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CI-groups with respect to ternary relational structures: new examples

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Abstract

We find a sufficient condition to establish that certain abelian groups are not CI-groups with respect to ternary relational structures, and then show that the groups $\mathbb{Z}_3 \times \mathbb{Z}_2^2$, $\mathbb{Z}_7 \times \mathbb{Z}_2^3$, and $\mathbb{Z}_5 \times \mathbb{Z}_2^4$ satisfy this condition. Then we completely determine which groups $\mathbb{Z}_2^3 \times \mathbb{Z}_p$, p a prime, are CI-groups with respect to color binary and ternary relational structures. Finally, we show that $\mathbb{Z}_2^5$ is not a CI-group with respect to ternary relational structures.

Keywords: CI-group, ternary relation.

Math. Subj. Class.: 05C15, 05C10

1 Introduction

In recent years, there has been considerable interest in which groups G have the property that any two Cayley graphs of G are isomorphic if and only if they are isomorphic by a group automorphism of G. Such a group is a called a *CI-group with respect to graphs*, and this problem is often referred to as the Cayley isomorphism problem. The interested

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reader is referred to [11] for a survey on CI-groups with respect to graphs. Of course, the Cayley isomorphism problem can and has been considered for other types of objects (see, for example, [9, 14, 16] for work on this problem on codes and on designs). Before proceeding we give the relevant definitions. (There are several equivalent definitions of combinatorial object [1, 15], here we follow [13].)

Definition 1.1. A *k-ary relational structure* is an ordered pair $X = (V, E)$, with V a set and E a subset of V^k. Furthermore, a *color k-ary relational structure* is an ordered pair $X = (V, (E_1, \ldots, E_c))$, with V a set and $E_1, \ldots, E_c$ pairwise disjoint subsets of V^k. If $k = 2, 3$, or 4, we simply say that X is a (color) binary, ternary, or quaternary relational structure. A *combinatorial object* is a pair $X = (V, E)$, with V a set and E a subset of $\bigcup_{i=1}^{\infty} V^i$.

The following two definitions are due to Babai [1].

Definition 1.2. For a group G, define $g_L : G \to G$ by $g_L(h) = gh$, and let $G_L = \{g_L : g \in G\}$. Then G_L is a permutation group on G, called the *left regular representation of G*. We will say that a (color) k-ary relational structure X is a *Cayley (color) k-ary relational structure of G* if $G_L \leq \mathrm{Aut}(X)$ (note that this implies $V = G$). In general, a combinatorial object X will be called a *Cayley object of G* if $G_L \leq \mathrm{Aut}(X)$.

Definition 1.3. For a class $\mathcal{C}$ of Cayley objects of G, we say that G is a *CI-group with respect to $\mathcal{C}$* if whenever $X, Y \in \mathcal{C}$, then X and Y are isomorphic if and only if they are isomorphic by a group automorphism of G.

It is clear that if G is a CI-group with respect to *color* k-ary relational structures, then G is a CI-group with respect to k-ary relational structures.

Perhaps the most significant result in this area is a well-known theorem of Pálfy [15] which states that a group G of order n is a CI-group with respect to every class of combinatorial objects if and only if $n = 4$ or $\gcd(n, \varphi(n)) = 1$, where φ is the Euler phi function. In fact, in proving this result, Pálfy showed that if a group G is not a CI-group with respect to some class of combinatorial objects, then G is not a CI-group with respect to quaternary relational structures. As much work has been done on the case of binary relational structures (i.e., digraphs), until recently there was a "gap" in our knowledge of the Cayley isomorphism problem for k-ary relational structures with $k = 3$. As additional motivation to study this problem, we remark that a group G that is a CI-group with respect to ternary relational structures is necessarily a CI-group with respect to binary relational structures, see [5, page 227].

Although Babai [1] showed in 1977 that the dihedral group of order $2p$ is a CI-group with respect to ternary relational structures, no additional work was done on this problem until the first author considered the problem in 2003 [5]. Indeed, in [5] a relatively short list of groups is given and it is proved that every CI-group with respect to ternary relational structures lies in this list (although not every group in this list is necessarily a CI-group with respect to ternary relational structures). Additionally, several groups in the list were shown to be CI-groups with respect to ternary relational structures. Recently, the second author [17] has shown that two groups given in [5] are not CI-groups with respect to ternary relational structures, namely $\mathbb{Z}_3 \ltimes Q_8$ and $\mathbb{Z}_3 \times Q_8$. In this paper, we give a sufficient condition to ensure that certain abelian groups are not CI-groups with respect to ternary relational structures (Theorem 2.1), and then show that $\mathbb{Z}_2^2 \times \mathbb{Z}_3$, $\mathbb{Z}_2^3 \times \mathbb{Z}_7$, and $\mathbb{Z}_2^4 \times \mathbb{Z}_5$

satisfy this condition in Corollary 2.4 (and so are not CI-groups with respect to ternary relational structures). We then show that $\mathbb{Z}_2^3 \times \mathbb{Z}_5$ is a CI-group with respect to ternary relational structures. As the first author has shown [6] that $\mathbb{Z}_2^3 \times \mathbb{Z}_p$ is a CI-group with respect to ternary relational structures provided that $p \geq 11$, we then have a complete determination of which groups $\mathbb{Z}_2^3 \times \mathbb{Z}_p$, p a prime, are CI-groups with respect to ternary relational structures.

Theorem A. The group $\mathbb{Z}_2^3 \times \mathbb{Z}_p$ is a CI-group with respect to color ternary relational structures if and only if $p \notin \{3, 7\}$.

We will show that both $\mathbb{Z}_2^3 \times \mathbb{Z}_3$ and $\mathbb{Z}_2^3 \times \mathbb{Z}_7$ are CI-groups with respect to color binary relational structures. As it is already known that $\mathbb{Z}_2^4$ is a CI-group with respect to binary relational structures [11], we have the following result.

Corollary A. The group $\mathbb{Z}_2^3 \times \mathbb{Z}_p$ is a CI-group with respect to color binary relational structures for all primes p.

We are then left in the situation of knowing whether or not any subgroup of $\mathbb{Z}_2^3 \times \mathbb{Z}_p$ is a CI-group with respect to color binary or ternary relational structures, with the exception of $\mathbb{Z}_2^2 \times \mathbb{Z}_7$ with respect to color ternary relational structures (as $\mathbb{Z}_2^2 \times \mathbb{Z}_7$ is a CI-group with respect to color binary relational structures [10]). We show that $\mathbb{Z}_2^2 \times \mathbb{Z}_7$ is a CI-group with respect to color ternary relational structures (which generalizes a special case of the main result of [10]) and we prove the following.

Corollary B. The group $\mathbb{Z}_2^2 \times \mathbb{Z}_p$ is a CI-group with respect to color ternary relational structures if and only if $p \neq 3$.

Finally, using Magma [2] and GAP [8], we show that $\mathbb{Z}_2^5$ is not a CI-group with respect to ternary relational structures. We conclude this introductory section by recalling the following.

Definition 1.4. For g, h in G, we denote the commutator $g^{-1}h^{-1}gh$ of g and h by $[g, h]$.

2 The main ingredient and Theorem A

We start by proving the main ingredient for our proof of Theorem A.

Theorem 2.1. *Let G be an abelian group and p an odd prime. Assume that there exists an automorphism α of G of order p fixing only the zero element of G. Then $\mathbb{Z}_p \times G$ is not a* CI-*group with respect to color ternary relational structures. Moreover, if there exists a ternary relational structure Z on G with* $\mathrm{Aut}(Z) = \langle G_L, \alpha \rangle$*, then $\mathbb{Z}_p \times G$ is not a* CI-*group with respect to ternary relational structures.*

Proof. Since α fixes only the zero element of G, we have $|G| \equiv 1 \pmod p$ and so $\gcd(p, |G|) = 1$.

For each $g \in G$, define $\hat{g} : \mathbb{Z}_p \times G \to \mathbb{Z}_p \times G$ by $\hat{g}(i, j) = (i, j + g)$. Additionally, define $\tau, \gamma, \bar{\alpha} : \mathbb{Z}_p \times G \to \mathbb{Z}_p \times G$ by $\tau(i, j) = (i + 1, j)$, $\gamma(i, j) = (i, \alpha^i(j))$, and $\bar{\alpha}(i, j) = (i, \alpha(j))$. Then $(\mathbb{Z}_p \times G)_L = \langle \tau, \hat{g} : g \in G \rangle$.

