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17 Properties of Fermions With Integer Spin Described in Grassmann Space ⋆

D. Lukman and N.S. Mankoč Borštnik[1]

[1]Department of Physics, University of Ljubljana,
Jadranska 19, SI-1000 Ljubljana, Slovenia

Abstract. In Ref. [1] one of the authors (N.S.M.B.) study the second quantization of fermions with integer spin while describing the internal degrees of freedom of fermions in Grassmann space. In this contribution we study the representations in Grassmann space of the groups $SO(5,1)$, $SO(3,1)$, $SU(3)\times U(1)$, and $SO(4)$, which are of particular interest as the subgroups of the group $SO(13,1)$. The second quantized integer spin fermions, appearing in Grassmann space, not observed so far, could be an alternative choice to the half integer spin fermions, appearing in Clifford space. The *spin-charge-family* theory, using two kinds of Clifford operators — γ^a and $\tilde{\gamma}^a$ — for the description of spins and charges (frst) and family quantum numbers (second), offers the explanation for not only the appearance of femilies but also for all the properties of quarks and leptons, the gauge fields, scalar fields and others [2–5]. In both cases the gauge fields in $d \geq (13+1)$ — the spin connections $\omega_{ab\alpha}$ (of the two kinds in Clifford case and of one kind in Grassmann case) and the vielbeins $f^{\alpha}{}_{\alpha}$ — determine in $d=(3+1)$ scalars, those with the space index $\alpha = (5,6,\cdots,d)$, and gauge fields, those with the space index $\alpha=(0,1,2,3)$. While states of the Lorentz group and all its subgroups (in any dimension) are in Clifford space in the fundamental representations of the groups, with the family degrees of freedom included [2,3,1], states in Grassmann space manifest with respect to the Lorentz group adjoint representations, allowing no families.

Povzetek. V članku [1], ki uporabi za opis notranjih prostostnih stopenj fermionov Grassmannov prostor, predstavi eden od avtorjev (N.S.M.B.) drugo kvantizacijo fermionov s celoštevilskimi spini. Prispevek predstavi lastnosti upodobitev grup $SO(5,1)$, $SO(3,1)$, $SU(3)\times U(1)$ in $SO(4)$ v Grassmannovem prostoru. Te grupe so posebej zanimive kot podgrupe grupe $SO(13,1)$. Kreacijski in anihilacijski operatorji, ki ustrežejo komutacijskim relacijam za fermione, nosijo v Grassmannovem prostoru celoštevilčni spin. Fermioni s celoštevilčnim spinom ponudijo alternativni opis fermionom v Cliffordovem prostoru, ki nosijo polštevilčni spin. Opaženi so le fermioni s polštevilčnim spinom. Teorija *spinov-nabojev-družin*, ki uporabi dve vrsti operatorjev γ v Cliffordovem prostoru — γ^a in $\tilde{\gamma}^a$ — prvega za opis spina in vseh nabojev in drugega za opis družinskega kvantnega števila, ponuja razlago ne samo za pojav družin, ampak tudi pojasni vse lastnosti kvarkov in leptonov, umeritvenih polj, skalarnih polj in drugo [2–5]. Umeritvena polja v $d \geq (13+1)$ — spinske povezave $\omega_{ab\alpha}$ (dveh vrst v Cliffordovem primeru in ene vrste v Grassmannovem primeru) in "vielbeini" $f^{\alpha}{}_{\alpha}$ — določajo v obeh primerih v $d=(3+1)$ skalarje, če nosijo prostorski indeks $\alpha = (5,6,\cdots,d)$, ter umeritvena polja, kadar imajo prostorski indeks

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$\alpha = (0, 1, 2, 3)$. Stanja Lorentzove grupe in vseh njenih podgrup so za poljubno dimenzijo v Cliffordovem prostoru v fundamentalni upodobitvi in vključujejo družinske prostostne stopnje [2,3,1], v Grassmannovem prostoru pa so glede na Lorentzovo grupo v adjungirani upodobitvi in ne dopuščajo družin.

Keywords: Spinor representations in Grassmann space, Second quantization of fermion fields in Grassmann space, Higher dimensional spaces, Kaluza-Klein theories, Beyond the standard model
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17.1 Introduction

In Ref. [2] the representations in Grassmann and in Clifford space were discussed. In Ref. ([1] and the references therein) the second quantization procedure in both spaces — in Clifford space and in Grassmann space — were discussed in order to try to understand "why nature made a choice of Clifford rather than Grassmann space" during the expansion of our universe, although in both spaces the creation operators $\hat{b}^\dagger_j$ and the annihilation operators $\hat{b}_j$ exist fulfilling the anticommutation relations required for fermions [1]

$$
\begin{aligned}
\{\hat{b}_i, \hat{b}^\dagger_j\}_+ |\phi_o> &= \delta_{ij}\, |\psi_o>\,,\\
\{\hat{b}_i, \hat{b}_j\}_+ |\psi_o> &= 0\, |\psi_o>\,,\\
\{\hat{b}^\dagger_i, \hat{b}^\dagger_j\}_+ |\psi_o> &= 0\, |\psi_o>\,,\\
\hat{b}^\dagger_j |\psi_o> &= |\psi_j>\\
\hat{b}_j |\psi_o> &= 0\, |\psi_o>\ .
\end{aligned}
\tag{17.1}
$$

$|\psi_o>$ is the vacuum state. We use $|\psi_o>= |1>$.

The creation operators can be expressed in both spaces by products of eigenstates of the Cartan subalgebra, Eq. (17.33), of the Lorentz algebra, Eqs. (17.3, 17.11). Starting with one state (Ref. [1]) all the other states of the same representation are reachable by the generators of the Lorentz transformations (which do not belong to the Cartan subalgebra), with $\mathbf{S^{ab}}$ presented in Eq. (17.32) in Grassmann space and with either S^{ab} or $\tilde{S}^{ab}$, Eq. (17.34), in Clifford space.

But while there are in Clifford case two kinds of the generators of the Lorentz transformations — S^{ab} and $\tilde{S}^{ab}$, the first transforming members of one family among themselves, and the second transforming one member of a particular family into the same member of other families — there is in Grassmann space only one kind of the Lorentz generators — $\mathbf{S^{ab}}$. Correspondingly are all the states in Clifford space, which can be second quantized as products of nilpotents and projectors [9,10,1], reachable with one of the two kinds of the operators S^{ab} and $\tilde{S}^{ab}$, while different representations are in Grassmann space disconnected.

On the other hand the vacuum state is in Grassmann case simple — $|\psi_o>= |1>$ — while in Clifford case is the sum of products of projectors, Eq. (17.17).

In Grassmann space states are in the adjoint representations with respect to the Lorentz group, while states in Clifford space belong to the fundamental representations with respect to both generators, S^{ab} and $\tilde{S}^{ab}$, or they are singlets. Correspondingly are properties of fermions, described with the *spin-charge-family* theory [3,4,6,5,8,7], which uses the Clifford space to describe fermion degrees of freedom, in agreement with the observations, offering explanation for all the assumptions of the *standard model* (with families included) and also other observed phenomena.

In Grassmann case the spins manifest, for example, in the case of $SO(6)$ or $SO(5,1)$ decuplets or singlets — triplets and singlets in Clifford case, Table 17.2 — while with respect to the subgroups $SU(3)$ and $U(1)$ of $SO(6)$ the states belong to either singlets, or triplets or sextets, Tables 17.3, 17.4 — triplets and singlets in the Clifford case.

In what follows we discuss representations, manifesting as charges and spins of fermions, of subgroups of $SO(13,1)$, when internal degrees of freedom of fermions are described in Grassmann space and compare properties of these representations with the properties of the corresponding representations appearing in Clifford space. We assume, as in the *spin-charge-family* theory, that both spaces, the internal and the ordinary space, have $d = 2(2n+1)$-dimensions, n is positive integer, $d \geq 14$ and that all the degrees of freedom of fermions and bosons originate in $d = 2(2n+1)$, in which fermions interact with gravity only.

After the break of the starting symmetry $SO(13,1)$ into $SO(7,1)\times SU(3)\times U(1)$, and further to $SO(3,1) \times SU(2) \times SU(2) \times SU(3) \times U(1)$, fermions manifest in $d = (3+1)$ the spin and the corresponding charges and interact with the gauge fields, which are indeed the spin connections with the space index $m = (0,1,2,3)$, originating in $d = (13,1)$ [7]. Also scalar fields originate in gravity: Those spin connections with the space index $a = (5,6,7,8)$ determine masses of fermions, those with the space index $a = (9,10,\ldots,14)$ contribute to particle/antiparticle asymmetry in our universe [4].

We pay attention on fermion fields, the creation and annihilation operators of which fulfill the anticommutation relations of Eq. (17.1).

17.1.1 Creation and annihilation operators in Grassmann space

In Grassmann $d = 2(2n+1)$-dimensional space the creation and annihilation operators follow from the starting two creation and annihilation operators, both with an odd Grassmann character, since those with an even Grassmann character do not obey the anticommutation relations of Eq. (17.1) [1]

$$\begin{aligned}
\hat{b}^{\theta 1\dagger}_{1} &= (\frac{1}{\sqrt{2}})^{\frac{d}{2}}\,(\theta^0 - \theta^3)(\theta^1 + i\theta^2)(\theta^5 + i\theta^6)\cdots(\theta^{d-1} + i\theta^d)\,,\\
\hat{b}^{\theta 1}_{1} &= (\frac{1}{\sqrt{2}})^{\frac{d}{2}}\,(\frac{\partial}{\partial\theta^{d-1}} - i\frac{\partial}{\partial\theta^{d}})\cdots(\frac{\partial}{\partial\theta^{0}} - \frac{\partial}{\partial\theta^{3}})\,,\\
\hat{b}^{\theta 2\dagger}_{1} &= (\frac{1}{\sqrt{2}})^{\frac{d}{2}}\,(\theta^0 + \theta^3)(\theta^1 + i\theta^2)(\theta^5 + i\theta^6)\cdots(\theta^{d-1} + i\theta^d)\,,\\
\hat{b}^{\theta 2}_{1} &= (\frac{1}{\sqrt{2}})^{\frac{d}{2}}\,(\frac{\partial}{\partial\theta^{d-1}} - i\frac{\partial}{\partial\theta^{d}})\cdots(\frac{\partial}{\partial\theta^{0}} + \frac{\partial}{\partial\theta^{3}})\,. \qquad (17.2)
\end{aligned}$$

All the creation operators are products of the eigenstates of the Cartan subalgebra operators,Eq. (17.33)

$$\begin{aligned}
&\mathbf{S}^{ab}(\theta^a \pm \epsilon\theta^b) = \mp i\frac{\eta^{aa}}{\epsilon}(\theta^a \pm \epsilon\theta^b)\,,\\
&\epsilon = 1\,,\ \text{for}\ \eta^{aa} = 1\,, \quad \epsilon = i\,,\ \text{for}\ \eta^{aa} = -1\,,\\
&\mathbf{S}^{ab}\,(\theta^a\theta^b \pm \epsilon\theta^c\theta^d) = 0\,, \quad \mathbf{S}^{cd}\,(\theta^a\theta^b \pm \epsilon\theta^c\theta^d) = 0\,. \qquad (17.3)
\end{aligned}$$

The two creation operators, $\hat{b}^{\theta 1\dagger}_1$ and $\hat{b}^{\theta 2\dagger}_1$, if applied on the vacuum state, form the starting two states ϕ^1_1 and ϕ^2_1 of the two representations, respectively. The vacuum state is chosen to be the simplest one [1] — $|\phi_0>=|1>$. The rest of creation operators of each of the two groups, $\hat{b}^{\theta 1\dagger}_i$ and $\hat{b}^{\theta 2\dagger}_i$, follow from the starting one by the application of the generators of the Lorentz transformations in Grassmann space $\mathbf{S}^{ab}$, Eq. (17.32), which do not belong to the Cartan subalgebra, Eq. (17.33), of the Lorentz algebra. They generate either $|\phi^1_j>$ of the first group or $|\phi^2_j>$ of the second group.

