H}(r, q^2)$ in a degenerate Hermitian variety $P\mathcal{H}(r - 2, q^2)$, that is in a Hermitian cone having as vertex the point P and as base a non-singular Hermitian variety of $\Theta \cong \mathrm{PG}(r - 2, q^2)$ contained in $P^\perp$ with $P \notin \Theta$.

Every hyperplane of $\mathrm{PG}(r, q^2)$ that is not tangent meets $\mathcal{H}(r, q^2)$ in a non-singular Hermitian variety $\mathcal{H}(r-1, q^2)$, and is called a *secant hyperplane* of $\mathcal{H}(r, q^2)$. In particular, a tangent hyperplane contains

$$1 + q^2(q^{r-1} + (-1)^r)(q^{r-2} - (-1)^r)/(q^2 - 1)$$

$\mathrm{GF}(q^2)$-rational points of $\mathcal{H}(r, q^2)$, whereas a secant hyperplane contains

$$(q^r + (-1)^{r-1})(q^{r-1} - (-1)^{r-1})/(q^2 - 1)$$

$\mathrm{GF}(q^2)$-rational points of $\mathcal{H}(r, q^2)$.

We now recall several results which shall be used in the course of this paper.

Lemma 2.1 ([15]). *Let d be an integer with $1 \le d \le q + 1$ and let $\mathcal{C}$ be a curve of degree d in $\mathrm{PG}(2, q)$ defined over $\mathrm{GF}(q)$, which may have $\mathrm{GF}(q)$-linear components. Then the number of its rational points is at most $dq + 1$ and $N_q(\mathcal{C}) = dq + 1$ if and only if $\mathcal{C}$ is a pencil of d lines of $\mathrm{PG}(2, q)$.*

Lemma 2.2 ([10]). *Let d be an integer with $2 \le d \le q + 2$, and $\mathcal{C}$ a curve of degree d in $\mathrm{PG}(2, q)$ defined over $\mathrm{GF}(q)$ without any $\mathrm{GF}(q)$-linear components. Then $N_q(\mathcal{C}) \le (d - 1)q + 1$, except for a class of plane curves of degree 4 over $\mathrm{GF}(4)$ having 14 rational points.*

Lemma 2.3 ([11]). *Let $\mathcal{S}$ be a surface of degree d in $\mathrm{PG}(3, q)$ over $\mathrm{GF}(q)$. Then*

$$N_q(\mathcal{S}) \le dq^2 + q + 1$$

Lemma 2.4 ([8]). *Suppose $q \ne 2$. Let $\mathcal{C}$ be a plane curve over $\mathrm{GF}(q^2)$ of degree $q + 1$ without $\mathrm{GF}(q^2)$-linear components. If $\mathcal{C}$ has $q^3 + 1$ rational points, then $\mathcal{C}$ is a Hermitian curve.*

Lemma 2.5 ([7]). *A subset of points of* $\mathrm{PG}(r, q^2)$ *having the same intersection numbers with respect to hyperplanes and spaces of codimension* 2 *as non-singular Hermitian varieties is a non-singular Hermitian variety of* $\mathrm{PG}(r, q^2)$.

From [9, Theorem 23.5.1, Theorem 23.5.3] we have the following.

Lemma 2.6. *If* $\mathcal{W}$ *is a set of* $q^7 + q^4 + q^2 + 1$ *points of* $\mathrm{PG}(4, q^2)$, $q > 2$ *such that every line of* $\mathrm{PG}(4, q^2)$ *meets* $\mathcal{W}$ *in* $1, q + 1$ *or* $q^2 + 1$ *points, then* $\mathcal{W}$ *is a Hermitian cone with vertex a line and base a unital.*

Finally, we recall that a *blocking set with respect to lines* of $\mathrm{PG}(r, q)$ is a point set which blocks all the lines, i.e., intersects each line of $\mathrm{PG}(r, q)$ in at least one point.

3 Proof of Theorem 1.1

We first provide an estimate on the number of points of a curve of degree $q+1$ in $\mathrm{PG}(2, q^2)$, where q is any prime power.

Lemma 3.1. *Let* $\mathcal{C}$ *be a plane curve over* $\mathrm{GF}(q^2)$, *without* $\mathrm{GF}(q^2)$*-lines as components and of degree* $q + 1$. *If the number of* $\mathrm{GF}(q^2)$*-rational points of* $\mathcal{C}$ *is* $N < q^3 + 1$, *then*

$$N \leq \begin{cases} q^3 - (q^2 - 2) & \text{if } q > 3 \\ 24 & \text{if } q = 3 \\ 8 & \text{if } q = 2. \end{cases} \tag{3.1}$$

Proof. We distinguish the following three cases:

(a) $\mathcal{C}$ has two or more $\mathrm{GF}(q^2)$-components;

(b) $\mathcal{C}$ is irreducible over $\mathrm{GF}(q^2)$, but not absolutely irreducible;

(c) $\mathcal{C}$ is absolutely irreducible.

Suppose first $q \neq 2$.
Case (a) Suppose $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$. Let d_i be the degree of $\mathcal{C}_i$, for each $i = 1, 2$. Hence $d_1 + d_2 = q + 1$. By Lemma 2.2,

$$N \leq N_{q^2}(\mathcal{C}_1) + N_{q^2}(\mathcal{C}_2) \leq [(q + 1) - 2]q^2 + 2 = q^3 - (q^2 - 2)$$

Case (b) Let $\mathcal{C}'$ be an irreducible component of $\mathcal{C}$ over the algebraic closure of $\mathrm{GF}(q^2)$. Let $\mathrm{GF}(q^{2t})$ be the minimum defining field of $\mathcal{C}'$ and σ be the Frobenius morphism of $\mathrm{GF}(q^{2t})$ over $\mathrm{GF}(q^2)$. Then

$$\mathcal{C} = \mathcal{C}' \cup \mathcal{C}'^{\sigma} \cup \mathcal{C}'^{\sigma^2} \cup \ldots \cup \mathcal{C}'^{\sigma^{t-1}},$$

and the degree of $\mathcal{C}'$, say e, satisfies $q + 1 = te$ with $e > 1$. Hence any $\mathrm{GF}(q^2)$-rational point of $\mathcal{C}$ is contained in $\cap_{i=0}^{t-1} \mathcal{C}'^{\sigma^i}$. In particular, $N \leq e^2 \leq (\frac{q+1}{2})^2$ by Bezout's Theorem and $(\frac{q+1}{2})^2 < q^3 - (q^2 - 2)$.
Case (c) Let $\mathcal{C}$ be an absolutely irreducible curve over $\mathrm{GF}(q^2)$ of degree $q + 1$. Either $\mathcal{C}$ has a singular point or not.

In general, an absolutely irreducible plane curve $\mathcal{M}$ over $\mathrm{GF}(q^2)$ is q^2-Frobenius non-classical if for a general point $P(x_0, x_1, x_2)$ of $\mathcal{M}$ the point $P^{q^2} = P^{q^2}(x_0^{q^2}, x_1^{q^2}, x_2^{q^2})$ is on the tangent line to $\mathcal{M}$ at the point P. Otherwise, the curve $\mathcal{M}$ is said to be Frobenius classical. A lower bound of the number of $\mathrm{GF}(q^2)$-points for q^2-Frobenius non-classical curves is given by [4, Corollary 1.4]: for a q^2-Frobenius non-classical curve $\mathcal{C}'$ of degree d, we have $N_{q^2}(\mathcal{C}') \geq d(q^2 - d + 2)$. In particular, if $d = q + 1$, the lower bound is just $q^3 + 1$.

Going back to our original curve $\mathcal{C}$, we know that $\mathcal{C}$ is Frobenius classical because $N < q^3 + 1$. Let $F(x, y, z) = 0$ be an equation of $\mathcal{C}$ over $\mathrm{GF}(q^2)$. We consider the curve $\mathcal{D}$ defined by $\frac{\partial F}{\partial x} x^{q^2} + \frac{\partial F}{\partial y} y^{q^2} + \frac{\partial F}{\partial z} z^{q^2} = 0$. Then $\mathcal{C}$ is not a component of $\mathcal{D}$ because $\mathcal{C}$ is Frobenius classical. Furthermore, any $\mathrm{GF}(q^2)$-point P lies on $\mathcal{C} \cap \mathcal{D}$ and the intersection multiplicity of $\mathcal{C}$ and $\mathcal{D}$ at P is at least 2 by Euler's theorem for homogeneous polynomials. Hence by Bézout's theorem, $2N \leq (q + 1)(q^2 + q)$. Hence

$$N \leq \frac{1}{2} q(q + 1)^2.$$

This argument is due to Stöhr and Voloch [18, Theorem 1.1]. This Stöhr and Voloch's bound is lower than the estimate for N in case (a) for $q > 4$ and it is the same for $q = 4$. When $q = 3$ the bound in case (a) is smaller than the Stöhr and Voloch's bound.
Finally, we consider the case $q = 2$. Under this assumption, $\mathcal{C}$ is a cubic curve and neither case (a) nor case (b) might occur. For a degree 3 curve over $\mathrm{GF}(q^2)$ the Stöhr and Voloch's bound is loose, thus we need to change our argument. If $\mathcal{C}$ has a singular point, then $\mathcal{C}$ is a rational curve with a unique singular point. Since the degree of $\mathcal{C}$ is 3, singular points are either cusps or ordinary double points. Hence $N \in \{4, 5, 6\}$. If $\mathcal{C}$ is nonsingular, then it is an elliptic curve and, by the Hasse-Weil bound, see [19], $N \in I$ where $I = \{1, 2, \dots, 9\}$ and for each number N belonging to I there is an elliptic curve over $\mathrm{GF}(4)$ with N points, from [14, Theorem 4.2]. This completes the proof. □

Henceforth, we shall always suppose $q > 2$ and we denote by $\mathcal{S}$ an algebraic hypersurface of $\mathrm{PG}(6, q^2)$ satisfying the following hypotheses of Theorem 1.1:

(S1) $\mathcal{S}$ is an algebraic hypersurface of degree $q + 1$ defined over $\mathrm{GF}(q^2)$;

(S2) $|\mathcal{S}| = q^{11} + q^9 + q^7 + q^4 + q^2 + 1$;

(S3) $\mathcal{S}$ does not contain projective 3-spaces (solids);

(S4) there exists a solid Σ_3 such that $|\mathcal{S} \cap \Sigma_3| = q^4 + q^2 + 1$.

We first consider the behavior of $\mathcal{S}$ with respect to the lines.

Lemma 3.2. *An algebraic hypersurface $\mathcal{T}$ of degree $q + 1$ in $\mathrm{PG}(r, q^2)$, $q \neq 2$, with $|\mathcal{T}| = |\mathcal{H}(r, q^2)|$ is a blocking set with respect to lines of* $\mathrm{PG}(r, q^2)$

Proof. Suppose on the contrary that there is a line ℓ of $\mathrm{PG}(r, q^2)$ which is disjoint from $\mathcal{T}$. Let α be a plane containing ℓ. The algebraic plane curve $\mathcal{C} = \alpha \cap \mathcal{T}$ of degree $q + 1$ cannot have $\mathrm{GF}(q^2)$-linear components and hence it has at most $q^3 + 1$ points because of Lemma 2.2. If $\mathcal{C}$ had $q^3 + 1$ rational points, then from Lemma 2.4, $\mathcal{C}$ would be a Hermitian curve with an external line, a contradiction since Hermitian curves are blocking sets. Thus $N_{q^2}(\mathcal{C}) \leq q^3$. Since $q > 2$, by Lemma 3.1, $N_{q^2}(\mathcal{C}) < q^3 - 1$ and hence every plane through r meets $\mathcal{T}$ in at most $q^3 - 1$ rational points. Consequently, by considering all planes through r, we can bound the number of rational points of $\mathcal{T}$ by $N_{q^2}(\mathcal{T}) \leq (q^3 - 1)\frac{q^{2r-4} - 1}{q^2 - 1} =$

$q^{2r-3} + \cdots < |\mathcal{H}(r, q^2)|$, which is a contradiction. Therefore there are no external lines to $\mathcal{T}$ and so $\mathcal{T}$ is a blocking set w.r.t. lines of $\mathrm{PG}(r, q^2)$. □

Remark 3.3. The proof of [3, Lemma 3.1] would work perfectly well here under the assumption $q > 3$. The alternative argument of Lemma 3.2 is simpler and also holds for $q = 3$.

By the previous Lemma and assumptions (S1) and (S2), $\mathcal{S}$ is a blocking set for the lines of $\mathrm{PG}(6, q^2)$ In particular, the intersection of $\mathcal{S}$ with any 3-dimensional subspace Σ of $\mathrm{PG}(6, q^2)$ is also a blocking set with respect to lines of Σ and hence it contains at least $q^4 + q^2 + 1$ $\mathrm{GF}(q^2)$-rational points; see [5].

Lemma 3.4. *Let Σ_3 be a solid of $\mathrm{PG}(6, q^2)$ satisfying condition (S4), that is Σ_3 meets $\mathcal{S}$ in exactly $q^4 + q^2 + 1$ points. Then, $\Pi := \mathcal{S} \cap \Sigma_3$ is a plane.*

Proof. $\mathcal{S} \cap \Sigma_3$ must be a blocking set for the lines of $\mathrm{PG}(3, q^2)$; also it has size $q^4 + q^2 + 1$. It follows from [5] that $\Pi := \mathcal{S} \cap \Sigma_3$ is a plane. □

Lemma 3.5. *Let Σ_3 be a solid of satisfying condition (S4). Then, any 4-dimensional projective space Σ_4 through Σ_3 meets $\mathcal{S}$ in a Hermitian cone with vertex a line and basis a Hermitian curve.*

Proof. Consider all of the $q^6 + q^4 + q^2 + 1$ subspaces $\overline{\Sigma}_3$ of dimension 3 in $\mathrm{PG}(6, q^2)$ containing $\Pi = \mathcal{S} \cap \Sigma_3$.

From Lemma 2.3 and condition (S3) we have $|\overline{\Sigma}_3 \cap \mathcal{S}| \leq q^5 + q^4 + q^2 + 1$. Hence,

$$|\mathcal{S}| = (q^7 + 1)(q^4 + q^2 + 1) \leq (q^6 + q^4 + q^2)q^5 + q^4 + q^2 + 1 = |\mathcal{S}|.$$

Consequently, $|\overline{\Sigma}_3 \cap \mathcal{S}| = q^5 + q^4 + q^2 + 1$ for all $\overline{\Sigma}_3 \neq \Sigma_3$ such that $\Pi \subset \overline{\Sigma}_3$.

Let $C := \Sigma_4 \cap \mathcal{S}$. Counting the number of rational points of C by considering the intersections with the $q^2 + 1$ subspaces Σ_3' of dimension 3 in Σ_4 containing the plane Π we get

$$|C| = q^2 \cdot q^5 + q^4 + q^2 + 1 = q^7 + q^4 + q^2 + 1.$$

In particular, $C \cap \Sigma_3'$ is a maximal surface of degree $q + 1$; so it must split in $q + 1$ distinct planes through a line of Π; see [17]. So C consists of $q^3 + 1$ distinct planes belonging to distinct q^2 pencils, all containing Π ; denote by $\mathcal{L}$ the family of these planes. Also for each $\Sigma_3' \neq \Sigma_3$, there is a line ℓ' such that all the planes of $\mathcal{L}$ in Σ_3' pass through ℓ'. It is now straightforward to see that any line contained in C must necessarily belong to one of the planes of $\mathcal{L}$ and no plane not in $\mathcal{L}$ is contained in C.

In order to get the result it is now enough to show that a line of Σ_4 meets C in either 1, $q + 1$ or $q^2 + 1$ points. To this purpose, let ℓ be a line of Σ_4 and suppose $\ell \not\subseteq C$. Then, by Bezout's theorem,

$$1 \leq |\ell \cap C| \leq q + 1.$$

Assume $|\ell \cap C| > 1$. Then we can distinguish two cases:

1. $\ell \cap \Pi \neq \emptyset$. If ℓ and Π are incident, then we can consider the 3-dimensional subspace $\Sigma_3' := \langle \ell, \Pi \rangle$. Then ℓ must meet each plane of $\mathcal{L}$ in Σ_3' in different points (otherwise ℓ passes through the intersection of these planes and then $|\ell \cap C| = 1$). As there are $q + 1$ planes of $\mathcal{L}$ in Σ_3', we have $|\ell \cap C| = q + 1$.

2. $\ell \cap \Pi = \emptyset$. Consider the plane Λ generated by a point $P \in \Pi$ and ℓ. Clearly $\Lambda \notin \mathcal{L}$. The curve $\Lambda \cap S$ has degree $q+1$ by construction, does not contain lines (for otherwise $\Lambda \in \mathcal{L}$) and has q^3+1 $\mathrm{GF}(q^2)$-rational points (by a counting argument). So from Lemma 2.4 it is a Hermitian curve . It follows that ℓ is a $q+1$ secant.

We can now apply Lemma 2.6 to see that C is a Hermitian cone with vertex a line. □

Lemma 3.6. *Let Σ_3 be a space satisfying condition (S4) and take Σ_5 to be a 5-dimensional projective space with $\Sigma_3 \subseteq \Sigma_5$. Then $\mathcal{S} \cap \Sigma_5$ is a Hermitian cone with vertex a point and basis a Hermitian hypersurface $\mathcal{H}(4, q^2)$.*

Proof. Let

$$\Sigma_4 := \Sigma_4^1, \Sigma_4^2, \ldots, \Sigma_4^{q^2+1}$$

be the 4-spaces through Σ_3 contained in Σ_5. Put $C_i := \Sigma_4^i \cap \mathcal{S}$, for all $i \in \{1, \ldots, q^2+1\}$ and $\Pi = \Sigma_3 \cap C_1$. From Lemma 3.5 C_i is a Hermitian cone with vertex a line, say ℓ_i. Furthermore $\Pi \subseteq \Sigma_3 \subseteq \Sigma_4^i$ where Π is a plane. Choose a plane $\Pi' \subseteq \Sigma_4^1$ such that $m := \Pi' \cap C_1$ is a line m incident with Π but not contained in it. Let $P_1 := m \cap \Pi$. It is straightforward to see that in Σ_4^1 there are exactly 1 plane through m which is a (q^4+q^2+1)-secant, q^4 planes which are (q^3+q^2+1)-secant and q^2 planes which are (q^2+1)-secant. Also P_1 belongs to the line ℓ_1. There are now two cases to consider:

(a) There is a plane $\Pi'' \neq \Pi'$ not contained in Σ_4^i for all $i = 1, \ldots, q^2+1$ with $m \subseteq \Pi'' \subseteq S \cap \Sigma_5$.

We first show that the vertices of the cones C_i are all concurrent. Consider $m_i := \Pi'' \cap \Sigma_4^i$. Then $\{m_i : i = 1, \ldots, q^2+1\}$ consists of q^2+1 lines (including m) all through P_1. Observe that for all i, the line m_i meets the vertex ℓ_i of the cone C_i in $P_i \in \Pi$. This forces $P_1 = P_2 = \cdots = P_{q^2+1}$. So $P_1 \in \ell_1, \ldots, \ell_{q^2+1}$.

Now let $\overline{\Sigma}_4$ be a 4-dimensional space in Σ_5 with $P_1 \notin \overline{\Sigma}_4$; in particular $\Pi \nsubseteq \overline{\Sigma}_4$. Put also $\overline{\Sigma}_3 := \Sigma_4^1 \cap \overline{\Sigma}_4$. Clearly, $r := \overline{\Sigma}_3 \cap \Pi$ is a line and $P_1 \notin r$. So $\overline{\Sigma}_3 \cap \mathcal{S}$ cannot be the union of $q+1$ planes, since if this were to be the case, these planes would have to pass through the vertex ℓ_1. It follows that $\overline{\Sigma}_3 \cap \mathcal{S}$ must be a Hermitian cone with vertex a point and basis a Hermitian curve. Let $\mathcal{W} := \overline{\Sigma}_4 \cap \mathcal{S}$. The intersection $\mathcal{W} \cap \Sigma_4^i$, as i varies, is a Hermitian cone with basis a Hermitian curve, so, the points of $\mathcal{W}$ are

$$|\mathcal{W}| = (q^2+1)q^5 + q^2 + 1 = (q^2+1)(q^5+1);$$

in particular, $\mathcal{W}$ is a hypersurface of $\overline{\Sigma}_4$ of degree $q+1$ such that there exists a plane of $\overline{\Sigma}_4$ meeting $\mathcal{W}$ in just one line (such planes exist in $\overline{\Sigma}_3$). Also suppose $\mathcal{W}$ to contain planes and let $\Pi''' \subseteq \mathcal{W}$ be such a plane. Since $\Sigma_4^i \cap \mathcal{W}$ does not contain planes, all Σ_4^i meet Π''' in a line t_i. Also Π''' must be contained in $\bigcup_{i=1}^{q^2+1} t_i$. This implies that the set $\{t_i\}_{i=1,\ldots,q^2+1}$ consists of q^2+1 lines through a point $P \in \Pi \setminus \{P_1\}$.

Furthermore each line t_i passing through P must meet the radical line ℓ_i of the Hermitian cone $\mathcal{S} \cap \Sigma_4^i$ and this forces P to coincide with P_1, a contradiction. It follows that $\mathcal{W}$ does not contain planes.

So by the characterization of $\mathcal{H}(4, q^2)$ of [3] we have that $\mathcal{W}$ is a Hermitian variety $\mathcal{H}(4, q^2)$.

We also have that $|\mathcal{S}\cap\Sigma_5| = |P_1\mathcal{H}(4,q^2)|$. Let now r be any line of $\mathcal{H}(4,q^2) = \mathcal{S}\cap\overline{\Sigma}_4$ and let Θ be the plane $\langle r, P_1\rangle$. The plane Θ meets Σ_4^i in a line $q_i \subseteq \mathcal{S}$ for each $i = 1,\dots,q^2+1$ and these lines are concurrent in P_1. It follows that all the points of Θ are in S. This completes the proof for the current case and shows that $\mathcal{S}\cap\Sigma_5$ is a Hermitian cone $P_1\mathcal{H}(4,q^2)$.

(b) All planes Π'' with $m \subseteq \Pi'' \subseteq \mathcal{S}\cap\Sigma_5$ are contained in Σ_4^i for some $i = 1,\dots,q^2+1$. We claim that this case cannot happen. We can suppose without loss of generality $m\cap\ell_1 = P_1$ and $P_1 \notin \ell_i$ for all $i = 2,\dots,q^2+1$. Since the intersection of the subspaces Σ_4^i is Σ_3, there is exactly one plane through m in Σ_5 which is (q^4+q^2+1)-secant, namely the plane $\langle\ell_1, m\rangle$. Furthermore, in Σ_4^1 there are q^4 planes through m which are (q^3+q^2+1)-secant and q^2 planes which are (q^2+1)-secant. We can provide an upper bound to the points of $\mathcal{S}\cap\Sigma_5$ by counting the number of points of $\mathcal{S}\cap\Sigma_5$ on planes in Σ_5 through m and observing that a plane through m not in Σ_5 and not contained in $\mathcal{S}$ has at most q^3+q^2+1 points in common with $\mathcal{S}\cap\Sigma_5$. So

$$|\mathcal{S}\cap\Sigma_5| \le q^6\cdot q^3 + q^7 + q^4 + q^2 + 1.$$

As $|\mathcal{S}\cap\Sigma_5| = q^9+q^7+q^4+q^2+1$, all planes through m which are neither (q^4+q^2+1)-secant nor (q^2+1)-secant are (q^3+q^2+1)-secant. That is to say that all of these planes meet $\mathcal{S}$ in a curve of degree $q+1$ which must split into $q+1$ lines through a point because of Lemma 2.1.

Take now $P_2 \in \Sigma_4^2\cap\mathcal{S}$ and consider the plane $\Xi := \langle m, P_2\rangle$. The line $\langle P_1, P_2\rangle$ is contained in Σ_4^2; so it must be a $(q+1)$-secant, as it does not meet the vertex line ℓ_2 of C_2 in Σ_4^2. Now, Ξ meets every of Σ_4^i for $i = 2,\dots,q^2+1$ in a line through P_1 which is either a 1-secant or a $q+1$-secant; so

$$|\mathcal{S}\cap\Xi| \le q^2(q) + q^2 + 1 = q^3 + q^2 + 1.$$

It follows that $|\mathcal{S}\cap\Xi| = q^3+q^2+1$ and $\mathcal{S}\cap\Xi$ is a set of $q+1$ lines all through the point P_1. This contradicts our previous construction.

□

Lemma 3.7. *Every hyperplane of* $\mathrm{PG}(6,q^2)$ *meets* $\mathcal{S}$ *either in a non-singular Hermitian variety* $\mathcal{H}(5,q^2)$ *or in a cone with vertex a point over a Hermitian hypersurface* $\mathcal{H}(4,q^2)$.

Proof. Let Σ_3 be a solid satisfying condition (S4). Denote by Λ a hyperplane of $\mathrm{PG}(6,q^2)$. If Λ contains Σ_3 then, from Lemma 3.6 it follows that $\Lambda\cap\mathcal{S}$ is a Hermitian cone $P\mathcal{H}(4,q^2)$.

Now assume that Λ does not contain Σ_3. Denote by S_5^j, with $j = 1,\dots,q^2+1$ the q^2+1 hyperplanes through Σ_4^1, where as before, Σ_4^1 is a 4-space containing Σ_3. By Lemma 3.6 again we get that $S_5^j\cap\mathcal{S} = P^j\mathcal{H}(4,q^2)$. We count the number of rational points of $\Lambda\cap\mathcal{S}$ by studying the intersections of $S_5^j\cap\mathcal{S}$ with Λ for all $j\in\{1,\dots,q^2+1\}$. Setting $\mathcal{W}_j := S_5^j\cap\mathcal{S}\cap\Lambda$, $\Omega := \Sigma_4^1\cap\mathcal{S}\cap\Lambda$ then

$$|\mathcal{S}\cap\Lambda| = \sum_j |\mathcal{W}_j\setminus\Omega| + |\Omega|.$$

If Π is a plane of Λ then Ω consists of $q+1$ planes of a pencil. Otherwise let m be the line in which Λ meets the plane Π. Then Ω is either a Hermitian cone $P_0\mathcal{H}(2,q^2)$, or $q+1$

planes of a pencil, according as the vertex $P^j \in \Pi$ is an external point with respect to m or not.

In the former case $\mathcal{W}_j$ is a non singular Hermitian variety $\mathcal{H}(4, q^2)$ and thus $|\mathcal{S} \cap \Lambda| = (q^2+1)(q^7) + q^5 + q^2 + 1 = q^9 + q^7 + q^5 + q^2 + 1$.

In the case in which Ω consists of $q+1$ planes of a pencil then $\mathcal{W}_j$ is either a $P_0\mathcal{H}(3, q^2)$ or a Hermitian cone with vertex a line ℓ and basis a Hermitian curve $\mathcal{H}(2, q^2)$.

If there is at least one index j such that $\mathcal{W}_j = \ell\mathcal{H}(2, q^2)$, then there must be a 3-dimensional space Σ'_3 of $S_5^j \cap \Lambda$ meeting $\mathcal{S}$ in a generator. Hence, from Lemma 3.6 we get that $\mathcal{S} \cap \Lambda$ is a Hermitian cone $P'\mathcal{H}(4, q^2)$.

Assume that for all $j \in \{1, \dots, q^2+1\}$, $\mathcal{W}_j$ is a $P_0\mathcal{H}(3, q^2)$. In this case

$$|\mathcal{S} \cap \Lambda| = (q^2+1)q^7 + (q+1)q^4 + q^2 + 1 = q^9 + q^7 + q^5 + q^4 + q^2 + 1 = |\mathcal{H}(5, q^2)|.$$

We are going to prove that the intersection numbers of $\mathcal{S}$ with hyperplanes are only two that is $q^9 + q^7 + q^5 + q^4 + q^2 + 1$ or $q^9 + q^7 + q^4 + q^2 + 1$.

Denote by x_i the number of hyperplanes meeting $\mathcal{S}$ in i rational points with $i \in \{q^9 + q^7 + q^4 + q^2 + 1, q^9 + q^7 + q^5 + q^2 + 1, q^9 + q^7 + q^5 + q^4 + q^2 + 1\}$. Double counting arguments give the following equations for the integers x_i:

$$\begin{cases} \sum_i x_i = q^{12} + q^{10} + q^8 + q^6 + q^4 + q^2 + 1 \\ \sum_i i x_i = |\mathcal{S}|(q^{10} + q^8 + q^6 + q^4 + q^2 + 1) \\ \sum_{i=1} i(i-1) x_i = |\mathcal{S}|(|\mathcal{S}| - 1)(q^8 + q^6 + q^4 + q^2 + 1). \end{cases} \tag{3.2}$$

Solving (3.2) we obtain $x_{q^9+q^7+q^5+q^2+1} = 0$. In the case in which $|\mathcal{S} \cap \Lambda| = |\mathcal{H}(5, q^2)|$, since $\mathcal{S} \cap \Lambda$ is an algebraic hypersurface of degree $q+1$ not containing 3-spaces, from [19, Theorem 4.1] we get that $\mathcal{S} \cap \Lambda$ is a Hermitian variety $\mathcal{H}(5, q^2)$ and this completes the proof. □

Proof of Theorem 1.1. The first part of Theorem 1.1 follows from Lemma 3.4. From Lemma 3.7, $\mathcal{S}$ has the same intersection numbers with respect to hyperplanes and 4-spaces as a non-singular Hermitian variety of $\mathrm{PG}(6, q^2)$, hence Lemma 2.5 applies and $\mathcal{S}$ turns out to be a $\mathcal{H}(6, q^2)$. □

Remark 3.8. The characterization of the non-singular Hermitian variety $\mathcal{H}(4, q^2)$ given in [3] is based on the property that a given hypersurface is a blocking set with respect to lines of $\mathrm{PG}(4, q^2)$, see [3, Lemma 3.1]. This lemma holds when $q > 3$. Since Lemma 3.2 extends the same property to the case $q = 3$ it follows that the result stated in [3] is also valid in $\mathrm{PG}(4, 3^2)$.

4 Conjecture

We propose a conjecture for the general $2n$-dimensional case.

Let $\mathcal{S}$ be a hypersurface of $\mathrm{PG}(2d, q^2)$, $q > 2$, defined over $\mathrm{GF}(q^2)$, not containing d-dimensional projective subspaces. If the degree of $\mathcal{S}$ is $q+1$ and the number of its rational points is $|\mathcal{H}(2d, q^2)|$, then every d-dimensional subspace of $\mathrm{PG}(2d, q^2)$ meets $\mathcal{S}$ in at least $\theta_{q^2}(d-1) := (q^{2d-2} - 1)/(q^2 - 1)$ rational points. If there is at least a d-dimensional *subspace* Σ_d *such that* $|\Sigma_d \cap \mathcal{S}| = |\mathrm{PG}(d-1, q^2)|$*, then* $\mathcal{S}$ *is a non-singular Hermitian variety of* $\mathrm{PG}(2d, q^2)$.

Lemma 3.1 and Lemma 3.2 can be a starting point for the proof of this conjecture since from them we get that $\mathcal{S}$ is a blocking set with respect to lines of $\mathrm{PG}(2d, q^2)$.

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Achromatic numbers of Kneser graphs*

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Abstract

Complete vertex colorings have the property that any two color classes have at least an edge between them. Parameters such as the Grundy, achromatic and pseudoachromatic numbers come from complete colorings, with some additional requirement. In this paper, we estimate these numbers in the Kneser graph $K(n,k)$ for some values of n and k. We give the exact value of the achromatic number of $K(n,2)$.

Keywords: Achromatic number, pseudoachromatic number, Grundy number, block designs, geometric type Kneser graphs.

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1 Introduction

Since the beginning of the study of colorings in graph theory, many interesting results have appeared in the literature, for instance, the chromatic number of Kneser graphs. Such graphs give an interesting relation between finite sets and graphs.

Let V be the set of all k-subsets of $[n] := \{1, 2, \ldots, n\}$, where $1 \leq k \leq n/2$. The *Kneser graph* $K(n, k)$ is the graph with vertex set V such that two vertices are adjacent if and only if the corresponding subsets are disjoint. Lovász [18] proved that $\chi(K(n, k)) = n - 2(k - 1)$ via the Borsuk-Ulam theorem, see Chapter 38 of [2].

Some results on the Kneser graphs and parameters of colorings have appeared since then, for instance [4, 8, 10, 13, 16, 19].

An l-*coloring* of a graph G is a surjective function ς that assigns a number from the set $[l]$ to each vertex of G. An l-coloring of G is *proper* if any two adjacent vertices have different colors. An l-coloring ς is *complete* if for each pair of different colors $i, j \in [l]$ there exists an edge $xy \in E(G)$ such that $\varsigma(x) = i$ and $\varsigma(y) = j$.

The largest value of l for which G has a complete l-coloring is called the *pseudoachromatic number* of G [12], denoted $\psi(G)$. A similar invariant, which additionally requires an l-coloring to be proper, is called the *achromatic number* of G and denoted by $\alpha(G)$ [15]. Note that $\alpha(G)$ is at least $\chi(G)$ since the *chromatic number* $\chi(G)$ *of* G is the smallest number l for which there exists a proper l-coloring of G and then such an l-coloring is also complete. Therefore, for any graph G, $\chi(G) \leq \alpha(G) \leq \psi(G)$.

In this paper, we estimate these parameters arising from complete colorings of Kneser graphs. The paper is organized as follows. In Section 2 we recall notions of block designs.

Section 3 is devoted to the achromatic number $\alpha(K(n, 2))$ of the Kneser graph $K(n, 2)$. It is proved that $\alpha(K(n, 2)) = \left\lfloor \binom{n+1}{2}/3 \right\rfloor$ for $n \neq 3$.

In Section 4 it is shown that $\psi(K(n, 2))$ satisfies

$$\left\lfloor \binom{n}{2}/2 \right\rfloor \leq \psi(K(n, 2)) \leq \left\lfloor (\binom{n}{2} + \left\lfloor \frac{n}{2} \right\rfloor)/2 \right\rfloor$$

for $n \geq 7$ and that the upper bound is tight.

The Section 5 establishes that the Grundy number $\Gamma(K(n, 2))$ equals $\alpha(K(n, 2))$. The *Grundy number* $\Gamma(G)$ of a graph G is determined by the worst-case result of a greedy proper coloring applied on G. A *greedy* l-coloring technique operates as follows. The vertices (listed in some particular order) are colored according to the algorithm that assigns to a vertex under consideration the smallest available color. Therefore, greedy proper colorings are also complete.

Section 6 gives a natural upper bound for the pseudoachromatic number of $K(n, k)$ and a lower bound for the achromatic number of $K(n, k)$ in terms of the b-chromatic number of $K(n, k)$, another parameter arising from complete colorings.

Section 7 is about the achromatic numbers of some geometric type Kneser graphs. A *complete geometric graph* of n points is an embedding of the complete graph K_n in the Euclidean plane such that its vertex set is a set V of points in general position, and its edges are straight-line segments connecting pairs of points in V. We study the achromatic numbers of graphs $D_V(n)$ whose vertex set is the set of edges of a complete geometric graph of n points and adjacency is defined in terms of geometric disjointness.

To end, in Section 8, we discuss the case of the odd graphs $K(2k + 1, k)$.

2 Preliminaries

All graphs in this paper are finite and simple. Note that the complement of the line graph of the complete graph on n vertices is the Kneser graph $K(n, 2)$. We use this model of the Kneser graph $K(n, 2)$ in Sections 3, 4, 5 and 7.

Let n, b, k, r and λ be positive integers with $n > 1$. Let $D = (P, B, I)$ be a triple consisting of a set P of n distinct objects, called points of D, a set B of b distinct objects, called blocks of D (with $P \cap B = \emptyset$), and an incidence relation I, a subset of $P \times B$. We say that v is incident to u if exactly one of the ordered pairs (u, v) and (v, u) is in I; then v is incident to u if and only if u is incident to v. D is called a 2-(n, b, k, r, λ) *block design* (for short, 2-(n, b, k, r, λ) *design*) if it satisfies the following axioms.

1. Each block of D is incident to exactly k distinct points of D.
2. Each point of D is incident to exactly r distinct blocks of D.
3. If u and v are distinct points of D, then there are exactly λ blocks of D incident to both u and v.

A 2-(n, b, k, r, λ) design is called a balanced incomplete block design BIBD; it is called an (n, k, λ)-design, too, since the parameters of a 2-(n, b, k, r, λ) design are not all independent. The two basic equations connecting them are $nr = bk$ and $r(k - 1) = \lambda(n - 1)$. For a detailed introduction to block designs we refer to [5, 6].

A design is *resolvable* if its blocks can be partitioned into r sets so that b/r blocks of each part are point-disjoint and each part is called a *parallel class*.

A *Steiner triple system* $STS(n)$ is an $(n, 3, 1)$-design. It is well-known that an $STS(n)$ exists if and only if $n \equiv 1, 3 \bmod 6$. A resolvable $STS(n)$ is called a *Kirkman triple system* and denoted by $KTS(n)$ and exists if and only if $n \equiv 3 \bmod 6$, see [21].

An $(n, 5, 1)$-design exists if and only if $n \equiv 1, 5 \bmod 20$, see [6].

An $(n, k, 1)$-design can naturally be regarded as an edge partition into K_k subgraphs, of the complete graph K_n.

Finally, we recall that the concepts of a 1-factor and a 1-factorization represent, for the case of K_n, a parallel class and a resolubility of an $(n, 2, 1)$-design, respectively.

3 The exact value of $\alpha(K(n, 2))$

In this section, we prove that $\alpha(K(n, 2)) = \left\lfloor \binom{n+1}{2}/3 \right\rfloor$ for every $n \neq 3$. The proof is about the upper bound and the lower bound have the same value.

Theorem 3.1. *The achromatic number $\alpha(K(n, 2))$ of $K(n, 2)$ equals $\left\lfloor \binom{n+1}{2}/3 \right\rfloor$ for $n \neq 3$ and $\alpha(K(3, 2)) = 1$.*

Proof. First, we prove the upper bound $\alpha(K(n, 2)) \leq \left\lfloor \binom{n+1}{2}/3 \right\rfloor$.

Let ς be a proper and complete coloring of $K(n, 2)$. Consider the graph $K(n, 2)$ as the complement of $L(K_n)$. Note that vertices corresponding to a color class of ς of size two induce a P_3 subgraph, say abc, of the complete graph K_n with $V(K_n) = [n]$; then no color class of ς of size one is a pair containing b. Therefore, if ς has x color classes of size one (they form a matching in K_n of size x) and y color classes of size two, then $y \leq n - 2x$,

$$\alpha(K(n, 2)) \leq \frac{\binom{n}{2} - x - 2(n - 2x)}{3} + x + (n - 2x) = \frac{\binom{n}{2} + 2x + (n - 2x)}{3} = \frac{\binom{n}{2} + n}{3}$$

and we get $\alpha(K(n,2)) \leq \lfloor \binom{n+1}{2}/3 \rfloor$. For the case of $n = 3$, $K(3,2)$ is an edgeless graph, hence $\alpha(K(3,2)) = 1$.

Next, we exhibit a proper and complete edge coloring of the complement of $L(K_n)$ that uses $\lfloor \binom{n+1}{2}/3 \rfloor$ colors. We remark that in order to obtain such a tight coloring it suffices to achieve that all color classes are of size at most three, while the number of *exceptional* vertices of K_n (that are involved neither in a color class of size one nor in the role of the "center" of a color class of size two) is at most one. We shall refer to this condition as the condition (C).