Clearly, $\langle G_L, \alpha \rangle = G_L \rtimes \langle \alpha \rangle$ is a subgroup of $\mathrm{Sym}(G)$ (where G_L acts on G by left multiplication and α acts as an automorphism). Note that the stabilizer of 0 in $\langle G_L, \alpha \rangle$ is $\langle \alpha \rangle$. As α fixes only 0, we conclude that for every $g \in G$ with $g \neq 0$, the point-wise stabilizer of 0 and g in $\langle G_L, \alpha\rangle$ is 1. Therefore, by [18, Theorem 5.12], there exists a color Cayley ternary relational structure Z of G such that $\mathrm{Aut}(Z) = \langle G_L, \alpha\rangle$. If there exists also a ternary relational structure with automorphism group $\langle G_L, \alpha\rangle$, then we let Z be one such ternary relational structure.

Let

$$\begin{aligned} U &= \{((0_{\mathbb{Z}_p}, g), (0_{\mathbb{Z}_p}, h)) : (0_G, g, h) \in E(Z)\}, \text{ and} \\ S &= \{([\hat{g}, \gamma](1, 0_G), [\hat{g}, \gamma](2, 0_G)) : g \in G\} \cup U \end{aligned}$$

and define a (color) ternary relational structure X by

$$V(X) = \mathbb{Z}_p \times G \quad \text{and} \quad E(X) = \{k(0_{\mathbb{Z}_p\times G}, s_1, s_2) : (s_1, s_2) \in S, k \in (\mathbb{Z}_p \times G)_L\}.$$

If Z is a color ternary relational structure, then we assign to the edge $k(0_{\mathbb{Z}_p\times G}, s_1, s_2)$ the color of the edge $(0_G, g, h)$ in Z if $(s_1, s_2) \in U$ and $(s_1, s_2) = ((0_{\mathbb{Z}_p}, g), (0_{\mathbb{Z}_p}, h))$, and otherwise we assign a fixed color distinct from those used in Z. By definition of X we have $(\mathbb{Z}_p \times G)_L \le \mathrm{Aut}(X)$ and so X is a (color) Cayley ternary relational structure of $\mathbb{Z}_p \times G$.

We claim that $\bar{\alpha} \in \mathrm{Aut}(X)$. As $\bar{\alpha}$ is an automorphism of $\mathbb{Z}_p \times G$, we see that $\bar{\alpha} \in \mathrm{Aut}(X)$ if and only if $\bar{\alpha}(S) = S$ and $\bar{\alpha}$ preserves colors (if X is a color ternary relational structure). By definition of Z and U, we have $\bar{\alpha}(U) = U$ and $\bar{\alpha}$ preserves colors (if X is a color ternary relational structure). So, it suffices to consider the case $s \in S - U$, i.e., $s = ([\hat{g}, \gamma](1, 0), [\hat{g}, \gamma](2, 0))$ for some $g \in G$. Note that now we need not consider colors as all the edges in $S - U$ are of the same color. Then $\bar{\alpha}\hat{g}(i, j) = (i, \alpha(j) + \alpha(g)) = \widehat{\alpha(g)}\bar{\alpha}(i, j)$. Thus $\bar{\alpha}\hat{g} = \widehat{\alpha(g)}\bar{\alpha}$. Similarly, $\bar{\alpha}\hat{g}^{-1} = \widehat{\alpha(g)}^{-1}\bar{\alpha}$. Clearly $\bar{\alpha}$ commutes with γ, and so $\bar{\alpha}[\hat{g}, \gamma] = [\widehat{\alpha(g)}, \gamma]\bar{\alpha}$. As $\bar{\alpha}$ fixes $(1, 0)$ and $(2, 0)$, we see that

$$\begin{aligned} \bar{\alpha}(s) = \bar{\alpha}([\hat{g}, \gamma](1, 0), [\hat{g}, \gamma](2, 0)) &= (\bar{\alpha}[\hat{g}, \gamma](1, 0), \bar{\alpha}[\hat{g}, \gamma](2, 0)) \\ &= ([\widehat{\alpha(g)}, \gamma]\bar{\alpha}(1, 0), [\widehat{\alpha(g)}, \gamma]\bar{\alpha}(2, 0)) \\ &= ([\widehat{\alpha(g)}, \gamma](1, 0), [\widehat{\alpha(g)}, \gamma](2, 0)) \in (S - U). \end{aligned}$$

Thus $\bar{\alpha}(S) = S$, $\bar{\alpha}$ preserves colors (if X is a color ternary relational structure) and $\bar{\alpha} \in \mathrm{Aut}(X)$.

We claim that $\gamma^{-1}(\mathbb{Z}_p \times G)_L\gamma$ is a subgroup of $\mathrm{Aut}(X)$. We set $\tau' = \gamma^{-1}\tau\gamma$ and $g' = \gamma^{-1}\hat{g}\gamma$, for $g \in G$. Note that $\tau' = \tau\bar{\alpha}^{-1}$. As $\bar{\alpha} \in \mathrm{Aut}(X)$, we have $\tau' \in \mathrm{Aut}(X)$. Therefore it remains to prove that $\langle g' : g \in G\rangle$ is a subgroup of $\mathrm{Aut}(X)$. Let $e \in E(X)$ and $g \in G$. Then $e = k((0, 0), s)$, where $s \in S$ and $k = \tau^a\widehat{l}$, for some $a \in \mathbb{Z}_p$, $l \in G$. We have to prove that $g'(e) \in E(X)$ and has the same color as e (if X is a color ternary relational structure).

Assume that $s \in U$. As $g'(i, j) = (i, j + \alpha^{-i}(g))$, by definition of U, we have $g'[k((0, 0), s)] \in E(X)$ and has the same color of e (if X is a color ternary relational structure). So, it remains to consider the case $s \in S - U$, i.e., $s = ([\widehat{x}, \gamma](1, 0), [\widehat{x}, \gamma](2, 0))$ for some $x \in G$. As before, we need not concern ourselves with colors because all the edges in $S - U$ are of the same color.

Set $m = k\widehat{\alpha^{-a}(g)}$. Since $\bar{\alpha}\widehat{g} = \widehat{\alpha(g)}\bar{\alpha}$ and $\bar{\alpha}, \gamma$ commute, we get $\bar{\alpha}g' = (\alpha(g))'\bar{\alpha}$. Also observe that as G is abelian, g' commutes with $\widehat{h}$ for every $g, h \in G$. Hence

$$\begin{aligned} g'k &= \gamma^{-1}\widehat{g}\gamma\tau^a\widehat{l} = \gamma^{-1}\widehat{g}\tau^a\gamma\bar{\alpha}^a\widehat{l} = \gamma^{-1}\tau^a\widehat{g}\bar{\alpha}^a\gamma\widehat{l} \\ &= \tau^a\gamma^{-1}\bar{\alpha}^{-a}\widehat{g}\bar{\alpha}^a\gamma\widehat{l} = \tau^a(\alpha^{-a}(g))'\widehat{l} = \tau^a\widehat{l}(\alpha^{-a}(g))' \\ &= k\widehat{\alpha^{-a}(g)}\widehat{\alpha^{-a}(g)}^{-1}\gamma^{-1}\widehat{\alpha^{-a}(g)}\gamma = m[\widehat{\alpha^{-a}(g)},\gamma] \end{aligned}$$

and

$$\begin{aligned} g'[k((0,0),s)] &= g'k((0,0),[\widehat{x},\gamma](1,0),[\widehat{x},\gamma](2,0)) \\ &= m[\widehat{\alpha^{-a}(g)},\gamma]((0,0),[\widehat{x},\gamma](1,0),[\widehat{x},\gamma](2,0)) \\ &= m((0,0),[\widehat{\alpha^{-a}(g)},\gamma][\widehat{x},\gamma](1,0),[\widehat{\alpha^{-a}(g)},\gamma][\widehat{x},\gamma](2,0)) \\ &= m((0,0),[\widehat{\alpha^{-a}(g)x},\gamma](1,0),[\widehat{\alpha^{-a}(g)x},\gamma](2,0)) \in E(X). \end{aligned}$$

This proves that $g' \in \mathrm{Aut}(X)$. Since g is an arbitrary element of G, we have $\gamma^{-1}G_L\gamma \subseteq \mathrm{Aut}(X)$. As claimed, $\gamma^{-1}(\mathbb{Z}_p \times G)_L\gamma$ is a regular subgroup of $\mathrm{Aut}(X)$ conjugate in $\mathrm{Sym}(\mathbb{Z}_p \times G)$ to $(\mathbb{Z}_p \times G)_L$.

