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7 How Far has so Far the Spin-Charge-Family Theory Succeeded To explain the Standard Model Assumptions, the Matter-Antimatter Asymmetry, the Appearance of the Dark Matter, the Second Quantized Fermion Fields. . . , Making Several Predictions

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Abstract. The assumptions of the *standard model*, which 50 years ago offered an elegant new step towards understanding basic fermion and boson fields, are still waiting for an explanation. The *spin-charge-family* theory is promising not only in explaining the *standard model* postulates but also in explaining the cosmological observations, like there are the appearance of the *dark matter*, of the *matter-antimatter asymmetry*, making several predictions. This theory assumes that the internal degrees of freedom of fermions (spins, handedness and all the charges) are described by the Clifford algebra objects in $d \geq (13+1)$-dimensional space. Fermions interact with only the gravity (the vielbeins and the two kinds of the spin connection fields, which manifest in $d = (3+1)$ as all the vector gauge fields as well as the scalar fields - the higgs and the Yukawa couplings). The theory describes the internal space of fermions with the Clifford objects which are products of odd numbers of γ^a objects, what offers the explanation for quantum numbers of quarks and leptons and anti-quarks and ani-leptons, with family included. In this talk I overview shortly the achievements of the *spin-charge-family* theory so far and in particular the explanation of the second quantization procedure offered by the description of the internal space of fermions with the anticommuting Clifford algebra objects of the odd character. The theory needs still to answer many open questions that it could be accepted as the next step beyond the *standard model*.

Povzetek. Privzetki *Standardnega Modela*, ki je pred 50 leti ponudil eleganten opis osnovnih fermionskh in bozonskih polj, so še vedno nepojasnjeni. *Teorija spinov-nabojev-družin* ponuja, poleg razlage privzetkov *Standardnega Modela*, tudi razlago nekaterih kozmoloških opažanj, kot je pojav *temne snovi*, *asimetrije snovi in antisnovi*, ponudi pa tudi več napovedi. Teorija privzame, da so notranje prostostne stopnje fermionov (spin, ročnost in vsi naboji) opisane z objekti Cliffordove algebre v prostoru z razsežnostjo $d \geq (13+1)$. Fermioni interagirajo samo z gravitacijskim poljem (s tetradami in dvema vrstama spinskih povezav), ki se v prostoru $d = (3+1)$ predstavi kot običajna gravitacija, kot vsa poznana vektorska umeritvena polja ter kot skalarna umeritvena polja, ki pojacnijo pojav Higgsovega skalarja in Yukawinih sklopitev. Notranje prostostne stopnje fermionov opisuje avtorica teorije s Cliffordovo algebro, ki ponudi razumevanje privzetkov za lastnosti kvarkov in leptonov in njihovih družin, v *Standardnem Modelu*. V predavanju avtorica na kratko predstavi dosedanje dosežke *Teorije spinov-nabojev-družin*, napovedi teorije ter tudi odprta vprašanja.

Poudarek predavanja je na ponudbi drugačne poti do druge kvantizacije fermionov kot je splošno privzeta Diracova. Opis notranjega prostora fermionov z objekti, ki antikomutirajo, pojasni antikomutacjske lastnosti fermionov v drugi kvantizaciji. Predstavi tudi odprta vprašanja, ki jih je potrebno rešiti, da bo teorija lahko sprejeta kot nov korak k razumevanju vesolja in osnovnih gradnikov vesolja.

Keywords: Beyond the standard model, Gravity as the only gauge fields, Kaluza-Klein-like theories, Higher dimensional spaces, Dark matter, Matter/antimatter asymmetry, Four families of quarks and leptons, Second quantization of fermion fields in Clifford and in Grassmann space, Explanation of the Dirac postulates

7.1 Introduction

Let us start with the motivation for the *spin-charge-family* theory.

The *standard model* offered an elegant new step towards understanding elementary fermion and boson fields by postulating (the inspiration came from the experiments):

a. The existence of massless fermion family members with the spins and charges in the fundamental representation of the groups, **a.i.** the quarks as colour triplets and colouress leptons, **a.ii** the left handed members as the weak doublets, **a.ii.** the right handed weak chargeless members, **a.iii.** the left handed quarks differing from the right handed leptons in the hyper charge, **a.iv.** all the right handed members differing among themselves in hyper charges, **a.v.** antifermions carry the corresponding anticharges of fermions and opposite handedness, **a.vi.** the number of massless families, determined by experiments (there is no right handed neutrino postulated, since it would carry none of the so far observed charges, and correspondingly there is also no left handed antineutrino allowed).

b. The existence of massless vector gauge fields to the observed charges of quarks and leptons, carrying charges in the adjoint representations of the corresponding charged groups.

c. The existence of the massive weak doublet scalar higgs, **c.i.** carrying the weak charge $\pm\frac{1}{2}$ and the hypercharge $\pm\frac{1}{2}$ (as it would be in the fundamental representation of the two groups), **c.ii.** gaining at some step of the expanding universe the nonzero vacuum expectation value, **c.iii.** breaking the weak and the hyper charge and correspondingly breaking the mass protection, **c.iv.** taking care of the properties of fermions and of the weak bosons masses, **c.v.** as well as of the Yukawa couplings.

d. The presentation of fermions and bosons as second quantized fields.

e. The gravitational field in $d = (3 + 1)$ as independent gauge field.

The *standard model* assumptions have been confirmed without raising any doubts so far, but also by offering no explanations for the assumptions. The last among the fields postulated by the *standard model*, the scalar higgs, was detected in June 2012, the gravitational waves were detected in February 2016.

The *standard model* has in the literature several explanations, mostly with many new not explained assumptions. The most popular seem to be the grand unifying theories [14–30]. At least $SO(10)$ offers the explanation for the potulates from **a.i.** to **a.iv**, partly to **b.** — but does not explain the assumptions **a.v.** up to **a.vi.**, **c.** and **d.**, and does not connect gravity with gauge vector and scalar fields.

What questions should one ask to be able to find a trustworthy next step beyond the *standard models* of elementary particle physics and cosmology, which would offer understanding of not yet understood phenomena?

i. Where do fermions, quarks and leptons, originate and why do they differ from the boson fields in spins, charges and statistics?

ii. How can one describe the internal degrees of fermions to explain the Dirac's postulates of the second quantization?

iii. Why are charges of quarks and leptons so different, why have the left handed family members so different charges from the right handed ones and why does the handedness relate charges to anticharges?

iv. Where do families of quarks and leptons originate and how many families do exist?

v. Why do family members – quarks and leptons — manifest so different masses if they all start as massless?

vi. How is the origin of the scalar field (the Higgs's scalar) and the Yukawa couplings connected with the origin of families and how many scalar fields determine properties of the so far (and others possibly be) observed fermions and masses of weak bosons? (The Yukawa couplings certainly speak for the existence of several scalar fields with the properties of Higgs's scalar.) Why is the Higgs's scalar, or are all scalar fields of similar properties as the higgs, if there are several, doublets with respect to the weak and the hyper charge? Do possibly exist also scalar fields with the colour charges in the fundamental representation and where, if they are, do they manifest?

vii. Where do the so far observed (and others possibly non observed) vector gauge fields originate? Do they have anything in common with the scalar fields and the gravitational fields?

viii. Where does the *dark matter* originate?

ix. Where does the "ordinary" matter-antimatter asymmetry originate?

x. Where does the dark energy originate and why is it so small?

xi. What is the dimension of space? $(3+1)$?, $((d-1)+1)$?, ∞?

And many others.

My working hypotheses is that a trustworthy next step must offer answers to several open questions, the more answers to the above open questions the step covers the greater the possibilities of the theory being the right next step.

I am proposing the *spin-charge-family* theory [1–10], offering so far the answers from **i.** to **ix.** of the above questions; The more work is invested in this theory the more answers to the above open questions the theory offers.

Let me make in what follows a short introduction into the *spin-charge-family* theory to show briefly up the way the theory is offering the answers to the above mentioned open questions. A more detailed presentation of the theory and its achievements are presented in Sect. 7.2.

The *spin-charge-family* theory is a kind of the Kaluza-Klein like theories [8, 31–38] due to the assumption that in $d \geq 5$ — in the *spin-charge-family* theory

$d \geq (13+1)$ — fermions interact with the gravity only [1], treating consequently all the vector gauge fields, the scalar gauge fields, and the gravity in an equivalent way, offering answers to the above questions **vi.** and **vii.**.

In the *spin-charge-family* theory the fermion internal space is described by the "basis vectors", which are the superposition of the odd products of the Clifford algebra objects. There are two kinds of the Clifford algebra objects [1,2,12,45,46]. In $d = (13+1)$-dimensional space the odd Clifford algebra objects of one kind offer in $d = (3+1)$ the description of the spins and all the charges of fermions and antifermions, since both — fermions and antifermions — appear in the same irreducible representation of one of the two Lorentz groups in the internal space of fermions, what consequently explains the connection among the spins, handedness and charges of fermions, answering the questions **i.** and **iii.**.

The other kind takes care of the family quantum numbers of fermions, distinguishing among different irreducible representations [3,4,7,9], and offering a part answer to **iv.**.

The creation operators, creating the single particle states, are tensor products of the superposition of the finite number of the Clifford odd "basis vectors" of the internal space and of the infinite basis in the momentum space. The "basis vectors" of the internal space transfer their oddness to the creation operators and correspondingly guarantees the oddness of the single fermion states, since the vacuum state has an even Clifford character.

The Hilbert space of fermions is formed from all possible tensor products of any number of single fermion creation operators, operating on the vacuum state [12].

The *spin-charge-family* theory offers correspondingly answers to the questions from **i.** to **iv.**, explaining the common origin of spins and charges of fermions and antifermions, of all the quantum numbers of quarks and leptons and antiquarks and antileptons postulated by the *standard model*, as well as of the origin of families. The theory explains as well the Dirac postulates of the second quantization of the ferrmion fields.

Fermions interact with the vielbeins and the two kinds of the spin connection fields, the gauge fields of the momenta and of the two kinds of the generators of the Lorentz transformations, determined by the two kinds of the Clifford algebra objects [3–10,12].

The spin connection fields of one kind manifest in $d = (3+1)$ as the vector gauge fields of the charges of fermions, as the gravitational fields and also as the scalar gauge fields [5], to which also the scalar fields which are the gauge field of the second kind of the spin connection fields contribute. These offer answers to the questions **vi.** and **vii.**, while explaining the common origin of the gravity, the vector gauge fields of the charges and the scalar gauge fields. The scalar gauge fields of

[1] Correspondingly the *spin-charge-family* theory shares with the Kaluza-Klein like theories their weak points, at least: **a.** Not yet solved the quantization problem of the gravitational field. **b.** The spontaneous break of the starting symmetry, which would at low energies manifest the observed almost massless fermions [32]. Concerning this second point we proved on the toy model of $d = (5+1)$ that the break of symmetry can lead to (almost) massless fermions [68–70].

both origins — of both generators of the Lorentz transformations in internal space of fermions — determine the scalar higgs and the Yukawa couplings, which all are in the *standard model* postulated.

The two kinds of the Clifford algebra objects require the existence of the two groups of four families of quarks and leptons and antiquarks and antileptons. The two groups distinguish from each other with respect to the family quantum numbers and correspondingly with respect to the interaction with the different two groups of the scalar gauge fields, which determine masses of these two groups of families after the break of the weak and hyper charge symmetries. Consequently: **a.** To the observed three families of quarks and leptons and antiquarks and antileptons there must exist the fourth family [3,9,49,51,53,54]. **b.** The second group of the four families offers the explanation for the existence of the *d*ark matter [52,61].

The quantum numbers of the weak charge and the hyper charge of the scalar fields, taking care of the masses of the two groups of four families, depend on the space index of the scalar fields. The scalar fields with the space indexes 7 and 8 do carry the weak and the hyper charge as assumed by the *standard model*, explaining the origin of scalar higgs and Yukawa couplings [3,9,49,51,53,54], what adds the explanation to the question **vi.**.

There appear in the *spin-charge-family* the scalar fields with the space indexes $9-14$, which are the colour triplets [4,61]. They cause the transitions of antiquarks and antileptons into quarks and back. In the expanding universe under the non equilibrium conditions they offer the explanation of today's dominance of ordinary matter in the observed part of the universe.

It remains to tell how does in the *spin-charge-family* appear the spontaneous breaking of the starting symmetry in $d = (13+1)$, first with the appearance of the condensate of two right handed neutrinos [3,4,9], and then when scalar fields with space index $(7,8)$ obtain nonzero vacuum expectation values.

The detailed, although still short, presentation of the *spin-charge-family* theory is presented in Sects. 7.2and 7.2.1.

7.2 Short presentation of the *spin-charge-family* theory

The *spin-charge-family* theory assumes a simple starting action for fermions, coupled to only gravitational field in $d \geq (13+1)$-dimensional space through the vielbeins $f^{\alpha}{}_{a}$, the gauge fields of momenta, and the two kinds of the spin connection fields, $\omega_{ab\alpha}$ and $\tilde{\omega}_{ab\alpha}$, the gauge fields of the two kinds of the generators of the Lorentz transformations of the Clifford algebras, and with the internal space of fermions described by the anticommuting "basis vectors" of one of the two

Clifford algebras

$$
\begin{aligned}
\mathcal{A} &= \int d^d x \; E \; \frac{1}{2} \, (\bar{\psi} \, \gamma^a p_{0a} \psi) + h.c. + \\
&\quad \int d^d x \; E \; (\alpha \, R + \tilde{\alpha} \, \tilde{R}) \,, \\
p_{0a} &= f^{\alpha}{}_a p_{0\alpha} + \frac{1}{2E} \, \{p_\alpha, E f^{\alpha}{}_a\}_- \,, \\
p_{0\alpha} &= p_\alpha - \frac{1}{2} S^{ab} \omega_{ab\alpha} - \frac{1}{2} \tilde{S}^{ab} \tilde{\omega}_{ab\alpha} \,, \\
R &= \frac{1}{2} \, \{f^{\alpha[a} f^{\beta b]} \; (\omega_{ab\alpha,\beta} - \omega_{ca\alpha} \, \omega^c{}_{b\beta})\} + h.c. \,, \\
\tilde{R} &= \frac{1}{2} \, \{f^{\alpha[a} f^{\beta b]} \; (\tilde{\omega}_{ab\alpha,\beta} - \tilde{\omega}_{ca\alpha} \, \tilde{\omega}^c{}_{b\beta})\} + h.c. \,. \qquad (7.1)
\end{aligned}
$$

Here [2] $f^{\alpha[a} f^{\beta b]} = f^{\alpha a} f^{\beta b} - f^{\alpha b} f^{\beta a}$.

As written in the introduction, the tensor products of the superposition of the finite number of anticommuting "basis vectors" and of the infinite basis in the momentum space offer the description of the fermion creation and annihilation anticommuting operators. The creation and annihilation operators explain the Dirac postulates of the second quantized fermions, Sect. (7.2.1,7.2.1, 7.2.1).

The single fermion states manifest in $d = (3+1)$ space the spins and all the charges of the observed quarks and leptons and antiquarks and antileptons, Table 7.3, as well as families, Table 7.4, predicting the fourth family [49–51,53,54, 57,58] to the observed three families and offering the explanation for the observed *dark matter* [52,61].

The spin connection gauge fields manifest in $d = (3+1)$ as the ordinary gravity, the known vector gauge fields, the scalar gauge fields [5] with the properties of higgs explaining the higgses and the Yukawa couplings, predicting new vector and scalar fields, which offer explanation for the *dark matter* [52] and for *matter/antimatter asymmetry* [4].

To be in agreement with the observations in $d = (3+1)$ the manifold $M^{(13+1)}$ must break first into $M^{(7+1)} \times M^{(6)}$ (which manifests as $SO(7,1) \times SU(3) \times U(1)$), affecting both internal degrees of freedom - the one represented by γ^a and the one represented by $\tilde{\gamma}^a$ [3].

There is a scalar condensate (Table 7.5) of two right handed neutrinos with the family quantum numbers of the group of four families (the one which does not include the observed three families), Table 7.4, which bring masses of the scale $\propto 10^{16}$ GeV or higher to all the vector and scalar gauge fields, which interact with the condensate [4].

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[2] $f^{\alpha}{}_a$ are inverted vielbeins to $e^a{}_\alpha$ with the properties $e^a{}_\alpha f^\alpha{}_b = \delta^a{}_b$, $e^a{}_\alpha f^\beta{}_a = \delta^\beta_\alpha$, $E = \det(e^a{}_\alpha)$. Latin indices $a, b, .., m, n, .., s, t, ..$ denote a tangent space (a flat index), while Greek indices $\alpha, \beta, .., \mu, \nu, ..\sigma, \tau, ..$ denote an Einstein index (a curved index). Letters from the beginning of both the alphabets indicate a general index ($a, b, c, ..$ and $\alpha, \beta, \gamma, ..$), from the middle of both the alphabets the observed dimensions $0, 1, 2, 3$ ($m, n, ..$ and $\mu, \nu, ..$), indexes from the bottom of the alphabets indicate the compactified dimensions ($s, t, ..$ and $\sigma, \tau, ..$). We assume the signature $\eta^{ab} = diag\{1, -1, -1, \cdots, -1\}$.

Since the left handed spinors couple differently (with respect to $M^{(7+1)}$) to scalar fields than the right handed ones, the break can leave massless and mass protected $2^{((7+1)/2-1)}$ families [68]. The rest of families get heavy masses [3].

There is additional breaking of symmetry: The manifold $M^{(7+1)}$ breaks further to $M^{(3+1)}\times SU(2)\times SU(2)$ included in $M^{(4)}$. These electroweak break is caused by the scalar fields with the space index $(7,8)$. They carry due to the space index the weak charge and hyper charge [3,4].

I shall shortly present the influence of the breaks with the condensate and with the scalar fields (the electroweak break) when presenting properties of fermions and vector and scalar gauge fields in $d=(3+1)$.

7.2.1 Properties of fermion fields in the *spin-charge-family* theory

Let us start with the properties of the fermion fields in the *spin-charge-family* theory.

Fermion fields, which are the superposition of tensor products of the anticommuting "basis vectors" describing fermions internal degrees of freedom and of commuting basis in the momentum (coordinate) space, manifest the anticommuting properties already on the single fermion level [13], demonstrating that the first quantized fermions are the approximation to the second quantized fields.