Figure 1 documents the equality for $n \leq 5$. For the remainder of this proof, we need to

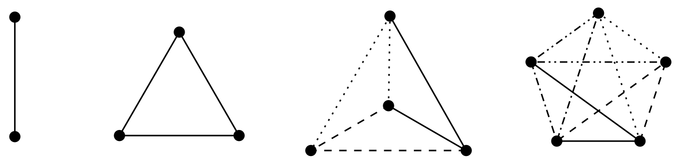

Figure 1: $\alpha(K(n,2)) = \lfloor \binom{n+1}{2}/3 \rfloor$ for $n = 2,4,5$ and $\alpha(K(3,2)) = 1$.

distinguish four cases, namely, when $n = 6k, 6k+2$; $n = 6k+3, 6k+5$; $n = 6k+4$ and $n = 6k+1$ for $k \geq 1$.

1. Case $n = 6k$ or $n = 6k+2$. Since $n+1 \equiv 1,3 \bmod 6$ there exists an $STS(n+1)$. We can think of $K(n,2)$ as having the vertex set equal to the set of points of $STS(n+1)$ other than v. Then each vertex of $K(n,2)$ is a subset of exactly one block of $STS(n+1) - v$; the blocks of $STS(n+1) - v$ are (3-element) blocks of $STS(n+1)$ not containing v, and (2-element) blocks $B \setminus \{v\}$, where B is a block of $STS(n+1)$ with $v \in B$. Consider a vertex coloring of $K(n,2)$ that is defined in the following way: Color classes of size three are triangles of $STS(n+1) - v$ (we use this simplified expression to indicate that all vertices of $K(n,2)$, that are subsets of a fixed triangle of $STS(n+1)-v$, receive the same color). All remaining color classes are of size one; they are formed by 2-element blocks of $STS(n+1) - v$. (They can also be regarded as edges of a perfect matching of the "underlying" complete graph on points of $STS(n+1) - v$.) The coloring is obviously proper. It is complete, too (see Figure 2), and satisfies the condition (C), hence $\alpha(K(n,2)) = \lfloor \binom{n+1}{2}/3 \rfloor$.

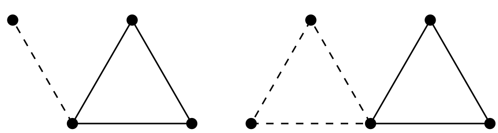

Figure 2: Every two classes have two disjoint edges in the complement of $L(K_n)$.

2. Case $n = 6k+3$ or $n = 6k+5$. Add two points u, v to the points of $STS(n-2)$, and color $K(n,2)$ as follows. Color classes of size three are triangles of $STS(n-2)$ except for one with points a, b, c. The remaining color classes are of size two. Five of them correspond to the optimum coloring of $K(5,2)$ depicted in Figure 1 (with

points a, b, c, u, v). Finally, every point x of $STS(n-2)$, $x \notin \{a, b, c\}$, gives rise to the color class $\{ux, xv\}$, see Figure 3 (Left). The coloring is proper and complete, see Figure 3 (Right), and it satisfies the condition (C).

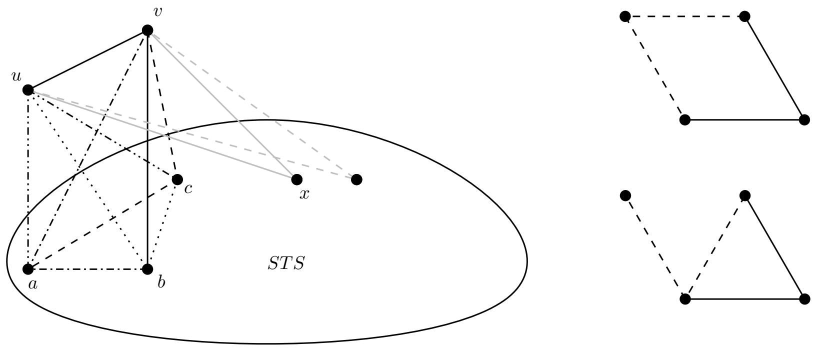

Figure 3: (Left) Color classes of size two in the proof of Case 2. (Right) Every two distinct color classes of size two contain disjoint edges in the complement of $L(K_n)$.

3. Case $n = 6k + 4$. Add a point v to the points of a resolvable $STS(n-1)$, for instance a $KTS(n-1)$. Color classes of size three are triangles of $STS(n-1)$ except for the triangles of a parallel class $P = \{T_i : i = 0, 1, \ldots, \frac{n-4}{3}\}$, where $T_i = \{v_{3i+1}, v_{3i+2}, v_{3i+3}\}$. For each triangle T_i of P color vertices of $K(4, 2)$ with the vertex set $T_i \cup \{v\}$ according to the optimum coloring of Figure 1, see Figure 4. The resulting coloring is proper and complete, and it fulfills the condition (C). Indeed, the number of color classes of size two is $n - 1$; since "centers" of those color classes are pairwise disjoint, v is the only exceptional vertex.

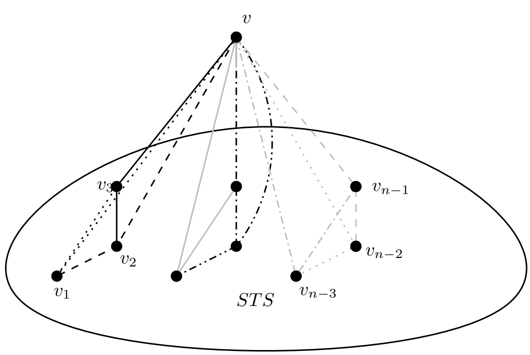

Figure 4: Color classes of size two in the proof of Case 3.

4. Case $n = 6k + 1$. First, we analyze the case of $k = 1$. Delete two points of $STS(9)$ presented in Figure 5 (Left) to finish with points $v_1, v_2, \ldots, v_7$. The "survived" triangles are color classes of size three, see Figure 5 (Center). The remaining six pairs of points are divided into four color classes $\{v_1v_2, v_2v_3\}$, $\{v_3v_4\}$,

$\{v_4v_5, v_5v_6\}$ and $\{v_6v_1\}$, see Figure 5 (Right). The obtained coloring shows that $\alpha(K(7,2)) = 9 = \lfloor \binom{7+1}{2}/3 \rfloor$.

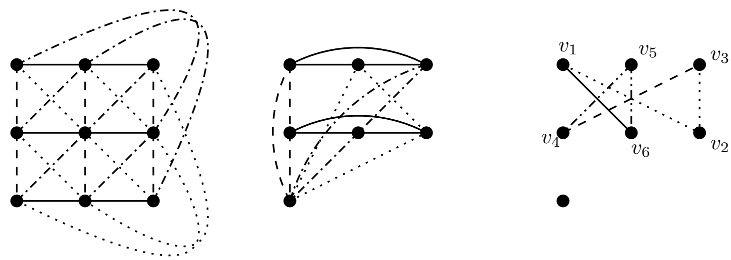

Figure 5: (Left) $STS(9)$. (Center) The 5 color classes of size three of K_7. (Right) Color classes of size one and two in $K(7,2)$.

If $k \geq 2$, consider an $STS(n-4)$ with points $v_1, v_2, \ldots, v_{n-4}$ that has a parallel class $P = \{T_i \colon i = 0, 1, \ldots, \frac{n-7}{3}\}$, where $T_i = \{v_{3i+1}, v_{3i+2}, v_{3i+3}\}$. Add to points of $STS(n-4)$ the points a, b, c, d. Every triangle of $STS(n-4)$ except for the triangles of P is a color class of size three. Let H_i denote the join of T_i with the complement of K_4 on vertices a, b, c, d; the join of two vertex disjoint graphs G and H has the vertex set $V(G) \cup V(H)$ and the edge set $E(G) \cup E(H) \cup \{xy \colon x \in V(G), y \in V(H)\}$. Pairs of points corresponding to edges of H_i, $i = 0, 1, \ldots, \frac{n-10}{3}$, form color classes of size three determined by point triples $\{v_{3i+1}, v_{3i+2}, a\}$, $\{v_{3i+2}, v_{3i+3}, b\}$ and $\{v_{3i+1}, v_{3i+3}, c\}$, and color classes of size two $\{av_{3i+3}, v_{3i+3}d\}$, $\{bv_{3i+1}, v_{3i+1}d\}$ and $\{cv_{3i+2}, v_{3i+2}d\}$, see Figure 6. Finally, pairs of points from the set $S = \{v_{n-6}, v_{n-5}, v_{n-4}, a, b, c, d\}$ are colored so that nine color classes are created just as in the coloring of $K(7,2)$ described above for the case $k = 1$. The coloring is proper and complete, and the condition (C) is fulfilled, since the number of exceptional vertices in the "underlying" K_n is one (exceptional is the vertex of S that is involved only in color classes of size three); so, $\alpha(K(n,2)) = \lfloor \binom{n+1}{2}/3 \rfloor$ in this case, too.

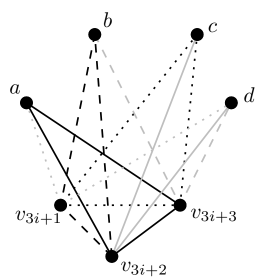

Figure 6: The 6-coloring of H_i.

By the four cases, the theorem follows. □

4 About the value of $\psi(K(n,2))$

In this section, we determine bounds for $\psi(K(n,2))$. The gap between the bounds is $\Theta(n)$, however, the upper bound is tight for an infinite number of values of n.

Theorem 4.1. $\psi(K(n,2)) = \alpha(K(n,2))$ *for* $2 \le n \le 6$ *and*

$$\left\lfloor \frac{\binom{n}{2}}{2} \right\rfloor \le \psi(K(n,2)) \le \left\lfloor \frac{\binom{n}{2} + \lfloor \frac{n}{2} \rfloor}{2} \right\rfloor$$

for $n \ge 7$*. Moreover, the upper bound is tight if* $n \equiv 0, 4$ *mod* 20*.*

Proof. For $n = 2, 3$, the graph $K(n,2)$ is edgeless and then $\psi(K(n,2)) = \alpha(K(n,2)) = 1$.

For $n = 4, 5, 6$, $\alpha(K(n,2))$ is $3, 5, 7$, respectively (by Theorem 3.1). Note that any complete coloring having a color class of size one uses at most $k = 2, 4, 7$ colors, respectively. And any complete coloring without color classes of size one uses at most $k = 3, 5, 7$ colors, respectively. Therefore, $\psi(K(n,2))$ is at most $3, 5, 7$, respectively. Hence $\psi(K(n,2)) = \alpha(K(n,2))$.

For $n \ge 7$, any complete coloring of $K(n,2)$ has at most $\omega(K(n,2)) = \lfloor \frac{n}{2} \rfloor$ classes of size 1 ($\omega(G)$ is the clique number of the garph G, that is, the largest order of a complete subgraph of G), then

$$\psi(K(n,2)) \le \left\lfloor \frac{\binom{n}{2} - \lfloor \frac{n}{2} \rfloor}{2} + \left\lfloor \frac{n}{2} \right\rfloor \right\rfloor = \left\lfloor \frac{\binom{n}{2} + \lfloor \frac{n}{2} \rfloor}{2} \right\rfloor .$$

Such an upper bound is proved in [1].

To see the lower bound, we use a 1-factorization F of K_{2t} such that no component induced by two distinct 1-factors of F is a 4-cycle, see [17, 20]. We need to distinguish four cases, namely, when $n = 4k-1$, $4k$, $n = 4k+1$ and $n = 4k+2$ for $k \ge 1$.

1. Case $n = 4k$. Consider F for $t = 2k$. Since each 1-factor contains t edges, we have k color classes of size two for each 1-factor, therefore the lower bound follows.
2. Case $n = 4k+1$. Consider F for $t = 2k+1$ and delete a vertex of K_{4k+2}. Since each maximal matching arising from a 1-factor of F contains $t-1$ edges, we have k color classes of size two for each such maximal matching, hence the lower bound follows.
3. Case $n = 4k+2$. Consider F for $t = 2k$ and add two new vertices a and b to $V(K_{4k})$ to obtain K_{4k+2}. Color the subgraph K_{4k} as above, and the remaining edges as follows. For each vertex x of K_{4k}, we have the classes $\{ax, xb\}$. Finally, color the edge ab in a greedy way and the result follows.
4. Case $n = 4k-1$. Consider F for $t = 2k-1$ and adding a new vertex b to obtain K_{4k-1}. Color the subgraph $K_{4k-3} = K_{4k-1} - \{a,b\}$ as in the case $n \equiv 1$ mod 4, and form for each vertex x of K_{4k-3} the color class $\{ax, xb\}$. Finally, choose for the edge ab greedily a color that is already used; the result then follows.

Now, to verify that the upper bound is tight, consider an $(n+1, 5, 1)$-design D, see [6]. Therefore $n+1 \equiv 1, 5$ mod 20. Choose a point v of D and let $P = \{Q_i \colon i =$

$0, 1, \ldots, \frac{n-4}{4}\}$ with $Q_i = \{v_{4i+1}, v_{4i+2}, v_{4i+3}, v_{4i+4}\}$ be the set of 4-blocks of $D - v$. Pairs of points of every 5-block of $D - v$ are colored so that five color classes of size two are created, see Figure 7; the coloring is not proper, since all those color classes induce a K_2 subgraph of $K(n, 2)$.

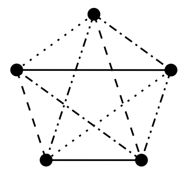

Figure 7: A complete coloring of $K(5, 2)$ using five colors that is not proper.

Label the edges of each block Q_i of P as $f_{2i} = v_{4i+1}v_{4i+2}$, $f_{2i+1} = v_{4i+3}v_{4i+4}$, e_i, $e_{n/4+i}$, $e_{n/2+i}$ and $e_{3n/4+i}$. Remaining vertices of $K(n, 2)$ are colored to form color classes $\{f_i\}$ of size one for $i = 0, 1, \ldots, n/2 - 1$, and color classes $\{e_{2i}, e_{2i+1}\}$ of size two for $i = 0, 1, \ldots, n/4 - 1$. The coloring is complete, hence the result follows. □

5 On the Grundy number of $K(n, 2)$

In this section, we observe that the coloring used in Theorem 3.1 is also a greedy coloring.

An l-coloring of G is called *Grundy*, if it is a proper coloring having the property that for every two colors i and j with $i < j$, every vertex colored j has a neighbor colored i (consequently, every Grundy coloring is a complete coloring). Moreover, a coloring ς of a graph G is a Grundy coloring of G if and only if ς is a greedy coloring of G, see [7]. Therefore, the Grundy number $\Gamma(G)$ is the largest l for which a Grundy l-coloring of G exists. Any graph G satisfies, $\chi(G) \leq \Gamma(G) \leq \alpha(G) \leq \psi(G)$.

Consider the coloring used in Theorem 3.1. Divide colors into *small*, *medium* and *high* (recall that colors used in Theorem 3.1 are positive integers), and use them for color classes of size three, two and one, respectively. We only need to verify that if i and j are colors with $i < j$, then for every edge e of color j there exists an edge of color i that is disjoint with e. This is certainly true if j is a high color, since the coloring is complete. If the color j is not high, the required condition is satisfied because of the following facts: (i) (3-element) vertex sets corresponding to color classes i and j have at most one vertex in common; (ii) the centers of involved P_3 subgraphs are distinct if both i and j are medium colors.

Consider the coloring used in Theorems 3.1. Taking the highest colors as the color class of size 1 and the smallest colors as the color classes of size 3. We only need to verify that for every two color classes with colors i and j, $i < j$, and every edge of color j there always exist a disjoint edge of color i. This is true if the color classes are triangles because they only share at most one vertex. If the color classes are an triangle K_3 with color i and a path P_3 with color j this is also true.

Theorem 5.1. $\Gamma(K(n, 2)) = \left\lfloor \binom{n+1}{2}/3 \right\rfloor$ *for* $n \neq 3$ *and* $\Gamma(K(3, 2)) = 1$.

6 About general upper bounds

The known upper bound for the pseudoachromatic number states, for $K(n,k)$ (see [7]), that

$$\psi(K(n,k)) \leq \frac{1}{2} + \sqrt{\frac{1}{4} + \binom{n}{k}\binom{n-k}{k}} = O\left(\frac{n^{k/2}(n-k)^{k/2}}{k!}\right). \tag{6.1}$$

A slightly improved upper bound is the following. Let ς be a complete coloring of $K(n,k)$ using l colors with $l = \psi(K(n,k))$. Let $x = \min\{\left|\varsigma^{-1}(i)\right| : i \in [l]\}$, that is, x is the cardinality of the smallest color class of ς; without loss of generality we may suppose that $x = \left|\varsigma^{-1}(l)\right|$. Since ς defines a partition of the vertex set of $K(n,k)$ it follows that $l \leq f(x) := \binom{n}{k}/x$.

Additionally, since $K(n,k)$ is $\binom{n-k}{k}$-regular, there are at most $\binom{n-k}{k}$ vertices adjacent in $K(n,k)$ to a vertex of $\varsigma^{-1}(l)$. With $\mathcal{X} := \bigcup_{X \in \varsigma^{-1}(l)} X$ we have $|[n] \setminus \mathcal{X}| \geq n - kx$. If $n - kx \geq k$ and $Y \subseteq [n] \setminus \mathcal{X}$, $|Y| = k$, then each of x edges XY, $X \in \varsigma^{-1}(l)$, corresponds to the pair of colors $l, \varsigma(Y)$. Therefore, $\psi(K(n,k)) \leq g(x)$, where $g(x) := 1 + x\binom{n-k}{k} - (x-1)\binom{n-kx}{k}$, if $n - kx \geq k$, and $g(x) := 1 + x\binom{n-k}{k}$ otherwise. Consequently, we have:

$$\psi(K(n,k)) \leq \max\{\min\{f(x), g(x)\} \colon x \in \mathbb{N}\}.$$

Hence, we conclude that:

$$\psi(K(n,k)) \leq \lfloor \max\{\min\{f(x), g(x)\} \colon x \in \mathbb{N}\} \rfloor$$

and then

$$\psi(K(n,k)) \leq \left\lfloor \max\left\{\min\{f(x), g(x)\} \colon x \in \mathbb{R}^+\right\} \right\rfloor.$$

It is not hard to see that $\max\{\min\{f(x), g(x)\} \colon x \in \mathbb{R}^+\} \leq \frac{1}{2} + \sqrt{\frac{1}{4} + \binom{n}{k}\binom{n-k}{k}}$.

On a general lower bound. An l-coloring ς is called *dominating* if every color class contains a vertex that has a neighbor in every other color class. The *b-chromatic number* $\varphi(G)$ *of* G is defined as the largest number l for which there exists a dominating l-coloring of G (see [16]). Since a dominating coloring is also complete, hence, for any graph G, $\varphi(G) \leq \alpha(G)$. The following theorem was proved in [13]:

Theorem 6.1 (Hajiabolhassan [13])**.** *Let $k \geq 3$ an integer. If $n \geq 2k$, then $2\binom{\lfloor \frac{n}{2} \rfloor}{k} \leq \varphi(K(n,k))$.*

In consequence, for any n, k satisfying $n \geq 2k \geq 6$, we have

$$\alpha(K(n,k)) \geq 2\binom{\lfloor \frac{n}{2} \rfloor}{k} = \Omega\left(\frac{n^k}{2^{k-1}k^k}\right).$$

7 The achromatic numbers of $D_V(n)$

Let V be a set of n points in general position in the plane, i.e., no three points of V are collinear. The segment disjointness graph $D_V(n)$ has the vertex set equal to the set of all straight line segments with endpoints in V, and two segments are adjacent in $D_V(n)$ if and only if they are disjoint. Each graph $D_V(n)$ is a spanning subgraph of $K(n,2)$. The chromatic number of the graph $D_V(n)$ is bounded in [3] where it is proved that $\chi(D_V(n)) = \Theta(n)$.

In this subsection, we prove bounds for $\alpha(D_V(n))$ and $\psi(D_V(n))$. Having in mind the fact that $\psi(H) \leq \psi(G)$ if H is a subgraph of G, Theorem 4.1 yields

$$\psi(D_V(n)) \leq \left\lfloor \frac{\binom{n}{2} + \lfloor \frac{n}{2} \rfloor}{2} \right\rfloor \leq \frac{n^2}{4}.$$

For the lower bound, we use the following results. A *straight line thrackle* is a set S of straight line segments such that any two distinct segments of S either meet at a common endpoint or they cross each other (see [9]).

Theorem 7.1 (Erdős [9] (see also the proof of Theorem 1 of [22]))**.** *If $d_1(n)$ denotes the maximum number of edges of a straight line thrackle of n vertices then $d_1(n) = n$.*

Lemma 7.2. *Any two triangles T_1 and T_2 with points in V, that share at most one point, contain two disjoint edges.*

Proof. Case 1. T_1 has a point in common with T_2: Since $T_1 \cup T_2$ have five points and six edges, then two of its edges are disjoint due to $d_1(5) = 5$.

Case 2. T_1 has no points in common with T_2: Let e be an edge of T_2. Let us suppose that $T_1 \cup T_2$ does not contain two disjoint edges, then T_1 and e is a straight line thrackle. Therefore, a vertex of e and a vertex of T_1 have to be the same, which is impossible because T_1 has no points in common with T_2. □

Now, if we identify a Steiner triple system $STS(n)$ with the complete geometric graph of n points and we color each triangle with a different color, by Lemma 7.2, we have the following.

Lemma 7.3. *If $n \equiv 1, 3$ mod 6 and V is a set of n points in general position, then*

$$\frac{n^2}{6} - \frac{n}{6} = \frac{1}{3}\binom{n}{2} \leq \alpha(D_V(n))$$

Therefore, we have the following theorem.

Theorem 7.4. *For any natural number n and any set of n points V in general position,*

$$\frac{n^2}{6} - \Theta(n) \leq \alpha(D_V(n)).$$

Further, if K_n has an even number of vertices, then there is a set $F \subseteq E(K_n)$ such that $E(K_n) \setminus F$ can be partitioned into triangles. More precisely, if $n \equiv 0, 2$ mod 6,then F is a perfect matching in K_n, and if $n \equiv 4$ mod 6, then F induces a spanning forest of $n/2 + 1$ edges in K_n with all vertices having an odd degree, see [11, 14].

A set V of n points in *convex position* is a set of n points in general position such that they are the vertices of a convex polygon (each internal angle is strictly less than 180 degrees).

Theorem 7.5. *For any even natural number n and any set of n points V in convex position,*

$$\frac{n^2}{6} + \Theta(n) \leq \alpha(D_V(n))$$

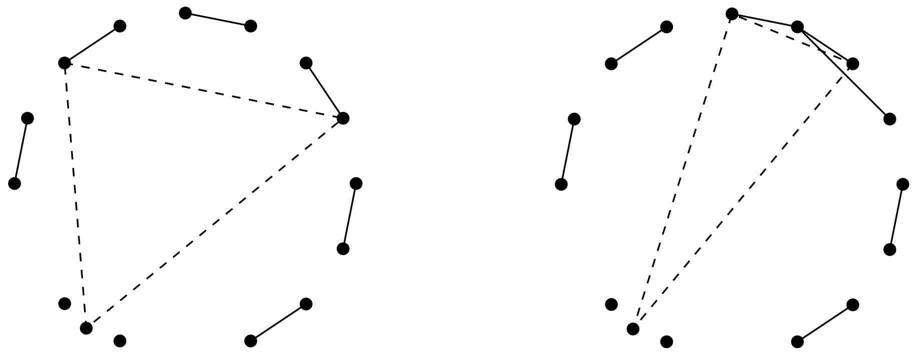

Figure 8: Configurations of the color classes of size one arising from F and a dashed triangle of $K_n - F$ in the proof of Theorem 7.5.

Proof. Take the edges of F in the convex hull of V, except for one in the case of $n \equiv 4 \mod 6$, see Figure 8. Each component of F is a color class. Each triangle of $K_n - F$ is a color class. Essentially we use $\binom{n+1}{2}/3$ triangles, and the result follows. □

Finally, the geometric type Kneser graph $D_V(n,k)$ for $k \geq 2$ whose vertex set consists of all subsets of k points in V. Two such sets X and Y are adjacent if and only if their convex hulls are disjoint. Given a point set V, for a line dividing V into two sets V_1 and V_2 of $n/2$ points, having a coloring such that each color class has sets $X \subseteq V_1$ and $Y \subseteq V_2$, we have that

$$\psi(D_V(n,k)) \geq \binom{n/2}{k} = \Omega\left(\frac{n^k}{2^k k^k}\right)$$

8 On odd graphs

It is obvious to prove that the achromatic and the pseudoachromatic number as well of (the graph induced by) a matching of size $\binom{k}{2}$ is equal to k. Therefore, a matching of size m has achromatic and pseudoachromatic number equal to $\left\lfloor \frac{1}{2} + \sqrt{\frac{1}{4} + 2m} \right\rfloor$, which means that in the case $n = 2k$ the upper bound of (1) for $\psi(K(2k,k))$ is equal to the lower bound for $\alpha(K(2k,k))$; in other words,

$$\alpha(K(2k,k)) = \psi(K(2k,k)) = \left\lfloor \frac{1}{2} + \sqrt{\frac{1}{4} + \binom{2k}{k}} \right\rfloor$$

However, the situation is different in the case of $K(2k+1,k)$, the Kneser graphs that are called *odd graphs*. The better lower bound we have is

$$\Omega\left(2^{k/2}\right) = \left\lfloor \frac{1}{2} + \sqrt{\frac{1}{4} + \binom{2k}{k}} \right\rfloor \leq \psi(K(2k+1,k)),$$

due to the fact that $K(2k,k)$ is a subgraph of the odd graph $K(2k+1,k)$.

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Coarse distinguishability of graphs with symmetric growth*

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Abstract

Let X be a connected, locally finite graph with symmetric growth. We prove that there is a vertex coloring $\phi\colon X \to \{0,1\}$ and some $R \in \mathbb{N}$ such that every automorphism f preserving ϕ is R-close to the identity map; this can be seen as a coarse geometric version of symmetry breaking. We also prove that the infinite motion conjecture is true for graphs where at least one vertex stabilizer S_x satisfies the following condition: for every non-identity automorphism $f \in S_x$, there is a sequence x_n such that $\lim d(x_n, f(x_n)) = \infty$.

Keywords: Graph, coloring, distinguishing, coarse, growth, symmetry.

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1 Introduction

A (not necessarily proper) vertex coloring ϕ of a graph is *distinguishing* if the only automorphism that preserves ϕ is the identity. This notion was first introduced in [4] under the name *asymmetric coloring*, where it was proved that 2 colors suffice to produce a distinguishing coloring of a regular tree. Later, Albertson and Collins [1] defined the *distinguishing number* $D(X)$ of a graph X as the least number of colors needed to produce a distinguishing coloring. The problem of calculating $D(X)$ and variants thereof has accumulated an extensive literature in the last 20 years, see e.g. [2, 14, 16, 17, 18, 22] and references therein.

One of most important open problems in graph distinguishability is the Infinite Motion Conjecture of T. Tucker. Let us introduce some preliminaries: The *motion* $m(f)$ of a graph automorphism f is the cardinality of the set of points that are not fixed by f. For a graph X and a subset $A \subset \operatorname{Aut}(X)$, the motion of A is $m(A) = \inf\{m(f) \mid f \in A,\ f \neq \mathrm{id}\}$, and the motion of X is $m(X) = m(\operatorname{Aut}(X))$. A probabilistic argument yields the following result for finite graphs.

Lemma 1.1 (Motion Lemma, [20])**.** *If X is a finite graph and $2^{m(X)} \geq |\operatorname{Aut}(X)|^2$, then $D(X) \leq 2$.*

We always have $|\operatorname{Aut}(X)|^2 \leq 2^{\aleph_0}$ when X is countable, which motivates the following generalization.

Conjecture 1.2 (Infinite motion conjecture, [22])**.** *If X is a connected, locally finite graph with infinite motion, then $D(X) \leq 2$.*

The condition of local finiteness cannot be omitted [17]; note also that every connected, locally finite graph is countable. This conjecture has been confirmed for special classes of graphs: F. Lehner proved it in [16] for graphs with growth at most $\mathcal{O}(2^{(1-\epsilon)\frac{\sqrt{n}}{2}})$ for some $\epsilon > 0$,[1] and later, together with M. Pilśniak and M. Stawiski [18], for graphs with degree less or equal to five.

The aim of this paper is to introduce a large-scale-geometric version of distinguishability for colorings, and to prove the existence of such colorings in graphs whose growth functions are large-scale symmetric. This will result in a proof of Conjecture 1.2 for graphs with a vertex stabilizer S_x satisfying that, for every automorphism $f \in S_x \setminus \{\mathrm{id}\}$, there is a sequence x_n such that $d(x_n, f(x_n)) \to \infty$; we can regard this condition as a geometric refinement of having infinite motion.

Let X and Y be connected graphs, endowed with their canonical $\mathbb{N}$-valued[2] metric. In the context of coarse geometry (see [19] for a nice exposition on the subject), two functions $f, g\colon X \to Y$ are *R-close* ($R \geq 0$) if $d(f(x), g(x)) \leq R$ for all $x \in X$, and we say that f and g are *close* if they are R-close for some $R \geq 0$. Let $\mathrm{QI}(X)$ denote the group of closeness classes of quasi-isometries (in the sense of Gromov) $f\colon X \to X$, and let $\iota\colon \operatorname{Aut}(X) \to \mathrm{QI}(X)$ denote the natural map that sends every automorphism to its closeness class. We can adapt the notion of distinguishing coloring to this setting as follows:

Definition 1.3. A coloring $\phi\colon X \to \mathbb{N}$ is *coarsely distinguishing* if every $f \in \operatorname{Aut}(X, \phi)$ is close to the identity; that is, $\iota(\operatorname{Aut}(X, \phi)) = \{[\mathrm{id}_X]\}$.

[1] The notation $f = \mathcal{O}(g)$ is used if there are C, N such that $f(x) \leq C g(x)$ for all $x > N$.

[2] We will use the convention that $0 \in \mathbb{N}$.

This new definition begs the following question: which connected, locally finite graphs admit a coarsely distinguishing coloring by two colors? In Section 5.1 we present a simple example of a graph that does not admit such a coloring. The first main result of this paper shows that graphs with *symmetric growth* admit coarsely distinguishing colorings by two colors; this condition is satisfied by vertex-transitive graphs and, more generally, coarsely quasi-symmetric graphs [3, Corollary 4.17]. The intuitive ideas behind these notions are as follows: A connected, locally finite graph has the same growth type at all vertices (see Section 2). If all of those growth types can be compared using the same constants, then the graph is said to have *symmetric growth* (see Definition 2.3). Similarly, given any pair of vertices, there is a quasi-isometry mapping one of them to the other one. If all of those quasi-isometries can be obtained with the same distortion bounds, then the graph is called *coarsely quasi-symmetric* [3, Definition 3.16]. This can be thought of as the coarse-geometric analogue of being vertex-transitive.

Theorem 1.4. *Let X be a connected, locally finite graph of symmetric growth. Then there are $R \in \mathbb{N}$ and $\phi\colon X \to \{0,1\}$ such that every $f \in \operatorname{Aut}(X,\phi)$ satisfies $d(x,f(x)) \leq R$ for all $x \in X$.*

Note that we obtain a uniform closeness parameter R for all $f \in \operatorname{Aut}(X,\phi)$; furthermore, we make no assumption on the motion of the graph. A slight modification of the proof of Theorem 1.4 proves the infinite motion conjecture for graphs X containing a vertex $x \in X$ such that the restriction $\iota\colon S_x \to \operatorname{QI}(X)$ is injective. Let us rephrase this condition in a language closer to the statement of Conjecture 1.2. Let X be a connected graph and let $f \in \operatorname{Aut}(X)$. The *geometric motion* of f is then $\operatorname{gm}(f) = \sup\{d(x,f(x)) \mid x \in X\}$; for a subset $A \subset \operatorname{Aut}(X)$, the geometric motion of A is $\operatorname{gm}(A) = \sup\{\operatorname{gm}(f) \mid f \in A,\ f \neq \operatorname{id}\}$. The definition of the "closeness" relation for functions yields that the restriction $\iota\colon A \to \operatorname{QI}(X)$ is injective if and only if $\operatorname{gm}(A) = \infty$. The second main result of the paper therefore reads as follows.

Theorem 1.5. *Let X be a connected, locally finite graph with symmetric growth. If $m(X) = \infty$ and there exists $x \in X$ such that $\operatorname{gm}(S_x) = \infty$, then $D(X) \leq 2$.*

In Sections 5.3 and 5.4 we present two families of graphs satisfying the hypothesis of Theorem 1.5: the *Diestel-Leader graphs* $\operatorname{DL}(p,q)$, $p,q \geq 2$, and graphs with *bounded cycle length*. The origin of Diestel-Leader graphs goes back to the following question, posed in [21, 23] by W. Woess:

Question 1.6. Is there a locally finite vertex-transitive graph that is not quasi-isometric to the Cayley graph of some finitely generated group?

R. Diestel and I. Leader introduced in [10] the graph $\operatorname{DL}(2,3)$ and conjectured that it satisfies the conditions of Question 1.6. A. Eskin, D. Fisher, and K. Whyte proved in [11, 12, 13] that in fact all graphs $\operatorname{DL}(p,q)$ with $p \neq q$ answer Question 1.6 positively. On the other hand, graphs with bounded cycle length are hyperbolic (in the sense of Gromov) and contain as examples free products of finite graphs.

A preliminary version of this paper stated that the authors did not know of any proof in the literature for the existence of distinguishing colorings by 2 colors for these families of graphs. An anonymous referee has pointed to us that, in the case of Diestel-Leader graphs, this actually follows from the fact that they satisfy the Distinct Spheres Condition

(DSC) [15, Theorem 4]. A connected graph X satisfies the DSC if there is a vertex $v \in X$ such that, for all distinct $u, w \in X$,

$$d(v,u) = d(v,w) \quad \Longrightarrow \quad S(u,n) \neq S(w,n) \quad \text{for infinitely many } n. \tag{1.1}$$

Since both symmetric growth and the DSC prove the existence of distinguishing colorings by 2 colors for the same family of graphs, it is natural to ask if there is any relation between these two notions; in Section 5 we present simple examples showing that all four possible Boolean combinations of these two conditions can be realized. This shows to some extent that our results and those in [15] are independent.

We can sketch the idea behind the proofs of Theorems 1.4 and 1.5 as follows: Choose a suitable $R > 0$ and a subset $Y \subset X$ such that $d(x, Y) \leq R$ for all $x \in X$. Suppose that there is a partial coloring ψ by two colors such that, if $\phi\colon X \to \{0,1\}$ is an extension of ψ and f is an automorphism of X preserving ϕ, then $f(Y) = Y$. Thus we can regard every extension ϕ of ψ as a coloring $\bar{\phi}\colon Y \to \mathbb{N}$ by more than two colors. The hypothesis of symmetric growth ensures that, for R large enough, we have sufficiently many local extensions of ψ around every point $y \in Y$ so that, gluing them, we can find a global extension ϕ with $\bar{\phi}$ distinguishing. Theorems 1.4 and 1.5 then follow from a simple geometrical argument. In general, we cannot find a partial coloring ψ as above, but the same idea works with minor modifications; this technique is similar to that used in [2].

The outline of the paper is as follows: In the next section we introduce some preliminaries to be used in the proof of the main theorems, which comprises Sections 3 and 4. Finally, Section 5 contains several examples illustrating some of the concepts that appear in the paper.

2 Preliminaries

In what follows we only consider undirected, simple graphs, so there are no loops and no multiple edges. We identify a graph with its vertex set, and by abuse of notation we write $X = (X, E_X)$. The *degree* of a vertex $x \in X$, $\deg x$, is the number of edges incident to x, and the degree of X is $\deg X = \sup\{\deg x \mid x \in X\}$. A graph X is *locally finite* if $\deg x < \infty$ for all $x \in X$. A *path* γ in X of *length* $l \in \mathbb{N}$ is a finite sequence $x_0, x_1, \ldots, x_l$ of vertices such that $x_{i-1} E_X x_i$ for all $i = 1, \ldots, l$; when the sequence of vertices is infinite, we call γ a *ray*. We may also think of a path (respectively, a ray) as a function $\sigma\colon \{0, \ldots, n\} \to X$ (respectively, $\sigma\colon \mathbb{N} \to X$). A graph is *connected* if every two vertices can be joined by a path. All graphs in this paper are assumed to be connected and locally finite, hence countable. We consider every graph to be endowed with its canonical $\mathbb{N}$-valued metric, where $d(x,y)$ is the length of the shortest path joining x and y; a length-minimizing path is termed a *geodesic path*.

A *partial coloring* of a graph X is a map $\psi\colon Y \to \mathbb{N}$, where $Y \subset X$; if $Y = X$, we simply call ψ a *coloring*. We use the term (partial) 2-*coloring* when ψ takes values in $\{0,1\}$. For every graph X and coloring $\phi\colon X \to \mathbb{N}$, let $\operatorname{Aut}(X,\phi)$ denote the group of automorphisms f of X satisfying $\phi = \phi \circ f$. A coloring $\phi\colon X \to \mathbb{N}$ is *distinguishing* if $\operatorname{Aut}(X,\phi) = \{\mathrm{id}\}$.

For a graph X, $x \in X$, and $r \in \mathbb{N}$, let

$$D(x,r) = \{\, y \in X \mid d(y,x) \leq r \,\}, \qquad S(x,r) = \{\, y \in X \mid d(y,x) = r \,\}$$

denote the *disk* and the *sphere* of center x and radius r, respectively. We may write $D_X(x,r)$ for $D(x,r)$ when the ambient space X is not clear from context. A subset Y

of X is *R-separated* ($R > 0$) if $d(y, y') \geq R$ for all $y, y' \in Y$ with $y \neq y'$; it is *R-coarsely dense* if, for every $x \in X$, there is some $y \in Y$ with $d(x, y) \leq R$.

Lemma 2.1 (E.g. [2, Corollary 2.2.]). *Let X be a graph and let $R > 0$. For every $x \in X$, there is a $(2R+1)$-separated, $2R$-coarsely dense subset $Y \subset X$ containing x.*

Remark 2.2. The proof in [2, Corollary 2.2.] makes use of Zorn's Lemma, but the result can be proved for countable graphs without assuming the Axiom of Choice: First, note that the proof in [2, Corollary 2.2.] does not require the Axiom of Choice for finite graphs. Let X be a countable graph, and let A_n be an increasing and exhausting sequence of finite subsets of X. Since we can use Lemma 2.1 with finite subsets, there is a sequence of $(2R+1)$-separated, $2R$-coarsely dense subsets $S_n \subset A_n$. The space 2^X is sequentially compact with the topology of pointwise convergence[3], so there is a convergent subsequence $S_{n_i} \to S$. It is now elementary to check that S is a $(2R+1)$-separated, $2R$-coarsely dense subset of X.

Let $\beta_x \colon \mathbb{N} \to \mathbb{N}$ and $\sigma_x \colon \mathbb{N} \to \mathbb{N}$ be the functions defined by

$$\beta_x(r) = |D(x, r)|, \qquad \sigma_x(r) = |S(x, r)|.$$

Given two non-decreasing functions $f, g \colon \mathbb{N} \to \mathbb{R}^+$, f is *dominated* by g if there are integers k, l, m such that $f(r) \leq kg(lr)$ for all $r \geq m$. Two functions have the same *growth type* if they dominate one another. The growth type of β_x does not depend on the choice of point $x \in X$, so every graph has a well-defined growth type. The functions β_x, $x \in X$, however, may not dominate one another with a uniform choice of constants, which motivates the following definition.

Definition 2.3 ([3, Definition 4.13]). A graph X has *symmetric growth* if there are $k, l, m \in \mathbb{N}$ such that $\beta_x(r) \leq k\beta_y(lr)$ for all $r \geq m$ and $x, y \in X$.

Lemma 2.4. *If X has symmetric growth, then* $\deg X < \infty$.