We now see that $Y = \gamma(X)$ is a Cayley (color) ternary relational structure of $\mathbb{Z}_p \times G$ as $\mathrm{Aut}(Y) = \gamma\mathrm{Aut}(X)\gamma^{-1}$. We will next show that $Y \neq X$. Assume by way of contradiction that $Y = X$. As $\gamma(0,g) = (0,g)$ for every $g \in G$, the permutation γ must map edges of U to themselves, so that $\gamma(S-U) = S-U$. We will show that $\gamma(S-U) \neq S-U$. Note that we need not concern ourselves with colors as all the edges derived from $S-U$ via translations of $(\mathbb{Z}_p \times G)_L$ have the same color. Observing that

$$\begin{aligned} ([\hat{g},\gamma](1,0),[\hat{g},\gamma](2,0)) &= (\hat{g}^{-1}\gamma^{-1}\hat{g}\gamma(1,0),\hat{g}^{-1}\gamma^{-1}\hat{g}\gamma(2,0)) \\ &= (\hat{g}^{-1}\gamma^{-1}\hat{g}(1,0),\hat{g}^{-1}\gamma^{-1}\hat{g}(2,0)) \\ &= (\hat{g}^{-1}\gamma^{-1}(1,g),\hat{g}^{-1}\gamma^{-1}(2,g)) \\ &= (\hat{g}^{-1}(1,\alpha^{-1}(g),\hat{g}^{-1}(2,\alpha^{-2}(g)) \\ &= ((1,\alpha^{-1}(g)-g),(2,\alpha^{-2}(g)-g)), \end{aligned}$$

we see that $\gamma(S-U) = \{((1,g-\alpha(g)),(2,g-\alpha^2(g))) : g \in G\}$. Moreover, as $S-U = \{(1,\alpha^{-1}(g)-g),(2,\alpha^{-2}(g)-g) : g \in G\}$, we conclude that for each $g \in G$, there exists $h_g \in G$ such that

$$g-\alpha(g) = \alpha^{-1}(h_g)-h_g \quad \text{and} \quad g-\alpha^2(g) = \alpha^{-2}(h_g)-h_g.$$

Setting $\iota : G \to G$ to be the identity permutation, we may rewrite the above equations as

$$(\iota-\alpha)(g) = (\alpha^{-1}-\iota)(h_g) \quad \text{and} \quad (\iota-\alpha^2)(g) = (\alpha^{-2}-\iota)(h_g).$$

Computing in the endomorphism ring of the abelian group G, we see that $(\alpha^{-2}-\iota) = (\alpha^{-1}+\iota)(\alpha^{-1}-\iota)$. Applying the endomorphism $(\alpha^{-1}+\iota)$ to the first equation above, we then have

$$(\alpha^{-1}+\iota)(\iota-\alpha)(g) = (\alpha^{-1}+\iota)(\alpha^{-1}-\iota)(h_g) = (\alpha^{-2}-\iota)(h_g) = (\iota-\alpha^2)(g).$$

Hence $(\alpha^{-1}+\iota)(\iota-\alpha)=\iota-\alpha^2$, and so

$$0=(\alpha^{-1}+\iota)(\iota-\alpha)-(\iota-\alpha^2)=((\alpha^{-1}+\iota)-(\iota+\alpha))(\iota-\alpha)=(\alpha^{-1}-\alpha)(\iota-\alpha),$$

(here 0 is the endomorphism of G that maps each element of G to 0). As α fixes only 0, the endomorphism $\iota-\alpha$ is invertible, and so we see that $\alpha^{-1}-\alpha=0$, and $\alpha=\alpha^{-1}$. However, this implies that $p=|\alpha|=2$, a contradiction. Thus $\gamma(S-U)\neq S-U$ and so $Y\neq X$.

We set $T=\gamma(S)$, so that $((0,0),t)\in E(Y)$ for every $t\in T$, where if X is a color ternary relational structure we assume that γ preserves colors. Now suppose that there exists $\beta\in\mathrm{Aut}(\mathbb{Z}_p\times G)$ such that $\beta(X)=Y$. Since $\gcd(p,|G|)=1$, we obtain that $\mathbb{Z}_p\times 1_G$ and $1_{\mathbb{Z}_p}\times G$ are characteristic subgroups of $\mathbb{Z}_p\times G$. Therefore $\beta(i,j)=(\beta_1(i),\beta_2(j))$, where $\beta_1\in\mathrm{Aut}(\mathbb{Z}_p)$ and $\beta_2\in\mathrm{Aut}(G)$.

As β fixes $(0,0)$, we must have $\beta(S)=T$. Observe that every element of S and of T is of the form $((0,g),(0,h))$ or $((1,g),(2,h))$, for some $g,h\in G$. In particular, we must have $\beta_1(1)=1$ and hence $\beta_1=1$. As $\bar{\alpha}\in\mathrm{Aut}(X)$ and $X\neq Y$, we have $\beta_2\notin\langle\alpha\rangle$. Now observe that $\beta(U)=U$. Thus $\beta_2\in\mathrm{Aut}(Z)=\langle G_L,\alpha\rangle$. We conclude that $\beta_2\in\langle\alpha\rangle$, a contradiction. Thus X,Y are not isomorphic by a group automorphism of $\mathbb{Z}_p\times G$, and the result follows. □

The following two lemmas, which in our opinion are of independent interest, will be used (together with Theorem 2.1) in the proof of Corollary 2.4.

Lemma 2.2. *Let G be a transitive permutation group on Ω. If $x\in\Omega$ and $\mathrm{Stab}_G(x)$ in its action on $\Omega-\{x\}$ is the automorphism group of a k-ary relational structure with vertex set $\Omega-\{x\}$, then G is the automorphism group of a $(k+1)$-ary relational structure.*

Proof. Let Y be a k-ary relational structure with vertex set $\Omega-\{x\}$ and automorphism group $\mathrm{Stab}_G(x)$ in its action on $\Omega-\{x\}$. Let $W=\{(x,v_1,\ldots,v_k):(v_1,\ldots,v_k)\in E(Y)\}$, and define a $(k+1)$-ary relational structure X by $V(X)=\Omega$ and $E(X)=\{g(w):w\in W \text{ and } g\in G\}$. We claim that $\mathrm{Aut}(X)=G$. First, observe that $\mathrm{Stab}_G(x)$ maps W to W. Also, if $e\in E(X)$ and $e=(x,v_1,\ldots,v_k)$ for some $v_1,\ldots,v_k\in\Omega$, then there exists $(x,u_1,\ldots,u_k)\in W$ and $g\in G$ with $g(x,u_1,\ldots,u_k)=(x,v_1,\ldots,v_k)$. We conclude that $g(x)=x$ and $g(u_1,\ldots,u_k)=(v_1,\ldots,v_k)$. Hence $g\in\mathrm{Stab}_G(x)$ and $(v_1,\ldots,v_k)\in E(Y)$. Then W is the set of all edges of X with first coordinate x.

By construction, $G\leq\mathrm{Aut}(X)$. For the reverse inclusion, let $h\in\mathrm{Aut}(X)$. As G is transitive, there exists $g\in G$ such that $g^{-1}h\in\mathrm{Stab}_{\mathrm{Aut}(X)}(x)$. Note that as $g\in G$, the element $g^{-1}h\in G$ if and only if $h\in G$. We may thus assume without loss of generality that $h(x)=x$. Then h stabilizes set-wise the set of all edges of X with first coordinate x, and so $h(W)=W$ and h induces an automorphism of Y. As $\mathrm{Aut}(Y)=\mathrm{Stab}_G(x)\leq G$, the result follows. □

Lemma 2.3. *Let $m\geq 2$ be an integer and $\rho\in\mathrm{Sym}(\mathbb{Z}_{ms})$ be a semiregular element of order m with s orbits. Then there exists a digraph Γ with vertex set $\mathbb{Z}_{ms}$ and with $\mathrm{Aut}(\Gamma)=\langle\rho\rangle$.*

Proof. For each $i\in\mathbb{Z}_s$, set

$$\rho_i=(0,1,\ldots,m-1)\cdots(im,im+1,\ldots,im+m-1)\quad\text{and}\quad V_i=\{im+j:j\in\mathbb{Z}_m\}.$$

We inductively define a sequence of graphs $\Gamma_0, \ldots, \Gamma_{s-1} = \Gamma$ such that the subgraph of Γ induced by $\mathbb{Z}_{(i+1)m}$ is Γ_i, the indegree in Γ of a vertex in V_i is $i+1$, and $\mathrm{Aut}(\Gamma_i) = \langle \rho_i \rangle$, for each $i \in \mathbb{Z}_s$.