There are two kinds of the anticommuting objects [1–3,9,12] — the Grassmann coordinates and correspondingly the Grassmann operators, θ^a and $\frac{\partial}{\partial\theta_a}$, and the Clifford coordinates/operators, γ^a and $\tilde{\gamma}^a$, expressible with one another. Either the Grassmann or the two Clifford algebras offer in d-dimensional space $2\cdot 2^d$ operators (the Grassmann algebra has 2^d-1 products of θ^a's and 2^d-1 products of $\frac{\partial}{\partial\theta_a}$'s and the identity, the two Clifford algebras have each 2^d-1 products of γ^a' and 2^d-1 products of $\tilde{\gamma}^a$'s and the identity) with the properties [12,13]

$$
\begin{aligned}
&\{\theta^a,\theta^b\}_+=0\,,\quad \{\frac{\partial}{\partial\theta_a},\frac{\partial}{\partial\theta_b}\}_+=0\,,\quad \{\theta_a,\frac{\partial}{\partial\theta_b}\}_+=\delta_{ab}\,,\\
&(\theta^a)^\dagger=\eta^{aa}\frac{\partial}{\partial\theta_a}\,,\quad (\frac{\partial}{\partial\theta_a})^\dagger=\eta^{aa}\theta^a\,,\\
&\{\gamma^a,\gamma^b\}_+=2\eta^{ab}=\{\tilde{\gamma}^a,\tilde{\gamma}^b\}_+\,,\quad \{\gamma^a,\tilde{\gamma}^b\}_+=0\,,\\
&(\gamma^a)^\dagger=\eta^{aa}\gamma^a\,,\quad (\tilde{\gamma}^a)^\dagger=\eta^{aa}\tilde{\gamma}^a\,,\\
&(a,b)=(0,1,2,3,5,\cdots,d)\,. \qquad (7.2)
\end{aligned}
$$

The identity is the self adjoint member. The signature $\eta^{ab}=diag\{1,-1,-1,\cdots,-1\}$ is assumed.

The two algebras are expressible with one another

[3] A toy model [68,69] was studied in $d=(5+1)$ with the same action as in Eq. (7.1). The break from $d=(5+1)$ to $d=(3+1)\times$ an almost S^2 was studied. For a particular choice of vielbeins and for a class of spin connection fields the manifold $M^{(5+1)}$ breaks into $M^{(3+1)}$ times an almost S^2, while $2^{((3+1)/2-1)}$ families remain massless and mass protected. Equivalent assumption, although not yet proved how does it really work, is made in the $d=(13+1)$ case. This study is in progress quite some time.

$$\begin{aligned}
\gamma^a &= (\theta^a + \frac{\partial}{\partial\theta_a})\,, \quad \tilde{\gamma}^a = i\,(\theta^a - \frac{\partial}{\partial\theta_a})\,,\\
\theta^a &= \frac{1}{2}\,(\gamma^a - i\tilde{\gamma}^a)\,, \quad \frac{\partial}{\partial\theta_a} = \frac{1}{2}\,(\gamma^a + i\tilde{\gamma}^a)\,.
\end{aligned} \tag{7.3}$$

Let me add the generators of the Lorentz transformations in both algebras

$$\begin{aligned}
\mathbf{S}^{ab} &= i\,(\theta^a \frac{\partial}{\partial\theta_b} - \theta^b \frac{\partial}{\partial\theta_a})\,, \quad (\mathbf{S}^{ab})^\dagger = \eta^{aa}\eta^{bb}\mathbf{S}^{ab}\,,\\
S^{ab} &= \frac{i}{4}(\gamma^a\gamma^b - \gamma^b\gamma^a)\,, \quad \tilde{S}^{ab} = \frac{i}{4}(\tilde{\gamma}^a\tilde{\gamma}^b - \tilde{\gamma}^b\tilde{\gamma}^a)\,,\\
\mathbf{S}^{ab} &= S^{ab} + \tilde{S}^{ab}\,, \quad \{S^{ab}, \tilde{S}^{ab}\}_- = 0\,,\\
\{S^{ab}, \gamma^c\}_- &= i(\eta^{bc}\gamma^a - \eta^{ac}\gamma^b)\,, \quad \{S^{ab}, \tilde{\gamma}^c\}_- = 0\,,\\
\{\tilde{S}^{ab}, \tilde{\gamma}^c\}_- &= i(\eta^{bc}\tilde{\gamma}^a - \eta^{ac}\tilde{\gamma}^b)\,, \quad \{\tilde{S}^{ab}, \gamma^c\}_- = 0\,,
\end{aligned} \tag{7.4}$$

The Grassmann algebra offers the description of the integer spin fermions, with the charges in the adjoint representations. Both Clifford algebras offer the description of the half integer spin fermions with charges in the fundamental representations. Both algebras, the Grassmann algebra and the two Clifford algebras, can be separated into odd and even parts with odd and even products of algebra elements.

While in the Grassmann algebra the Hermitian conjugated partners of products of θ^a's are the corresponding products of $\frac{\partial}{\partial\theta^a}$'s, Eq. (7.2), and opposite, in the Clifford algebras the Hermitian conjugated partners are less transparent, due to the relations $\gamma^{a\dagger} = \eta^{aa}\gamma^a$ and $\tilde{\gamma}^{a\dagger} = \eta^{aa}\tilde{\gamma}^a$, Eq. (7.2).

In order to resolve the problem of the Hermitian conjugated partners in the Clifford case and also to be able to make predictions of the theory to be compared with the experimental results, let us arrange products of θ^a's as well as products of either γ^a's or $\tilde{\gamma}^a$'s into irreducible representations with respect to the Lorentz group with the generators [2] presented in Eq. (7.4) and to arrange the members of each irreducible representation to be eigenstates of the Cartan subalgebra

$$\begin{aligned}
&\mathbf{S}^{03}, \mathcal{S}^{12}, \mathcal{S}^{56}, \cdots, \mathcal{S}^{d-1\;d}\,,\\
&S^{03}, S^{12}, S^{56}, \cdots, S^{d-1\;d}\,,\\
&\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56}, \cdots, \tilde{S}^{d-1\;d}\,.
\end{aligned} \tag{7.5}$$

The easiest way to achieve this is to find the eigenstates of each member of the Cartan subalgebras separately.

The observed fermions have the half integer spin and charges in the fundamental representations, and there are no fermions observed yet with the integer spins and charges in the adjoint representations. The *spin-charge-family* theory must correspondingly use the Clifford algebras. However, there are also no experimental evidences that there is any need for two independent representations offered by the two kinds of the Clifford algebra objects, γ^a's and $\tilde{\gamma}^a$'s.

Let us therefore start the discussion about the description of the internal space of fermions by taking into account the two Clifford algebras and let us leave the discussion on the Grassmann algebra for later, Ref. [13].

We can make a choice for the members of the irreducible representations of the two Lorentz groups to be the "eigenvectors" of the corresponding Cartan subalgebras of Eq. (7.5), taking into account Eq. (7.2). If S^{ab} and $\tilde{S}^{ab}$ represents each one of the ($\frac{d}{2}$ for even d) members of the Cartan subalgebra elements, we easily check that

$$\begin{aligned}
S^{ab} \stackrel{ab}{(k)} &= \frac{k}{2} \stackrel{ab}{(k)}\,, \quad \stackrel{ab}{(k)}:= \frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b)\,, \quad (\stackrel{ab}{(k)})^2 = 0\,, \quad \stackrel{ab}{(k)}^{\dagger} = \eta^{aa} \stackrel{ab}{(-k)}\,,\\
S^{ab} \stackrel{ab}{[k]} &= \frac{k}{2} \stackrel{ab}{[k]}\,, \quad \stackrel{ab}{[k]}:= \frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b)\,, \quad (\stackrel{ab}{[k]})^2 = \stackrel{ab}{[k]}\,, \quad \stackrel{ab}{[k]}^{\dagger} = \stackrel{ab}{[k]}\,,\\
\tilde{S}^{ab} \stackrel{ab}{(\tilde{k})} &= \frac{k}{2} \stackrel{ab}{(\tilde{k})}\,, \quad \stackrel{ab}{(\tilde{k})}:= \frac{1}{2}(\tilde{\gamma}^a + \frac{\eta^{aa}}{ik}\tilde{\gamma}^b)\,, \quad (\stackrel{ab}{(\tilde{k})})^2 = 0\,, \quad \stackrel{ab}{(\tilde{k})}^{\dagger} = \eta^{aa} \stackrel{ab}{(\tilde{-k})}\,,\\
\tilde{S}^{ab} \stackrel{ab}{[\tilde{k}]} &= \frac{k}{2} \stackrel{ab}{[\tilde{k}]}\,, \quad \stackrel{ab}{[\tilde{k}]}:= \frac{1}{2}(1 + \frac{i}{k}\tilde{\gamma}^a\tilde{\gamma}^b)\,, \quad (\stackrel{ab}{[\tilde{k}]})^2 = \stackrel{ab}{[\tilde{k}]}\,, \quad \stackrel{ab}{[\tilde{k}]}^{\dagger} = \stackrel{ab}{[\tilde{k}]}\,.
\end{aligned} \tag{7.6}$$

The notation $\stackrel{ab}{(k)}, \stackrel{ab}{[k]}, \stackrel{ab}{(\tilde{k})}$ and $\stackrel{ab}{[\tilde{k}]}$ is introduced to simplify the discussions. The Clifford "vectors" — nilpotents ($\stackrel{ab}{(k)}\stackrel{ab}{(k)}= 0, \stackrel{ab}{(\tilde{k})}\stackrel{ab}{(\tilde{k})}= 0$) and projectors ($\stackrel{ab}{[k]}\stackrel{ab}{[k]}=\stackrel{ab}{[k]}$, $\stackrel{ab}{[\tilde{k}]}\stackrel{ab}{[\tilde{k}]}=\stackrel{ab}{[\tilde{k}]}$ — of both algebras are normalized up to a phase [2,12,13].

Both have half integer spins. The "eigenvalues" of the operator S^{03} for the "eigenvectors" $\frac{1}{2}(\gamma^0 \mp \gamma^3)$, for example, are equal to $\pm \frac{i}{2}$, respectively, for the "vectors" $\frac{1}{2}(1 \pm \gamma^0\gamma^3)$ are $\pm \frac{i}{2}$, respectively, while all the rest "eigenvectors" have "eigenvalues" $\pm \frac{1}{2}$. One finds equivalently for the "eigenvectors" of the operator $\tilde{S}^{03}$: for $\frac{1}{2}$ $(\tilde{\gamma}^0 \mp \tilde{\gamma}^3)$ the "eigenvalues" $\pm \frac{i}{2}$, respectively, and for the "eigenvectors" $\frac{1}{2}(1 \pm \tilde{\gamma}^0\tilde{\gamma}^3)$ the "eigenvalues" $k = \pm \frac{i}{2}$, respectively, while all the rest "eigenvectors" have $k = \pm \frac{1}{2}$.

It is useful to know some additional relations among nilpotents and projectors, taken from Ref. [3]

$$\begin{aligned}
\stackrel{ab}{(k)}\stackrel{ab}{[k]} &= 0\,, & \stackrel{ab}{[k]}\stackrel{ab}{(k)} &= \stackrel{ab}{(k)}\,,\\
\stackrel{ab}{(k)}\stackrel{ab}{[-k]} &= \stackrel{ab}{(k)}\,, & \stackrel{ab}{[k]}\stackrel{ab}{(-k)} &= 0\,,\\
\stackrel{ab}{(k)}\stackrel{ab}{(-k)} &= \eta^{aa} \stackrel{ab}{[k]}\,, & \stackrel{ab}{[k]}\stackrel{ab}{[-k]} &= 0\,.
\end{aligned} \tag{7.7}$$

The same relations are valid also if one replaces $\stackrel{ab}{(k)}$ with $\stackrel{ab}{(\tilde{k})}$ and $\stackrel{ab}{[k]}$ with $\stackrel{ab}{[\tilde{k}]}$.

The "basis vectors" are products of $\frac{d}{2}$ eigenvectors of all the Cartan subalgebra members. For the description of the internal space of fermions only those "basis vectors" which are products of an odd number of nilpotents, the rest can be projectors, are acceptable, since they *anticommute algebraically*, what we expect for the single fermion states of the second quantized fields.

To make clear what the anticommutation of the basis vectors mean, let us start with the first "basic vector", denoting it as $\hat{b}^{m=1\dagger}_{f=1}$, with f defining different irreducible representations and m a member in the representation f. Then its

Hermitian conjugated partner is $\hat{b}^{m}_{f} = (\hat{b}^{m\dagger}_{f})^{\dagger}$. Let us make a choice of the starting "basic vector" for the Clifford algebra of the kind γ^{a}'s with an odd products of the nilpotents

$$
\begin{aligned}
\hat{b}^{m=1\dagger}_{f=1} &:= \stackrel{03}{(+i)}\stackrel{12}{[+]}\stackrel{56}{[+]}\stackrel{78}{(+)}\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}\cdots\stackrel{d-3\,d-2}{[-]}\stackrel{d-1\,d}{[-]}\,,\\
(\hat{b}^{m=1\dagger}_{f=1})^{\dagger} = \hat{b}^{m=1}_{f=1} &= \stackrel{d-1\,d}{[-]}\stackrel{d-3\,d-2}{[-]}\cdots\stackrel{13\,14}{[-]}\stackrel{11\,12}{[-]}\stackrel{9\,10}{(-)}\stackrel{78}{(-)}\stackrel{56}{[+]}\stackrel{12}{[+]}\stackrel{03}{(-i)}\,,
\end{aligned}
\tag{7.8}
$$

the rest products in $\cdots\stackrel{d-3\,d-2}{[-]}\stackrel{d-1\,d}{[-]}$ are assumed to be all projectors with $k=-1$, $[-]$. All the rest members of this irreducible representation are reachable by S^{ab}.

Let us see how do S^{ab}'s transform the "basis vectors".

$$
\begin{aligned}
S^{ac}\,\stackrel{ab}{(k)}\stackrel{cd}{(k)} &= -\frac{i}{2}\eta^{aa}\eta^{cc}\,\stackrel{ab}{[-k]}\stackrel{cd}{[-k]}\,,\\
S^{ac}\,\stackrel{ab}{[k]}\stackrel{cd}{[k]} &= \frac{i}{2}\,\stackrel{ab}{(-k)}\stackrel{cd}{(-k)}\,,\\
S^{ac}\,\stackrel{ab}{(k)}\stackrel{cd}{[k]} &= -\frac{i}{2}\eta^{aa}\,\stackrel{ab}{[-k]}\stackrel{cd}{(-k)}\,,\\
S^{ac}\,\stackrel{ab}{[k]}\stackrel{cd}{(k)} &= \frac{i}{2}\eta^{cc}\,\stackrel{ab}{(-k)}\stackrel{cd}{[-k]}\,,
\end{aligned}
\tag{7.9}
$$

We learn from Eq. (7.50) that S^{01} transforms $\hat{b}^{m=1\dagger}_{f=1}$ into, let us call it $\hat{b}^{m=2\dagger}_{f=1}$, $\hat{b}^{m=2\dagger}_{f=1} = \stackrel{03}{[-i]}\stackrel{12}{(+)}\stackrel{56}{[+]}\stackrel{78}{(+)}\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}\cdots\stackrel{d-3\,d-2}{[-]}\stackrel{d-1\,d}{[-]}$.

Application of all possible S^{dg} generates $2^{\frac{d}{2}-1}$ members of each Clifford odd irreducible representation. To each irreducible representation the Hermitian conjugated irreducible representation belongs.

The Hermitian conjugated partner of the starting "basic vector" of an odd product of nilpotents obviously belong to another irreducible representation, since it is not reachable by S^{ab}. Each S^{cd} namely transforms a pair of projectors into a pair of nilpotents, a pair of nilpotents into a pair of projectors, and a pair of a nilpotent and a projector into a pair of a projector and a nilpotens, changing in each member of a pair its k into $-k$. The Hermitian conjugation transforms in $\hat{b}^{m\dagger}_{f}$ an odd number of nilpotents, each carrying its own k, into the same number of nilpotents, each carrying then $-k$ [4].

From Eq. (7.50) we learn that the starting member $\hat{b}^{m=1\dagger}_{f=2}$ of the next irreducible representation can be obtained from $\hat{b}^{m=1\dagger}_{f=1}$ by replacing, for example, $\stackrel{03}{(+i)}\stackrel{12}{[+]}$ in $\hat{b}^{m=1\dagger}_{f=1}$ with $\stackrel{03}{[+i]}\stackrel{12}{(+)}$. This new "basis vector" does not belong to either the starting irreducible representation, or to the Hermitian conjugated partners of the starting irreducible representation, due to the way how it is creating: S^{01} transforms $\stackrel{03}{(+i)}\stackrel{12}{[+]}$ into $\stackrel{03}{[-i]}\stackrel{12}{(-)}$, the Hermitian conjugation transforms $\stackrel{03}{(-i)}\stackrel{12}{[+]}$.

[4] The "basis vectors" with an even number of nilpotents have in even dimensional spaces the property that there is one member of each representation which is self adjoint, the one which is the product of only projectors.

Exchanging all possible pairs in the starting "basis vector" by keeping the same k's while transforming a pair of nilpotents into a pair of projectors, a pair of projectors into a pair of nilpotents and a pair of a nilpotent and a projector into a pair of the projector and the nilpotent, we generate $2^{\frac{d}{2}-1}$ irreducible representations with $2^{\frac{d}{2}-1}$ members each.

The Hemitian conjugation then generates $2^{\frac{d}{2}-1}\cdot 2^{\frac{d}{2}-1}$ partners to the $2^{\frac{d}{2}-1}$ members of each of the $2^{\frac{d}{2}-1}$ irreducible representations.

One can find that the algebraic product of $\hat{b}^{m}_{f} *_A \hat{b}^{m\dagger}_{f}$ is the same for all m of a particular irreducible representation f (since $\hat{b}^{m}_{f}(2\,S^{ab})^{\dagger} *_A (2\,S^{ab})\hat{b}^{m\dagger}_{f} = \hat{b}^{m}_{f} *_A \hat{b}^{m\dagger}_{f}$, due to the relation $(-2iS^{ab})^{\dagger}(-2iS^{ab}) = 1$).