Proof. Let $x \in X$, then we have $\deg y < \beta_y(1) \leq k\beta_x(lm) < \infty$ for every $y \in X$. □

Let X be a graph with $\Delta := \deg X < \infty$, then the following holds for all $x \in X$ and $r \geq 1$ [2, Lemma 2.12]:

$$\sigma_x(1) \leq \Delta, \tag{2.1}$$
$$\sigma_x(r+1) \leq \sigma_x(r)(\Delta - 1), \tag{2.2}$$
$$\sigma_x(r+1) \leq \Delta(\Delta - 1)^r. \tag{2.3}$$

We will later fix a graph with $\Delta > 2$; note that in this case $\Delta/(\Delta - 2) \leq 3$, so

$$\begin{aligned}\beta_x(r) \leq 1 + \Delta \sum_{s=0}^{r-1} (\Delta - 1)^s &= 1 + \frac{\Delta((\Delta - 1)^r - 1)}{\Delta - 2} \\ &\leq 1 + 3(\Delta - 1)^r - 1 \\ &= 3(\Delta - 1)^r. \end{aligned} \tag{2.4}$$

We say that X has *exponential growth* if $\liminf \frac{\log \beta_x(r)}{r} > 0$ for some, and hence all $x \in X$, else it has *subexponential growth*. The following lemmas have elementary proofs.

[3]It is well-known that, for a countable product of compact subsets of the real line, the Tychonoff theorem can be proved without using the Axiom of Choice.

Lemma 2.5. *Let X be a graph with symmetric exponential growth. Then there are $k, l, m \in \mathbb{N}$ such that $e^r \leq k\beta_x(lr)$ for all $x \in X$ and $r \geq m$.*

Lemma 2.6. *If X has symmetric subexponential growth, then, for every $a, b > 0$, there is some $m \in \mathbb{N}$ such that $\beta_x(r) \leq ae^{br}$ for all $x \in X$ and $r \geq m$.*

3 Construction of the coloring

Let R be a large enough odd number, to be determined later. Let Y be a $(2R+1)$-separated, $2R$-coarsely dense subset of X; we define a graph structure E_Y on Y as follows:

$$yE_Yy' \quad \text{if and only if} \quad 0 < d(y, y') \leq 4R + 1. \tag{3.1}$$

Lemma 3.1. *The graph (Y, E_Y) is connected with $\deg_Y y \leq |D_X(y, 4R + 1)| - 1$ for all $y \in Y$.*

Proof. The inequality follows trivially from (3.1), so let us prove that Y is connected. Let $y, y' \in Y$, and let $(y, x_1, \dots, x_{n-1}, y')$ be a path in X. Since Y is $2R$-coarsely dense, for every $i = 1, \dots, n$ there is some $y_i \in Y$ with $d_X(x_i, y_i) \leq 2R$. The triangle inequality and (3.1) then yield that $(y, y_1, \dots, y_{n-1}, y')$ is a path on (Y, E_Y). □

Recall that R is a large enough odd number, so assume $R \geq 5$. Let

$$A = \{\, 2n \mid 2 \leq n \leq \frac{R-1}{2} \,\}, \qquad B = \{\, 2n + 1 \mid 1 \leq n \leq \frac{R-1}{2} \,\}, \tag{3.2}$$

and, for $r \leq R$, let

$$D(Y, r) = \bigcup_{y \in Y} D(y, r), \qquad S(Y, r) = D(Y, r) \setminus D(Y, r - 1) = \bigcup_{y \in Y} S(y, r),$$

where the last equality holds because Y is $(2R + 1)$-separated. Let us define a partial coloring

$$\psi\colon X \setminus \bigcup_{r \in B} S(Y, r) \to \{0, 1\}$$

as follows (Cf. [9, Lemma 3.2], see Figure 1 for an illustration):

$$\psi(x) = \begin{cases} 0, & x \in \bigcup_{r=0,1} S(Y, r), \\ 1, & x \in S(Y, 2), \\ 1, & x \in \bigcup_{r \in A} S(Y, r), \\ 1, & x \notin D(Y, R). \end{cases} \tag{3.3}$$

Note that the vertices that are not colored by this formula are precisely those in $S(y, r)$ for $r \in B$.

Lemma 3.2 (Cf. [9, Lemma 3.2.]). *Let $\phi\colon X \to \{0, 1\}$ be an extension of ψ, and let $f \in \mathrm{Aut}(X, \phi)$. For each $y \in Y$, there is some $\bar{y} \in Y$ such that $d(\bar{y}, f(y)) \leq 1$ and $d(z, \bar{y}) = d(z, f(y))$ for all $z \in X \setminus \{\bar{y}, f(y)\}$.*

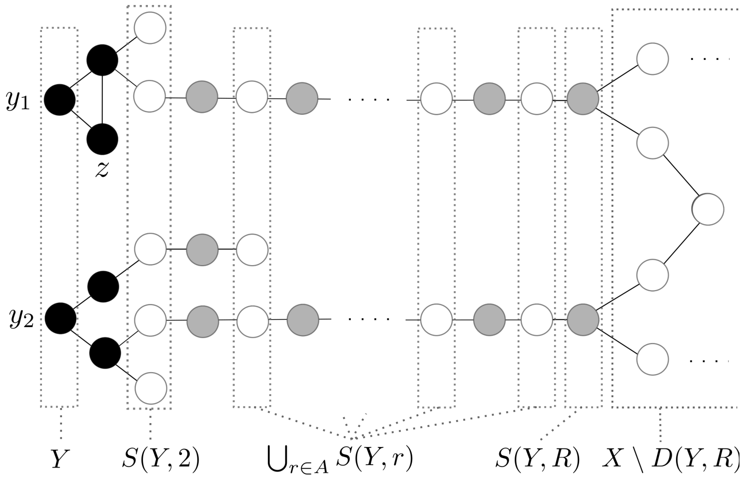

Figure 1: An illustration of the coloring ψ, where $y_1, y_2 \in Y$, black represents the color 0, and white represents 1. The grey vertices are those where ψ is not defined.

Proof. Let

$$Y' = \{\, z \in X \mid \phi(z') = 0 \text{ for all } z' \in D(z,1) \,\},$$

then (3.3) yields $Y' \subset D(Y,1)$, and clearly $f(Y') = Y'$ for all $f \in \mathrm{Aut}(X,\phi)$. For $y \in Y$, let $\bar{y}$ be the unique vertex in Y which is adjacent to $f(y)$. We have $\phi(z) = 0$ for every vertex $z \in D(f(y),1)$ and $D(f(y),1) \subset D(\bar{y},2)$, so $D(f(y),1) \subset D(\bar{y},1)$ by (3.3). Since $D(\bar{y},1) \subset D(f(y),2)$, we also get $D(\bar{y},1) \subset D(f(y),1)$, and the result follows. □

Corollary 3.3. *If X has infinite motion, then $f(Y) = Y$.*

Proof. Let $f \in \mathrm{Aut}(X,\phi)$ and suppose $f(y) \neq \bar{y}$. By the previous lemma we have $D(f(y),1) = D(\bar{y},1)$, so there is a non-trivial automorphism exchanging $f(y)$ and $\bar{y}$ and leaving all other vertices in X fixed. This contradicts the assumption that X has infinite motion. □

Remark 3.4. Note that there might be automorphisms $f \in \mathrm{Aut}(X,\phi)$ with $f(Y) \neq Y$ when $m(X) < \infty$. The graph in Figure 1 provides such an example: the map f that interchanges y_1 and z and leaves the rest of vertices fixed is an automorphism preserving ψ, but $f(Y) \neq Y$.

Since $\operatorname{dom}\psi = X \setminus \bigcup_{r\in B} S(Y,r)$, an extension of ψ to X is the same thing as a coloring of $\bigcup_{r\in B} S(Y,r)$; for such an extension ϕ, let $\bar{\phi}$ denote the induced coloring $Y \to \prod_B \mathbb{N}$ defined by

$$\bar{\phi}(y) = (\bar{\phi}_r(y))_{r\in B}, \quad \text{where } \bar{\phi}_r(y) = |S(y,r) \cap \phi^{-1}(1)|. \tag{3.4}$$

Lemma 3.5. *If $\xi := (\xi_r)_{r\in B} \colon Y \to \prod_B \mathbb{N}$ is such that $\xi_r(y) \leq \sigma_y(r)$ for every $y \in Y$, then there is at least one extension ϕ satisfying $\bar{\phi} = \xi$.*

Proof. Since Y is $(2R+1)$-separated, the spheres $S(y,r)$, $y \in Y$, $r \in B$, are pairwise disjoint. Thus we can define ϕ independently over each sphere $S(y,r)$ by coloring $\xi_r(y)$ vertices with the color 1 and the rest with the color 0. □

Lemma 3.6. *For each extension $\phi\colon X \to \{0,1\}$ of ψ and every automorphism $f \in \operatorname{Aut}(X,\phi)$, there is a unique automorphism $\bar{f} \in \operatorname{Aut}(Y,\bar{\phi})$ such that $d(\bar{f}(y), f(y)) \leq 1$ for all $y \in Y$.*

Proof. Let $\bar{f}$ be defined by the formula $\bar{f}(y) = \bar{y}$, where $\bar{y} \in Y$ denotes the point given by Lemma 3.2. This point satisfies $d(\bar{f}(y), z) = d(f(y), z)$ for all $z \in X \setminus \{f(y), \bar{f}(y)\}$, so

$$d(y,y') = d(f(y), f(y')) = d(\bar{f}(y), \bar{f}(y'))$$

for every $y, y' \in Y$, $y \neq y'$. This equation and (3.1) yield that $\bar{f}$ is an automorphism of Y; moreover,

$$f(S(y,r)) = S(f(y), r) = S(\bar{f}(y), r)$$

for $r \geq 1$ by Lemma 3.2, so $\bar{f}$ preserves ξ by (3.4). □

Proposition 3.7. *If X has symmetric growth, then we can choose R large enough so that $\prod_{r\in B}(\sigma_x(r)+1) > \beta_x(4R+1)$ for all $x \in X$.*

In order to keep with the flow of the argument, we defer the proof of Proposition 3.7 to Section 4. Assume for the remainder of this section that X has symmetric growth and that R has been chosen satisfying the statement of Proposition 3.7.

Proposition 3.8. *There is a distinguishing coloring $\xi := (\xi_r)_{r\in B}\colon Y \to \prod_B \mathbb{N}$ such that $\xi_r(y) \leq \sigma_y(r) + 1$.*

Proof. Choose a spanning tree T for (Y, E_Y) and a root $y_0 \in Y$. In order to define ξ, first let $\xi(y_0) = (0,\dots,0)$. Every $y \in Y$ with $y \neq y_0$ has at most $|D_X(y, 4R+1)| - 1$ siblings in T by Lemma 3.1. Using Proposition 3.7, we can define ξ so that $\xi(y) \neq (0,\dots,0)$ for all $y \neq y_0$, and every vertex is colored differently from its siblings in T. It can be easily checked that such a coloring is distinguishing [8, Lemma 4.1]. □

Proof of Theorem 1.4. Lemma 3.5 and Proposition 3.8 prove the existence of some $\phi\colon X \to \{0,1\}$ extending ψ and such that $\bar{\phi}\colon Y \to \mathbb{N}$ is distinguishing. By Lemma 3.6, every $f \in \operatorname{Aut}(X,\phi)$ satisfies $d(f(y), y) \leq 1$ for all $y \in Y$. Since Y is $2R$-coarsely dense, the triangle inequality yields $d(x, f(x)) \leq 4R+1$ for all $x \in X$. □

Proof of Theorem 1.5. Let X have infinite motion and pick $x \in X$ so that S_x has infinite geometric motion; Lemma 2.1 ensures that we can choose Y so that $x \in Y$. Using Lemma 3.5 and Proposition 3.8, we construct a coloring $\phi\colon X \to \{0,1\}$ extending ψ and such that $\bar{\phi}$ is distinguishing. Since X has infinite motion, Corollary 3.3 yields $f(Y) = Y$ for every $f \in \operatorname{Aut}(X,\psi)$. Moreover, Lemma 3.6 and the fact that $\bar{\phi}$ is distinguishing show that $f|_Y = \mathrm{id}_Y$, so $\operatorname{Aut}(X,\phi) \subset S_x$. Since $\operatorname{gm}(S_x) = \infty$ by hypothesis, $\operatorname{gm}(\operatorname{Aut}(X,\phi)) = \infty$. But Y is a $2R$-coarsely dense subset and is fixed pointwise by every automorphism f, so the triangle inequality yields $d(x, f(x)) \leq 4R$ for all $x \in X$, a contradiction. □

4 Growth estimates

In this section we assume that X is a graph with symmetric growth. We will derive Proposition 3.7 from the following result:

Proposition 4.1. *For R large enough, we have $\prod_{r=3}^{R}(\sigma_x(r)+1) > (\Delta-1)[\beta_x(4R+1)]^2$ for all $x \in X$.*

Proof. First, note that this result is trivial in the case where X is a graph of symmetric subexponential growth. Indeed, since X is infinite, we have $\sigma_x(r) \geq 1$ for all $x \in X$, $r \geq 0$, so

$$\prod_{r=3}^{R}(\sigma_x(r)+1) \geq 2^{R-2} = \frac{1}{4}e^{R\log 2}. \tag{4.1}$$

Using Lemma 2.6, we have that, for R large enough,

$$\beta_x(4R+1) \leq \frac{1}{8(\Delta-1)}e^{[(4R+1)\log 2]/10} \leq \frac{1}{8(\Delta-1)}e^{(R\log 2)/2} \tag{4.2}$$

for every $x \in X$. Combining now (4.1) and (4.2), we get

$$(\Delta-1)[\beta_x(4R+1)]^2 \leq \frac{1}{8}e^{R\log 2} \leq \prod_{r=3}^{R}(\sigma_x(r)+1),$$

as desired. So, for the purposes of this proof, we will assume from now on that X is a graph with symmetric exponential growth.

In order to obtain lower bounds for the function $\prod_{r=3}^{R}(\sigma_x(r)+1)$, let us consider the following optimization problem: given $\Delta, Q, R \in \mathbb{N}$ with

$$\Delta > 2, \qquad R > 3, \qquad Q > \Delta^2 + R - 1, \tag{4.3}$$

minimize the function

$$f(a_1,\dots,a_R) = \prod_{i=3}^{R}(a_i+1) \tag{4.4}$$

for $a = (a_1,\dots,a_R) \in (\mathbb{Z}^+)^R$ satisfying

$$a_1 \leq \Delta, \tag{C1}$$

$$a_i \leq a_{i-1}(\Delta-1), \tag{C2}$$

$$\sum_{i=1}^{R} a_i = Q-1 \tag{C3}$$

for $i = 1,\dots,R$.

Claim 4.2. *The above problem has a minimizer $(a_1,\dots,a_R)$ satisfying:*

(i) $a_1 = \Delta$, *and* $a_2 = \Delta(\Delta-1)$.

(ii) *There is $0 \leq I \leq R-2$ such that the sequence $a_2,\dots,a_{2+I}$ is increasing and $a_i < \Delta(\Delta-1)$ for $i > 2+I$.*

(iii) *For* $3 \leq i \leq 2 + I$, *we have* $a_i + 1 > (a_{i-1} - 1)(\Delta - 1)$.

Suppose that $(a_1, \dots, a_R)$ is a minimizer that does not satisfy (i), let $n \in \{1, 2\}$ be the first index such that $a_n < \Delta(\Delta - 1)^{n-1}$, and let $m \geq 3$ be such that $a_m = \max\{a_i \mid i \geq 3\}$. Conditions (C1) and (C2) yield

$$a_1 + a_2 \leq \Delta + \Delta(\Delta - 1) = \Delta^2. \tag{4.5}$$

If $a_i = 1$ for all $i \geq 3$, then

$$\sum_{i=1}^{R} a_i = a_1 + a_2 + \sum_{i=3}^{R} a_i \leq \Delta^2 + R - 2 < Q - 1$$

by (4.3), contradicting (C3); this shows that $a_m > 1$. The sequence $(a'_1, \dots, a'_R)$ given by

$$a'_i = \begin{cases} a_i + 1 & \text{for } i = n, \\ a_i - 1 & \text{for } i = m, \\ a_i & \text{otherwise.} \end{cases}$$

still satifies (C1)–(C3), and clearly $f(a'_1, \dots, a'_R) < f(a_1, \dots, a_R)$ since the index n does not appear in (4.4). It follows that every minimizer has to satisfy (i).

Let us prove that we can obtain a minimizer satisfying both (i) and (ii). Let $(a_1, \dots, a_R)$ be a minimizer, and let s be a permutation of $\{1, \dots, R\}$ so that $s(1) = 1$, $s(2) = 2$, and

$$(a'_1, \dots, a'_R) = (a_{s(1)}, \dots, a_{s(R)})$$

satisfies (ii); it is obvious that such a permutation always exists. Since s leaves the subset $\{3, \dots, R\}$ invariant and the function f is symmetric in those indices, $(a'_1, \dots, a'_R)$ is also a minimizer if it satisfies (C1)–(C3).

Let us prove that $(a'_1, \dots, a'_R)$ satisfies (C1)–(C3): Condition (C1) holds because $s(1) = 1$. In order to prove (C2), we begin by showing the following claim.

Claim 4.3. *For every* $i \in \{3, \dots, R\}$ *with* $a_i > a_2$, *there is some* $j \in \{2, \dots R\}$ *such that* $j \neq i$ *and* $a_2 \leq a_j < a_i \leq (\Delta - 1)a_j$.

Let l be an integer to be determined later, we are going to define a sequence of indices $m_1, \dots, m_l$ in $\{2, \dots, R\}$. Let

$$m_1 = \inf\{\, i \in \{2, \dots, R\} \mid a_i \geq a_j \text{ for all } 2 \leq j \leq R \,\},$$

and assume $a_{m_1} > a_2$, since otherwise the claim is vacuously true. Suppose now that, for $i > 1$, we have defined m_j for $1 \leq j < i$. If $a_{m_{i-1}} = a_2$, then let $l = i - 1$, so that m_{i-1} is the last element in the sequence. If $a_{m_{i-1}} > a_2$, then let

$$m_i = \inf\{\, i \in \{2, \dots, m_{i-1}\} \mid a_i \geq a_j \text{ for all } 2 \leq j \leq m_{i-1} \,\}.$$

The claim is again vacuously true if $l = 1$, so assume $l \geq 2$. It follows easily from the definition of m_i that $a_{m_i - 1} = a_{m_{i+1}}$ for all $1 \leq i < l$, and thus (C2) yields

$$a_{m_i} \leq (\Delta - 1)a_{m_i - 1} = (\Delta - 1)a_{m_{i+1}}. \tag{4.6}$$

Observe that, for every $i \in \{3, \dots, R\}$ such that $a_2 < a_i$, there is some $j \in \{1, \dots, l-1\}$ such that $a_{m_{j+1}} \leq a_i \leq a_{m_j}$, which combined with (4.6) gives

$$a_{m_{j+1}} \leq a_i \leq a_{m_j} \leq (\Delta - 1) a_{m_{j+1}}.$$

This concludes the proof of Claim 4.3.

We resume the proof of (C2), so let I be the largest non-negative integer so that $a'_2, \dots a'_{2+I}$ is increasing. Recall that $a'_2 = a_2$, and let $3 \leq i \leq 2 + I$. If $a'_i = a'_2$, then $a'_{i-1} = a'_2 = a'_i$, so (C2) is satisfied. If $a'_i > a'_2$, then by Claim 4.3 there is some $j \in \{2, \dots, R\}$ such that $a_2 \leq a_j < a_{s(i)} \leq (\Delta - 1) a_j$. Since $a_j > a_2$, we have $2 \leq s^{-1}(j) \leq 2 + I$ by (ii). Also, the sequence $a'_2, \dots, a'_{2+I}$ is increasing, so $a_j \leq a'_{i-1}$ and therefore $a'_i \leq (\Delta - 1) a'_{i-1}$. Thus Condition (C3) is satisfied because the sum $\sum_{i=1}^{R} a_i$ is invariant by permutations, and we have obtained a minimizer $(a'_1, \dots, a'_R)$ that satisfies (i) and (ii).

Finally, suppose that $(a_1, \dots, a_R)$ is a minimizer satisfying (i) and (ii), but not (iii). Let n be an index such that $3 \leq n \leq R - 1$ and $a_n + 1 \leq (a_{n-1} - 1)(\Delta - 1)$, then one can easily check that the solution $(a'_1, \dots, a'_R)$ given by

$$a'_i = \begin{cases} a_i - 1 & \text{for } i = n - 1, \\ a_i + 1 & \text{for } i = n, \\ a_i & \text{otherwise.} \end{cases}$$

still satifies (C1)–(C3). Furthermore, $a_{n+1} \geq a_n$ implies $(a_{n+1} + 1)(a_n - 1) < a_{n+1} a_n$, so $f(a'_1, \dots, a'_R) < f(a_1, \dots, a_R)$, contradicting the assumption that $(a_1, \dots, a_R)$ was a minimizer. This completes the proof of Claim 4.2.

One can easily check that, for every graph X of bounded degree Δ, every $x \in X$, and every $R > 3$, the sequence $(\sigma_x(1), \dots, \sigma_x(R))$ satisfies (C1)–(C3) for $Q = \beta_x(R)$. Then Claim 4.2 shows that, for every $x \in X$, there is a sequence $(a_{x,1}, \dots, a_{x,R})$ satisfying Claim 4.2(i)–(iii) for $Q = \beta_x(R)$ and such that

$$\prod_{r=3}^{R} (\sigma_x(r) + 1) \geq \prod_{r=3}^{R} (a_{x,r} + 1) \tag{4.7}$$

Fix such a sequence $a_{x,r}$ for every point $x \in X$. Now (4.5) and Claim 4.2(ii) yield

$$\begin{aligned} \sum_{r=3}^{2+I} a_{x,r} &= \sum_{r=1}^{R} a_{x,r} - \sum_{r=3+I}^{R} a_{x,r} - \sum_{r=1}^{2} a_{x,r} \\ &\geq \beta_x(R) - (R - 2 - I)\Delta(\Delta - 1) - \Delta^2 \\ &\geq \beta_x(R) - R\Delta(\Delta - 1) - (\Delta - 1)^2. \end{aligned} \tag{4.8}$$

By (C2), we have $a_{x,2+r} \leq a_{x,2}(\Delta - 1)^r$ for $r = 1, \dots, I$, so

$$\begin{aligned} \sum_{r=3}^{2+I} a_{x,r} \leq \sum_{r=1}^{I} a_{x,2}(\Delta - 1)^r = a_{x,2}(\Delta - 1)\frac{(\Delta - 1)^I - 1}{\Delta - 2} &\leq a_{x,2}\Delta(\Delta - 1)^I \\ &\leq \Delta^3(\Delta - 1)^I. \end{aligned} \tag{4.9}$$

Since X has symmetric exponential growth, by Lemma 2.5 we have

$$R\Delta(\Delta-1)+(\Delta-1)^2<\beta_x(R)/2$$

for R large enough and all $x\in X$, so

$$\sum_{r=3}^{2+I} a_{x,r}\geq \beta_x(R)/2 \tag{4.10}$$

by (4.8), and now (4.9) and (4.10) yield

$$(\Delta-1)^I\geq \beta_x(R)/2\Delta^3. \tag{4.11}$$

From Claim 4.2(iii) we obtain by induction the following inequality for $r=1,\dots,I$.

$$\begin{aligned} a_{x,2+r} &\geq a_{x,2}(\Delta-1)^r-1-2\sum_{i=1}^{r-1}(\Delta-1)^i\\ &\geq a_{x,2}(\Delta-1)^r-1-2(\Delta-1)\frac{(\Delta-1)^{r-1}-1}{\Delta-2}\\ &\geq (\Delta-1)^r(a_{x,2}-\frac{2}{\Delta-2})-1. \end{aligned}$$

Since $a_{x,2}=\Delta(\Delta-1)>2/(\Delta-2)+1$, we have

$$a_{x,2+r}\geq(\Delta-1)^r.$$

Letting $C=1/2\Delta^3$, (4.11) yields

$$\begin{aligned} \prod_{r=3}^{R}(a_{x,r}+1)\geq\prod_{r=3}^{2+I}(a_{x,r}+1)\geq\prod_{r=1}^{I}(\Delta-1)^r &= ((\Delta-1)^{I+1})^{I/2}\\ &\geq [C\beta_x(R)]^{(\log_{\Delta-1}C\beta_x(R))/2}. \end{aligned} \tag{4.12}$$

Since X has symmetric exponential growth, by Lemma 2.5 there are $k,l,m\in\mathbb{N}$ such that $k\beta_x(ln)\geq e^n$ for all $x\in X$ and $n\geq m$. So, if $R\geq lm$, then (4.12) yields

$$\prod_{r=3}^{R}(a_{x,r}+1)\geq(Ck^{-1}e^{\lfloor R/l\rfloor})^{(\lfloor R/l\rfloor+\log Ck^{-1})/2}.$$

Since $(Ck^{-1}e^{\lfloor R/l\rfloor})^{(\lfloor R/l\rfloor+\log Ck^{-1})/2}$ grows faster than Δ^{8R+7}, we can assume that R is large enough so that

$$\prod_{r=3}^{R}(a_{x,r}+1)>\Delta^{8R+7}$$

for all $x\in X$. Noting that $(\Delta-1)^2>3$, equations (2.4) and (4.7) yield

$$\prod_{r=3}^{R}(\sigma_x(r)+1)\geq\prod_{r=3}^{R}(a_{x,r}+1)\Delta[(\Delta-1)^{4R+3}]^2\geq(\Delta-1)[\beta_x(4R+1)]^2. \qquad \square$$

Proof of Proposition 3.7. The definitions of A and B in (3.2) yield

$$\prod_{r=3}^{R}(\sigma_x(r)+1) = \left[\prod_{r\in A}(\sigma_x(r)+1)\right]\left[\prod_{r\in B}(\sigma_x(r)+1)\right]. \tag{4.13}$$

We have $r-1\in B$ for every $r\in A$, so

$$\prod_{r\in A}(\sigma_x(r)+1) \le (\Delta-1)\prod_{r\in B}(\sigma_x(r)+1) \tag{4.14}$$

because $\sigma_x(r)\le(\Delta-1)\sigma_x(r-1)$ by (2.2). The combination of (4.13) and (4.14) then yields

$$\prod_{r\in B}(\sigma_x(r)+1) \ge \sqrt{\frac{\prod_{r=3}^{R}(\sigma_x(r)+1)}{\Delta-1}},$$

and the result follows from Proposition 4.1. □

5 Examples

5.1 A connected, locally finite graph with no coarsely distinguishing 2-coloring

For $n\in\mathbb{Z}^+$, let $I_n=\{v_0,\dots,v_n\}$ be a graph with edges $\{v_m,v_{m+1}\}$ for $m=0,\dots,n-1$, and let $X=\{u_m\}_{m=1}^{\infty}$ be a graph with edges $\{u_m,u_{m+1}\}$ for $m\in\mathbb{Z}^+$. For every $n\in\mathbb{Z}^+$, take 2^n+1 copies of I_n and denote them by

$$I_n^i=\{\,v_m^i \mid i=0,\dots,n\,\},\qquad i=1,\dots,2^n+1.$$

For every n and i, glue the graph I_n^i to X by identifying the points u_n and v_0^i; denote the resulting graph by Y (see Figure 2), and let Y_n be the full subgraph whose vertex set is the image of $\bigcup_i I_n^i$ by the quotient map.

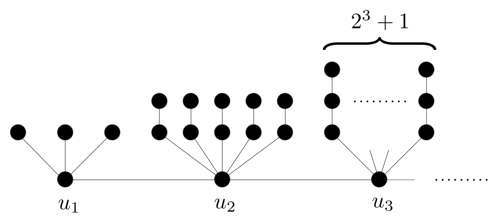

Figure 2: A graph without coarsely distinguishing 2-colorings

Let ϕ be an arbitrary 2-coloring of Y. Since we have 2^n+1 copies of I_n glued to u_n $(n\in\mathbb{Z}^+)$, by the pigeonhole principle there are at least two indices $i(n)\ne j(n)$ such that the restrictions of ϕ to $I_n^{i(n)}$ and $I_n^{j(n)}$ are equal. So there exists an isomorphism f_n of Y_n that preserves ϕ and maps $I_n^{i(n)}$ to $I_n^{j(n)}$, and therefore $d(f(v_n^{i(n)}),v_n^{i(n)})=2n$. Choose such an isomorphism f_n for every $n\in\mathbb{Z}^+$, and combine them into an isomorphism f of Y preserving ϕ. Since $d(f(v_n^{i(n)}),v_n^{i(n)})=2n$ for all $n\in\mathbb{Z}^+$, the map f is not close to the identity. Note that the vertex u_n has degree $4+2^n$, so $\deg Y=\infty$ and hence Y does not have symmetric growth.

5.2 Graphs with infinite motion but finite geometric motion

Perhaps the simplest example of a connected locally finite graph X with $m(X) = \infty$ and $\mathrm{gm}(X) < \infty$ is shown in Figure 3. This graph has symmetric linear growth. The only non-trivial automorphism f is the obvious one interchanging the horizontal rays starting at y and z, and it is easy to check that $d(x, f(x)) \leq 1$ for all $x \in X$.

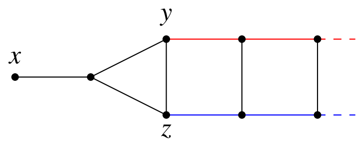

Figure 3: Example of a graph X with $m(X) = \infty$ and $\mathrm{gm}(X) < \infty$

We can modify this example to obtain graphs with infinite motion, finite geometric motion, and faster growth. For example, let T_3 be the regular tree of degree 4, and let $\phi\colon T_3 \to \{0, 1\}$ be an distinguishing coloring. Substitute each edge in T_3 by a "gadget" depending on the colors of the incident vertices (see Figure 4). In this way we obtain a graph Y with $\mathrm{Aut}(Y) = \{\mathrm{id}_Y\}$ and symmetric exponential growth. Moreover, we can identify T_3 with the subset $\overline{Y}$ of Y consisting of vertices of degree 4. Gluing one copy of X to each vertex $y \in \overline{Y}$ by identifying it with x, we obtain a graph with infinite motion, finite geometric motion, and exponential (but not symmetric) growth.

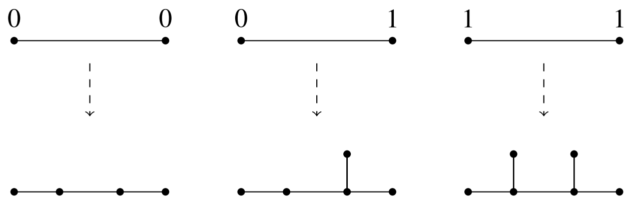

Figure 4: Substituting each edge in T_4 by a graph

5.3 Diestel-Leader graphs

The Diestel-Leader graphs $\mathrm{DL}(p_1, \ldots, p_n)$ are defined for $n, p_1, \ldots, p_n \geq 2$. For the sake of simplicity, however, we will restrict our attention to the case $n = 2$; at any rate, the following discussion can be easily adapted to include the case $n > 2$. In order to define $\mathrm{DL}(p, q)$, let T_p and T_q be the regular trees of degree $p + 1$ and $q + 1$, respectively. For $i = p, q$, choose a root $o_i \in T_i$ and fix an end ω_i of T_i. These choices induce *height* or *Busemann functions* $\mathfrak{h}_i\colon T_i \to \mathbb{Z}$, and then

$$\mathrm{DL}(p, q) := \{\, (x, y) \in T_p \times T_q \mid \mathfrak{h}_p(x) + \mathfrak{h}_q(y) = 0 \,\}.$$

Let us write $(x, y) \in \mathrm{DL}(p, q)$ as xy for the sake of clarity, and let xE_iy denote that x and y are adjacent in T_i, then the graph structure E in $\mathrm{DL}(p, q)$ is defined by

$$xyEx'y' \quad \text{if and only if} \quad xE_px' \text{ and } yE_qy'.$$

This yields

$$d_{\mathrm{DL}(p,q)}(xy, x'y') \geq \max\{d_{T_p}(x,x'), d_{T_q}(y,y')\}$$
$$\geq \max\{|\,\mathfrak{h}(x) - \mathfrak{h}(x')|, |\,\mathfrak{h}(y) - \mathfrak{h}(y')|\}. \quad (5.1)$$

For $i = p, q$, let $\mathrm{Aff}(T_i)$ be the subgroup of automorphisms of T_i that fix ω_i. For every $f \in \mathrm{Aff}(T_i)$, the quantity $\mathfrak{h}(f(x)) - \mathfrak{h}(x)$ is independent of $x \in T_i$, and we will denote it by $\mathfrak{h}(f)$. Let

$$\mathcal{A}_{p,q} = \{\, (f, f') \in \mathrm{Aff}(T_p) \times \mathrm{Aff}(T_q) \mid \mathfrak{h}_p(f) + \mathfrak{h}_q(f') = 0 \,\}.$$

Lemma 5.1 ([5, Theorem 2.7.], [6, Prop. 3.3])**.** *If $p \neq q$, then $\mathrm{Aut}(\mathrm{DL}(p,q)) \cong \mathcal{A}_{p,q}$. For $p = q$, the group $\mathrm{Aut}(\mathrm{DL}(p,p))$ is generated by $\mathcal{A}_{p,p}$ and the map $\sigma\colon xy \mapsto yx$.*

Let us prove that $\mathrm{DL}(p,q)$ satisfies the hypothesis of Theorem 1.5.

Lemma 5.2. *The group $\mathrm{Aut}(\mathrm{DL}(p,q))$ has infinite motion, and the stabilizer $S_{o_p o_q}$ has infinite geometric motion.*

Proof. Let $a = (f, f') \in \mathcal{A}_{p,q}$. If $a \neq \mathrm{id}$, then at least one of f, f' is non-trivial, say f. Therefore f is a non-trivial automorphism of a regular tree, hence $m(f) = m(a) = \infty$. If moreover $a \in S_{o_p o_q}$, then $f(o_p) = o_p$, and therefore $\mathrm{gm}(f) = \infty$ when considered as an automorphism of T_p (it is elementary to check that stabilizers in regular tres have infinite geometric motion). Now (5.1) yields $\mathrm{gm}(a) = \infty$, proving the result when $p \neq q$ by Lemma 5.1.

If $p = q$, then every automorphism which is not in $\mathcal{A}_{p,q}$ can be written as σa, where $a = (f, f') \in \mathcal{A}_{p,p}$ and σ is the map $xy \mapsto yx$. Since $f(o_p) = f'(o_p) = o_p$, we have $\mathfrak{h}(f) = \mathfrak{h}(f') = 0$. Let $x_n y_n$ be a sequence in $\mathrm{DL}(p,p)$ with $\mathfrak{h}_p(x_n) = -\,\mathfrak{h}_p(y_n) = n$. Then

$$\begin{aligned} d(x_n y_n, \sigma a(x_n y_n)) = d(x_n y_n, f'(y_n) f(x_n)) &\geq |\,\mathfrak{h}_p(x_n) - \mathfrak{h}_p(f'(y_n))| \\ &= |\,\mathfrak{h}_p(x_n) - \mathfrak{h}_p(y_n) - \mathfrak{h}_p(f)| \\ &\geq 2n - \mathfrak{h}_p(f), \end{aligned}$$

so $\mathrm{gm}(a) = m(a) = \infty$. □

5.4 Graphs with bounded cycle length

A *cycle* of length $n \in \mathbb{N}$ in a graph is a path σ of length n with $\sigma(0) = \sigma(n)$ and $\sigma(i) \neq \sigma(j)$ for $0 \leq i < j < n$. A graph X has *bounded cycle length* if there is $L \in \mathbb{N}$ such that every cycle in X has length $\leq L$. It is not difficult to prove that all graphs of bounded cycle length are hyperbolic in the sense of Gromov. There are in the literature several non-equivalent definitions of the *free product* of graphs, see e.g. [7]; one can easily check, however, that the following result holds for any of the definitions: The free product of a finite family of graphs of bounded cycle length has bounded cycle length. In particular, the free product of a finite family of finite graphs has bounded cycle length.

Lemma 5.3 (Cf. [16, Lemma 3.6])**.** *Let X be a connected locally finite graph with infinite motion, let $x \in X$, and let $f \in S_x$. Then there is a ray $\gamma\colon \mathbb{N} \to X$ such that $\gamma(0) = f(\gamma(0))$ and $\mathrm{im}(\gamma) \cap \mathrm{im}(f \circ \gamma) = \{\gamma(0)\}$.*

Proof. See the proof of [16, Lemma 3.6]. □

Proposition 5.4. *If X has infinite motion and bounded cycle length, then every vertex stabilizer has infinite geometric motion.*

Proof. Let $x \in X$ and let $f \in S_x$. By Lemma 5.3, there is a ray γ such that, if we let $\gamma' = f(\gamma)$, then $\gamma(0) = \gamma'(0)$ and $\mathrm{im}(\gamma) \cap \mathrm{im}(\gamma') = \{\gamma(0)\}$. For $n \in \mathbb{Z}^+$, choose geodesic paths σ_n from $\gamma(n)$ to $\gamma'(n)$. Let m_n be the largest integer such that $\sigma_n(m_n) \in \mathrm{im}\,\gamma$, and let m'_n be the least integer such that $\sigma_n(m'_n) \in \mathrm{im}\,\gamma'$; clearly $m_n, m'_n \leq d(\gamma(n), \gamma'(n))$. The triangle Z_n with sides

$$\begin{aligned}&(\gamma(0), \ldots, \gamma(i) = \sigma(m_n)),\\ &(\sigma(m_n), \sigma(m_n+1), \ldots, \sigma(m'_n)), \quad \text{and}\\ &(\gamma'(j) = \sigma(m'_n), \gamma'(j-1), \ldots, \gamma'(0))\end{aligned}$$

determines a cycle of length $\geq 2n - 2d(\gamma(n), \gamma'(n))$. Now the assumption that X has bounded cycle length yields $\lim d(\gamma(n), \gamma'(n)) = d(\gamma(n), f(\gamma(n)) = \infty$, and the result follows. □

5.5 Symmetric growth and the distinct spheres condition

In this section we show, using examples and a short argument, that all four possible Boolean combinations of the conditions "having symmetric growth" and "satisfying the DSC" can be realized in very simple graphs. Recall that X satisfies the DSC if there is a vertex $v \in X$ such that, for all distinct $u, w \in X$,

$$d(v, u) = d(v, w) \implies S(u, n) \neq S(w, n) \quad \text{for infinitely many } n. \tag{5.2}$$

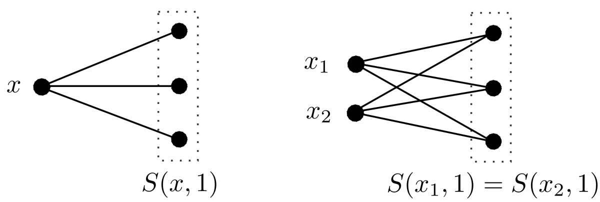

Figure 5: We substitute a vertex x by two copies x_1, x_2 with the same sphere of radius one

We will begin by showing how to modify a graph X to obtain a similar graph X' that does not satisfy the DSC. Let X be any connected graph, and take two different points $x, y \in X$. Using the substitution shown in Figure 5 on x and y, we can obtain a graph X' that has two pairs of vertices x_i, and y_i ($i = 1, 2$), instead of x and y, and so that, for any points $u, v \in X$ with $u, v \neq x, y$ and $i \in \{1, 2\}$,

$$d_{X'}(x_i, u) = d_X(x, u), \quad d_{X'}(y_i, u) = d_X(y, u), \quad d_{X'}(u, v) = d_X(u, v), \tag{5.3}$$

where by abuse of notation we are identifying the points of $X \setminus \{x, y\}$ with those of $X' \setminus \{x_1, x_2, y_1, y_2\}$. It follows immediately from (5.3) that X' shares the same coarse-geometric properties of X; in particular, X' has symmetric growth if and only if X does.

