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On the rank two geometries of the groups $\mathrm{PSL}(2, q)$: part II*

Francis Buekenhout , Julie De Saedeleer , Dimitri Leemans †
Université Libre de Bruxelles, Département de Mathématiques - C.P.216
Boulevard du Triomphe, B-1050 Bruxelles

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Abstract

We determine all firm and residually connected rank 2 geometries on which $\mathrm{PSL}(2, q)$ acts flag-transitively, residually weakly primitively and locally two-transitively, in which one of the maximal parabolic subgroups is isomorphic to A_4, S_4, A_5, $\mathrm{PSL}(2, q')$ or $\mathrm{PGL}(2, q')$, where q' divides q, for some prime-power q.

Keywords: Projective special linear groups, coset geometries, locally s-arc-transitive graphs.

Math. Subj. Class.: 51E24, 05C25

1 Introduction

In [5], we started the classification of the residually weakly primitive and locally two-transitive coset geometries of rank two for the groups $\mathrm{PSL}(2, q)$. The aim of this paper is to finish this classification. It remains to focus on the cases in which one of the maximal parabolic subgroups is isomorphic to A_4, S_4, A_5, $\mathrm{PSL}(2, q')$ or $\mathrm{PGL}(2, q')$ where q' divides q. For motivation, basic definitions, notations and context of the work we refer to [5].

In Section 3, we sketch the proof of our main result:

Theorem 1.1. *Let $G \cong \mathrm{PSL}(2, q)$ and $\Gamma(G; \{G_0, G_1, G_0 \cap G_1\})$ be a locally two-transitive RWPRI coset geometry of rank two. If G_0 is isomorphic to one of A_4, S_4, A_5, $\mathrm{PSL}(2, q')$ or $\mathrm{PGL}(2, q')$, where q' divides q, then Γ is isomorphic to one of the geometries appearing in Table 1, Table 2, Table 3, Table 4, Table 5, and Table 6.*

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E-mail addresses: fbueken@ulb.ac.be (Francis Buekenhout), judesaed@ulb.ac.be (Julie De Saedeleer), dleemans@ulb.ac.be (Dimitri Leemans)

		$G_0 \cong A_5$			$q = 4^r$ with r prime	
	$G_0 \cap G_1$	G_1	$\sharp$ Geom. up to conj.	$\sharp$ Geom. up to isom.	Extra conditions on q	loc. (G, s)-arc-trans. g.
Γ_1	D_{10}	D_{30}	1	1	$\frac{q\pm1}{15}$ odd	$s = 3$
Γ_2	A_4	$E_{16}:3$	1	1	$q = 16$	$s = 3$
Γ_3	A_4	$E_{16}:3$	5	2	$q = 64$	$s = 3$
Γ_4	A_4	$E_{16}:3$	$\frac{4^{r-1}-1}{3}$	$\frac{2(4^{r-2}-1)+3.2^{r-2}}{3r}$	$r > 3$, r odd prime	$s = 3$
		$G_0 \cong A_5$			$q = p = \pm1(5)$ with p odd prime	
	$G_0 \cap G_1$	G_1	$\sharp$ Geom. up to conj.	$\sharp$ Geom. up to isom.	Extra conditions on q	loc. (G, s)-arc-trans. g.
Γ_5	D_{10}	D_{20}	2	1	$q = \pm1(20)$	$s = 3$
Γ_6	D_{10}	D_{30}	2	1	$q = \pm1(30)$	$s = 3$
Γ_7	D_{10}	A_5	2	1	$\frac{q\pm1}{10}$ even	$s = 2$
Γ_8	D_{10}	A_5	1	1	$\frac{q\pm1}{10}$ odd	$s = 2$
Γ_9	A_4	S_4	2	1	$q = \pm1(40)$ or $q = \pm9(40)$	$s = 3$
Γ_{10}	A_4	A_5	2	1	$q = \pm1(40)$ or $q = \pm9(40)$	$s = 2$
Γ_{11}	A_4	A_5	1	1	$q = \pm11(40)$ or $q = \pm19(40)$	$s = 2$
		$G_0 \cong A_5$			$q = p^2 = -1(5)$ with p odd prime	
	$G_0 \cap G_1$	G_1	$\sharp$ Geom. up to conj.	$\sharp$ Geom. up to isom.	Extra conditions on q	loc. (G, s)-arc-trans. g.
Γ_{12}	D_{10}	D_{20}	2	1	$q = -1(20)$	$s = 3$
Γ_{13}	D_{10}	D_{30}	2	1	$q = -1(30)$	$s = 3$
Γ_{14}	D_{10}	A_5	2	1	$\frac{q+1}{10}$ even	$s = 2$
Γ_{15}	D_{10}	A_5	1	1	$\frac{q+1}{10}$ odd	$s = 2$
Γ_{16}	A_4	S_4	2	1	$q = -1(40)$ or $q = 9(40)$	$s = 3$
Γ_{17}	A_4	A_5	2	1	$q = -1(40)$ or $q = 9(40)$	$s = 2$
Γ_{18}	A_4	A_5	1	1	$q = -11(40)$ or $q = 19(40)$	$s = 2$

Table 1: The RWPRI and $(2T)_1$ geometries with $G_0 \cong A_5$.

		$G_0 \cong A_4$			$q = p > 3$ and $q = 3, 13, 27, 37(40)$ or $q = 5$	
	$G_0 \cap G_1$	G_1	$\sharp$ Geom. up to conj.	$\sharp$ Geom. up to isom.	Extra conditions on q	locally(G, s)-arc-transitive graphs
Γ_1	3	Z_6	1	1	$q = 13, 37, 83, 107(120)$	$s = 3$
Γ_2	3	D_6	$\frac{q+1}{6}$	$\frac{\frac{q+1}{6}+1}{2}$	$\frac{q+1}{6}$ odd	$s = 3$
Γ_3	3	D_6	$\frac{q-1}{6}$	$\frac{\frac{q-1}{6}+1}{2}$	$\frac{q-1}{6}$ odd	$s = 3$
Γ_4	3	D_6	$\frac{q+1}{6}$	$\frac{q+1}{12}$	$\frac{q+1}{6}$ even	$s = 3$
Γ_5	3	D_6	$\frac{q-1}{6}$	$\frac{q-1}{12}$	$\frac{q-1}{6}$ even	$s = 3$
Γ_6	3	A_4	$\frac{q+1}{3} - 1$	$\frac{q+1}{6}$	$3 \mid q+1$	$s = 2$
Γ_7	3	A_4	$\frac{q-1}{3} - 1$	$\frac{q-1}{6}$	$3 \mid q-1$	$s = 2$

Table 2: The RWPRI and $(2T)_1$ geometries with $G_0 \cong A_4$

		$G_0 \cong S_4$			$q = p > 2$ and $q = \pm 1(8)$	
	$G_0 \cap G_1$	G_1	$\sharp$ Geom. up to conj.	$\sharp$ Geom. up to isom.	Extra conditions on q	locally(G,s)-arc-transitive graphs
Γ_1	D_6	D_{12}	2	1	$q = \pm 1(24)$	$s = 3$
Γ_2	D_6	D_{18}	2	1	$q = \pm 1(72)$ or $q = \pm 17(72)$	$s = 3$
Γ_3	D_6	S_4	2	1	$\frac{q \pm 1}{6}$ even	$s = 2$
Γ_4	D_6	S_4	1	1	$\frac{q \pm 1}{6}$ odd	$s = 2$
Γ_5	D_8	D_{16}	2	1	$q = \pm 1(16)$	$s = 7$
Γ_6	D_8	D_{24}	2	1	$q = \pm 1(24)$	$s = 3$
Γ_7	D_8	S_4	2	1	$\frac{q \pm 1}{8}$ even	$s = 4$
Γ_8	D_8	S_4	1	1	$\frac{q \pm 1}{8}$ odd	$s = 4$
Γ_9	A_4	A_5	2	1	$q = \pm 1(40)$ or $q = \pm 9(40)$	$s = 3$

Table 3: The RWPRI and $(2T)_1$ geometries with $G_0 \cong S_4$.

		$G_0 \cong \mathrm{PSL}(2,2^n)$			$q = 2^{nm}$, with m prime	
	$G_0 \cap G_1$	G_1	$\sharp$ Geom. up to conj.	$\sharp$ Geom. up to isom.	Extra conditions on q	loc.(G,s)-arc-trans. graphs
Γ_1	$E_{2^n} : (2^n - 1)$	$E_{2^{mn}} : (2^n - 1)$	1	1	$m = 2, n \neq 1$	$s = 3$
Γ_2	2	D_6	1	1	$q = 4; n = 1, m = 2$	$s = 2$
Γ_3	2	2^2	1	1	$q = 4; n = 1, m = 2$	$s = 3$
Γ_4	3	A_4	1	1	$q = 4; n = 1, m = 2$	$s = 3$
Γ_5	D_{10}	D_{30}	1	1	$q = 4^m; n = 2; \frac{q \pm 1}{15} odd$	$s = 3$

Table 4: The RWPRI and $(2T)_1$ geometries with $G_0 \cong \mathrm{PSL}(2,2^n)$.

		$G_0 \cong \mathrm{PSL}(2,p^n)$			$q = p^{nm}$, p and m odd primes	
	$G_0 \cap G_1$	G_1	$\sharp$ Geom. up to conj.	$\sharp$ Geom. up to isom.	Extra conditions on q	locally(G,s)-arc-transitive graphs
Γ_1	3	A_4	$3^{m-1} - 1$	$\frac{3^{m-1}-1}{2m}$	$q = 3^m; n = 1, m \neq 3$	$s = 2$
Γ_2	3	A_4	8	2	$q = 27; n = 1, m = 3$	$s = 2$

Table 5: The RWPRI and $(2T)_1$ geometries with $G_0 \cong \mathrm{PSL}(2,q')$, q' odd.

		$G_0 \cong \mathrm{PGL}(2,p^n)$			$q = p^{2n}$, with p odd prime	
	$G_0 \cap G_1$	G_1	$\sharp$ Geom. up to conj.	$\sharp$ Geom. up to isom.	Extra conditions on q	loc. (G,s)-arc-transitive graphs
Γ_1	$E_{p^n} : (p^n - 1)$	$E_{p^{2n}} : (p^n - 1)$	2	1	none	$s = 3$
Γ_2	$\mathrm{PSL}(2,p^n)$	A_5	2	1	$q = 9$	$s = 3$
Γ_3	D_8	$\mathrm{PGL}(2,3)$	1	1	$q = 9$	$s = 4$

Table 6: The RWPRI and $(2T)_1$ geometries with $G_0 \cong \mathrm{PGL}(2,q')$.