Each irreducible representation contributes different algebraic product $\hat{b}^{m}_{f} *_A \hat{b}^{m\dagger}_{f}$.

For the representation of Eq. (7.8) the product $\hat{b}^{m=1}_{f=1} *_a \hat{b}^{m=1\dagger}_{f=1}$ is equal to
$|\psi_{oc}>|_{f=1} = \overset{03}{[-i]}\overset{12}{[+]}\overset{56}{[+]}\overset{78}{[-]}\overset{910}{[-]}\;\overset{1112}{[-]}\;\overset{1314}{[-]}\cdots\overset{d-3\,d-2}{[-]}\;\overset{d-1\,d}{[-]}\,.$
This can be checked by using Eq. (7.7).

Defining the vacuum state $|\psi_{oc}>$ for the vector space determined by γ^a's as a sum of all different products of $\sum_{f=1}^{2^{\frac{d}{2}-1}} \hat{b}^{m}_{f} *_A \hat{b}^{m\dagger}_{f}$, $\forall$ m, and for $d = 2n+1$, one obtains

$$
\begin{aligned}
|\psi_{oc}> = &\ \overset{03}{[-i]}\overset{12}{[-]}\overset{56}{[-]}\cdots\overset{d-1\;d}{[-]} + \overset{03}{[+i]}\overset{12}{[+]}\overset{56}{[-]}\cdots\overset{d-1\;d}{[-]}\\
+ &\ \overset{03}{[+i]}\overset{12}{[-]}\overset{56}{[+]}\cdots\overset{d-1\;d}{[-]} + \cdots |1>,\\
&\ \text{for } d = 2(2n+1)\,.
\end{aligned}
\tag{7.10}
$$

Let me add that the application of any member of the Cartan subalgebras on the vacuum state, $S^{ab}|\psi_{oc}>= 0, \tilde{S}^{ab}|\psi_{oc}>= 0$, $\forall$ S^{ab} and $\tilde{S}^{ab}$ belonging to Cartan subalgeras of Eq. (7.5).

One finds that all the members of all the irreducible representations fulfill together with their Hermitian conjugated partners the relations

$$
\begin{aligned}
\hat{b}^{m}_{f}{}_{*_A}|\psi_{oc}> &= 0\cdot|\psi_{oc}>\,,\\
\hat{b}^{m\dagger}_{f}{}_{*_A}|\psi_{oc}> &= |\psi^{m}_{f}>\,,\\
\{\hat{b}^{m}_{f}, \hat{b}^{m'}_{f'}\}_{*_A+}|\psi_{oc}> &= 0\,|\psi_{oc}>\,,\\
\{\hat{b}^{m}_{f}, \hat{b}^{m'\dagger}_{f}\}_{*_A+}|\psi_{oc}> &= \delta^{mm'}|\psi_{oc}>\,,\\
\{\hat{b}^{m\dagger}_{f}, \hat{b}^{m'\dagger}_{f'}\}_{*_A+}|\psi_{oc}> &= 0\cdot|\psi_{oc}>\,,
\end{aligned}
\tag{7.11}
$$

for each f. $*_A$ represents the algebraic multiplication of $\hat{b}^{m\dagger}_{f}$'s and $\hat{b}^{m'}_{f'}$'s among themselves and with the vacuum state $|\psi_{oc}>$ of Eq.(7.10).

The relations of Eq. (7.11) *almost* manifest the anticommutation relations for the second quantized fermion fields postulated by Dirac [67]. It is pointed out *almost*, since the relation

$$
\{\hat{b}^{m}_{f}, \hat{b}^{m'\dagger}_{f'}\}_{*_A+}|\psi_{oc}> = \delta^{mm'}\delta^{ff'}|\psi_{oc}>
\tag{7.12}
$$

is not fulfilled. There are, namely, besides $\hat{b}^{m}_{f}$, $2^{\frac{d}{2}-1}-1$ members of the Hermitian conjugated partners belonging each to a different irreducible representation, which

give a nonzero contribution — not an identity as $\hat{b}^{m}_{f}$ does — when multiplying $\hat{b}^{m\dagger}_{f}$ from the left hand side. $\hat{b}^{m}_{f'} *_{A} \hat{b}^{m\dagger}_{f} \neq 0$
for $2^{\frac{d}{2}-1} - 1$ different $f' \neq f$, while $\hat{b}^{m}_{f} *_{A} \hat{b}^{m\dagger}_{f} = 1$.

Let me illustrate this on the example of $\hat{b}^{m=1\dagger}_{f=1}$ of Eq. (7.8). Besides $(\hat{b}^{m=1\dagger}_{f=1})^{\dagger} = \hat{b}^{m=1}_{f=1} = \stackrel{d-1\, d}{[-]}\; \stackrel{d-3\, d-2}{[-]} \cdots \stackrel{13\,14}{[-]}\; \stackrel{11\,12}{[-]}\; \stackrel{9\,10}{(-)}\stackrel{7\,8}{(-)}\stackrel{56}{[+]}\stackrel{12}{[+]}\stackrel{03}{(-i)}$ also

$$\begin{aligned}
&\stackrel{d-1\, d}{[-]}\; \stackrel{d-3\, d-2}{[-]} \cdots \stackrel{13\,14}{[-]}\; \stackrel{11\,12}{[-]}\; \stackrel{9\,10}{(-)}\stackrel{7\,8}{(-)}\stackrel{56}{[+]}\stackrel{12}{(-)}\stackrel{03}{[+i]}\,,\\
&\stackrel{d-1\, d}{[-]}\; \stackrel{d-3\, d-2}{[-]} \cdots \stackrel{13\,14}{[-]}\; \stackrel{11\,12}{[-]}\; \stackrel{9\,10}{(-)}\stackrel{7\,8}{(-)}\stackrel{56}{(-)}\stackrel{12}{[+]}\stackrel{03}{[+i]}\,,\\
&\text{etc}
\end{aligned} \tag{7.13}$$

applied on $\hat{b}^{m=1\dagger}_{f=1}$, give a nonzero contributions.

But index f determine different irreducible representations and we can not expect that the algebraic anticommutation relations will be fulfilled also among different irreducible representations. Different irreducible representations should be treated in tensor products.

All the discussions about the Clifford algebra with γ^{a}'s, appearing after Eq. (7.7), can be as well repeated also for the Clifford algebra with $\tilde{\gamma}^{a}$'s.

The Dirac's postulates for the second quantized fermion fields include the infinite basis in momentum space, while we treated so far the finite dimensional internal space of fermions. Before extending the vector basis space by making the tensor product of internal space and the momentum space let us recognize that the observed quarks and leptons and antiquarks and antileptons do not at all suggest that there might be two different internal spaces, which could be described by two kinds of the Clifford algebra objects. Let us therefore first reduce the Clifford space by the postulate, which leave only γ^{a}'s as the algebra describing the internal degrees of freedom of fermions, while $\tilde{\gamma}^{a}$'s are used to give quantum numbers to different irreducible representations.

Reduction of the Clifford space It is needed to give to each irreducible representation of the Lorentz transformations in the internal space of fermions the quantum number, which will distinguish among the $2^{\frac{d}{2}-1}$ different irreducible representations. If we keep the Clifford algebra with γ^{a}'s to describe the internal space of fermions, then $\tilde{\gamma}^{a}$'s, or rather $\tilde{S}^{ab}$'s, can be used to determine "family" quantum number of each irreducible representation of the Lorentz algebra in the Clifford space of γ^{a}'s.

We want that all the relations among γ^{a}'s and $\tilde{\gamma}^{a}$'s, presented in Eq. (7.2), remain unchanged, while the eigenvalues of the Cartan subalgebra of $\tilde{S}^{ab}$ are expected to be changed.

The postulate [2,7,9,10,12,46]

$$\tilde{\gamma}^{a} B = (-)^{B}\, i\, B \gamma^{a}\,, \tag{7.14}$$

with $(-)^{B} = -1$, if B is a function of an odd product of γ^{a}'s, otherwise $(-)^{B} = 1$ [46], does just that [5]

[5] Eq. (7.14) requires that $\tilde{\gamma}^{a}(a_0 + a_b\gamma^{b} + a_{bc}\gamma^{b}\gamma^{c} + \cdots) = (i a_0 \gamma^{a} + (-i) a_b \gamma^{b}\gamma^{a} + i a_{bc}\gamma^{b}\gamma^{c}\gamma^{a} + \cdots)$, what means that the relation $\tilde{\gamma}^{a}\, a_0 = i a_0 \gamma^{a}$ is only one of the

It is not difficult to check that the relations in Eq. (7.2), concerning $\tilde{\gamma}^a$'s are still valid: $\{\gamma^a, \gamma^b\}_+ = 2\eta^{ab} = \{\tilde{\gamma}^a, \tilde{\gamma}^b\}_+ \,, \{\gamma^a, \tilde{\gamma}^b\}_+ = 0 \,, (\gamma^a)^\dagger = \eta^{aa}\,\gamma^a \,, (\tilde{\gamma}^a)^\dagger = \eta^{aa}\,\tilde{\gamma}^a$.

After this postulate the vector space of $\tilde{\gamma}^a$'s is "frozen out". And also the Grassmann algebra space is now reduced to $\theta^a = \gamma^a$ and $\frac{\partial}{\partial\theta_a} = 0$ [6]. No vector space of $\tilde{\gamma}^a$'s exists any longer, what is in agreement with the observed properties of fermions. While the anticommutation relations among γ^a's and $\tilde{\gamma}^a$'s remain the same as in Eq. (7.2), it follows for the eigenvalues of $\tilde{S}^{ab}$

$$\begin{aligned} S^{ab} \stackrel{ab}{(k)} &= \frac{k}{2} \stackrel{ab}{(k)}\,, & \tilde{S}^{ab} \stackrel{ab}{(k)} &= \frac{k}{2} \stackrel{ab}{(k)}\,, \\ S^{ab} \stackrel{ab}{[k]} &= \frac{k}{2} \stackrel{ab}{[k]}\,, & \tilde{S}^{ab} \stackrel{ab}{[k]} &= -\frac{k}{2} \stackrel{ab}{[k]}\,, \end{aligned} \tag{7.15}$$

showing that the eigenvalues of S^{ab} on the nilpotents and projectors of γ^a's differ from the eigenvalues of $\tilde{S}^{ab}$ on the nilpotents and projectors of γ^a's. The members of the Cartan subalgebra of $\tilde{S}^{ab}$, Eq. (7.5), can now be used to give to the irreducible representations of S^{ab} the "family" quantum numbers.

Let me mention that if one arranges the space of odd products of γ^a's with respect to $\mathcal{S}^{ab} (= S^{ab} + \tilde{S}^{ab})$, these new "basis vector" will form multiplets with integer spins and charges in adjoint representations. Like the "basis vectors" expressed by Grassmann algebra do in Ref. [13], Table I, but this time with θ^a's replaced by γ^a's.

---

relations included into Eq. (7.14). Another relation, for example, is $\tilde{\gamma}^a\gamma^a = (-i)\gamma^a\gamma^a = -i\eta^{aa}$. One correspondingly finds $\{\tilde{\gamma}^a, \tilde{\gamma}^b\}_+ = 2\eta^{ab} = \tilde{\gamma}^a\tilde{\gamma}^b + \tilde{\gamma}^b\tilde{\gamma}^a = \tilde{\gamma}^a i\gamma^b + \tilde{\gamma}^b i\gamma^a = i\gamma^b(-i)\gamma^a + i\gamma^a(-i)\gamma^b = 2\eta^{ab}$. $\{\tilde{\gamma}^a, \gamma^b\}_+ = 0 = \tilde{\gamma}^a\gamma^b + \gamma^b\tilde{\gamma}^a = \gamma^b(-i)\gamma^a + \gamma^b i\gamma^a = 0$. $\{\tilde{\gamma}^a, \gamma^a\}_+ = 0 = \tilde{\gamma}^a\gamma^a + \gamma^a\tilde{\gamma}^a = \gamma^a(-i)\gamma^a + \gamma^a i\gamma^a = 0$.

[6] Let me show how does the Grassmann space loose the Hermitian conjugated partners to θ^a's, while θ^a's become equal to γ^a's. My statement that Eq. (7.14) requires $\theta^a = \gamma^a$ and $\frac{\partial}{\partial\theta_a} = 0$ can be proved as follows. There are only two requirements which have to be analyzed in details, $\tilde{\gamma^a}(\alpha) = i\alpha\gamma^a$, α is any constant and $\tilde{\gamma}^a\gamma^a = -i\gamma^a\gamma^a$. Both relations apply on $|\psi_{oc}>$: In the Grassmann case the vacuum state is identity $|\,1>$, while in the Clifford algebra the vacuum state is the sum of even products of γ^a's as seen in Eq. (7.10), which applies on identity. Let us express γ^a's, $\tilde{\gamma}^a$'s and $|\psi_{oc}>$ in terms of θ^a's and $\frac{\partial}{\partial\theta_a}$ as written in Eq. (7.3). Eq. (7.3) requires that $\gamma^a = (\theta^a + \frac{\partial}{\partial\theta_a})$, $\tilde{\gamma}^a = i(\theta^a - \frac{\partial}{\partial\theta_a})$. Let us put these expressions into Eq. (7.14) and let $|\psi_{oc}>$ be expressed in terms of θ^a's. Taking into account that θ^a's applying on identity gives θ^a's back while $\frac{\partial}{\partial\theta_a}$ applying on identity gives zero, it follows that $|\psi_{oc}> = a_0 + a_{ab}\theta^a\theta^b + \cdots$, the rest of expansion is irrelevant for the proof. The constant α can be skipped, since constants appear in $|\psi_{oc}> = a_0 + a_{ab}\theta^a\theta^b + \cdots$ anyhow. The first relation $[\tilde{\gamma^a} = i\gamma^a]|\psi_o c>$, expressed with θ^a's and $\frac{\partial}{\partial\theta_a}$, reads: $i(\theta^a - \frac{\partial}{\partial\theta_a})(a_0 + a_{ab}\theta^a\theta^b + \cdots) = i(\theta^a + \frac{\partial}{\partial\theta_a})(a_0 + a_{ab}\theta^a\theta^b + \cdots)$. From this we find $i\theta^a a_0 = i a_0\theta^a$ and $i(-\frac{\partial}{\partial\theta_a})a_{ab}\theta^a\theta^b = i\frac{\partial}{\partial\theta_a}a_b\theta^a\theta^b$, requiring that $\frac{\partial}{\partial\theta_a} = 0$ (as an operator Hermitian conjugated to θ^a for $\forall\, a$). These relation requires that the derivatives should not exist any longer, if the relation should hold. Then it follows from $\gamma^a = (\theta^a + \frac{\partial}{\partial\theta_a})$ that $\theta^a = \gamma^a$, which means that the Grassmann space has no meaning any longer, the only remaining space is the space of the Clifford products of odd number of γ^a's, on which γ^a's and $\tilde{\gamma^a}$'s operate: $[\tilde{\gamma^a} = i\gamma^a]|\psi_o c>$ and $[\tilde{\gamma^a}\gamma^b = -i\gamma^b\gamma^a]|\psi_{oc}>$. This complicates the proof .

It is useful to notice that γ^a transform $\stackrel{ab}{(k)}$ into $\stackrel{ab}{[-k]}$, never to $\stackrel{ab}{[k]}$, while $\tilde{\gamma}^a$ transform $\stackrel{ab}{(k)}$ into $\stackrel{ab}{[k]}$, never to $\stackrel{ab}{[-k]}$

$$\begin{aligned}
&\gamma^a \stackrel{ab}{(k)}= \eta^{aa} \stackrel{ab}{[-k]}, \quad \gamma^b \stackrel{ab}{(k)}= -ik \stackrel{ab}{[-k]}, \\
&\gamma^a \stackrel{ab}{[k]}=\stackrel{ab}{(-k)}, \quad \gamma^b \stackrel{ab}{[k]}= -ik\eta^{aa} \stackrel{ab}{(-k)}, \\
&\tilde{\gamma^a} \stackrel{ab}{(k)}= -i\eta^{aa} \stackrel{ab}{[k]}, \quad \tilde{\gamma^b} \stackrel{ab}{(k)}= -k \stackrel{ab}{[k]}, \\
&\tilde{\gamma^a} \stackrel{ab}{[k]}= \; i \stackrel{ab}{(k)}, \quad \tilde{\gamma^b} \stackrel{ab}{[k]}= -k\eta^{aa} \stackrel{ab}{(k)} \; .
\end{aligned} \tag{7.16}$$

Some additional applications of $\tilde{\gamma}^a$'s and $\tilde{S}^{ab}$'s on nilpotents and projectors expressed by the γ^a's can be found in App. 7.4.

Each irreducible representation has now the "family" quantum number, determined by $\tilde{S}^{ab}$ of the Cartan subalgebra of Eq. (7.5). Now we can replace the fourth equation in Eq. (7.11) — $\{\hat{b}^m_f, \hat{b}^{m'\dagger}_f\}_{*_A+}|\psi_{oc}>= \delta^{mm'}|\psi_{oc}>$ — with the relation in Eq. (7.12) — $\{\hat{b}^m_f, \hat{b}^{m'\dagger}_{f'}\}_{*_A+}|\psi_{oc}>= \delta^{mm'}\delta_{ff'}|\psi_{oc}>$.

Each family contributes in even dimensional spaces one summand of $\frac{d}{2}$ projectors to the vacuum state $|\psi_{oc}>$ of fermions.

Correspondingly the "basic vectors" and their Hermitian conjugated partners fulfill algebraically the anticommutation relations of Dirac's second quantized fermions: Different irreducible representations carry different "family" quantum numbers and to each "family" quantum number only one Hermitian conjugated partner with the same "family" quantum number belongs. Also each summand of the vacuum state, Eq. (7.10), belongs to a particular "family".

One can easily check that each "basic vector" $\hat{b}^{m\dagger}_f$, applied algebraically on $|\psi_{oc}>$, gives nonzero contribution on the summand in the odd number of γ^a's, determined by $\hat{b}^m_f \hat{b}^{m\dagger}_f$, which is the same for all m of particular f, representing therefore the corresponding state $|\psi^f_m>$, while on all other summands $\hat{b}^{m\dagger}_f$ gives zero, $\hat{b}^m_f$ applying on $|\psi_{oc}>$ gives zero for all f and all m.