Let us show that X' never satisfies the DSC: Let $v \in X'$ be arbitrary, then at least one pair of the new vertices does not contain v, assume $v \notin \{x_1, x_2\}$. Now (5.3) yields that $d(v, x_1) = d(v, x_2)$, but $S(x_1, n) = S(x_2, n)$ for every $n > 0$, so X' does not satisfy the DSC. This procedure can be used to obtain examples of graphs of symmetric and non-symmetric growth that do not satisfy the DSC.

Regarding graphs with symmetric growth that satisfy the DSC, as stated in the introduction, the Diestel-Leader graphs constitute a family of such examples, but even simpler examples like the Cayley graph of the integers satisfy this conditions.

Finally, as for graphs with non-symmetric growth that satisfy the DSC, let X denote the (unmarked, undirected) Cayley graph of $\mathbb{Z}^2$ with respect to the generating set $\{(0,1),(1,0)\}$, and let Y be a semi-infinite ray; that is, the vertex set of Y is $\{y_i\}_{i=0}^{\infty}$ and there is an edge $y_i \sim y_{i+1}$ for every $i \geq 0$. It is elementary to check that X satisfies the DSC. Let Z be the graph obtained by gluing Y to X by identifying y_0 and $(0,0)$, and let us see that Z still satisfies the DSC: Let $v = (0,0)$, and let u, w be distinct vertices in Z with $d(v,u) = d(v,u)$. If $u, w \in X \subset Z$ (we can obviously identify X and Y with subsets of Z), then

$$S(u,n) \cap X \neq S(w,n) \cap S \quad \text{for infinitely many } n$$

because X satisfies the DSC. If $u \in X$ and $w = y_i \in Y$ for some $i > 0$, then, for every $n > 0$, we have

$$y_{i+n} \in S(w,n) \quad \text{but} \quad y_{i+n} \notin S(u,n)$$

because $d(u,Y) > 0$, so Z also satisfies the DSC. Moreover, since Y has linear growth and X has quadratic growth, it is easy to check that Z has non-symmetric growth.

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On complete multipartite derangement graphs*

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Abstract

Given a finite transitive permutation group $G \leq \mathrm{Sym}(\Omega)$, with $|\Omega| \geq 2$, the derangement graph Γ_G of G is the Cayley graph $\mathrm{Cay}(G, \mathrm{Der}(G))$, where $\mathrm{Der}(G)$ is the set of all derangements of G. Meagher et al. [On triangles in derangement graphs, *J. Combin. Theory Ser. A*, 180:105390, 2021] recently proved that $\mathrm{Sym}(2)$ acting on $\{1, 2\}$ is the only transitive group whose derangement graph is bipartite and any transitive group of degree at least three has a triangle in its derangement graph. They also showed that there exist transitive groups whose derangement graphs are complete multipartite.

This paper gives two new families of transitive groups with complete multipartite derangement graphs. In addition, we prove that if p is an odd prime and G is a transitive group of degree $2p$, then the independence number of Γ_G is at most twice the size of a point-stabilizer of G.

Keywords: Derangement graph, cocliques, Erdős-Ko-Rado theorem, Cayley graphs.

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1 Introduction

This paper is concerned with Erdős-Ko-Rado (EKR) type theorems for finite transitive groups. The classical EKR Theorem is stated as follows.

Theorem 1.1 (Erdős-Ko-Rado [9]). *Suppose that $n, k \in \mathbb{N}$ such that $2k \leq n$. If $\mathcal{F}$ is a family of k-subsets of $[n] := \{1, 2, \ldots, n\}$ such that $A \cap B \neq \varnothing$ for all $A, B \in \mathcal{F}$, then $|\mathcal{F}| \leq \binom{n-1}{k-1}$. Moreover, if $2k < n$, then equality holds if and only if $\mathcal{F}$ consists of all the k-subsets which contain a fixed element of $[n]$.*

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The EKR theorem has been well studied and generalized for numerous combinatorial objects in the past 50 years [6, 7, 10, 11, 13, 19, 20, 22, 25]. Of interest to us is the generalization of Theorem 1.1 for the symmetric group by Deza and Frankl in [10].

Given a finite transitive permutation group $G \leq \mathrm{Sym}(\Omega)$, we say that the permutations $\sigma, \pi \in G$ are *intersecting* if $\omega^\sigma = \omega^\pi$, for some $\omega \in \Omega$. A subset or family $\mathcal{F}$ of G is *intersecting* if any two permutations of $\mathcal{F}$ are intersecting.

Theorem 1.2 (Deza-Frankl, [10])**.** *Let Ω be a set of size $n \geq 2$. If $\mathcal{F} \subset \mathrm{Sym}(\Omega)$ is an intersecting family, then $|\mathcal{F}| \leq (n-1)!$.*

The characterization of the maximum intersecting families of $\mathrm{Sym}(\Omega)$ was solved almost three decades later by Cameron and Ku [6], and independently by Larose and Malvenuto [15].

Theorem 1.3 ([6, 15])**.** *Let Ω be a set of size $n \geq 2$. If $\mathcal{F} \subset \mathrm{Sym}(\Omega)$ is an intersecting family of maximum size, that is $|\mathcal{F}| = (n-1)!$, then $\mathcal{F}$ is a coset of a stabilizer of a point of $\mathrm{Sym}(\Omega)$. In particular, there exist $i, j \in \Omega$ such that*

$$\mathcal{F} = \{\sigma \in \mathrm{Sym}(\Omega) \mid i^\sigma = j\}.$$

The natural question that arises is whether analogues of Theorem 1.2 and Theorem 1.3 hold for different subgroups of $\mathrm{Sym}(\Omega)$, i.e., permutation groups of degree n. All groups considered in this paper are finite. We are interested in the following extremal problem.

Problem 1.4. Let $G \leq \mathrm{Sym}(\Omega)$ be transitive.

(1) What is the largest size of an intersecting family of G?

(2) If $\mathcal{F}$ is an intersecting family of G of maximum size, then describe the structure of $\mathcal{F}$.

Not surprisingly, the answer to this problem depends on the structure of the subgroup of $\mathrm{Sym}(\Omega)$. For instance, if $\sigma_1 = (1\ 2)(3\ 4)$, $\sigma_2 = (3\ 4)(5\ 6)$ and $\tau = (1\ 3\ 5)(2\ 4\ 6)$ are permutations of $\Omega = \{1, 2, 3, 4, 5, 6\}$, then $\langle \sigma_1, \sigma_2, \tau \rangle$ has its point-stabilizers of size 2 but

$$\mathcal{F} = \{id, (1\ 2)(3\ 4), (3\ 4)(5\ 6), (1\ 2)(5\ 6)\}$$

is a larger intersecting family. More examples of transitive permutation groups having larger intersecting families than point-stabilizers are given in [3, 16, 18]. Due to this, we consider the following definitions. We say that the group G has the *EKR property* if any intersecting family of G has size at most $\frac{|G|}{|\Omega|}$ and G has the *strict-EKR property* if it has the EKR property and an intersecting family of size $\frac{|G|}{|\Omega|}$ is a coset of a stabilizer of a point.

A typical approach in solving EKR-type problems is reducing it into a problem on a graph theoretical invariant. The *derangement graph* Γ_G of $G \leq \mathrm{Sym}(\Omega)$ is the graph whose vertex set is G and two permutations σ, π are adjacent if and only if they are not intersecting; that is, $\omega^\sigma \neq \omega^\pi$, for every $\omega \in \Omega$. In other words, Γ_G is the Cayley graph $\mathrm{Cay}(G, \mathrm{Der}(G))$, where $\mathrm{Der}(G)$ is the set of all derangements of G. Then, a family $\mathcal{F} \subset G$ is intersecting if and only if $\mathcal{F}$ is an *independent set* or a *coclique* of the derangement graph Γ_G. Therefore, Problem 1.4 is equivalent to finding the size of the maximum cocliques $\alpha(\Gamma_G)$ and the structures of the cocliques of size $\alpha(\Gamma_G)$.

Our long term objective is to classify the transitive permutation groups that have the EKR property and strict-EKR property. A big step toward this classification is the result of Meagher, Spiga and Tiep [20], which says that every finite 2-transitive group has the EKR property. More examples of primitive groups having the EKR property are given in [1, 2, 5, 8, 17, 19, 22].

We are motivated to find more transitive groups that do not have the EKR property. The group $\langle \sigma_1, \sigma_2, \tau \rangle$ given above is special in the sense that its derangement graph is a complete tri-partite graph. A recent result by Meagher, Spiga and the author [18] brought to light the existence of many transitive groups that do not have the EKR property. The most important of these are the transitive groups whose derangement graphs are complete multipartite graphs. If $G \leq \mathrm{Sym}(\Omega)$ is transitive and Γ_G is a complete multipartite graph, then it is easy to see that the part H of Γ_G, which contains the identity element id, consists of the elements with at least one fixed point. Moreover, every element of $G \setminus H$ is a derangement. Therefore, H is a maximum coclique of Γ_G and H is the union of all the point-stabilizers of G. Thus, G does not have the EKR property unless $H = \{id\}$. An important result on the structure of derangement graphs of transitive groups is given in the next theorem.

Theorem 1.5 ([18]). *Let $G \leq \mathrm{Sym}(\Omega)$ be transitive. Then, Γ_G is bipartite if and only if $|\Omega| \leq 2$. Further, if $|\Omega| \geq 3$, then Γ_G contains a triangle.*

Our motivation for this work is to find more transitive groups having complete multipartite derangement graphs. In this paper, we give two infinite families of transitive groups whose derangement graphs are complete multipartite. Our main results are stated as follows.

Theorem 1.6. *Let p be a prime and let $q = p^k$, for some $k \geq 1$. Then, there exists a transitive group G_q, of degree $q(q+1)$, such that Γ_{G_q} is a complete $(q+1)$-partite graph.*

The following was conjectured in [18] on the existence of complete multipartite derangement graphs.

Conjecture 1.7. *If n is even but not a power of 2, then there is a transitive group G of degree n such that Γ_G is a complete multipartite graph with $n/2$ parts.*

A transitive group of degree $n = 2\ell$, where ℓ is odd, with a complete ℓ-partite derangement graph was given in [18, Lemma 5.3]. We generalize this construction to find another family of transitive groups with complete multipartite derangement graphs. This result further reinforces Conjecture 1.7.

Theorem 1.8. *For any odd ℓ, there exists a transitive permutation group of degree 4ℓ whose derangement graph is a complete 2ℓ-partite graph.*

The *intersection density* $\rho(G)$ of a permutation group G was introduced in [16, 18] as the ratio between the size of the largest intersecting families of G and the size of the largest point-stabilizer of G. That is, if $G \leq \mathrm{Sym}(\Omega)$, then

$$\rho(G) := \frac{\max\{|\mathcal{F}| \ : \mathcal{F} \subset G \text{ is intersecting}\}}{\max_{\omega \in \Omega} |G_\omega|}. \tag{1.1}$$

For any $n \in \mathbb{N}$, we define $\mathcal{I}_n := \{\rho(G) \mid G \text{ is transitive of degree } n\}$ and $I(n) := \max \mathcal{I}_n$. The following was conjectured in [18].

Conjecture 1.9 ([18]). (1) *If $n = pq$ where p and q are distinct odd primes, then $I(n) = 1$.*

(2) *If $n = 2p$ where p is prime, then $I(n) = 2$.*

In this paper, we also prove that Conjecture 1.9(2) holds.

Theorem 1.10. *If p is an odd prime, then $I(2p) = 2$.*

This paper is organized as follows. In Section 2, we give some background results on complete multipartite derangement graphs and some properties of the intersection density of transitive groups. In Section 3, Section 4, and Section 5, we give the proof of Theorem 1.6, Theorem 1.8, and Theorem 1.10, respectively.

2 Background

Throughout this section, we let $G \leq \mathrm{Sym}(\Omega)$ be a transitive group and $|\Omega| = n$.

2.1 Bound on maximum cocliques

We recall that the problem of finding the size of the maximum intersecting families of G is equivalent to finding the size of the maximum cocliques of Γ_G. We give a classical upper bound on the size of the largest cocliques in vertex-transitive graphs (i.e., graphs whose automorphism groups act transitively on their vertex sets). As the derangement graph of an arbitrary finite permutation group is a Cayley graph, the right-regular representation of G acts regularly on $V(\Gamma_G)$. In other words, Γ_G is vertex transitive.

Lemma 2.1 ([13]). *If $X = (V, E)$ is a vertex-transitive graph, then $\alpha(X) \leq \frac{|V(X)|}{\omega(X)}$. Moreover, equality holds if and only if a maximum coclique of X intersects each maximum clique at exactly one vertex.*

Lemma 2.1 can be used to prove the EKR property of groups. For instance, one can prove that $\mathrm{Sym}(n)$, for $n \geq 3$, has the EKR property [6, 10, 12] by showing first that $\omega(\Gamma_{\mathrm{Sym}(n)}) = n$ (a clique of $\Gamma_{\mathrm{Sym}(n)}$ is induced by a Latin square of size n) and applying Lemma 2.1. A subset $S \subset G$ with $|S| = n$ that forms a clique in Γ_G is called a *sharply 1-transitive set*. It is well-known that a transitive group need not have a sharply 1-transitive set. Therefore, Lemma 2.1 does not hold with equality for the derangement graphs of many transitive groups.

2.2 Intersection density

By (1.1), the intersection density of the transitive group G is the rational number

$$\rho(G) := \frac{\max |\{\mathcal{F} \subseteq G \mid \mathcal{F} \text{ is intersecting}\}|}{|G_\omega|},$$

where $\omega \in \Omega$.

The major result in [18] (see also Theorem 1.5) asserts that the intersection density of the transitive group G cannot be equal to $\frac{n}{2}$. This is equivalent to saying that the derangement graph of transitive groups cannot be bipartite if $n \geq 3$ (see [18]). It is also proved in [18] that for any transitive group K of degree n, $\rho(K)$ is in the interval $\left[1, \frac{n}{3}\right]$. We note that $\rho(K) = 1$ if and only if K has the EKR property. Moreover, the upper bound $\frac{n}{3}$ is sharp since there are transitive groups whose derangement graphs are complete tri-partite graphs [18, Theorem 5.1]. It is conjectured that the only transitive groups that attain the upper bound are those with complete tri-partite derangements graphs.

The study of the intersection density (see [16, 18]) of a transitive group was mainly motivated by studying how far from having the EKR property a transitive group can be. The intersection density, therefore, is a measure of the EKR property for transitive groups.

We make the following conjecture based on computer search using `Sagemath` [23].

Conjecture 2.2. *For any $n \geq 3$, almost all elements of the set $\mathcal{I}_n$ are integers. That is,*

$$\frac{|\{\rho(G) \mid \ G \text{ is transitive of degree } n\} \cap \mathbb{N}|}{|\mathcal{I}_n|} \xrightarrow[n\to\infty]{} 1.$$

Note that the intersection density of a transitive group can be non-integer. For example, the transitive groups of degree n and number k in the `TransitiveGroup` function of `Sagemath`, with $(n, k) \in \{(12, 122), (12, 93)\}$, have non-integer intersection densities. `TransitiveGroup`(12,122) and `TransitiveGroup`(12,93) have intersection density equal to $\frac{3}{2}$ and $\frac{17}{16}$, respectively.

Proposition 2.3. *If the derangement Γ_G has a clique of size k, then $\rho(G) \leq \frac{n}{k}$.*

Proof. The proof follows by applying Lemma 2.1. □

2.3 Complete multipartite derangement graphs

The transitive groups with complete multipartite derangement graphs are the most natural examples of groups that do not have the EKR property. In this subsection, we give some properties of transitive groups whose derangement graphs are complete multipartite.

The following lemma is a straightforward observation on the intersecting subgroups of G.

Lemma 2.4 ([16, 18])**.** *Let $G \leq \mathrm{Sym}(\Omega)$ and let $H \leq G$. Then, H is intersecting if and only if H does not have any derangement.*

The next lemma illustrates that transitive groups with complete multipartite derangement graphs have a very distinct algebraic structure.

Lemma 2.5 ([18])**.** *If $G \leq \mathrm{Sym}(\Omega)$ is transitive such that Γ_G is a complete multipartite graph, then G is imprimitive.*

A transitive group whose derangement graph is a complete multipartite graph is uniquely determined by a specific subgroup of G. We define $\mathrm{F}(G)$ to be the subgroup of G generated by all the permutations of G with at least one fixed point. That is,

$$\mathrm{F}(G) := \left\langle \bigcup_{\omega\in\Omega} G_\omega \right\rangle.$$

Proposition 2.6. *The subgroup $\mathrm{F}(G)$ is a normal subgroup of G.*

Proof. The proof follows from the fact that $\mathrm{F}(G)$ is generated by all point-stabilizers. □

Note that Lemma 2.5 follows from the normality of $\mathrm{F}(G)$ as its orbits form a non-trivial system of imprimitivity of G acting on Ω.

A characterization of transitive groups with complete multipartite derangement graphs is given in the next lemma.

Lemma 2.7 ([18]). *Let $G \leq \mathrm{Sym}(\Omega)$ be transitive. The graph Γ_G is complete multipartite if and only if $\mathrm{F}(G)$ is intersecting. Moreover, if Γ_G is a complete multipartite graph, then the number of parts of Γ_G is $[G : \mathrm{F}(G)]$.*

Suppose that Γ_G is a complete multipartite graph. When the subgroup $\mathrm{F}(G)$ is the trivial group $\{id\}$, then Γ_G is the complete multipartite graph that has $|G|$ parts of size 1. In other words, Γ_G is the complete graph $K_{|G|}$. When $\mathrm{F}(G) = G$, then $\mathrm{F}(G)$ cannot be intersecting since by Lemma 2.4, this would contradict the celebrated theorem of Jordan [14, 21] on the existence of derangements in finite transitive groups. Hence, we say that Γ_G is a *non-trivial complete multipartite graph* if $1 < |\mathrm{F}(G)| < |G|$. In this paper, we are only interested in transitive groups with non-trivial complete multiplartite derangement graphs.

Next, we study the structure of $\mathrm{F}(G)$. If $\mathrm{F}(G)$ is intersecting, then by Lemma 2.4, $\mathrm{F}(G)$ is derangement-free. Thus,

$$\mathrm{F}(G) = \bigcup_{\omega \in \Omega} G_\omega.$$

Recall that if $K \leq \mathrm{Sym}(\Omega)$ and $\omega \in \Omega$, then the orbit of K containing ω is denoted by ω^K. Moreover, if $S \subset \Omega$, then the setwise stabilizer of S in K is denoted by $K_{\{S\}}$.

The following lemma is a standard result in the theory of permutation groups.

Lemma 2.8. *Let $G \leq \mathrm{Sym}(\Omega)$ and $\omega \in \Omega$. If H is a non-trivial subgroup of G containing G_ω, then $G_{\{\omega^H\}} = H$.*

Corollary 2.9. *Let $G \leq \mathrm{Sym}(\Omega)$ be transitive and let K be the subgroup of G fixing the system of imprimitivity $\left\{\omega^{\mathrm{F}(G)} \mid \omega \in \Omega\right\}$. Then $K = \mathrm{F}(G)$.*

Proof. Since $\mathrm{F}(G)$ is generated by the point-stabilizers, by the previous lemma, we have

$$K = \bigcap_{\omega \in \Omega} G_{\{\omega^{\mathrm{F}(G)}\}} = \bigcap_{\omega \in \Omega} \mathrm{F}(G) = \mathrm{F}(G).$$

□

Remark 2.10. A representation of the derangement graph of the transitive group G as a complete mutlipartite graph is unique. This is due to the fact that the part of Γ_G, which contains the identity element, must be equal to $F(G)$.

3 Proof of Theorem 1.6

In this section, we describe the action of $\mathrm{AGL}(2, q)$ on the lines and give some basic results. Then, we prove Theorem 1.6.

3.1 An action of $\mathrm{AGL}(2, q)$ on the lines

Let $q = p^k$ be a prime power, where $k \geq 1$. For $b \in \mathbb{F}_q^2$ and $A \in \mathrm{GL}(2, q)$, we let $(b, A) : \mathbb{F}_q^2 \to \mathbb{F}_q^2$ be the affine transformation such that $(b, A)(v) := Av + b$. The *affine group* $\mathrm{AGL}(2, q)$ is the permutation group

$$\left\{(b, A) \mid A \in \mathrm{GL}(2, q),\ b \in \mathbb{F}_q^2\right\},$$

with the multiplication $(a, A)(b, B) := (a + Ab, AB)$.

Hence, $\mathrm{AGL}(2, q)$ acts naturally on the vectors of $\mathbb{F}_q^2$. This action induces an action of $\mathrm{AGL}(2, q)$ on the set Ω of all lines of $\mathbb{F}_q^2$ (i.e., the collection of all sets of the form $L_{u,v} := \{u + tv \mid t \in \mathbb{F}_q\}$, where $u, v \in \mathbb{F}_q^2$ and $v \neq 0$). Recall that $\mathrm{PG}(1, \mathbb{F}_q) := \mathrm{PG}(1, q)$ is the set of all 1-dimensional subspaces of the $\mathbb{F}_q$-vector space $\mathbb{F}_q^2$. The elements of $\mathrm{PG}(1, q)$ are exactly the lines containing $0 \in \mathbb{F}_q^2$. By a simple counting argument, each vector of $\mathbb{F}_q^2 \setminus \{0\}$ determines a line, and each line passing through 0 has $q - 1$ points (excluding 0). So there are $\frac{q^2-1}{q-1} = q + 1$ subspaces in $\mathrm{PG}(1, q)$. For any line $\ell \in \mathrm{PG}(1, q)$, we define $\Omega_\ell := \left\{\ell + b \mid b \in \mathbb{F}_q^2\right\}$. The set Ω_ℓ consists of $\mathbb{F}_q^2$-shifts of the 1-dimensional subspace ℓ, thus its elements are affine lines of $\mathbb{F}_q^2$ that are parallel to ℓ. Therefore, $\Omega := \bigcup_{\ell \in \mathrm{PG}(1,q)} \Omega_\ell$ is exactly the set of lines of $\mathbb{F}_q^2$. Note that we can also view Ω as the lines of the incidence structure $\left(\mathbb{F}_q^2, L, \sim\right)$, where $L = \{L_{u,v} \mid u, v \in \mathbb{F}_q^2,\ v \neq 0\}$ and $v \sim \ell$, for $v \in \mathbb{F}_q^2$ and $\ell \in L$, if and only if $v \in \ell$. This incidence structure is the affine plane $\mathrm{AG}(2, q)$.

As $\mathrm{GL}(2, q)$ acts transitively on $\mathrm{PG}(1, q)$, it is easy to see that $\mathrm{AGL}(2, q)$ acts transitively on Ω. Since the elements of $\mathrm{GL}(2, q) \leq \mathrm{AGL}(2, q)$ leave $\mathrm{PG}(1, q)$ invariant, for any $\ell \in \mathrm{PG}(1, q)$, the set Ω_ℓ is either invariant by the action of an element of $\mathrm{AGL}(2, q)$ or is mapped to some other $\Omega_{\ell'}$, where $\ell' \in \mathrm{PG}(1, q) \setminus \{\ell\}$. That is, Ω_ℓ is a block for the action of $\mathrm{AGL}(2, q)$ on Ω. Therefore, $\mathrm{AGL}(2, q)$ acts imprimitively on Ω.

As the elements of $\mathrm{AGL}(2, q)$ are affine transformations, the pair of parallel lines $(l, l') \in \Omega_\ell \times \Omega_\ell$ can be mapped by $\mathrm{AGL}(2, q)$ to any other pair of parallel lines. However, if $(l, l') \in \Omega_\ell \times \Omega_{\ell'}$, for distinct $\ell, \ell' \in \mathrm{PG}(1, q)$, then no element of $\mathrm{AGL}(2, q)$ can map (ℓ, ℓ') to a pair of parallel lines. In addition, one can prove that any pair of non-parallel lines can be mapped to any other pair of non-parallel lines. In other words, $\mathrm{AGL}(2, q)$ acting on Ω^2 has exactly 3 orbits. We formulate this result as the following lemma.

Lemma 3.1. *The group* $\mathrm{AGL}(2, q)$ *acting on* Ω *is a rank 3 imprimitive group.*

3.2 Action of Singer subgroups of $\mathrm{GL}(2, q)$ as subgroups of $\mathrm{AGL}(2, q)$

We recall that for $n \geq 1$, $\mathrm{GL}(n, q)$ admits elements of order $q^n - 1$. These elements are called *Singer cycles*, and a subgroup of order $q^n - 1$ generated by a Singer cycle is called a *Singer subgroup.* We recall the following observation about Singer cycles.

Proposition 3.2. *If A is a Singer cycle of* $\mathrm{GL}(2, q)$*, then the subgroup $\langle A \rangle$ acts regularly on $\mathbb{F}_q^2 \setminus \{0\}$.*

For any matrix $C \in \mathrm{GL}(2, q)$, we define

$$G_q(C) := \left\{(b, B) \mid B \in \langle C \rangle,\ b \in \mathbb{F}_q^2\right\}.$$

Now, let A be an arbitrary Singer cycle of $\mathrm{GL}(2, q)$. By Proposition 3.2, it is easy to see that the action of

$$H_q := \{(0, B) \in \mathrm{AGL}(2, q) \mid B \in \langle A \rangle\}$$

on $\mathrm{PG}(1,q)$ is transitive. The latter implies that the action of the subgroup $G_q(A)$ on Ω is transitive. To see this, let $\ell = \ell_0 + b$ and $\ell' = \ell'_0 + b'$ be two lines in Ω such that ℓ_0 and ℓ'_0 are 1-dimensional subspaces and $b, b' \in \mathbb{F}_q^2$. By transitivity of H_q on $\mathrm{PG}(1,q)$, there exists $(0,B) \in H_q$ such that $(0,B)(\ell_0) = \ell'_0$. Hence,

$$(b' - Bb, B)(\ell) = (b' - Bb, B)(\ell_0 + b) = B\ell_0 + Bb + b' - Bb = \ell'_0 + b' = \ell'.$$

Thus, $G_q(A)$ is transitive. It is straightforward to verify that for any $\ell \in \mathrm{PG}(1,q)$, Ω_ℓ is a block of $G_q(A)$. Therefore, we have the following.

Proposition 3.3. *The group $G_q(A)$ acts imprimitively on Ω and Ω_ℓ is a block of $G_q(A)$, for any $\ell \in \mathrm{PG}(1,q)$.*

3.3 Kernel of the action of $G_q(A)$

In this subsection, we study the kernel of the action of $G_q(A)$ on the system of imprimitivity $\{\Omega_\ell \mid \ell \in \mathrm{PG}(1,q)\}$.

To avoid any confusion, we use the notation $\mathrm{Stab}_{G_q(A)}(l)$ in the remainder of Section 3 to denote the point-stabilizer of $\ell \in \Omega$ in $G_q(A)$, instead of the standard notation used in the theory of permutation groups. Similarly, for any $S \subset \Omega$, we use the notation $\mathrm{Stab}(G_q(A), S)$ for the setwise stabilizer of S in $G_q(A)$.

By Lemma 3.1, the action of $\mathrm{AGL}(2,q)$ on Ω has a unique system of imprimitivity, namely the set $\{\Omega_\ell \mid \ell \in \mathrm{PG}(1,q)\}$. Define

$$M_q := \bigcap_{\ell \in \mathrm{PG}(1,q)} \mathrm{Stab}(G_q(A), \Omega_\ell).$$

We prove the following lemma.

Lemma 3.4. *The affine transformation $(b,B) \in M_q$ if and only if there exists $k \in \mathbb{F}_q^*$ such that $B = kI$, where I is the 2×2 identity matrix.*

Proof. It is easy to see that if $B = kI$, for some $k \in \mathbb{F}_q^*$, then $(0,B)$ fixes every element of $\mathrm{PG}(1,q)$. Therefore, (b,B) leaves Ω_ℓ invariant for any $\ell \in \mathrm{PG}(1,q)$.

If $(b,B) \in M_q$, then $(0,B)$ fixes every element of $\mathrm{PG}(1,q)$. In particular, there exists $k_1, k_2 \in \mathbb{F}_q^*$ such that

$$(0,B)\begin{bmatrix}1\\0\end{bmatrix} = B\begin{bmatrix}1\\0\end{bmatrix} = k_1\begin{bmatrix}1\\0\end{bmatrix}, \text{ and } \quad (0,B)\begin{bmatrix}0\\1\end{bmatrix} = B\begin{bmatrix}0\\1\end{bmatrix} = k_2\begin{bmatrix}0\\1\end{bmatrix}.$$

Therefore, the matrix $B = \mathrm{diag}(k_1, k_2)$. The 1-dimensional subspace generated by the vector $u = \begin{bmatrix}1\\1\end{bmatrix}$ forces $k_1 = k_2$, since $Bu = ku$ for some $k \in \mathbb{F}_q^*$. Hence $B = kI$. □

We present an immediate corollary of this.

Corollary 3.5. *The subgroup M_q of $G_q(A)$ is intersecting.*

Proof. It suffices to prove that any element of M_q has a fixed point. Let $(b, kI_2) \in M_q$. If $k = 1$, then it is obvious that (b, I) fixes every line in the block Ω_ℓ, where $\ell \in \mathrm{PG}(1,q)$ such that $b \in \ell$.

If $k \neq 1$, then we prove that there exist $\beta \in \mathbb{F}_q^2$ such that for any $\ell \in \mathrm{PG}(1,q)$, (b, kI) fixes the line $\ell + \beta$. If (b, kI) fixes this line, then we must have

$$\begin{aligned}(b, kI)(\ell + \beta) &= k\ell + k\beta + b \\ &= \ell + k\beta + b = \ell + \beta.\end{aligned}$$

In other words, we should find β such that $(1-k)\beta - b \in \ell$, for any $\ell \in \mathrm{PG}(1,q)$. For $\beta = (1-k)^{-1}b$, we have $(1-k)\beta - b = 0 \in \ell$. Moreover, the solution $\beta = (1-k)^{-1}b$ does not depend on ℓ since every element of $\mathrm{PG}(1,q)$ contains 0.

We conclude that when $k = 1$, then (b, kI) fixes every line of the block Ω_ℓ, with $b \in \ell$ and if $k \neq 1$, then (b, kI) fixes every line of the form $\ell + (1-k)^{-1}b \in \Omega$, for any $\ell \in \mathrm{PG}(1,q)$. □

We prove the following lemma about the relation between the kernel of the action of $G_q(A)$ on $\{\Omega_\ell \mid \ell \in \mathrm{PG}(1,q)\}$ and the subgroup $\mathrm{F}(G_q(A))$ generated by the non-derangements of $G_q(A)$.

Lemma 3.6. *The subgroup* $\mathrm{F}(G_q(A))$ *is equal to* M_q.

Proof. Let $(b, kI) \in M_q$. In the proof of Corollary 3.5, we showed that a transformation of (b, kI) either fixes every element of Ω_ℓ, for some $\ell \in \mathrm{PG}(1,q)$, or it fixes exactly one line in each Ω_ℓ. Therefore, $M_q \leq \mathrm{F}(G_q(A))$.

Next, we will prove that the point-stabilizer $\mathrm{Stab}_{G_q(A)}(\ell)$ of ℓ in $G_q(A)$ is a subgroup of M_q, for any $\ell \in \Omega$. First, let $\ell \in \mathrm{PG}(1,q)$ be the line that contains the point $\begin{bmatrix}1\\0\end{bmatrix} \in \mathbb{F}_q^2$. Observe that for $b \in \mathbb{F}_q^2$, $\ell + b = \ell$ if and only if $b \in \ell$. Therefore, the affine transformation $(b, kI) \in \mathrm{Stab}_{G_q(A)}(\ell)$, for any $b \in \ell$ and $k \in \mathbb{F}_q^*$. There are $q(q-1)$ affine transformations of this form in $\mathrm{Stab}_{G_q(A)}(\ell)$. Arguing by the size of the stabilizer of ℓ in $G_q(A)$, we have

$$|\mathrm{Stab}_{G_q(A)}(\ell)| = \frac{q^2(q^2-1)}{q(q+1)} = q(q-1).$$

We conclude that the point-stabilizer of ℓ in $G_q(A)$ is

$$\mathrm{Stab}_{G_q(A)}(\ell) = \left\{(b, B) \in G_q \mid B = kI,\ b \in \ell,\ k \in \mathbb{F}_q^*\right\} \leq M_q.$$

Since $M_q \lhd G_q(A)$ and $G_q(A)$ is transitive, we have $\mathrm{Stab}_{G_q(A)}(\ell) \leq M_q$ for any $\ell \in \Omega$. Therefore, $\mathrm{F}(G_q(A)) \leq M_q$. This completes the proof. □

3.4 Proof of Theorem 1.6

We prove that the derangement graph $\Gamma_{G_q(A)}$ of $G_q(A)$ is a complete $(q+1)$-partite graph. By Corollary 3.5, M_q is intersecting, and by Lemma 3.6, we have $M_q = \mathrm{F}(G_q(A))$. Therefore, $\Gamma_{G_q(A)}$ is a complete k-partite graph, where $k = [G_q(A) : M_q] = \frac{q^2(q^2-1)}{q^2(q-1)} = q+1$.

Note that Lemma 3.6 is crucial to our proof. Indeed, the subgroup generated by the permutations with fixed points in $\mathrm{AGL}(2,q)$ acting on Ω, i.e., $\mathrm{F}(\mathrm{AGL}(2,q))$, is the whole of $\mathrm{AGL}(2,q)$; whereas the stabilizer of its unique system of imprimitivity is the proper subgroup M_q.

4 Proof of Theorem 1.8

We will construct a transitive permutation group G of degree $n = 4\ell$ acting on $[n]$, where ℓ is an odd natural number. The derangement Γ_G of this group G will be a complete multipartite graph with $\frac{n}{2}$ parts. The group G that we will construct is isomorphic to

$$\underbrace{(C_2 \times C_2 \times \ldots \times C_2)}_{\ell-1} \rtimes D_\ell,$$

where D_ℓ is the dihedral group of order 2ℓ.

4.1 Kernel of the action

We would like to construct G so that it will have a system of imprimitivity

$$\mathcal{B} = \{\{i, i+1\} \mid \text{ for } i \in [n] \cap (2\mathbb{Z}+1)\}.$$

For any $i, j \in (4\mathbb{Z}+1) \cap [n]$, define $\sigma_i := (i \;\; i+1)(i+2 \;\; i+3)$ and $\pi_j := \sigma_j \sigma_{4\ell-3}$. Let $S = \{\pi_j \mid j \in (4\mathbb{Z}+1) \cap [n]\}$. Notice that $|S| = \ell$, however, $\pi_{4\ell-3} = id \in S$. We consider the permutation group $H = \langle S \rangle$. It is easy to see that

$$H \cong \underbrace{C_2 \times C_2 \times \ldots \times C_2}_{\ell-1}.$$

Moreover, for any fixed $k \in [n]\cap(4\mathbb{Z}+1)$, any subset of the form $\{\sigma_i\sigma_k \mid i \in [n] \cap (4\mathbb{Z}+1)\}$ generates H.

A permutation of H either fixes, pointwise, an element of $\mathcal{B}$ or interchanges the pair of elements in a set of $\mathcal{B}$. Therefore, H leaves $\mathcal{B}$ invariant. Any $g \in H$ can be written in the form

$$g = \prod_{j \in [n]\cap(4\mathbb{Z}+1)} \pi_j^{k_j}, \tag{4.1}$$

for some $k_j \in \{0, 1\}$. Since $\pi_{4\ell-3} = id$, there are at most $\ell - 1$ (which is even) permutations of the form π_j in the expression of g in (4.1). If the number of non-identity terms in (4.1) is even, then g fixes the points $4\ell-3,\ 4\ell-2,\ 4\ell-1,$ and 4ℓ. If the number of non-identity terms in (4.1) is odd, then there exists $j \in [n]\cap(4\mathbb{Z}+1)$, $j \neq 4\ell-3$, such that $k_j = 0$ (because $\ell - 1$ is even). Therefore, g fixes the elements $j,\ j+1,\ j+2,$ and $j+4$. We conclude that

$$H \text{ is an intersecting subgroup of degree } n = 4\ell. \tag{4.2}$$

The group G will be defined so that $H = \mathrm{F}(G)$.

4.2 Action of a dihedral group on H

First, we give a permutation c, which is a product of four disjoint ℓ-cycles. Then, we construct a transposition τ so that $\tau c \tau^{-1} = c^{-1}$. In other words, $\langle c, \tau\rangle = D_\ell$. This subgroup will act on H so that $\langle H, c, \tau\rangle$ is transitive.

For any $i \in \mathbb{Z}$, define $A_i := (i \;\; i{+}4 \;\; \ldots \;\; i{+}4k \;\; \ldots \;\; i{+}4(\ell{-}1))$ to be the permutation of order ℓ, whose entries in the cycle notation are those of an arithmetic progression of step 4, and with initial value i. Let

$$c := A_1 A_2 A_3 A_4.$$

We note that $A_1, A_2, A_3,$ and A_4 are pairwise disjoint ℓ-cycles. Consider the permutation

$$\tau := (1\ 3)(2\ 4) \prod_{i\in\{1\ 2\ \dots\ \ell-1\}} (1+4i\ \ 3+4(\ell-i))\,(2+4i\ \ 4+4(\ell-i))\,.$$

The transpositions in the expression of τ are also pairwise disjoint. Moreover, τ is a derangement of $\mathrm{Sym}(n)$. The following conditions are satisfied by τ

$$\begin{aligned} \tau A_1 \tau^{-1} &= A_3^{-1},\\ \tau A_2 \tau^{-1} &= A_4^{-1},\\ \tau A_3 \tau^{-1} &= A_1^{-1},\\ \tau A_4 \tau^{-1} &= A_2^{-1}. \end{aligned} \tag{4.3}$$

From (4.3), we deduce that $\tau c \tau^{-1} = c^{-1}$. We conclude that $\langle \tau, c\rangle \cong D_\ell$.

Next, we see how the subgroup $\langle c, \tau\rangle$ acts on H. For $i \in [n]\cap(4\mathbb{Z}+1)$ with $i \neq 4\ell-3$, we have

$$\nu_i := c\pi_i c^{-1} = c\sigma_i\sigma_{4\ell-3}c^{-1} = \sigma_{i+4}\sigma_1.$$

Since $\{\nu_i \mid i \in [n]\cap(4\mathbb{Z}+1)\}$ also generates H, we conclude that $cHc^{-1} = H$. In addition, for any $i \in [n]\cap(4\mathbb{Z}+1)$, we have

$$\mu_i := \tau\pi_i\tau^{-1} = \tau\sigma_i\sigma_{4\ell-3}\tau^{-1} = \sigma_{\tau(i+2)}\sigma_5.$$

Since $\{\mu_i \mid i \in [n]\cap(4\mathbb{Z}+1)\}$ also generates H, we have $\tau H\tau^{-1} = H$.

We conclude that $G := H\langle\tau, c\rangle$ is a permutation group of degree 4ℓ. In addition, it is easy to see that $H\cap\langle\tau, c\rangle = \{id\}$, so we have $G = H \rtimes \langle\tau, c\rangle$. Furthermore, G is transitive because

(1) the orbits of $H\langle c\rangle$ are $\{1+4i \mid i \in \{0,1,2,\dots,\ell-1\}\}\cup\{2+4i \mid i \in \{0,1,2,\dots,\ell-1\}\}$ and $\{3+4i \mid i \in \{0,1,2,\dots,\ell-1\}\}\cup\{4+4i \mid i \in \{0,1,2,\dots,\ell-1\}\}$, and

(2) the orbits of $\langle\tau\rangle$ are the sets of the form $\{1+4i, 3+4(\ell-i)\}$, $\{2+4i, 4+4(\ell-i)\}$ where $i \in \{0,1,\dots,\ell-1\}$, $\{2,4\}$, and $\{1,3\}$.