Observe that, geometry Γ_5 in Table 4 is exactly geometry Γ_1 in Table 1.

In Section 4, we recall the subgroup lattice of $\mathrm{PSL}(2,q)$, and we give the two-transitive representations of the maximal subgroups. In Section 5, we prove Theorem 1.1, which is based on the proof of Propositions 5.5, 5.6, 5.10, 5.12, 5.16 and 5.21. For that purpose, we determine the rank two RWPRI and $(2T)_1$ geometries of $\mathrm{PSL}(2,q)$ and their number, up to isomorphism and up to conjugacy. The existence of such geometries is equivalent to the existence of a locally 2-arc transitive bipartite graph for which the action of G is primitive on one of the bipartite halves (see [8]). Our result is also a part of the program initiated in [8].

These graphs are interesting in their own right because of the numerous connections they have with other fields of mathematics (see [8] for more details). We also refer to the classification of these graphs for almost simple groups with socle a Ree simple group $\mathrm{Ree}(q)$ (see [7]). In terms of locally 2-arc-transitive graphs, we obtain here the classification of these graphs with one vertex-stabilizer maximal in $\mathrm{PSL}(2,q)$ and isomorphic to A_4, S_4, A_5, $\mathrm{PSL}(2,q')$ or $\mathrm{PGL}(2,q')$. The last column of Table 1, Table 2, Table 3, Table 4, Table 5 and Table 6 gives, for each geometry Γ, the value of s such that Γ is a locally s-arc-transitive but not a locally $(s+1)$-arc-transitive graph. In section 6, we determine the exact value of s in all cases that are not current by the method of Leemans.

In Tables 1, 2, 3, 4, 5, 6 and 9 most values are $s = 2$ or $s = 3$, but there are some spectacular examples with larger values of s. Indeed we obtain a locally 4-arc transitive graph and a locally 7-arc transitive graph. As one of the referees pointed out, the $(G,2)$-arc transitive graphs with $L_2(q) \leq G \leq P\Gamma L_2(q)$ were classified by Hassani, Nochefranca and Praeger in [9]. Therefore, they already classified the geometries of Theorem 1.1 in which $G_0 \cap G_1$ is of index two in one of G_0 or G_1. Our proof of Theorem 1.1 uses a completely different approach. In cases where our work overlaps with [9], the results are the same.

Also, in Table 3, geometry Γ_5 is due to Wong in [22] and geometries Γ_7 and Γ_8 are the Biggs-Hoare graphs in [1] (see also [14], Table 1).

1.1 Aknowledgement

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2 Definitions and notation

For basic notions on coset geometries and locally s-arc-transitive graphs needed to understand this paper we shall freely use the definitions from Section 2 in [5].

Let us nevertheless recall concepts related to isomorphism. Let G be a group and $Aut(G)$ be its automorphism group. The coset geometries $\Gamma(G;\{G_0,G_1\})$ and $\Gamma(G;\{G_0',G_1'\})$ are *conjugate* (resp. *isomorphic*) provided there exists an element $g \in G$ (resp. $g \in Aut(G)$) such that $\{G_0^g, G_1^g\} = \{G_0', G_1'\}$ (resp. $\{g(G_0), g(G_1)\} = \{G_0', G_1'\}$). We classify geometries up to conjugacy and up to isomorphism. That is, for each triple $\{G_0, G_1, G_0 \cap G_1\}$, we give the number of corresponding classes of geometries with respect to conjugacy and isomorphism.

3 Sketch of the proof of Theorem 1.1

Let $G \cong \mathrm{PSL}(2,q)$. Let G_0 and G_1 be subgroups of G and let $G_{01} = G_0 \cap G_1$. The RWPRI condition in rank two requires that either G_0 or G_1 is a maximal subgroup of G and that G_{01} is a maximal subgroup of G_0 and G_1. The $(2T)_1$ condition requires that both G_0 and G_1 act two-transitively on the respective cosets of G_{01}.

We break down the task by classifying those geometries with a fixed subgroup G_0. Since we may assume without loss of generality that G_0 is maximal in G, we follow Tables 7 and 8 that give all maximal subgroups of $\mathrm{PSL}(2,q)$. The number of RWPRI and $(2T)_1$ geometries of rank 2 depends on the value of $q = p^n$. More precisely, it usually depends on whether $p = 2$ or $p \neq 2$. Knowing that $q = p^n$ with p a prime, the two cases are $q = 2^n$ or q odd.

The way we work to determine the RWPRI and $(2T)_1$ geometries of rank two always follows the same path. To achieve our goal we first choose a subgroup G_0, which is a maximal subgroup of $G \cong \mathrm{PSL}(2,q)$. Then, using the results obtained in Proposition 4.6, we determine the possibilities for $G_{01} := G_0 \cap G_1$. They are the two-transitive pairs (G_0, G_{01}). At last, in Section 5 we determine the possible subgroups G_1 of $\mathrm{PSL}(2,q)$ such that (G_1, G_{01}) is a two-transitive pair. Finally, we determine for each triple (G_0, G_1, G_{01}) the number of geometries it gives rise to, up to conjugacy and up to isomorphism.

4 Structure of subgroups of $\mathrm{PSL}(2, q)$

To follow the approach described above, we first recall the list of subgroups of the projective special linear groups $\mathrm{PSL}(2,q)$. We then give the list of maximal subgroups of $\mathrm{PSL}(2,q)$. Finally we determine the two-transitive representations of the maximal subgroups of $\mathrm{PSL}(2,q)$ in order to be able to check the $(2T)_1$ property easily.

4.1 The subgroups of $\mathrm{PSL}(2, q)$

We recall the complete subgroup structure of $\mathrm{PSL}(2,q)$ for which we refer to Dickson [6], Moore [15], Huppert [10] and Suzuki [16]. In the statement of Lemma 4.1, we make use of the phrasing due to O. H. King [11].

Lemma 4.1. *[Dickson-Moore] The group* $\mathrm{PSL}(2,q)$ *of order* $\frac{q(q^2-1)}{(2,q-1)}$*, where* $q = p^n$ *(*p *prime), contains exactly the following subgroups:*

1. *The identity subgroup.*
2. *A single class of* $q+1$ *conjugate elementary abelian subgroups of order* q*, denoted by* E_q*.*
3. *A single class of* $\frac{q(q+1)}{2}$ *conjugate cyclic subgroups of order* d*, denoted by either* Z_d *or* d*; for every divisor* d *of* $q-1$ *for* q *even and* $\frac{q-1}{2}$ *for* q *odd, with* $d > 1$*.*
4. *A single class of* $\frac{q(q-1)}{2}$ *conjugate cyclic subgroups of order* d*,denoted by either* Z_d *or* d*; for every divisor* d *of* $q+1$ *for* q *even and* $\frac{q+1}{2}$ *for* q *odd, with* $d > 1$*.*
5. - *For* q *odd, a single class of* $\frac{q(q^2-1)}{4d}$ *dihedral groups of order* $2d$*, denoted by* D_{2d}*, for every divisor* d *of* $\frac{q-1}{2}$ *with* $\frac{q-1}{2d}$ *odd, with* $d > 1$*;*

- *For q odd, two classes each of* $\frac{q(q^2-1)}{8d}$ *dihedral groups of order* $2d$*, denoted by* D_{2d}*, for every divisor* $d > 2$ *of* $\frac{q-1}{2}$ *with* $\frac{q-1}{2d}$ *even;*
- *For q even, a single class of* $\frac{q(q^2-1)}{2d}$ *dihedral groups of order* $2d$*, denoted by* D_{2d}*, for every divisor* d *of* $q-1$*, with* $d > 1$*;*
- *For q odd, a single class of* $\frac{q(q^2-1)}{4d}$ *dihedral groups of order* $2d$*, denoted by* D_{2d}*, for every divisor* d *of* $\frac{q+1}{2}$ *with* $\frac{q+1}{2d}$ *odd, with* $d > 1$*;*
- *For q odd, two classes each of* $\frac{q(q^2-1)}{8d}$ *dihedral groups of order* $2d$*, denoted by* D_{2d}*, for every divisor* $d > 2$ *of* $\frac{q+1}{2}$ *with* $\frac{q+1}{2d}$ *even;*
- *For q even, a single class of* $\frac{q(q^2-1)}{2d}$ *dihedral groups of order* $2d$*, denoted by* D_{2d}*, for every divisor* d *of* $q+1$*, with* $d > 1$*.*

6.
   - *A single class of* $\frac{q(q^2-1)}{24}$ *conjugate dihedral groups of order* 4 *denoted by* 2^2 *when* $q = \pm 3(8)$*;*
   - *Two classes each of* $\frac{q(q^2-1)}{48}$ *conjugate dihedral groups of order* 4 *denoted by* 2^2 *when* $q = \pm 1(8)$*;*
   - *When q is even, the groups* 2^2 *are in the case 7.*

7. *A number of classes of* $\frac{q^2-1}{(2,1,1)(p^k-1)}$ *conjugate elementary abelian subgroups of order* p^m*, denoted by* E_{p^m} *for every natural number* m*, such that* $1 \leq m \leq n-1$*, where* k *is a common divisor of* n *and* m *and* $(2,1,1)$ *is equal to 2 (resp. 1, 1) if* $p > 2$ *and* $\frac{n}{k}$ *is even (resp.* $p > 2$ *and* $\frac{n}{k}$ *is odd,* $p = 2$*).*

8. *A number of classes of* $\frac{(q^2-1)p^{n-m}}{(2,1,1)(p^k-1)}$ *conjugate subgroups* $E_{p^m}:d$ *which are semidirect products of an elementary abelian group* E_{p^m} *and a cyclic group of order* d*,* $d > 1$*, for every natural number* m *such that* $1 \leq m \leq n$ *and every natural number* d *dividing* $\frac{p^k-1}{(1,2,1)}$*, where* k *is a common divisor of* n *and* m *and* $(1,2,1)$ *is one of*

   - *1 for* $p > 2$ *and* $\frac{n}{k}$ *is even*
   - *2 for* $p > 2$ *and* $\frac{n}{k}$ *is odd*
   - *1 for* $p = 2$

   These subgroups are Frobenius groups.