To define creation and annihilation operators, which determine on the vacuum state the single fermion states, we ought to make the tensor products of the $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ "basis vectors", describing the internal space of fermions and of infinite basis of momenta.

The oddness of the products of the odd number of γ^a's guarantees the anticommuting properties of all the objects which include an odd number of γ^a's.

The creation and annihilation operators, derived as tensor products of the "basis vectors" and the basis in momenum space, will fulfill the Dirac's postulates of the second quantized fermions without postulating them, as Dirac did. They follow by themselves from the fact that the creation and annihilation operators are superposition of odd products of γ^a's.

Second quantized fermion fields Since the nonrelativistic quantum theory is an approximation of the relativistic second quantized field theory — as the relativistic classical physics is an approximation of the quantum physics, and as the nonrelativistic classical physics, which we use the most of time, is the approximation of the relativistic classical physics — let us try to recognize what properties should the single particle states have to form the Hilbert space of second quantized fields.

In the references [10, 12, 13] the properties of the single fermion states, the tensor products among which form the Hilbert space, are discussed in details. In this talk I am presenting this topic from the point of view of the *spin-charge-family*. This theory offers, as written in the introduction, the explanation for the appearance of the spin (and handedness in the case of massless fermions), of all the charges, as well as of the families fermions. The number of families depends on the way how does the symmetry of the space breaks from $d = (13 + 1)$ to $d = (3 + 1)$.

In Table 7.3 one irreducible representation of $SO(13 + 1)$ of one family (belonging to the one of the two groups of four families which includes the so far observed three families) is presented. The first "basis vector" describes the internal degrees of freedom of the right handed quark $\hat{u}_R^{c1\dagger}$, of the first family with $(\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56}, \tilde{S}^{78})$ equal to $(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$, presented in Table 7.4 as $\hat{u}_{R1}^{c1\dagger}$. The "basis vector" $\hat{b}_{f=1}^{m=1\dagger}$, Eq. 7.8, represents for $d = (13 + 1)$ just this $\hat{u}_{R1}^{c1\dagger}$ quark, and $\hat{b}_{f=1}^{m=1}$ is its Hermitian conjugated partner.

The "basis vector" $\hat{b}_{f=2}^{m=1\dagger}$ represents for $d = (13+1)$ the right handed u-quark with all the properties of $\hat{u}_{R1}^{c1\dagger}$ except for the family quantum numbers, which are now equal to $(-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$. One can read in Table 7.3 that the spin of this right handed quark $\hat{u}_R^{c1\dagger}$ is $+\frac{1}{2}$, the weak $SU(2)$ charge is zero, the colour charge is $(\frac{1}{2}, \frac{1}{2\sqrt{3}})$. It carries the additional $SU(2)$ charge equal to $\frac{1}{2}$ and the "fermion" quantum number — τ^4 charge — equal to $\frac{1}{6}$.

When solving the equations of motion for free massless fermions, which follow from the action, presented in Eq. (7.1), under the assumption, that at low energies the momentum of this right handed quark is $p^a = (p^0, p^1, p^2, p^3, 0, \cdots, 0)$, the solution $s = 1$ is the superposition

$$\hat{\mathbf{u}}_R^{sf=1\dagger}(\vec{p}) = \beta(\hat{u}_{R\uparrow}^{c1\dagger} + \frac{p^1 + ip^2}{|p^0| + |p^3|}\hat{u}_{R\downarrow}^{c1\dagger}), \tag{7.17}$$

with $|p^0| = |\vec{p}|$, with $\uparrow, \downarrow$ denoting spin $\pm\frac{1}{2}$, respectively, and with $\beta^*\beta = \frac{|p^0|+|p^3|}{2|p^0|}$ normalizing the state.

There are steps from the $d = (13 + 1)$ dimensional space to the step where momentum in higher dimensions do not contribute to dynamics in $d = (3 + 1)$, while the break of symmetry makes the internal degrees of freedom (spins and families) to manifest as the spin and charges as presented in Table 7.3 and families as presented at Table 7.4. One finds the detailed presentations in Ref.([3–5, 9, 49, 52, 70] and the references therein).

Let us here represent the general solutions of equations of motion for free massless fermions with the internal space of fermions described by the "basis vectors" $\hat{b}_f^{m\dagger}$, fulfilling the relations of Eq. (7.11), for each family f separately, and also with respect to different families, $\hat{b}^m_f {}_{*A} \hat{b}^{m'\dagger}_f = \delta^{mm'} \delta_{ff'}$,

$$\begin{aligned}
\hat{\mathbf{b}}^{sf\dagger}(\vec{p})|_{p^0=|\vec{p}|} &\stackrel{\text{def}}{=} \sum_m c^{sf}{}_m\,(\vec{p}, |p^0| = |\vec{p}|)\; \hat{b}^{m\dagger}_f\,, \\
\underline{\hat{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p}, \vec{x}) &\stackrel{\text{def}}{=} (\hat{\mathbf{b}}^{sf\dagger}(\vec{p})\, e^{-i(p^0x^0 - \vec{p}\cdot\vec{x})})|_{|p^0|=|\vec{p}|}\,, \\
\sum_m (c^{sf*}{}_m(\vec{p}) \cdot c^{s'f'}{}_m(\vec{p}))|_{|p^0|=|\vec{p}|} &= \delta^{ss'} \delta_{ff'}\,,
\end{aligned} \tag{7.18}$$

s represents different orthonormalized solutions of the equations of motion, $c^{sf}{}_m(\vec{p}, |p^0| = |\vec{p}|)$ are coefficients, depending on momentum $|\vec{p}|$ with $|p^0| = |\vec{p}|$. For the case of the right handed u-quarks of Eq. (7.17) the two nonzero coefficients are β and $\beta\frac{p^1+ip^2}{|p^0|+|p^3|}$.

Creation operators of an odd Clifford character $\underline{\hat{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p})$ create the single particle states, $< x|\psi^{\mathbf{sf}}(\tilde{\mathbf{p}}, \mathbf{p^0}) > |_{\mathbf{p^0}=|\tilde{\mathbf{p}}|}$, manifesting the oddness of the creation operators

$$\begin{aligned}
< x|\psi^{\mathbf{sf}}(\tilde{\mathbf{p}}, \mathbf{p^0}) > |_{\mathbf{p^0}=|\tilde{\mathbf{p}}|} &= \int dp^0 \delta(p^0 - |\vec{p}|)\, \hat{\mathbf{b}}^{sf\dagger}(\vec{p})\, e^{-ip_a x^a} \;{}_{*A}\, |\psi_{oc} > \\
&= (\hat{\mathbf{b}}^{sf\dagger}(\vec{p}) \cdot\, e^{-i(p^0x^0 - \varepsilon\vec{p}\cdot\vec{x})})|_{p^0=|\vec{p}|} \;{}_{*A}\; |\psi_{oc} >\,,
\end{aligned} \tag{7.19}$$

with the property

$$\begin{aligned}
&\int \frac{d^{d-1}x}{(\sqrt{2\pi})^{d-1}}\; < \psi^{\mathbf{s'f'}}(\vec{p'}, p'^0 = |\vec{p'}|)|x >< x||\, \psi^{\mathbf{sf}}(\vec{p}, p^0 = |\vec{p}|) >= \\
&\int \frac{d^{d-1}x}{(\sqrt{2\pi})^{d-1}} e^{ip'_a x^a}|_{p'^0=|\vec{p'}|}\, e^{-ip_a x^a}|_{p^0|=|\vec{p}|} \\
&\cdot < \psi_{oc}|\, (\hat{\mathbf{b}}^{s'f'}(\vec{p'})\; \hat{\mathbf{b}}^{sf\dagger}(\vec{p}))\,{}_{*A}\, |\psi_{oc} >= \delta ss' \delta^{ff'} \delta(\vec{p} - \vec{p'})\,.
\end{aligned} \tag{7.20}$$

One further finds the single particle fermion states in the coordinate representation

$$\begin{aligned}
&|\psi^{\mathbf{sf}}(\tilde{\mathbf{x}}, \mathbf{x^0}) >= \int_{-\infty}^{+\infty} \frac{\mathbf{d^{d-1}p}}{(\sqrt{\mathbf{2\pi}})^{\mathbf{d-1}}}\, (\hat{\mathbf{b}}^{\mathbf{sf}\dagger}(\tilde{\mathbf{p}})\, \mathbf{e}^{-\mathbf{i}(\mathbf{p^0x^0} - \varepsilon\tilde{\mathbf{p}}\cdot\tilde{\mathbf{x}})}|_{\mathbf{p^0}=|\tilde{\mathbf{p}}|} \;{}_{*\mathbf{A}}\; |\psi_{\mathbf{oc}} >= \\
&\sum_m \hat{b}^{m\dagger}_f |\psi_{oc}> \int_{-\infty}^{+\infty} \frac{d^{d-1}p}{(\sqrt{2\pi})^{d-1}}\, (c^{sf}{}_m(\vec{p})\, e^{-i(p^0x^0 - \varepsilon\vec{p}\cdot\vec{x})})|_{p^0=|\vec{p}|} = \\
&\sum_m \hat{b}^{m\dagger}_f |\psi_{oc} >\; c^{sf}{}_m(-i\frac{\partial}{\partial x_a}, |p^0| = |(-i\frac{\partial}{\partial x_a}|)\, \delta(\vec{x})\,,
\end{aligned} \tag{7.21}$$

where it is taken into account that $\hat{\mathbf{b}}^{sf\dagger}(\vec{p})|_{p^0=|\vec{p}|}\, |\psi_{oc} >= \sum_m c^{sf}{}_m\; (\vec{p}, |p^0| = |\vec{p}|)\; \hat{b}^{m\dagger}_f$, Eq. (7.18), and that $\int \frac{d^{d-1}x}{(\sqrt{2\pi})^{d-1}}\, e^{ip'_a x^a}\, e^{-ip_a x^a} = \delta(\vec{p} - \vec{p'})$. $\varepsilon = \pm 1$, depending on handedness and spin of solutions.

Taking into account the above derivations, leading to

$$\int dp^0 \delta(p^0 - |\vec{p}|) e^{i(p^0x^0 - p^0x^0)} = 1$$

and $<\psi_{oc}|\,\hat{\mathbf{b}}^{sf}(\vec{p},p^0)_{*_A}\,\hat{\mathbf{b}}^{s'f'\dagger}(\vec{p},p^0)\,|\psi_{oc}>=\delta^{ss'}\delta^{ff'}$, one finds

$$\begin{aligned}
&<\psi^{\mathbf{sf}}(\vec{x},x^0)|\,\psi^{\mathbf{s'f'}}(\vec{x}\,',x^0)>=\\
&=\int_{-\infty}^{+\infty}\frac{d^{d-1}p}{(\sqrt{2\pi})^{d-1}}\int_{-\infty}^{+\infty}\delta(p^0-|\vec{p}|)<\psi^{\mathbf{sf}}(\vec{x},x^0)|\vec{p}><\vec{p}|\psi^{\mathbf{s'f'}}(\vec{x}\,',x^0)>\\
&=\int_{-\infty}^{+\infty}\frac{d^{d-1}p}{(\sqrt{2\pi})^{d-1}}\,e^{-i\vec{p}\cdot\vec{x}}\,e^{i\vec{p}\cdot\vec{x}'}\int dp^0\,\delta(p^0-|\vec{p}|)\\
&<\psi_{oc}|\hat{\mathbf{b}}^{sf}\,(\vec{p},p^0)_{*_A}\hat{\mathbf{b}}^{s'f'\dagger}\,(\vec{p},p^0)_{\,*_A}\,|\psi_{oc}>=\\
&=\delta^{ss'}\,\delta_{ff'}\,\delta(\vec{x}-\vec{x}\,')\,. \qquad (7.22)
\end{aligned}$$

The scalar product $<\psi^{\mathbf{sf}}(\vec{x},x^0)\,|\,\psi^{\mathbf{s'f'}}(\vec{x}\,',x^0)>$ has obviously the desired properties of the second quantized states.

The new creation operators $\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p},\vec{x})$, which are generated on the tensor products of both spaces, internal and momentum, fulfill obviously the below anticommutation relations when applied on $|\psi_{oc}>$

$$\begin{aligned}
\{\hat{\underline{\mathbf{b}}}^{sf}_{tot}(\vec{p},\vec{x})\,,\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p}\,',\vec{x})\}_{+\,*_T}|\psi_{oc}>&=\delta^{ss'}\,\delta_{ff'}\,\delta(\vec{p}-\vec{p}')\,|\psi_{oc}>\,,\\
\{\hat{\underline{\mathbf{b}}}^{sf}_{tot}(\vec{p},\vec{x})\,,\hat{\underline{\mathbf{b}}}^{s'f'}_{tot}\,(\vec{p}',\vec{x})\}_{+\,*_T}|\psi_{oc}>&=0\cdot|\psi_{oc}>\,,\\
\{\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p},\vec{x})\,,\hat{\underline{\mathbf{b}}}^{s'f'\dagger}_{tot}(\vec{p}\,',\vec{x})\}_{+\,*_T}|\psi_{oc}>&=0\cdot|\psi_{oc}>\,,\\
\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p},\vec{x})_{\,*_T}|\psi_{oc>}&=|\psi^{sf}(\vec{p},,\vec{x})>\,,\\
\hat{\underline{\mathbf{b}}}^{sf}_{tot}(\vec{p},,\vec{x})_{\,*_T}|\psi_{oc}>&=0\cdot|\psi_{oc}>\,,\\
|p^0|&=|\vec{p}|\,. \qquad (7.23)
\end{aligned}$$

It is not difficult to show that $\hat{\underline{\mathbf{b}}}^{sf}_{tot}(\vec{p},\vec{x})$ and $\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p},\vec{x})$ manifest the same anticommutation relations also on tensor products of an arbitrary chosen products of sets of single fermion states [13].

Hilbert space of fermion fields The tensor products of any number of any sets of the single fermion creation operators $\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p},\vec{x})$ (fulfilling together with their Hermitian conjugated partners annihilation operators $\hat{\underline{\mathbf{b}}}^{sf}_{tot}(\vec{p},\vec{x})$ the anticommutation relations of Eq. (7.23)) form the Hilbert space of the second quantized fermion fields. The number of the sets is infinite. The internal space, defined by $\hat{b}^m_f$, contributes in d-dimensional space for each chosen momentum $\vec{p}$ (and for a parameter $\vec{x}$) the finite number, $2^{2^{\frac{d}{2}-1}\cdot 2^{\frac{d}{2}-1}}$, of such sets, the total Hilbert space has, due to the infinite basis in the momentum (or coordinate) space, the infinite number of sets

$$N_{\mathcal{H}}=\prod_{\vec{p}}^{\infty}2^{2^{d-2}}\,. \qquad (7.24)$$

The number operator is defined as

$$\begin{aligned}
N^{sf}_{\vec{p}}&=\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p},\vec{x})\;*_T\;\hat{\underline{\mathbf{b}}}^{sf}_{tot}(\vec{p},\vec{x})\,,\\
N^{sf}_{\vec{p}}\,|\psi_{oc}>&=0\cdot|\psi_{oc}>\;. \qquad (7.25)
\end{aligned}$$

The vacuum state contains no fermions.

The Clifford odd objects $\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p},\vec{x})$ demonstrate their oddness also with respect to the whole Hilbert space $\mathcal{H}$, that is with respect to any tensor product of members of any sets of creation operators $\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p}\,',\vec{x})$). Correspondingly the anticommutation relations follow also for the application of $\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p},\vec{x})$ and $\hat{\underline{\mathbf{b}}}^{sf}_{tot}(\vec{p},\vec{x})$ on $\mathcal{H}$

$$\begin{aligned}
&\{\hat{\underline{\mathbf{b}}}^{sf}_{tot}(\vec{p},\vec{x})\,,\hat{\underline{\mathbf{b}}}^{s'f'\dagger}_{tot}(\vec{p}\,',\vec{x})\}_{*_T+}\,\mathcal{H}=\delta^{ss'}\,\delta_{ff'}\,\delta(\vec{p}-\vec{p}\,')\,\mathcal{H}\,,\\
&\{\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p},\vec{x})\,,\hat{\mathbf{b}}^{s'f'\dagger}_{tot}(\vec{p}\,',\vec{x})\}_{*_T+}\,\mathcal{H}=0\,\cdot\mathcal{H}\,,\\
&\{\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p},\vec{x}),\hat{\mathbf{b}}^{s'f'\dagger}_{tot}(\vec{p}\,',\vec{x})\}_{*_T+}\,\mathcal{H}=0\,\cdot\mathcal{H}\,.
\end{aligned}\tag{7.26}$$

I presented in this talk the derivation of the creation and annihilation operator of the second quantized fermion fields, which obey the Dirac's postulates for the second quantized fermion fields without postulating them, just by analyzing properties of creation and annihilation operators obtained as tensor products of the "basis vectors" of an odd Clifford algebra and of the basis in either momentum or coordinate space. In Ref. [10–13] the relation between the creation and annihilation operators, postulated by Dirac and the ones presented in this talk are discussed.

Properties of fermions in $\mathbf{d} = (3+1)$ This section follows quite a lot Refs. [3,4]. With respect to the last years I have not succeeded to improve much the part presented in this subsection. I have been working on the symmetries of the *spin-charge-family* theory and in particular on how can the theory, using the Clifford algebra to describe all the internal properties of fermions — spins, charges and families — help to explain the assumptions of the second quantized fermion fields. I shall therefore review the other achievements of the theory very briefly.

In Eq. (7.1) the starting action is presented for fermion and boson fields in $d = (13+1)$. In order that predictions of the *spin-charge-family* theory are in agreement with the observed properties of quarks and leptons and antiquarks and antileptons, of the vector gauge fields and of the scalar gauge fields (manfesting as the higgs and Yukawa couplings), the manifold $M^{(13+1)}$ ought to break first into $M^{(7+1)} \times M^{(6)}$ (which manifests as $SO(7,1) \times SU(3) \times U(1)$), affecting fermions, vector gauge fields and scalar gauge fields.

This first break is caused by the scalar condensate of two right handed neutrinos, presented in Table 7.5, Sect. 7.5 which interact with all the scalar gauge fields (with the gauge fields with the space index $(5,6,7,\cdots,14)$, as well as with those vector gauge fields (with the gauge fields with the space index $(0,1,2,3)$, which couple to the condensate. The only vector gauge fields which do not interact with the condensate and remain consequently massless are the weak charge, colour charge and hyper charge vector gauge fields.