4.3 Derangement graph of *G*

The derangement graph of G is a complete multipartite graph with 2ℓ parts. To prove this, we need to show that H is intersecting and $\mathrm{F}(G) = H$. We only need to prove the latter since H is an intersecting subgroup (see (4.2)).

On one hand, as the elements of S all have fixed points, it is easy to see that $\langle S\rangle = H \leq \mathrm{F}(G)$. On the other hand, the subgroup $K = \langle \pi_5, \pi_9, \dots, \pi_{4i+1}, \dots, \pi_{4\ell-7}\rangle \leq H$ fixes 1; that is, $K \leq G_1$. Since $|K| = 2^{\ell-2}$ and $|G_1| = \frac{|G|}{4\ell} = 2^{\ell-2}$, we conclude that $G_1 = K \leq H$. As G is transitive, the point-stabilizers of G are conjugate. Moreover, since $H \lhd G$ (because $G = H \rtimes \langle\tau, c\rangle$) and $G_1 \leq H$, we can conclude that $G_i \leq H$, for any $i \in [n]$. Therefore, $\mathrm{F}(G) \leq H$.

In conclusion, we know that $\mathrm{F}(G) = H$ is intersecting. This is equivalent to Γ_G being a complete multipartite graph, with $[G : H] = 2\ell$ parts.

5 Proof of Theorem 1.10

We will prove that every transitive group of degree $2p$, for any odd prime p, has intersection density at most 2 (Theorem 1.10) by showing that there is a clique of size p in the derangement graph of G. In this case, we have $\rho(G) \leq \frac{|\Omega|}{p} = 2$. Therefore, $1 \leq \rho(G) \leq 2$ for any transitive group G of degree $2p$. It is proved in [18, Lemma 5.3] that for any odd ℓ, there is a transitive group of degree 2ℓ, whose intersection density is 2. Therefore, we will have $I(2p) = 2$, for any odd prime p.

As $p \mid |G|$, by Cauchy's theorem, there exists $\sigma \in G$ whose order is p. Therefore, σ is either a p-cycle or the product of two disjoint p-cycles. If the latter holds, then σ is a derangement of G and $\langle \sigma \rangle$ is then a clique of size p in Γ_G. So, we may suppose that σ is a p-cycle.

5.1 Imprimitive case

Since $G \leq \mathrm{Sym}(\Omega)$ is imprimitive of degree $2p$, a non-trivial block of imprimitivity of G has size 2 or p. Assume that

$$\sigma = (x_1 \ x_2 \ x_3 \ \ldots \ x_p).$$

As p is an odd prime and $\sigma \in G$, it is easy to see that G cannot have a system of imprimitivity consisting of sets of size 2. We suppose that G has a set of imprimitivity $\mathcal{Q}$ consisting of two subsets of size p of Ω. It is easy to see that σ cannot interchange the two blocks of $\mathcal{Q}$ since the support of σ only has p elements. Thus, σ is in the setwise stabilizer of $\mathcal{Q}$. Suppose that $\mathcal{Q} = \{B, B'\}$, where $B = \{x_1, x_2, \ldots, x_p\}$ and $B' = \{y_1, y_2, \ldots, y_p\}$. As G_{y_1} and G_{x_1} are conjugate, there exists an element $\sigma' \in G_{x_1}$, which is a p-cycle. As σ' is a p-cycle, it must fix B pointwise and act as a p-cycle on B'.

We conclude that the permutation $\sigma\sigma' \in G$ is a product of two disjoint p-cycles. The subgroup $\langle \sigma\sigma' \rangle$ is a clique of size p of Γ_G.

5.2 Primitive case

Suppose that $G \leq \mathrm{Sym}(\Omega)$ is primitive of degree $2p$. We derive the result of Theorem 1.10 from the following lemma.

Lemma 5.1 ([24]). *Suppose that p is an odd prime. A primitive group of degree $2p$ is either 2-transitive or every non-identity element of a Sylow p-subgroup of G is a product of two disjoint p-cycles.*

By Lemma 5.1, we conclude that G is 2-transitive or G contains a derangement of order p. Hence, either G has the EKR property [20] (in which case $\rho(G) = 1$) or $\rho(G) \leq 2$.

This completes the proof of Theorem 1.10.

6 Further work

We finish this paper by posing some open questions. In Section 5, we proved that for any odd prime p, a transitive group G of degree $2p$ has intersection density $1 \leq \rho(G) \leq 2$. It follows from the classification of finite simple groups that the only simply primitive groups (i.e., primitive groups that are not 2-transitive) of degree $2p$ are $\mathrm{Alt}(5)$ and $\mathrm{Sym}(5)$, both of degree 10. Using `Sagemath` [23], the largest intersecting family of $\mathrm{Alt}(5)$ is of size

12, whereas its stabilizer of a point has size 6. The largest intersecting family of $\mathrm{Sym}(5)$ is 12, which equals the size of its point-stabilizers. We conclude that the group $\mathrm{Alt}(5)$ of degree 10 has the largest intersection density among all primitive groups of degree $2p$, for every odd prime p.

For the imprimitive case, there are infinitely many examples of transitive groups with intersection density equal to 2. In [18, Lemma 5.3], the authors gave a family of transitive groups of degree 2ℓ, for any odd ℓ, whose derangement graphs are ℓ-partite and whose intersection densities are equal to 2. Based on a non-exhaustive search on the small transitive groups of degree $2p$ (where p is an odd prime) available on `Sagemath`, we are inclined to believe that the intersection density of a transitive group of degree $2p$, where p is an odd prime, is an integer. We ask the following question.

Question 6.1. Does there exist an odd prime p and a transitive group G of degree $2p$ such that $\rho(G)$ is not an integer?

In Theorem 1.8, we proved that there exists a family of transitive groups of degree 4ℓ, for any odd ℓ, with complete 2ℓ-partite derangement graphs. This further confirms the validity of [18, Conjecture 6.6 (1)] (see also Conjecture 1.7) about the existence of transitive groups of any degree n which is even but not a power of 2, with a complete $\frac{n}{2}$-partite derangement graph.

Problem 6.2. For any odd ℓ and an integer $i \geq 3$, find a transitive group of degree $2^i\ell$ whose derangement graph is a complete $2^{i-1}\ell$-partite graph.

In Section 3, we gave an example of a transitive group of degree $q(q+1)$, where q is a prime power, whose intersection density is equal q. A non-exhaustive search on small transitive groups of degree $q(q+1)$, which are available on `Sagemath`, shows that the largest intersection density for these groups is q. We ask the following question.

Question 6.3. Does there exist a transitive group G of degree $q(q+1)$, where q is a prime power, such that $\rho(G) > q$?

Our motivation to work on the EKR property for the transitive group in Section 3 comes from studying the EKR property for $\mathrm{AGL}(2,q)$ acting on the lines of $\mathrm{AG}(2,q)$ (see Section 3), where q is a prime power. Observe that if H and G are transitive permutation groups acting on Ω and $H \leq G$, then Γ_H is an induced subgraph of Γ_G. Using the No-Homomorphism Lemma [4], one can prove that $\alpha(\Gamma_G) \leq \alpha(\Gamma_H)\frac{|G|}{|H|}$. We deduce from this inequality that if H has the EKR property, then so does G. Moreover, $\rho(G) \leq \rho(H)$.

Recall that the subgroup $G_q(A)$ defined in Section 3 is a subgroup of $\mathrm{AGL}(2,q)$ acting on the lines of $\mathrm{AG}(2,q)$. Using the result from the previous paragraph, we know that

$$\rho(\mathrm{AGL}(2,q)) \leq \rho(G_q(A)) = \frac{q^2(q-1)}{q(q-1)} = q,$$

where q is a prime power and A is a Singer cycle of $\mathrm{GL}(2,q)$. However, we believe that this bound is not sharp. Indeed, from the observation of the behavior of the intersection density of $\mathrm{AGL}(2,q)$ ($q \in \{3,4,5,7,8\}$) acting on the lines of $\mathrm{AG}(2,q)$, we make the following conjecture.

Conjecture 6.4. *For any $\varepsilon > 0$, there exists a prime power q_0, such that for any prime power $q \geq q_0$, $0 \leq \rho(\mathrm{AGL}(2,q)) - 1 \leq \varepsilon$. In particular, $\rho(\mathrm{AGL}(2,q)) \in \mathbb{Q} \setminus \mathbb{N}$, for any prime power q.*

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On 2-closures of rank 3 groups

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Abstract

A permutation group G on Ω is called a *rank* 3 group if it has precisely three orbits in its induced action on $\Omega \times \Omega$. The largest permutation group on Ω having the same orbits as G on $\Omega \times \Omega$ is called the 2-*closure* of G. A description of 2-closures of rank 3 groups is given. As a special case, it is proved that the 2-closure of a primitive one-dimensional affine rank 3 group of sufficiently large degree is also affine and one-dimensional.

Keywords: 2*-closure, rank* 3 *group, rank* 3 *graph, permutation group.*

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1 Introduction

Let G be a permutation group on a finite set Ω. Recall that the *rank* of G is the number of orbits in the induced action of G on $\Omega \times \Omega$; these orbits are called 2-*orbits*. If a rank 3 group has even order, then its non-diagonal 2-orbit induces a strongly regular graph on Ω, which is called a *rank* 3 *graph*. It is readily seen that a rank 3 group acts on the corresponding rank 3 graph as an automorphism group. Notice that an arc-transitive strongly regular graph need not be a rank 3 graph, since its automorphism group might be intransitive on non-arcs.

Related to this is the notion of a 2-*closure* of a permutation group [31]. The 2-closure $G^{(2)}$ of a permutation group G is the largest permutation group having the same 2-orbits as G. Clearly $G \leq G^{(2)}$, the 2-closure of $G^{(2)}$ is equal to $G^{(2)}$, and $G^{(2)}$ has the same rank as G. Note also that given a rank 3 graph Γ corresponding to the rank 3 group G, we have $\mathrm{Aut}(\Gamma) = G^{(2)}$.

The rank 3 groups are completely classified. A primitive rank 3 group either stabilizes a nontrivial product decomposition, or is almost simple or is an affine group. The rank 3 groups stabilizing a nontrivial product decomposition are given by the classification of the

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2-transitive almost simple groups, see Theorem 4.1 (ii)(a) and Section 5 in [6]. Almost simple rank 3 groups were determined in [1] when the socle is an alternating group, in [16] when the socle is a classical group and in [21] when the socle is an exceptional or sporadic group. The classification of affine rank 3 groups was completed in [19].

In order to describe the 2-closures of rank 3 groups (or, equivalently, the automorphism groups of rank 3 graphs), it is essential to know which rank 3 groups give rise to isomorphic graphs. Despite all the rank 3 groups being known, it is not a trivial task (considerable progress in describing rank 3 graphs was made in [4]). In the present paper we give a detailed description of the 2-closures of rank 3 groups.

Theorem 1.1. *Let G be a rank 3 permutation group on a set Ω. Then either G is one of the groups from Table 8, or exactly one of the following is true.*

(i) *G is imprimitive, i.e. it preserves a nontrivial decomposition $\Omega = \Delta \times X$. Then $G^{(2)} = \mathrm{Sym}(\Delta) \wr \mathrm{Sym}(X)$.*

(ii) *G is primitive and preserves a product decomposition $\Omega = \Delta^2$. Then $G^{(2)} = \mathrm{Sym}(\Delta) \uparrow \mathrm{Sym}(2)$.*

(iii) *G is primitive almost simple with socle L, i.e. $L \trianglelefteq G \leq \mathrm{Aut}(L)$. Then $G^{(2)} = N_{\mathrm{Sym}(\Omega)}(L)$, and $G^{(2)}$ is almost simple with socle L.*

(iv) *G is a primitive affine group which does not stabilize a product decomposition. Then $G^{(2)}$ is also an affine group. More precisely, there exist an integer $a \geq 1$ and a prime power q such that $G \leq \mathrm{A\Gamma L}_a(q)$, and exactly one of the following holds (setting $F = \mathrm{GF}(q)$).*

   (a) $G \leq \mathrm{A\Gamma L}_1(q)$. Then $G^{(2)} \leq \mathrm{A\Gamma L}_1(q)$.

   (b) $G \leq \mathrm{A\Gamma L}_{2m}(q)$ preserves the bilinear forms graph $H_q(2, m)$, $m \geq 3$. Then

$$G^{(2)} = F^{2m} \rtimes ((\mathrm{GL}_2(q) \circ \mathrm{GL}_m(q)) \rtimes \mathrm{Aut}(F)).$$

   (c) $G \leq \mathrm{A\Gamma L}_{2m}(q)$ preserves the affine polar graph $\mathrm{VO}^{\epsilon}_{2m}(q)$, $m \geq 2$, $\epsilon = \pm$. Then

$$G^{(2)} = F^{2m} \rtimes \Gamma\mathrm{O}^{\epsilon}_{2m}(q).$$

   (d) $G \leq \mathrm{A\Gamma L}_{10}(q)$ preserves the alternating forms graph $A(5, q)$. Then

$$G^{(2)} = F^{10} \rtimes ((\Gamma\mathrm{L}_5(q)/\{\pm 1\}) \times (F^{\times}/(F^{\times})^2)).$$

   (e) $G \leq \mathrm{A\Gamma L}_{16}(q)$ preserves the affine half spin graph $\mathrm{VD}_{5,5}(q)$. Then $G^{(2)} \leq \mathrm{A\Gamma L}_{16}(q)$ and

$$G^{(2)} = F^{16} \rtimes ((F^{\times} \circ \mathrm{Inndiag}(D_5(q))) \rtimes \mathrm{Aut}(F)).$$

   (f) $G \leq \mathrm{A\Gamma L}_4(q)$ preserves the Suzuki-Tits ovoid graph $\mathrm{VSz}(q)$, $q = 2^{2e+1}$, $e \geq 1$. Then

$$G^{(2)} = F^4 \rtimes ((F^{\times} \times \mathrm{Sz}(q)) \rtimes \mathrm{Aut}(F)).$$

Up arrow symbol in (ii) denotes the primitive wreath product (see Section 2), notation for graphs in the affine case is explained in Section 3. Table 8 contains finitely many permutation groups and can be found in Appendix. We note that the largest degree of a permutation group from Table 8 is 3^{12}.

We also remark that the value of a in (iv) of Theorem 1.1 is not necessarily minimal subject to $G \leq \mathrm{A\Gamma L}_a(q)$, since it is not completely defined by the corresponding rank 3 graph and may depend on the group-theoretical structure of G. Minimal values of a can be found in Table 1.

The proof of Theorem 1.1 can be divided into three parts. First we reduce the study to the case when $G^{(2)}$ has the same socle as G, and deal with cases (i)–(iii) (Proposition 2.8). In the affine case (iv) we apply the classification of affine rank 3 groups [19], and compare subdegrees of groups from various classes (Lemma 3.5 and Proposition 3.6); that allows us to deal with case (a). Finally, we invoke known results on automorphisms of some families of strongly regular graphs to cover cases (b)–(d), while cases (e) and (f) are treated separately.

The case (iv), (a) of Theorem 1.1 can be formulated as a standalone result that may be of the independent interest.

Theorem 1.2. *Let G be a primitive affine permutation group of rank 3 and suppose that* $G \leq \mathrm{A\Gamma L}_1(q)$ *for some prime power q. Then* $G^{(2)} \leq \mathrm{A\Gamma L}_1(q)$, *unless the degree and the smallest subdegree of G are as in Table 7.*

It is important to stress that the group G in Theorem 1.2 can have a nonsolvable 2-closure; such an example of degree 2^6 has been found in [28].

The main motivation for the present study is the application of Theorem 1.1 to the computational 2-closure problem. Namely, the problem asks if given generators of a rank 3 group one can find generators of its 2-closure in polynomial time. This task influenced the scope of the present paper considerably, for instance, while one can determine the structure of the normalizer in Theorem 1.1 (iii) explicitly depending on the type of the corresponding rank 3 graph, this is not required for the computational problem as this normalizer can be computed in polynomial time [22]. In other cases it is possible to compute automorphism groups of associated rank 3 graphs directly (for example, of Hamming graphs), but in many situations a more detailed study of relevant groups is required. The author plans to turn to the computational problem in his future work.

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2 Reduction to affine case

We will prove Theorem 1.1 by dealing with rank 3 groups on a case by case basis. Recall the following well-known general classification of rank 3 groups.

Proposition 2.1. *Let G be a rank* 3 *group with socle L. Then G is transitive and one of the following holds:*

(i) *G is imprimitive,*

(ii) *L is a direct product of two isomorphic simple groups, and G preserves a nontrivial product decomposition,*

(iii) *L is nonabelian simple,*

(iv) *L is elementary abelian.*

Proof. Transitivity part is clear. If G is primitive, Theorem 4.1 and Proposition 5.1 from [6] imply that G belongs to one of the last three cases from the statement. □

Suppose that $G \leq \mathrm{Sym}(\Omega)$. Observe that G acts imprimitively on Ω if and only if it preserves a nontrivial decomposition $\Omega = \Delta \times X$, i.e. the action domain Ω can be identified with a nontrivial Cartesian product $\Delta \times X$, $|\Delta| > 1$, $|X| > 1$, where G permutes blocks of the form $\Delta \times \{x\}$, $x \in X$. Denote by $\mathrm{Sym}(\Delta) \wr \mathrm{Sym}(X) \leq \mathrm{Sym}(\Omega)$ the wreath product of $\mathrm{Sym}(\Delta)$ and $\mathrm{Sym}(X)$ in the imprimitive action, so $G \leq \mathrm{Sym}(\Delta) \wr \mathrm{Sym}(X)$.

Proposition 2.2. *Let G be an imprimitive rank 3 permutation group on Ω. Let Δ be a nontrivial block of imprimitivity of G, so Ω can be identified with $\Delta \times X$ for some set X. Then $G^{(2)} = \mathrm{Sym}(\Delta) \wr \mathrm{Sym}(X)$.*

Proof. Set $H = \mathrm{Sym}(\Delta) \wr \mathrm{Sym}(X)$. Then $G \leq H$ and since G and H are both groups of rank 3, we have $G^{(2)} = H^{(2)}$. By [15, Lemma 2.5] (see also [8, Proposition 3.1]), we have

$$(\mathrm{Sym}(\Delta) \wr \mathrm{Sym}(X))^{(2)} = \mathrm{Sym}(\Delta)^{(2)} \wr \mathrm{Sym}(X)^{(2)} = \mathrm{Sym}(\Delta) \wr \mathrm{Sym}(X),$$

so H is 2-closed. Hence $G^{(2)} = H^{(2)} = H$, as claimed. □

Suppose that the action domain is a Cartesian power of some set: $\Omega = \Delta^m$, $m \geq 2$ and $|\Delta| > 1$. Denote by $\mathrm{Sym}(\Delta) \uparrow \mathrm{Sym}(m)$ the wreath product of $\mathrm{Sym}(\Delta)$ and $\mathrm{Sym}(m)$ in the product action, i.e. the base group acts on Δ^m coordinatewise, while $\mathrm{Sym}(m)$ permutes the coordinates. We say that $G \leq \mathrm{Sym}(\Omega)$ preserves a nontrivial product decomposition $\Omega = \Delta^m$ if $G \leq \mathrm{Sym}(\Delta) \uparrow \mathrm{Sym}(m)$.

If G preserves a nontrivial product decomposition $\Omega = \Delta^m$, then G induces a permutation group $G_0 \leq \mathrm{Sym}(\Delta)$. Recall that we can identify G with a subgroup of $G_0 \uparrow \mathrm{Sym}(m)$. We need the following well-known formula for the rank of a primitive wreath product; the proof is provided for completeness (see also [18]).

Lemma 2.3. *Let G be a transitive group of rank r. Then $G \uparrow \mathrm{Sym}(m)$ has rank $\binom{r+m-1}{m}$.*

Proof. Let $G \leq \mathrm{Sym}(\Delta)$, and recall that $\Gamma = G \uparrow \mathrm{Sym}(m)$ acts on Δ^m. Choose $\alpha_1 \in \Delta$ and set $\overline{\alpha_1} = (\alpha_1, \dots, \alpha_1) \in \Delta^m$. Let $\alpha_1, \dots, \alpha_r$ be representatives of orbits of G_{α_1} on Δ. Since the point stabilizer $\Gamma_{\overline{\alpha_1}}$ is equal to $G_{\alpha_1} \uparrow \mathrm{Sym}(m)$, the points $(\alpha_{i_1}, \dots, \alpha_{i_m})$, $1 \leq i_1 \leq \dots \leq i_m \leq r$, form a set of representatives of orbits of $\Gamma_{\overline{\alpha_1}}$ on Δ^m. The number of indices $i_1, \dots, i_m$ satisfying $1 \leq i_1 \leq \dots \leq i_m \leq r$ is equal to the number of weak compositions of m into r parts, hence the claim is proved. □

Observe that in the particular case when $\Omega = \Delta^2$, the wreath product $\mathrm{Sym}(\Delta) \uparrow \mathrm{Sym}(2)$ has rank 3 and its 2-orbit of size $|\Delta|^2(|\Delta| - 1)$ is the edge set of the Hamming graph $H(2, |\Delta|)$.

Proposition 2.4. *Let G be a primitive rank 3 permutation group on Ω, preserving a nontrivial product decomposition $\Omega = \Delta^m$, $m \geq 2$. Then $m = 2$, a 2-orbit of G induces a Hamming graph and $G^{(2)} = \mathrm{Sym}(\Delta) \uparrow \mathrm{Sym}(2)$.*

Proof. Set $H = \mathrm{Sym}(\Delta) \uparrow \mathrm{Sym}(m)$, and recall that by Lemma 2.3, H has rank $\binom{2+m-1}{m} = m + 1$ as a permutation group. Since $G \leq H$, we have $m + 1 \leq 3$. Therefore $m = 2$ and H is a rank 3 group. Then $G^{(2)} = H^{(2)}$ and it suffices to show that H is 2-closed.

A 2-orbit of H induces the Hamming graph $H(2, q)$ on Ω, where $q = |\Delta|$. By [3, Theorem 9.2.1], $\mathrm{Aut}(H(2, q)) = \mathrm{Sym}(q) \uparrow \mathrm{Sym}(2)$. It readily follows that $H^{(2)} = \mathrm{Aut}(H(2, q)) = H$, completing the proof. □

In order to find 2-closures in the last two cases of Proposition 2.1, we need to show that 2-closure almost always preserves the socle of a rank 3 group.

Lemma 2.5. *Let G be a primitive rank 3 group and suppose that G and $G^{(2)}$ have different socles. Then either G preserves a nontrivial product decomposition, or G is an almost simple group with socle and degree as in Table 8.*

Proof. From [25, Theorem 2] it follows that either G preserves a nontrivial product decomposition, or G and $G^{(2)}$ are almost simple groups. By [20, Theorem 1], the latter situation applies only to a finite number of rank 3 groups, namely, either G is one of exceptional examples from [20, Table 1], or the socle of G is $G_2(q)$, $q \geq 3$, or the socle is $\Omega_7(q)$. Since rank 3 graphs are distance-transitive, [20, Proposition 1] implies $q \in \{3, 4, 8\}$ in the case of $G_2(q)$, while [20, Proposition 2] yields $q \in \{2, 3\}$ in the case of $\Omega_7(q)$. □

Lemma 2.6. *Let G be a primitive rank 3 group with nonabelian simple socle. Then G does not preserve a nontrivial product decomposition.*

Proof. Let $G \leq \mathrm{Sym}(\Omega)$ and L be the socle of G. Suppose on the contrary that G preserves a nontrivial product decomposition. Since G is primitive, and L is a nonabelian simple minimal normal subgroup of G, [26, Theorem 8.21] implies that either L is A_6 and $|\Omega| = 36$, or $L = M_{12}$ and $|\Omega| = 144$, or $L = \mathrm{Sp}_4(q)$, $q \geq 4$, q even and $|\Omega| = q^4(q^2 - 1)^2$. One can easily check that neither of these situations occurs in rank 3 by inspecting the classification of almost simple rank 3 groups. The reader is referred to [5, Table 5] for alternating socles, [5, Table 9] for sporadic socles and [5, Tables 6 and 7] for classical socles. □

It should be noted that an almost simple group with rank larger than 3 might preserve a nontrivial product decomposition, see [26, Section 1.3].

Proposition 2.7. *Let G be a primitive rank 3 permutation group on Ω with nonabelian simple socle L. Then either G appears in Table 8, or $G^{(2)}$ has socle L and $G^{(2)} = N_{\mathrm{Sym}(\Omega)}(L)$.*

Proof. By Lemma 2.6, G does not preserve a nontrivial product decomposition, hence by Lemma 2.5, either G belongs to Table 8, or 2-closure $G^{(2)}$ has the same socle as G. Set $N = N_{\mathrm{Sym}(\Omega)}(L)$. Clearly $G^{(2)} \leq N$, and to establish equality it suffices to show that N is a rank 3 group.

Suppose that this not the case and N is 2-transitive. By [6, Proposition 5.2], N has a unique minimal normal subgroup, and since L is a minimal normal subgroup of N, the socle of N must be equal to L. Hence N is an almost simple 2-transitive group with socle L.

The possibilities for a socle of a 2-transitive almost simple group are all known and moreover, such a socle is a 2-transitive group itself, unless G acts on 28 points and $L =$

$\mathrm{PSL}_2(8)$ (see Theorem 5.3 (S) and the following notes in [6]). By [16, Theorem 1.2], there is no rank 3 group of degree 28 with socle $\mathrm{PSL}_2(8)$, hence L and thus G are 2-transitive, which is a contradiction. Therefore N is a rank 3 group and $G^{(2)} = N$. □

We summarize the results of this section in the following.

Proposition 2.8. *Let G be a rank 3 permutation group on Ω. Then either G appears in Table 8, or exactly one of the following holds.*

(i) *G is imprimitive, i.e. it preserves a nontrivial decomposition $\Omega = \Delta \times X$. Then $G^{(2)} = \mathrm{Sym}(\Delta) \wr \mathrm{Sym}(X)$.*

(ii) *G is primitive and preserves a product decomposition $\Omega = \Delta^2$. Then $G^{(2)} = \mathrm{Sym}(\Delta) \uparrow \mathrm{Sym}(2)$.*

(iii) *G is a primitive almost simple group with socle L, i.e. $L \trianglelefteq G \leq \mathrm{Aut}(L)$. Then $G^{(2)} = N_{\mathrm{Sym}(\Omega)}(L)$, and $G^{(2)}$ is almost simple with socle L.*

(iv) *G is a primitive affine group which does not stabilize a product decomposition. Then $G^{(2)}$ is also an affine group.*

3 Affine case

In the previous section we reduced the task of describing the 2-closures of rank 3 groups to the case when the group in question is affine. Recall that a primitive permutation group $G \leq \mathrm{Sym}(\Omega)$ is called *affine*, if it has a unique minimal normal subgroup V equal to its socle, such that V is an elementary abelian p-group for some prime p and $G = V \rtimes G_0$ for some $G_0 < G$. The permutation domain Ω can be identified with V in such a way that V acts on it by translations, and G_0 acts on it as a subgroup of $\mathrm{GL}(V)$. Clearly G_0 is the stabilizer of the zero vector in V under such identification.

If G_0 acts semilinearly on V as a $\mathrm{GF}(q)$-vector space, where q is a power of p, then we write $G_0 \leq \Gamma\mathrm{L}_m(q)$, where $\Gamma\mathrm{L}_m(q)$ is the full semilinear group and $V \simeq \mathrm{GF}(q)^m$. If the field is clear from the context, we may use $\Gamma\mathrm{L}(V) = \Gamma\mathrm{L}_m(q)$ instead. We write $\mathrm{A}\Gamma\mathrm{L}_m(q)$ for the full affine semilinear group.

Now we are ready to state the classification of affine rank 3 groups.

Theorem 3.1 ([19]). *Let G be a finite primitive affine permutation group of rank 3 and degree $n = p^d$, with socle $V \simeq \mathrm{GF}(p)^d$ for some prime p, and let G_0 be the stabilizer of the zero vector in V. Then G_0 belongs to one of the following classes.*

(A) *Infinite classes. These are:*

(1) $G_0 \leq \Gamma\mathrm{L}_1(p^d)$;

(2) *G_0 is imprimitive as a linear group;*

(3) *G_0 stabilizes the decomposition of $V \simeq \mathrm{GF}(q)^{2m}$ into $V = V_1 \otimes V_2$, where $p^d = q^{2m}$, $\dim V_1 = 2$ and $\dim V_2 = m$;*

(4) *$G_0 \trianglerighteq \mathrm{SL}_m(\sqrt{q})$ and $p^d = q^m$, where 2 divides $\frac{d}{m}$;*

(5) *$G_0 \trianglerighteq \mathrm{SL}_2(\sqrt[3]{q})$ and $p^d = q^2$, where 3 divides $\frac{d}{2}$;*

(6) *$G_0 \trianglerighteq \mathrm{SU}_m(q)$ and $p^d = q^{2m}$;*

(7) $G_0 \trianglerighteq \Omega^{\pm}_{2m}(q)$ *and* $p^d = q^{2m}$;
(8) $G_0 \trianglerighteq \mathrm{SL}_5(q)$ *and* $p^d = q^{10}$;
(9) $G_0 \trianglerighteq B_3(q)$ *and* $p^d = q^8$;
(10) $G_0 \trianglerighteq D_5(q)$ *and* $p^d = q^{16}$;
(11) $G_0 \trianglerighteq \mathrm{Sz}(q)$ *and* $p^d = q^4$.

(B) *'Extraspecial' classes.*

(C) *'Exceptional' classes.*

Moreover, classes (B) and (C) consist of finitely many groups.

Observe that the only case when a primitive affine rank 3 group can lie in some other class from the statement of Proposition 2.8 is when it preserves a nontrivial product decomposition. This is precisely case (A2) of the classification, and this situation does occur.

Recall that each rank 3 group gives rise to a rank 3 graph. By [4, Table 11.4], the groups from case (A) of Theorem 3.1 correspond to the following series of graphs:

- One-dimensional affine graphs (i.e. those arising from case (A1)). These graphs are either Van Lint–Schrijver, Paley or Peisert graphs [23];
- Hamming graphs. These graphs correspond to linearly imprimitive groups;
- Bilinear forms graph $H_q(2, m)$, where $m \geq 2$ and q is a prime power. These graphs correspond to groups fixing a nontrivial tensor decomposition;
- Affine polar graph $\mathrm{VO}^{\epsilon}_{2m}(q)$, where $m \geq 2$, $\epsilon = \pm$, and q is a prime power;
- Alternating forms graph $A(5, q)$, where q is a prime power;
- Affine half spin graph $\mathrm{VD}_{5,5}(q)$, where q is a prime power;
- Suzuki-Tits ovoid graph $\mathrm{VSz}(q)$, where $q = 2^{2e+1}$, $e \geq 1$.

The reader is referred to [4] for the construction and basic properties of the mentioned graphs.

It should be noted that different cases of Theorem 3.1 may lead to isomorphic graphs. Table 3 lists affine rank 3 groups from case (A) and indicates the corresponding rank 3 graphs. In Tables 1 and 2 we provide degrees and subdegrees of affine rank 3 groups in case (A). These and some other relevant tables and comments on sources of data used are collected in Appendix.

Our first goal is to show that almost all pairs of affine rank 3 graphs can be distinguished based on their subdegrees. We start with the class (A1). The following lemma summarizes some of the arithmetical conditions for the subdegrees of the corresponding groups.

Lemma 3.2. *Let G be a primitive affine rank 3 group from class (A1), having degree $n = p^d$, where p is a prime. Denote by m_1, m_2 the subdegrees of G and suppose that $m_1 < m_2$. Then m_1 divides m_2 and $\frac{m_2}{m_1}$ divides d.*

Proof. See [10, Proposition 3.3] for the first claim and [10, Theorem 3.7, (4)] for the second. □

The following lemmas apply conditions from Lemma 3.2 to groups from classes (B), (C) and (A).

Lemma 3.3. *Let G be a primitive affine rank 3 group from class (B). Suppose that G has the same subdegrees as a group from class (A1). Then the degree and subdegrees of G are one of the following:* $(7^2, 24, 24)$, $(17^2, 96, 192)$, $(23^2, 264, 264)$, $(3^6, 104, 624)$, $(47^2, 1104, 1104)$, $(3^4, 16, 64)$, $(7^4, 480, 1920)$.

Proof. Let n denote the degree of G, and let $m_1 \leq m_2$ be the subdegrees. In Table 5 all possible subdegrees of groups from class (B) are listed. We apply Lemma 3.2. For instance, if $n = 29^2$ then $m_1 = 168$, $m_2 = 672$. The quotient $\frac{m_2}{m_1} = 4$ does not divide 2, hence this case cannot happen. The other cases are treated in the same manner. □

Lemma 3.4. *Let G be a primitive affine rank 3 group from class (C). Suppose that G has the same subdegrees as a group from class (A1). Then the degree and subdegrees of G are* $(3^4, 40, 40)$ *or* $(89^2, 2640, 5280)$.

Proof. Follows from Lemma 3.2 and Table 6. □

Lemma 3.5. *Let G be a primitive affine rank 3 group from class (A) and suppose that G has the same subdegrees as a group from class (A1). Then either G lies in (A1) or degree and subdegrees of G are one of the following:* $(3^2, 4, 4)$, $(3^4, 16, 64)$, $(3^6, 104, 624)$, $(2^4, 5, 10)$, $(2^6, 21, 42)$, $(2^8, 51, 204)$, $(2^{10}, 93, 930)$, $(2^{12}, 315, 3780)$, $(2^{16}, 3855, 61680)$, $(5^2, 8, 16)$.

Proof. Suppose that G does not lie in class (A1), but shares subdegrees with some group from (A1). Notice that in cases (A3) through (A11), exactly one of the subdegrees is divisible by p, so the subdegrees are not equal (see Table 1). In case (A2) subdegrees are the same if and only if $p^m = 3$, and consequentially $n = 9$. This situation is the first example in our list of parameters, hence from now on we may assume that the subdegrees of G are not equal.

Let m_1 and m_2 denote the subdegrees of G, where, as shown earlier, we may assume $m_1 < m_2$. Since m_1 and m_2 are subdegrees of some group from the class (A1), Lemma 3.2 yields that m_1 divides m_2 and the number $u = \frac{m_2}{m_1}$ divides d, where $n = p^d$.

Now, since G belongs to one of the classes (A2)–(A11), we apply the above arithmetical conditions in each case. We consider some classes together, since they give rise to isomorphic rank 3 graphs and hence have the same formulae for subdegrees. The reader is referred to Table 1 for the list of subdegrees in question.

(A2) Subdegrees in this case are $2(p^m-1)$ and $(p^m-1)^2$. If $2(p^m-1) > (p^m-1)^2$, then $p^m = 2$ and $n = 4$. It can be easily seen that G is not primitive in this case, contrary to our hypothesis. Therefore we can assume that $2(p^m - 1) < (p^m - 1)^2$.

Then $u = \frac{p^m-1}{2}$ and since u divides $d = 2m$, we have $p^m - 1 \leq 4m$. It follows that (n, m_1, m_2) is one of $(3^2, 4, 4)$, $(3^4, 16, 64)$ or $(5^2, 8, 16)$.

(A3)–(A5) We write r for the highest power of p dividing m_2, so the second subdegree is equal to $r(r^m - 1)(r^{m-1} - 1)$ for some $m \geq 2$.

We have $u = r\frac{r^{m-1}-1}{r+1}$ and hence $u \geq \frac{r^{m-1}-1}{2}$. Now $r^{2m} = p^d \geq p^{\frac{r^{m-1}-1}{2}}$. Using inequalities $m \geq 2$ and $p \geq 2$, we obtain $2r^{8(m-1)} \geq 2^{r^{m-1}}$. Therefore

$r^{m-1} \leq 44$ and there are finitely many choices for r and m. Checking these values of r and m against original divisibility conditions we yield the following possibilities for (n, m_1, m_2): $(2^6, 21, 42)$, $(2^{10}, 93, 930)$, $(2^{12}, 315, 3780)$, $(3^6, 104, 624)$.

(A6), (A7) $u = q^{m-1}\frac{q-1}{q^{m-1}\pm 1}$. Numbers q^{m-1} and $q^{m-1} \pm 1$ are coprime, so $q^{m-1} \pm 1$ divides $q-1$. That is possible only when $m = 2$, so we have $u = q$. Now $2^q \leq p^q \leq p^d = q^4$, so $q \leq 16$. Hence we have the following possibilities for n, m_1, m_2 in this case: $(2^4, 5, 10)$, $(2^8, 51, 204)$, $(2^{16}, 3855, 61680)$.

(A8) $u = q^3 - q^2\frac{q+1}{q^2+1}$. Since q^2 and q^2+1 are coprime, q^2+1 must divide $q+1$. This can not happen, so this case does not occur.

(A9) $u = q^3\frac{q-1}{q^3+1}$. Since q^3+1 does not divide $q-1$, this case does not occur.

(A10) $u = q^5 - q^3\frac{q^2+1}{q^3+1}$. Since q^3+1 does not divide q^2+1, this case does not occur.

(A11) $u = q$ and $p^d = q^4$. Hence we obtain the same possible parameters as in cases (A6), (A7).

In all cases considered we either got a contradiction or got one of the possible exceptions recorded in the statement. The claim is proved. □

As an immediate corollary we derive that 2-closures of primitive rank 3 subgroups of $\mathrm{A\Gamma L}_1(q)$ also lie in $\mathrm{A\Gamma L}_1(q)$ (Theorem 1.2), apart from a finite number of exceptions.

Proof of Theorem 1.2. Suppose that G and $G^{(2)}$ have different socles. Since G is not almost simple, Lemma 2.5 implies that $G^{(2)}$ and thus G must preserve a nontrivial product decomposition. In that situation G has subdegrees of the form $2(\sqrt{n}-1)$, $(\sqrt{n}-1)^2$, in particular, G has subdegrees as a group from class (A2) and hence parameters of G are listed in Lemma 3.5. We may assume that G does not preserve a nontrivial product decomposition and so G and $G^{(2)}$ have equal socles. The claim now follows from Theorem 3.1 and Lemmas 3.3–3.5. □

Note that Lemmas 3.3–3.5 list degrees and subdegrees of possible exceptions to Theorem 1.2; in Table 7 of Appendix we collect these data in one place.

Now we move on to establish a partial analogue of Lemma 3.5 for classes (A2)–(A11). First we need to recall some notions related to quadratic and bilinear forms.

Let V be a vector space over a field F. Given a symmetric bilinear form $f : V \times V \to F$, the radical of f is $\mathrm{rad}(f) = \{x \in V \mid f(x,y) = 0 \text{ for all } y\}$; we say that f is *non-singular*, if $\mathrm{rad}(f) = 0$. If $\kappa : V \times V \to F$ is a quadratic form with an associated bilinear form f, then the radical of κ is $\mathrm{rad}(\kappa) = \mathrm{rad}(f) \cap \{x \in V \mid \kappa(x) = 0\}$. We say that κ is *non-singular*, if $\mathrm{rad}(\kappa) = 0$, and we say that κ is *non-degenerate*, if $\mathrm{rad}(f) = 0$.

If F has odd characteristic, then $\mathrm{rad}(\kappa) = \mathrm{rad}(f)$. If F has even characteristic and κ is non-singular, then the dimension of $\mathrm{rad}(f)$ is at most one, f induces a non-singular alternating form on $V/\mathrm{rad}(f)$ and, hence, the dimension of $V/\mathrm{rad}(f)$ is even (see [32, Section 3.4.7]). Therefore if the dimension of V is even, then the notions of non-singular and non-degenerate quadratic forms coincide regardless of the characteristic.