9.
   - *Two classes each of* $\frac{q(q^2-1)}{48}$ *conjugates of* A_4 *when* $q = \pm 1(8)$*;*
   - *A single class of* $\frac{q(q^2-1)}{24}$ *conjugates of* A_4 *when* $q = \pm 3(8)$*;*
   - *A single class of* $\frac{q(q^2-1)}{12}$ *conjugates of* A_4 *when q is an even power of* 2*.*

10. *Two classes each of* $\frac{q(q^2-1)}{48}$ *conjugates of* S_4 *when* $q = \pm 1(8)$*.*

11. *Two classes each of* $\frac{q(q^2-1)}{120}$ *conjugate alternating groups* A_5 *when* $q = \pm 1(10)$*.*

12.
    - *Two classes each of* $\frac{q(q^2-1)}{2q'(q'^2-1)}$ *groups* $PSL(2,q')$*, where q is an even power of* q'*, for q odd;*
    - *A single class of* $\frac{q(q^2-1)}{q'(q'^2-1)}$ *groups* $PSL(2,q')$*, where q is an odd power of* q'*, for q odd;*

- *A single class of* $\frac{q(q^2-1)}{q'(q'^2-1)}$ *groups* $PSL(2,q')$*, where* q *is a power of* q'*, for* q *even.*

13. *Two classes each of* $\frac{q(q^2-1)}{2q'(q'^2-1)}$ *groups* $PGL(2,q')$*, where* q *is an even power of* q'*, for* q *odd.*
14. $\mathrm{PSL}(2,q)$ *itself.*

Remark 4.2. Subgroups A_5 are given either by case 11 (when $q = \pm 1(5)$) or by case 12 (when $q = 0(5)$ and $q = 4^m$) of Lemma 4.1. Also, if q is even, the $PGL(2,q')$ are given by case 12, since $PGL(2,q') \cong PSL(2,q')$ provided q is even.

Remark 4.3. Let us mention that in the cases 7 and 8 of Lemma 4.1, the number of conjugacy classes is not given. The number of conjugacy classes of the elementary abelian subgroups E_{p^m} given by Dickson (see [6], §260) is incorrect. For an example we refer to [5] Remark 7.

Notice that Dickson does not give the number of conjugacy classes of the subgroups $E_{p^m} : d$, except in the particular case where $m = n$ and $d = \frac{p^n-1}{(2,q-1)}$. There are $q+1$ subgroups $E_q : \frac{q-1}{(2,q-1)}$, all conjugate.

4.2 Maximal subgroups of $\mathbf{PSL(2,q)}$

In this section, we list the maximal subgroups of $\mathrm{PSL}(2,q)$. As the classification of geometries usually depends on whether q is even or odd, we give in Table 7 and Table 8 the maximal subgroups of $\mathrm{PSL}(2,q)$ in these two cases. We borrowed this result from Suzuki [16], page 417. Notice that the subgroups A_5 appear both as A_5 and $\mathrm{PSL}(2,q')$ for $q' = 5$.

Let us mention that a little error in Suzuki [16] was detected and corrected by Patricia Vanden Cruyce [19] in her thesis: *Indeed the subgroup* A_5 *is maximal if* r *is an odd prime. Because if* $r = 2$ *we have that* $A_5 < \mathrm{PGL}(2,5) < \mathrm{PSL}(2,25)$. However there remains a missing case in Suzuki [16] because, A_4 is maximal if $q = 5$. We include it in Table 8.

4.3 Two-transitive representations of the maximal subgroups of $\mathbf{PSL(2,q)}$

The first lemma is obvious but used often in the next section as a necessary condition to have a two-transitive action.

Lemma 4.4. *Let* G *be a group and let* H *be a subgroup of* G*. If* G *acts two-transitively on the cosets of* H *in* G*, then* $|G|$ *must be divisible by* $[G:H]([G:H]-1)$.

A group G is said to act *regularly* on a set Ω if G is transitive on Ω and the stabilizer in G of a point $x \in \Omega$ is the identity.

Lemma 4.5. *[21] Let* (G,Ω) *be a permutation group which is transitive over* Ω *and let* G *be abelian. Then* G *is regular. Moreover, if* G *is two-transitive then* $|\Omega| = 2$.

In order to simplify notation used throughout this section and the following one, we need another basic definition (borrowed from [2]). In a group G, an ordered pair of subgroups (A,B) is called *two-transitive* provided that B is a maximal subgroup of A and that the action of A on the left cosets of B is two-transitive.

Structure	Order	Index
$E_q:(q-1)$	$q(q-1)$	$q+1$
$D_{2(q+1)}$ $q \neq 2$	$2(q+1)$	$\frac{q(q-1)}{2}$
$D_{2(q-1)}$	$2(q-1)$	$\frac{q(q+1)}{2}$
A_5 $q = 4^r$ r *is prime*	60	$\frac{q(q^2-1)}{60}$
$\mathrm{PSL}(2,q') \cong \mathrm{PGL}(2,q')$ $q' > 4$, $q = q'^m$, m is *prime* or $q' = 2$, $q = q'^2$	$q'(q'^2-1)$	$\frac{q(q^2-1)}{q'(q'^2-1)}$

Table 7: The maximal subgroups of $\mathrm{PSL}(2,q)$, for q even

Structure	Order	Index
$E_q:\frac{q-1}{2}$	$\frac{q(q-1)}{2}$	$q+1$
$D_{(q+1)}$ $q \neq 7,9$	$q+1$	$\frac{q(q-1)}{2}$
$D_{(q-1)}$ $q \neq 3,5,7,9,11$	$q-1$	$\frac{q(q+1)}{2}$
A_4 *if* $q = p > 3$ *and* $q = 3,13,27,37(40)$ or $q = 5$	12	$\frac{q(q^2-1)}{12\times 2}$
S_4 *if* $q = p > 2$ *and* $q = \pm 1(8)$	24	$\frac{q(q^2-1)}{24\times 2}$
A_5 if $\begin{cases} q = 5^r & r\ odd\ prime & or \\ p = q = \pm 1(5) & p\ prime & or \\ q = p^2 = -1(5) & p\ prime & or \end{cases}$	60	$\frac{q(q^2-1)}{60\times 2}$
$\mathrm{PSL}(2,q')$ $q' \neq 5$, $q = q'^m$ m *odd prime*	$\frac{q'(q'^2-1)}{2}$	$\frac{q(q^2-1)}{q'(q'^2-1)}$
$\mathrm{PGL}(2,q')$ $q = q'^2$	$q'(q'^2-1)$	$\frac{q(q^2-1)}{q'(q'^2-1)}$

Table 8: The maximal subgroups of $\mathrm{PSL}(2,q)$, for q odd

We now provide the classification (existence and uniqueness) of all two-transitive representations of every maximal subgroup of $\mathrm{PSL}(2,q)$, a result borrowed from [2].

For the time being, let U be a group acting 2-transitively on a set Ω. Let $Ker\ U$ be the kernel of the representation, namely, the set of all $u \in U$ such that $u(x) = x$ for every $x \in \Omega$. Let U_0 be the stabilizer in U of some element 0 in Ω.

Proposition 4.6. *[2] Let $G \cong \mathrm{PSL}(2,q)$ for some power q of a prime p. Let (U, U_0) be a 2-transitive pair of subgroups of G with U maximal in G. Then one of the following holds:*

1. *$U \cong E_q : \frac{q-1}{2}$, $q = 1(4)$, $Ker\ U$ is the unique subgroup of index 2 of U, $|\Omega| = 2$, $U_0 = Ker\ U$ (unique up to conjugacy);*
2. *$U \cong E_q : (q-1)$, q even, $|\Omega| = q$, $Ker\ U = 1$, U_0 is a cyclic subgroup of order $(q-1)$ (unique up to conjugacy);*
3. *$U \cong \mathrm{PSL}(2,2) \cong S_3$, $|\Omega| = 2$, $Ker\ U = Z_3 = U_0$ (unique up to conjugacy);*
4. *$U \cong \mathrm{PSL}(2,2) \cong S_3$, $|\Omega| = 3$, $Ker\ U = 1$, $U_0 \cong Z_2$ (unique up to conjugacy);*
5. *$U \cong \mathrm{PSL}(2,3) \cong A_4$, $|\Omega| = 4$, $Ker\ U = 1$, $U_0 \cong Z_3$ (unique up to conjugacy);*
6. *$U \cong A_5 \cong \mathrm{PSL}(2,5) \cong \mathrm{PSL}(2,4)$, $p \neq 2$, $p \neq 5$, either $q = p = \pm 1(5)$ or $q = p^2 = -1(5)$. Here $|\Omega| = 5$, $Ker\ U = 1$, $U_0 \cong A_4$; (two such representations, up to conjugacy; they are fused in $\mathrm{PGL}(2,q)$); or $|\Omega| = 6$, $Ker\ U = 1$, $U_0 \cong D_{10}$.*
7. *$U \cong \mathrm{PSL}(2,11)$, $|\Omega| = 11$, $Ker\ U = 1$, $U_0 \cong A_5$ (two such representations, up to conjugacy; they are fused in $\mathrm{PGL}(2,11) = Aut(U)$);*
8. *$U \cong \mathrm{PSL}(2,9) \cong A_6$, $|\Omega| = 6$, $Ker\ U = 1$, $U_0 \cong A_5$ (two such representations, up to conjugacy; they are fused in $PGL(2,9)$);*
9. *$U \cong \mathrm{PSL}(2,7) \cong \mathrm{PSL}(3,2)$, $|\Omega| = 7$, $Ker\ U = 1$, $U_0 \cong S_4$ (two such representations, up to conjugacy; they are fused in $\mathrm{PGL}(2,7)$);*
10. *$U \cong \mathrm{PSL}(2,r)$ for every $r = p^s$, $s \geq 1$, $r > 3$ with $q = r^m$ and m prime. Moreover, for $p > 2$ we also require $m > 2$. Here $|\Omega| = r+1$, $Ker\ U = 1$, $U_0 \cong E_r : \frac{r-1}{(2,r-1)}$ (unique up to conjugacy for given r);*
11. *$U \cong \mathrm{PGL}(2,r)$, r odd, $r = p^s$, $q = r^2$, $|\Omega| = 2$, $Ker\ U = U_0 \cong \mathrm{PSL}(2,r)$ (unique up to conjugacy);*
12. *$U \cong \mathrm{PGL}(2,r)$, r odd, $r = p^s$, $s \geq 1$ with $q = r^2$. Here $|\Omega| = r+1$, $Ker\ U = 1$, $U_0 \cong E_r : (r-1)$ (unique up to conjugacy);*
13. *$U \cong \mathrm{PGL}(2,3) \cong S_4$, $q = \pm 1(8)$, $|\Omega| = 3$, $Ker\ U = E_4$, $U/Ker\ U \cong S_3$, $U_0 \cong D_8$ (two such representations, up to conjugacy; they are fused in $\mathrm{PGL}(2,q)$);*
14. *U is dihedral of order $2(q-1)$ or $2(q+1)$, q even. $|\Omega| = 2$, $Ker\ U = U^+ = U_0$ where U^+ is the cyclic subgroup of index 2 of U, (unique up to conjugacy for each of the two possible values of $|U|$);*
15. *U is dihedral of order $(q-1)$ or $(q+1)$, q odd. $|\Omega| = 2$, $Ker\ U = U^+ = U_0$ where U^+ is the cyclic subgroup of index 2 of U, (unique up to conjugacy for each of the two possible values of $|U|$). In the particular case where $q = 3$, the case of $(q+1)$ provides $U = E_4$, $|\Omega| = 2$. Then U_0 is one of the three subgroups of order 2 in U (unique up to conjugacy);*