Since the left handed fermions couple differently to scalar fields than the right handed ones, the break can leave massless and mass protected $2^{((7+1)/2-1)}$ families [68]. The rest of families get heavy masses [7].

[7] A toy model [68, 69] was studied in $d = (5+1)$ with the same action as in Eq. (7.1). The break from $d = (5+1)$ to $d = (3+1)\times$ an almost S^2 was studied. For a particular

The fermion families are arranged into twice two groups of massless four families, with respect to family quantum numbers as presented in Table 7.4 in Sect. 7.5, each group manifesting $SU(2)_{\subset SO(3,1)} \times SU(2)_{\subset SO(4)}$ symmetry, one group manifesting $SU(2)_L \times SU(2)_L$ symmetry, the other $SU(2)_R \times SU(2)_R$ symmetry.

The nonzero vacuum expectation values of the scalar fields with the space index $(7, 8)$, which carry the weak and hyper charges, break the mass protection and make family massive [7,9].

The breaks of the staring symmetry make the spins in higher dimensions to manifest as charges in $d = (3 + 1)$.

The superposition of the Lorentz members of the Clifford algebra, manifesting in $d = (3 + 1)$ the spins, Eq. (7.52), charges, Eqs. (7.53, 7.54) and families, Eqs (7.55, 7.56). are presented in Sect. 7.5.

Let me rewrite the fermion part of the action, Eq. (7.1), by taking into account the degrees of freedom the action manifests in $d = (3 + 1)$ in the way that we can clearly see that the action does manifest in the low energy regime by the *standard model* required properties of fermions, of vector gauge fields and of scalar gauge fields [1–3,7,9,51–53,71,72].

$$\begin{aligned}\mathcal{L}_f = \bar{\psi}\gamma^m(p_m - \sum_{A,i} g^{Ai}\tau^{Ai}A^{Ai}_m)\psi + \\ \{\sum_{s=7,8} \bar{\psi}\gamma^s p_{0s}\,\psi\} + \\ \{\sum_{t=5,6,9,\dots,14} \bar{\psi}\gamma^t p_{0t}\,\psi\},\end{aligned} \tag{7.27}$$

where $p_{0s} = p_s - \frac{1}{2}S^{s's"}\omega_{s's"s} - \frac{1}{2}\tilde{S}^{ab}\tilde{\omega}_{abs}$, $p_{0t} = p_t - \frac{1}{2}S^{t't"}\omega_{t't"t} - \frac{1}{2}\tilde{S}^{ab}\tilde{\omega}_{abt}$, with $m \in (0, 1, 2, 3)$, $s \in (7, 8)$, $(s', s") \in (5, 6, 7, 8)$, (a, b) (appearing in $\tilde{S}^{ab}$) run within either $(0, 1, 2, 3)$ or $(5, 6, 7, 8)$, t runs $\in (5, \dots, 14)$, $(t', t")$ run either $\in (5, 6, 7, 8)$ or $\in (9, 10, \dots, 14)$. The spinor function ψ represents all family members of all the $2^{\frac{7+1}{2}-1} = 8$ families.

a. The first line of Eq. (7.27) determines in $d = (3 + 1)$ the kinematics and dynamics of fermion fields, coupled to the vector gauge fields [3,5,9]. The vector gauge fields are the superposition of the spin connection fields ω_{stm}, $m = (0, 1, 2, 3)$, $(s, t) = (5, 6, \cdots, 13, 14)$, the gauge fields of S^{st}. They are shortly presented in Sect. 7.34.

The operators τ^{Ai} ($\tau^{Ai} = \sum_{a,b} c^{Ai}{}_{ab}\, S^{ab}$, S^{ab} are the generators of the Lorentz transformations in the Clifford space of γ^a's) are presented in Eqs. (7.53, 7.54) of Sect. 7.5. They represent the colour charge, $\vec{\tau}^3$, the weak charge, $\vec{\tau}^1$, and the hyper charge, $Y = \tau^4 + \tau^{23}$, τ^4 is the fermion charge, originating in $SO(6) \subset SO(13, 1)$, τ^{23} belongs together with $\vec{\tau^1}$ of $SU(2)_{weak}$ to $SO(4)$ group ($\subset SO(13 + 1)$).

choice of vielbeins and for a class of spin connection fields the manifold $M^{(5+1)}$ breaks into $M^{(3+1)}$ times an almost S^2, while $2^{((3+1)/2-1)}$ families remain massless and mass protected. Equivalent assumption, although not yet proved how does it really work, is made in the $d = (13 + 1)$ case. This study is in progress.

One fermion irreducible representation of the Lorentz group contains, as seen in Table 7.3, *quarks and leptons and antiquarks and antileptons*, belonging to the first family in Table 7.4. One can notice that the $SO(7,1)$ subgroup content of the $SO(13,1)$ group is the same for the quarks and leptons and the same for the antiquarks and antileptons. Quarks distinguish from leptons, and antiquarks from antileptons, only in the $SO(6) \subset SO(13,1)$ part, that is in the colour (τ^{33}, τ^{38}) part and in the fermion quantum number τ^4. The quarks distinguish from antiquarks, and leptons from antileptons, in the handedness, in the colour part and in the τ^4 part, explaining the relation between handedness and charges of fermions and antifermions [8].

The vector gauge fields, which interact with the condensate, presented in Table 7.5, become massive. The *vector gauge fields not interacting with the condensate — the weak, colour and hyper charged vector gauge fields — remain massless*, in agreement with by the *standard model* assumed gauge fields before the electroweak break of the mass protection,

After the electroweak break, caused by the scalar fields, the only conserved charges are the colour and the electromagnetic charge $Q = \tau^{13} + Y$, $Y = \tau^4 + \tau^{23}$.

b. The second line of Eq. (7.27) is the mass term, responsible in $d = (3+1)$ for the masses of fermions. The interaction of fermions with the superposition of the spin connection fields with the space index $s = (7,8)$, which gain nonzero vacuum expectation values, cause the electroweak break, bringing masses to fermions and antifermions and to the weak vector gauge fields. They are superposition of either $\omega_{s't's}$ or $\tilde{\omega}_{abs}$. *These scalar fields explain the appearance of the higgs and Yukawa couplings* of the *standard model*. Their properties are shortly presented in Subsect. 7.2.2.

These scalar gauge fields split into two groups of four families, one group manifesting the symmetry — $\widetilde{SU}(2)_{(\widetilde{SO}(3,1),L)} \times \widetilde{SU}(2)_{(\widetilde{SO}(4),L)} \times U(1)$ — and the other the symmetry — $\widetilde{SU}(2)_{(\widetilde{SO}(3,1),R)} \times \widetilde{SU}(2)_{(\widetilde{SO}(4),R)} \times U(1)$, Eq. (7.37). The scalar gauge fields, manifesting $SU(2)_{L,R} \times SU(2)_{L,R}$, are the superposition of the gauge fields $\tilde{\omega}_{abs}$, $s = (7,8)$, $(a,b) =$ either $(0,1,2,3)$ or $(5,6,7,8)$, manifesting as twice two triplets interacting each with one of the two groups of four families, presented in Table 7.4. The three $U(1)$ singlet scalar gauge fields are superposition of $\omega_{s't's}$, $s = (7,8)$, $(s',t') = (5,6,7,8,9,\cdots,14)$, with the sum of $S^{s't'}$ arranged into superposition of τ^{13}, τ^{23} and τ^4. The three triplets interact with both groups of quarks and leptons and antiquarks and antileptons.

Each of *the two groups have well defined symmetry of mass matrices, what limits the number of free parameters*.

To one of the groups of four families the observed quarks and leptons belong [51,54,57,58].

We predict the mixing matrices for quarks, taking as the input the masses of the fourth family, since the elements for the 3×3 submatrix of the 4×4

[8] Ref. [8] points out that the connection between handedness and charges for fermions and antifermions, both appearing in the same irreducible representation, explains the triangle anomalies in the *standard model* with no need to connect "by hand" the handedness and charges of fermions and antifermions.

mixing matrix are (far) not accurately enough measured, that we could predict masses of the fourth family quarks [8, 51, 54]. The newer are the experimental data the better is the agreement of the measured mixing matrix elements with our predictions [54, 58] at least so far.

The stable of the upper four families offers the explanation for the *dark matter*appearance and it is so far in agreement with experimental evidences of the dark matter [52, 61].

I discuss predictions of the *spin-charge-family* theory for the properties of the lower four families and of the *dark matter* in Sect. 7.3.

c. The third line of Eq. (7.27) represents the scalar fields, which cause transitions from antileptons and antiquarks into quarks and leptons and back, offering the explanation for the matter/antimatter asymmetry in the expanding universe at non equilibrium conditions [4]. They are colour triplets with respect to the space index equal to $(9, 10, 11, 12, 13, 14)$, while they carry the quantum numbers with respect to the superposition of $\mathcal{S}^{ab}$ in adjoint representations, as can be seen in Table 7.2 and in Fig. 7.1 of Subsect. 7.2.2. I discuss properties of these scalar fields, offered by the *spin-charge-family* theory, in Sect. 7.3.

7.2.2 Properties of vector and scalar gauge fields in *spin-charge-family* theory

In the starting action, Eq. (7.1), the second line represents the action for gauge fields in $d = (13 + 1)$-dimensional space, with the index $_{gf}$ denoting gauge fields, vector or scalar,

$$\begin{aligned}
\mathcal{A}_{gf} &= \int d^d x \, E \, (\alpha \, R + \tilde{\alpha} \, \tilde{R}) \,, \\
R &= \frac{1}{2} \{ f^{\alpha [a} f^{\beta b]} \, (\omega_{ab\alpha,\beta} - \omega_{ca\alpha} \, \omega^{c}{}_{b\beta}) \} + h.c. \,, \\
\tilde{R} &= \frac{1}{2} \{ f^{\alpha [a} f^{\beta b]} \, (\tilde{\omega}_{ab\alpha,\beta} - \tilde{\omega}_{ca\alpha} \, \tilde{\omega}^{c}{}_{b\beta}) \} + h.c. \,,
\end{aligned} \tag{7.28}$$

which in the *spin-charge-family* theory manifests after the break of the starting symmetry in $d = (3 + 1)$ as the action for all observed vector and scalar gauge fields. Here $f^{\beta}{}_{a}$ and $e^{a}{}_{\alpha}$ are vielbeins and inverted vielbeins respectively

$$e^{a}{}_{\alpha} f^{\beta}{}_{a} = \delta^{\beta}_{\alpha} \,, \quad e^{a}{}_{\alpha} f^{\alpha}{}_{b} = \delta^{a}_{b} \,, \tag{7.29}$$

$E = \det(e^{a}{}_{\alpha})$.

Varying the action of Eq. (7.28) with respect to the spin connection fields, the expression for the spin connection fields ω^{e}_{ab} follows

$$\begin{aligned}
\omega_{ab}{}^{e} = & \frac{1}{2E} \{ e^{e}{}_{\alpha} \, \partial_{\beta} (E f^{\alpha}{}_{[a} f^{\beta}{}_{b]}) - e_{a\alpha} \, \partial_{\beta} (E f^{\alpha}{}_{[b} f^{\beta e]}) \\
& - e_{b\alpha} \partial_{\beta} (E f^{\alpha [e} f^{\beta}{}_{a]}) \} \\
& + \frac{1}{4} \{ \bar{\Psi} (\gamma^{e} \, S_{ab} - \gamma_{[a} S_{b]}{}^{e}) \Psi \} \\
& - \frac{1}{d-2} \{ \delta^{e}_{a} [\frac{1}{E} \, e^{d}{}_{\alpha} \partial_{\beta} (E f^{\alpha}{}_{[d} f^{\beta}{}_{b]}) + \bar{\Psi} \gamma_{d} S^{d}{}_{b} \, \Psi] \\
& - \delta^{e}_{b} [\frac{1}{E} e^{d}{}_{\alpha} \partial_{\beta} (E f^{\alpha}{}_{[d} f^{\beta}{}_{a]}) + \bar{\Psi} \gamma_{d} S^{d}{}_{a} \, \Psi] \} \,.
\end{aligned} \tag{7.30}$$

If replacing S^{ab} in Eq. (7.30) with $\tilde{S}^{ab}$, the expression for the spin connection fields $\tilde{\omega}_{ab}{}^{e}$ follows.

In Ref. [5] it is proven that in spaces with the desired symmetry the vielbein can be expressed with the gauge fields, if only one of the two spin connection fields are present

$$f^{\sigma}{}_{m} = \sum_{A} \vec{\tau}^{A\sigma} \vec{\mathcal{A}}^{A}_{m}\,, \tag{7.31}$$

with

$$\begin{aligned}
\mathcal{A}^{Ai}_{m} &= \sum_{st} c^{Ai}{}_{st}\, \omega^{st}{}_{m}\,,\\
\tau^{Ai\sigma} &= \sum_{st} c^{Ai}{}_{st}\, (e_{s\tau}\, f^{\sigma}{}_{t} - e_{t\tau}\, f^{\sigma}{}_{s}) x^{\tau}\,,\\
\tau^{Ai} &= \sum_{st} c^{Ai}{}_{st}\, S^{st}\,.
\end{aligned} \tag{7.32}$$

If fermions are not present them spin connections of both kinds are uniquely determined by vielbeins, as can be noticed from Eq. (7.30). If fermions are present, carrying both — family members and family quantum numbers — then vielbeins and both kinds of spin connections are influenced by the presence of fermions, which carry different family and family members quantum numbers.

The scalar (gauge) fields, carrying the space index $s = (5, 6, \ldots, d)$, offer in the *spin-charge-family* for $s = (7, 8)$ the explanation for the origin of the Higgs's scalar and the Yukawa couplings of the *standard model*, while scalars with the space index $s = (9, 10, \ldots, 14)$ offer the explanation for the proton decay, as well as for the matter/antimatter asymmetry in the universe.

We use the notation

$$\begin{aligned}
\tau^{Ai} &= \sum_{a,b} c^{Ai}{}_{ab}\; S^{ab}\,,\\
\{\tau^{Ai}, \tau^{Bj}\}_{-} &= i\delta^{AB} f^{Aijk} \tau^{Ak}\,,\\
A^{Ai}_{a} &= \sum_{s,t} c^{Ai}{}_{st}\, \omega^{st}{}_{a}\,,
\end{aligned} \tag{7.33}$$

$a = m = (0, 1, 2, 3)$ for vector gauge fields and $a = s = (5, 6, \ldots, 14)$ for scalar aguge fields.

The explicit expressions for $c^{Ai}{}_{ab}$, and correspondingly for τ^{Ai}, and A^{Ai}_{a}, are written in Sect. 7.5.

Vector gauge fields in $d = (3 + 1)$ In the *spin-charge-family* theory there are besides the gravity, the colour and the weak $SU(2)_{I}$ vector gauge fields, also the second $SU(2)_{II}$ and the $U(1)_{\tau^4}$ vector gauge fields. The $U(1)_{\tau^4}$ vector gauge field is the vector gauge field of $\tau^4 (= -\frac{1}{3}(S^{9\,10} + S^{11\,12} + S^{13\,14}))$ - the fermion charge. The hyper charge vector gauge field of the *standard model* is the superposition of the third component of the second $SU(2)_{II}$ vector gauge fields and the $U(1)_{\tau^4}$ vector gauge field ($A^{Y}_{m} = \cos\theta_2 A^{\tau^4}_{m} + \sin\theta_2 A^{23}_{m}$, θ_2 is the angle of the break of the $SU(2)_{II} \times U(1)_{\tau^4}$ symmetry to $U(1)_Y$ at the scale $\geq 10^{16}$ or higher, [9] and references therein). After the appearance of the condensate, presented in Table 7.5, there are namely only the gravity, the colour, the weak $SU(2)_I$ and the $U(1)_Y$ hyper charge vector gauge fields, which remain massless. The two components of the second $SU(2)_{II}$ vector gauge fields and the superposition $A^{Y'}_m = -\sin\theta_2 A^{\tau^4}_m + \cos\theta_2 A^{23}_m$, which is the gauge field of $Y'(= -\tan^2\theta_2 \tau^4 + \tau^{23})$ gain high masses due to the interaction with the condensate. All the vector gauge fields are expressible with the spin connection fields ω_{stm},

$$A^{Ai}_m = \sum_{s,t} c^{Ai}{}_{st}\, \omega^{st}{}_m \,. \tag{7.34}$$

Let me present expressions for the two $SU(2)$ vector gauge fields, $SU(2)_I$ and $SU(2)_{II}$

$$\begin{aligned} \vec{\mathcal{A}}^1_m &= \vec{A}^1_m = (\omega_{58m} - \omega_{67m}, \omega_{57m} + \omega_{68m}, \omega_{56m} - \omega_{78m})\,, \\ \vec{\mathcal{A}}^2_m &= \vec{A}^2_m = (\omega_{58m} + \omega_{67m}, \omega_{57m} - \omega_{68m}, \omega_{56m} + \omega_{78m})\,. \end{aligned} \tag{7.35}$$

The reader can similarly construct all the other vector gauge fields from the coefficients for the corresponding charges, or find the expressions in Refs. [4,7,9] and references therein.

The electroweak break, caused by the non zero expectation values of the scalar gauge fields, carrying the space index $s = (7, 8)$, makes the weak and the hyper charge massive. The only vector gauge fields which remains massless are the electromagnetic and the colour vector gauge fields — the observed two.

Scalar gauge fields in d $= (3+1)$ There are in the *spin-charge-family* theory scalar fields taking care of the masses of quarks and leptons: They have the space index $s = (7, 8)$ and carry with respect to the space index the weak charge $\tau^{13} = \pm\frac{1}{2}$ and the hyper charge $Y = \mp\frac{1}{2}$. With respect to $\tau^{Ai} = \sum_{ab} c^{Ai}{}_{ab} S^{ab}$ and $\tilde{\tau}^{Ai} = \sum_{ab} c^{Ai}{}_{ab} \tilde{S}^{ab}$ they carry charges and family charges in adjoint representations, Table 7.1, Eq. (7.39).