We set Γ_0 to be the directed cycle of length m with edges $\{(j, j+1) : j \in \mathbb{Z}_m\}$ and with automorphism group $\langle \rho_0 \rangle$. Inductively assume that Γ_{s-2}, with the above properties, has been constructed. We construct Γ_{s-1} as follows. First, the subgraph of Γ_{s-1} induced by $\mathbb{Z}_{(s-1)m}$ is Γ_{s-2}. Then we place the directed m cycle $\{((s-1)m+j, (s-1)m+j+1) : j \in \mathbb{Z}_m\}$ whose automorphism group is $\langle ((s-1)m, (s-1)m+1, \ldots, (s-1)m+m-1) \rangle$ on the vertices in V_{s-1}. Additionally, we declare the vertex $(s-1)m$ to be outadjacent to $(s-2)m$ and to every vertex that $(s-2)m$ is outadjacent to that is not contained in V_{s-2}. Finally, we add to Γ_{s-1} every image of one of these edges under an element of $\langle \rho_{s-1} \rangle$.

By construction, ρ_{s-1} is an automorphism of Γ_{s-1} and the subgraph of Γ_{s-1} induced by $\mathbb{Z}_{(s-1)m}$ is Γ_{s-2}. Then each vertex in $\Gamma_{s-1} \cap V_i$ has indegree $i+1$ for $0 \le i \le s-2$, while it is easy to see that each vertex of V_{s-1} has indegree s. Finally, if $\delta \in \mathrm{Aut}(\Gamma_{s-1})$, then δ maps vertices of indegree $i+1$ to vertices of indegree $i+1$, and so δ fixes setwise V_i, for every $i \in \mathbb{Z}_s$. Additionally, the action induced by $\langle \delta \rangle$ on V_{s-1} is necessarily $\langle ((s-1)m, (s-1)m+1, \ldots, (s-1)m+m-1) \rangle$ as this is the automorphism group of the subgraph of Γ_{s-1} induced by V_{s-1}. Moreover, arguing by induction, we may assume that the action induced by δ on $V(\Gamma_{s-1}) - V_{s-1}$ is given by an element of $\langle \rho_{s-2} \rangle$. If $\delta \notin \langle \rho_{s-1} \rangle$, then $\mathrm{Aut}(\Gamma_{s-1})$ has order at least m^2, and there is some element of $\mathrm{Aut}(\Gamma_{s-1})$ that is the identity on $V(\Gamma_{s-2})$ but not on V_{s-1} and vice versa. This however is not possible as each vertex of V_{s-2} is inadjacent to exactly one vertex of V_{s-1}. Then $\mathrm{Aut}(\Gamma_{s-1}) = \langle \rho_{s-1} \rangle$ and the result follows. □

Corollary 2.4. *None of the groups $\mathbb{Z}_3 \times \mathbb{Z}_2^2$, $\mathbb{Z}_7 \times \mathbb{Z}_2^3$, or $\mathbb{Z}_5 \times \mathbb{Z}_2^4$ are CI-groups with respect to ternary relational structures.*

Proof. Observe that $\mathbb{Z}_2^2$ has an automorphism α_3 of order 3 that fixes 0 and acts regularly on the remaining 3 elements, and similarly, $\mathbb{Z}_2^3$ has an automorphism α_7 of order 7 that fixes 0 and acts regularly on the remaining 7 elements. As a regular cyclic group is the automorphism group of a directed cycle, we see that $\langle (\mathbb{Z}_3 \times \mathbb{Z}_2^2)_L, \alpha_3 \rangle$ and $\langle (\mathbb{Z}_7 \times \mathbb{Z}_2^3)_L, \alpha_7 \rangle$ are the automorphism groups of ternary relational structures by Lemma 2.2. The result then follows by Theorem 2.1.

Now $\mathbb{Z}_2^4$ has an automorphism α_5 of order 5 that fixes 0 and acts semiregularly on the remaining 15 points. Then $\langle \alpha_5 \rangle$ in its action on $\mathbb{Z}_2^4 - \{0\}$ is the automorphism group of a binary relational structure by Lemma 2.3. By Lemma 2.2, there exists a ternary relational structure with automorphism group $\langle (\mathbb{Z}_5 \times \mathbb{Z}_2^4)_L, \alpha_5 \rangle$. The result then follows by Theorem 2.1. □

Before proceeding, we will need terms and notation concerning complete block systems.

Let $G \le \mathrm{Sym}(n)$ be a transitive permutation group (acting on $\mathbb{Z}_n$, say). A subset $B \subseteq \mathbb{Z}_n$ is a *block for* G if $g(B) = B$ or $g(B) \cap B = \emptyset$ for every $g \in G$. Clearly $\mathbb{Z}_n$ and its singleton subsets are always blocks for G, and are called *trivial blocks*. If B is a block, then $g(B)$ is a block for every $g \in G$, and the set $\mathcal{B} = \{g(B) : g \in G\}$ is called a *complete block system for* G, and we say that G *admits* $\mathcal{B}$. A complete block system is *nontrivial* if its blocks are nontrivial. Observe that a complete block system is a partition of $\mathbb{Z}_n$, and any two blocks have the same size. If G admits $\mathcal{B}$ as a complete block system, then each $g \in G$

induces a permutation of $\mathcal{B}$, which we denote by $g/\mathcal{B}$. We set $G/\mathcal{B} = \{g/\mathcal{B} : g \in G\}$. The kernel of the action of G on $\mathcal{B}$, denoted by $\mathrm{fix}_G(\mathcal{B})$, is then the subgroup of G which fixes each block of $\mathcal{B}$ set-wise. That is, $\mathrm{fix}_G(\mathcal{B}) = \{g \in G : g(B) = B \text{ for all } B \in \mathcal{B}\}$. For fixed $B \in \mathcal{B}$, we denote the set-wise stabilizer of B in G by $\mathrm{Stab}_G(B)$. That is $\mathrm{Stab}_G(B) = \{g \in G : g(B) = B\}$. Note that $\mathrm{fix}_G(\mathcal{B}) = \cap_{B\in\mathcal{B}}\mathrm{Stab}_G(B)$. Finally, for $g \in \mathrm{Stab}_G(B)$, we denote by $g|_B$ the action induced by g on $B \in \mathcal{B}$.

Note that Corollary 2.4, together with the fact that $\mathbb{Z}_2^3 \times \mathbb{Z}_p$, $p \geq 11$, is a CI-group with respect to color ternary relational structures [6], settles the question of which groups $\mathbb{Z}_2^3 \times \mathbb{Z}_p$ are CI-groups with respect to color ternary relational structures except for $p = 5$. Our next goal is to show that $\mathbb{Z}_2^3 \times \mathbb{Z}_5$ is a CI-group with respect to color ternary relational structures. From a computational point of view, the number of points is too large to enable a computer to determine the answer without some additional information. Lemma 6.1 in [6] is the only result that uses the hypothesis $p \geq 11$. For convenience, we report [6, Lemma 6.1].

Lemma 2.5. *Let $p \geq 11$ be a prime and write $H = \mathbb{Z}_2^3 \times \mathbb{Z}_p$. For every $\phi \in \mathrm{Sym}(H)$, there exists $\delta \in \langle H_L, \phi^{-1}H_L\phi\rangle$ such that $\langle H_L, \delta^{-1}\phi^{-1}H_L\phi\delta\rangle$ admits a complete block system consisting of 8 blocks of size p.*

In particular, to prove that $\mathbb{Z}_2^3 \times \mathbb{Z}_5$ is a CI-group with respect to color ternary relational structures, it suffices to prove that Lemma 2.5 holds true also for the prime $p = 5$. We begin with some intermediate results which accidentally will also help us to prove that $\mathbb{Z}_2^3 \times \mathbb{Z}_7$ is a CI-group with respect to color binary relational structures. (Here we denote by $\mathrm{Alt}(X)$ the alternating group on the set X and by $\mathrm{Alt}(n)$ the alternating group on $\{1, \dots, n\}$.)