16. *U is dihedral of order either $2(q-1)$ or $2(q+1)$, q even, and $3 \mid |U|$; $|\Omega| = 3$, $Ker\ U$ is the unique cyclic subgroup of index 6 in U. Then U_0 is one of the three dihedral subgroups of index 3 in U, $U/Ker\ U \cong S_3$ (unique up to conjugacy);*
17. *U is dihedral of order either $(q-1)$ or $(q+1)$, q odd, and $3 \mid |U|$. Here $|\Omega| = 3$, $Ker\ U$ is the unique cyclic subgroup of index 6 in U. Then U_0 is one of the three dihedral subgroups of index 3 in U, $U/Ker\ U \cong S_3$ (unique up to conjugacy);*
18. *U is dihedral of order either $(q-1)$ or $(q+1)$, q odd, $q > 5$, and $4 \mid |U|$. Here $|\Omega| = 2$, $Ker\ U = U_0$ is one of the two dihedral subgroups of index 2 in U (two such representations, up to conjugacy; they are fused in $\mathrm{PGL}(2,q)$); $Ker\ U$ is dihedral, U_0 is dihedral of index 2;*
19. *U is dihedral of order 4, q is one of 3,5; $|\Omega| = 2$, $Ker\ U = U_0$ is one of the three dihedral subgroups of index 2 in U (unique up to conjugacy);*
20. *$U \cong \mathrm{PGL}(2,5) \cong S_5$, $|\Omega| = 5$, $Ker\ U = 1$, $U_0 \cong S_4$ (unique up to conjugacy).*

4.4 Some other useful results

An observation used in our proofs is that $\mathrm{PGL}(2,q)$ can be viewed as a subgroup of $\mathrm{PSL}(2,q^2)$ and also that $\mathrm{PGL}(2,q)$ has a unique subgroup isomorphic to $\mathrm{PSL}(2,q)$. This lets us extract the list of subgroups of $\mathrm{PGL}(2,q)$ from the list of subgroups of $\mathrm{PSL}(2,q^2)$. Therefore we require the properties of the subgroup lattice of $\mathrm{PGL}(2,q)$ for which we refer to [4] (see also [13] and [15]). The next lemma is often used to count the geometries up to isomorphism.

Lemma 4.7.
- *Assume that $\frac{q\pm1}{d(2,q-1)}$ is even. In this case both conjugacy classes of D_{2d} for every $d > 2$ dividing $\frac{q\pm1}{(2,q-1)}$ fuse in $\mathrm{PGL}(2,q)$ and thus also in $P\Gamma L(2,q)$.*
- *Assume that $q = \pm1(8)$. In this case both conjugacy classes of S_4 and A_4 fuse in $\mathrm{PGL}(2,q)$ and thus also in $P\Gamma L(2,q)$.*
- *Assume that $q = \pm1(5)$. In this case both conjugacy classes of A_5 fuse in $\mathrm{PGL}(2,q)$ and thus also in $P\Gamma L(2,q)$.*
- *Assume that $q = p^{2n}$ is odd. In this case both conjugacy classes of $\mathrm{PGL}(2,p^n)$ fuse in $\mathrm{PGL}(2,p^{2n})$ and thus also in $P\Gamma L(2,q)$.*

5 Proof of Theorem 1.1

In this section, we prove the Classification Theorem 1.1 by a case analysis. We determine the rank 2 RWPRI and $(2T)_1$ geometries of $\mathrm{PSL}(2,q)$.

In order to structure this work we introduce a subsection for each type of G_0. There are 5 such subsections left to consider, which are the different types of maximal subgroups of $G \cong \mathrm{PSL}(2,q)$, listed in section 4.2. The cases $E_q : \frac{(q-1)}{(2,q-1)}$, $D_{2\frac{(q-1)}{(2,q-1)}}$ and $D_{2\frac{(q+1)}{(2,q-1)}}$ have been treated in [5].

The various cases for the two-transitive pairs (G_0, G_{01}) with G_0 maximal in G are provided by Proposition 4.6. Those situations are analysed in order to detect the admissible G_1 in a series of Lemmas. During this analysis, candidates for G_1 are represented by the symbol H. They become G_1 only when they resist the analysis.

5.1 The case where $G_0 = A_5$

Recall that following Table 7 and Table 8, the subgroup A_5 is maximal in $\mathrm{PSL}(2,q)$ if

$$\begin{cases} q = 5^r & r \text{ odd prime} & \text{or} \\ q = 4^r & r \text{ prime} & \text{or} \\ q = p = \pm 1(5) & p \text{ odd prime} & \text{or} \\ q = p^2 = -1(5) & p \text{ odd prime} & . \end{cases}$$

In this section we assume these conditions on q. Observe that if $q = 0(5)$ the group A_5 is isomorphic to $\mathrm{PSL}(2,5)$ which is a particular case of the family $\mathrm{PSL}(2,5^n)$ with $q = 5^{nm}$ for m an odd prime. In this section we treat this particular situation. The general situation is treated in Proposition 5.16. If $q = 0(4)$ the group A_5 is isomorphic to $\mathrm{PSL}(2,4)$ which is a particular case of the family $\mathrm{PSL}(2,4^n)$ with $q = 4^{nm}$ for m prime. In this section we analyse this particular situation. The general situation is treated in Proposition 5.12.

In view of (6) in Proposition 4.6 there are two cases for G_{01}, namely the case of D_{10} and A_4. For each of these G_{01} we look for the various possible groups H in one of the two following Lemmas. Remember that H is any subgroup of G such that (H, G_{01}) is a two-transitive pair. In order to determine all H candidates we scan the list of maximal subgroups. For each maximal subgroup we analyse its subgroup lattice.

Lemma 5.1. *Let $G \cong \mathrm{PSL}(2,q)$ with q as required in this section. If H is a subgroup of G such that (H, D_{10}) is a two-transitive pair then one of the three following statements holds:*

- $H \cong D_{20}$ *provided* $10 \mid \frac{q\pm1}{(2,q-1)}$;
- $H \cong D_{30}$ *provided* $15 \mid \frac{q\pm1}{(2,q-1)}$;
- $H \cong \mathrm{PSL}(2,5) \cong A_5$.

Proof. Left to the reader. See Appendix pg 1. (The Appendix contains details for this and several other results to follow.) □

Lemma 5.2. *Let $G \cong \mathrm{PSL}(2,q)$ with q as required in this section. If H is a subgroup of G such that (H, A_4) is a two-transitive pair then one of the five following statements holds:*

- $H \cong E_{16}:3$ *provided* $q = 4^r$;
- $H \cong \mathrm{PSL}(2,4) \cong A_5$ *provided* $q = 4^r$;
- $H \cong \mathrm{PSL}(2,5)$ *provided* $q = 5^r$;
- $H \cong S_4$ *provided* $q = \pm1(5)$ *and* $q = \pm1(8)$;
- $H \cong A_5$.

Proof. Left to the reader. See Appendix pg 2. □

In Remark 4.3 of section 4.1. we mention that the number of conjugacy classes of cases 7 and 8 are not given in Lemma 4.1. To prove the following Proposition we need the number of conjugacy classes of a particular situation, treated in the next two Lemmas.

Lemma 5.3. *The number of conjugacy classes of $E_{16}:3$ in $\mathrm{PSL}(2,4^r)$, for an odd prime r, is equal to $\frac{4^{r-1}-1}{15}$.*

Proof. Step 1: We must count the number of conjugacy classes of subgroups $E_{16}:3$ in $\mathrm{PSL}(2,4^r)$. Therefore we first count the total number of subgroups $E_{16}:3$ in $\mathrm{PSL}(2,4^r)$ and divide this number by the length of the conjugacy classes. We shall indeed see that this number is constant.

Step 2: We consider $G \cong \mathrm{PSL}(2,4^r)$ as a permutation group acting on the projective line $PG(1,4^r)$. This group is sharply 3-transitive on 4^r+1 points. Given a point ∞, its stabilizer is $E_{4^r} : 4^r-1 \cong AGL(1,4^r)$. The latter contains our $E_{16}:3$. Let H be any subgroup $E_4:3 \cong A_4 \cong AGL(1,4)$. It is contained in a subgroup $K := \mathrm{PGL}(2,4) \cong A_5$ which has an orbit of length five namely $PG(1,4)$.

Step 3: Let us see $AG(1,4^r) = PG(1,4^r)\backslash\{\infty\}$ as an affine space V of dimension r over the field $GF(4)$. The subgroup H stabilizes a line l of V namely $AG(1,4)$. Hence, l contains the points 0 and 1. The space V endowed with the point 0 is a vector space of dimension r on $GF(4)$.

Observe that H fixes a unique point namely ∞. In A_5 there are four conjugate subgroups $E_4:3$ say X_1,X_2,X_3,X_4 other than H, each fixing a unique point which belongs to l. Moreover, H stabilizes no other line l' in V since otherwise $l' \cup \{\infty\}$ is an orbit of length five of A_5 and so each of X_1,X_2,X_3,X_4 fixes a point on l' while this point is on l implying $l=l'$. Therefore, H stabilizes a unique line of V which is l.

Step 4: Observe that $AG(1,4^r)$ is transitive on the lines of V. There are $\frac{4^r(4^r-1)}{12}$ lines in V and, taking the point ∞ into account, we see that the conjugacy class of H in G consists of $\frac{4^r(4^r-1)}{12}$ subgroups $E_{16}:3$.

Step 5: Coming back to the beginning of Step 3, the multiplicative group of $GF(4)$ is a cyclic group Z_3 which is a subgroup of H and so also a subgroup of A_5 namely $E_{16}:3$.