Annihilation operators $\hat{b}^{\theta 1}_i$ and $\hat{b}^{\theta 2}_i$ follow from the creation ones by the Hermitian conjugation [1], when taking into account the assumption

$$(\theta^a)^\dagger = \frac{\partial}{\partial\theta_a}\eta^{aa} = -i\,p^{\theta a}\eta^{aa}\,, \qquad (17.4)$$

from where it follows

$$(\frac{\partial}{\partial\theta_a})^\dagger = \eta^{aa}\,\theta^a\,, \quad (p^{\theta a})^\dagger = -i\eta^{aa}\theta^a\,. \qquad (17.5)$$

The annihilation operators $\hat{b}^{\theta 1}_i$ and $\hat{b}^{\theta 2}_i$ annihilate states $|\phi^1_i>$ and $|\phi^2_i>$, respectively.

The application of $\mathbf{S}^{01}$ on $\hat{b}^{\theta 1\dagger}_1$, for example, transforms this creation operator into $\hat{b}^{\theta 1\dagger}_2 = (\frac{1}{\sqrt{2}})^{\frac{d}{2}-1}\,(\theta^0\theta^3 + i\theta^1\theta^2)\,(\theta^5 + i\theta^6)\cdots(\theta^{d-1} - i\theta^d)$. Correspondingly its Hermitian conjugate annihilation operator is equal to $\hat{b}^{\theta 1}_2 = (\frac{1}{\sqrt{2}})^{\frac{d}{2}-1}\,(\frac{\partial}{\partial\theta^{d-1}} - i\frac{\partial}{\partial\theta^d})\cdots(\frac{\partial}{\partial\theta^3}\frac{\partial}{\partial\theta^0} - i\frac{\partial}{\partial\theta^2}\frac{\partial}{\partial\theta^1})$.

All the states are normalized with respect to the integral over the Grassmann coordinate space [2]

$$\begin{aligned}
<\phi^a_i|\phi^b_j> &= \int d^{d-1}x d^d\theta^a\ \omega <\phi^a_i|\theta><\theta|\phi^b_j>= \delta^{ab}\,\delta_{ij}\,,\\
\omega &= \Pi^d_{k=0}(\frac{\partial}{\partial\theta_k} + \theta^k)\,, \qquad (17.6)
\end{aligned}$$

where ω is a weight function, defining the scalar product $<\phi^a_i|\phi^b_j>$, and we require that [2]

$$\begin{aligned}
&\{d\theta^a, \theta^b\}_+ = 0, \quad \int d\theta^a = 0\,, \quad \int d\theta^a\theta^a = 1\,,\\
&\int d^d\theta\ \theta^0\theta^1\cdots\theta^d = 1\,,\\
&d^d\theta = d\theta^d\ldots d\theta^0\,, \qquad (17.7)
\end{aligned}$$

with $\frac{\partial}{\partial \theta_a} \theta^c = \eta^{ac}$.

There are $\frac{1}{2} \frac{d!}{\frac{d}{2}! \frac{d}{2}!}$ in each of these two groups of creation operators of an odd Grassmann character in $d = 2(2n+1)$-dimensional space.

The rest of creation operators (and the corresponding annihilation operators) would have rather opposite Grassmann character than the ones studied so far: like **a.** $\theta^0\theta^1$ for the creation operator and $[\frac{\partial}{\partial\theta^1}\frac{\partial}{\partial\theta^0}]$ for the corresponding annihilation operator in $d = (1+1)$ (since $\{\theta^0\theta^1, \frac{\partial}{\partial\theta^1}\frac{\partial}{\partial\theta^0}\}_+$ gives $(1+(1+1)\theta^0\theta^1 \frac{\partial}{\partial\theta^1}\frac{\partial}{\partial\theta^0})$), and like **b.** $(\theta^0 \mp \theta^3)(\theta^1 \pm i\theta^2)$ for creation operator and $[(\frac{\partial}{\partial\theta^1} \mp i\frac{\partial}{\partial\theta^2})(\frac{\partial}{\partial\theta^0} \mp \frac{\partial}{\partial\theta^3})]$ for the annihilation operator, or $\theta^0\theta^3\theta^1\theta^2$ for the creation operator and $[\frac{\partial}{\partial\theta^2}\frac{\partial}{\partial\theta^1}\frac{\partial}{\partial\theta^3}\frac{\partial}{\partial\theta^0}]$ for the annihilation operator in $d = (3+1)$ (since, let say, $\{\frac{1}{2}(\theta^0 - \theta^3)(\theta^1 + i\theta^2), \frac{1}{2}(\frac{\partial}{\partial\theta^1} - i\frac{\partial}{\partial\theta^2})(\frac{\partial}{\partial\theta^0} - \frac{\partial}{\partial\theta^3})\}_+$ gives $(1+\frac{1}{4}(1+1)(\theta^0-\theta^3)(\theta^1+i\theta^2)(\frac{\partial}{\partial\theta^1} - i\frac{\partial}{\partial\theta^2})(\frac{\partial}{\partial\theta^0} - \frac{\partial}{\partial\theta^3})$ and equivalently for other cases), but applied on a vacuum states some of them still fulfill some of the relations of Eq. (17.1), but not all (like $\{\frac{1}{2}(\theta^0 - \theta^3)(\theta^1 + i\theta^2), \frac{1}{2}(\theta^0 + \theta^3)(\theta^1 - i\theta^2)\}_+ = i\theta^0\theta^1\theta^2\theta^3$, while it should be zero).

Let us add that, like in Clifford case, one can simplify the scalar product in Grassmann case by recognizing that the scalar product is equal to $\delta^{ab}\,\delta_{ij}$

$$< \phi^a_i|\theta >< \theta|\phi^b_j > = \delta^{ab}\,\delta_{ij}\,, \tag{17.8}$$

without integration over the Grassmann coordinates. Let us manifest this in the case of $d = (1+1)$:$< 1|\frac{1}{\sqrt{2}}(\frac{\partial}{\partial\theta^0} - \frac{\partial}{\partial\theta^1})\frac{1}{\sqrt{2}}(\theta^0 - \theta^1)|1 >= 1$, $|1>$ is the normalized vacuum state, $< 1|1 >= 1$. It is true in all dimensions, what can easily be understood for all the states, which are defined by the creation operators $\hat{b}^\dagger_i$ on the vacuum state $|1>$, $|\phi^b_i >= \hat{b}^\dagger_i|1>$, fulfilling the anticommutation relations of Eq. (17.1).

17.1.2 Creation and annihilation operators in Clifford space

There are two kinds of Clifford objects [2], ([3] and Refs. therein), γ^a and $\tilde{\gamma}^a$, both fulfilling the anticommutation relations

$$\begin{aligned}
\{\gamma^a, \gamma^b\}_+ &= 2\eta^{ab} = \{\tilde{\gamma}^a, \tilde{\gamma}^b\}_+\,, \\
\{\gamma^a, \tilde{\gamma}^b\}_+ &= 0\,.
\end{aligned} \tag{17.9}$$

Both Clifford algebra objects are expressible with θ^a and $\frac{\partial}{\partial\theta^a}$ [2,1], ([3] and Refs. therein)

$$\begin{aligned}
\gamma^a &= (\theta^a + \frac{\partial}{\partial\theta_a})\,, \\
\tilde{\gamma}^a &= i\,(\theta^a - \frac{\partial}{\partial\theta a})\,, \\
\theta^a &= \frac{1}{2}(\gamma^a - i\tilde{\gamma}^a)\,, \\
\frac{\partial}{\partial\theta_a} &= \frac{1}{2}\,(\gamma^a + i\tilde{\gamma}^a)\,,
\end{aligned} \tag{17.10}$$

from where it follows: $(\gamma^a)^\dagger = \gamma^a\eta^{aa}$, $(\tilde{\gamma}^a)^\dagger = \tilde{\gamma}^a\eta^{aa}$, $\gamma^a\gamma^a = \eta^{aa}$, $\gamma^a(\gamma^a)^\dagger = 1$, $\tilde{\gamma}^a\tilde{\gamma}^a = \eta^{aa}$, $\tilde{\gamma}^a(\tilde{\gamma}^a)^\dagger = 1$.

Correspondingly we can use either γ^a or $\tilde{\gamma}^a$ instead of θ^a to span the internal space of fermions. Since both, γ^a and $\tilde{\gamma}^a$, are expressible with θ^a and the derivatives with respect to θ^a, the norm of vectors in Clifford space can be defined by the same integral as in Grassmann space, Eq.(17.6), or we can simplify the scalar product (as in the Grassmann case, Eq. (17.8) by introducing the Clifford vacuum state $|\psi_{oc}>$, Eq. (17.17), instead of $|1>$ in Grassmann case.

We make use of γ^a to span the vector space. As in the case of Grassmann space we require that the basic states are eigenstates of the Cartan subalgebra operators of S^{ab} and $\tilde{S}^{ab}$, Eq. (17.33).

$$
\begin{aligned}
&\stackrel{ab}{(k)}:= \frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b)\,, \quad && \stackrel{ab}{(k)}^{\dagger} = \eta^{aa} \stackrel{ab}{(-k)}\,,\\
&\stackrel{ab}{[k]}:= \frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b)\,, \quad && \stackrel{ab}{[k]}^{\dagger} = \stackrel{ab}{[k]}\,,\\
S^{ab} &\stackrel{ab}{(k)} = \frac{1}{2}k \stackrel{ab}{(k)}\,, && S^{ab} \stackrel{ab}{[k]} = \frac{1}{2}k \stackrel{ab}{[k]}\,,\\
\tilde{S}^{ab} &\stackrel{ab}{(k)} = \frac{1}{2}k \stackrel{ab}{(k)}\,, && \tilde{S}^{ab} \stackrel{ab}{[k]} = -\frac{1}{2}k \stackrel{ab}{[k]}\,,
\end{aligned}
\tag{17.11}
$$

with $k^2 = \eta^{aa}\eta^{bb}$. To calculate $\tilde{S}^{ab} \stackrel{ab}{(k)}$ and $\tilde{S}^{ab} \stackrel{ab}{[k]}$ we use [10,9] the relation on any Clifford algebra object A as follows

$$
(\tilde{\gamma^a} A = i(-)^{(A)} A\gamma^a)|\psi_{oc}>\,, \tag{17.12}
$$

where A is any Clifford algebra object and $(-)^{(A)} = -1$, if A is an odd Clifford algebra object and $(-)^{(A)} = 1$, if A is an even Clifford algebra object, $|\psi_{oc}>$ is the vacuum state, replacing the vacuum state $|\psi_o> = |1>$, used in Grassmann case, with the one of Eq. (17.17), in accordance with the relation of Eqs. (17.10, 17.6, 17.7), Ref. [1].

We can define now the creation and annihilation operators in Clifford space so that they fulfill the requirements of Eq. (17.1). We write the starting creation operator and its Hermitian conjugate one (in accordance with Eq. (17.11) and Eq.(17.33)) in $2(2n+1)$-dimensional space as follows [1]

$$
\begin{aligned}
\hat{b}^{1\dagger}_1 &= \stackrel{03}{(+i)}\stackrel{12}{(+)}\stackrel{56}{(+)} \cdots \stackrel{d-1\ d}{(+)}\,,\\
\hat{b}^{1}_1 &= \stackrel{d-1\ d}{(-)} \cdots \stackrel{56}{(-)}\stackrel{12}{(-)}\stackrel{03}{(-i)}\,.
\end{aligned}
\tag{17.13}
$$

The starting creation operator $\hat{b}^{1\dagger}_1$, when applied on the vacuum state $|\psi_{oc}>$, defines the starting family member of the starting "family". The corresponding starting annihilation operator is its Hermitian conjugated one, Eq. (17.11).