Lemma 2.6. *Let p be an arbitrary divisor of n with $p \neq 1$ and let P_1 and P_2 be partitions of $\mathbb{Z}_n$ where each block in P_1 and P_2 has size p. Then there exists $\phi \in \mathrm{Alt}(\mathbb{Z}_n)$ such that $\phi(P_1) = P_2$.*

Proof. Let $P_1 = \{\Delta_1, \dots, \Delta_{n/p}\}$ and $P_2 = \{\Omega_1, \dots, \Omega_{n/p}\}$. As $\mathrm{Alt}(n)$ is $(n-2)$-transitive, there exists $\phi \in \mathrm{Alt}(n)$ such that $\phi(\Delta_i) = \Omega_i$, for $i \in \{1, \dots, n/p - 1\}$. As both P_1 and P_2 are partitions, we see that $\phi(\Delta_{n/p}) = \Omega_{n/p}$ as well. □

Lemma 2.7. *Let p be a prime, let $G = \mathbb{Z}_2^3 \times \mathbb{Z}_p$ and let $\delta \in \mathrm{Sym}(G)$. Suppose that $\langle G_L, \delta^{-1}G_L\delta\rangle$ admits a complete block system $\mathcal{C}$ with p blocks of size 8 such that $\mathrm{Alt}(C) \leq \mathrm{Stab}_{\langle G_L,\delta^{-1}G_L\delta\rangle}(C)|_C$, where $C \in \mathcal{C}$. Then there exists $\gamma \in \langle G_L, \delta^{-1}G_L\delta\rangle$ such that $\langle G_L, \gamma^{-1}\delta^{-1}G_L\delta\gamma\rangle$ admits a complete block system $\mathcal{B}$ with $4p$ blocks of size 2.*

Proof. Write $H = \langle G_L, \delta^{-1}G_L\delta\rangle$, $N = \mathrm{fix}_H(\mathcal{C})$ and $M = [N, N]$. Clearly both G_L and $\delta^{-1}G_L\delta$ are regular, and so both $\mathrm{fix}_{G_L}(\mathcal{C})$ and $\mathrm{fix}_{\delta^{-1}G_L\delta}(\mathcal{C})$ are semiregular of order 8. Moreover, as $\mathrm{fix}_{G_L}(\mathcal{C})|_C$ and $\mathrm{fix}_{\delta^{-1}G_L\delta}(\mathcal{C})|_C$ have exponent 2, we see that they are both consist of even permutations and hence they are contained in $\mathrm{Alt}(C)$, for each $C \in \mathcal{C}$.

From the previous paragraph, as $\mathrm{Alt}(8)$ is simple and $1 \neq N|_C \lhd \mathrm{Stab}_{\langle G,\delta^{-1}G\delta\rangle}(\mathcal{C})|_C$, we have $\mathrm{Alt}(C) = M|_C$, for every $C \in \mathcal{C}$. In particular, M is isomorphic to a subgroup of $\mathrm{Alt}(8)^p$.

Denote by $M_{(C)}$ the pointwise stabilizer of $C \in \mathcal{C}$. Define an equivalence relation $\equiv$ on $\mathcal{C}$ by $C \equiv C'$ if and only if $M_{(C)} = M_{(C')}$. Clearly, $\equiv$ is an H-invariant equivalence relation because $M \lhd H$. As $|\mathcal{C}| = p$, we see that $\equiv$ is either the identity or the universal relation. From this, we infer that either $M \cong \mathrm{Alt}(8)$ (when $\equiv$ is the identity relation) or $M \cong \mathrm{Alt}(8)^p$ (when $\equiv$ is the universal relation). Observe further that, when $M \cong \mathrm{Alt}(8)$, since $\mathrm{Alt}(8)$ has only one permutation representation of degree 8 [3, Theorem 5.3], the group M induces equivalent actions on C and on C', for every C and C' in $\mathcal{C}$. In particular, in both cases, given a subgroup I of G_L and J of $\delta^{-1}G_L\delta$ both of order 2, there exists $\gamma \in M$ with $I = \gamma^{-1}J\gamma$.

Write $K = \langle G_L, \gamma^{-1}\delta^{-1}G_L\delta\gamma\rangle$. Clearly, I is centralized by G_L and by $\gamma^{-1}\delta^{-1}G_L\delta\gamma$ because $I \le G_L$ and $I \le \gamma^{-1}\delta^{-1}G_L\delta\gamma$. So I is centralized by K. As $I \lhd K$, the orbits of I form a complete block system for K with $4p$ blocks of size 2. □

The proof of the following result is similar to the proof of [6, Lemma 6.1], and generalizes it.

Lemma 2.8. *Let H be an abelian group of order ℓp, where $\ell < p$ and p is prime. Let $\phi \in \mathrm{Sym}(H)$. Then there exists $\delta \in \langle H_L, \phi^{-1}H_L\phi\rangle$ such that $\langle H_L, \delta^{-1}\phi^{-1}H_L\phi\delta\rangle$ admits a complete block system with blocks of size p.*

Proof. Let $\rho \in H$ be of order p. Then H_L admits a complete block system $\mathcal{B}$ of ℓ blocks of size p formed by the orbits of $\langle\rho\rangle$. Note that as $\ell < p$, a Sylow p-subgroup of $\mathrm{Sym}(H)$ has order p^ℓ. In particular, $\langle \rho|_B : B \in \mathcal{B}\rangle$ is a Sylow p-subgroup of $\mathrm{Sym}(H)$ isomorphic to $\mathbb{Z}_p^\ell$, an elementary abelian p-group of order p^ℓ. Let P and P_1 be Sylow p-subgroups of $\langle H_L, \phi^{-1}H_L\phi\rangle$ containing ρ and $\phi^{-1}\rho\phi$, respectively. Then there exists $\delta \in \langle H_L, \phi^{-1}H_L\phi\rangle$ such that $\delta^{-1}P_1\delta = P$. Now, every element of H_L normalizes $\langle\rho\rangle$, and so normalizes $\langle \rho|_B : B \in \mathcal{B}\rangle$. This then implies that H_L normalizes P because $P = \langle \rho|_B : B \in \mathcal{B}\rangle \cap \langle H_L, \phi^{-1}H_L\phi\rangle$.

Let $\mathcal{B}'$ be the complete block system of $\delta^{-1}\phi^{-1}H_L\phi\delta$ formed by the orbits of the cyclic group $\delta^{-1}\phi^{-1}\langle\rho\rangle\phi\delta$. Arguing as above, we see that $\delta^{-1}\phi^{-1}H_L\phi\delta$ normalizes $M = \langle (\delta^{-1}\phi^{-1}\rho\phi\delta)|_{B'} : B' \in \mathcal{B}'\rangle \cap \langle H_L, \delta^{-1}\phi^{-1}H_L\phi\delta\rangle$. However, M is the Sylow p-subgroup of $\langle H_L, \delta^{-1}\phi^{-1}H_L\phi\delta\rangle$ containing $\delta^{-1}\phi^{-1}\langle\rho\rangle\delta\phi$, which is P. Thus we have $P \lhd \langle H_L, \delta^{-1}\phi^{-1}H_L\phi\delta\rangle$, and the orbits of P form the required complete block system. □

Lemma 2.9. *Let $p \ge 5$, $H = \mathbb{Z}_2^3 \times \mathbb{Z}_p$, and $\phi \in \mathrm{Sym}(H)$. Then either there exists $\delta \in \langle H_L, \phi^{-1}H_L\phi\rangle$ such that $\langle H_L, \delta^{-1}\phi^{-1}H_L\phi\delta\rangle$ admits a complete block system with blocks of size p or $\langle H_L, \phi^{-1}H_L\phi\rangle$ admits a complete block system $\mathcal{B}$ with blocks of size 8 and $\mathrm{fix}_{\langle H_L, \phi^{-1}H_L\phi\rangle}(\mathcal{B})|_B$ is isomorphic to a primitive subgroup of $\mathrm{AGL}(3,2)$, for $B \in \mathcal{B}$.*

Proof. Set $K = \langle H_L, \phi^{-1}H_L\phi\rangle$. As H has a cyclic Sylow p-subgroup, we have by [4, Theorem 3.5A] that K is doubly-transitive or imprimitive. If K is doubly-transitive, then by [12, Theorem 1.1] we have $\mathrm{Alt}(H) \le K$. Now Lemma 2.6 reduces this case to the imprimitive case. Thus we may assume that K is imprimitive with a complete block system $\mathcal{C}$.