Now we can describe the affine polar graph $\mathrm{VO}^{\epsilon}_{2m}(q)$, $m \geq 2$. Let V be a $2m$-dimensional vector space over $\mathrm{GF}(q)$, and let $\kappa : V \to \mathrm{GF}(q)$ be a non-singular quadratic form of type ϵ. Vertices of the graph $\mathrm{VO}^{\epsilon}_{2m}(q)$ are identified with vectors from V, and two distinct vertices $u, v \in V$ are joined by an edge if $\kappa(u - v) = 0$. Up to isomorphism, $\mathrm{VO}^{\epsilon}_{2m}(q)$ does not depend on the form κ.

Allowing some abuse of terminology, we say that subdegrees of a rank 3 graph are simply subdegrees of the respective rank 3 group.

Proposition 3.6. *If two affine rank 3 graphs have the same subdegrees, then they are isomorphic apart from the following exceptions:*

- *graphs arising from affine groups from Table 8,*
- $\mathrm{VSz}(q)$ *and* $\mathrm{VO}^{-}_{4}(q)$ *for* $q = 2^{2e+1}$, $e \geq 1$,
- *Paley and Peisert graphs.*

In particular, graphs $H_q(2,2)$ *and* $\mathrm{VO}^{+}_{4}(q)$ *are isomorphic.*

Proof. Since classes (B) and (C) of Theorem 3.1 and all exceptional parameter sets of Lemma 3.5 are included in Table 8, we may assume that our graphs come from the case (A) and their subdegrees are not among the exceptions from Lemma 3.5.

By Lemma 3.5, if one of the graphs in question arises from the case (A1), then the second graph also comes from (A1). By Table 2, Van Lint-Schrijver graph has unequal subdegrees, while Paley and Peisert graphs have equal subdegrees, hence in this case graphs are either isomorphic or it is a Paley graph and a Peisert graph. We may now assume that our graphs do not come from (A1).

Notice that given $n = p^d$ for p prime, the largest subdegree of graphs from classes (A3)–(A11) is divisible by p. This is not the case in class (A2), unless $n = 4$ with subdegrees 2 and 1. The corresponding rank 3 group is imprimitive in that situation, contrary to our assumptions. Thus we may assume that none of the two graphs comes from (A2).

We compare subdegrees of classes (A3)–(A11) and collect the relevant information in Table 4. Let us explain the procedure in the case $H_q(2,m)$ vs. $\mathrm{VO}^{\pm}_{2\overline{m}}(\overline{q})$ only, since other cases are treated similarly.

Consider the graph $H_q(2,m)$. The number of its vertices is equal to $n = q^{2m}$ and the second subdegree is equal to $q(q^m - 1)(q^{m-1} - 1)$. Recall that $n = p^d$ for some prime p, and the largest power of p dividing the second subdegree is q. In the case of the graph $\mathrm{VO}^{\epsilon}_{2\overline{m}}(\overline{q})$, we have $n = \overline{q}^{2\overline{m}}$ and the largest power of p dividing the second subdegree is $\overline{q}^{\overline{m}-1}$. We obtain a system of equations

$$q^{2m} = \overline{q}^{2\overline{m}},\ q = \overline{q}^{\overline{m}-1},$$

which is written in the relevant cell of Table 4. We derive that $m = \frac{\overline{m}}{\overline{m}-1}$, and hence $m = \overline{m} = 2$, $q = \overline{q}$. Now, the second subdegree for $\mathrm{VO}^{\epsilon}_{4}(q)$ is $q(q-1)(q^2 + (-1)^{\epsilon})$. Therefore $\epsilon = +$, which gives us the first example of affine rank 3 graphs with same subdegrees. Other cases are dealt with in the same way.

Now, Table 4 lists two cases when graphs from different classes have the same subdegrees, namely, $H_q(2,2)$, $\mathrm{VO}^{+}_{4}(q)$ and $\mathrm{VSz}(q)$, $\mathrm{VO}^{-}_{4}(q)$. To finish the proof of the proposition, we show that graphs $H_q(2,2)$ and $\mathrm{VO}^{+}_{4}(q)$ are in fact isomorphic.

Identify vertices of $H_q(2,2)$ with 2×2 matrices over $\mathrm{GF}(q)$, and recall that two vertices are connected by an edge if the rank of their difference is 1. A nonzero 2×2 matrix has rank 1 precisely when its determinant is zero:

$$\mathrm{rk}\begin{pmatrix} u_1 & u_3 \\ u_4 & u_2 \end{pmatrix} = 1 \iff u_1u_2 - u_3u_4 = 0.$$

It can be easily seen that $u_1u_2 - u_3u_4$ is a non-degenerate quadratic form on $\mathrm{GF}(q)^4$, so $H_q(2,2)$ is isomorphic to the affine polar graph $\mathrm{VO}_4^{\epsilon}(q)$. By comparing subdegrees we derive that $\epsilon = +$, and we are done. $\square$

It should be noted that $\mathrm{VSz}(q)$ and $\mathrm{VO}_4^-(q)$ in fact have the same parameters as strongly regular graphs (see [5, Table 24]). In Lemma 3.13 we will see that these graphs are actually not isomorphic since they have non-isomorphic automorphism groups.

Paley and Peisert graphs are generally not isomorphic (see [24]), but have the same parameters since they are strongly regular and self-complementary (i.e. isomorphic to their complements).

Recall that in order to describe 2-closures of rank 3 groups it suffices to find full automorphism groups of corresponding rank 3 graphs. Hamming graphs were dealt with in Proposition 2.4, and graphs arising in the case (A1) were covered in Theorem 1.2. We are left with five cases: bilinear forms graph, affine polar graph, alternating forms graph, affine half spin graph and the Suzuki-Tits ovoid graph. In most of these cases the full automorphism group was described earlier in some form, and we state relevant results here.

For two groups G_1 and G_2 let $G_1 \circ G_2$ denote their central product. Note that the central product $\mathrm{GL}(U) \circ \mathrm{GL}(W)$ has a natural action on the tensor product $U \otimes W$.

Proposition 3.7 ([3, Theorem 9.5.1]). *Let q be a prime power and $m \geq 2$. Set $G = \mathrm{Aut}(H_q(2,m))$ and $F = \mathrm{GF}(q)$. If $m > 2$, then*

$$G = F^{2m} \rtimes ((\mathrm{GL}_2(q) \circ \mathrm{GL}_m(q)) \rtimes \mathrm{Aut}(F)).$$

If $m = 2$, then

$$G = F^4 \rtimes (((\mathrm{GL}_2(q) \circ \mathrm{GL}_2(q)) \rtimes \mathrm{Aut}(F)) \rtimes C_2),$$

where the additional automorphism of order 2 exchanges components of simple tensors.

Let V be a vector space endowed with a quadratic form κ. We say that a nonzero vector $v \in V$ is *isotropic* if $\kappa(v) = 0$.

Lemma 3.8 ([27]). *Let V be a vector space over some (possibly finite) field F, and suppose that $\dim V \geq 3$. Let $\kappa : V \to F$ be a non-singular quadratic form, possessing an isotropic vector. If f is a permutation of V with the property that*

$$\kappa(x - y) = 0 \Leftrightarrow \kappa(x^f - y^f) = 0,$$

then $f \in \mathrm{A\Gamma L}(V)$ and $f : x \mapsto x^{\phi} + v$, $v \in V$, where $\phi \in \Gamma\mathrm{L}(V)$ is a semisimilarity of κ, i.e. there exist $\lambda \in F^{\times}$ and $\alpha \in \mathrm{Aut}(F)$ such that $\kappa(x^{\phi}) = \lambda\kappa(x)^{\alpha}$ for all $x \in V$.

Denote by $\Gamma\mathrm{O}^{\epsilon}_{2m}(q)$ the group of all semisimilarities of a non-degenerate quadratic form of type ϵ on the vector space of dimension $2m$ over the finite field of order q. The reader is referred to [17, Sections 2.7 and 2.8] for the structure and properties of groups $\Gamma\mathrm{O}^{\epsilon}_{2m}(q)$.

Proposition 3.9. *Let q be a prime power and $m \geq 2$. Set $F = \mathrm{GF}(q)$. Then*

$$\mathrm{Aut}(\mathrm{VO}^{\epsilon}_{2m}(q)) = F^{2m} \rtimes \Gamma\mathrm{O}^{\epsilon}_{2m}(q),\ \epsilon = \pm.$$

Proof. Recall that the graph $\mathrm{VO}^{\epsilon}_{2m}(q)$ is defined by a vector space $V = F^{2m}$ over F and a non-singular (or, equivalently, non-degenerate) quadratic form $\kappa : V \to F$. Since $m \geq 2$, we have $\dim V \geq 3$ and κ possesses an isotropic vector. The claim now follows from Lemma 3.8. □

Proposition 3.10 ([3, Theorem 9.5.3])**.** *Let q be a prime power and set $F = \mathrm{GF}(q)$. Then*

$$\mathrm{Aut}(A(5,q)) = F^{10} \rtimes ((\Gamma\mathrm{L}_5(q)/\{\pm 1\}) \times (F^{\times}/(F^{\times})^2)).$$

Denote by $D_5(q)$ an orthogonal group of universal type, in particular, recall the formula $|Z(D_5(q))| = \gcd(4, q^5-1)$ (see [7, Table 5]).

Lemma 3.11. *Let q be a prime power, $q^{16} = p^d$, let $F = \mathrm{GF}(q)$, $V = F^{16}$ and set $G = \mathrm{Aut}(\mathrm{VD}_{5,5}(q))$. Then $G = V \rtimes G_0$, and*

$$F^{\times} \circ D_5(q) \leq G_0 = N_{\mathrm{GL}_d(p)}(D_5(q)),$$

where $D_5(q)$ acts on the spin module. Moreover, $G_0/F^{\times}$ is an almost simple group and $G_0 \leq \Gamma\mathrm{L}_{16}(q)$.

Proof. Set $H = V \rtimes (F^{\times} \circ D_5(q))$. By [19, Lemma 2.9], $D_5(q)$ has two orbits on the set of lines $P_1(V)$, so H is an affine rank 3 group of type (A10). Clearly $G = H^{(2)}$ so by Lemma 2.5, G is an affine rank 3 group. By Proposition 3.6, G belongs to class (A10) and the main result of [19] implies that $G_0 \leq N_{\mathrm{GL}_d(p)}(D_5(q))$. By [19, (1.4)], the generalized Fitting subgroup of $G_0/F^{\times}$ is simple, hence this quotient group is almost simple. By Hering's theorem [12] (see also [19, Appendix 1]), the normalizer $N_{\mathrm{GL}_d(p)}(D_5(q))$ cannot be transitive on the nonzero vectors of V, so $G_0 = N_{\mathrm{GL}_d(p)}(D_5(q))$ as claimed.

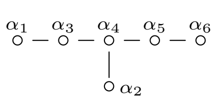
Finally, let a be the minimal integer such that $G_0 \leq \Gamma\mathrm{L}_a(p^{d/a})$. By Table 1, $a = 16$, so the last inclusion follows. □

We write $\mathrm{Inndiag}(D_5(q))$ for the overgroup of $D_5(q)$ in $\mathrm{Aut}(D_5(q))$, containing all diagonal automorphisms.

Proposition 3.12. *Let q be a prime power, and set $F = \mathrm{GF}(q)$. Then*

$$\mathrm{Aut}(\mathrm{VD}_{5,5}(q)) = F^{16} \rtimes ((F^{\times} \circ \mathrm{Inndiag}(D_5(q))) \rtimes \mathrm{Aut}(F)).$$

Proof. We follow [19, Lemma 2.9]. Take $K = E_6(q)$ to be of universal type, so that $|Z(K)| = \gcd(3, q-1)$. The Dynkin diagram of K is:



Let Σ be the set of roots and let $x_\alpha(t)$, $h_\alpha(t)$ be Chevalley generators of K. Write $X_\alpha = \{x_\alpha(t) | t \in F\}$. Let P be a parabolic subgroup of K corresponding to the set of roots $\{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$, and let $P = UL$ be its Levi decomposition. Moreover, $L = MH$ and we may choose P such that

$$U = \langle X_\alpha \mid \alpha \in \Sigma^+,\ \alpha \text{ involves } \alpha_1\rangle,$$

$$M = \langle X_{\pm\alpha_i} \mid 2 \leq i \leq 6\rangle,$$

where M is of universal type and $H = \langle h_{\alpha_i}(t) | t \in F,\ 1 \leq i \leq 6\rangle$ is the Cartan subgroup. In [19, Lemma 2.9] it was shown that $M \simeq D_5(q)$, the group U is elementary abelian of order q^{16} and in fact, it is a spin module for M. By [11, Theorem 2.6.5 (f)], H induces diagonal automorphisms on M, and by [30, Section 1, B] it induces the full group of diagonal automorphisms. Recall that for an element h of H we have $x_\alpha(t)^h = x_\alpha(k \cdot t)$ for some $k \in F$. In particular, diagonal automorphisms of $D_5(q)$ commute with the action of the field F on U.

Let ϕ be a generator of the field automorphisms group of K, and note that one can identify that group with $\mathrm{Aut}(F)$; in particular, ϕ acts on F under such an identification. By [11, Theorem 2.5.1 (c)], generators $x_\alpha(t)$ and $h_\alpha(t)$ are carried to $x_\alpha(t^\phi)$ and $h_\alpha(t^\phi)$ by ϕ, so field automorphisms normalize U, M and H. Furthermore, ϕ induces the full group of field automorphisms on M.

Set $T = L \rtimes \langle\phi\rangle$. We have $M \trianglelefteq T$ and T induces all field and diagonal automorphisms on M. Set $\overline{T} = T/Z(K)$ and $\overline{M} = MZ(K)/Z(K)$. By [11, Theorem 2.6.5 (e)], the centralizer $C_{\mathrm{Aut}(K)}(U)$ is the image of $Z(U)$ in $\mathrm{Aut}(K)$. Therefore $\overline{T}$ acts faithfully on U, and since $|Z(M)|$ is coprime to $|Z(K)|$, we derive that $\overline{M} \simeq M \simeq D_5(q)$. Hence we have an embedding $\overline{T} \leq \mathrm{GL}_d(p)$, where $|U| = p^d$, and, with some abuse of notation, $\overline{T} \leq N_{\mathrm{GL}_d(p)}(D_5(q))$. By Lemma 3.11, the latter normalizer is an almost simple group (modulo scalars), and thus we have shown that it contains all field and diagonal automorphisms of $D_5(q)$. It is left to show that it does not contain graph automorphisms.

Suppose that a graph automorphism ψ lies in $G_0 = N_{\mathrm{GL}_d(p)}(D_5(q))$, and recall that $\overline{M} \simeq D_5(q)$. By [19, Lemma 2.9], there is an orbit Δ of G_0 on the nonzero vectors of U, such that the point stabilizer $\overline{M}_\delta$, $\delta \in \Delta$ is a parabolic subgroup of type A_4. Since ψ preserves the orbit Δ and normalizes $\overline{M}$, it must take a point stabilizer $\overline{M}_\delta$ to the point stabilizer $\overline{M}_{\delta'}$ for some $\delta' \in \Delta$, in particular, it takes $\overline{M}_\delta$ to a conjugate subgroup. That is impossible, since by [11, Theorem 2.6.5 (c)], automorphism ψ interchanges conjugacy classes of parabolic subgroups of type A_4, so the final claim is proved. □

Recall the construction of the graph $\mathrm{VSz}(q)$, $q = 2^{2e+1}$, $e \geq 1$. Set $F = \mathrm{GF}(q)$, $V = F^4$ and let σ be an automorphism of F acting as $\sigma(x) = x^{2^{e+1}}$. Define the subset O of the projective space $P_1(V)$ by

$$O = \{(0,0,1,0)\} \cap \{(x,y,z,1) \mid z = xy + x^2x^\sigma + y^\sigma\},$$

where vectors are written projectively. The vertex set of $\mathrm{VSz}(q)$ is V and two vectors are connected by an edge, if a line connecting them has a direction in O.

Recall that $\mathrm{Sz}(q) \leq \mathrm{GL}_4(q)$ is faithfully represented on $P_1(V)$ and induces the group of all collineations which preserve the Suzuki-Tits ovoid O (see [14, Chapter XI, Theorem 3.3]). Clearly scalar transformations preserve the preimage of O in V, and it can be easily seen that $O^\alpha = O$ for any $\alpha \in \mathrm{Aut}(F)$. Hence the following group

$$H = V \rtimes ((F^\times \times \mathrm{Sz}(q)) \rtimes \mathrm{Aut}(F))$$

acts as a group of automorphisms of $\mathrm{VSz}(q)$. By [13, Lemma 16.4.6], $\mathrm{Sz}(q)$ acts transitively on $P_1(V) \setminus O$, hence H is a rank 3 group.

We will show that H is the full automorphism group of $\mathrm{VSz}(q)$, but first we need to note the following basic fact.

Lemma 3.13. *If $q = 2^{2e+1}$, $e \geq 1$, then there is no subgroup of $\mathrm{Aut}(\mathrm{VO}_4^-(q))$ isomorphic to $\mathrm{Sz}(q)$. In particular, graphs $\mathrm{VO}_4^-(q)$ and $\mathrm{VSz}(q)$ are not isomorphic.*

Proof. Suppose the contrary, so that $\mathrm{Sz}(q)$ is a subgroup of $\mathrm{Aut}(\mathrm{VO}_4^-(q))$. By Proposition 3.9, we have $\mathrm{Aut}(\mathrm{VO}_4^-(q)) \simeq V \rtimes \Gamma\mathrm{O}_4^-(q)$ for some elementary abelian group V. Recall that the orthogonal group $\Omega_4^-(q)$ is a normal subgroup of $\Gamma\mathrm{O}_4^-(q)$, and the quotient $\Gamma\mathrm{O}_4^-(q)/\Omega_4^-(q)$ is solvable. Clearly V is also solvable, and since $\mathrm{Sz}(q)$ is simple, we obtain an embedding of $\mathrm{Sz}(q)$ into $\Omega_4^-(q)$. Yet that is impossible, as can be easily seen by inspection of maximal subgroups of $\Omega_4^-(q)$, see, for instance, [2, Table 8.17]. That is a contradiction, so the first claim is proved.

The second claim follows from the fact that $\mathrm{Sz}(q)$ lies in $\mathrm{Aut}(\mathrm{VSz}(q))$. □

Proposition 3.14. *Let $q = 2^{2e+1}$, where $e \geq 1$, and set $F = \mathrm{GF}(q)$. Then*

$$\mathrm{Aut}(\mathrm{VSz}(q)) = F^4 \rtimes ((F^\times \times \mathrm{Sz}(q)) \rtimes \mathrm{Aut}(F)).$$

Proof. Let $H = F^4 \rtimes ((F^\times \times \mathrm{Sz}(q)) \rtimes \mathrm{Aut}(F))$ be a rank 3 group acting on $\mathrm{VSz}(q)$ by automorphisms. Set $G = \mathrm{Aut}(\mathrm{VSz}(q))$ and recall that $G = H^{(2)}$. By Lemma 2.5, G is an affine group with the same socle as H, and by Proposition 3.6 and Table 3, it follows that G lies in class (A7) or (A11), or it is one of the groups from Table 8. It can be easily checked that there is no group with degree q^4 and subdegrees $(q^2+1)(q-1)$, $q(q^2+1)(q-1)$ in Table 8, so the last possibility does not happen. By Lemma 3.13, G does not lie in (A7), so it is a group from class (A11). Denote by H_0 and G_0 zero stabilizers in H and G respectively. Notice that $H_0 \leq G_0$.

By Theorem 3.1 and Table 1, we have $G_0 \leq \Gamma\mathrm{L}_4(q)$ and $\mathrm{Sz}(q) \trianglelefteq G_0$. By [19, (1.4)], given $Z = Z(\mathrm{GL}_4(q)) \simeq F^\times$, the generalized Fitting subgroup of $G_0/(G_0 \cap Z)$ is a simple group. Hence $G_0/(G_0 \cap Z)$ is an almost simple group with socle $\mathrm{Sz}(q)$.

The outer automorphisms group of $\mathrm{Sz}(q)$ consists of field automorphisms only (see [7, Table 5]), so

$$|G_0| \leq |Z| \cdot |\mathrm{Aut}(\mathrm{Sz}(q))| \leq |F^\times||\mathrm{Sz}(q)||\mathrm{Aut}(F)|.$$

Since $H_0 \simeq (F^\times \times \mathrm{Sz}(q)) \rtimes \mathrm{Aut}(F)$, the order of H_0 coincides with the value on right-hand side of the inequality. Now $H_0 = G_0$ and the claim is proved. □

Proof of Theorem 1.1. Let G be a rank 3 group, and suppose that G is not listed in Table 8. By Proposition 2.8, we may assume that G is a primitive affine group which does not stabilize a product decomposition and, moreover, $G^{(2)}$ is also an affine group. By Theorem 3.1, G is either a one-dimensional affine group (class (A1)), or preserves a bilinear forms graph $H_q(2,m)$, $m \geq 2$, an affine polar graph $\mathrm{VO}_{2m}^\epsilon(q)$, $\epsilon = \pm$, $m \geq 2$, alternating forms graph $A(5,q)$, affine half-spin graph $\mathrm{VD}_{5,5}(q)$, Suzuki-Tits ovoid graph $\mathrm{VSz}(q)$ or lies in class (B) or (C).

The full automorphism groups of these graphs (i.e. 2-closures of respective groups) are described in Theorem 1.2 (one-dimensional affine groups), Proposition 3.7 (bilinear forms graph), Proposition 3.9 (affine polar graph), Proposition 3.10 (alternating forms graph), Proposition 3.12 (affine half-spin graph) and Proposition 3.14 (Suzuki-Tits ovoid graph). Notice that we do not need to consider classes (B) and (C) as they are included in Table 8.

Since by Proposition 3.6, the graph $H_q(2,2)$ is isomorphic to $\mathrm{VO}_4^+(q)$, we may exclude it from the bilinear forms case. Now it is easy to see that cases considered in Theorem 1.1 (iv) are mutually exclusive. Indeed, it suffices to prove that graphs from different cases are not isomorphic. By Proposition 3.6, if two affine rank 3 graphs have the same subdegrees, then they belong to the same case except for $\mathrm{VSz}(q)$ and $\mathrm{VO}_4^-(q)$, $q = 2^{2e+1}$, $e \geq 1$ (note that we group one-dimensional affine graphs into one case). By Lemma 3.13, graphs $\mathrm{VSz}(q)$ and $\mathrm{VO}_4^-(q)$ are not isomorphic, which proves the claim.

Finally, inclusions of the form $G \leq \mathrm{A\Gamma L}_a(q)$ can be read off Table 1. Notice that in some cases we do not give the minimal value of a, for example, if $\mathrm{SU}_m(q) \leq G$ lies in class (A6), then $G \leq \mathrm{A\Gamma L}_m(q^2)$, but we list the inclusion $G \leq \mathrm{A\Gamma L}_{2m}(q)$. This completes the proof of the theorem. □

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A Appendix

In this section we collect some relevant tabular data. Table 1 lists information on affine rank 3 groups from class (A), namely, for each group G it provides rough group-theoretical structure (column "Type of G"), degree n and subdegrees. Column "a" gives the smallest integer a such that the stabilizer of the zero vector G_0 lies in $\Gamma\mathrm{L}_a(p^{d/a})$. Most of the information in Table 1 is taken from [19, Table 12], see also [5, Table 10] for the values of a.

Table 2 lists the subdegrees of one-dimensional affine rank 3 groups. The first column specifies the type of graph associated to the group in question, next two columns provide degree and subdegrees, and the last column lists additional constraints on parameters involved. By [23], these graphs turn out to be either Van Lint–Schrijver, Paley or Peisert graphs. See [29, Section 2] for the parameters of the Van Lint–Schrijver graph; parameters of Paley and Peisert graphs are computed using the fact that these graphs are self-complementary.

Table 3 lists rank 3 graphs corresponding to rank 3 groups from classes (A1)–(A11), cf. [4, Table 11.4]. Terminology and graph notation is mostly consistent with [4], see also [5, Table 10].

Table 4 records information on when some families of affine rank 3 graphs can have identical subdegrees, the procedure for building this table being described in Proposition 3.6. Trivial cases (when two graphs are the same) are not listed, also graphs from cases (A1) and (A2) are omitted, since they are dealt with separately.

Tables 5 and 6 list degrees and subdegrees of affine rank 3 groups from classes (B) and (C), without repetitions (i.e. parameter sets are listed only once, regardless of whether several groups possess same parameters). If the smaller subdegree divides the largest, the last column gives the respective quotient; otherwise a dash is placed. Information in Table 5 is taken from [9, Theorem 1.1] and [19, Table 13], see also [5, Table 11]. Information in Table 6 before the horizontal line is taken from [10, Theorem 5.3], but notice that we exclude the case of 119^2, since 119 is not a prime number (that error was observed by Liebeck in [19]). Information in Table 6 after the horizontal line is taken from [19, Table 14], with the correction for the case of $\mathrm{Alt}(9)$, where subdegrees should be 120, 135 instead of 105, 150, as noted in [5, Table 12].

Table 7 lists parameters of possible exceptions to Theorem 1.2. The table consists of three subtables, corresponding to classes (A), (B) and (C) of Theorem 3.1, i.e. values for the first subtable are taken from Lemma 3.5, for the second from Lemma 3.3, and for the third from Lemma 3.4. Each subtable lists degrees and smallest subdegrees of possible exceptions. Notice that parameters of one-dimensional affine rank 3 groups stabilizing a nontrivial product decomposition are collected in the subtable for the class (A).

Table 8 lists possible exceptions to Theorem 1.1. The first column references the statement where a possible exception first appears, the second column describes the structure of the group, and the third column gives its degree, either explicitly or by referencing another table. Notice that we include classes (B) and (C) of Theorem 3.1 in Table 8; corresponding groups can be found in [19, Table 1 and 2].

Finally, we mention that Tables 7 and 8 list *potential* exceptions to Theorems 1.2 and 1.1 respectively, in particular, it might be possible to remove some parameter sets and groups by a more careful analysis.

Table 1: Class (A) in the classification of affine rank 3 groups

	Type of G	$n = p^d$	a	Subdegrees
(A1):	$G_0 < \Gamma L_1(p^d)$	p^d	1	See Table 2
(A2):	G_0 imprimitive	p^{2m}	$2m$	$2(p^m - 1), (p^m - 1)^2$
(A3):	tensor product	q^{2m}	$2m$	$(q + 1)(q^m - 1), q(q^m - 1)(q^{m-1} - 1)$
(A4):	$G_0 \trianglerighteq \mathrm{SL}_m(\sqrt{q})$	q^m	m	$(\sqrt{q} + 1)(\sqrt{q}^m - 1), \sqrt{q}(\sqrt{q}^m - 1)(\sqrt{q}^{m-1} - 1)$
(A5):	$G_0 \trianglerighteq \mathrm{SL}_2(\sqrt[3]{q})$	q^2	2	$(\sqrt[3]{q} + 1)(q - 1), \sqrt[3]{q}(q - 1)(\sqrt[3]{q}^2 - 1)$
(A6):	$G_0 \trianglerighteq \mathrm{SU}_m(q)$	q^{2m}	m	$\begin{cases} (q^m - 1)(q^{m-1} + 1), q^{m-1}(q - 1)(q^m - 1), & m \text{ even} \\ (q^m + 1)(q^{m-1} - 1), q^{m-1}(q - 1)(q^m + 1), & m \text{ odd} \end{cases}$
(A7):	$G_0 \trianglerighteq \Omega^{\epsilon}_{2m}(q)$	q^{2m}	$2m$	$\begin{cases} (q^m - 1)(q^{m-1} + 1), q^{m-1}(q - 1)(q^m - 1), & \epsilon = + \\ (q^m + 1)(q^{m-1} - 1), q^{m-1}(q - 1)(q^m + 1), & \epsilon = - \end{cases}$
(A8):	$G_0 \trianglerighteq \mathrm{SL}_5(q)$	q^{10}	10	$(q^5 - 1)(q^2 + 1), q^2(q^5 - 1)(q^3 - 1)$
(A9):	$G_0 \trianglerighteq B_3(q)$	q^8	8	$(q^4 - 1)(q^3 + 1), q^3(q^4 - 1)(q - 1)$
(A10):	$G_0 \trianglerighteq D_5(q)$	q^{16}	16	$(q^8 - 1)(q^3 + 1), q^3(q^8 - 1)(q^5 - 1)$
(A11):	$G_0 \trianglerighteq \mathrm{Sz}(q)$	q^4	4	$(q^2 + 1)(q - 1), q(q^2 + 1)(q - 1)$

Table 2: Subdegrees of one-dimensional affine rank 3 groups

Graph	Degree	Subdegrees	Comments
Van Lint–Schrijver	$q = p^{(e-1)t}$	$\frac{1}{e}(q - 1), \frac{1}{e}(e - 1)(q - 1)$	$e > 2$ is prime, p is primitive $\pmod{e}$
Paley	q	$\frac{1}{2}(q - 1), \frac{1}{2}(q - 1)$	$q \equiv 1 \pmod 4$
Peisert	$q = p^{2t}$	$\frac{1}{2}(q - 1), \frac{1}{2}(q - 1)$	$p \equiv 3 \pmod 4$

Table 3: Rank 3 graphs in class (A)

	Type of G	Graph	Comments
(A1):	$G_0 < \Gamma\mathrm{L}_1(p^d)$	Van Lint–Schrijver, Paley or Peisert graph	
(A2):	G_0 imprimitive	Hamming graph	
(A3):	tensor product	bilinear forms graph $H_q(2, m)$	
(A4):	$G_0 \trianglerighteq \mathrm{SL}_m(\sqrt{q})$	bilinear forms graph $H_{\sqrt{q}}(2, m)$	$\mathrm{SL}_m(\sqrt{q})$ stabilizes an m-dimensional subspace over $\mathrm{GF}(\sqrt{q})$
(A5):	$G_0 \trianglerighteq \mathrm{SL}_2(\sqrt[3]{q})$	bilinear forms graph $H_{\sqrt[3]{q}}(2, 3)$	$\mathrm{SL}_2(\sqrt[3]{q})$ stabilizes a 2-dimensional subspace over $\mathrm{GF}(\sqrt[3]{q})$
(A6):	$G_0 \trianglerighteq \mathrm{SU}_m(q)$	affine polar graph $\mathrm{VO}^{\epsilon}_{2m}(q)$, $\epsilon = (-1)^m$	
(A7):	$G_0 \trianglerighteq \Omega^{\epsilon}_{2m}(q)$	affine polar graph $\mathrm{VO}^{\epsilon}_{2m}(q)$	
(A8):	$G_0 \trianglerighteq \mathrm{SL}_5(q)$	alternating forms graph $A(5, q)$	
(A9):	$G_0 \trianglerighteq B_3(q)$	affine polar graph $\mathrm{VO}^{+}_{8}(q)$	
(A10):	$G_0 \trianglerighteq D_5(q)$	affine half spin graph $\mathrm{VD}_{5,5}(q)$	
(A11):	$G_0 \trianglerighteq \mathrm{Sz}(q)$	Suzuki-Tits ovoid graph $\mathrm{VSz}(q)$	

Table 4: Intersections between classes based on subdegrees

	$\mathrm{VO}^{\pm}_{2\overline{m}}(\overline{q})$	$A(5, \overline{q})$	$\mathrm{VD}_{5,5}(\overline{q})$	$\mathrm{VSz}(\overline{q})$
$H_q(2, m)$	$q^{2m} = \overline{q}^{2\overline{m}}$ $q = \overline{q}^{\overline{m}-1}$ $m = \frac{\overline{m}}{\overline{m}-1}$ $m = \overline{m} = 2, q = \overline{q}$	$q^{2m} = \overline{q}^{10}$ $q = \overline{q}^2$ $m = \frac{10}{4}$ Impossible	$q^{2m} = \overline{q}^{16}$ $q = \overline{q}^3$ $m = \frac{8}{3}$ Impossible	$q^{2m} = \overline{q}^4$ $q = \overline{q}$ $m = 2$ $q(q^2 - 1)(q - 1) = q(q^2 + 1)(q - 1)$ Impossible
$\mathrm{VO}^{\pm}_{2m}(q)$		$q^{2m} = \overline{q}^{10}$ $q^{m-1} = \overline{q}^2$ $m = \frac{5}{3}$ Impossible	$q^{2m} = \overline{q}^{16}$ $q^{m-1} = \overline{q}^3$ $m = \frac{8}{5}$ Impossible	$q^{2m} = \overline{q}^4$ $q^{m-1} = \overline{q}$ $m = 2, q = \overline{q}$
$A(5, q)$			$q^{10} = \overline{q}^{16}$ $q^2 = \overline{q}^3$ Impossible	$q^{10} = \overline{q}^4$ $q^2 = \overline{q}$ Impossible
$\mathrm{VD}_{5,5}(q)$				$q^{16} = \overline{q}^4$ $q^3 = \overline{q}$ Impossible

Table 5: Subdegrees of rank 3 groups in class (B)

$n = p^d$	Subdegrees m_1, m_2	$\frac{m_2}{m_1}$ if it is an integer
2^6	$27, 36$	—
3^4	$32, 48$	—
7^2	$24, 24$	1
13^2	$72, 96$	—
17^2	$96, 192$	2
19^2	$144, 216$	—
23^2	$264, 264$	1
3^6	$104, 624$	6
29^2	$168, 672$	4
31^2	$240, 720$	3
47^2	$1104, 1104$	1
3^4	$16, 64$	4
5^4	$240, 384$	—
7^4	$480, 1920$	4
3^8	$1440, 5120$	—

Table 6: Subdegrees of rank 3 groups in class (C)

$n = p^d$	Subdegrees m_1, m_2	$\frac{m_2}{m_1}$ if it is an integer
3^4	$40, 40$	1
31^2	$(31-1)\cdot 12, (31-1)\cdot 20$	—
41^2	$(41-1)\cdot 12, (41-1)\cdot 30$	—
7^4	$(7^2-1)\cdot 20, (7^2-1)\cdot 30$	—
71^2	$(71-1)\cdot 12, (71-1)\cdot 60$	5
79^2	$(79-1)\cdot 20, (79-1)\cdot 60$	3
89^2	$(89-1)\cdot 30, (89-1)\cdot 60$	2
2^6	$18, 45$	—
5^4	$144, 480$	—
2^8	$45, 210$	—
7^4	$720, 1680$	—
2^8	$120, 135$	—
2^8	$102, 153$	—
3^6	$224, 504$	—
7^4	$240, 2160$	9
3^5	$22, 220$	10
3^5	$110, 132$	—
2^{11}	$276, 1771$	—
2^{11}	$759, 1288$	—
3^{12}	$65520, 465920$	—
2^{12}	$1575, 2520$	—
5^6	$7560, 8064$	—

Table 7: Possible exceptions to Theorem 1.2

(A)	Degree	2^4	2^6	2^8	2^{10}	2^{12}	2^{16}	3^2	3^4	3^6	5^2
	Subdegree	5	21	51	93	315	3855	4	16	104	8
(B)	Degree	3^4	3^6	7^2	7^4	17^2	23^2	47^2			
	Subdegree	16	104	24	480	96	264	1104			
(C)	Degree	3^4	89^2								
	Subdegree	40	2640								

Table 8: Possible exceptions to Theorem 1.1

Appearance	Type of group	Degree
Lemma 2.5	$\mathrm{P\Gamma L}_2(8)$	36
	M_{11}	55
	M_{12}	66
	M_{23}	253
	M_{24}	276
	$\mathrm{Alt}(9)$	120
	$G_2(q) \trianglelefteq G$	$q^3(q^3-1)/2$, where $q \in \{3,4,8\}$
	$\Omega_7(q) \trianglelefteq G$	$q^3(q^4-1)/\gcd(2,q-1)$, where $q \in \{2,3\}$
Theorem 3.1	(B) and (C)	Tables 5 and 6
Theorem 1.2	$G \leq \mathrm{A\Gamma L}_1(q)$	Table 7

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Nonlinear maps preserving the elementary symmetric functions

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Abstract

Let $\mathcal{M}_n$ be the algebra of all $n \times n$ complex matrices, and for a natural number $2 \leq k \leq n$ denote by $E_k(x)$ the kth elementary symmetric function on the eigenvalues of $x \in \mathcal{M}_n$. For two maps $\varphi, \psi\colon \mathcal{M}_n \to \mathcal{M}_n$, one of them being surjective, we prove that if $E_k(\lambda x + y) = E_k(\lambda\varphi(x) + \psi(y))$ for each $\lambda \in \mathbf{C}$ and $x, y \in \mathcal{M}_n$, then $\varphi = \psi$ on $\mathcal{M}_n$, the common value being a linear map from $\mathcal{M}_n$ into itself. In particular, for $3 \leq k \leq n$ the general form of φ and ψ can be computed explicitly.

Keywords: Elementary symmetric function, nonlinear, preserver.

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1 Introduction and statement of the result

For a natural number n, let us denote by $\mathcal{M}_n$ the algebra of all $n \times n$ matrices over the complex field $\mathbf{C}$. By $I_n \in \mathcal{M}_n$ we shall denote the $n \times n$ identity matrix. For $x \in \mathcal{M}_n$, by tr(x) we shall denote its usual trace, and by $\det(x)$ its determinant. Also, by $x^t \in \mathcal{M}_n$ we shall denote the transpose of x.

For $k \in \{1, ..., n\}$, a k-by-k principal submatrix of $x \in \mathcal{M}_n$ is the submatrix of x which lies in the rows and columns of x indexed by $J \subseteq \{1, ..., n\}$ with $|J| = k$. Equivalently, we eliminate from the matrix x the rows and the columns which are not in J. The determinant of the k-by-k principal submatrix given by $J \subseteq \{1, ..., n\}$ is called a k-by-k principal minor, and shall be denoted by $\Delta_J(x)$. There are $\binom{n}{k}$ different k-by-k principal minors, and put

$$E_k(x) = \sum_{|J|=k} \Delta_J(x) \qquad (x \in \mathcal{M}_n,\ k = \overline{1, n}). \tag{1.1}$$

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In particular, $k = 1$ in (1.1) gives $E_1(x) = \mathrm{tr}(x)$ and $k = n$ gives $E_n(x) = \det(x)$ for each $x \in \mathcal{M}_n$.

For $k \in \{1, ..., n\}$, the kth elementary symmetric function on the complex numbers $\lambda_1, ..., \lambda_n$ is

$$S_k(\lambda_1, ..., \lambda_n) = \sum_{1 \le i_1 < \cdots < i_k \le n} \prod_{j=1}^{k} \lambda_{i_j}. \tag{1.2}$$

(We have a sum of $\binom{n}{k}$ products in (1.2).) For $x \in \mathcal{M}_n$, if the spectrum $\sigma(x)$ of x (taking into account multiplicities) is $\{\alpha_1, ..., \alpha_n\}$, the equality

$$\det(\lambda I_n + x) = \lambda^n + \lambda^{n-1} E_1(x) + \cdots + E_n(x) \qquad (x \in \mathcal{M}_n,\ \lambda \in \mathbf{C})$$

implies that

$$E_k(x) = S_k(\sigma(x)) \qquad (x \in \mathcal{M}_n,\ k = \overline{1,n}).$$

Thus, for each $k \in \overline{1,n}$ and $x \in \mathcal{M}_n$ we have that $E_k(x)$ is the kth elementary symmetric function on the eigenvalues of x.