All the other creation operators of the same family can be obtained by the application of the generators of the Lorentz transformations S^{ab}, Eq. (17.34), which do not belong to the Cartan subalgebra of $SO(2(2n+1)-1,1)$, Eq. (17.33).

$$
\begin{aligned}
\hat{b}^{1\dagger}_i &\propto S^{ab}..S^{ef}\hat{b}^{1\dagger}_1\,,\\
\hat{b}^{1}_i &\propto \hat{b}^{1}_1 S^{ef}..S^{ab}\,,
\end{aligned}
\tag{17.14}
$$

with $S^{ab\dagger} = \eta^{aa}\eta^{bb}S^{ab}$. The proportionality factors are chosen so, that the corresponding states $|\psi^1_1> = \hat{b}^{1\dagger}_i|\psi_{oc}>$ are normalized, where $|\psi_{oc}>$ is the normalized vacuum state, $<\psi_{oc}|\psi_{oc}> = 1$.

The creation operators creating different "families" with respect to the starting "family", Eq. (17.13), can be obtained from the starting one by the application of $\tilde{S}^{ab}$, Eq. (17.34), which do not belong to the Cartan subalgebra of $\widetilde{SO}(2(2n+1)-1,1)$, Eq. (17.33). They all keep the "family member" quantum number unchanged.

$$\hat{b}^{\alpha\dagger}_i \propto \tilde{S}^{ab}\cdots\tilde{S}^{ef}\,\hat{b}^{1\dagger}_i\,. \tag{17.15}$$

Correspondingly we can define (up to the proportionality factor) any creation operator for any "family" and any "family member" with the application of S^{ab} and $\tilde{S}^{ab}$ [1]

$$\begin{aligned}\hat{b}^{\alpha\dagger}_i &\propto \tilde{S}^{ab}\cdots\tilde{S}^{ef}S^{mn}\cdots S^{pr}\hat{b}^{1\dagger}_1\\ &\propto S^{mn}\cdots S^{pr}\hat{b}^{1\dagger}_1 S^{ab}\cdots S^{ef}\,.\end{aligned} \tag{17.16}$$

All the corresponding annihilation operators follow from the creation ones by the Hermitian conjugation.

There are $2^{\frac{d}{2}-1}\times 2^{\frac{d}{2}-1}$ creation operators of an odd Clifford character and the same number of annihilation operators, which fulfill the anticommutation relations of Eq. (17.1) on the vacuum state $|\psi_{oc}>$ with $2^{\frac{d}{2}-1}$ summands

$$\begin{aligned}&|\psi_{oc}> =\\ &\alpha\,(\overset{03}{[-i]}\overset{12}{[-]}\overset{56}{[-]}\cdots\overset{d-1\;d}{[-]} + \overset{03}{[+i]}\overset{12}{[+]}\overset{56}{[-]}\cdots\overset{d-1\;d}{[-]} + \overset{03}{[+i]}\overset{12}{[-]}\overset{56}{[+]}\cdots\overset{d-1\;d}{[-]} + \cdots)|0>\,,\\ &\alpha = \frac{1}{\sqrt{2^{\frac{d}{2}-1}}}\,,\\ &\text{for } d = 2(2n+1)\,,\end{aligned} \tag{17.17}$$

n is a positive integer. For a chosen $\alpha = \frac{1}{\sqrt{2^{\frac{d}{2}-1}}}$ the vacuum is normalized: $<\psi_{oc}|\psi_{oc}> = 1$.

It is proven in Ref. [1] that the creation and annihilation operators fulfill the anticommutation relations required for fermions, Eq. (17.1).

17.2 Properties of representations of the Lorentz group $SO(2(2n+1))$ and of subgroups in Grassmann and in Clifford space

The purpose of this contribution is to compare properties of the representations of the Lorentz group $SO(2(2n+1))$, $n \geq 3$, when for the description of the internal degrees of freedom of fermions either **i.** Grassmann space or **ii.** Clifford space is used. The *spin-charge-family* theory ([6,5,3,4,8,7,11] and the references therein) namely predicts that all the properties of the observed either quarks and leptons or vector gauge fields or scalar gauge fields originate in $d \geq (13+1)$, in which massless fermions interact with the gravitational field only — with its spin connections and vielbeins.

However, both — Clifford space and Grassmann space — allow second quantized states, the creation and annihilation operators of which fulfill the anticommutation relations for fermions of Eq. (17.1).

But while Clifford space offers the description of spins, charges and families of fermions in $d = (3+1)$, all in the fundamental representations of the Lorentz group $SO(13,1)$ and the subgroups of the Lorentz group, in agreement with the observations, the representations of the Lorentz group are in Grassmann space the adjoint ones, in disagreement with what we observe.

We compare properties of the representations in Grassmann case with those in Clifford case to be able to better understand "the choice of nature in the expanding universe, making use of the Clifford degrees of freedom", rather than Grassmann degrees of freedom.

In introduction we briefly reviewed properties of creation and annihilation operators in both spaces, presented in Ref. [1] (and the references therein). We pay attention on spaces with $d = 2(2n+1)$ of ordinary coordinates and $d = 2(2n+1)$ internal coordinates, either of Clifford or of Grassmann character.

i. In Clifford case there are $2^{\frac{d}{2}-1}$ creation operators of an odd Clifford character, creating "family members" when applied on the vacuum state. We choose them to be eigenstates of the Cartan subalgebra operators, Eq.(17.33), of the Lorentz algebra. All the members can be reached from any of the creation operators by the application of S^{ab}, Eq. (17.34). Each "family member" appears in $2^{\frac{d}{2}-1}$ "families", again of an odd Clifford character, since the corresponding creation operators are reachable by $\tilde{S}^{ab}$, Eq. (17.34), which are Clifford even objects.

There are correspondingly $2^{\frac{d}{2}-1} \cdot 2^{\frac{d}{2}-1}$ creation and the same number ($2^{\frac{d}{2}-1} \cdot 2^{\frac{d}{2}-1}$) of annihilation operators. Also the annihilation operators, annihilating states of $2^{\frac{d}{2}-1}$ "family members" in $2^{\frac{d}{2}-1}$ "families", have an odd Clifford character, since they are Hermitian conjugate to the creation ones.

The rest of $2 \cdot 2^{\frac{d}{2}-1} \cdot 2^{\frac{d}{2}-1}$ members of the Lorentz representations have an even Clifford character, what means that the corresponding creation and annihilation operators can not fulfill the anticommutation relations required for fermions, Eq. (17.1). Among these $2^{\frac{d}{2}-1}$ products of projectors determine the vacuum state, Eq. (17.17).

ii. In Grassmann case there are $\frac{d!}{\frac{d}{2}!\frac{d}{2}!}$ operators of an odd Grassmann character, which form the creation operators, fulfilling with the corresponding annihilation operators the requirements of Eq. (17.1). All the creation operators are chosen to be products of the eigenstates of the Cartan subalgebra $\mathbf{S}^{ab}$, Eq. (17.33). The corresponding annihilation operators are the Hermitian conjugated values of the creation operators, Eqs. (17.4, 17.5, 17.2). The creation operators form, when applied on the simple vacuum state $|\phi_o>=|1>$, two independent groups of states. The members of each of the two groups are reachable from any member of a group by the application of $\mathbf{S}^{ab}$, Eq. (17.32). All the states of any of the two decuplets are orthonormalized.

We comment in what follows the representations in $d = (13+1)$ in Clifford and in Grassmann case. In *spin-charge family* theory there are breaks of the starting symmetry from $SO(13,1)$ to $SO(3,1) \times SU(2) \times SU(3) \times U(1)$ in steps, which lead to the so far observed quarks and leptons, gauge and scalar fields and gravity. One of the authors (N.S.M.B.), together with H.B. Nielsen, defined the discrete symmetry operators for Kaluza-Klein theories for spinors in Clifford space [19]. In Ref. [1] the same authors define the discrete symmetry operators in the case that for the description of fermion degrees of freedom Grassmann space is used. Here we comment symmetries in both spaces for some of subgroups of the $SO(13,1)$ group, as well as the appearance of the Dirac sea.

17.2.1 Equations of motion in Grassmann and Clifford space

We define [1] the action in Grassmann space, for which we require — similarly as in Clifford case — that the action for a free massless object

$$\mathcal{A} = \frac{1}{2}\{\int d^d x \, d^d\theta \, \omega \, (\phi^\dagger (1 - 2\theta^0 \frac{\partial}{\partial\theta^0}) \, \frac{1}{2} \, (\theta^a p_a + \eta^{aa}\theta^{a\dagger} p_a)\phi\}, \quad (17.18)$$

is Lorentz invariant. The corresponding equation of motion is

$$\frac{1}{2}[(1 - 2\theta^0 \frac{\partial}{\partial\theta^0}) \, \theta^a + ((1 - 2\theta^0 \frac{\partial}{\partial\theta^0}) \, \theta^a)^\dagger] \; p_a \, |\phi^\theta_i > \, = 0 \, , \quad (17.19)$$

$p_a = i \, \frac{\partial}{\partial x^a}$, leading to the Klein-Gordon equation

$$\{(1 - 2\theta^0 \frac{\partial}{\partial\theta^0})\theta^a p_a\}^\dagger \, \theta^b p_b |\phi^\theta_i > = p^a p_a |\phi^\theta_i >= 0 \, . \quad (17.20)$$

In the Clifford case the action for massless fermions is well known

$$\mathcal{A} = \int d^d x \, \frac{1}{2} \, (\psi^\dagger \gamma^0 \, \gamma^a p_a \psi) + h.c. \, , \quad (17.21)$$

leading to the equations of motion

$$\gamma^a p_a |\psi^\alpha > = 0 \, , \quad (17.22)$$

which fulfill also the Klein-Gordon equation

$$\gamma^a p_a \gamma^b p_b |\psi^\alpha_i > = p^a p_a |\psi^\alpha_i >= 0 \, . \quad (17.23)$$

17.2.2 Discrete symmetries in Grassmann and Clifford space

We follow also here Ref. [1] and the references therein. We distinguish in d-dimensional space two kinds of dicsrete operators $\mathcal{C}, \mathcal{P}$ and $\mathcal{T}$ operators with respect to the internal space which we use.

In the Clifford case [19], when the whole d-space is treated equivalently, we have

$$\begin{aligned}
&\mathcal{C}_{\mathcal{H}} = \prod_{\gamma^a \in \Im} \gamma^a \; K \, , \quad \mathcal{T}_{\mathcal{H}} = \gamma^0 \prod_{\gamma^a \in \Re} \gamma^a \; K \, I_{x^0} \;\; , \quad \mathcal{P}^{(d-1)}_{\mathcal{H}} = \gamma^0 \, I_{\vec{x}} \, , \\
&I_x x^a = -x^a \, , \quad I_{x^0} x^a = (-x^0, \vec{x}) \, , \quad I_{\vec{x}} \vec{x} = -\vec{x} \, , \\
&I_{\vec{x}_3} x^a = (x^0, -x^1, -x^2, -x^3, x^5, x^6, \ldots, x^d) \, . \qquad (17.24)
\end{aligned}$$

The product $\prod \gamma^a$ is meant in the ascending order in γ^a.

In the Grassmann case we correspondingly define

$$\mathcal{C}_G = \prod_{\gamma^a_G \in \Im \gamma^a} \gamma^a_G \, K \,, \quad \mathcal{T}_G = \gamma^0_G \prod_{\gamma^a_G \in \Re \gamma^a} \gamma^a_G \, K \, I_{x^0} \,, \quad \mathcal{P}^{(d-1)}_G = \gamma^0_G \, I_{\vec{x}} \,, \tag{17.25}$$

with γ^a_G defined as

$$\gamma^a_G = (1 - 2\theta^a \eta^{aa} \frac{\partial}{\partial \theta_a}) \,, \tag{17.26}$$

while I_x, $I_{\vec{x}_3}$ is defined in Eq. (17.24). Let be noticed, that since γ^a_G $(= -i\eta^{aa}\,\gamma^a \tilde{\gamma}^a)$ is always real as there is $\gamma^a i \tilde{\gamma}^a$, while γ^a is either real or imaginary, we use in Eq. (17.25) γ^a to make a choice of appropriate γ^a_G. In what follows we shall use the notation as in Eq. (17.25).