Suppose that the blocks of $\mathcal{C}$ have size ℓp, where $\ell = 2$ or 4. Notice that $p > \ell$. As H is abelian, $\mathrm{fix}_{H_L}(\mathcal{C})$ is a semiregular group of order ℓp and $\mathrm{fix}_{\phi^{-1}H_L\phi}(\mathcal{C})$ is also a semiregular group of order ℓp. Then, for $C \in \mathcal{C}$, both $\mathrm{fix}_{H_L}(\mathcal{C})|_C$ and $\mathrm{fix}_{\phi^{-1}H_L\phi}(\mathcal{C})|_C$ are regular groups of order ℓp. Let $C \in \mathcal{C}$. By Lemma 2.8, there exists $\delta \in \langle \mathrm{fix}_{H_L}(\mathcal{C}), \mathrm{fix}_{\phi^{-1}H_L\phi}(\mathcal{C})\rangle$ such that $\langle \mathrm{fix}_{H_L}(\mathcal{C}), \mathrm{fix}_{\delta^{-1}\phi^{-1}H_L\phi\delta}(\mathcal{C})\rangle|_C$ admits a complete block system $\mathcal{B}_C$ consisting of blocks of size p. Let $C' \in \mathcal{C}$ with $C' \neq C$. Arguing as above, there exists $\delta' \in \langle \mathrm{fix}_{H_L}(\mathcal{C}), \mathrm{fix}_{\delta^{-1}\phi^{-1}H_L\phi\delta}(\mathcal{C})\rangle$ such that $\langle \mathrm{fix}_{H_L}(\mathcal{C}), \mathrm{fix}_{\delta'^{-1}\delta^{-1}\phi^{-1}H_L\phi\delta\delta'}(\mathcal{C})\rangle|_{C'}$ admits a complete block system $\mathcal{B}_{C'}$ consisting of blocks of size p. Note that the restriction $\delta'|_C$ is in $\langle \mathrm{fix}_{H_L}(\mathcal{C}), \mathrm{fix}_{\delta^{-1}\phi^{-1}H_L\phi\delta}(\mathcal{C})\rangle|_C$ and so $\langle \mathrm{fix}_{H_L}(\mathcal{C}), \mathrm{fix}_{\delta'^{-1}\delta^{-1}\phi^{-1}H_L\phi\delta\delta'}(\mathcal{C})\rangle|_C$ admits $\mathcal{B}_C$ as a complete block system. Repeating this argument for every block in $\mathcal{C}$, we find that there exists $\delta \in \langle \mathrm{fix}_{H_L}(\mathcal{C}), \mathrm{fix}_{\phi^{-1}H_L\phi}(\mathcal{C})\rangle$ such that $\langle \mathrm{fix}_{H_L}(\mathcal{C}), \mathrm{fix}_{\delta^{-1}\phi^{-1}H_L\phi\delta}(\mathcal{C})\rangle|_C$ admits a complete block system $\mathcal{B}_C$ consisting of blocks of size p. Let $\mathcal{B} = \cup_C \mathcal{B}_C$. We claim that $\mathcal{B}$ is a complete block system for $\langle H_L, \delta^{-1}\phi^{-1}H_L\phi\delta\rangle$, which will complete the argument in this case.

Let $\rho \in H_L$ be of order p. By construction, $\rho \in \mathrm{fix}_{H_L}(\mathcal{B})$. As H is abelian, $\mathrm{fix}_{H_L}(\mathcal{C})|_C$ is abelian, for every $C \in \mathcal{C}$. Then $\mathcal{B}_C$ is formed by the orbits of some subgroup of $\mathrm{fix}_{H_L}(\mathcal{C})|_C$ of order p, and as $\langle\rho\rangle|_C$ is the unique subgroup of $\mathrm{fix}_{H_L}(\mathcal{C})|_C$ of order p, we obtain that $\mathcal{B}_C$ is formed by the orbits of $\langle\rho\rangle|_C$. Then $\mathcal{B}$ is formed by the orbits of $\langle\rho\rangle \lhd H_L$ and $\mathcal{B}$ is a complete block system for H_L. An analogous argument for $\delta^{-1}\phi^{-1}\langle\rho\rangle\phi\delta$ gives that $\mathcal{B}$ is a complete block system for $\delta^{-1}\phi^{-1}H_L\phi\delta$. Then $\mathcal{B}$ is a complete block system for $\langle H_L, \delta^{-1}\phi^{-1}H_L\phi\delta\rangle$ with blocks of size p, as required.

Suppose that the blocks of $\mathcal{C}$ have size 8. Now $H_L/\mathcal{C}$ and $\phi^{-1}H_L\phi/\mathcal{C}$ are cyclic of order p, and as $\mathbb{Z}_p$ is a CI-group [1, Theorem 2.3], replacing $\phi^{-1}H_L\phi$ by a suitable conjugate, we may assume that $\langle H_L, \phi^{-1}H_L\phi\rangle/\mathcal{C} = H_L/\mathcal{C}$. Then $K/\mathcal{C}$ is regular and $\mathrm{Stab}_K(C) = \mathrm{fix}_K(\mathcal{C})$, for every $C \in \mathcal{C}$.

Suppose that $\mathrm{Stab}_K(\mathcal{C})|_C$ is imprimitive, for $C \in \mathcal{C}$. By [4, Exercise 1.5.10], the group K admits a complete block system $\mathcal{D}$ with blocks of size 2 or 4. Then $K/\mathcal{D}$ has degree $2p$ or $4p$ and, by Lemma 2.8, there exists $\delta \in K$ such that $\langle H_L, \delta^{-1}\phi^{-1}H_L\phi\delta\rangle/\mathcal{D}$ admits a complete block system $\mathcal{B}'$ with blocks of size p. In particular, $\mathcal{B}'$ induces a complete block system $\mathcal{B}''$ for $\langle H_L, \delta^{-1}\phi^{-1}H_L\phi\delta\rangle$ with blocks of size $2p$ or $4p$, and we conclude by the case previously considered applied with $\mathcal{C} = \mathcal{B}''$. Suppose that $\mathrm{Stab}_K(\mathcal{C})|_C$ is primitive, for $C \in \mathcal{C}$. If $\mathrm{Stab}_K(\mathcal{C})|_C \geq \mathrm{Alt}(C)$, then the result follows by Lemma 2.7, and so we may assume this is not the case. By [12, Theorem 1.1], we see that $\mathrm{Stab}_K(\mathcal{C})|_C \leq \mathrm{AGL}(3,2)$. The result now follows with $\mathcal{B} = \mathcal{C}$. □

Corollary 2.10. *Let $H = \mathbb{Z}_2^3 \times \mathbb{Z}_5$ and $\phi \in \mathrm{Sym}(H)$. Then there exists $\delta \in \langle H_L, \phi^{-1}H_L\phi\rangle$ such that $\langle H_L, \delta^{-1}\phi^{-1}H_L\phi\delta\rangle$ admits a complete block system with blocks of size 5.*

Proof. Set $K = \langle H_L, \phi^{-1}H_L\phi\rangle$. By Lemma 2.9, we may assume that K admits a complete block system $\mathcal{B}$ with blocks of size 8 and with $\mathrm{Stab}_K(\mathcal{B})|_B \leq \mathrm{AGL}(3,2)$, for $B \in \mathcal{B}$. As $|\mathrm{AGL}(3,2)| = 8 \cdot 7 \cdot 6 \cdot 4$, we see that a Sylow 5-subgroup of K has order 5. Let $\langle\rho\rangle$ be the subgroup of H_L of order 5. So $\langle\rho\rangle$ is a Sylow 5-subgroup of K. Then $\phi^{-1}\langle\rho\rangle\phi$ is also a Sylow 5-subgroup of K, and by a Sylow theorem there exists $\delta \in K$ such that $\delta^{-1}\phi^{-1}\langle\rho\rangle\phi\delta = \langle\rho\rangle$. We then see that $\langle H_L, \delta^{-1}\phi^{-1}H_L\phi\delta\rangle$ has a unique Sylow 5-subgroup, whose orbits form the required complete block system $\mathcal{B}$. □

We are finally ready to prove Theorem A.