Step 6: The group Z_3 stabilizes the point 0 and every line on 0 in the space V. Therefore, it also stabilizes every plane on 0 in this space, in particular every plane containing l. There are $\frac{4^r-4}{16-4} = \frac{4^{r-1}-1}{3}$ such planes on l.

Step 7: Let π be a plane of V containing l. It is invariant under 16 translations and Z_3. Thus π is invariant under a subgroup $E_{16}:3$ containing H. Conversely, every $E_{16}:3$, say L, containing H also contains Z_3 which fixes the point 0. The orbit of 0 under L is its orbit under E_{16}. And Z_3 acts on this orbit, hence this orbit is a plane. In conclusion, the subgroups $E_{16}:3$ containing H and the planes containing l are in one-to-one correspondence.

Step 8: Combining Steps 3, 6 and 7 we see that the number of conjugacy classes of subgroups $E_{16}:3$ containing H and fixing ∞ is $\frac{4^{r-1}-1}{3}.\frac{1}{5}$ as required. □

For the particular situation of Lemma 5.3, we count the number of geometries up to conjugacy and up to isomorphism in the following Lemma.

Lemma 5.4. *Let r be an odd prime. Let $\alpha_C(r)$ (resp. $\alpha_I(r)$) be the number of geometries of type $\Gamma(\mathrm{PSL}(2,4^r),A_5,A_4,E_{16}:3)$ up to conjugacy (resp. isomorphism). Then the following hold:*

1. *$\alpha_C(3)=5$;*
2. *$\alpha_I(3)=2$;*
3. *if $r>3$, then $\alpha_C(r)=\frac{4^{r-1}-1}{3}$;*

4. *if* $r > 3$, *then* $\alpha_I(r) = \frac{2(4^{r-2}-1)+3.2^{r-2}}{3r}$.

Proof. Step 1: Lemma 5.3 gives the number of conjugacy classes of $E_{16}:3$ for a given $\mathrm{PSL}(2,4^r)$. Every $E_{16}:3$ has five conjugacy classes of subgroups $E_4:3$. Moreover, each $E_4:3$ is contained in a unique A_5. Therefore, we get the number of triples consisting of a representative G_1 of every conjugacy class of $E_{16}:3$, a representative G_{01} of every conjugacy class of $E_4:3$ in G_1 and the unique subgroup $G_0 \cong A_5$ containing G_{01}. Hence $\alpha_C(r) = \frac{4^{r-1}-1}{15} \cdot 5 \cdot 1$. In particular $\alpha_C(3) = 5$. This is proving respectively (3) and (1).

Step 2: Let ∞, V, H and l be defined as in the proof of Lemma 5.3, Steps 2 and 3. Recall that l contains 0 and 1. To get $\alpha_I(r)$, we still have to figure out how $N_{P\Gamma L(2,4^r)}(H)$ acts on the subgroups $E_{16}:3$ containing H. In other words, how does $N_{P\Gamma L(2,4^r)}(H)$ act on the planes of V containing l ?

Step 3: To answer the question of Step 2 we shall show that $N_{P\Gamma L(2,4^r)}(H) = H:K$, where K is the group of field automorphisms of $GF(4^r)$. Recall the fact that the group $P\Gamma L(2,4^r)$ is $PSL(2,4^r):K$. Recall also that K is a cyclic group of order $2r$. The group K leaves every subfield of $GF(4^r)$ invariant. Hence K leaves $GF(4)$ invariant, thus also the line l, and it normalizes H. Applying Lemma 4.1 we see that $N_{PSL(2,4^r)}(H) = H$ in view of the fact that $H \cong A_4$ and of the restrictions on the values taken by q. We get that $N_{P\Gamma L(2,4^r)}(H)$ is a group of order $H.K.\epsilon$ and we want now to show that $\epsilon = 1$. Let N_1 (resp. N_2) be the number of conjugate subgroups of H in G (resp. $P\Gamma L(2,4^r)$). Then $N_1 \leq N_2$, $N_1 = \frac{|G|}{|H|}$, $N_2 = \frac{|P\Gamma L(2,4^r)|}{|H|.|K|.\epsilon} = \frac{|G|}{|H|.\epsilon}$ and so $\epsilon = 1$. Therefore $N_{P\Gamma L(2,4^r)}(H) = H:K$.

Step 4: In our count of triples, we may assume that G_0 and G_{01} are fixed because, up to isomorphism, the chain of subgroups $PSL(2,4^r) - A_5 - A_4$ is unique. Moreover, without loss of generality, we suppose that G_{01} is H.

Step 5: We consider $G \cong PSL(2,4^r)$ and H in it. We recall the $\frac{4^{r-1}-1}{15}$ conjugacy classes of subgroups $E_{16}:3$ containing H as found in Lemma 5.3. Let Ω be the set of these $\frac{4^{r-1}-1}{15}$ conjugacy classes. Recall that $N_{PSL(2,4^r)}(H) = H$ and so the action of H on Ω is the identity. Next we consider the action of K on Ω which is also the action of $H:K$. The number of orbits of this K-action on Ω is the number $\alpha_I(r)$ we have to determine.

Step 6: As in the proof of Lemma 5.3, Step 2 we consider $G \cong \mathrm{PSL}(2,4^r)$ as a triply transitive permutation group acting on the projective line $PG(1,4^r)$. For every t dividing $2r$ there is a subfield $GF(2^t)$ of $GF(4^r)$. It fixes 2^t+1 points on $PG(1,4^r)$. This set of points is called a circle as well as all of its transforms under G. Every triple of distinct points on $PG(1,4^r)$ is contained in one and only circle of 2^t+1 points.

Step 7: Given three points ∞, 0 and 1, there is a unique circle C_5 of five points, namely $PG(1,4) = \{\infty\} \cup l$ and there is a unique circle of 2^r+1 points C_{2^r+1}, namely $PG(1,2^r)$. The involution $\beta \in K$ fixes all the points of C_{2^r+1}. On C_5, it fixes ∞, 0 and 1, and it permutes the remaining 2 points that we call a and $\beta(a)$. The group induced on C_5 by the stabilizer of C_5 in $PSL(2,4^r)$ is $A_5 \cong \mathrm{PSL}(2,4)$ extended by β, that is, S_5.

The unique subgroup K^+ of K, of order r fixes all points of C_5 and splits the remaining points of $PG(1,4^r)$ in orbits of length r. Therefore, $(2^{2r}+1)-5$ must be divisible by r. Indeed, $(2^{2r}+1)-5 = 4^r-4 = 4(4^{r-1}-1)$, the latter being divisible by r thanks to Fermat's little theorem.

The subgroup $H \cong E_4:3$ fixes ∞. Every cyclic subgroup of order 3 of H fixes two points of C_5. This gives ten conjugate subgroups of order 3 in A_5.

The group K^+ fixes C_5 point-wise. Suppose K^+ stabilizes a plane π of V containing l. Then it must decompose the $16 - 4 = 12$ points of $\pi \backslash l$ in orbits of length r.

If $r = 3$, this may occur and K^+ indeed stabilizes two of the five planes containing l, hence it normalizes two of the $E_{16}:3$ containing H. Moreover, it fuses the other three. The two $E_{16}:3$ normalized by K^+ are swapped by β, giving $\alpha_I(3) = 2$. This is proving (2).

If $r > 3$, no plane of V that contains l can be stabilized by K^+. Hence K^+ fuses the $\frac{4^{r-1}-1}{3}$ subgroups $E_{16}:3$ in $\frac{4^{r-1}-1}{3r}$ orbits of length r.

Step 8: It remains to look at the action of β on these orbits. In $GF(4^r)$, there are three proper subfields, namely $GF(2)$, $GF(4)$ and $GF(2^r)$. The involution β fixes all the elements of $GF(2^r)$. Let us show that β stabilizes $\frac{2^r-2}{2}$ planes containing l. Given an element $x \in GF(2^r)$, the plane π containing 0, 1 and x is stabilized since 0, 1 and x are fixed by β. Moreover, π contains the point $x + 1 \in GF(2^r)$. Hence, there are at least four points fixed in π by β. If there are more, there must be at least 8 points fixed and the whole plane π is fixed point-wise, a contradiction with the fact that $a \in \pi$ and a is not fixed by β. Therefore, the elements of $GF(2^r)$ give $\frac{2^r-2}{2}$ distinct planes that are stabilized by β.

Step 9: We claim that the remaining planes of V that contain l are fused in pairs by β. Indeed, suppose that there exists a plane π containing l and no other element of $GF(2^r)$ in V, and such that $\beta(\pi) = \pi$. In π, the only fixed points are thus 0 and 1. For every $x \in \pi \backslash C_5$, the line $x\beta(x)$ is stabilized by β. It is either secant or parallel to l. Suppose first that it is secant. Then, it intersects l in either 0 or 1 and the fourth point of $x\beta(x)$ must be fixed, a contradiction. Suppose then that it is parallel. The other two points of $x\beta(x)$ may be written as y and $\beta(y)$. Let us recall that we denote the points of l as 0, 1, a and $\beta(a)$. The lines ax and $\beta(a)\beta(x)$ are swapped and parallel. One of the lines $1y$ or $1\beta(y)$ must also be parallel to ax. Its image by β is not parallel to ax. This is a contradiction. Therefore, no other plane of V containing l can be stabilized by β.

Step 10: In conclusion, we get $\frac{2^{r-1}-1}{r}$ sets of r isomorphic geometries and $\frac{1}{2r}(\frac{4^{r-1}-1}{3} - (2^{r-1}-1))$ sets of $2r$ isomorphic geometries. Finally, we obtain $\alpha_I(r) = \frac{2^{r-1}-1}{r} + \frac{1}{2r}(\frac{4^{r-1}-1}{3} - (2^{r-1}-1))$ and the formula given in the Lemma is obtained by a straightforward simplification. This is proving (4). □

Proposition 5.5. *Let $G \cong \mathrm{PSL}(2, q)$ with q as required in this section. Every RWPRI and $(2T)_1$ geometry of rank two $\Gamma(G; G_0, G_1, G_0 \cap G_1)$ in which $G_0 \cong A_5$ is isomorphic to one of the geometries appearing in Table 1.*

Proof. Let $G_0 \cong A_5$.
We subdivide our discussion in two cases, namely the two G_{01}-candidates in view of (6) in Proposition 4.6 which are: D_{10} and A_4. In each of these two cases we review all possibilities for G_1 given in the previous Lemmas 5.1 and 5.2, as well as the number of classes of geometries with respect to conjugacy (resp. isomorphism).