We define, according to Ref. [1] (and the references therein) in both cases — Clifford Grassmann case — the operator "emptying" [6,5] (arxiv:1312.1541) the Dirac sea, so that operation of "emptying$_N$" after the charge conjugation $\mathcal{C}_\mathcal{H}$ in the Clifford case and "emptying$_G$" after the charge conjugation $\mathcal{C}_G$ in the Grassmann case (both transform the state put on the top of either the Clifford or the Grassmann Dirac sea into the corresponding negative energy state) creates the anti-particle state to the starting particle state, both put on the top of the Dirac sea and both solving the Weyl equation, either in the Clifford case, Eq. (17.22), or in the Grassmann case, Eq. (17.19), for free massless fermions

$$\begin{aligned} \text{"emptying}_N\text{"} &= \prod_{\Re \gamma^a} \gamma^a \, K \quad \text{in Clifford space} \,, \\ \text{"emptying}_G\text{"} &= \prod_{\Re \gamma^a} \gamma^a_G \, K \quad \text{in Grassmann space} \,, \end{aligned} \tag{17.27}$$

although we must keep in mind that indeed the anti-particle state is a hole in the Dirac sea from the Fock space point of view. The operator "emptying" is bringing the single particle operator $\mathcal{C}_\mathcal{H}$ in the Clifford case and $\mathcal{C}_G$ in the Grassmann case into the operator on the Fock space in each of the two cases. Then the anti-particle state creation operator — $\underline{\Psi}^\dagger_a[\Psi_p]$ — to the corresponding particle state creation operator — can be obtained also as follows

$$\begin{aligned} \underline{\Psi}^\dagger_a[\Psi_p] \, |vac> &= \mathbb{C}_\mathcal{H} \, \underline{\Psi}^\dagger_p[\Psi_p] \, |vac> = \int \Psi^\dagger_a(\vec{x}) \, (\mathbb{C}_\mathcal{H} \, \Psi_p(\vec{x})) \, d^{(d-1)}x \; |vac> \,, \\ \mathbb{C}_\mathcal{H} &= \text{"emptying}_N\text{"} \cdot \mathcal{C}_\mathcal{H} \end{aligned} \tag{17.28}$$

in both cases.

The operators $\mathbb{C}_\mathcal{H}$ and $\mathbb{C}_G$

$$\begin{aligned} \mathbb{C}_\mathcal{H} &= \text{"emptying}_N\text{"} \cdot \mathcal{C}_\mathcal{H} \,, \\ \mathbb{C}_G &= \text{"emptying}_{NG}\text{"} \cdot \mathcal{C}_G \,, \end{aligned} \tag{17.29}$$

operating on $\Psi_p(\vec{x})$ transforms the positive energy spinor state (which solves the corresponding Weyl equation for a massless free fermion) put on the top of the Dirac sea into the positive energy anti-fermion state, which again solves the corresponding Weyl equation for a massless free anti-fermion put on the top of the Dirac sea. Let us point out that either the operator "emptying$_{N}$" or the operator "emptying$_{NG}$" transforms the single particle operator either $\mathcal{C}_{\mathcal{H}}$ or $\mathcal{C}_{G}$ into the operator operating in the Fock space.

We use the Grassmann even, Hermitian and real operators γ^a_G, Eq. (17.26), to define discrete symmetry in Grassmann space, first we did in $((d+1)-1)$ space, Eq. (17.25), now we do in $(3+1)$ space, Eq. (17.30), as it is done in [19] in the Clifford case. In the Grassmann case we do this in analogy with the operators in the Clifford case [19]

$$
\begin{aligned}
\mathcal{C}_{NG} &= \prod_{\gamma^m_G \in \Re\gamma^m} \gamma^m_G \, K \, I_{x^6 x^8 \dots x^d} \,, \\
\mathcal{T}_{NG} &= \gamma^0_G \prod_{\gamma^m_G \in \Im\gamma^m} K \, I_{x^0} I_{x^5 x^7 \dots x^{d-1}} \,, \\
\mathcal{P}^{(d-1)}_{NG} &= \gamma^0_G \prod_{s=5}^{d} \gamma^s_G I_{\vec{x}} \,, \\
\mathbb{C}_{NG} &= \prod_{\gamma^s_G \in \Re\gamma^s} \gamma^s_G \,, I_{x^6 x^8 \dots x^d} \,, \\
\mathbb{C}_{NG}\mathcal{P}^{(d-1)}_{NG} &= \gamma^0_G \prod^{d}_{\gamma^s_G \in \Im\gamma^s, s=5} \gamma^s_G \, I_{\vec{x}_3} \, I_{x^6 x^8 \dots x^d} \,, \\
\mathbb{C}_{NG}\mathcal{T}_{NG}\mathcal{P}^{(d-1)}_{NG} &= \prod_{\gamma^s_G \in \Im\gamma^a} \gamma^a_G \, I_x K \,.
\end{aligned}
\tag{17.30}
$$

17.2.3 Representations in Grassmann and in Clifford space in d = (13 + 1)

In the *spin-charge-family* theory the starting dimension of space must be $\geq (13+1)$, in order that the theory manifests in $d = (3+1)$ all the observed properties of quarks and leptons, gauge and scalar fields (explaining the appearance of higgs and the Yukawa couplings), offering as well the explanations for the observations in cosmology.

Let us therefore comment properties of representations in both spaces when $d = (13+1)$, if we analyze one group of "family members" of one of families in Clifford space, and one of the two representations of $\frac{1}{2}\frac{d!}{\frac{d}{2}!\frac{d}{2}!}$.

a. Let us start with Clifford space [3,5,4,6,13,12,2]. Each "family" representation has $2^{\frac{d}{2}-1} = 64$ "family members". If we analyze this representation with respect to the subgroups $SO(3,1)$, $(SU(2)\times SU(2))$ of $SO(4)$ and $(SU(3)\times U(1))$ of $SO(6)$ of the Lorentz group $SO(13,1)$, we find that the representations have quantum numbers of all the so far observed quarks and leptons and antiquarks and antileptons, all with spin up and spin down, as well as of the left and right handedness, with the right handed neutrino included as the member of this representation.

Let us make a choice of the "family", which follows by the application of $\tilde{S}^{15}$ on the "family", for which the creation operator of the right-handed neutrino with spin $\frac{1}{2}$ would be $\stackrel{03}{(+i)}\stackrel{12}{(+)} \mid \stackrel{56}{(+)}\stackrel{78}{(+)} \parallel \stackrel{9\,10}{(+)}\;\;\stackrel{11\,12}{(+)}\;\;\stackrel{13\,14}{(+)}$. (The corresponding annihilation operator of this creation operator is $\stackrel{13\,14}{(-)}\;\;\stackrel{11\,12}{(-)}\;\;\stackrel{9\,10}{(-)} \parallel \stackrel{78}{(-)}\stackrel{56}{(-)} \mid \stackrel{12}{(-)}\stackrel{03}{(-i)}$). In Table 6.3 (see pages 112–113 in this volume) presented creation operators for all the "family members" of this family follow by the application of S^{ab} on $\tilde{S}^{15}$ $\stackrel{03}{(+i)}\stackrel{12}{(+)} \mid \stackrel{56}{(+)}\stackrel{78}{(+)} \parallel \stackrel{9\,10}{(+)}\;\;\stackrel{11\,12}{(+)}\;\;\stackrel{13\,14}{(+)}$. (The annihilation operator of $\tilde{S}^{15}$ $\stackrel{03}{(+i)}\stackrel{12}{(+)} \mid \stackrel{56}{(+)}\stackrel{78}{(+)} \parallel \stackrel{9\,10}{(+)}\;\;\stackrel{11\,12}{(+)}\;\;\stackrel{13\,14}{(+)}$ is $\stackrel{13\,14}{[-]}\;\;\stackrel{11\,12}{[-]}\;\;\stackrel{9\,10}{(-)} \parallel \stackrel{78}{(-)}\stackrel{56}{[+]} \mid \stackrel{12}{[+]}\stackrel{03}{(-i)}$.)

This is the representation of Table 6.3 (see pages 112–113 in this volume), in which all the 'family members" of one "family" are classified with respect to the subgroups $SO(3,1) \times SU(2) \times SU(2) \times SU(3) \times U(1)$. The vacuum state on which the creation operators, represented in the third column, apply is defined in Eq. (17.17). All the creation operators of all the states are of an odd Clifford character, fulfilling together with the annihilation operators (which have as well the equivalent odd Clifford character, since the Hermitian conjugation do not change the Clifford character) the requirements of Eq. (17.1). Since the Clifford even operators S^{ab} and $\tilde{S}^{ab}$ do not change the Clifford character, all the creation and annihilation operators, obtained by products of S^{ab} or $\tilde{S}^{ab}$ or both, fulfill the requirements of Eq. (17.1).

We recognize in Table 6.3 (see pages 112–113 in this volume) that quarks distinguish from leptons only in the $SO(6)$ part of the creation operators. Quarks belong to the colour ($SU(3)$) triplet carrying the "fermion" $(U(1))$ quantum number $\tau^4 = \frac{1}{6}$, antiquarks belong to the colour antitriplet, carrying the "fermion" quantum number $\tau^4 = -\frac{1}{6}$. Leptons belong to the colour ($SU(3)$) singlet, carrying the "fermion" $(U(1))$ quantum number $\tau^4 = -\frac{1}{2}$, while antileptons belong to the colour antisinglet, carrying the "fermion" quantum number $\tau^4 = \frac{1}{2}$.

Let us also comment that the oddness and evenness of part of states in the subgroups of the $SO(13,1)$ group change: While quarks and leptons have in the part of $SO(6)$ an odd Clifford character, have antiquarks and antileptons in this part an even odd Clifford character. Correspondingly the Clifford character changes in the rest of subgroups.

Families are generated by $\tilde{S}^{ab}$ applying on any one of the "family members". Again all the "family members" of this "family" follow by the application of all S^{ab} (not belonging to Cartan subalgebra).

The spontaneous break of symmetry from $SO(13,1)$ to $SO(7,1)\times SU(3)\times U(1)$, Refs. [3–5], makes in the *spin-charge-family* theory all the families, generated by $\tilde{S}^{mt}$ and $\tilde{S}^{st}$, $[m = (0,1,2,3),\, s = (5,6,7,8),\, t = (9,10,11,12,13,14)]$, massive of the scale of $\geq 10^{16}$ GeV [14–16]. Correspondingly there are only eight families of quarks and leptons, which split into two groups of four families, both manifesting the symmetry $\widetilde{SU}(2) \times \widetilde{SU}(2) \times U(1)$. (The fourth of the lower four families is predicted to be observed at the LHC, the stable of the upper four families contributes to the dark matter [17].)

In the *spin-charge-family* theory fermions interact with only gravity, which manifests after the break of the starting symmetry in $d = (3+1)$ as all the known vector gauge fields, ordinary gravity and the higgs and the Yukawa couplings [7,3–

5,11]. There are scalar fields which bring masses to family members. The theory explains not only all the assumptions of the *standard model* with the appearance of families, the vector gauge fields and the scalar fields, it also explains appearance of the dark matter [17], matter/antimatter asymmetry [4] and other phenomena, like the miraculous cancellation of the triangle anomalies in the *standard model* [8].

b. We compare representations of $SO(13,1)$ in Clifford space with those in Grassmann space. We have **no "family" quantum numbers in Grassmann space**. We only have two groups of creation operators, defining — when applied on the vacuum state $|1>$ — $\frac{1}{2}\,\frac{d!}{\frac{d}{2}!\frac{d}{2}!}$ equal in $d=(13+1)$ to 1716 members in each of the two groups in comparison in Clifford case with 64 "family members" in one "family" and 64 "families", which the breaks of symmetry reduce to 8 "families", making all the $(64-8)$ "families" massive and correspondingly not observable at low energies ([5,14] and the references therein).