There are scalar fields transforming antileptons and antiquarks into quarks and leptons and back. They carry space index $s = (9, 10, \ldots, 14)$, They are with respect to the space index colour triplets, while they carry charges τ^{Ai} and $\tilde{\tau}^{Ai}$ in adjoint representations.

The infinitesimal generators $\mathcal{S}^{ab}$, which apply on the spin connections ω_{bde} $(= f^{\alpha}{}_e\ \omega_{bd\alpha})$ and $\tilde{\omega}_{\tilde{b}\tilde{d}e}$ $(= f^{\alpha}{}_e\ \tilde{\omega}_{\tilde{b}\tilde{d}\alpha})$, on either the space index e or any of the indices $(b, d, \tilde{b}, \tilde{d})$, as follows

$$\mathcal{S}^{ab}\, A^{d\ldots e\ldots g} = i\,(\eta^{ae}\, A^{d\ldots b\ldots g} - \eta^{be}\, A^{d\ldots a\ldots g})\,, \tag{7.36}$$

(see Section IV. and Appendix B in Ref. [9]).

Scalar gauge fields determining scalar higgs and Yukawa couplings

Let me introduce a common notation A^{Ai}_s for all the scalar gauge fields with $s = (7, 8)$, independently of whether they originate in ω_{abs} — in this case Ai

$= (Q, Q', Y')$ - or in $\tilde{\omega}_{\tilde{a}\tilde{b}s}$ — in this case all the family quantum numbers of all eight families contribute. All these gauge fields contribute to the masses of the quarks and leptons and the antiquarks and antileptons after gaining nonzero vacuum expectation values.

$$\begin{aligned}
&A^{Ai}_{s} \text{ represents } (\,A^{Q}_{s}\,, A^{Q'}_{s}\,, A^{Y'}_{s}\,, \vec{\tilde{A}}^{\tilde{1}}_{s}\,, \vec{\tilde{A}}^{\tilde{N}_{\tilde{L}}}_{s}\,, \vec{\tilde{A}}^{\tilde{2}}_{s}\,, \vec{\tilde{A}}^{\tilde{N}_{\tilde{R}}}_{s}\,)\,,\\
&\tau^{Ai} \text{ represents } (Q,\, Q',\, Y',\, \vec{\tilde{\tau}}^{1},\, \vec{\tilde{N}}_{L},\, \vec{\tilde{\tau}}^{2},\, \vec{\tilde{N}}_{R})\,. \qquad (7.37)
\end{aligned}$$

Here τ^{Ai} represent all the operators, which apply on the fermions. These scalars, the gauge scalar fields of the generators τ^{Ai} and $\tilde{\tau}^{Ai}$, are expressible in terms of the spin connection fields (Ref. [9], Eqs. (10, 22, A8, A9)).

Let me demonstrate [9] that all the scalar fields with the space index $(7, 8)$ carry with respect to this space index the weak and the hyper charge $(\mp\frac{1}{2}, \pm\frac{1}{2})$, respectively. This means that all these scalars have properties as required for the Higgs in the *standard model*.

We need to know the application of the operators τ^{13} $(= \frac{1}{2}(\mathcal{S}^{56} - \mathcal{S}^{78})$, Y $(= \tau^{4} + \tau^{23})$ and Q $(= \tau^{13} + Y)$, Eq (7.53, 7.54, 7.58), with $\mathcal{S}^{ab}$ defined in Eq. (7.36), on the scalar fields with the space index $s = (7, 8)$.

To compare the properties of the scalar fields with those of the Higgs's scalar of the *standard model* let the scalar fields be eigenstates of $\tau^{13} = \frac{1}{2}(\mathcal{S}^{56} - \mathcal{S}^{78})$.

I rewrite for this purpose the second line of Eq. (7.27) as follows, ignoring the momentum p_s, $s = (5, 6, \ldots, d)$, since it is expected that solutions with nonzero momenta in higher dimensions do not contribute to the masses of fermion fields at low energies in $d = (3+1)$. We pay correspondingly no attention to the momentum p_s , $s \in (5, \ldots, 8)$, when having in mind the lowest energy solutions, manifesting at low energies.)

$$\begin{aligned}
&\sum_{s=(7,8),A,i} \bar{\psi}\,\gamma^{s}\,(-\tau^{Ai}\,A^{Ai}_{s}\,)\,\psi =\\
&-\bar{\psi}\{\stackrel{78}{(+)}\;\tau^{Ai}\,(A^{Ai}_{7} - i\,A^{Ai}_{8}) + \stackrel{78}{(-)}\;(\tau^{Ai}\,(A^{Ai}_{7} + i\,A^{Ai}_{8})\}\psi\,,\\
&\stackrel{78}{(\pm)} = \frac{1}{2}\,(\gamma^{7} \pm\, i\gamma^{8}\,)\,, \quad A^{Ai}_{\stackrel{78}{(\pm)}} := (A^{Ai}_{7} \mp i\,A^{Ai}_{8})\,, \qquad (7.38)
\end{aligned}$$

with the summation over A and i performed, since A^{Ai}_{s} represent the scalar fields $(A^{Q}_{s}, A^{Q'}_{s}, A^{Y'}_{s})$ determined by $\omega_{s',s'',s}$ and those determined by $(\tilde{\omega}_{a,b,s}\; \tilde{A}^{\tilde{4}}_{s}, \vec{\tilde{A}}^{\tilde{1}}_{s}$, $\vec{\tilde{A}}^{\tilde{2}}_{s}$, $\vec{\tilde{A}}^{\tilde{N}_{R}}_{s}$ and $\vec{\tilde{A}}^{\tilde{N}_{L}}_{s})$.

The application of the operators τ^{13}, Y $(Y = \frac{1}{2}(\mathcal{S}^{56} + \mathcal{S}^{78}) - \frac{1}{3}(\mathcal{S}^{9\,10} + \mathcal{S}^{11\,12} + \mathcal{S}^{13\,14}))$ and Q on the scalar fields $(A^{Ai}_{7} \mp i\,A^{Ai}_{8})$ with respect to the space index $s = (7, 8)$, by taking into account Eq. (7.36) to make the application of the generators $\mathcal{S}^{ab}$ on the space indexes, gives

$$\begin{aligned}
\tau^{13}\,(A^{Ai}_{7} \mp i\,A^{Ai}_{8}) &= \pm\,\frac{1}{2}\,(A^{Ai}_{7} \mp i\,A^{Ai}_{8})\,,\\
Y\,(A^{Ai}_{7} \mp i\,A^{Ai}_{8}) &= \mp\,\frac{1}{2}\,(A^{Ai}_{7} \mp i\,A^{Ai}_{8})\,,\\
Q\,(A^{Ai}_{7} \mp i\,A^{Ai}_{8}) &= 0\,. \qquad (7.39)
\end{aligned}$$

Since τ^4, Y, τ^{13} and τ^{1+}, τ^{1-} give zero if applied on (A^Q_s, $A^{Q'}_s$ and $A^{Y'}_s$) with respect to the quantum numbers (Q, Q', Y'), and since Y and τ^{13} commute with the family quantum numbers, one sees that the scalar fields A^{Ai}_s (=(A^Q_s, A^Y_s, $A^{Y'}_s$, $\tilde{A}^{\tilde{4}}_s$, $\tilde{A}^{\tilde{Q}}_s$, $\vec{\tilde{A}}^{\tilde{1}}_s$, $\vec{\tilde{A}}^{\tilde{2}}_s$, $\vec{\tilde{A}}^{\tilde{N}_R}_s$, $\vec{\tilde{A}}^{\tilde{N}_L}_s$)), rewritten as $A^{Ai}_{\substack{78\\(\pm)}} = (A^{Ai}_7 \mp i\, A^{Ai}_8)$, are eigenstates of τ^{13} and Y, having the quantum numbers of the *standard model* Higgs' scalar.

These superposition of $A^{Ai}_{\substack{78\\(\pm)}}$ are presented in Table 7.1 as two doublets with respect to the weak charge τ^{13}, with the eigenvalue of τ^{23} (the second $SU(2)_{II}$ charge) equal to either $-\frac{1}{2}$ or $+\frac{1}{2}$, respectively. The operators $\tau^{1\boxed{\pm}} = \tau^{11} \pm i\tau^{12}$

name	superposition	τ^{13}	τ^{23}	spin	τ^4	Q
$A^{Ai}_{\substack{78\\(-)}}$	$A^{Ai}_7 + iA^{Ai}_8$	$+\frac{1}{2}$	$-\frac{1}{2}$	0	0	0
$A^{Ai}_{\substack{56\\(-)}}$	$A^{Ai}_5 + iA^{Ai}_6$	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	-1
$A^{Ai}_{\substack{78\\(+)}}$	$A^{Ai}_7 - iA^{Ai}_8$	$-\frac{1}{2}$	$+\frac{1}{2}$	0	0	0
$A^{Ai}_{\substack{56\\(+)}}$	$A^{Ai}_5 - iA^{Ai}_6$	$+\frac{1}{2}$	$+\frac{1}{2}$	0	0	+1

Table 7.1. The two scalar weak doublets, one with $\tau^{23} = -\frac{1}{2}$ and the other with $\tau^{23} = +\frac{1}{2}$, both with the "fermion" quantum number $\tau^4 = 0$, are presented. In this table all the scalar fields carry besides the quantum numbers determined by the space index also the quantum numbers A and i from Eq. (7.37). The table is taken from Ref. [9].

$$\tau^{1\boxed{\pm}} = \frac{1}{2}[(\mathcal{S}^{58} - \mathcal{S}^{67})\ \boxed{\mp}\ i\,(\mathcal{S}^{57} + \mathcal{S}^{68})]\,, \tag{7.40}$$

transform one member of a doublet from Table 7.1 into another member of the same doublet, keeping τ^{23} ($= \frac{1}{2}\,(\mathcal{S}^{56} + \mathcal{S}^{78})$) unchanged, clarifying the above statement.

It is not difficult to show that the scalar fields $A^{Ai}_{\substack{78\\(\pm)}}$ are *triplets* as the gauge fields of the *family quantum numbers* ($\vec{\tilde{N}}_R$, $\vec{\tilde{N}}_L$, $\vec{\tilde{\tau}}^2$, $\vec{\tilde{\tau}}^1$; Eqs. (7.55, 7.56, 7.36)) or singlets as the gauge fields of $Q = \tau^{13} + Y$, $Q' = -\tan^2\vartheta_1 Y + \tau^{13}$ and $Y' = -\tan^2\vartheta_2 \tau^4 + \tau^{23}$.

Let us do this for $\tilde{A}^{\tilde{N}_L i}_{\substack{78\\(\pm)}}$ and for $A^{Q}_{\substack{78\\(\pm)}}$, taking into account Eq. (7.52) (where we replace S^{ab} by $\mathcal{S}^{ab}$) and Eq. (7.36), and recognizing that $\tilde{A}^{\tilde{N}_L \boxed{\pm}}_{\substack{78\\(\pm)}} = \tilde{A}^{\tilde{N}_L 1}_{\substack{78\\(\pm)}}\ \boxed{\mp}\ i\,\tilde{A}^{\tilde{N}_L 2}_{\substack{78\\(\pm)}}$.

$$\begin{aligned}
\tilde{A}^{\tilde{N}_L \boxed{\pm}}_{\substack{78\\(\pm)}} &= \{(\tilde{\omega}_{23\substack{78\\(\pm)}} + i\,\tilde{\omega}_{01\substack{78\\(\pm)}})\ \boxed{\mp}\ i\,(\tilde{\omega}_{31\substack{78\\(\pm)}} + i\,\tilde{\omega}_{02\substack{78\\(\pm)}})\}\,,\\
\tilde{A}^{\tilde{N}_L 3}_{\substack{78\\(\pm)}} &= (\tilde{\omega}_{12\substack{78\\(\pm)}} + i\,\tilde{\omega}_{03\substack{78\\(\pm)}})\,,\\
A^{Q}_{\substack{78\\(\pm)}} &= \omega_{56\substack{78\\(\pm)}} - (\omega_{9\,10\substack{78\\(\pm)}} + \omega_{11\,12\substack{78\\(\pm)}} + \omega_{13\,14\substack{78\\(\pm)}})\,.
\end{aligned}$$

One finds

$$
\begin{aligned}
\tilde{N}^{3}_{L}\, \tilde{A}^{\tilde{N}_{L} \boxplus}_{\substack{78\\(\pm)}} &= \boxplus\, \tilde{A}^{\tilde{N}_{L} \boxplus}_{\substack{78\\(\pm)}}\,, \quad \tilde{N}^{3}_{L}\, \tilde{A}^{\tilde{N}_{L} 3}_{\substack{78\\(\pm)}} = 0\,,\\
Q\, A^{Q}_{\substack{78\\(\pm)}} &= 0\,.
\end{aligned} \tag{7.41}
$$

with $Q = \mathcal{S}^{56} + \tau^{4} = \mathcal{S}^{56} - \frac{1}{3}(\mathcal{S}^{9\,10} + \mathcal{S}^{11\,12} + \mathcal{S}^{13\,14})$, and with τ^4 defined in Eq. (7.54), if replacing S^{ab} by $\mathcal{S}^{ab}$ from Eq. (7.36). Similarly one finds properties with respect to the Ai quantum numbers for all the scalar fields $A^{Ai}_{\substack{78\\(\pm)}}$.

After the appearance of the condensate (Table 7.5), which breaks the $SU(2)_{II}$ symmetry and brings masses to all the scalar fields, the weak $\vec{\tau}^1$ and the hyper charge Y remain the conserved charges.

At the electroweak scale the scalar gauge fields with the space index $(7,8)$, with the Lagrange density

$$
\begin{aligned}
\mathcal{L}_{sg} = E \sum_{A,i} \{ & (p_m\, A^{Ai}_{s})^{\dagger}\, (p^m\, A^{Ai}_{s}) - (-\lambda^{Ai} + (m'_{Ai})^2))\, A^{Ai\dagger}_{s} A^{Ai}_{s} \\
& + \sum_{B,j} \Lambda^{AiBj}\, A^{Ai\dagger}_{s} A^{Ai}_{s}\, A^{Bj\dagger}_{s} A^{Bj}_{s}\},
\end{aligned} \tag{7.42}
$$

gain nonzero vacuum expectation values and cause the electroweak break [9]. The above Lagrange density needs to be studied. At this stage is just postulated.

The two groups of four families became massive. The mass matrices manifest either $\widetilde{SU}(2)_{\widetilde{SO}(3,1)L} \times \widetilde{SU}(2)_{\widetilde{SU}(4)L} \times U(1)$ symmetry, this is the case for the lower four families of the eight families, presented in Table 7.4, or $\widetilde{SU}(2)_{\widetilde{SO}(3,1)R} \times \widetilde{SU}(2)_{\widetilde{SU}(4)R} \times U(1)$ symmetry, this is the case for the higher four families, presented in Table 7.4. The same three $U(1)$ singlet fields contribute to the masses of both groups, the two $SU(2)$ triplet fields are for each of the two groups different, although manifesting the same symmetries.

The mass matrix of family member — quarks and leptons — are 4×4 matrices. The observed three families of quarks and leptons form the 3×3 submatrices of the 4×4 matrices. The symmetry of the mass matrices, manifesting in all orders [57], limits the number of free parameters.

All the scalars, the two triplets and the three singlets, are doublets with respect to the weak charge, contributing to the weak and the hyper charge of the fermions so that they transform the right handed members into the left handed onces.

$$
\mathcal{M}^{\alpha} = \begin{pmatrix} -a_1 - a & e & d & b \\ e & -a_2 - a & b & d \\ d & b & a_2 - a & e \\ b & d & e & a_1 - a \end{pmatrix}^{\alpha}, \tag{7.43}
$$

with α representing family members — quarks and leptons of left and right handedness [49–51, 53, 54, 58].

[9] The expression for the Lagrange density of Eq. (7.42) is only estimated, more or less guessed, I have no estimate yet for the constants.

The mass matrices of the upper four families have the same symmetry as the mass matrices of the lower four families, but the scalar fields determining the masses of the upper four families have different properties (nonzero vacuum expectation values, masses and coupling constants) than those of the lower four, giving to quarks and leptons of the upper four families much higher masses in comparison with the lower four families of quarks and leptons, what offers the explanation for the appearance of the *dark matter*, studied at Refs. [52, 61].

Scalar fields transforming antiquarks and antileptons into quarks and leptons

I follow in this part to a great deal similar part in Ref. [3].

To the matter-antimatter asymmetry the terms contribute, which cause transitions from antileptons into quarks and from antiquarks into quarks and back. These are terms included into the third line of Eq. (7.27). Let me rewrite this part of the fermion action

$$\mathcal{L}_{f'} = \psi^{\dagger} \gamma^0 \gamma^t \Big\{ \sum_{t=(9,10,\dots 14)} \big[p_t - (\frac{1}{2} S^{s's"} \omega_{s's"t} + \frac{1}{2} S^{t't''} \omega_{t't"t} + \frac{1}{2} \tilde{S}^{ab} \tilde{\omega}_{abt})\big] \Big\} \psi,$$

as follows

$$\begin{aligned}
\mathcal{L}_{f"} = \psi^{\dagger} \gamma^0 (-) \{ \sum_{+,-} \sum_{(t\,t')} \overset{tt'}{(\oplus)} \cdot \\
[\tau^{2+} A^{2+}_{\overset{tt'}{(\oplus)}} + \tau^{2-} A^{2-}_{\overset{tt'}{(\oplus)}} + \tau^{23} A^{23}_{\overset{tt'}{(\oplus)}} + \tau^{1+} A^{1+}_{\overset{tt'}{(\oplus)}} + \tau^{1-} A^{1-}_{\overset{tt'}{(\oplus)}} + \tau^{13} A^{13}_{\overset{tt'}{(\oplus)}} \\
+ \tilde{\tau}^{2+} \tilde{A}^{2+}_{\overset{tt'}{(\oplus)}} + \tilde{\tau}^{2-} \tilde{A}^{2-}_{\overset{tt'}{(\oplus)}} + \tilde{\tau}^{23} \tilde{A}^{23}_{\overset{tt'}{(\oplus)}} + \tilde{\tau}^{1+} \tilde{A}^{1+}_{\overset{tt'}{(\oplus)}} + \tilde{\tau}^{1-} \tilde{A}^{1-}_{\overset{tt'}{(\oplus)}} + \tilde{\tau}^{13} \tilde{A}^{13}_{\overset{tt'}{(\oplus)}} \\
+ \tilde{N}^{+}_{R} \tilde{A}^{N_R +}_{\overset{tt'}{(\oplus)}} + \tilde{N}^{-}_{R} \tilde{A}^{N_R -}_{\overset{tt'}{(\oplus)}} + \tilde{N}^{3}_{R} \tilde{A}^{N_R 3}_{\overset{tt'}{(\oplus)}} + \tilde{N}^{+}_{L} \tilde{A}^{N_L +}_{\overset{tt'}{(\oplus)}} + \tilde{N}^{-}_{L} \tilde{A}^{N_L -}_{\overset{tt'}{(\oplus)}} + \tilde{N}^{3}_{L} \tilde{A}^{N_L 3}_{\overset{tt'}{(\oplus)}} \\
+ \sum_{i} \tau^{3i} A^{3i}_{\overset{tt'}{(\oplus)}} + \tau^{4} A^{4}_{\overset{tt'}{(\oplus)}} + \sum_{i} \tilde{\tau}^{3i} \tilde{A}^{3i}_{\overset{tt'}{(\oplus)}} + \tilde{\tau}^{4} \tilde{A}^{4}_{\overset{tt'}{(\oplus)}}] \} \psi \,,
\end{aligned} \tag{7.44}$$

where (t, t') run in pairs over $[(9, 10), (11, 12), (13, 14)]$ and the summation must go over $+$ and $-$ of $\overset{tt'}{(\oplus)}$.