Subcase 1: $G_{01} = G_0 \cap G_1 \cong D_{10}$.
This is dealt with in the appendix, (pg 2-6).

Subcase 2: $G_{01} = G_0 \cap G_1 \cong A_4$.
By Lemma 5.2 the possibilities for G_1 are $E_{16}:3$ if $q = 4^r$, $\mathrm{PSL}(2, 4) \cong A_5$ if $q = 4^r$, $\mathrm{PSL}(2, 5) \cong A_5$ if $q = 5^r$, S_4 if $q = \pm 1(5)$ as well as $q = \pm 1(8)$ and A_5.

2.1 We consider the case where $G_1 \cong E_{16}:3$.
The condition on q is $q = 4^r$ with r prime. In this situation there is only one conjugacy class of A_5 and one of A_4 in $\mathrm{PSL}(2, q)$. Notice that there are 5 conjugacy classes of A_4 in $E_{16}:3$. Since $\mathrm{PSL}(2, 16)$ is simple and A_5 maximal, A_5 is self-normalized. Moreover, A_4 is self-normalized in $\mathrm{PSL}(2, 4^r)$. The normalizer of $E_{16}:3$ depends on whether $r = 2$ or not. We distinguish three cases namely: $r = 2$, $r = 3$ and $r > 3$. In the latter two, notice that since $r \neq 2$, $E_{16}:3$ is self-normalized in $\mathrm{PSL}(2, 4^r)$.

• Let us first consider the particular case where $r = 2$. In this situation there exists only one conjugacy class of $E_{16}:3$ in $\mathrm{PSL}(2, 16)$. We also have that $N_{\mathrm{PSL}(2,16)}(E_{16}:3) = E_{16}:15$. Therefore the number of subgroups $E_{16}:3$ containing a given subgroup A_4 in $\mathrm{PSL}(2, 16)$ is equal to

$$\frac{|\ \mathrm{PSL}(2,16)\ |}{|\ E_{16}:15\ |} \cdot \frac{|\ E_{16}:3\ |}{|\ A_4\ |} \cdot 5 \cdot \frac{|\ A_4\ |}{|\ \mathrm{PSL}(2,16)\ |} = 1.$$

Thus the RWPRI and $(2T)_1$ geometry $\Gamma_2 = \Gamma\left(\mathrm{PSL}(2, 16); A_5, E_{16}:3, A_4\right)$ exists and is unique up to conjugacy and also up to isomorphism.

• In view of Lemma 5.3 and Lemma 5.4 we know that if $r = 3$ there exist up to conjugacy exactly five RWPRI and $(2T)_1$ geometries $\Gamma_3 := \Gamma(\mathrm{PSL}(2, 64), A_5, A_4, E_{16}:3)$ and exactly two up to isomorphism.

• In view of Lemma 5.3 and Lemma 5.4 we know that if $r > 3$ there exist up to conjugacy exactly $\frac{4^{r-1}-1}{3}$ RWPRI and $(2T)_1$ geometries $\Gamma_4 := \Gamma(\mathrm{PSL}(2, q), A_5, A_4, E_{16}:3)$ and exactly $\frac{2(4^{r-2}-1)+3.2^{r-2}}{3r}$ up to isomorphism.

This geometry is new and the number of classes up to conjugacy (resp. isomorphism) is confirmed by MAGMA for $q = 16, 64$. For $q = 16$, it is also confirmed by [20].

2.2 We consider the case where $G_1 \cong S_4$.
The conditions given on q are $q = \pm 1(5)$ and $q = \pm 1(8)$. They imply that there are two conjugacy classes of S_4, two of A_5 and also two of A_4 in $\mathrm{PSL}(2, q)$. Therefore we consider two situations: either $q = p = \pm 1(5)$ or $q = p^2 = -1(5)$, with p an odd prime. We distinguish these two cases in the discussion below.

• Assume $q = p = \pm 1(5)$ with p prime. All conditions given on q imply that either $q = \pm 1(40)$ or $q = \pm 9(40)$. In both situations we know that S_4 is a maximal subgroup of $\mathrm{PSL}(2, q)$. Therefore $N_{\mathrm{PSL}(2,q)}(A_4) = S_4 = N_{S_4}(A_4)$ and $N_{A_5}(A_4) = A_4$. Now all A_4 in an S_4 are conjugate and this is also the case for all A_4 in an A_5. The number of subgroups A_5 containing a given subgroup A_4 in $\mathrm{PSL}(2, q)$ is equal to

$$\frac{|\ \mathrm{PSL}(2,q)\ |}{|\ A_5\ |} \cdot \frac{|\ A_5\ |}{|\ A_4\ |} \cdot \frac{|\ S_4\ |}{|\ \mathrm{PSL}(2,q)\ |} = 2.$$

To count the geometries up to conjugacy we need to know whether the S_4 normalizes each of the A_5. This is not the case because $|N_{\mathrm{PSL}(2,q)}(A_4) \cap N_{\mathrm{PSL}(2,q)}(S_4)| = |S_4| = 2|A_4|$. Hence, there exist exactly two RWPRI and $(2T)_1$ geometries $\Gamma_9 = \Gamma(\mathrm{PSL}(2, q); A_5, S_4, A_4)$ up to conjugacy, provided $q = \pm 1(40)$ or $q = \pm 9(40)$.

Let us deal with the fusion of non-conjugate classes. Following Lemma 4.7 the two classes of S_4, A_4 and A_5 are fused under the action of $\mathrm{PGL}(2, q)$ and thus also under the action of $P\Gamma L(2, q)$. Therefore, there exists exactly one RWPRI and $(2T)_1$ geometry $\Gamma_9 = \Gamma\left(\mathrm{PSL}(2, q); A_5, S_4, A_4\right)$ up to isomorphism provided $q = \pm 1(40)$ or $q = \pm 9(40)$.

• Assume $q = p^2 = -1(5)$ with p prime. All conditions given on q imply that either $q = -1(40)$ or $q = 9(40)$. All A_4 in an S_4 are conjugate and $N_{\mathrm{PSL}(2,q)}(A_4) = S_4 = N_{S_4}(A_4)$ and $N_{A_5}(A_4) = A_4$. We also know that $N_{\mathrm{PSL}(2,q)}(S_4) = S_4$. Therefore the number of S_4 containing a given A_4 is one.

To count the geometries up to conjugacy we need to know whether the S_4 normalizes each of the A_5. This is not the case because $|N_{\mathrm{PSL}(2,q)}(A_4) \cap N_{\mathrm{PSL}(2,q)}(S_4)| = |S_4| = 2|A_4|$. Therefore, up to conjugacy there exist exactly two RWPRI and $(2T)_1$ geometries $\Gamma_{16} = \Gamma\,(\mathrm{PSL}(2,q); A_5, S_4, A_4)$ provided either $q = -1(40)$ or $q = 9(40)$, with $q = p^2$.

Let us deal with the fusion of non-conjugate classes. Following Lemma 4.7 the two classes of A_4, S_4 and A_5 are fused under the action of $\mathrm{PGL}(2,q)$ and thus also under the action of $P\Gamma L(2,q)$. Therefore, there exists exactly one RWPRI and $(2T)_1$ geometry $\Gamma_{16} = \Gamma\,(\mathrm{PSL}(2,q); A_5, S_4, A_4)$ up to isomorphism, provided either $q = -1(40)$ or $q = 9(40)$, with $q = p^2$.

This geometry is new and the number of classes up to conjugacy (resp. isomorphism) is confirmed by MAGMA for $q = 9, 31, 41, 49$. For $q = 9$, it is also confirmed by [3].

2.3 Consider the case where $G_0 \cong G_1 \cong A_5$.
With the given conditions on q there are three cases to consider:

• If $q = 4^r$ with r prime, there is only one conjugacy class of A_5 and also one of A_4. Since every A_4 is contained in only one A_5, there is no such geometry.

• Assume $q = 5^r$ with r an odd prime. The number of conjugacy classes of A_4 in $\mathrm{PSL}(2,q)$ depends on whether $q = \pm 1(8)$ or $q = \pm 3(8)$. If $q = \pm 1(8)$ there is a contradiction with r odd in $q = 5^r$. Now $q = \pm 3(8)$ implies that there is one conjugacy class of A_4 and also one of A_5. Since every A_4 is contained in only one A_5, there exists no such geometry.

• Assume $q = p = \pm 1(5)$ or $q = p^2 = -1(5)$ with p an odd prime.
There are two conjugacy classes of A_5 in $\mathrm{PSL}(2,q)$. The number of conjugacy classes of A_4 in $\mathrm{PSL}(2,q)$ depends on whether $q = \pm 1(8)$ or $q = \pm 3(8)$. We distinguish these two cases.

If $q = \pm 1(8)$ there are two classes of A_4, all A_4 in an A_5 are conjugate, and the normalizer of A_4 in $\mathrm{PSL}(2,q)$ is S_4. All conditions on q imply that if $q = p = \pm 1(5)$ either $q = \pm 1(40)$ or $q = \pm 9(40)$; and if $q = p^2 = -1(5)$ either $q = -1(40)$ or $q = +9(40)$.
The number of subgroups A_5 containing a given subgroup A_4 in $\mathrm{PSL}(2,q)$ is equal to

$$\frac{|\,\mathrm{PSL}(2,q)\,|}{|\,A_5\,|} \cdot \frac{|\,A_5\,|}{|\,A_4\,|} \cdot \frac{|\,S_4\,|}{|\,\mathrm{PSL}(2,q)\,|} = 2.$$

Therefore, there exist exactly two RWPRI and $(2T)_1$ geometries $\Gamma_{10} = \Gamma(\mathrm{PSL}(2,q); A_5, A_5, A_4)$ up to conjugacy, provided either $q = \pm 1(40)$ or $q = \pm 9(40)$, with q prime, one for each class of A_5. Also, there exist exactly two RWPRI and $(2T)_1$ geometries $\Gamma_{17} = \Gamma\,(\mathrm{PSL}(2,q); A_5, A_5, A_4)$ up to conjugacy, provided either $q = -1(40)$ or $q = +9(40)$, with $q = p^2$, one for each class of A_5.

Let us deal with the fusion of non-conjugate classes. Following Lemma 4.7 the two classes of A_5 are fused under the action of $\mathrm{PGL}(2,q)$ and thus also under the action of $P\Gamma L(2,q)$. Therefore there exists exactly one RWPRI and $(2T)_1$ geometry $\Gamma_{10} = \Gamma\,(\mathrm{PSL}(2,q); A_5, A_5, A_4)$ up to isomorphism provided either $q = \pm 1(40)$ or $q = \pm 9(40)$.