Since the 1716 members are hard to be mastered, let us look therefore at each subgroup — $SU(3)\times U(1)$, $SO(3,1)$ and $SU(2)\times SU(2)$ of $SO(13,1)$ — separately.

Let us correspondingly analyze the subgroups: $SO(6)$ from the point of view of the two subgroups $SU(3)\times U(1)$, and $SO(7,1)$ from the point of view of the two subgroups $SO(3,1)\times SO(4)$, and let us also analyze $SO(4)$ as $SU(2)\times SU(2)$.

17.2.4 Examples of second quantizable states in Grassmann and in Clifford space

We compare properties of representations in Grassmann and in Clifford space for several choices of subgroups of $SO(13,1)$ in the case that in both spaces creation and annihilation operators fulfill requirements of Eq. (17.1), that is that both kinds of states can be second quantized. Let us again point out that in Grassmann case fermions carry integer spins, while in Clifford case they carry half integer spin.

States in Grassmann and in Clifford space for $\mathbf{d=(5+1)}$ We study properties of representations of the subgroup $SO(5,1)$ (of the group $SO(13,1)$), in Clifford and in Grassmann space, requiring that states can be in both spaces second quantized, fulfilling therefore Eq. (17.1).

a. In Clifford space there are $2^{\frac{d}{2}-1}$, each with $2^{\frac{d}{2}-1}$ family members, that is 4 families, each with 4 members. All these sixteen states are of an odd Clifford character, since all can be obtained by products of S^{ab}, $\tilde{S}^{ab}$ or both from an Clifford odd staring states and are correspondingly second quantizable as required in Eq. (17.1). All the states are the eigenstates of the Cartan subalgebra of the Lorentz algebra in Clifford space, Eq. (17.33), solving the Weyl equation for free massless spinors in Clifford space, Eq. (17.22). The four familes, with four members each, are presented in Table 17.1. All of these 16 states are reachable from the first one in each of the four families by S^{ab}, or by $\tilde{S}^{ab}$ if aplied on any family member.

Each of these four families have positive and negative energy solutions, as presented in [19], in Table I.. We present in Table 17.1 only states of a positive energy, that is states above the Dirac sea. The antiparticle states are reachable from the particle states by the application of the operator $\mathbb{C}_{\mathcal{N}}\,\mathcal{P}^{(d-1)}_{\mathcal{N}}=\gamma^0\gamma^5 I_{\vec{x}_3} I_{x^6}$, keeping the spin $\frac{1}{2}$, while changing the charge from $\frac{1}{2}$ to $-\frac{1}{2}$. All the states above the Dirac sea are indeed the hole in the Dirac sea, as explained in Ref. [19].

	ψ	S^{03}	S^{12}	S^{56}	$\tilde{S}^{03}$	$\tilde{S}^{12}$	$\tilde{S}^{56}$
ψ^{I}_1	$\overset{03}{(+i)}\overset{12}{(+)}\overset{56}{(+)}$	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
ψ^{I}_2	$\overset{03}{[-i]}\overset{12}{[-]}\overset{56}{(+)}$	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
ψ^{I}_3	$\overset{03}{[-i]}\overset{12}{(+)}\overset{56}{[-]}$	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
ψ^{I}_4	$\overset{03}{(+i)}\overset{12}{[-]}\overset{56}{[-]}$	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
ψ^{II}_1	$\overset{03}{[+i]}\overset{12}{[+]}\overset{56}{(+)}$	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
ψ^{II}_2	$\overset{03}{(-i)}\overset{12}{(-)}\overset{56}{(+)}$	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
ψ^{II}_3	$\overset{03}{(-i)}\overset{12}{[+]}\overset{56}{[-]}$	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
ψ^{II}_4	$\overset{03}{[+i]}\overset{12}{(-)}\overset{56}{[-]}$	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
ψ^{III}_1	$\overset{03}{[+i]}\overset{12}{(+)}\overset{56}{[+]}$	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
ψ^{III}_2	$\overset{03}{(-i)}\overset{12}{[-]}\overset{56}{[+]}$	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
ψ^{III}_3	$\overset{03}{(-i)}\overset{12}{(+)}\overset{56}{(-)}$	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
ψ^{III}_4	$\overset{03}{[+i]}\overset{12}{[-]}\overset{56}{(-)}$	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
ψ^{IV}_1	$\overset{03}{(+i)}\overset{12}{[+]}\overset{56}{[+]}$	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
ψ^{IV}_2	$\overset{03}{[-i]}\overset{12}{(-)}\overset{56}{[+]}$	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
ψ^{IV}_3	$\overset{03}{[-i]}\overset{12}{[+]}\overset{56}{(-)}$	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
ψ^{IV}_4	$\overset{03}{(+i)}\overset{12}{(-)}\overset{56}{(-)}$	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$

Table 17.1. The four families, each with four members. For the choice $p^a = (p^0, 0, 0, p^3, 0, 0)$ have the first and the second member the space part equal to $e^{-i|p^0|x^0 + i|p^3|x^3}$ and $e^{-i|p^0|x^0 - i|p^3|x^3}$, representing the particles with spin up and down, respectively. The third and the fourth member represent the antiparticle states, with the space part equal to $e^{-i|p^0|x^0 - i|p^3|x^3}$ and $e^{-i|p^0|x^0 + i|p^3|x^3}$, with the spin up and down respectively. The antiparticle states follow from the particle state by the application of $\mathbb{C}_{\mathcal{N}}\, \mathcal{P}^{(d-1)}_{\mathcal{N}} = \gamma^0 \gamma^5 I_{\vec{x}_3} I_{x^6}$. The charge of the particle states is $\frac{1}{2}$, for antiparticle states $-\frac{1}{2}$.

b.0 In Grassmann space there are $\frac{d!}{\frac{d}{2}!\frac{d}{2}!}$ second quantizable states as required in Eq. (17.1), forming in $d = (5+1)$ two decuplets — each with $\frac{1}{2}\, \frac{d!}{\frac{d}{2}!\frac{d}{2}!}$ states — all are the eigenstates of the Cartan subalgebra of the Lorentz algebra in (internal) Grassmann space. All the states of one (anyone of the two) decuplets are reachable by the application of the operators $\mathbf{S}^{ab}$ on a starting state. The two decouplets are presented in Table 17.2

Let us first find the solution of the equations of motion for free massless fermions, Eq. (17.19), with the momentum $p^a = (p^0, p^1, p^2, p^3, 0, 0)$. One obtains for $\psi_I = \alpha(\theta^0 - \theta^3)(\theta^1 + i\theta^2)(\theta^5 + i\theta^6) + \beta(\theta^0\theta^3 + i\theta^1\theta^2)(\theta^5 + i\theta^6) + \gamma(\theta^0 +$

$\theta^3)(\theta^1 - i\theta^2)(\theta^5 + i\theta^6)$ the solution

$$\begin{aligned}
&\beta = \frac{2\gamma(p^1 - ip^2)}{(p^0 - p^3)} = \frac{2\gamma(p^0 + p^3)}{(p^1 + ip^2)} = -\frac{2\alpha(p^0 - p^3)}{(p^1 - ip^2)} = -\frac{2\alpha(p^1 + ip^2)}{(p^0 + p^3)}\,,\\
&(p^0)^2 = (p^1)^2 + (p^2)^2 + (p^3)^2\,,\\
&\frac{\beta}{-\alpha} = \frac{2(p^0 - p^3)}{(p^1 - ip^2)}\,, \quad \frac{\gamma}{-\alpha} = \frac{(p^0 - p^3)^2}{(p^1 - ip^2)^2}\,.
\end{aligned} \tag{17.31}$$

One has for $p^0 = |p^0|$ the positive energy solution, describing a fermion above the "Dirac sea", and for $p^0 = -|p^0|$ the negative energy solution, describing a fermion in the "Dirac sea". The "charge" of the "fermion" is 1. Similarly one finds the solution for the other three states with the negative "charge" -1, again with the positive and negative energy. The space part of the "fermion" state is for "spin up" equal to $e^{-i|p^0|x^0 + i\vec{p}\vec{x}}$, for his antiparticle for the same internal spin $e^{-i|p^0|x^0 - i\vec{p}\vec{x}}$.

The discrete symmetry operator $\mathbb{C}_{NG}\,\mathcal{P}^{(d-1)}_{NG}$, which is in our case equal to $\gamma^0_G\gamma^5_G I_{\vec{x}_3} I_{x^6}$, transforms the first state in Table 17.2 into the sixth, the second state into the fifth, the third state into the fourth, keeping the same spin while changing the "charge" of the superposition of the three states ψ_{Ip}. Both superposition of states, Eq. (17.31) represent the positive energy states put on the top of the "Dirac" sea, the first describing a particle with "charge" 1 and the second superposition of the second three states ψ_{Ia}, describing the antiparticle with the"charge" -1. We namely apply $\underline{\mathbb{C}}_{\mathcal{NG}}\,\mathcal{P}^{(d-1)}_{\mathcal{NG}}$ on $\underline{\Psi}^{\dagger}_{p}[\Psi^{pos}_{I}]$ by applying $\mathbb{C}_{\mathcal{NG}}\,\mathcal{P}^{(d-1)}_{\mathcal{NG}}$ on Ψ^{pos}_{I} as follows: $\underline{\mathbb{C}}_{\mathcal{NG}}\,\mathcal{P}^{(d-1)}_{\mathcal{NG}}\;\underline{\Psi}^{\dagger}_{p}[\Psi^{pos}_{I}]\;(\underline{\mathbb{C}}_{\mathcal{NG}}\;\mathcal{P}^{(d-1)}_{\mathcal{NG}})^{-1} = \underline{\Psi}^{\dagger}_{aNG}[\mathbb{C}_{\mathcal{NG}}\,\mathcal{P}^{(d-1)}_{\mathcal{NG}}\Psi^{pos}_{1}]$. One recognizes that it is $\mathbb{C}_{\mathcal{NG}}\,\mathcal{P}^{(d-1)}_{\mathcal{NG}}\;\Psi^{pos}_{I} = \Psi^{pos}_{II}$ (Table 17.2), which must be put on the top of the "Dirac" sea, representing the hole in the particular state in the "Dirac" sea, which solves the corresponding equation of motion for the negative energy.

Properties of $SO(6)$ in Grassmann and in Clifford space when $SO(6)$ is embedded into $SO(13,1)$ **a.** Let us first repeat properties of the $SO(6)$ part of the $SO(13,1)$ representation of 64 "family members" in Clifford space, presented in Table 6.3 (see pages 112–113 in this volume). As seen in Table 6.3 (see pages 112–113 in this volume) there are one quadruplet ($2^{\frac{d}{2}-1} = 4$) — ($\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$, $\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$, $\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$, $\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$), representing quarks and leptons — and one antiquadruplet — ($\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$, $\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$, $\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$, $\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$), representing antiquarks and antileptons, which both belong to the 64^{th}-plet, if $SO(6)$ is embedded into $SO(13,1)$. The creation operators (and correspondingly their annihilation operators) have for 32 members (representing quarks and leptons) the $SO(6)$ part of an odd Clifford character (and can be correspondingly second quantized (by itselves [1] or) together with the rest of space, manifesting $SO(7,1)$ (since it has an even Clifford character). The rest of 32 creation operators (representing antiquarks and antileptons) has in the $SO(6)$ part an even Clifford character and correspondingly in the rest of the Clifford space in $SO(7,1)$ an odd Clifford character.