Frobenius studied in [5] linear maps on $\mathcal{M}_n$ which preserve the determinant. He proved that if $\phi : \mathcal{M}_n \to \mathcal{M}_n$ is a bijective linear map such that $\det(\phi(x)) = \det(x)$ for each $x \in \mathcal{M}_n$, there exist then invertible matrices $a, b \in \mathcal{M}_n$ satisfying $\det(ab) = 1$ such that either $\phi(x) = axb$ for each $x \in \mathcal{M}_n$, or $\phi(x) = ax^t b$ for each $x \in \mathcal{M}_n$. One way to generalize this result is to relax the linearity assumption on the map ϕ. In [4], Dolinar and Šemrl proved that we arrive at the same conclusion if we merely suppose that $\phi : \mathcal{M}_n \to \mathcal{M}_n$ is a surjective map such that

$$\det(\lambda x + y) = \det(\lambda\phi(x) + \phi(y)) \qquad (x, y \in \mathcal{M}_n,\ \lambda \in \mathbf{C}). \tag{1.3}$$

(In this case, the linearity for the map ϕ is part of the conclusion.) Another way to generalize the result of Frobenius is to consider the elementary symmetric functions E_k instead of the determinant. The following theorem was proved by Marcus and Purves in [7] for $4 \le k < n$ and by Beasley in [1] for $3 = k < n$.

Theorem 1.1 ([7, Theorem 3.1] and [1, Theorem 1.1]). *Let* $3 \le k < n$ *and let* $\phi : \mathcal{M}_n \to \mathcal{M}_n$ *be a linear map such that*

$$E_k(x) = E_k(\phi(x)) \qquad (x \in \mathcal{M}_n).$$

There exist then an invertible matrix $a \in \mathcal{M}_n$ *and* $\eta \in \mathbf{C}$ *satisfying* $\eta^k = 1$ *such that either*

$$\phi(x) = \eta a x a^{-1} \qquad (x \in \mathcal{M}_n),$$

or

$$\phi(x) = \eta a x^t a^{-1} \qquad (x \in \mathcal{M}_n).$$

The result of Frobenius shows that the conclusion of Theorem 1.1 does not hold if $k = n$. The same happens if $k = 1$ (see, for example, [6, Section 2]) or $k = 2$ (see, for example, [7, Section 4] or [6, Section 3].) However, if the linear map $\phi : \mathcal{M}_n \to \mathcal{M}_n$ preserves both the trace and the determinant, then there exists an invertible matrix $a \in \mathcal{M}_n$ such that either

$$\phi(x) = a x a^{-1} \qquad (x \in \mathcal{M}_n), \tag{1.4}$$

or

$$\phi(x) = ax^t a^{-1} \qquad (x \in \mathcal{M}_n). \tag{1.5}$$

(See, for example, [8, Theorem 1] or [9, Theorem 3].) Thus, if $E_k(x) = E_k(\phi(x))$ for each $k \in \{1, n\}$ and $x \in \mathcal{M}_n$, then ϕ is of the form given by (1.4) or (1.5). Also, if the linear map $\phi : \mathcal{M}_n \to \mathcal{M}_n$ satisfies $E_k(x) = E_k(\phi(x))$ for each $k \in \{2, n\}$ and $x \in \mathcal{M}_n$, there exists then an invertible matrix $a \in \mathcal{M}_n$ such that either

$$\phi(x) = \eta a x a^{-1} \qquad (x \in \mathcal{M}_n), \tag{1.6}$$

or

$$\phi(x) = \eta a x^t a^{-1} \qquad (x \in \mathcal{M}_n), \tag{1.7}$$

where $\eta = 1$ if n is odd, and $\eta = -1$ or $\eta = 1$ if n is even. (See, for example, [9, Theorem 4].)

The aim of this article is to improve the results of Theorem 1.1 in a way that is similar to [4, Theorem 1.1]. Thus, we eliminate the linearity assumption on the map ϕ and we impose a strengthened preservation condition which is suggested by (1.3). Incidentally, the result holds with a preservation condition which is stated for two maps φ and ψ on $\mathcal{M}_n$ instead of a single map ϕ.

Theorem 1.2. *Let $2 \leq k \leq n$ and consider two maps $\varphi, \psi : \mathcal{M}_n \to \mathcal{M}_n$, one of them being surjective, such that*

$$E_k(\lambda x + y) = E_k(\lambda\varphi(x) + \psi(y)) \qquad (x, y \in \mathcal{M}_n,\ \lambda \in \mathbf{C}). \tag{1.8}$$

Then $\varphi = \psi$ on $\mathcal{M}_n$, the common value being a linear map of $\mathcal{M}_n$ into itself.

As a corollary, we obtain the following generalization of Theorem 1.1.

Corollary 1.3. *Let $3 \leq k < n$ and consider two maps $\varphi, \psi : \mathcal{M}_n \to \mathcal{M}_n$, one of them being surjective, such that (1.8) holds. There exist then an invertible matrix $a \in \mathcal{M}_n$ and $\eta \in \mathbf{C}$ satisfying $\eta^k = 1$ such that either*

$$\varphi(x) = \psi(x) = \eta a x a^{-1} \qquad (x \in \mathcal{M}_n),$$

or

$$\varphi(x) = \psi(x) = \eta a x^t a^{-1} \qquad (x \in \mathcal{M}_n).$$

Of course, if we suppose that (1.8) holds for $k \in \{1, n\}$, then $\varphi = \psi$ on $\mathcal{M}_n$, the common value being a linear map of the form (1.4) or (1.5), and if we suppose that (1.8) holds for $k \in \{2, n\}$, then $\varphi = \psi$ on $\mathcal{M}_n$, the common value being a linear map of the form (1.6) or (1.7).

Since Theorem 1.2 also holds for $k = n$, we obtain a different proof for the following slight generalization of [4, Theorem 1.1]. (See also [2, Theorem 1] and [3, Theorem 1].)

Corollary 1.4. *Consider two maps $\varphi, \psi : \mathcal{M}_n \to \mathcal{M}_n$, one of them being surjective, such that*

$$\det(\lambda x + y) = \det(\lambda\varphi(x) + \psi(y)) \qquad (x, y \in \mathcal{M}_n,\ \lambda \in \mathbf{C}).$$

There exist then invertible matrices $a, b \in \mathcal{M}_n$ satisfying $\det(ab) = 1$ such that either

$$\varphi(x) = \psi(x) = axb \qquad (x \in \mathcal{M}_n),$$

or

$$\varphi(x) = \psi(x) = ax^t b \qquad (x \in \mathcal{M}_n).$$

2 Preliminary lemmas

Let $2 \leq k \leq n$. For $x, y \in \mathcal{M}_n$, consider the complex polynomial (with respect to λ) given by

$$\lambda \mapsto E_k(\lambda x + y).$$

This section is devoted to the study of these polynomials. As a general property, let us observe that its degree is always at most k, the coefficient of λ^k being exactly $E_k(x)$. In particular, the degree of the polynomial is also bounded by the rank of the matrix x. Also, if we fix $x \in \mathcal{M}_n$, then the coefficient of λ^{k-1} is linear with respect to $y \in \mathcal{M}_n$. Indeed, this comes from (1.1) and the fact that for $J \subseteq \{1, ..., n\}$ with $|J| = k$ we have

$$\Delta_J(\lambda x + y) = \lambda^k \Delta_J(x) + \lambda^{k-1}\mathrm{tr}(\mathrm{adj}(x_J)y_J) + \cdots + \Delta_J(y),$$

where x_J (respectively, y_J) is the principal submatrix of x (respectively, y) corresponding to J, and for $z \in \mathcal{M}_k$ by $\mathrm{adj}(z) \in \mathcal{M}_k$ we have denoted the (classical) adjoint of the matrix z, obtained from its cofactors.

This coefficient will play an important role in our approach to prove Theorem 1.1, and shall be studied thoroughly in this section.

Lemma 2.1. *Let $2 \leq k \leq n$, and let $1 \leq i, j \leq n$, with $i \neq j$. There exists then a matrix $x_0 \in \mathcal{M}_n$ such that, for each $y \in \mathcal{M}_n$ we have that the coefficient of λ^k for the polynomial $\lambda \mapsto E_k(\lambda x_0 + y)$ is zero, and the coefficient of λ^{k-1} for the same polynomial is y_{ij}.*

Proof. Suppose, without lost of generality, that $i = 1$ and $j = 2$. Put then $J_0 = \{1, ..., k\}$, and let

$$x_0 = \begin{bmatrix} \begin{matrix} 0 & 0 \\ -1 & 0 \end{matrix} & 0 & 0 \\ 0 & I_{k-2} & 0 \\ 0 & 0 & 0_{n-k} \end{bmatrix} \in \mathcal{M}_n.$$

For $y \in \mathcal{M}_n$, let us observe that

$$\begin{aligned} \Delta_{J_0}(\lambda x_0 + y) &= \det \begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1k} \\ -\lambda + y_{21} & y_{22} & \cdots & y_{2k} \\ \vdots & \vdots & \multicolumn{2}{c}{\lambda I_{k-2} + (y_{st})_{3 \leq s,t \leq k}} \\ y_{k1} & y_{k2} & & \end{bmatrix} \\ &= 0 \cdot \lambda^k + y_{12}\lambda^{k-1} + \cdots . \end{aligned}$$

Also, if $|J| = k$ and $J \neq J_0$, then the degree of $\lambda \mapsto \Delta_J(\lambda x_0 + y)$ with respect to λ is at most $k-2$, since we have at most $k-2$ appearances of λ in the principal submatrix of $\lambda x_0 + y$ corresponding to J. We use now (1.1) to finish the proof. □

The remaining of this section is devoted to prove that the same type of statement as the one of Lemma 2.1 also holds for $i = j$. This requires a little extra work.

Lemma 2.2. *Let $2 \leq k \leq n$, and let $J_0 \subseteq \{1, ..., n\}$ with $|J_0| = k - 1$. There exists then a matrix $x_0 \in \mathcal{M}_n$ such that, for each $y \in \mathcal{M}_n$ we have that the coefficient of λ^k for the polynomial $\lambda \mapsto E_k(\lambda x_0 + y)$ is zero, and the coefficient of λ^{k-1} for the same polynomial is $\sum_{j \notin J_0} y_{jj}$.*

Proof. Suppose, without lost of generality, that $J_0 = \{1, ..., k-1\}$. Let then

$$x_0 = \begin{bmatrix} I_{k-1} & 0 \\ 0 & 0_{n-k+1} \end{bmatrix} \in \mathcal{M}_n.$$

For $y \in \mathcal{M}_n$, let us observe that for each $j \in \{1, ..., n\} \backslash J_0$ we have that

$$\Delta_{J_0 \cup \{j\}}(\lambda x_0 + y) = 0 \cdot \lambda^k + y_{jj}\lambda^{k-1} + \cdots .$$

Also, if $|J| = k$ and $|J \cap J_0| \leq k-2$, then the degree of $\lambda \mapsto \Delta_J(\lambda x_0 + y)$ with respect to λ is at most $k-2$, since we have at most $k-2$ appearances of λ in the principal submatrix of $\lambda x_0 + y$ corresponding to J. We use again (1.1) to finish the proof. □

Corollary 2.3. *Let $2 \leq k \leq n$, and let $1 \leq i, j \leq n$, with $i \neq j$. There exist then matrices $x_1, x_2 \in \mathcal{M}_n$ such that, for each $y \in \mathcal{M}_n$ we have that the coefficient of λ^k for the polynomial $\lambda \mapsto E_k(\lambda x_1 + y) - E_k(\lambda x_2 + y)$ is zero, and the coefficient of λ^{k-1} for the same polynomial is $y_{jj} - y_{ii}$.*

Proof. Consider $J \subseteq \{1, ..., n\} \backslash \{i, j\}$ such that $|J| = k-2$. We apply then Lemma 2.2 to $J \cup \{i\}$ to find a matrix $x_1 \in \mathcal{M}_n$ such that, for each $y \in \mathcal{M}_n$ we have that the coefficient of λ^k for the polynomial $\lambda \mapsto E_k(\lambda x_1 + y)$ is zero, and the coefficient of λ^{k-1} for the same polynomial is $\sum_{t \notin (J \cup \{i\})} y_{tt}$, and we apply the same lemma to $J \cup \{j\}$ to find a matrix $x_2 \in \mathcal{M}_n$ such that, for each $y \in \mathcal{M}_n$ we have that the coefficient of λ^k for the polynomial $\lambda \mapsto E_k(\lambda x_2 + y)$ is zero, and the coefficient of λ^{k-1} for the same polynomial is $\sum_{t \notin (J \cup \{j\})} y_{tt}$. To finish the proof, observe that $(\sum_{t \notin (J \cup \{i\})} y_{tt}) - (\sum_{t \notin (J \cup \{j\})} y_{tt}) = y_{jj} - y_{ii}$. □

Lemma 2.4. *Let $2 \leq k \leq n$, and let $j \in \{1, ..., n\}$. There exist then $q > 0$, $m, p \geq 0$ in $\mathbf{Z}$ and matrices $x_0, x_1, .., x_{m+p} \in \mathcal{M}_n$ such that, for each $y \in \mathcal{M}_n$ we have that the coefficient of λ^k for the polynomial*

$$\lambda \mapsto \sum_{i=0}^{m} E_k(\lambda x_i + y) - \left(\sum_{i=m+1}^{m+p} E_k(\lambda x_i + y) \right)$$

is zero, and the coefficient of λ^{k-1} for the same polynomial is $q \cdot y_{jj}$.

Proof. Consider $J \subseteq \{1, ..., n\} \backslash \{j\}$ such that $|J| = k-1$. We apply Lemma 2.2 to J to find a matrix $x_0 \in \mathcal{M}_n$ such that, for each $y \in \mathcal{M}_n$ we have that the coefficient of λ^k for the polynomial $\lambda \mapsto E_k(\lambda x_0 + y)$ is zero, and the coefficient of λ^{k-1} for the same polynomial is $\sum_{t \notin J} y_{tt} = y_{jj} + \sum_{t \notin (J \cup \{j\})} y_{tt}$. For each $t \notin (J \cup \{j\})$ in $\{1, ..., n\}$, we apply Corollary 2.3 to $t \neq j$ to find matrices $x_t^{(1)}, x_t^{(2)} \in \mathcal{M}_n$ such that, for each $y \in \mathcal{M}_n$ we have that the coefficient of λ^k for the polynomial $\lambda \mapsto E_k(\lambda x_t^{(1)} + y) - E_k(\lambda x_t^{(2)} + y)$ is zero, and the coefficient of λ^{k-1} for the same polynomial is $y_{jj} - y_{tt}$. To finish the proof, observe that

$$\sum_{t \notin J} y_{tt} + \left(\sum_{t \notin (J \cup \{j\})} (y_{jj} - y_{tt}) \right) = q \cdot y_{jj},$$

for some strictly positive integer q. □

3 Proof of the main result

Let $2 \leq k \leq n$. Let us first observe that if $\varphi, \psi : \mathcal{M}_n \to \mathcal{M}_n$ satisfy (1.8), dividing by $\lambda \in \mathbf{C}\backslash\{0\}$ we obtain that $E_k(x + \mu y) = E_k(\varphi(x) + \mu\psi(y))$ for all $x, y \in \mathcal{M}_n$ and $\mu \in \mathbf{C}\backslash\{0\}$. By continuity, the same holds for $\mu = 0$, too. Thus

$$E_k(x + \mu y) = E_k(\varphi(x) + \mu\psi(y)) \qquad (x, y \in \mathcal{M}_n,\ \mu \in \mathbf{C}). \tag{3.1}$$

That is, the same type of equalities as the ones in (1.8) hold, with the role of φ and ψ interchanged. Thus, without lost of generality, we may suppose for the remaining of the paper that the map φ is surjective. (If not, then ψ must be surjective, and we work with (3.1) instead of (1.8).)

Another immediate observation is the fact that if φ and ψ satisfy (1.8), for $\lambda = 0$ in (1.8) and $\mu = 0$ in (3.1) we see that $E_k(y) = E_k(\psi(y))$ for all $y \in \mathcal{M}_n$, respectively $E_k(x) = E_k(\varphi(x))$ for all $x \in \mathcal{M}_n$.

As a corollary of Lemma 2.1 and Lemma 2.4, the following result holds.

Theorem 3.1. *Suppose $2 \leq k \leq n$, and let $i, j \in \{1, ..., n\}$. There exist then a nonzero scalar α, positive integers m and p and matrices $x_0, x_1, .., x_{m+p} \in \mathcal{M}_n$ such that, for each $y \in \mathcal{M}_n$ we have that the coefficient of λ^k for the polynomial*

$$\lambda \mapsto \sum_{s=0}^{m} E_k(\lambda x_s + y) - \left(\sum_{s=m+1}^{m+p} E_k(\lambda x_s + y) \right)$$

is zero, and the coefficient of λ^{k-1} for the same polynomial is $\alpha \cdot y_{ij}$.

As a direct corollary of Theorem 3.1, we obtain the following test for the equality to $0 \in \mathcal{M}_n$ in terms of the functions E_k. (See also [7, Lemma 3.1].)

Corollary 3.2. *Suppose $2 \leq k \leq n$. Let $y \in \mathcal{M}_n$ such that*

$$E_k(x + y) = E_k(x) \qquad (x \in \mathcal{M}_n). \tag{3.2}$$

Then $y = 0$.

Proof. Observe that (3.2) gives

$$E_k(\lambda x + y) = \lambda^k E_k(x) \qquad (x \in \mathcal{M}_n,\ \lambda \in \mathbf{C}). \tag{3.3}$$

Let $i, j \in \{1, ..., n\}$. By Theorem 3.1, there exist $\alpha \neq 0$, positive integers m and p and matrices $x_0, x_1, .., x_{m+p} \in \mathcal{M}_n$ such that, for all $\lambda \in \mathbf{C}$,

$$\sum_{s=0}^{m} E_k(\lambda x_s + y) - \left(\sum_{s=m+1}^{m+p} E_k(\lambda x_s + y) \right) = 0 \cdot \lambda^k + (\alpha y_{ij}) \cdot \lambda^{k-1} + \cdots .$$

Using (3.3), for all $\lambda \in \mathbf{C}$ we have that

$$\begin{aligned} \sum_{s=0}^{m} E_k(\lambda x_s + y) - \left(\sum_{s=m+1}^{m+p} E_k(\lambda x_s + y) \right) &= \lambda^k \left(\sum_{s=0}^{m} E_k(x_s) - \left(\sum_{s=m+1}^{m+p} E_k(x_s) \right) \right) \\ &= 0. \end{aligned}$$

Thus $\alpha y_{ij} = 0$, and therefore $y_{ij} = 0$. Since this holds for any i and j, we obtain that $y = 0 \in \mathcal{M}_n$. □

Theorem 3.1 gives us also linearity for the maps φ and ψ from the statement of Theorem 1.1.

Proof of Theorem 1.1. Let us prove first that (1.8) and the surjectivity of φ implies that ψ is linear on $\mathcal{M}_n$. To see this, consider $i, j \in \{1, ..., n\}$ and let us prove that $\psi_{ij} : \mathcal{M}_n \to \mathbf{C}$ is linear, where ψ_{ij} is the (i, j) entry of the map ψ. By Theorem 3.1, there exist $\alpha \neq 0$ in $\mathbf{C}$, natural numbers m and p and matrices $x_0, x_1, .., x_{m+p} \in \mathcal{M}_n$ such that, for each $y \in \mathcal{M}_n$ we have that the coefficient of λ^k for the polynomial

$$\lambda \mapsto \sum_{s=0}^{m} E_k(\lambda x_s + y) - \left(\sum_{s=m+1}^{m+p} E_k(\lambda x_s + y) \right)$$

is zero, and the coefficient of λ^{k-1} for the same polynomial is αy_{ij}. Since φ is supposed surjective, let $w_0, w_1, .., w_{m+p} \in \mathcal{M}_n$ such that $\varphi(w_j) = x_j$ for $j = 0, ..., m + p$. Then for each $z \in \mathcal{M}_n$, we have that the coefficient of λ^k for the polynomial

$$\lambda \mapsto \sum_{s=0}^{m} E_k(\lambda\varphi(w_s) + \psi(z)) - \left(\sum_{s=m+1}^{m+p} E_k(\lambda\varphi(w_s) + \psi(z)) \right)$$

is zero, and the coefficient of λ^{k-1} for the same polynomial is $\alpha\psi_{ij}(z)$. Using (1.8), for all $\lambda \in \mathbf{C}$ we have that

$$\sum_{s=0}^{m} E_k(\lambda\varphi(w_s) + \psi(z)) - \left(\sum_{s=m+1}^{m+p} E_k(\lambda\varphi(w_s) + \psi(z)) \right)$$

equals

$$\sum_{s=0}^{m} E_k(\lambda w_s + z) - \left(\sum_{s=m+1}^{m+p} E_k(\lambda w_s + z) \right).$$

The remark at the beginning of Section 2 shows that the coefficient of λ^{k-1} for the polynomial $\lambda \mapsto \sum_{s=0}^{m} E_k(\lambda w_s + z) - (\sum_{s=m+1}^{m+p} E_k(\lambda w_s + z))$ is linear with respect to $z \in \mathcal{M}_n$. Therefore, ψ_{ij} is linear with respect to $z \in \mathcal{M}_n$.

Thus $\psi : \mathcal{M}_n \to \mathcal{M}_n$ is linear and $E_k(x) = E_k(\psi(x))$ for each $x \in \mathcal{M}_n$. If $\psi(y) = 0$, then for each $x \in \mathcal{M}_n$ we have that

$$\begin{aligned} E_k(x) &= E_k(\psi(x)) = E_k(\psi(x) + \psi(y)) = E_k(\psi(x + y)) \\ &= E_k(x + y). \end{aligned}$$

Then Corollary 3.2 gives $y = 0$. Thus the linear map ψ is injective on $\mathcal{M}_n$, and therefore bijective. Using (1.8), the linearity of ψ and the fact that $E_k(z) = E_k(\psi^{-1}(z))$ for each z, then for each $x, y \in \mathcal{M}_n$ we have that

$$\begin{aligned} E_k(x + y) &= E_k(\varphi(x) + \psi(y)) = E_k(\psi^{-1}(\varphi(x) + \psi(y))) \\ &= E_k((\psi^{-1} \circ \varphi)(x) + y). \end{aligned}$$

Denoting $z = x + y$, we conclude that $E_k(z) = E_k(((\psi^{-1} \circ \varphi)(x) - x) + z)$ for each $x, z \in \mathcal{M}_n$. Then Corollary 3.2 gives $(\psi^{-1} \circ \varphi)(x) - x = 0$, equality which holds for every $x \in \mathcal{M}_n$. Thus $\varphi = \psi$ on $\mathcal{M}_n$. $\square$

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A characterization of exceptional pseudocyclic association schemes by multidimensional intersection numbers

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Abstract

Recent classification of $\frac{3}{2}$-transitive permutation groups leaves us with three infinite families of groups which are neither 2-transitive, nor Frobenius, nor one-dimensional affine. The groups of the first two families correspond to special actions of $\mathrm{PSL}(2,q)$ and $\mathrm{P\Gamma L}(2,q)$, whereas those of the third family are the affine solvable subgroups of $\mathrm{AGL}(2,q)$ found by D. Passman in 1967. The association schemes of the groups in each of these families are known to be pseudocyclic. It is proved that apart from three particular cases, each of these exceptional pseudocyclic schemes is characterized up to isomorphism by the tensor of its 3-dimensional intersection numbers.

Keywords: Association schemes, groups, coherent configurations.

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1 Introduction

In the late 1960s, H. Wielandt proposed a method for studying permutation groups via invariant relations. Later, D. Higman axiomatized a part of this method (connected with binary relations) by introducing a new object called a coherent configuration [6]. The coherent configuration of a permutation group G is formed by the orbits of the induced action of G on the Cartesian square of the underlying set of points (for exact definitions, see Section 2). Looking only at the parameters of this coherent configuration, the so-called intersection numbers, one can easily determine whether the original group is transitive, primitive, 2-transitive, etc. For example, the transitivity of a group G means exactly that the coherent configuration of G is an association scheme.

The concept of *pseudocyclic* (association) scheme goes back to research of D. Mesner [11], related with constructing of designs and strongly regular graphs; the defining property of such a scheme is that the ratio of the multiplicity and degree of its nonprincipal irreducible character does not depend on the choice of the character. It was proved in [13, Theorem 3.2] that this is equivalent to a certain relation for intersection numbers, see Subsection 2.6.

The class of pseudocyclic schemes contains all Frobenius schemes, i.e., the coherent configurations of the Frobenius groups, and, moreover, every pseudocyclic scheme of rank sufficiently large comparing with its degree is Frobenius [13, Theorem 1.1]. Thus the pseudocyclic schemes can be considered as combinatorial analogs of the Frobenius groups. It should be mentioned that the analogy is not complete, because there exist schurian (i.e., those associated with permutation groups) pseudocyclic schemes which are not Frobenius, as well as non-schurian pseudocyclic schemes [2, Example 2.6.15].

It is well known that every Frobenius group is $\frac{3}{2}$-transitive, i.e., is transitive and all the orbits of the stabilizer of a point α, other than $\{\alpha\}$, have the same size greater than 1. It immediately follows that so is the automorphism group of any Frobenius scheme. Moreover, the above mentioned relation for intersection numbers implies that the automorphism group of a schurian pseudocyclic scheme is also $\frac{3}{2}$-transitive. A recent classification of $\frac{3}{2}$-transitive permutation groups shows that in most cases the coherent configuration of a $\frac{3}{2}$-transitive group is pseudocyclic, see Subsection 6.1. We cite a part of this classification in the following theorem, see [10, Corollaries 2,3].

Theorem 1.1. *Let G be a $\frac{3}{2}$-transitive permutation group of degree n. Assume that neither G is 2-transitive or Frobenius nor $G \leq \mathrm{A\Gamma L}(1,q)$ for some prime power q. Then apart from finitely many cases,*

(1) *$n = q(q-1)/2$, $q = 2^d \geq 8$, and either $G = \mathrm{PSL}(2,q)$, or d is prime and $G = \mathrm{P\Gamma L}(2,q)$,*

(2) *$n = q^2$, q is odd, and $G \leq \mathrm{AGL}(2,q)$ is the affine group with point stabilizer of order $4(q-1)$, consisting of all monomial matrices of determinant ± 1.*

Remark 1.2. The finitely many cases mentioned in Theorem 1.1 include some affine groups of degree at most 13^4 and the groups $\mathrm{Alt}(7)$ and $\mathrm{Sym}(7)$ both of degree 21.

The association schemes of the groups in statement (1) of Theorem 1.1 appeared in Master Thesis of H. Hollmann (1982); such a scheme is called a *large* or *small Hollmann* *scheme* depending on whether $G = \mathrm{PSL}(2,q)$ or $G = \mathrm{P\Gamma L}(2,q)$.[1] These schemes have been studied in [8]; in particular, it was proved there that both are pseudocyclic. The group in statement (2) of Theorem 1.1 appeared in D. Passman's characterization of solvable $\frac{3}{2}$-transitive groups [16]. The association scheme of this group, the *Passman scheme*, is also pseudocyclic [13].

The goal of the present paper is to establish combinatorial characterizations of the Hollmann and Passman schemes; from the point of view of Theorem 1.1, they can naturally be considered as exceptional. One of the best possible combinatorial characterization of an association scheme is obtained when the scheme in question is determined up to (combinatorial) isomorphism by its intersection numbers; in this case the scheme is called separable. However, most of association schemes are not separable. In [4], multidimensional intersection numbers and separability number $s(\mathcal{X})$ of a coherent configuration $\mathcal{X}$ have been introduced and studied (see also [2, Section 3.5 and 4.2]). According to the definition, $s(\mathcal{X}) \le m$ if and only if $\mathcal{X}$ is determined up to isomorphism by its m-dimensional intersection numbers; thus $s(\mathcal{X}) = 1$ if and only if $\mathcal{X}$ is separable. It was proved in [4] that $s(\mathcal{X}) = 1$ or 2 if $\mathcal{X}$ is the scheme of a Hamming, Johnson, or Grassmann graph; later, the estimate $s(\mathcal{X}) \le 3$ has been established in [5] for any cyclotomic scheme $\mathcal{X}$ over finite field.

Theorem 1.3. *Let $\mathcal{X}$ be a large Hollmann scheme. Then $s(\mathcal{X}) \le 2$.*

The proof of Theorem 1.3 is given in Section 3. The difficult step in the proof is to verify that the one point extension of the scheme $\mathcal{X}$ (which is a combinatorial analog of a one point stabilizer of a permutation group) is a coherent configuration of the stabilizer of this point in $\mathrm{Aut}(\mathcal{X})$. In proving this fact we use the formulas for the intersection numbers of $\mathcal{X}$, which were calculated in [8].

The proofs of the following two theorems are based on Theorem 4.1 (see Section 4), giving a sufficient condition for an arbitrary coherent configuration $\mathcal{X}$ to be *partly regular*, i.e., to be the coherent configuration of a permutation group having a faithful regular orbit. Using this sufficient condition we are able to show that if $\mathcal{X}$ is the small Hollmann scheme (apart from several exceptions) or the Passman scheme, then a two point extension of $\mathcal{X}$ is partly regular. Modulo known results, this immediately implies that $s(\mathcal{X}) \le 3$.

Theorem 1.4. *Let $\mathcal{X}$ be a small Hollmann scheme of degree $q(q-1)/2$, where $q = 2^d$ with prime $d \ne 7, 11, 13$. Then $s(\mathcal{X}) \le 3$.*

In the three exceptional cases of Theorem 1.4, the sufficient condition given in Theorem 4.1 does not work. It seems that the conclusion of Theorem 1.4 is also true for them. However, the corresponding schemes are too large to check this statement by a direct calculation.

Theorem 1.5. *Let $\mathcal{X}$ be a Passman scheme. Then $s(\mathcal{X}) \le 3$.*

Throughout the paper, we actively use the notation, concepts, and statements from the theory of coherent configurations. All of them can be found in the monograph [2]. In Section 2, we give a brief extract from the theory of coherent configurations that is relevant for this paper.

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[1] According to the Galois correspondence between permutation groups and coherent configurations [2, Section 2.2], the smaller groups correspond to larger coherent configurations

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Notation.

- For a prime power q, $\mathbb{F}_q$ is a finite field of order q.
- Throughout the paper, Ω is a finite set.
- The diagonal of the Cartesian product $\Omega \times \Omega$ is denoted by 1_Ω; for $\alpha \in \Omega$, we set $1_\alpha := 1_{\{\alpha\}}$.
- For $r \subseteq \Omega \times \Omega$, we set $r^* = \{(\beta, \alpha) : (\alpha, \beta) \in r\}$ and $\alpha r = \{\beta \in \Omega : (\alpha, \beta) \in r\}$, $\alpha \in \Omega$.
- For relations $r, s \subseteq \Omega \times \Omega$, we set $r \cdot s = \{(\alpha, \beta) : (\alpha, \gamma) \in r, (\gamma, \beta) \in s\}$.
- For a set S of relations on Ω, we define $S^* = \{s^* : s \in S\}$ and put $S^\cup$ to be the set of all unions of the relations of S.

2 Coherent configurations

2.1 Rainbows

Let Ω be a finite set and S a partition of $\Omega \times \Omega$. A pair $\mathcal{X} = (\Omega, S)$ is called a *rainbow* on Ω if

$$1_\Omega \in S^\cup, \text{ and } S^* = S.$$

The elements of the sets Ω, $S = S(\mathcal{X})$, and $S^\cup$ are called, respectively, the *points*, *basis relations*, and *relations* of $\mathcal{X}$. The numbers $|\Omega|$ and $|S|$ are called the *degree* and *rank* of $\mathcal{X}$, respectively. The unique basis relation containing a pair $(\alpha, \beta) \in \Omega \times \Omega$ is denoted by $r_\mathcal{X}(\alpha, \beta)$; we omit the subscript $\mathcal{X}$ wherever it does not lead to misunderstanding.

A set $\Delta \subseteq \Omega$ is called a *fiber* of the rainbow $\mathcal{X}$ if $1_\Delta \in S$; the set of all fibers is denoted by $F := F(\mathcal{X})$. The point set Ω is the disjoint union of fibers. If Δ is a union of fibers, then the pair

$$\mathcal{X}_\Delta = (\Delta, S_\Delta)$$

is a rainbow, where S_Δ consists of all $s_\Delta = s \cap (\Delta \times \Delta)$, $s \in S$.

Let $\mathcal{X} = (\Omega, S)$ and $\mathcal{X}' = (\Omega', S')$ be rainbows. A bijection $f : \Omega \to \Omega'$ is called a *combinatorial isomorphism* (or simply isomorphism) from $\mathcal{X}$ to $\mathcal{X}'$ if $S^f = S'$, where $S^f = \{s^f : s \in S\}$. When $\mathcal{X} = \mathcal{X}'$, the set of all these isomorphisms form a permutation group on Ω. This group has a (normal) subgroup

$$\mathrm{Aut}(\mathcal{X}) = \{f \in \mathrm{Sym}(\Omega) : s^f = s \text{ for all } s \in S\},$$

called the *automorphism group* of $\mathcal{X}$.

2.2 Coherent configurations

A rainbow $\mathcal{X} = (\Omega, S)$ is called a *coherent configuration* if for any $r, s, t \in S$, the number

$$c_{rs}^t = |\alpha r \cap \beta s^*|$$

does not depend on the choice of $(\alpha, \beta) \in t$; the numbers c_{rs}^t are called the *intersection numbers* of $\mathcal{X}$. If, in addition, $1_\Omega \in S$, then the coherent configuration $\mathcal{X}$ is said to be *homogeneous*, an *association scheme*, or just a *scheme*. A scheme $\mathcal{X}$ is called *symmetric* if $s = s^*$ for all $s \in S$.

Let $\mathcal{X}$ be a coherent configuration. Then for any $s \in S$, there exist uniquely determined $\Delta, \Gamma \in F$ such that $s \subseteq \Delta \times \Gamma$. Denote by $S_{\Delta,\Gamma}$ the set of all s contained in $\Delta \times \Gamma$. Then the union

$$S = \bigcup_{\Delta,\Gamma \in F} S_{\Delta,\Gamma}$$

is disjoint. The positive integer $|\delta s|$, $\delta \in \Delta$, equals the intersection number $c_{ss^*}^{1_\Delta}$, and hence does not depend on the choice of δ. It is called the *valency* of s and denoted by n_s. In homogeneous case, $n_s = n_{s^*}$ and also

$$n_t c_{rs}^{t^*} = n_r c_{st}^{r^*} = n_s c_{tr}^{s^*}, \qquad r, s, t \in S. \tag{2.1}$$

A basis relation $s \in S$ is called a *matching* if $n_s = n_{s^*} = 1$; note that s is not necessarily symmetric like in theory of undirected graphs. The matching $s \in S_{\Delta,\Gamma}$ defines a bijection from Δ to Γ, taking $\delta \in \Delta$ to the unique point of the singleton δs. Furthermore, one can see that if $r \in S$ and $t = s \cdot r$ (respectively, $t = r \cdot s$) is nonempty, then $t \in S$.

Let G be a permutation group on Ω. Denote by $(\alpha, \beta)^G$ the orbit of the induced action of G on $\Omega \times \Omega$, that contains the pair (α, β). Then

$$\mathrm{Inv}(G) = \mathrm{Inv}(G, \Omega) = (\Omega, \{(\alpha, \beta)^G : \alpha, \beta \in \Omega\})$$

is a coherent configuration; we say that $\mathrm{Inv}(G)$ is the coherent configuration associated with G. A coherent configuration $\mathcal{X}$ is said to be *schurian* if $\mathcal{X} = \mathrm{Inv}(\mathrm{Aut}(\mathcal{X}))$.

2.3 Separability

Let $\mathcal{X} = (\Omega, S)$ and $\mathcal{X}' = (\Omega', S')$ be coherent configurations. A bijection $\varphi : S \to S',\ s \mapsto s'$, is called an *algebraic isomorphism* from $\mathcal{X}$ to $\mathcal{X}'$ if

$$c_{rs}^t = c_{r's'}^{t'}, \qquad r, s, t \in S. \tag{2.2}$$

When $\mathcal{X} = \mathcal{X}'$, the set of all such φ forms a subgroup of $\mathrm{Sym}(S)$, denoted by $\mathrm{Aut}_{\mathrm{alg}}(\mathcal{X})$.

Each isomorphism f from $\mathcal{X}$ to $\mathcal{X}'$ induces an algebraic isomorphism from $\mathcal{X}$ to $\mathcal{X}'$, which maps $r \in S$ to $r^f \in S'$. A coherent configuration $\mathcal{X}$ is said to be *separable* if every algebraic isomorphism from $\mathcal{X}$ is induced by a suitable bijection (which is an isomorphism of the coherent configurations in question).

The algebraic isomorphism φ induces a bijection from $S^\cup$ to $(S')^\cup$: the union $r \cup s \cup \cdots$ of basis relations of $\mathcal{X}$ is mapped to $r' \cup s' \cup \cdots$. This bijection is also denoted by φ. It preserves the dot product, i.e., $\varphi(r \cdot s) = \varphi(r) \cdot \varphi(s)$ for all $r, s \in S$.

2.4 Coherent closure

There is a natural partial order $\leq$ on the set of all rainbows on the same set Ω. Namely, given two such rainbows $\mathcal{X}$ and $\mathcal{X}'$, we set

$$\mathcal{X} \leq \mathcal{X}' \;\Leftrightarrow\; S(\mathcal{X})^{\cup} \subseteq S(\mathcal{X}')^{\cup}.$$

The minimal and maximal elements with respect to this order are the *trivial* and *discrete* coherent configurations, respectively: the basis relations of the former are the reflexive relation 1_Ω and (if $|\Omega| > 1$) its complement in $\Omega \times \Omega$, whereas the basis relations of the latter are singletons.

The functors $\mathcal{X} \to \mathrm{Aut}(\mathcal{X})$ and $G \to \mathrm{Inv}(G)$ form a Galois correspondence between the posets of coherent configurations and permutation groups on the same set, i.e.,

$$\mathcal{Y} \leq \mathcal{X} \Rightarrow \mathrm{Aut}(\mathcal{Y}) \geq \mathrm{Aut}(\mathcal{X}) \quad \text{and} \quad L \leq K \Rightarrow \mathrm{Inv}(L) \geq \mathrm{Inv}(K),$$

and

$$\mathrm{Aut}(\mathrm{Inv}(\mathrm{Aut}(\mathcal{X}))) = \mathrm{Aut}(\mathcal{X}) \quad \text{and} \quad \mathrm{Inv}(\mathrm{Aut}(\mathrm{Inv}(G))) = \mathrm{Inv}(G).$$

The *coherent closure* $\mathrm{WL}(T)$ of a set T of relations on Ω, is defined to be the smallest coherent configuration on Ω, for which T is a set of relations. The *point extension* $\mathcal{X}_{\alpha,\beta,\ldots}$ of the rainbow $\mathcal{X}$ with respect to the points $\alpha, \beta, \ldots \in \Omega$ is defined to be $\mathrm{WL}(T)$, where T consists of $S(\mathcal{X})$ and the relations $1_\alpha, 1_\beta, \ldots$. In other words, $\mathcal{X}_{\alpha,\beta,\ldots}$ is the smallest coherent configuration on Ω that is larger than or equal to $\mathcal{X}$ and has singletons $\{\alpha\}, \{\beta\}, \ldots$ as fibers.

2.5 Multidimensional intersection numbers

The theory of multidimensional extensions of coherent configurations has been developed in [4], see also [2, Section 3.5].

Let $m \geq 1$ be an integer. The m-extension of a coherent configuration $\mathcal{X}$ on Ω is defined to be the smallest coherent configuration on Ω^m, which contains the Cartesian m-power of $\mathcal{X}$ and for which the set $\mathrm{Diag}(\Omega^m)$ is a union of fibers. The intersection numbers of the m-extension are called the *m-dimensional intersection numbers* of the configuration $\mathcal{X}$. If $m = 1$, then the m-extension of $\mathcal{X}$ coincides with $\mathcal{X}$ and the m-dimensional intersection numbers of $\mathcal{X}$ are the ordinary intersection numbers.