Also, there exists exactly one RWPRI and $(2T)_1$ geometry $\Gamma_{17} = \Gamma\left(\mathrm{PSL}(2,q); A_5, A_5, A_4\right)$ up to isomorphism provided either $q = -1(40)$ or $q = +9(40)$.

If $q = \pm 3(8)$, there is one conjugacy class of A_4 in $\mathrm{PSL}(2,q)$. All conditions on q imply that if $q = p = \pm 1(5)$ either $q = \pm 11(40)$ or $q = \pm 19(40)$; and if $q = p^2 = -1(5)$ either $q = -11(40)$ or $q = +19(40)$. Every A_4 is contained in exactly one A_5, and there are two conjugacy classes of A_5 in $\mathrm{PSL}(2,q)$.
Hence, there exists exactly one RWPRI and $(2T)_1$ geometry $\Gamma_{11} = \Gamma(\mathrm{PSL}(2,q); A_5, A_5, A_4)$ up to conjugacy and thus also exactly one up to isomorphism provided either $q = \pm 11(40)$ or $q = \pm 19(40)$, with q prime.
This geometry is new and the number of classes up to conjugacy (resp. isomorphism) is confirmed by MAGMA for $q = 11, 19, 29, 31, 41, 61$. For $q = 11, 19$, it is also confirmed by [20].
Also, there exists exactly one RWPRI and $(2T)_1$ geometry $\Gamma_{18} = \Gamma(\mathrm{PSL}(2,q); A_5, A_5, A_4)$ up to conjugacy and thus also exactly one up to isomorphism provided either $q = -11(40)$ or $q = +19(40)$, with $q = p^2$. This geometry is new and the number of classes up to conjugacy (resp. isomorphism) is confirmed by MAGMA for $q = 9, 49$. □

5.2 The case where $G_0 = A_4$

Recall that following Table 8, the subgroup A_4 is maximal in $\mathrm{PSL}(2,q)$ provided q is prime, $q > 3$ and either $q = 3, 13, 27, 37(40)$ or $q = 5$. Therefore $q = \pm 3(8)$ and there exists only one conjugacy class of subgroups isomorphic to A_4. In view of (5) in Proposition 4.6 there is only one case for G_{01}, namely the cyclic subgroup of order 3.

The proof of all following propositions are very similar to that of Proposition 5.5. Therefore we do not give the details and we refer to the Appendix. The proof of proposition 5.6 may be found in the Appendix (pg. 6-9).

Proposition 5.6. *Let $G \cong \mathrm{PSL}(2,q)$ with q prime, $q > 3$ and either $q = 3, 13, 27, 37(40)$ or $q = 5$. Every RWPRI and $(2T)_1$ geometry of rank two $\Gamma(G; G_0, G_1, G_0 \cap G_1)$ in which $G_0 \cong A_4$ is isomorphic to one of the geometries appearing in Table 2.*

5.3 The case where $G_0 = S_4$

Recall that following Table 7 and Table 8, the subgroup S_4 is maximal in $\mathrm{PSL}(2,q)$ if q is an odd prime and $q = \pm 1(8)$. In this section we assume these conditions on q. Moreover, there are two conjugacy classes of subgroups isomorphic to S_4 in G.

In view of (11), (12) and (13) in Proposition 4.6 there are three cases for G_{01}, namely the case of $D_6 \cong E_3 : 2$, the case of D_8 and the case of A_4. For each of these G_{01} we look for the various possible groups H in one of the three following Lemmas, whose proofs are left to the reader. The proof of proposition 5.10 may be found in the Appendix (pg. 8-12).

Lemma 5.7. *Let $G \cong \mathrm{PSL}(2,q)$ with q an odd prime and $q = \pm 1(8)$ as required in this section. If H is a subgroup of G such that (H, D_6) is a two-transitive pair then one of the three following statements holds: $H \cong D_{12}$ provided $6 \mid \frac{q\pm 1}{2}$; $H \cong D_{18}$ provided $9 \mid \frac{q\pm 1}{2}$; or $H \cong S_4$.*

Lemma 5.8. *Let $G \cong \mathrm{PSL}(2,q)$ with q an odd prime and $q = \pm 1(8)$ as required in this section. Then the following statement holds: If H is a subgroup of G such that (H, D_8) is a two-transitive pair then $H \cong D_{16}$ provided $8 \mid \frac{q\pm 1}{2}$, $H \cong D_{24}$ provided $12 \mid \frac{q\pm 1}{2}$; or$H \cong S_4$.*

Lemma 5.9. *Let $G \cong \mathrm{PSL}(2,q)$ with q an odd prime and $q = \pm 1(8)$ as required in this section. Then the following statement holds: If H is a subgroup of G such that (H, A_4) is a two-transitive pair then $H \cong S_4$; or $H \cong A_5$ provided $q = \pm 1(5)$.*

Proposition 5.10. *Let $G \cong \mathrm{PSL}(2,q)$ with q an odd prime and $q = \pm 1(8)$. Every RWPRI and $(2T)_1$ geometry of rank two $\Gamma(G; G_0, G_1, G_0 \cap G_1)$ in which $G_0 \cong S_4$ is isomorphic to one of the geometries appearing in Table 3.*

5.4 The case where $G_0 = \mathrm{PSL}(2, q')$

In this section we make a distinction between the cases q odd and q even with $q = p^{nm}$. The subgroups $\mathrm{PSL}(2, q')$ and $\mathrm{PGL}(2, q')$ with $q' = p^n$ are isomorphic provided q is even and they are distinct provided q is odd.

5.4.1 The case q even

Since q is even, $\mathrm{PSL}(2, q') \cong \mathrm{PGL}(2, q')$. Recall that following Table 7, the subgroup $\mathrm{PSL}(2, q') \cong \mathrm{PGL}(2, q')$ is maximal in $\mathrm{PSL}(2, q)$ provided $q' = 2^n$ and $q = q'^m = 2^{n.m}$ for m prime; moreover for $n = 1$ we need $m = 2$. In this section we assume these conditions on q.

In view of (3), (4), (6) and (10) in Proposition 4.6 there are three cases for G_{01}, namely: the case of the cyclic subgroup of order 3 provided $q' = 2$, the case of D_{10} provided $q' = 4$ and the case of $E_{2^n} : (2^n - 1)$.

For each of these G_{01} we look for the various possible groups H; the case of $E_{2^n} : (2^n - 1)$ is treated in the following Lemma, whose proof is left to the reader. The proof of proposition 5.12 may be found in the Appendix (pg. 13-14).

Lemma 5.11. *Assume $q = 2^{nm}$ with m prime and $n \neq 1$ and let $G \cong \mathrm{PSL}(2,q)$. If H is a subgroup of G such that $(H, E_{2^n} : 2^n - 1)$ is a two-transitive pair then one of the two following statements holds: $H \cong E_{2^{2n}} : 2^n - 1$ provided $m = 2$ or $H \cong \mathrm{PSL}(2, 2^n)$.*

Notice that if $n = 2$, $\mathrm{PSL}(2, 2^n) \cong A_5$.

Proposition 5.12. *Assume $q' = 2^n$ and $q = q'^m = 2^{n.m}$ for m prime ; moreover for $n = 1$ we need $m = 2$. Let $G \cong \mathrm{PSL}(2, 2^{n.m})$. Every RWPRI and $(2T)_1$ geometry of rank two $\Gamma(G; G_0, G_1, G_0 \cap G_1)$ in which $G_0 \cong \mathrm{PSL}(2, q')$ is isomorphic to one of the geometries appearing in Table 4.*

5.4.2 The case q odd

Since q is odd we need to consider two distinct maximal subgroups which are $\mathrm{PSL}(2, p^n)$ provided $q = p^{mn}$ where m and p are odd primes and $\mathrm{PGL}(2, p^n)$ provided $q = p^{2n}$ where p is an odd prime. The latter will be treated in section 5.5.

Recall that following Table 8, the subgroup $\mathrm{PSL}(2, p^n)$ is maximal in $\mathrm{PSL}(2, q)$ provided $q = p^{mn}$ with m and p odd primes. In this section we assume these conditions on q.

In view of (5)-(10) in Proposition 4.6 there are four possibilities for G_{01}, namely: A_4 provided $q' = 5$, S_4 provided $q' = 7$, A_5 provided $q' = 9, 11$ and $E_{q'} : \frac{q'-1}{2}$. For each of these G_{01} we look for the various possible groups H in the three following Lemmas, whose proofs are left to the reader. The proof of proposition 5.16 may be found in the Appendix (pg. 14-17). The case of A_4 provided $q' = 5$, will be treated directly in the proof of the Proposition.

Lemma 5.13. *Assume q odd, $q = p^{nm}$ with m prime and let $G \cong \mathrm{PSL}(2,q)$; then the following statement holds: If H is a subgroup of G such that $\left(H, E_{p^n} : \frac{p^n-1}{2}\right)$ is a two-transitive pair then $H \cong \mathrm{PSL}(2,p^n)$.*

Notice that if $p^n = 3$, $\mathrm{PSL}(2,p^n) \cong A_4$ and if $p^n = 5$, $\mathrm{PSL}(2,p^n) \cong A_5$. They are particular cases of $\mathrm{PSL}(2,p^n)$.

Lemma 5.14. *Assume q is either 11^m or 9^m, with m an odd prime and let $G \cong \mathrm{PSL}(2,q)$. Then the following statement holds: If H is a subgroup of G such that (H, A_5) is a two-transitive pair then $H \cong \mathrm{PSL}(2,q')$ provided $q' = 9$ or 11.*

Lemma 5.15. *Assume $q = 7^m$, with m odd prime and let $G \cong \mathrm{PSL}(2,7^m)$. Then the following statement holds: If H is a subgroup of G such that (H, S_4) is a two-transitive pair then $H \cong \mathrm{PSL}(2,7)$.*

Proposition 5.16. *Assume $q = p^{nm}$ with p and m odd primes and let $G \cong \mathrm{PSL}(2,q)$. Every RWPRI and $(2T)_1$ geometry of rank two $\Gamma(G; G_0, G_1, G_0 \cap G_1)$ in which $G_0 \cong \mathrm{PSL}(2,p^n)$ is isomorphic to one of the geometries appearing in Table 5.*

5.5 The case where $G_0 = \mathrm{PGL}(2,q')$

If q is even, $\mathrm{PGL}(2,q') \cong \mathrm{PSL}(2,q')$ and this situation has been treated in Section 5.4. Therefore, we assume in this section that q is odd. Recall that following Table 8, the subgroup $\mathrm{PGL}(2,q')$ is maximal in $\mathrm{PSL}(2,q)$ provided $q' = p^n$ and $q = q'^2 = p^{2n}$ with p an odd prime.