I		decuplet	$\mathbf{S}^{03}$	$\mathbf{S}^{12}$	$\mathbf{S}^{56}$	$\mathcal{C}_{GN}$	$\mathcal{C}_{GN}\mathcal{P}^{(d-1)}_{GN}$	$\mathbb{C}_{GN}\mathcal{P}^{(d-1)}_{GN}$
	1	$(\theta^0-\theta^3)(\theta^1+i\theta^2)(\theta^5+i\theta^6)$	i	1	1			
	2	$(\theta^0\theta^3+i\theta^1\theta^2)(\theta^5+i\theta^6)$	0	0	1			
	3	$(\theta^0+\theta^3)(\theta^1-i\theta^2)(\theta^5+i\theta^6)$	$-i$	-1	1			
	4	$(\theta^0-\theta^3)(\theta^1-i\theta^2)(\theta^5-i\theta^6)$	i	-1	-1			
	5	$(\theta^0\theta^3-i\theta^1\theta^2)(\theta^5-i\theta^6)$	0	0	-1			
	6	$(\theta^0+\theta^3)(\theta^1+i\theta^2)(\theta^5-i\theta^6)$	$-i$	1	-1			
	7	$(\theta^0-\theta^3)(\theta^1\theta^2+\theta^5\theta^6)$	i	0	0			
	8	$(\theta^0+\theta^3)(\theta^1\theta^2-\theta^5\theta^6)$	$-i$	0	0			
	9	$(\theta^0\theta^3+i\theta^5\theta^6)(\theta^1+i\theta^2)$	0	1	0			
	10	$(\theta^0\theta^3-i\theta^5\theta^6)(\theta^1-i\theta^2)$	0	-1	0			
II		decuplet	$\mathbf{S}^{03}$	$\mathbf{S}^{12}$	$\mathbf{S}^{56}$			
	1	$(\theta^0+\theta^3)(\theta^1+i\theta^2)(\theta^5+i\theta^6)$	$-i$	1	1			
	2	$(\theta^0\theta^3-i\theta^1\theta^2)(\theta^5+i\theta^6)$	0	0	1			
	3	$(\theta^0-\theta^3)(\theta^1-i\theta^2)(\theta^5+i\theta^6)$	i	-1	1			
	4	$(\theta^0+\theta^3)(\theta^1-i\theta^2)(\theta^5-i\theta^6)$	$-i$	-1	-1			
	5	$(\theta^0\theta^3+i\theta^1\theta^2)(\theta^5-i\theta^6)$	0	0	-1			
	6	$(\theta^0-\theta^3)(\theta^1+i\theta^2)(\theta^5-i\theta^6)$	i	1	-1			
	7	$(\theta^0+\theta^3)(\theta^1\theta^2+\theta^5\theta^6)$	$-i$	0	0			
	8	$(\theta^0-\theta^3)(\theta^1\theta^2-\theta^5\theta^6)$	i	0	0			
	9	$(\theta^0\theta^3-i\theta^5\theta^6)(\theta^1+i\theta^2)$	0	1	0			
	10	$(\theta^0\theta^3+i\theta^5\theta^6)(\theta^1-i\theta^2)$	0	-1	0			

Table 17.2. The creation operators of the decuplet and the antidecuplet of the orthogonal group $SO(5,1)$ in Grassmann space are presented. Applying on the vacuum state $|\phi_0>=|1>$ the creation operators form eigenstates of the Cartan subalgebra, Eq. (17.33), $(\mathbf{S}^{03}, \mathbf{S}^{12}, \mathbf{S}^{56})$. The states within each decuplet are reachable from any member by $\mathbf{S}^{ab}$. The product of the discrete operators $\mathbb{C}_{NG}$ $(= \prod_{\Re\gamma^s}\gamma^s_G\, I_{x^6x^8\dots x^d}$, denoted as $\mathbb{C}$ in the last column) $\mathcal{P}^{(d-1)}_{NG}$ $(= \gamma^0_G \prod_{s=5}^{d}\gamma^s_G I_{\vec{x}_3})$ transforms, for example, ψ^I_1 into ψ^I_6, ψ^I_2 into ψ^I_5 and ψ^I_3 into ψ^I_4. Solutions of the Weyl equation, Eq. (17.19), with the negative energies belong to the "Grassmann sea", with the positive energy to the particles and antiparticles. Also the application of the discrete operators $\mathcal{C}_{GN}$, Eq. (17.30) and $\mathcal{C}_{NG}$ $\mathcal{P}^{(d-1)}_{NG}$, Eq. (17.30) is demonstrated.

Let us discuss the case with the quadruplet of $SO(6)$ with an odd Clifford character. From the point of view of the subgroups $SU(3)$ (the colour subgroup) and $U(1)$ (the $U(1)$ subgroup carrying the "fermion" quantum number), the quadruplet consists of one $SU(3)$ singlet with the "fermion" quantum number $-\frac{1}{2}$ and one triplet with the "fermion" quantum number $\frac{1}{6}$. The Clifford even $SO(7,1)$ part of $SO(13,1)$ define together with the Clifford odd $SO(6)$ part the quantum numbers of the right handed quarks and leptons and of the left handed quarks and leptons of the *standard model*, the left handed weak charged and the right handed weak chargeless.

In the same representation of $SO(13,1)$ there is also one antiquadruplet, which has the even Clifford character of $SO(6)$ part and the odd Clifford character in the $SO(7,1)$ part of the $SO(13,1)$. The antiquadruplet of the $SO(6)$ part consists of one $SU(3)$ antisinglet with the "fermion" quantum number $\frac{1}{2}$ and one antitriplet with the "fermion" quantum number $-\frac{1}{6}$. The $SO(7,1)\times SO(6)$ antiquadruplet of $SO(13,1)$ carries quantum numbers of left handed weak chargeless antiquarks and antileptons and of the right handed weak charged antiquarks and antileptons of the *standard model*.

Both, quarks and leptons and antiquarks and antileptons, belong to the same representation of $SO(13,1)$, explaining the miraculous cancellation of the triangle anomalies in the *standard model* without connecting by hand the handedness and the charges of quarks and leptons [8], as it must be done in the $SO(10)$ models.

b. In Grassmann space there are one ($\frac{1}{2}\frac{d!}{\frac{d}{2}!\frac{d}{2}!}=10$) decuplet representation of $SO(6)$ and one antidecuplet, both presented in Table 17.3. To be able to second quantize the theory, the whole representation must be Grassmann odd. Both decuplets in Table 17.3 have an odd Grassmann character, what means that products of eigenstates of the Cartan subalgebra in the rest of Grassmann space must be of an Grassmann even character to be second quantizable. Both decuplets would, however, appear in the same representation of $SO(13,1)$, and one can expect also decuplets of an even Grassmann character, if $SO(6)$ is embedded into $SO(13,1)$ [1].

With respect to $SU(3)\times U(1)$ subgroups of the group $SO(6)$ the decuplet manifests as one singlet, one triplet and one sextet, while the antidecuplet manifests as one antisinglet, one antitriplet and one antisextet. All the corresponding quantum numbers of either the Cartan subalgebra operators or of the corresponding diagonal operators of the $SU(3)$ or $U(1)$ subgroups are presented in Table 17.3.

While in Clifford case the representations of $SO(6)$, if the group $SO(6)$ is embedded into $SO(13,1)$, are defining an Clifford odd quadruplet and an Clifford even antiquadruplet, the representations in Grassmann case define one decuplet and one antidecuplet, both of the same Grassmann character, the odd one in our case. The two quadruplets in Clifford case manifest with respect to the subgroups $SU(3)$ and $U(1)$ as a triplet and a singlet, and as an antitriplet and an antisinglet, respectively. In Grassmann case the two decuplets manifest with respect to the subgroups $SU(3)$ and $U(1)$ as a (triplet, singlet, sextet) and as an (antitriplet, antisinglet, antisextet), respectively. The corresponding multiplets are presented in Table 17.4. The "fermion" quantum number τ^4 has for either singlets or triplets in Grassmann space, Table 17.4, twice the value of the corresponding singlets and triplets in Clifford space, Table 6.3 (see pages 112–113 in this volume): $(-1,+1)$ in

[1] This can easily be understood, if we look at the subgroups of the group $SO(6)$. **i.** Let us look at the subgroup $SO(2)$. There are two creation operators of an odd Grassmann character, in this case $(\theta^9 - i\theta^{10})$ and $(\theta^9 + i\theta^{10})$. Both appear in either decuplet or in antidecuplet — together with $\theta^9\theta^{10}$ with an even Grassmann character — multiplied by the part appearing from the rest of space $d = (11,12,13,14)$. But if $SO(2)$ is not embedded in $SO(6)$, then the two states, corresponding to the creation operators, $(\theta^9 \mp i\theta^{10})$, belong to different representations, and so is $\theta^9\theta^{10}$. **ii.** Similarly we see, if we consider the subgroup $SO(4)$ of the group $SO(6)$. All six states, $(\theta^9+i\theta^{10})\cdot(\theta^{11}+i\theta^{12})$, $(\theta^9-i\theta^{10})\cdot(\theta^{11}-i\theta^{12})$, $(\theta^9\theta^{10}+\theta^{11}\theta^{12})$, $(\theta^9+i\theta^{10})\cdot(\theta^{11}-i\theta^{12})$, $(\theta^9-i\theta^{10})\cdot(\theta^{11}+i\theta^{12})$, $(\theta^9\theta^{10}-\theta^{11}\theta^{12})$, appear in the decuplet and in the antidecuplet, multiplied with the part appearing from the rest of space, in this case in $d=(13,14)$, if $SO(4)$ is embedded in $SO(6)$. But, in $d=4$ space there are two decoupled groups of three states [2]: $[(\theta^9+i\theta^{10})\cdot(\theta^{11}+i\theta^{12}), (\theta^9\theta^{10}+\theta^{11}\theta^{12}), (\theta^9-i\theta^{10})\cdot(\theta^{11}-i\theta^{12})]$ and $[(\theta^9-i\theta^{10})\cdot(\theta^{11}+i\theta^{12}), (\theta^9\theta^{10}-\theta^{11}\theta^{12}), (\theta^9+i\theta^{10})\cdot(\theta^{11}-i\theta^{12})]$. Neither of these six members could be second quantized in $d=4$ alone.