Proof of Theorem A. If p is odd, then the paragraph following the proof of Corollary 2.4 shows that it suffices to prove that Lemma 2.5 holds for the prime $p = 5$. This is done in Corollary 2.10. If $p = 2$, then the result can be verified using GAP or Magma. □

3 Proof of Corollaries A and B

Before proceeding to our next result we will need the following definitions.

Definition 3.1. Let G be a permutation group on Ω and $k \geq 1$. A permutation $\sigma \in \mathrm{Sym}(\Omega)$ lies in the k-closure $G^{(k)}$ of G if for every k-tuple $t \in \Omega^k$ there exists $g_t \in G$ (depending on t) such that $\sigma(t) = g_t(t)$. We say that G is k-closed if the permutations lying in the

k-closure of G are the elements of G, that is, $G^{(k)} = G$. The group G is k-closed if and only if there exists a color k-ary relational structure X on Ω with $G = \mathrm{Aut}(X)$, see [18].

Definition 3.2. For color digraphs Γ_1 and Γ_2, we define the *wreath product of* Γ_1 *and* Γ_2, denoted $\Gamma_1 \wr \Gamma_2$, to be the color digraph with vertex set $V(\Gamma_1) \times V(\Gamma_2)$ and edge set $E_1 \cup E_2$, where $E_1 = \{((x_1, y_1), (x_1, y_2)) : x_1 \in V(\Gamma_1), (y_1, y_2) \in E(\Gamma_2)\}$ and $E_2 = \{((x_1, y_1), (x_2, y_2)) : (x_1, x_2) \in E(\Gamma_1), y_1, y_2 \in V(\Gamma_2)\}$.

The edge $((x_1, y_1), (x_1, y_2)) \in E_1$ is colored with the same color as (y_1, y_2) in Γ_2 and the edge $((x_1, y_1), (x_2, y_2)) \in E_2$ is colored with the same color as (x_1, x_2) in Γ_1.

Definition 3.3. Let $G \le \mathrm{Sym}(X)$ and let $H \le \mathrm{Sym}(Y)$. We define the *wreath product of* G *and* H, denoted by $G \wr H$, to be the semidirect product $G \ltimes H^X$, where H^X is the direct product of $|X|$ copies of H (labeled by the elements of X) and where G acts on H^X as a group of automorphisms by permuting the coordinates according to its action on X. The group $G \wr H$ has a natural faithful action on $X \times Y$, where for $(x, y) \in X \times Y$ the element $g \in G$ acts via $(x, y) \mapsto (g(x), y)$ and the element $(h_z)_{z \in X} \in H^X$ acts via $(x, y) \mapsto (x, h_x(y))$. We refer the reader to [4, page 46] for more details on this construction.

The following very useful result (see [1, Lemma 3.1]) characterizes CI-groups with respect to a class of combinatorial objects.

Lemma 3.4. *Let H be a group and let $\mathcal{K}$ be a class of combinatorial objects. The following are equivalent.*

1. *H is a CI-group with respect to $\mathcal{K}$,*
2. *whenever X is a Cayley object of H in $\mathcal{K}$ and $\phi \in \mathrm{Sym}(H)$ such that $\phi^{-1} H_L \phi \le \mathrm{Aut}(X)$, then H_L and $\phi^{-1} H_L \phi$ are conjugate in $\mathrm{Aut}(X)$.*

Proof of Corollary A. From Theorem A, it suffices to show that $\mathbb{Z}_2^3 \times \mathbb{Z}_3$ and $\mathbb{Z}_2^3 \times \mathbb{Z}_7$ are CI-groups with respect to color binary relational structures. As the transitive permutation groups of degree 24 are readily available in GAP or Magma, it can be shown using a computer that $\mathbb{Z}_2^3 \times \mathbb{Z}_3$ is a CI-group with respect to color binary relational structures. It remains to consider $H = \mathbb{Z}_2^3 \times \mathbb{Z}_7$.

Fix $\phi \in \mathrm{Sym}(H)$ and set $K = \langle H_L, \phi^{-1} H_L \phi \rangle$. Assume that there exists $\delta \in K$ such that $\langle H_L, \delta^{-1}\phi^{-1} H_L \phi\delta \rangle$ admits a complete block system with blocks of size 7. Now, it follows by [6] (see the two paragraphs following the proof of Corollary 2.4) that H_L and $\delta^{-1}\phi^{-1} H_L \phi\delta$ are conjugate in $\langle H_L, \delta^{-1}\phi^{-1} H_L \phi\delta \rangle^{(3)}$. Since $\langle H_L, \delta^{-1}\phi^{-1} H_L \phi\delta \rangle^{(3)} \le \langle H_L, \delta^{-1}\phi^{-1} H_L \phi\delta \rangle^{(2)}$, the corollary follows from Lemma 3.4 (and from Definition 3.1).

Assume that there exists no $\delta \in K$ such that $\langle H_L, \delta^{-1}\phi^{-1} H_L \phi\delta \rangle$ admits a complete block system with blocks of size 7. By Lemma 2.9, the group K admits a complete block system $\mathcal{C}$ with blocks of size 8 and $\mathrm{fix}_K(\mathcal{C})|_C$ is isomorphic to a primitive subgroup of $\mathrm{AGL}(3, 2)$, for $C \in \mathcal{C}$. Suppose that 7 and $|\mathrm{fix}_K(\mathcal{C})|$ are relatively prime. So, a Sylow 7-subgroup of K has order 7. We are now in the position to apply the argument in the proof of Corollary 2.10. Let $\langle \rho \rangle$ be the subgroup of H_L of order 7. Then $\phi^{-1}\langle \rho \rangle \phi$ is a Sylow 7-subgroup of K, and by a Sylow theorem there exists $\delta \in K$ such that $\delta^{-1}\phi^{-1}\langle \rho \rangle \phi\delta = \langle \rho \rangle$. We then see that $\langle H_L, \delta^{-1}\phi^{-1} H_L \phi\delta \rangle$ has a unique Sylow 7-subgroup, whose orbits form a complete block system with blocks of size 7, contradicting our hypothesis on K. We thus assume that 7 divides $|\mathrm{fix}_K(\mathcal{C})|$ and so $\mathrm{fix}_K(\mathcal{C})$ acts doubly-transitively on C, for $C \in \mathcal{C}$.

Fix $C \in \mathcal{C}$ and let L be the point-wise stabilizer of C in $\mathrm{fix}_K(\mathcal{C})$. Assume that $L \neq 1$. Now, we compute $K^{(2)}$ and we deduce that H_L and $\phi^{-1}H_L\phi$ are conjugate in $K^{(2)}$, from which the corollary will follow from Lemma 3.4. As $L \triangleleft \mathrm{fix}_K(\mathcal{C})$, we have $L|_{C'} \triangleleft \mathrm{fix}_K(\mathcal{C})|_{C'}$, for every $C' \in \mathcal{C}$. As a nontrivial normal subgroup of a primitive group is transitive [19, Theorem 8.8], either $L|_{C'}$ is transitive or $L|_{C'} = 1$. Let Γ be a Cayley color digraph on H with $K^{(2)} = \mathrm{Aut}(\Gamma)$. Let $\mathcal{C} = \{C_i : i \in \mathbb{Z}_7\}$ where $C_i = \{(x_1, x_2, x_3, i) : x_1, x_2, x_3 \in \mathbb{Z}_2\}$, and assume without loss of generality that $C = C_0$. Suppose that there is an edge of color κ from some vertex of C_i to some vertex of C_j, where $i \neq j$. Then there is an edge of color κ from some vertex of C_0 to some vertex of C_{j-i}. Additionally, $j - i$ generates $\mathbb{Z}_7$, so there is a smallest integer s such that $L|_{C_{s(j-i)}} = 1$ while $L|_{C_{(s+1)(j-i)}}$ is transitive. As there is an edge of color κ from some vertex of $C_{s(j-i)}$ to some vertex of $C_{(s+1)(j-i)}$, we conclude that there is an edge of color κ from every vertex of $C_{s(j-i)}$ to every vertex of $C_{(s+1)(j-i)}$. This implies that there is an edge of color κ from every vertex of C_i to every vertex of C_j, and then Γ is the wreath product of a Cayley color digraph Γ_1 on $\mathbb{Z}_7$ and a Cayley color digraph Γ_2 on $\mathbb{Z}_2^3$. Since $\mathrm{fix}_K(\mathcal{C})$ is doubly-transitive on C, we have $\mathrm{Aut}(\Gamma_2) \cong \mathrm{Sym}(8)$. Therefore $K^{(2)} = \mathrm{Aut}(\Gamma_1) \wr \mathrm{Aut}(\Gamma_2) \cong \mathrm{Aut}(\Gamma_1) \wr \mathrm{Sym}(8)$. By [7, Corollary 6.8] and Lemma 3.4 H_L and $\phi^{-1}H_L\phi$ are conjugate in $K^{(2)}$. We henceforth assume that $L = 1$, that is, $\mathrm{fix}_K(\mathcal{C})$ acts faithfully on C, for each $C \in \mathcal{C}$.