In view of (11), (12), (13) and (20) in Proposition 4.6 there are four cases for G_{01}, namely the case of $E_{p^n} : (p^n - 1)$, the case of $\mathrm{PSL}(2,q')$, the case of D_8 provided $q = 3^2$ and the case of S_4 provided $q = 5^2$.

For each of these four G_{01} we look for the various possible groups H in one of the four following Lemmas, whose proofs are left to the reader. The proof of proposition 5.21 may be found in the Appendix (pg. 17-19).

Lemma 5.17. *Let $G \cong \mathrm{PSL}(2,3^2)$. Then the following statement holds:
If H is a subgroup of G such that (H, D_8) is a two-transitive pair then $H \cong \mathrm{PGL}(2,3)$.*

Lemma 5.18. *Assume q is odd and let $G \cong \mathrm{PSL}(2,p^{2n})$. Then the following statement holds:
If H is a subgroup of G such that $(H, E_{p^n} : (p^n - 1))$ is a two-transitive pair then $H \cong E_{p^{2n}} : (p^n - 1)$ or $H \cong \mathrm{PGL}(2,p^n)$.*

Lemma 5.19. *Assume q is odd and let $G \cong \mathrm{PSL}(2,p^{2n})$. Then the following statement holds:
If H is a subgroup of G such that $(H, \mathrm{PSL}(2,p^n))$ is a two-transitive pair then $H \cong A_5$ provided $p^n = 3$; or $H \cong \mathrm{PGL}(2,p^n)$.*

Notice that if $p^n = 3$, $\mathrm{PGL}(2,p^n) \cong S_4$.

Lemma 5.20. *Let $G \cong \mathrm{PSL}(2,5^2)$. Then the following statement holds:
If H is a subgroup of G such that (H, S_4) is a two-transitive pair then $H \cong \mathrm{PGL}(2,5)$.*

Proposition 5.21. *Assume $q' = p^n$ and $q = q'^2 = p^{2n}$ with p an odd prime. Let $G \cong \mathrm{PSL}(2,q)$. Every RWPRI and $(2T)_1$ geometry of rank two $\Gamma(G; G_0, G_1, G_0 \cap G_1)$ in which $G_0 \cong \mathrm{PGL}(2,q')$ is isomorphic to one of the geometries appearing in Table 6.*

The proof of Theorem 1.1 readily follows from Propositions 5.6, 5.10, 5.5, 5.12, 5.16 and 5.21.

The main Theorem of [5] and Theorem 1.1 complete the classification of rank two residually weakly primitive and locally two-transitive coset geometries for the groups $\mathrm{PSL}(2,q)$. We also give the number of classes of all such geometries with respect to conjugacy and isomorphism.

This classification includes infinite classes of geometries up to conjugacy and up to isomorphism. This number is dependent on the prime power $q = p^n$; it is a function of n and p.

6 Locally s-arc-transitive graphs

The construction of the $(G,2)$-arc-transitive graphs, using Tits' Theorem, is studied in full detail in Leemans [12]. This construction shows that the rank two incidence structures are also locally-2-arc-transitive graphs in the sense of [8].

All the RWPRI and $(2T)_1$ geometries we have obtained are bipartite graphs and also locally 2-arc-transitive graphs. Now we want the value of s such that the incidence graph of Γ is a locally s-arc-transitive but not a locally $(s+1)$-arc-transitive graph. We mainly use the method of D. Leemans [12] (Lemma 5.1). This provides the value of s in all cases given in Tables 1, 2, 3, 4, 5 and 6 (in the introduction) except those listed in Table 9. We don't give the details in the cases for which the Leemans' method works.

We now discuss the nine cases left over in Table 9. In every case if p is a vertex of the graph, we write $p^\perp$ for the set of neighbours of p which is also the residue of p.
We give the details for four of them, the other five are dealt with in the Appendix (pg. 19-20).

Case of Table 1, geometry Γ_1, case of Table 1, geometries Γ_6 and Γ_{13} and case of Table 4, geometry Γ_5.

We know that $s \geq 2$. Consider a path (a,b,c) such that a is of type 0, b is of type 1, c is of type 0. Here, $G_{abc} = Z_5$. This acts on the five 1-elements $d_1, ..., d_5$ other than b in $c^\perp$. The action is transitive since otherwise Z_5 would be in the kernel of the action of G_c on $c^\perp$ contradicting the simplicity of $G_0 = A_5 = G_c$. This provides $s \geq 3$ for paths starting at a $0-element$.

Next consider a path (h,i,j) such that h is of type 0, i is of type 1, j is of type 0. Here, $G_{hij} = Z_2$. This acts on the two 0-elements k_1, k_2 other than i in $j^\perp$. The action is transitive since otherwise Z_2 would be in the kernel of the action of G_j on $j^\perp$. This kernel for the action of D_{30} on the cosets of D_{10} is a group Z_5, a contradiction. Hence $s \geq 3$.

Applying Leemans' method we get $s = 2$ or 3. Thus $s = 3$.

Case of Table 3, geometry Γ_6.

We know that $s \geq 2$. Consider a path (a,b,c) as in the preceding case. Here, $G_{abc} = Z_4$. This acts on the two 1-elements d_1, d_2 other than b in $c^\perp$. The action is transitive since otherwise Z_4 would be in the kernel of the action of G_c on $c^\perp$. This kernel for the action

	$G_0 \cong A_5$	
G_{01}	G_1	
D_{10}	D_{30}	Table 1, Γ_1
D_{10}	D_{30}	Table 1, Γ_6 and Γ_{13}
	$G_0 \cong S_4$	
G_{01}	G_1	
D_6	D_{18}	Table 3, Γ_2
D_8	D_{16}	Table 3, Γ_5
D_8	D_{24}	Table 3, Γ_6
D_8	S_4	Table 3, Γ_7 and Γ_8
	$G_0 \cong \mathrm{PSL}(2,2^n)$	
G_{01}	G_1	
$E_{2^n} : (2^n-1)$	$E_{2^{mn}} : (2^n-1)$	Table 4, Γ_1
D_{10}	D_{30}	Table 4, Γ_5
	$G_0 \cong \mathrm{PGL}(2,p^n)$	
G_{01}	G_1	
$E_{p^n} : (p^n-1)$	$E_{p^{2n}} : (p^n-1)$	Table 6, Γ_1

Table 9: Cases in which s cannot be decided by Leemans' method.

of S_4 on the cosets of D_8 is 2^2, a contradiction. This provides $s \geq 3$ for paths starting at a $0-element$.

Next consider a path (h, i, j) as in the preceding case. Here, $G_{hij} = 2^2$. This acts on the two 0-elements k_1, k_2 other than i in $j^{\perp}$. The action is transitive since otherwise 2^2 would be in the kernel of the action of G_j on $j^{\perp}$. This kernel for the action D_{18} on the cosets of D_8 is a group Z_4, a contradiction. Hence $s \geq 3$. Applying Leemans' method we get that s equals 3 or 4.

We now prove that s cannot be equal to 4 thanks to the following argument due to an unknown referee: Given the path (a, b, c) starting at a 0-element we have shown that $G_{abc} = Z_4$ and that this is transitive on the two elements adjacent to c other than b. Thus $G_{abcd} = Z_2 = \langle x \rangle$, where x is the square of an element of order 4 in $G_{abc} < G_a \cong S_4$. Thus x lies in the normal subgroup of G_a of order 4 and so acts trivially on the set of neighbours of a. Thus G_{dcba} is not transitive on the set of 4-arcs starting with (d, c, b, a) and so the graph is not locally 4-arc transitive. Hence $s = 3$.

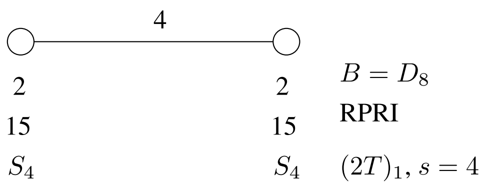
Let us make some observations on the results: In Tables 1, 2, 3, 4, 5, 6 and 9 most values are $s = 2$ or $s = 3$. There are some spectacular examples with larger values of s. Indeed we obtain a locally 4-arc transitive graph and a locally 7-arc transitive graph which are respectively

$$\Gamma\left(\mathrm{PSL}(2,q); S_4, S_4, D_8\right) \text{ due to Biggs-Hoare [1]}$$

$$\text{and} \quad \Gamma\left(\mathrm{PSL}(2,q); D_{16}, S_4, D_8\right) \text{ due to Wong [22]}$$

These examples also appear in Li [14].

However, let us pay more attention to the case $q = 9$. Here we are dealing with a geometry whose Buekenhout diagram is given by



This is the smallest thick generalised quadrangle. Its origin is the symplectic group $Sp_4(2)$; in that context it is known at least from [17]. It is also famous as Tutte's 8-cage [18]. Its incidence graph admits an automorphism group four times as big as group $\mathrm{PSL}(2, 9)$ which is $P\Gamma L(2, 9)$. Under the action of this group we check that the graph is actually 5-arc-transitive and this is also provided by Tutte.

Moreover, for the cases in which $q = 17, 23, 31, 41, 47, 71, 73, 79, 89$ the full automorphism group of the incidence graph is the group $\mathrm{PGL}(2, q)$. This group has a unique conjugacy class of subgroups S_4, according to E.H. Moore as we see in [4]. Thus $\mathrm{PGL}(2, q)$ fuses the two classes of S_4 in $\mathrm{PSL}(2, q)$ and so it cannot provide 5-arc-transitivity. Finally, for the case $\Gamma\left(\mathrm{PSL}(2, q); D_{16}, S_4, D_8\right)$ for $q = 17, 31, 79, 97$, there are two classes of S_4 in $\mathrm{PSL}(2, q)$ that are fused in $\mathrm{PGL}(2, q)$. There are two such geometries for each value of q and so the full automorphism group of Γ is $\mathrm{PSL}(2, q)$. (see Proposition 5.10).

7 Appendix

The Appendix contains details for several results of this paper, except the proofs of Lemmas 5.7, 5.8, 5.9, 5.11, 5.13, 5.14, 5.15, 5.17, 5.18, 5.19, 5.20 which are left to the reader. Appendix is available on-line at: `http://amc-journal.eu/index.php/amc/issue/view/17`.

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