I		decuplet	$\mathbf{S}^{9\,10}$	$\mathbf{S}^{11\,12}$	$\mathbf{S}^{13\,14}$	τ^4	τ^{33}	τ^{38}
	1	$(\theta^9+i\theta^{10})(\theta^{11}+i\theta^{12})(\theta^{13}+i\theta^{14})$	1	1	1	-1	0	0
	2	$(\theta^9+i\theta^{10})(\theta^{11}\theta^{12}+\theta^{13}\theta^{14})$	1	0	0	$-\frac{1}{3}$	$+\frac{1}{2}$	$+\frac{1}{2\sqrt{3}}$
	3	$(\theta^9+i\theta^{10})(\theta^{11}-i\theta^{12})(\theta^{13}-i\theta^{14})$	1	-1	-1	$+\frac{1}{3}$	$+1$	$+\frac{1}{\sqrt{3}}$
	4	$(\theta^9\theta^{10}+\theta^{11}\theta^{12})(\theta^{13}+i\theta^{14})$	0	0	1	$-\frac{1}{3}$	0	$-\frac{1}{\sqrt{3}}$
	5	$(\theta^9-i\theta^{10})(\theta^{11}-i\theta^{12})(\theta^{13}+i\theta^{14})$	-1	-1	-1	$+\frac{1}{3}$	0	$-\frac{2}{\sqrt{3}}$
	6	$(\theta^{11}+i\theta^{12})(\theta^9\theta^{10}+\theta^{13}\theta^{14})$	0	1	0	$-\frac{1}{3}$	$-\frac{1}{2}$	$+\frac{1}{2\sqrt{3}}$
	7	$(\theta^9-i\theta^{10})(\theta^{11}+i\theta^{12})(\theta^{13}-i\theta^{14})$	-1	1	-1	$+\frac{1}{3}$	-1	$+\frac{1}{\sqrt{3}}$
	8	$(\theta^9\theta^{10}-\theta^{11}\theta^{12})(\theta^{13}-i\theta^{14})$	0	0	-1	$+\frac{1}{3}$	0	$+\frac{1}{\sqrt{3}}$
	9	$(\theta^9\theta^{10}-\theta^{13}\theta^{14})(\theta^{11}-i\theta^{12})$	0	-1	0	$+\frac{1}{3}$	$+\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$
	10	$(\theta^9-i\theta^{10})(\theta^{11}\theta^{12}-\theta^{13}\theta^{14})$	-1	0	0	$+\frac{1}{3}$	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$
II		decuplet	$\mathbf{S}^{9\,10}$	$\mathbf{S}^{11\,12}$	$\mathbf{S}^{13\,14}$	τ^4	τ^{33}	τ^{38}
	1	$(\theta^9-i\theta^{10})(\theta^{11}-i\theta^{12})(\theta^{13}-i\theta^{14})$	-1	-1	-1	$+1$	0	0
	2	$(\theta^9-i\theta^{10})(\theta^{11}\theta^{12}+\theta^{13}\theta^{14})$	-1	0	0	$+\frac{1}{3}$	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$
	3	$(\theta^9-i\theta^{10})(\theta^{11}+i\theta^{12})(\theta^{13}+i\theta^{14})$	-1	1	1	$-\frac{1}{3}$	-1	$-\frac{1}{\sqrt{3}}$
	4	$(\theta^9\theta^{10}+\theta^{11}\theta^{12})(\theta^{13}-i\theta^{14})$	0	0	-1	$+\frac{1}{3}$	0	$+\frac{1}{\sqrt{3}}$
	5	$(\theta^9+i\theta^{10})(\theta^{11}+i\theta^{12})(\theta^{13}-i\theta^{14})$	1	1	-1	$-\frac{1}{3}$	0	$+\frac{2}{\sqrt{3}}$
	6	$(\theta^{11}-i\theta^{12})(\theta^9\theta^{10}+\theta^{13}\theta^{14})$	0	-1	0	$+\frac{1}{3}$	$+\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$
	7	$(\theta^9+i\theta^{10})(\theta^{11}-i\theta^{12})(\theta^{13}+i\theta^{14})$	1	-1	1	$-\frac{1}{3}$	$+1$	$-\frac{1}{\sqrt{3}}$
	8	$(\theta^9\theta^{10}-\theta^{11}\theta^{12})(\theta^{13}+i\theta^{14})$	0	0	1	$-\frac{1}{3}$	0	$-\frac{1}{\sqrt{3}}$
	9	$(\theta^9\theta^{10}-\theta^{13}\theta^{14})(\theta^{11}+i\theta^{12})$	0	1	0	$-\frac{1}{3}$	$-\frac{1}{2}$	$+\frac{1}{2\sqrt{3}}$
	10	$(\theta^9+i\theta^{10})(\theta^{11}\theta^{12}-\theta^{13}\theta^{14})$	1	0	0	$-\frac{1}{3}$	$+\frac{1}{2}$	$+\frac{1}{2\sqrt{3}}$

Table 17.3. The creation operators of the decuplet and the antidecuplet of the orthogonal group SO(6) in Grassmann space are presented. Applying on the vacuum state $|\phi_0>=|1>$ the creation operators form eigenstates of the Cartan subalgebra, Eq. (17.33), ($\mathbf{S}^{9\,10}$, $\mathbf{S}^{11\,12}$, $\mathbf{S}^{13\,14}$). The states within each decouplet are reachable from any member by $\mathbf{S}^{ab}$. The quantum numbers (τ^{33}, τ^{38}) and τ^4 of the subgroups SU(3) and U(1) of the group SO(6) are also presented, Eq. (17.38).

Grassmann case to be compared with $(-\frac{1}{2}, +\frac{1}{2})$ in Clifford case and $(+\frac{1}{3}, -\frac{1}{3})$ in Grassmann case to be compared with $(+\frac{1}{6}, -\frac{1}{6})$ in Clifford case.

When SO(6) is embedded into SO(13,1), the SO(6) representations of either even or odd Grassmann character contribute to both of the decupled, 1716 states of SO(13,1) representations contribute, provided that the SO(8) content has the opposite Grassmann character than the SO(6) content. The product of both representations must be Grassmann odd in order that the corresponding creation and annihilation operators fulfill the required anticommutation relations for fermions, Eq. (17.1).

Properties of the subgroups SO(3,1) and SO(4) of the group SO(8) in Grassmann and in Clifford space, when SO(8) is embedded into SO(13,1) a. Let us again repeat first properties of the SO(3,1) and SO(4) parts of the SO(13,1) representation of 64 "family members" in Clifford space, presented in Table 6.3 (see pages 112–113 in this volume). As seen in Table 6.3 (see pages 112–113 in

I			τ^4	τ^{33}	τ^{38}
singlet		$(\theta^9 + i\theta^{10})(\theta^{11} + i\theta^{12})(\theta^{13} + i\theta^{14})$	-1	0	0
triplet	1	$(\theta^9 + i\theta^{10})(\theta^{11}\theta^{12} + \theta^{13}\theta^{14})$	$-\frac{1}{3}$	$+\frac{1}{2}$	$+\frac{1}{2\sqrt{3}}$
	2	$(\theta^9\theta^{10} + \theta^{11}\theta^{12})(\theta^{13} + i\theta^{14})$	$-\frac{1}{3}$	0	$-\frac{1}{\sqrt{3}}$
	3	$(\theta^{11} + i\theta^{12})(\theta^9\theta^{10} + \theta^{13}\theta^{14})$	$-\frac{1}{3}$	$-\frac{1}{2}$	$+\frac{1}{2\sqrt{3}}$
sextet	1	$(\theta^9 + i\theta^{10})(\theta^{11} - i\theta^{12})(\theta^{13} - i\theta^{14})$	$\frac{1}{3}$	$+1$	$+\frac{1}{\sqrt{3}}$
	2	$(\theta^9 - i\theta^{10})(\theta^{11} - i\theta^{12})(\theta^{13} + i\theta^{14})$	$\frac{1}{3}$	0	$-\frac{2}{\sqrt{3}}$
	3	$(\theta^9 - i\theta^{10})(\theta^{11} + i\theta^{12})(\theta^{13} - i\theta^{14})$	$\frac{1}{3}$	-1	$+\frac{1}{\sqrt{3}}$
	4	$(\theta^9\theta^{10} - \theta^{11}\theta^{12})(\theta^{13} - i\theta^{14})$	$\frac{1}{3}$	0	$+\frac{1}{\sqrt{3}}$
	5	$(\theta^9\theta^{10} - \theta^{13}\theta^{14})(\theta^{11} - i\theta^{12})$	$\frac{1}{3}$	$+\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$
	6	$(\theta^9 - i\theta^{10})(\theta^{11}\theta^{12} - \theta^{13}\theta^{14})$	$\frac{1}{3}$	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$
II			τ^4	τ^{33}	τ^{38}
antisinglet		$(\theta^9 - i\theta^{10})(\theta^{11} - i\theta^{12})(\theta^{13} - i\theta^{14})$	$+1$	0	0
antitriplet	1	$(\theta^9 - i\theta^{10})(\theta^{11}\theta^{12} + \theta^{13}\theta^{14})$	$+\frac{1}{3}$	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$
	2	$(\theta^9\theta^{10} + \theta^{11}\theta^{12})(\theta^{13} - i\theta^{14})$	$+\frac{1}{3}$	0	$+\frac{1}{\sqrt{3}}$
	3	$(\theta^{11} - i\theta^{12})(\theta^9\theta^{10} + \theta^{13}\theta^{14})$	$+\frac{1}{3}$	$+\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$
antisextet	1	$(\theta^9 - i\theta^{10})(\theta^{11} + i\theta^{12})(\theta^{13} + i\theta^{14})$	$-\frac{1}{3}$	-1	$-\frac{1}{\sqrt{3}}$
	2	$(\theta^9 + i\theta^{10})(\theta^{11} + i\theta^{12})(\theta^{13} - i\theta^{14})$	$-\frac{1}{3}$	0	$+\frac{2}{\sqrt{3}}$
	3	$(\theta^9 + i\theta^{10})(\theta^{11} - i\theta^{12})(\theta^{13} + i\theta^{14})$	$-\frac{1}{3}$	$+1$	$-\frac{1}{\sqrt{3}}$
	4	$(\theta^9\theta^{10} - \theta^{11}\theta^{12})(\theta^{13} + i\theta^{14})$	$-\frac{1}{3}$	0	$-\frac{1}{\sqrt{3}}$
	5	$(\theta^9\theta^{10} - \theta^{13}\theta^{14})(\theta^{11} + i\theta^{12})$	$-\frac{1}{3}$	$-\frac{1}{2}$	$+\frac{1}{2\sqrt{3}}$
	6	$(\theta^9 + i\theta^{10})(\theta^{11}\theta^{12} - \theta^{13}\theta^{14})$	$-\frac{1}{3}$	$+\frac{1}{2}$	$+\frac{1}{2\sqrt{3}}$

Table 17.4. The creation operators in Grassmann space of the decuplet of Table 17.3 are arranged with respect to the $SU(3)$ and $U(1)$ subgroups of the group $SO(6)$ into a singlet, a triplet and a sextet. The corresponding antidecuplet manifests as an antisinglet, an antitriplet and an antisextet. $\tau^{33} = \frac{1}{2}(\mathbf{S}^{9\,10} - \mathbf{S}^{11\,12})$, $\tau^{38} = \frac{1}{2\sqrt{3}}(\mathbf{S}^{9\,10} + \mathbf{S}^{11\,12} - 2\mathbf{S}^{13\,14})$, $\tau^4 =$ - $\frac{1}{3}(\mathbf{S}^{9\,10} + \mathbf{S}^{11\,12} + \mathbf{S}^{13\,14})$; $\mathbf{S}^{ab} = i(\theta^a \frac{\partial}{\partial\theta_b} - \theta^b \frac{\partial}{\partial\theta_a})$.

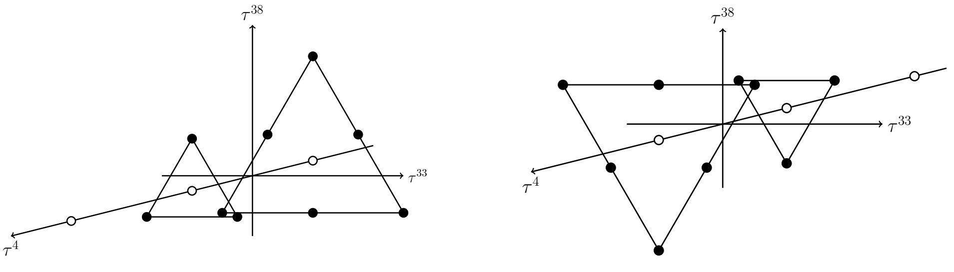

Fig. 17.1. Representations of the subgroups $SU(3)$ and $U(1)$ of the group $SO(6)$ in Grassmann space for two Grassmann odd representations of Table 17.4 are presented. On the abscissa axis and on the ordinate axis the values of the two diagonal operators, τ^{33} and τ^{38} of the coulour ($SU(3)$) subgroup are presented, respectively, with full circles. On the third axis the values of the subgroup of the "fermion number" $U(1)$ is presented with the open circles, the same for all the representations of each multiplet. There are one singlet, one triplet and one sextet on the left hand side and one antisinglet, one antitriplet and one antisextet on the right hand side.

this volume) there are four octets and four antioctets of $SO(8)$. All four octets, having an even Clifford character and forming 32 states when embedded into $SO(13,1)$, are the same for either quarks or for leptons, they distinguish only in the $SO(6)$ part (of an Clifford odd character) of the $SO(13,1)$ group, that is in the colour ($SU(3)$) part and the "fermion quantum number" ($U(1)$) part. Also the four antioctets, having an odd Clifford character, are all the same for the 32 family members of antiquarks and antileptons, they again distinguish only in the Clifford even $SO(6)$ part of $SO(13,1)$, that is in the anticolour ($SU(3)$) part and the "fermion quantum number" ($U(1)$) part.