In Eq. (7.44) the relations below are used

$$
\begin{aligned}
\sum_{t,s',s''} \gamma^t \, \frac{1}{2} S^{s's''} \omega_{s's''t} &= \sum_{+,-} \sum_{(tt')} \stackrel{tt'}{(\oplus)} \, \frac{1}{2} S^{s's''} \omega_{s''s''\stackrel{tt'}{(\oplus)}}\,, \\
\omega_{s''s''\stackrel{tt'}{(\oplus)}} &:= \omega_{s''s''\stackrel{tt'}{(\pm)}} = (\omega_{s's''t} \mp i\, \omega_{s's''t'})\,, \\
\stackrel{tt'}{(\oplus)} &:= = \frac{1}{2}\,(\gamma^t \pm \gamma^{t'})\,, \\
\stackrel{tt'}{(\oplus)} \, \frac{1}{2} S^{s's''} \omega_{s''s''\stackrel{tt'}{(\oplus)}} &= \stackrel{tt'}{(\oplus)} \, \{\tau^{2+} A^{2+}_{\stackrel{tt'}{(\oplus)}} + \tau^{2-} A^{2-}_{\stackrel{tt'}{(\oplus)}} + \tau^{23} A^{23}_{\stackrel{tt'}{(\oplus)}} \\
&\quad + \tau^{1+} A^{1+}_{\stackrel{tt'}{(\oplus)}} + \tau^{1-} A^{1-}_{\stackrel{tt'}{(\oplus)}} + \tau^{13} A^{13}_{\stackrel{tt'}{(\oplus)}}\}\,, \\
A^{2\boxed{\pm}}_{\stackrel{tt'}{(\oplus)}} &= (\omega_{58\stackrel{tt'}{(\oplus)}} + \omega_{67\stackrel{tt'}{(\oplus)}}) \boxed{\mp}\, i(\omega_{57\stackrel{tt'}{(\oplus)}} - \omega_{68\stackrel{tt'}{(\oplus)}})\,, \\
A^{23}_{\stackrel{tt'}{(\oplus)}} &= (\omega_{56\stackrel{tt'}{(\oplus)}} + \omega_{78\stackrel{tt'}{(\oplus)}})\,, \\
A^{1\boxed{\pm}}_{\stackrel{tt'}{(\oplus)}} &= (\omega_{58\stackrel{tt'}{(\oplus)}} - \omega_{67\stackrel{tt'}{(\oplus)}}) \boxed{\mp}\, i(\omega_{57\stackrel{tt'}{(\oplus)}} + \omega_{68\stackrel{tt'}{(\oplus)}})\,, \\
A^{13}_{\stackrel{tt'}{(\oplus)}} &= (\omega_{56\stackrel{tt'}{(\oplus)}} - \omega_{78\stackrel{tt'}{(\oplus)}})\,, \\
(t\,t') &\in ((9\,10), (11\,12), (13\,14))\,.
\end{aligned}
\tag{7.45}
$$

The rest of expressions in Eq. (7.45) are obtained in a similar way. They are presented in Eq. (7.62).

The scalar fields with the scalar index $s = (9, 10, \cdots, 14)$, presented in Table 7.2, carry one of the triplet colour charges and the "fermion" charge equal to twice the quark "fermion" charge, or the antitriplet colour charges and the "antifermion" charge. They carry in addition the quantum numbers of the adjoint representations originating in S^{ab} or in $\tilde{S}^{ab}$ [10].

If the antiquark $\bar{u}^{\bar{c}2}_L$, from the line 43 presented in Table 7.3, with the "fermion" charge $\tau^4 = -\frac{1}{6}$, the weak charge $\tau^{13} = 0$, the second $SU(2)_{II}$ charge $\tau^{23} = -\frac{1}{2}$, the colour charge $(\tau^{33}, \tau^{38}) = (\frac{1}{2}, -\frac{1}{2\sqrt{3}})$, the hyper charge $Y (= \tau^4 + \tau^{23} =) -\frac{2}{3}$ and the electromagnetic charge $Q (= Y + \tau^{13} =) -\frac{2}{3}$ submits the $A^{2\boxminus}_{\stackrel{9\,10}{(\oplus)}}$ scalar field, it transforms into u^{c3}_R from the line 17 of Table 7.3, carrying the quantum numbers $\tau^4 = \frac{1}{6}$, $\tau^{13} = 0$, $\tau^{23} = \frac{1}{2}$, $(\tau^{33}, \tau^{38}) = (0, -\frac{1}{\sqrt{3}})$, $Y = \frac{2}{3}$ and $Q = \frac{2}{3}$. These two quarks, d^{c1}_R and u^{c3}_R can bind together with u^{c2}_R from the 9^{th} line of the same table (at low enough energy, after the electroweak transition, and if they belong to a superposition with the left handed partners to the first family) -into the colour chargeless baryon - a proton. This transition is presented in Figure 7.1.

The opposite transition at low energies would make the proton decay.

[10] Although carrying the colour charge in one of the triplet or antitriplet states, these fields can not be interpreted as superpartners of the quarks since they do not have quantum numbers as required by, let say, the $N = 1$ supersymmetry. The hyper charges and the electromagnetic charges are namely not those required by the supersymmetric partners to the family members.

field	prop.	τ^4	τ^{13}	τ^{23}	(τ^{33},τ^{38})	Y	Q	$\tilde{\tau}^4$	$\tilde{\tau}^{13}$	$\tilde{\tau}^{23}$	$\tilde{N}^3_L$	$\tilde{N}^3_R$
$A^{1\boxplus}_{9\,10\,(\oplus)}$	scalar	$\oplus\,\frac{1}{3}$	$\boxplus\,1$	0	$(\oplus\,\frac{1}{2},\oplus\,\frac{1}{2\sqrt{3}})$	$\oplus\,\frac{1}{3}$	$\oplus\,\frac{1}{3}+\boxplus\,1$	0	0	0	0	0
$A^{13}_{9\,10\,(\oplus)}$	scalar	$\oplus\,\frac{1}{3}$	0	0	$(\oplus\,\frac{1}{2},\oplus\,\frac{1}{2\sqrt{3}})$	$\oplus\,\frac{1}{3}$	$\oplus\,\frac{1}{3}$	0	0	0	0	0
$A^{1\boxplus}_{11\,12\,(\oplus)}$	scalar	$\oplus\,\frac{1}{3}$	$\boxplus\,1$	0	$(\oplus\,\frac{1}{2},\oplus\,\frac{1}{2\sqrt{3}})$	$\oplus\,\frac{1}{3}$	$\oplus\,\frac{1}{3}+\boxplus\,1$	0	0	0	0	0
$A^{13}_{11\,12\,(\oplus)}$	scalar	$\oplus\,\frac{1}{3}$	0	0	$(\oplus\,\frac{1}{2},\oplus\,\frac{1}{2\sqrt{3}})$	$\oplus\,\frac{1}{3}$	$\oplus\,\frac{1}{3}$	0	0	0	0	0
$A^{1\boxplus}_{13\,14\,(\oplus)}$	scalar	$\oplus\,\frac{1}{3}$	$\boxplus\,1$	0	$(0,\oplus\,\frac{1}{\sqrt{3}})$	$\oplus\,\frac{1}{3}$	$\oplus\,\frac{1}{3}+\boxplus\,1$	0	0	0	0	0
$A^{13}_{13\,14\,(\oplus)}$	scalar	$\oplus\,\frac{1}{3}$	0	0	$(0,\oplus\,\frac{1}{\sqrt{3}})$	$\oplus\,\frac{1}{3}$	$\oplus\,\frac{1}{3}$	0	0	0	0	0
$A^{2\boxplus}_{9\,10\,(\oplus)}$	scalar	$\oplus\,\frac{1}{3}$	0	$\boxplus\,1$	$(\oplus\,\frac{1}{2},\oplus\,\frac{1}{2\sqrt{3}})$	$\oplus\,\frac{1}{3}+\boxplus\,1$	$\oplus\,\frac{1}{3}+\boxplus\,1$	0	0	0	0	0
$A^{23}_{9\,10\,(\oplus)}$	scalar	$\oplus\,\frac{1}{3}$	0	0	$(\oplus\,\frac{1}{2},\oplus\,\frac{1}{2\sqrt{3}})$	$\oplus\,\frac{1}{3}$	$\oplus\,\frac{1}{3}$	0	0	0	0	0
$\cdots$												
$\tilde{A}^{1\boxplus}_{910\,(\oplus)}$	scalar	$\oplus\,\frac{1}{3}$	0	0	$(\oplus\,\frac{1}{2},\oplus\,\frac{1}{2\sqrt{3}})$	$\oplus\,\frac{1}{3}$	$\oplus\,\frac{1}{3}$	0	$\boxplus\,1$	0	0	0
$\tilde{A}^{13}_{910\,(\oplus)}$	scalar	$\oplus\,\frac{1}{3}$	0	0	$(\oplus\,\frac{1}{2},\oplus\,\frac{1}{2\sqrt{3}})$	$\oplus\,\frac{1}{3}$	$\oplus\,\frac{1}{3}$	0	0	0	0	0
$\cdots$												
$\tilde{A}^{2\boxplus}_{910\,(\oplus)}$	scalar	$\oplus\,\frac{1}{3}$	0	0	$(\oplus\,\frac{1}{2},\oplus\,\frac{1}{2\sqrt{3}})$	$\oplus\,\frac{1}{3}$	$\oplus\,\frac{1}{3}$	0	0	$\boxplus\,1$	0	0
$\tilde{A}^{23}_{910\,(\oplus)}$	scalar	$\oplus\,\frac{1}{3}$	0	0	$(\oplus\,\frac{1}{2},\oplus\,\frac{1}{2\sqrt{3}})$	$\oplus\,\frac{1}{3}$	$\oplus\,\frac{1}{3}$	0	0	0	0	0
$\cdots$												
$\tilde{A}^{N_L\boxplus}_{910\,(\oplus)}$	scalar	$\oplus\,\frac{1}{3}$	0	0	$(\oplus\,\frac{1}{2},\oplus\,\frac{1}{2\sqrt{3}})$	$\oplus\,\frac{1}{3}$	$\oplus\,\frac{1}{3}$	0	0	0	$\boxplus\,1$	0
$\tilde{A}^{N_L 3}_{910\,(\oplus)}$	scalar	$\oplus\,\frac{1}{3}$	0	0	$(\oplus\,\frac{1}{2},\oplus\,\frac{1}{2\sqrt{3}})$	$\oplus\,\frac{1}{3}$	$\oplus\,\frac{1}{3}$	0	0	0	0	0
$\cdots$												
$\tilde{A}^{N_R\boxplus}_{910\,(\oplus)}$	scalar	$\oplus\,\frac{1}{3}$	0	0	$(\oplus\,\frac{1}{2},\oplus\,\frac{1}{2\sqrt{3}})$	$\oplus\,\frac{1}{3}$	$\oplus\,\frac{1}{3}$	0	0	0	0	$\boxplus\,1$
$\tilde{A}^{N_R 3}_{910\,(\oplus)}$	scalar	$\oplus\,\frac{1}{3}$	0	0	$(\oplus\,\frac{1}{2},\oplus\,\frac{1}{2\sqrt{3}})$	$\oplus\,\frac{1}{3}$	$\oplus\,\frac{1}{3}$	0	0	0	0	0
$\cdots$												
$A^{3i}_{9\,10\,(\oplus)}$	scalar	$\oplus\,\frac{1}{3}$	0	0	$(\boxplus\,1+\oplus\,\frac{1}{2},\oplus\,\frac{1}{2\sqrt{3}})$	$\oplus\,\frac{1}{3}$	$\oplus\,\frac{1}{3}$	0	0	0	0	0
$\cdots$												
$A^{4}_{9\,10\,(\oplus)}$	scalar	$\oplus\,\frac{1}{3}$	0	0	$(\oplus\,\frac{1}{2},\oplus\,\frac{1}{2\sqrt{3}})$	$\oplus\,\frac{1}{3}$	$\oplus\,\frac{1}{3}$	0	0	0	0	0
$\cdots$												
$\vec{A}^3_m$	vector	0	0	0	octet	0	0	0	0	0	0	0
A^4_m	vector	0	0	0	0	0	0	0	0	0	0	0

Table 7.2. Quantum numbers of the scalar gauge fields carrying the space index $t = (9, 10, \cdots, 14)$, appearing in Eq. (7.27), are presented. The space degrees of freedom contribute one of the triplets values to the colour charge of all these scalar fields. These scalars are with respect to the two $SU(2)$ charges, (τ^{13} and $\vec{\tau}^2$), and the two $\widetilde{SU}(2)$ charges, ($\vec{\tilde{\tau}}^1$ and $\vec{\tilde{\tau}}^2$), triplets (that is in the adjoint representations of the corresponding groups), and they all carry twice the "fermion" number (τ^4) of the quarks. The quantum numbers of the two vector gauge fields, the colour and the $U(1)_{II}$ ones, are added.

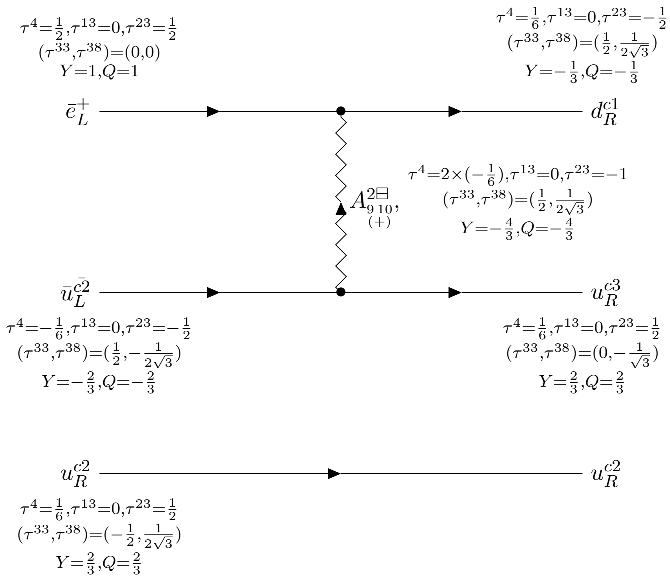

Fig. 7.1. The birth of a "right handed proton" out of an positron $\bar{e}_L{}^{+}$, antiquark $\bar{u}_L^{\bar{c2}}$ and quark (spectator) u_R^{c2}. The family quantum number can be any.

7.3 Achievements and conclusions

It remains to point out the achievements of the *spin-charge-family* theory so far and tell the open problems.

Achievements:

a. The simple starting action, Eq. (7.1), with the Clifford algebra used to describe the internal space of fermions, which in $d \geq (13+1)$ interact with the vielbeins and the two kinds of spin connection fields, offers **a.i.** that one irreducible representation of the Lorentz algebra in internal space manifests in $d = (3+1)$ all the fermions and antifermions with the spins and charges of the *standard model*, **a.ii.** that eight irreducible representations define in $d = (3+1)$ (after the reduction of the Clifford algebra from two kinds to only one kind) two times four families, **a.iii.** that the two kinds of the spin connection fields manifest in $d = (3+1)$ all the vector gauge fields of the *standard model*, **a.iv.** that the scalar fields with respect to $d = (3+1)$, carrying the weak and the hyper charge $\pm\frac{1}{2}$ and $\mp\frac{1}{2}$, respectively, forming two groups of scalar fields manifesting each the $SU(2) \times SU(2) \times U(1)$ symmetry, offer the explanation for the Higgs's scalar and Yukawa couplings of the *standard model* giving masses to two groups of four families — the lower four families predicting the fourth family of quarks and leptons to the observed three, the stable of the upper four families offering explanation for the *dark matter*, **a.v.** that both groups of four families together spread masses from almost zero to $\geq 10^{16}$ GeV, **a.vi.** that the scalar gauge fields manifesting as colour triplets and antitriplets offer the explanation for the matter/antimatter asymmetry of the ordinary matter.

b. The decision to describe the internal space of fermions with the Clifford odd algebra enables to define the creation operators as tensor products of finite number of "basis vectors" of internal space and infinite basis in ordinary space applying on the vacuum state, which fulfill together the their Hermitian conjugated annihilation operators the anticommutation relations postulated by Dirac for the second quantized fields. The single fermion states have therefore by themselves the anticommuting character. Tensor products of any number and any kind of the single fermion creation operators define the second quantized fermion fields forming the whole Hilbert space.

Predictions:

The *spin-charge-family* theory offers several explanations as discuss in Sects. 7.1 and 7.2 and also several predictions.
A. Prediction of the fourth family to the observed three families, Subsect. 7.2.1. Taking into account the experimental data for masses of the observed families of quarks and the corresponding mixing matrix we fit 6 parameters of the two quark mass matrices, presented in Eq. (7.43), to twice 3 measured massess of quarks and to 6 measured parameters of the mixing matrix.