An algebraic isomorphism φ from $\mathcal{X}$ to $\mathcal{X}'$ is said to be *m-dimensional* if it can be extended to an algebraic isomorphism from the m-extension of $\mathcal{X}$ to that of $\mathcal{X}'$, that takes $\mathrm{Diag}(\Omega^m)$ to $\mathrm{Diag}(\Omega'^m)$. The *separability number* $s(\mathcal{X})$ of the coherent configuration $\mathcal{X}$ is defined to be the smallest positive integer m for which every m-dimensional algebraic isomorphism from $\mathcal{X}$ is induced by some isomorphism. Thus, the equality $s(\mathcal{X}) = m$ expresses the fact that $\mathcal{X}$ is determined up to isomorphism by its tensor of the m-dimensional intersection numbers. The following statement was proved in [4, Theorem 4.6(1)].

Lemma 2.1. *Let $\mathcal{X}$ be a coherent configuration. Then $s(\mathcal{X}) \leq s(\mathcal{X}_\alpha) + 1$ for any point α of $\mathcal{X}$.*

2.6 Pseudocyclic schemes

Let $\mathcal{X} = (\Omega, S)$ be a coherent configuration. The *indistinguishing number* of a relation $s \in S(\mathcal{X})$ is defined to be the sum $c(s)$ of the intersection numbers $c_{rr^*}^s$, $r \in S$. For each pair $(\alpha,\beta)\in s$, we have $c(s)=|c(\alpha,\beta)|$, where

$$c(\alpha,\beta)=\{\gamma\in\Omega:\ r(\gamma,\alpha)=r(\gamma,\beta)\}. \tag{2.3}$$

The maximum $c(\mathcal{X})$ of the numbers $c(s)$, where s runs over the set of all irreflexive basis relations of $\mathcal{X}$, is called the *indistinguishing number* of $\mathcal{X}$. It is easily seen that $c=0$ if and only if $n_s=1$ for each $s\in S$.

Assume that $\mathcal{X}$ is a scheme. In accordance with [13, Theorem 3.2], $\mathcal{X}$ is *pseudocyclic* of valency k if the equalities

$$c(s)+1=k=n_s$$

hold for all irreflexive $s\in S$. The class of pseudocyclic schemes includes all (homogeneous) coherent configurations associated with regular or Frobenius groups.

2.7 Partly regular coherent configurations

A coherent configuration $\mathcal{X}$ is said to be *partly regular* if there exists a point $\alpha\in\Omega$ such that $|\alpha s|\le 1$ for all $s\in S$; the point α is said to be *regular*. When all the points of $\mathcal{X}$ are regular, we say that $\mathcal{X}$ is *semiregular*, and *regular* if $\mathcal{X}$ is a scheme. Thus, $\mathcal{X}$ is semiregular if and only if $c(\mathcal{X})=0$.

The following statement taken from [2, Theorem 3.3.19] shows, in particular, that the partly regular (respectively, semiregular, regular) coherent configurations are in one-to-one correspondence with those of the form $\mathrm{Inv}(G)$, where G is a permutation group having a faithful orbit (respectively, G is semiregular, regular).

Theorem 2.2. *Every partly regular coherent configuration $\mathcal{X}$ is schurian and separable. In particular, $s(\mathcal{X})=1$.*

The key point in the proof of Theorem 2.2 is the lemma below [2, Lemma 3.3.20]; it is also used in the proof of Theorem 1.3.

Lemma 2.3. *Let $\mathcal{X}$ be a coherent configuration and Δ a union of fibers of $\mathcal{X}$. Assume that for every $\Gamma\in F$ there exists $s\in S_{\Delta,\Gamma}$ such that $n_s=1$. Then*

(1) *the restriction mapping $\mathrm{Aut}(\mathcal{X})\to\mathrm{Aut}(\mathcal{X}_\Delta)$ is a group isomorphism,*

(2) *$\mathcal{X}$ is schurian and separable whenever $\mathcal{X}_\Delta$ is schurian and separable.*

3 Large Hollmann schemes

3.1 General properties

Throughout this section, $d\ge 3$ is an integer, $q=2^d$, and $G=\mathrm{PSL}(2,q)$ the permutation group of degree $n=q(q-1)/2$ from Theorem 1.1(1). The lemma below immediately follows from Theorem 1.2(iii) and Lemma 6.2 proved in [1].

Lemma 3.1. *For any point α, we have $G_\alpha=D_{2(q+1)}$. Moreover,*

$$|\Delta|=q+1,\qquad \Delta\in\mathrm{Orb}(G_\alpha),\ \Delta\ne\{\alpha\}. \tag{3.1}$$

To study the large Hollmann scheme $\mathcal{X}=\mathrm{Inv}(G)$, we make use of some results proved in [8]. However, formally, the definition of the scheme given there is different from the definition of $\mathcal{X}$. Thus our first goal is to verify that $\mathcal{X}$ is exactly the symmetric pseudocyclic scheme $\mathcal{X}'$ of degree n and valency $q+1$ defined in [13] and studied in [8].

First, we note that $\mathrm{Aut}(\mathcal{X})$ is a $\frac{3}{2}$-transitive group of the same degree as G; in particular, $\mathrm{Aut}(\mathcal{X})$ is not a subgroup of $\mathrm{A\Gamma L}(1,r)$ for some r. Moreover, it is neither 2-transitive, nor Frobenius (because $G \le \mathrm{Aut}(\mathcal{X})$). By Theorem 1.1 and Remark 1.2, this implies that $\mathrm{Aut}(\mathcal{X}) = \mathrm{PSL}(2,q)$ or $\mathrm{P\Gamma L}(2,q)$. The latter case is impossible by Lemma 3.1, because G and $\mathrm{Aut}(\mathcal{X})$ have the same subdegrees. Consequently, $G = \mathrm{Aut}(\mathcal{X})$. On the other hand, the scheme $\mathcal{X}'$ is also associated with G. Therefore a similar argument shows that $\mathrm{Aut}(\mathcal{X}') = G = \mathrm{Aut}(\mathcal{X})$. Thus,

$$\mathcal{X}' = \mathrm{Inv}(\mathrm{Aut}(\mathcal{X}')) = \mathrm{Inv}(\mathrm{Aut}(\mathcal{X})) = \mathcal{X}. \tag{3.2}$$

Proposition 3.2. *The large Hollmann scheme $\mathcal{X}$ is symmetric and pseudocyclic of degree $q(q-1)/2$, rank $q/2$, and valency $q+1$. Moreover,*

$$\mathrm{Aut}(\mathcal{X}) = G \quad \textit{and} \quad \mathrm{Aut}(\mathcal{X}_\alpha) = G_\alpha = D_{2(q+1)} \textit{ for all } \alpha. \tag{3.3}$$

Proof. By the remark before the proposition, we need to verify the second equality in (3.3) only. By [2, Proposition 3.3.3(1)], we have $\mathrm{Aut}(\mathcal{X}_\alpha) = \mathrm{Aut}(\mathcal{X})_\alpha$. Thus the required statement immediately follows from the first equality in (3.3) and Lemma 3.1. □

Equality (3.2) allows us to use formulas for the intersection numbers of the scheme $\mathcal{X}'$, given in [8, Theorem 2.2]. Namely, let $S = S(\mathcal{X})$ and

$$\mathbf{T}_0 = \{x \in \mathbb{F}_{2^d} : \mathrm{Tr}(x) = 0\},$$

where $\mathrm{Tr}(x)$ is the trace of x over the prime subfield of the field $\mathbb{F}_{2^d}$. Then there is a bijection $\mathbf{T}_0 \to S, x \mapsto s_x$, such that s_0 is reflexive and

$$c_{s_x,s_y}^{s_z} = 1 \quad \Leftrightarrow \quad \mathrm{Tr}(xz) = 0 \text{ and } x+y+z=0. \tag{3.4}$$

As is easily seen, $\mathbf{T}_0$ is a linear space of dimension $d-1$ over $\mathbb{F}_2$.

3.2 One point extension

Let us analyze the extension $\mathcal{X}_\alpha$ of the large Hollmann scheme $\mathcal{X}$ with respect to a point α. Since the scheme $\mathcal{X}$ is schurian, each fiber of the coherent configuration $\mathcal{X}_\alpha$ is of the form $\Delta = \alpha s$ for some $s \in S$ [2, Theorem 3.3.7]. When $s = s_x$ for some $x \in \mathbf{T}_0$, the fiber Δ is denoted by Δ_x. Thus,

$$F(\mathcal{X}_\alpha) = \{\Delta_x : x \in \mathbf{T}_0\}.$$

Theorem 3.3. *Let x and y be nonzero elements of $\mathbf{T}_0$. Then the set $S(\mathcal{X}_\alpha)_{\Delta_x,\Delta_y}$ contains a matching.*

Proof. For $d=3$, the statement has been checked with the help of the computer package COCO2P [9]. Assume that $d \ge 4$. We need auxiliary lemmas.

Lemma 3.4. *Theorem 3.3 holds whenever $\mathrm{Tr}(xy) = 0$.*

Proof. Let $z = x + y$. Then obviously $z \in \mathbf{T}_0$. Moreover

$$\mathrm{Tr}(xz) = \mathrm{Tr}(x^2 + xy) = \mathrm{Tr}(x^2) + \mathrm{Tr}(xy) = 0.$$

By formula (3.4), this implies that $c_{s_x,s_y}^{s_z} = 1$. If $z = 0$, then $x = y$ and 1_{Δ_x} is a desired matching. Assume that z is nonzero. Then $n_{s_x} = n_{s_z}$, because $\mathcal{X}$ is a pseudocyclic scheme (Proposition 3.2). Since $\mathcal{X}$ is also symmetric, we have

$$c_{s_y,s_z}^{s_x} = \frac{n_{s_z}}{n_{s_x}} c_{s_x,s_y}^{s_z} = 1,$$

see (2.1). Therefore if $r = s_z \cap (\alpha s_x \times \alpha s_y)$, then $|\beta r| = 1$ for all $\beta \in \Delta_x$, and $|\beta r^*| = 1$ for all $\beta \in \Delta_y$ (here we use the fact that $|\Delta_x| = |\Delta_y|$). Since r is a relation of $\mathcal{X}_\alpha$ [2, Lemma 3.3.5], this implies that r belongs to $S(\mathcal{X}_\alpha)_{\Delta_x,\Delta_y}$. Thus, r is a required matching. □

Let us define a graph $\mathfrak{X}$ with nonzero elements of $\mathbf{T}_0$ as the vertices and in which two distinct vertices x and y are adjacent if and only if $\mathrm{Tr}(xy) = 0$. One can see that $\mathfrak{X}$ is an undirected graph with exactly $|\mathbf{T}_0| - 1$ vertices.

Lemma 3.5. *The graph $\mathfrak{X}$ is connected.*

Proof. Given $x \in \mathbf{T}_0$, we define the linear mapping

$$f_x : \mathbf{T}_0 \to \mathbb{F}_2,\ y \mapsto \mathrm{Tr}(xy).$$

Now let $x \neq y$ be two vertices of the graph $\mathfrak{X}$. If $\mathrm{Tr}(xy) = 0$, then x and y are connected by an edge. Let $\mathrm{Tr}(xy) \neq 0$. Then both f_x and f_y are nonzero linear mappings, and $\ker(f_x)$ and $\ker(f_y)$ are subspaces of $\mathbf{T}_0$ of codimension one. Therefore, $\ker(f_x) \cap \ker(f_y)$ is a subspace of $\mathbf{T}_0$ of codimension at most two. Consequently,

$$\dim(\ker(f_x) \cap \ker(f_y)) \geq \dim(\mathbf{T}_0) - 2 = d - 3 \geq 1.$$

It follows that $\ker(f_x) \cap \ker(f_y)$ contains at least one nonzero vector, say z. Then $\mathrm{Tr}(xz) = \mathrm{Tr}(yz) = 0$. Since $\mathrm{Tr}(xy) \neq 0$, this implies that $z \neq x, y$. Thus the vertices x, y are at distance two in the graph $\mathfrak{X}$, in particular, $\mathfrak{X}$ is connected. □

Let us return to the proof of Theorem 3.3. By Lemma 3.5, the vertices x and y of the graph $\mathfrak{X}$ are connected by a path

$$x = x_0, x_1, \ldots, x_k = y,$$

where $k \geq 1$. For $i = 0, \ldots, k-1$, the vertices x_i and x_{i+1} are adjacent and hence $\mathrm{Tr}(x_i x_{i+1}) = 0$. Denote by s_i the matching in $S(\mathcal{X}_\alpha)_{\Delta_{x_i},\Delta_{x_{i+1}}}$ the existence of which is guaranteed by Lemma 3.4. Then the dot product

$$s = s_0 \cdot s_1 \cdots s_{k-1}$$

is a desired matching belonging to $S(\mathcal{X}_\alpha)_{\Delta_x,\Delta_y}$. □

Corollary 3.6. *Let $\Delta = \Delta_x$ for nonzero $x \in \mathbf{T}_0$. Then the coherent configuration $\mathcal{Y} = (\mathcal{X}_\alpha)_\Delta$ is schurian and separable. Moreover, the extension of $\mathcal{Y}$ with respect to at least one point is partly regular.*

Proof. By Proposition 3.2, we have $\mathrm{Aut}(\mathcal{X}_\alpha) = D_{2(q+1)}$. Furthermore, the hypothesis of Lemma 2.3 is satisfied for $\mathcal{X} = \mathcal{X}_\alpha$ by Theorem 3.3. Thus by statement (1) of that lemma, we have

$$H := \mathrm{Aut}(\mathcal{Y}) \cong \mathrm{Aut}(\mathcal{X}_\alpha) = D_{2(q+1)}. \tag{3.5}$$

On the other hand, $|\Delta| = q + 1$ by formula (3.1). Consequently, the group H contains a normal regular cyclic subgroup C of order $q + 1$. In terms of [2, Section 4.4], this means that $\mathcal{Y}$ is isomorphic to a normal circulant scheme. The radical of such a scheme, being a subgroup of the group C, is of order at most 2; this follows from the implication (1)⇔(3) in [5, Theorem 6.1]. Since the number $|C| = q + 1 = 2^d + 1$ is odd, the radical is trivial. Thus, the scheme $\mathcal{Y}$ is schurian by [2, Corollary 4.4.3], and every its extension with respect to at least one point is partly regular by [2, Theorem 4.4.7].

It remains to verify that $\mathcal{Y}$ is separable. Since $\mathcal{Y}$ is schurian by above, we have $\mathcal{Y} = \mathrm{Inv}(\mathrm{Aut}(\mathcal{Y})) = \mathrm{Inv}(H)$. By virtue of (3.5), this means that $\mathcal{Y}$ is the coherent configuration associated with $D_{2(q+1)}$. Thus, the required statement follows from [2, Exercise 2.7.33]. □

3.3 Proof of Theorem 1.3

By Lemma 2.1, it suffices to verify that a one point extension of a large Hollmann scheme is separable. But this immediately follows from Theorem 3.7 below.

Theorem 3.7. *Let $q = 2^d$ where $d \geq 3$. Then the extension of the large Hollmann scheme of degree $q(q-1)/2$ with respect to at least one point is schurian and separable.*

Proof. Let $\mathcal{X}'$ be the extension of the large Hollmann scheme $\mathcal{X}$ with respect to $m \geq 1$ points $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_m$. Let $x \in \mathbf{T}_0$ be nonzero and $\Delta = \Delta_x$. Then the hypothesis of Lemma 2.3 is satisfied for $\mathcal{X} = \mathcal{X}'$. Indeed, each $\Gamma \in F(\mathcal{X}')$ other than $\{\alpha\}$ is contained in Δ_y for some nonzero $y \in \mathbf{T}_0$. By Theorem 3.3, there is a matching $s' \in S(\mathcal{X}_\alpha)_{\Delta_x,\Delta_y}$, and as the required relation s one can take $s' \cap (\Delta \times \Gamma)$.

By Lemma 2.3(2), the coherent configuration $\mathcal{X}'$ is schurian and separable whenever so is $\mathcal{X}'_\Delta$. If $m = 1$, then $\mathcal{X}' = \mathcal{X}_\alpha$ and we are done by Corollary 3.6. Let $m > 1$. We claim that there exist $\beta_2, \ldots, \beta_m \in \Delta$ such that

$$\mathcal{X}' = \mathcal{X}_{\alpha,\beta_2,\ldots,\beta_m}. \tag{3.6}$$

Indeed, without loss of generality we may assume that none of the α_i, $i > 1$, equals α. By Theorem 3.3, there is a matching $s_i \in S(\mathcal{X}_\alpha)_{\Delta_i,\Delta}$, where Δ_i is the fiber of $\mathcal{X}_\alpha$, containing α_i. Then $\alpha_i s_i = \{\beta_i\}$ for some $\beta_i \in \Delta$. It follows that for any extension of $\mathcal{X}_\alpha$, each or none of the two singletons $\{\alpha_i\}$ and $\{\beta_i\}$ is a fiber of this extension, see [2, Corollary 3.3.6]. Thus,

$$\mathcal{X}' = \mathcal{X}_{\alpha,\alpha_2,\ldots,\alpha_m} = \mathcal{X}_{\alpha,\beta_2,\ldots,\beta_m},$$

which completes the proof of the claim.

Now from formula (3.6) and the fact that $\beta_i \in \Delta$ for all i, it easily follows that

$$\begin{aligned}\mathcal{X}'_\Delta &= (\mathcal{X}_{\alpha,\alpha_2,\ldots,\alpha_m})_\Delta = (\mathcal{X}_{\alpha,\beta_2,\ldots,\beta_m})_\Delta \\ &= ((\mathcal{X}_{\alpha,\beta_2,\ldots,\beta_m})_\Delta)_{\beta_2,\ldots,\beta_m} \geq ((\mathcal{X}_\alpha)_\Delta)_{\beta_2,\ldots,\beta_m}.\end{aligned}$$

The coherent configuration on the right-hand side of this relation is partly regular by Corollary 3.6. Therefore, the coherent configuration $\mathcal{X}'_\Delta$ is also partly regular. By Theorem 2.2, this implies that $\mathcal{X}'_\Delta$ is schurian and separable, as required. □

4 A lower bound for indistinguishing number

The main result of this section (Theorem 4.1 below) establishes a lower bound for the indistinguishing number of a coherent configuration which is not partly regular, cf. [17, Theorem 3.1]. This bound gives a sufficient condition for a coherent configuration to be partly regular, and is used to prove Theorems 1.4 and 1.5 in the next section.

Theorem 4.1. *Let $\mathcal{X}$ be a coherent configuration of degree n, k the maximal cardinality of a fiber of $\mathcal{X}$, and $c = c(\mathcal{X})$. If $\mathcal{X}$ is not partly regular, then*

$$(2k-1)c \geq n. \tag{4.1}$$

Proof. Let $\mathcal{X} = (\Omega, S)$. In the sequel, $\Delta \in F(\mathcal{X})$ and $|\Delta| = k$. The fiber Δ contains at least two points, for otherwise $k = 1$ and $\mathcal{X}$ is the discrete and hence partly regular configuration in contrast to the hypothesis of the theorem. Set

$$\Delta_\alpha = \{\delta \in \Delta :\ n_{r(\alpha,\delta)} = 1\} \quad \text{and} \quad \Omega_1 = \{\alpha \in \Omega :\ \Delta_\alpha = \Delta\}.$$

Lemma 4.2. *Ω_1 is a (possibly empty) union of fibers of cardinality k. Moreover, the coherent configuration $\mathcal{X}_{\Omega_1}$ is semiregular.*

Proof. Let Γ be a fiber containing a point of Ω_1. The set $S_{\Gamma,\Delta}$ consists of $s = r(\gamma, \delta)$, where $\gamma \in \Gamma \cap \Omega_1$ and δ runs over Δ. Consequently, $n_s = 1$ for all $s \in S_{\Gamma,\Delta}$. Therefore, $\Gamma \subseteq \Omega_1$. Thus, Ω_1 is a union of fibers of $\mathcal{X}$. Furthermore, given $s \in S_{\Gamma,\Delta}$ we have

$$k \geq |\Gamma| = n_s\,|\Gamma| = n_{s^*}\,|\Delta| \geq |\Delta| = k.$$

This proves the first statement. To prove the second one, let Δ' and Δ'' be fibers contained in Ω_1. By the definition of Ω_1 and the first statement, any relations $s' \in S_{\Delta',\Delta}$ and $s'' \in S_{\Delta,\Delta''}$ are matchings. Therefore, $s' \cdot s''$ is a matching contained in $S_{\Delta',\Delta''}$. Thus, S_{Ω_1} consists of matchings and we are done. □

By Lemma 4.2 and the hypothesis of the theorem, the complement Ω' of the set Ω_1 in Ω contains at least two distinct points.

Lemma 4.3. *For each $\gamma \in \Omega'$,*

$$\sum_{s \in S_\gamma} n_s \geq \frac{k}{2},$$

where $S_\gamma = \{r(\gamma, \delta) :\ \delta \in \Delta$ and $n_{r(\gamma,\delta)} > 1\}$.

Proof. We have

$$\sum_{s \in S_\gamma} n_s = \sum_{s \in S_\gamma} |\gamma s| = |\Delta| - |\Delta_\gamma|. \tag{4.2}$$

Since $|\Delta| = k$, this proves the required inequality if $\Delta_\gamma = \varnothing$. Let $\delta \in \Delta_\gamma$. It is easily seen that $r(\delta, \lambda) = r(\delta, \gamma) \cdot r(\gamma, \lambda)$ is a matching of $\mathcal{X}_\Delta$ for each $\lambda \in \Delta_\gamma$. Therefore,

$$\Delta_\gamma \subseteq \{\lambda \in \Delta :\ n_{r(\delta,\lambda)} = 1\}. \tag{4.3}$$

On the other hand, denote by e the union of all matchings of the scheme $\mathcal{X}_\Delta$. Then e is a relation of this scheme. Moreover, e is an equivalence relation on Δ (see [2, Theorem 2.1.25(4)]) and the set on the right-hand side of (4.3) is a class of e. In view of [2, Corollary 2.1.23], the cardinality a of this class divides $|\Delta| = k$. Furthermore, $a \neq k$, for otherwise, $\Delta_\gamma = \Delta$ and then $\gamma \in \Omega_1$, a contradiction. Thus,

$$|\Delta_\gamma| \leq a \leq \frac{k}{2}$$

and the required statement follows from (4.2). □

Lemma 4.4. *Let* $\varepsilon = \frac{2(k-1)}{2k-1}$. *Assume that* $|\Omega'| \geq \varepsilon n$. *Then inequality* (4.1) *holds.*

Proof. Denote by N the cardinality of the set

$$\{(\alpha, \beta, \gamma) \in \Delta \times \Delta \times \Omega' : \ \alpha \neq \beta, \ \gamma \in c(\alpha, \beta)\}, \tag{4.4}$$

where $c(\alpha, \beta)$ is as in formula (2.3). The number of $(\alpha, \beta) \in \Delta \times \Delta$ with $\alpha \neq \beta$ is equal to $k(k-1)$. Therefore there exists at least one such pair for which

$$|c(\alpha, \beta)| \geq \frac{N}{k(k-1)}. \tag{4.5}$$

On the other hand, let $\gamma \in \Omega'$, and let S_γ be as in Lemma 4.3. Then $n_s \geq 2$ for all $s \in S_\gamma$. For every such s there are exactly $n_s(n_s - 1)$ triples (α, β, γ) with distinct $\alpha, \beta \in \gamma s$, and all these triples belong to the set (4.4). By Lemma 4.3 this implies that

$$N = \sum_{\gamma \in \Omega'} \sum_{s \in S_{\Gamma,\Delta}} n_s(n_s - 1) \geq \sum_{\gamma \in \Omega'} \sum_{s \in S_\gamma} n_s \geq \sum_{\gamma \in \Omega'} \frac{k}{2} = \frac{|\Omega'|\, k}{2},$$

where Γ is the fiber containing γ. By formula (4.5) and the lemma assumption, we obtain

$$c \geq |c(\alpha, \beta)| \geq \frac{N}{k(k-1)} \geq \frac{|\Omega'|}{2(k-1)} \geq \frac{2(k-1)n}{2k-1} \cdot \frac{1}{2(k-1)} = \frac{n}{2k-1},$$

as required. □

By Lemma 4.4, we may assume that $|\Omega'| < \varepsilon n$. The coherent configuration $\mathcal{X}$ is not partly regular. Therefore no point $\delta \in \Omega_1$ is regular and there exist distinct $\alpha, \beta \in \Omega'$ such that $\delta \in c(\alpha, \beta)$. Since the coherent configuration $\mathcal{X}_{\Omega_1}$ is semiregular (Lemma 4.2), the relation $s = r(\delta, \lambda)$ is a matching for all $\lambda \in \Omega_1$. It follows that

$$r(\alpha, \lambda) = r(\alpha, \delta) \cdot s = r(\beta, \delta) \cdot s = r(\beta, \lambda).$$

Consequently, $\Omega_1 \subseteq c(\alpha, \beta)$. This implies that

$$c \geq |c(\alpha, \beta)| \geq |\Omega_1| = n - \varepsilon n > n\left(1 - \frac{2(k-1)}{2k-1}\right) = \frac{n}{2k-1}$$

which completes the proof of Theorem 4.1. □

Corollary 4.5. *Let* $\mathcal{X}$ *be a coherent configuration of degree* n, $c = c(\mathcal{X})$, *and* t *an irreflexive basis relation of* $\mathcal{X}$. *Assume that* $(2m_t - 1)\, c < n$, *where*

$$m_t = \max_{r,s \in S} c_{rs}^t. \tag{4.6}$$

Then the extension of $\mathcal{X}$ *with respect to any two points forming a pair from* t *is partly regular.*

Proof. Let $\mathcal{X}'$ be the extension of $\mathcal{X}$ with respect to the points α, β such that $(\alpha, \beta) \in t$. Then each fiber Δ of $\mathcal{X}'$ different from both $\{\alpha\}$ and $\{\beta\}$ is contained in the set $\alpha r \cap \beta s^*$ for appropriate $r, s \in S$. It follows that

$$|\Delta| \leq |\alpha r \cap \beta s^*| = c_{rs}^t \leq m_t.$$

Thus the maximal cardinality k' of a fiber of $\mathcal{X}'$ is less than or equal to m_t. Since obviously $c' = c(\mathcal{X}')$ is less than or equal to c, the condition of the corollary implies that

$$(2k' - 1)\, c' \leq (2m_t - 1)\, c < n.$$

Thus $\mathcal{X}'$ is partly regular by Theorem 4.1. □

5 Small Hollmann and Passman schemes

5.1 Algebraic fusion

Let $\mathcal{X} = (\Omega, S)$ be a coherent configuration, and let Φ be a group of algebraic automorphisms of $\mathcal{X}$. For each $s \in S$, set

$$s^\Phi = \bigcup_{\varphi \in \Phi} \varphi(s).$$

Clearly, $(1_\Omega)^\Phi = 1_\Omega$. Moreover the set $S^\Phi = \{s^\Phi : s \in S\}$ forms a partition of the Cartesian square Ω^2. According to [2, Lemma 2.3.26], the pair

$$\mathcal{X}^\Phi = (\Omega, S^\Phi)$$

is a coherent configuration called the algebraic fusion of $\mathcal{X}$ with respect to Φ. In the following lemma, we establish a simple upper bound for the intersection numbers of an algebraic fusion.

Lemma 5.1. *In the above notation, let $r, s, t \in S$ and m_t be as in* (4.6). *Then*

$$c_{r^\Phi s^\Phi}^{t^\Phi} \leq m_t\, |\Phi|^2.$$

Proof. We have $c_{r^\Phi s^\Phi}^{t^\Phi} = \sum\limits_{\varphi, \psi \in \Phi} c_{\varphi(r)\, \psi(s)}^t \leq m_t\, |\Phi|^2$. □

5.2 Proof of Theorem 1.4

Let $\mathcal{X}$ be a small Hollmann scheme. Then the degree of $\mathcal{X}$ is equal to $n = q(q-1)/2$, where $q = 2^d$ for a prime $d \geq 3$. Moreover, $\mathcal{X}$ is associated with the permutation group $G = \mathrm{P\Gamma L}(2, q)$ of degree n from Theorem 1.1(1). As in Subsection 3.1, one can see that $\mathcal{X}$ coincides with symmetric pseudocyclic scheme $\mathcal{X}'$ of degree n and valency $d(q+1)$, associated with the group $\mathrm{P\Gamma L}(2, q)$ and studied in [8]. In particular, $\mathcal{X}$ is obtained from the large Hollmann scheme of degree n by merging the basis relations via the Frobenius map $x \mapsto x^2$, $x \in \mathbb{F}_q$. In other words, $\mathcal{X}$ is the algebraic fusion of the large Hollmann scheme of degree n with respect the induced action of $\mathrm{Aut}(\mathbb{F}_q)$ on its basis relations.

Proposition 5.2. *Let $\mathcal{X}$ and $\mathcal{X}_q$ be the small and large Hollmann schemes of degree $q(q-1)/2$, respectively. Then*

(1) $\mathcal{X} = \mathcal{X}_q^{\Phi}$, *where* $\Phi \le \mathrm{Aut}_{alg}(\mathcal{X}_q)$ *is a group of order* d,

(2) $\mathcal{X}$ *is a pseudocyclic scheme of valency* $d(q+1)$,

(3) *for each irreflexive* $t \in S(\mathcal{X})$, *we have* $m_t \le 4d^2$, *where* m_t *is as in* (4.6).

Proof. Statements (1) and (2) follow from the above discussion. Next, from the formulas for the intersection numbers of the scheme $\mathcal{X}_q$, given in [8, Theorem 2.2], it follows that $c_{xy}^z \le 4$ for all irreflexive $x, y, z \in S(\mathcal{X}_q)$. In particular,

$$m_z \le 4.$$

On the other hand, by statement (1), each irreflexive $t \in S(\mathcal{X})$ is of the form z^{Φ} for some irreflexive z. Thus by Lemma 5.1,

$$m_t = \max_{x,y \in S(\mathcal{X}_q)} c_{x^{\Phi} y^{\Phi}}^{z^{\Phi}} \le m_z |\Phi|^2 \le 4d^2,$$

which proves statement (3). □

Let us prove Theorem 1.4. If $d = 3$, then a straightforward calculation shows that $\mathcal{X}$ is trivial scheme and hence $c(\mathcal{X}) = 1$. Let $d > 3$. By Lemma 2.1 and Theorem 2.2, it suffices to verify that the extension of $\mathcal{X}$ with respect to at least two points is partly regular.

Theorem 5.3. *Let* $q = 2^d$, *where* $d > 3$ *is a prime and* $d \ne 7, 11, 13$. *Then every extension of the small Hollmann scheme of degree* $q(q-1)/2$ *with respect to at least two points is partly regular.*

Proof. Let $\mathcal{X}$ be the small Hollmann scheme of degree $n = q(q-1)/2$. By Proposition 5.2(2), the number $c = c(\mathcal{X})$ is equal to $d(2^d + 1) - 1$. By statement (3) of the same proposition, $m_t \le 4\,d^2$ for any irreflexive $t \in S(\mathcal{X})$. Now if $d > 16$, then

$$(2m_t - 1)\,c < (8d^2 - 1)\,(d\,(2^d + 1) - 1) < 2^{d-1}(2^d - 1) = n.$$

By Corollary 4.5, this proves the required statement for all (prime) $d > 13$. In the remaining case, $d = 5$, the required statement has been checked with the help of the computer package COCO2P [9]. □

5.3 Proof of Theorem 1.5

Let q be an odd prime power. The permutation group $G \le \mathrm{AGL}(2, q)$ defined in Theorem 1.1(2) has a Frobenius subgroup H consisting of the permutations

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} b \\ c \end{pmatrix}, \qquad x, y \in \mathbb{F}_q, \tag{5.1}$$

where $a, b, c \in \mathbb{F}_q$ and $a \ne 0$. The group $D \le \mathrm{GL}(2, q)$ consisting of the eight matrices

$$\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{pmatrix}$$

is contained in G, normalizes H, and, moreover, $G = H \rtimes D$. In particular, D acts on the basis relations of the Frobenius scheme $\mathcal{Y} = \mathrm{Inv}(H)$ as a group Φ of algebraic automorphisms.

Proposition 5.4. *Let $\mathcal{X} = \mathrm{Inv}(G)$ be the Passman scheme of degree q^2. Then*

(1) *$\mathcal{X} = \mathcal{Y}^{\Phi}$, where $\Phi \le \mathrm{Aut}_{alg}(\mathcal{Y})$ is a group of order 2,*

(2) *$\mathcal{X}$ is a pseudocyclic scheme of valency $2(q-1)$,*

(3) *there is an irreflexive $t \in S(\mathcal{X})$ such that $m_t \le 8$, where m_t is as in (4.6).*

Proof. Statements (1) and (2) follow from [13, Section 4.3]. To prove statement (3), let u be the basis relation of $\mathcal{Y}$, containing the pair (α, β), where

$$\alpha = (0,0) \quad \text{and} \quad \beta = (1,1).$$

It suffices to verify that $m_u \le 2$; indeed, then $t = u^{\Phi}$ is the required relation by statement (1) and Lemma 5.1.

We need to verify that $c_{rs}^u \le 2$ for all $r, s \in S(\mathcal{Y})$. Without loss of generality, we may assume that r and s are such that $c_{rs}^u \ne 0$. Then there is $\gamma \in \alpha r \cap \beta s^*$. It follows that $\alpha r = \gamma^{H_\alpha}$, $\beta s^* = \gamma^{H_\beta}$, and

$$c_{rs}^u = |\gamma^{H_\alpha} \cap \gamma^{H_\beta}|. \tag{5.2}$$

Let us calculate the number on the right-hand side. Using the explicit form (5.1) of the elements of H, one can easily find that the groups H_α and H_β consist of permutations the parameters a, b, c of which satisfy the relations

$$b = c = 0 \quad \text{and} \quad a + b = a^{-1} + c = 1,$$

respectively. Consequently, assuming $\gamma = (x, y)$, we have

$$\gamma^{H_\alpha} = \left\{ \left(ax, \frac{y}{a} \right) : \ a \in \mathbb{F}_q^* \right\} \quad \text{and} \quad \gamma^{H_\beta} = \left\{ \left(a'x + 1 - a', \frac{y}{a'} + 1 - \frac{1}{a'} \right) : a' \in \mathbb{F}_q^* \right\}.$$

In view of (5.2), the intersection number c_{rs}^u is equal to the number of elements $a \in \mathbb{F}_q^*$ such that

$$ax = a'x + 1 - a' \quad \text{and} \quad \frac{y}{a} = \frac{y}{a'} + 1 - \frac{1}{a'}.$$

If exactly one of x, y equals 0, then these equations are satisfied for $a = 1 = a'$ only, and $c_{rs}^u = 1$. Assume that $x \ne 0 \ne y$. Then

$$a = a' \left(1 - \frac{1}{x} \right) + \frac{1}{x} \quad \text{and} \quad \frac{1}{a} = \frac{1}{a'} \left(1 - \frac{1}{y} \right) + \frac{1}{y},$$

and hence

$$1 = \left(1 - \frac{1}{x} \right) \left(1 - \frac{1}{y} \right) + \frac{a'}{y} \left(1 - \frac{1}{x} \right) + \frac{1}{a'x} \left(1 - \frac{1}{y} \right) + \frac{1}{xy}.$$

This gives a quadratic equation with unknown a', and at most two possible values for a'. Thus, $c_{rs}^u \le 2$. □

Let us prove Theorem 1.5. By Lemma 2.1 and Theorem 2.2, it suffices to verify that the extension of $\mathcal{X}$ with respect to at least two points is partly regular.

Theorem 5.5. *Let q be an odd prime power. Then every extension of the Passman scheme of degree q^2 with respect to at least two points is partly regular.*

Proof. Let $\mathcal{X}$ be the Passman scheme of degree $n = q^2$. By Proposition 5.4(2), the number $c = c(\mathcal{X})$ is equal to $2(q-1)-1$. Let t be the basis relation of $\mathcal{X}$, defined in Proposition 5.4(3). Then $m_t \leq 8$. Now if $q \geq 29$, then

$$(2m_t - 1)\,c \leq 15\,(2q-3) < q^2 = n.$$

By Corollary 4.5, this proves the required statement for all $q \geq 29$. In the remaining cases, the required statement has been checked with the help of the computer package COCO2P [9]. □

6 Concluding remarks and open problems

6.1 Pseudocyclic schemes

A scheme is said to be *k-equivalenced* (and just equivalenced if k is irrelevant) if all irreflexive basis relations of it have valency k. It is known that every k-equivalenced scheme is pseudocyclic for $1 \leq k \leq 4$; this follows from results obtained in [12, 14, 15] and [13, Theorem 3.1].

By Theorem 1.1, if $\mathcal{X}$ is a schurian equivalenced scheme of sufficiently large degree, which is not trivial or Frobenius, then either $\mathcal{X}$ is exceptional, i.e., the Hollmann or Passman scheme, or the inclusion $\mathrm{Aut}(\mathcal{X}) \leq \mathrm{A\Gamma L}(1,q)$ holds for some q. In all cases except for the last one, $\mathcal{X}$ is pseudocyclic, see Propositions 3.2, 5.2(2), and 5.4(2), and [13, Theorem 3.1]. In the latter case, the group $\mathrm{A\Gamma L}(1,q)$ can contain $\frac{3}{2}$-transitive subgroups which are not 2-equivalent to Frobenius groups (such a subgroup is always primitive). We do not know whether the scheme of at least one of these subgroups is not pseudocyclic.

6.2 Separability number

Finding the exact values of $s(\mathcal{X})$ for an exceptional scheme $\mathcal{X}$ is still an open problem. A direct calculation shows that these schemes are separable for small q.

6.3 Superschurian schemes

The following concept was first formulated many years ago in discussions of the third author with Sergei Evdokimov. A scheme $\mathcal{X}$ is said to be *superschurian* if the extension of $\mathcal{X}$ with respect to every set of points is schurian. In particular, all superschurian schemes are schurian. In fact, only a few families of superschurian schemes are known; these include partly regular schemes, cyclotomic schemes over finite fields, normal circulant schemes, and some TI-schemes [3, 5, 13]. Theorem 3.7 implies that any large Hollmann scheme is superschurian. We do not know whether other exceptional schemes are superschurian.

6.4 Base number

The base number $b(\mathcal{X})$ of a coherent configuration $\mathcal{X}$ is defined to be the smallest number of points such that the extension of $\mathcal{X}$ with respect to them is the discrete configuration, see [2, Section 3.3.2]. In general, the base number of the group $\mathrm{Aut}(\mathcal{X})$ is less than or equal to $b(\mathcal{X})$. The equality is attained, for example, if $\mathcal{X}$ is a partly regular coherent configuration. By virtue of this observation, Theorems 3.7, 5.3, and 5.5 imply that except, possibly, for several small Hollmann schemes the equality holds also for all exceptional pseudocyclic schemes.

In fact, the base number of an exceptional scheme of enough large degree is bounded by 3. This fact can be used to construct a polynomial-time algorithm recognizing whether or not a given scheme is exceptional. Taking the above discussion into account, this reduces the recognition problem for the class of schurian equivalenced schemes (see [18, p.281]) to the non-Frobenius schemes $\mathcal{X}$ for which $\mathrm{Aut}(\mathcal{X}) \leq \mathrm{A\Gamma L}(1, q)$.

6.5 Bound in Theorem 4.1

Denote by $f(n)$ the maximum of the ratio $\frac{n}{c(\mathcal{X})k(\mathcal{X})}$ taken over all non-partly-regular coherent configurations $\mathcal{X}$ of degree n, where $k(\mathcal{X})$ is the maximal cardinality of a fiber of $\mathcal{X}$. Clearly, $f(n) > 0$. Theorem 4.1 states that $f(n) < 2$. It would be interesting to find the function $f(n)$ explicitly. We have found a (schurian) non-partly-regular coherent configuration with parameters

$$n = 24, \quad k = 8, \quad c = 4,$$

which shows that $f(24) \geq 3/4$.

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