The 64^{th}-plet of creation operators has an odd Clifford character either for quarks and leptons or for antiquarks and antileptons — correspondingly have an odd Clifford character also their annihilation operators — and can be second quantized [1].

Let us analyze first the octet ($2^{\frac{8}{2}-1}=8$), which is the same for all 32 members of quarks and leptons. The octet has an even Clifford character. All the right handed u_R-quarks and ν_R-leptons have the $SO(4)$ part of $SO(8)$ equal to $\stackrel{56}{[+]}\stackrel{78}{(+)}$, while their left handed partners have the $SO(4)$ part of $SO(8)$ equal to $\stackrel{56}{[+]}\stackrel{78}{[-]}$. All the right handed d_R-quarks and e_R-leptons have the $SO(4)$ part of $SO(8)$ equal to $\stackrel{56}{(-)}\stackrel{78}{[-]}$, while their left handed partners have the $SO(4)$ part of $SO(8)$ equal to $\stackrel{56}{(-)}\stackrel{78}{(+)}$]. The left handed quarks and leptons are doublets with respect to $\vec{\tau}^1$ and singlets with $\vec{\tau}^2$, while the right handed quarks and leptons are singlets with respect to $\vec{\tau}^1$ and doublets with $\vec{\tau}^2$. The left and right handed quarks and lepton belong with respect to the $SO(3,1)$ group to either left handed or the right handed spinor representations, respectively.

b. In Grassmann space the $SO(8)$ group of an odd Grassmann character has $\frac{1}{2}\,\frac{8!}{4!4!}=35$ creation operators in each of the two groups and the same number of annihilation operators, obtained from the creation operators by Hermitian conjugation, Eq. (17.4). The corresponding states, created by the creation operators on the vacuum state $|\phi_o>$, can be therefore second quantized. But if embedded the group $SO(8)$ into the group $SO(13,1)$ the subgroup $SO(6)$ must have an even Grassmann character in oder that the states in $SO(13,1)$ can be second quantized according to Eq. (17.1).

According to what we learned in the case of the group $SO(6)$, each of the two independent representations of the group $SO(13,1)$ of an odd Grassmann character must include either the even $SO(7,1)$ part and the odd $SO(6)$ part or the odd $SO(7,1)$ part and the even $SO(6)$ part. To the even $SO(7,1)$ representation either the odd $SO(3,1)$ and the odd $SO(4)$ parts contribute or both must be of the Grassmann even character. In the case that the $SO(7,1)$ part has an odd Grassmann character (in this case the $SO(6)$ has an even Grassmann character) then one of the two parts $SO(3,1)$ and $SO(4)$ must be odd and the other even.

17.3 Concluding remarks

We learned in this contribution that although either Grassmann or Clifford space offer the second quantizable description of the internal degrees of freedom of fermions (Eq. (17.1)), the Clifford space offers more: It offers not only the description of all the "family members", explaining all the degrees of freedom of the observed quarks and leptons and antiquark and antileptons, but also the explanation for the appearance of families.

The interaction of fermions with the gravity fields — the vielbeins and the spin connections — in the $2(2n+1)$-dimensional space can be achieved, as suggested by the *spin-charge-family* theory ([5,4] and references therein), by replacing the momentum p_a in the Lagrange density function for a free particle by the covariant momentum, equally appropriate for both representations. In Grassmann space we have: $p_{0a} = f^{\alpha}{}_{a}\, p_{0\alpha}$, with $p_{0\alpha} = p_\alpha - \frac{1}{2}\, \mathbf{S}^{ab}\Omega_{ab\alpha}$, where $f^{\alpha}{}_{a}$ is the vielbein in $d = 2(2n+1)$-dimensional space and $\Omega_{ab\alpha}$ is the spin connection field of the Lorentz generators $\mathbf{S}^{ab}$. In Clifford space we have equivalently: $p_{0a} = f^{\alpha}{}_{a}\, p_{0\alpha}$, $p_{0\alpha} = p_\alpha - \frac{1}{2}S^{ab}\omega_{ab\alpha} - \frac{1}{2}\tilde{S}^{ab}\tilde{\omega}_{ab\alpha}$. Since $\mathbf{S}^{ab} = S^{ab} + \tilde{S}^{ab}$ we find that when no fermions are present either $\Omega_{ab\alpha}$ or $\omega_{ab\alpha}$ or $\tilde{\omega}_{ab\alpha}$ are uniquely expressible by vielbeins $f^{\alpha}{}_{a}$ ([5,4] and references therein). It might be that "our universe made a choice between the Clifford and the Grassmann algebra" when breaking the starting symmetry by making condensates of fermions, since that for breaking symmetries Clifford space offers better opportunity".

17.4 Appendix: Useful relations in Grassmann and Clifford space

The generator of the Lorentz transformation in Grassmann space is defined as follows [2]

$$\mathbf{S}^{ab} = (\theta^a p^{\theta b} - \theta^b p^{\theta a}) = S^{ab} + \tilde{S}^{ab}\,, \quad \{S^{ab}, \tilde{S}^{cd}\}_- = 0\,, \tag{17.32}$$

where S^{ab} and $\tilde{S}^{ab}$ are the corresponding two generators of the Lorentz transformations in the Clifford space, forming orthogonal representations with respect to each other.

We make a choice of the Cartan subalgebra of the Lorentz algebra as follows

$$\begin{aligned}
&\mathbf{S}^{03}, \mathbf{S}^{12}, \mathbf{S}^{56}, \cdots, \mathbf{S}^{d-1\; d}\,,\\
&S^{03}, S^{12}, S^{56}, \cdots, S^{d-1\; d}\,,\\
&\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56}, \cdots, \tilde{S}^{d-1\; d}\,,\\
&\text{if} \quad d = 2n\,.
\end{aligned} \tag{17.33}$$

We find the infinitesimal generators of the Lorentz transformations in Clifford space

$$\begin{aligned}
S^{ab} &= \frac{i}{4}(\gamma^a\gamma^b - \gamma^b\gamma^a)\,, \quad S^{ab\dagger} = \eta^{aa}\eta^{bb}S^{ab}\,,\\
\tilde{S}^{ab} &= \frac{i}{4}(\tilde{\gamma}^a\tilde{\gamma}^b - \tilde{\gamma}^b\tilde{\gamma}^a)\,, \quad \tilde{S}^{ab\dagger} = \eta^{aa}\eta^{bb}\tilde{S}^{ab}\,,
\end{aligned} \tag{17.34}$$

where γ^a and $\tilde{\gamma}^a$ are defined in Eq. (17.10). The commutation relations for either $\mathbf{S}^{ab}$ or S^{ab} or $\tilde{S}^{ab}$, $\mathbf{S}^{ab} = S^{ab} + \tilde{S}^{ab}$, are

$$
\begin{aligned}
\{S^{ab}, \tilde{S}^{cd}\}_- &= 0\,,\\
\{S^{ab}, S^{cd}\}_- &= i(\eta^{ad} S^{bc} + \eta^{bc} S^{ad} - \eta^{ac} S^{bd} - \eta^{bd} S^{ac})\,,\\
\{\tilde{S}^{ab}, \tilde{S}^{cd}\}_- &= i(\eta^{ad} \tilde{S}^{bc} + \eta^{bc} \tilde{S}^{ad} - \eta^{ac} \tilde{S}^{bd} - \eta^{bd} \tilde{S}^{ac})\,.
\end{aligned}
\tag{17.35}
$$

The infinitesimal generators of the two invariant subgroups of the group $SO(3,1)$ can be expressed as follows

$$
\vec{N}_{\pm}(= \vec{N}_{(L,R)}) := \frac{1}{2}(S^{23} \pm i S^{01}, S^{31} \pm i S^{02}, S^{12} \pm i S^{03})\,. \tag{17.36}
$$

The infinitesimal generators of the two invariant subgroups of the group $SO(4)$ are expressible with S^{ab}, $(a,b) = (5,6,7,8)$ as follows

$$
\begin{aligned}
\vec{\tau}^{1} &:= \frac{1}{2}(S^{58} - S^{67},\, S^{57} + S^{68},\, S^{56} - S^{78})\,,\\
\vec{\tau}^{2} &:= \frac{1}{2}(S^{58} + S^{67},\, S^{57} - S^{68},\, S^{56} + S^{78})\,,
\end{aligned}
\tag{17.37}
$$

while the generators of the $SU(3)$ and $U(1)$ subgroups of the group $SO(6)$ can be expressed by S^{ab}, $(a,b) = (9,10,11,12,13,14)$

$$
\begin{aligned}
\vec{\tau}^{3} :=\; & \frac{1}{2}\{S^{9\,12} - S^{10\,11}\,, S^{9\,11} + S^{10\,12},\, S^{9\,10} - S^{11\,12},\\
& S^{9\,14} - S^{10\,13},\, S^{9\,13} + S^{10\,14}\,, S^{11\,14} - S^{12\,13}\,,\\
& S^{11\,13} + S^{12\,14},\, \frac{1}{\sqrt{3}}(S^{9\,10} + S^{11\,12} - 2S^{13\,14})\}\,,\\
\tau^{4} :=\; & -\frac{1}{3}(S^{9\,10} + S^{11\,12} + S^{13\,14})\,.
\end{aligned}
\tag{17.38}
$$

The hyper charge Y can be defined as $Y = \tau^{23} + \tau^{4}$.

The equivalent expressions for the "family" charges, expressed by $\tilde{S}^{ab}$ follow if in Eqs. (17.36 - 17.38) S^{ab} are replaced by $\tilde{S}^{ab}$.

The breaks of the symmetries, manifesting in Eqs. (17.36, 17.37, 17.38), are in the *spin-charge-family* theory caused by the condensate and the nonzero vacuum expectation values (constant values) of the scalar fields carrying the space index $(7,8)$ (Refs. [5,3] and the references therein). The space breaks first to $SO(7,1)$ $\times SU(3) \times U(1)_{II}$ and then further to $SO(3,1) \times SU(2)_I \times U(1)_I \times SU(3) \times U(1)_{II}$, what explains the connections between the weak and the hyper charges and the handedness of spinors.

Let ius present some useful relations [3]

$$
\begin{array}{llll}
\stackrel{ab}{(k)}\stackrel{ab}{(k)} = 0\,, & \stackrel{ab}{(k)}\stackrel{ab}{(-k)} = \eta^{aa}\stackrel{ab}{[k]}\,, & \stackrel{ab}{(-k)}\stackrel{ab}{(k)} = \eta^{aa}\stackrel{ab}{[-k]}\,, & \stackrel{ab}{(-k)}\stackrel{ab}{(-k)} = 0\,,\\
\stackrel{ab}{[k]}\stackrel{ab}{[k]} = \stackrel{ab}{[k]}\,, & \stackrel{ab}{[k]}\stackrel{ab}{[-k]} = 0\,, & \stackrel{ab}{[-k]}\stackrel{ab}{[k]} = 0\,, & \stackrel{ab}{[-k]}\stackrel{ab}{[-k]} = \stackrel{ab}{[-k]}\,,\\
\stackrel{ab}{(k)}\stackrel{ab}{[k]} = 0\,, & \stackrel{ab}{[k]}\stackrel{ab}{(k)} = \stackrel{ab}{(k)}\,, & \stackrel{ab}{(-k)}\stackrel{ab}{[k]} = \stackrel{ab}{(-k)}\,, & \stackrel{ab}{(-k)}\stackrel{ab}{[-k]} = 0\,,\\
\stackrel{ab}{(k)}\stackrel{ab}{[-k]} = \stackrel{ab}{(k)}\,, & \stackrel{ab}{[k]}\stackrel{ab}{(-k)} = 0\,, & \stackrel{ab}{[-k]}\stackrel{ab}{(k)} = 0\,, & \stackrel{ab}{[-k]}\stackrel{ab}{(-k)} = \stackrel{ab}{(-k)}\,.
\end{array}
\tag{17.39}
$$

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