Define an equivalence relation on H by $h \equiv k$ if and only if it holds $\mathrm{Stab}_{\mathrm{fix}_K(\mathcal{C})}(h) = \mathrm{Stab}_{\mathrm{fix}_K(\mathcal{C})}(k)$. The equivalence classes of $\equiv$ form a complete block system $\mathcal{D}$ for K. As $\mathrm{fix}_K(\mathcal{C})|_C$ is primitive and not regular, each equivalence class of $\equiv$ contains at most one element from each block of $\mathcal{C}$. We conclude that $\mathcal{D}$ either consists of 8 blocks of size 7 or each block is a singleton. Since we are assuming that K has no block system with blocks of size 7, we see that each block of $\mathcal{D}$ is a singleton.

Fix C and D in $\mathcal{C}$ with $C \neq D$ and $h \in C$. Now, $\mathrm{Stab}_{\mathrm{fix}_K(\mathcal{C})}(h)$ is isomorphic to a subgroup of $\mathrm{GL}(3,2)$ and acts with no fixed points on D. From [4, Appendix B]), we see that $\mathrm{AGL}(3,2)$ is the only doubly-transitive permutation group of degree 8 whose point stabilizer admits a fixed-point-free action of degree 8. Therefore $\mathrm{fix}_K(\mathcal{C}) \cong \mathrm{AGL}(3,2)$. Additionally, $\mathrm{Stab}_{\mathrm{fix}_K(\mathcal{C})}(h)|_D$ is transitive on D.

Suppose that Γ is a color digraph with $K^{(2)} = \mathrm{Aut}(\Gamma)$ and suppose that there is an edge of color κ from h to $\ell \in E$, with $E \in \mathcal{C}$ and $E \neq D$. Then $\mathrm{Stab}_{\mathrm{fix}_K(\mathcal{C})}(h)|_E$ is transitive, and so there is an edge of color κ from h to every vertex of E. As $\mathrm{fix}_K(\mathcal{C})$ is transitive on both C and E, we see that there is an edge of color κ from every vertex of C to every vertex of D. We conclude that Γ is a wreath product of two color digraphs Γ_1 and Γ_2, where Γ_1 is a Cayley color digraph on $\mathbb{Z}_7$ and Γ_2 is either complete or the complement of a complete graph, and $K^{(2)} = \mathrm{Aut}(\Gamma_1) \wr \mathrm{Sym}(8)$. The result then follows by the same arguments as above. □

Proof of Corollary B. From Corollary 2.4 and Theorem A, it suffices to show that $\mathbb{Z}_2^2 \times \mathbb{Z}_7$ is a CI-group with respect to color ternary relational structures. As the transitive permutation groups of degree 28 are readily available in GAP or Magma, it can be shown using a computer that $\mathbb{Z}_2^2 \times \mathbb{Z}_7$ is a CI-group with respect to color ternary relational structures. (We note that a detailed analysis similar to the proof of Corollary A for the group $\mathbb{Z}_2^3 \times \mathbb{Z}_7$ also gives a proof of this theorem.) □

4 Concluding remarks

In the rest of this paper, we discuss the relevance of Theorem A to the study of CI-groups with respect to ternary relational structures. Using the software packages [2] and [8], we have determined that $\mathbb{Z}_2^5$ is not a CI-group with respect to ternary relational structures. Here we report an example witnessing this fact: the group G has order 2048, V and W are two *nonconjugate* elementary abelian regular subgroups of G, and $X = (\{1, \ldots, 32\}, E)$ is a ternary relational structure with $G = \mathrm{Aut}(X)$. The group V is generated by

(1,2)(3,4)(5,6)(7,8)(9,10)(11,12)(13,14)(15,16)(17,18)(19,20)(21,22)(23,24)(25,26)(27,28)(29,30)(31,32),

(1,3)(2,4)(5,7)(6,8)(9,11)(10,12)(13, 15)(14,16)(17,19)(18,20)(21,23)(22,24)(25,27)(26,28)(29,31)(30,32),

(1,5)(2,6)(3,7)(4,8)(9,13)(10,14)(11,15)(12,16)(17,21)(18,22)(19,23)(20,24)(25,29)(26,30)(27,31)(28,32),

(1,9)(2,10)(3,11)(4,12)(5,13)(6,14)(7,15)(8,16)(17,25)(18,26)(19,27)(20,28)(21,29)(22,30)(23,31)(24,32),

(1,17)(2,18)(3,19)(4,20)(5,21)(6,22)(7,23)(8,24)(9,25)(10,26)(11,27)(12,28)(13,29)(14,30)(15,31)(16,32),

the group W is generated by

(1,2)(3,4)(5,6)(7,8)(9,10)(11,12)(13,14)(15,16)(17,18)(19,20)(21,22)(23,24)(25,26)(27,28)(29,30)(31,32),

(1,3)(2,4)(5,7)(6,8)(9,11)(10,12)(13, 15)(14,16)(17,20)(18,19)(21,24)(22,23)(25,28)(26,27)(29,32)(30,31),

(1,5)(2,6)(3,7)(4,8)(9,14)(10,13)(11,16)(12,15)(17,22)(18,21)(19,24)(20, 23)(25,29)(26,30)(27,31)(28,32),

(1,9)(2,10)(3,11)(4,12)(5,14)(6,13)(7,16)(8,15)(17,27)(18,28)(19,25)(20,26)(21,32)(22,31)(23,30)(24,29),

(1,17)(2,18)(3,20)(4,19)(5,22)(6,21)(7,23)(8,24)(9,27)(10,28)(11,26)(12,25)(13,32)(14,31)(15,29)(16,30),

the group G is generated by

$V, W,$ (25,26)(27,28)(29,30)(31,32),(1,11)(2,12)(3,9)(4,10)(5,13)(6, 14)(7,15)(8,16)(17,19)(18,20)(25,27)(26,28),

the set E is defined by

$$\{g((1,3,9)), g((1,5,25)) : g \in G\}.$$

Definition 4.1. For a cyclic group $M = \langle g \rangle$ of order m and a cyclic group $\langle z \rangle$ of order 2^d, $d \geq 1$, we denote by $D(m, 2^d)$ the group $\langle z \rangle \ltimes M$ with $g^z = g^{-1}$.

Combining Theorem A with [5, Theorem 9], [5, Lemma 6], the construction given in [17] and the previous paragraph, we have the following result which lists every group that can be a CI-group with respect to ternary relational structures (although not every group on the list needs to be a CI-group with respect to ternary relational structures).

Theorem 4.2. *If G is a CI-group with respect to ternary relational structures, then all Sylow subgroups of G are of prime order or isomorphic to $\mathbb{Z}_4$, $\mathbb{Z}_2^d$, $1 \leq d \leq 4$, or Q_8. Moreover, $G = U \times V$, where $\gcd(|U|, |V|) = 1$, U is cyclic of order n, with $\gcd(n, \varphi(n)) = 1$, and V is one of the following:*

1. *$\mathbb{Z}_2^d$, $1 \leq d \leq 4$, $D(m, 2)$, or $D(m, 4)$, where m is odd and $\gcd(nm, \varphi(nm)) = 1$,*
2. *$\mathbb{Z}_4$, Q_8.*

Furthermore,

(a) if $V = \mathbb{Z}_4$, Q_8, *or* $D(m,4)$ *and* $p \mid n$ *is prime, then* $4 \nmid (p-1)$,
(b) if $V = \mathbb{Z}_2^d$, $d \geq 2$, *or* Q_8, *then* $3 \nmid n$,
(c) if $V = \mathbb{Z}_2^d$, $d \geq 3$, *then* $7 \nmid n$,
(d) if $V = \mathbb{Z}_2^4$, *then* $5 \nmid n$.

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