Althrough any accurate 3×3 submatrix of the 4×4 unitary matrix determines the 4×4 matrix completely, neither the quark nor the lepton mixing matrix is measured accurately enough that it would be possible to determine three complex phases of the 4×4 quark mixing matrix and the mixing matrix elements of the fourth family quarks to the other three family members. We therefore assume that mass matrices are symmetric and real, while making a choice for the masses of the fourth family.

Results are presented for two choices of $m_{u_4} = m_{d_4}$, Ref. [54], [arxiv:1412.5866]:
1. $m_{u_4} = 700$ GeV, $m_{d_4} = 700$ GeV.....new_1
2. $m_{u_4} = 1\,200$ GeV, $m_{d_4} = 1200$ GeV.....new_2

$$|V_{(ud)}| = \begin{pmatrix} exp_n & 0.97425 \pm 0.00022 & 0.2253 \pm 0.0008 & 0.00413 \pm 0.00049 & \\ \hline new_1 & 0.97423(4) & 0.22539(7) & 0.00299 & \mathit{0.00776(1)} \\ new_2 & 0.97423[5] & 0.22538[42] & 0.00299 & \mathbf{0.00793}[\mathbf{466}] \\ \hline exp_n & 0.225 \pm 0.008 & 0.986 \pm 0.016 & 0.0411 \pm 0.0013 & \\ \hline new_1 & 0.22534(3) & 0.97335 & 0.04245(6) & \mathit{0.00349(60)} \\ new_2 & 0.22531[5] & 0.97336[5] & 0.04248 & \mathbf{0.00002}[\mathbf{216}] \\ \hline exp_n & 0.0084 \pm 0.0006 & 0.0400 \pm 0.0027 & 1.021 \pm 0.032 & \\ \hline new_1 & 0.00667(6) & 0.04203(4) & 0.99909 & \mathit{0.00038} \\ new_2 & 0.00667 & 0.04206[5] & 0.99909 & \mathbf{0.00024}[\mathbf{21}] \\ \hline new_1 & \mathit{0.00677(60)} & \mathit{0.00517(26)} & \mathit{0.00020} & \mathit{0.99996} \\ new_2 & \mathbf{0.00773} & \mathbf{0.00178} & \mathbf{0.00022} & \mathbf{0.99997}[\mathbf{9}] \end{pmatrix} . \tag{7.46}$$

One can see that the above results for the mixing matrices of the lower three families are in agreement with what Ref. [55] requires, namely that
$V_{u_1 d_4} > V_{u_1 d_3}$, $V_{u_2 d_4} < V_{u_1 d_4}$, and $V_{u_3 d_4} < V_{u_1 d_4}$.

Since we have not yet fit the mass matrix of Eq. (7.43) to the newest experimental data [56], which appear after our Bled 2020, the evaluation for our 4×4 quark mixing matrix with the new data and correspondingly a new prediction is not yet offered.

Let me repeat the discussion of Ref. [58] that the existence of the fourth family to the observed three is still not in disagreement with the latest experimental data although some phenomenologists say different.

B. The *spin-charge-family* theory predicts in the low energy regime (up to 10^{16} GeV or higher) the existence of two decoupled groups of four families, which at the electroweak break become massive [52]. The stable family of the upper group of four families (with almost zero Yukawa couplings to the lower group of four families) is the candidate for the dark matter, Subsect. 7.2.1.

I review here briefly the estimations done in Ref. [52]. We used the simple hydrogen-like model to evaluate properties of the fifth family heavy baryons, taking into account that for masses of the order of a few TeV or larger the force among the constituents of the fifth family baryons is determined mostly by one gluon exchange. The fifth family neutron is estimated as the most stable nucleon. The "nuclear interaction" among these baryons is found to have very interesting properties. We studied scattering amplitudes among fifth family neutrons and with the ordinary matter.

We followed the behaviour of the fifth family quarks and antiquarks in the plasma of the expanding universe, through the freezing out procedure, solving the Boltzmann equations, through the colour phase transition, while forming neutrons, up to the present dark matter, taking into account the cosmological evidences, the direct experimental evidences and all others known properties of the dark matter.

The cosmological evolution suggested the limits for the masses of the fifth family quarks

$$10\ \text{TeV} < m_{q_5}\, c^2 < \text{a few} \cdot 10^2\ \text{TeV} \tag{7.47}$$

and for the scattering cross sections

$$10^{-8}\,\text{fm}^2 < \sigma_{c_5} < 10^{-6}\,\text{fm}^2\,, \tag{7.48}$$

while the measured density of the dark matter does not put much limitation on the properties of heavy enough clusters.

The direct measurements limit the fifth family quark mass to

$$\text{several}\, 10\,\text{TeV} < m_{q_5} c^2 < 10^5\,\text{TeV}\,. \tag{7.49}$$

We also find that our fifth family baryons of the mass of several 10 TeV/c^2 have for a factor more than 100 times too small scattering amplitude with the ordinary matter to cause a measurable heat flux on the Earth's surface.

C. The *spin-charge-family* theory predicts several scalar fields with the weak and the hyper charge of the Higg's scalar $(\pm\frac{1}{2}, \mp\frac{1}{2})$ — two triplets and three singlets — offering the explanation for the existence of the Higgs's scalar and Yukawa couplings, Subsect. 7.2.2.

The additional two triplets and the same three singlets determine properties of the upper four families of quarks and leptons, Subsect. 7.2.2.

D. The *spin-charge-family* theory predicts several scalar fields which are colour triplets or antitriplets, offering the explanation for the matter/antimatter asymmetry in the (nonequilibrium) expanding universe as well as the proton decay [4], Subsect. 7.2.2.
E. The mass matrices of the two fourth family groups are close to democratic one, causing spreading of the fermion masses from 10^{-8} MeV to 10^{16} GeV or even higher.

I conclude by saying that there are still a lot of open problems to be solved. Some of them are common to the other theories, like the Kaluza-Klein-like theories, the others require to extract as much as possible from the offer of the theory. We need collaborators, since the more work is put into the *spin-charge-family* theory the more explanations for the observed phenomena follow.

7.4 APPENDIX: Useful relations

From Eq. (7.16) it follows

$$
\begin{aligned}
S^{ac}\overset{ab}{(k)}\overset{cd}{(k)} &= -\frac{i}{2}\eta^{aa}\eta^{cc}\overset{ab}{[-k]}\overset{cd}{[-k]}\,, &\quad \tilde{S}^{ac}\overset{ab}{(k)}\overset{cd}{(k)} &= \frac{i}{2}\eta^{aa}\eta^{cc}\overset{ab}{[k]}\overset{cd}{[k]}\,,\\
S^{ac}\overset{ab}{[k]}\overset{cd}{[k]} &= \frac{i}{2}\overset{ab}{(-k)}\overset{cd}{(-k)}\,, &\quad \tilde{S}^{ac}\overset{ab}{[k]}\overset{cd}{[k]} &= -\frac{i}{2}\overset{ab}{(k)}\overset{cd}{(k)}\,,\\
S^{ac}\overset{ab}{(k)}\overset{cd}{[k]} &= -\frac{i}{2}\eta^{aa}\overset{ab}{[-k]}\overset{cd}{(-k)}\,, &\quad \tilde{S}^{ac}\overset{ab}{(k)}\overset{cd}{[k]} &= -\frac{i}{2}\eta^{aa}\overset{ab}{[k]}\overset{cd}{(k)}\,,\\
S^{ac}\overset{ab}{[k]}\overset{cd}{(k)} &= \frac{i}{2}\eta^{cc}\overset{ab}{(-k)}\overset{cd}{[-k]}\,, &\quad \tilde{S}^{ac}\overset{ab}{[k]}\overset{cd}{(k)} &= \frac{i}{2}\eta^{cc}\overset{ab}{(k)}\overset{cd}{[k]}\,.
\end{aligned}
\tag{7.50}
$$

By using Eq. (7.14) one finds the relations

$$
\begin{aligned}
&\overset{ab}{(\tilde{k})}\overset{ab}{(k)} = 0\,, \quad \overset{ab}{(\tilde{-k})}\overset{ab}{(k)} = -i\eta^{aa}\overset{ab}{[k]}\,,\\
&\overset{ab}{(\tilde{k})}\overset{ab}{[k]} = i\overset{ab}{(k)}\,, \quad \overset{ab}{(\tilde{k})}\overset{ab}{[-k]} = 0\,,\\
&\overset{ab}{[\tilde{k}]}\overset{ab}{(k)} = \overset{ab}{(k)}\,, \quad \overset{ab}{[\tilde{-k}]}\overset{ab}{(k)} = 0\,,\\
&\overset{ab}{[\tilde{k}]}\overset{ab}{[k]} = 0\,, \quad \overset{ab}{[\tilde{-k}]}\overset{ab}{[k]} = \overset{ab}{[k]}\,.
\end{aligned}
\tag{7.51}
$$

7.5 APPENDIX: One irreducible representation of the internal space and families described by the Clifford algebra γ^a

Below the subgroups of the starting groups $SO(13,1)$ and $\tilde{SO}(13,1)$ are presented, manifesting in $d = (3+1)$ the spins, charges and families of fermions in the *spin-charge-family* theory. Table 7.3, representing one $SO(13,1)$ irreducible representation of fermions — quarks and leptons and antiquarks and antileptons — uses these expressions.

a.i. The generators of the two $SU(2)$ ($\subset SO(3,1) \subset SO(7,1) \subset SO(13,1)$) groups, describing spins of fermions

$$\vec{N}_{\pm}(=\vec{N}_{(L,R)}) := \frac{1}{2}(S^{23} \pm iS^{01}, S^{31} \pm iS^{02}, S^{12} \pm iS^{03}), \tag{7.52}$$

are presented.

a.ii. The generators of the two $SU(2)$ ($SU(2) \subset SO(4) \subset SO(7,1) \subset SO(13,1)$) groups, describing the two kinds of weak charges of fermions

$$\begin{aligned} \vec{\tau}^{1} &:= \frac{1}{2}(S^{58} - S^{67},\, S^{57} + S^{68},\, S^{56} - S^{78}),\\ \vec{\tau}^{2} &:= \frac{1}{2}(S^{58} + S^{67},\, S^{57} - S^{68},\, S^{56} + S^{78}), \end{aligned} \tag{7.53}$$

are presented.

a.iii. The $SU(3)$ and $U(1)$ subgroups of $SO(6) \subset SO(13,1)$, describing the colour charge and the "fermion" charge of fermions

$$\begin{aligned} \vec{\tau}^{3} := & \frac{1}{2}\{S^{9\,12} - S^{10\,11}, S^{9\,11} + S^{10\,12}, S^{9\,10} - S^{11\,12},\\ & S^{9\,14} - S^{10\,13}, S^{9\,13} + S^{10\,14}, S^{11\,14} - S^{12\,13},\\ & S^{11\,13} + S^{12\,14}, \frac{1}{\sqrt{3}}(S^{9\,10} + S^{11\,12} - 2S^{13\,14})\},\\ \tau^{4} := & -\frac{1}{3}(S^{9\,10} + S^{11\,12} + S^{13\,14}), \end{aligned} \tag{7.54}$$

are presented.

b.i. The two $\widetilde{SU}(2)$ subgroups of $\tilde{SO}(3,1)$ ($\subset \widetilde{SO}(7,1) \subset \widetilde{SO}(13,1)$), describing families of fermions

$$\vec{\tilde{N}}_{L,R} := \frac{1}{2}(\tilde{S}^{23} \pm i\tilde{S}^{01}, \tilde{S}^{31} \pm i\tilde{S}^{02}, \tilde{S}^{12} \pm i\tilde{S}^{03}), \tag{7.55}$$

are presented.

b.ii. The two $\widetilde{SU}(2)$ subgroups of $\widetilde{SO}(4)$ ($\subset \widetilde{SO}(7,1) \subset \widetilde{SO}(13,1)$), describing families of fermions

$$\begin{aligned} \vec{\tilde{\tau}}^{1} &:= \frac{1}{2}(\tilde{S}^{58} - \tilde{S}^{67},\, \tilde{S}^{57} + \tilde{S}^{68},\, \tilde{S}^{56} - \tilde{S}^{78}),\\ \vec{\tilde{\tau}}^{2} &:= \frac{1}{2}(\tilde{S}^{58} + \tilde{S}^{67},\, \tilde{S}^{57} - \tilde{S}^{68},\, \tilde{S}^{56} + \tilde{S}^{78}), \end{aligned} \tag{7.56}$$

are presented.

b.iii. The group $\tilde{U}(1)$, the subgroup of $\widetilde{SO}(6)$ ($\subset \widetilde{SO}(13,1)$), describing family quantum numbers of fermions

$$\tilde{\tau}^{4} := -\frac{1}{3}(\tilde{S}^{9\,10} + \tilde{S}^{11\,12} + \tilde{S}^{13\,14}), \tag{7.57}$$

are presented.

c. Relations among the hyper, weak and the second $SU(2)$ charges

$$
\begin{aligned}
&Y := \tau^4 + \tau^{23}\,, \quad Y' := -\tau^4 \tan^2\vartheta_2 + \tau^{23}\,, \quad Q := \tau^{13} + Y\,, \quad Q' := -Y \tan^2\vartheta_1 + \tau^{13}\quad ,\\
&\tilde{Y} := \tilde{\tau}^4 + \tilde{\tau}^{23}\,, \quad \tilde{Y}' := -\tilde{\tau}^4 \tan^2\vartheta_2 + \tilde{\tau}^{23}\,, \quad \tilde{Q} := \tilde{Y} + \tilde{\tau}^{13}\,, \quad \tilde{Q}' = -\tilde{Y} \tan^2\vartheta_1 + \tilde{\tau}^{13}\quad ,
\end{aligned}
\tag{7.58}
$$

are presented.

Below are some of the above expressions written in terms of nilpotents and projectors

$$
\begin{aligned}
&N^{\pm}_{+} = N^{1}_{+} \pm i\, N^{2}_{+} = -\stackrel{03}{(\mp i)}\stackrel{12}{(\pm)}\,, \quad N^{\pm}_{-} = N^{1}_{-} \pm i\, N^{2}_{-} = \stackrel{03}{(\pm i)}\stackrel{12}{(\pm)}\,,\\
&\tilde{N}^{\pm}_{+} = -\stackrel{03}{\tilde{(\mp i)}}\stackrel{12}{\tilde{(\pm)}}\,, \quad \tilde{N}^{\pm}_{-} = \stackrel{03}{\tilde{(\pm i)}}\stackrel{12}{\tilde{(\pm)}}\,,\\
&\tau^{1\pm} = (\mp)\ \stackrel{56}{(\pm)}\stackrel{78}{(\mp)}\,, \quad \tau^{2\mp} = (\mp)\ \stackrel{56}{(\mp)}\stackrel{78}{(\mp)}\,,\\
&\tilde{\tau}^{1\pm} = (\mp)\ \stackrel{56}{\tilde{(\pm)}}\stackrel{78}{\tilde{(\mp)}}\,, \quad \tilde{\tau}^{2\mp} = (\mp)\ \stackrel{56}{\tilde{(\mp)}}\stackrel{78}{\tilde{(\mp)}}\ .
\end{aligned}
\tag{7.59}
$$

i		$^{a}\hat{b}^{\dagger}_{i}$ (Anti)octet, $\Gamma^{(7+1)} = (-1)\,1\,,\ \Gamma^{(6)} = (1)\,-1$ of (anti)quarks and (anti)leptons	$\Gamma^{(3+1)}$	S^{12}	τ^{13}	τ^{23}	τ^{33}	τ^{38}	τ^{4}	Y	Q
1	$\hat{u}^{c1\dagger}_{R}$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\mid\stackrel{56}{[+]}\stackrel{78}{(+)}\mid\mid\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}$
2	$\hat{u}^{c1\dagger}_{R}$	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\mid\stackrel{56}{[+]}\stackrel{78}{(+)}\mid\mid\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$	1	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}$
3	$\hat{d}^{c1\dagger}_{R}$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\mid\stackrel{56}{(-)}\stackrel{78}{[-]}\mid\mid\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$
4	$\hat{d}^{c1\dagger}_{R}$	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\mid\stackrel{56}{(-)}\stackrel{78}{[-]}\mid\mid\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$	1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$
5	$\hat{d}^{c1\dagger}_{L}$	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\mid\stackrel{56}{(-)}\stackrel{78}{(+)}\mid\mid\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$	-1	$\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$
6	$\hat{d}^{c1\dagger}_{L}$	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\mid\stackrel{56}{(-)}\stackrel{78}{(+)}\mid\mid\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$	-1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$
7	$\hat{u}^{c1\dagger}_{L}$	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\mid\stackrel{56}{[+]}\stackrel{78}{[-]}\mid\mid\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$	-1	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$
8	$\hat{u}^{c1\dagger}_{L}$	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\mid\stackrel{56}{[+]}\stackrel{78}{[-]}\mid\mid\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$	-1	$-\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$
9	$\hat{u}^{c2\dagger}_{R}$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\mid\stackrel{56}{[+]}\stackrel{78}{(+)}\mid\mid\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}$
10	$\hat{u}^{c2\dagger}_{R}$	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\mid\stackrel{56}{[+]}\stackrel{78}{(+)}\mid\mid\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$	1	$-\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}$
11	$\hat{d}^{c2\dagger}_{R}$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\mid\stackrel{56}{(-)}\stackrel{78}{[-]}\mid\mid\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$
12	$\hat{d}^{c2\dagger}_{R}$	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\mid\stackrel{56}{(-)}\stackrel{78}{[-]}\mid\mid\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$	1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$
13	$\hat{d}^{c2\dagger}_{L}$	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\mid\stackrel{56}{(-)}\stackrel{78}{(+)}\mid\mid\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$	-1	$\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$
14	$\hat{d}^{c2\dagger}_{L}$	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\mid\stackrel{56}{(-)}\stackrel{78}{(+)}\mid\mid\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$	-1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$
15	$\hat{u}^{c2\dagger}_{L}$	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\mid\stackrel{56}{[+]}\stackrel{78}{[-]}\mid\mid\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$	-1	$\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$
16	$\hat{u}^{c2\dagger}_{L}$	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\mid\stackrel{56}{[+]}\stackrel{78}{[-]}\mid\mid\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$	-1	$-\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$
17	$\hat{u}^{c3\dagger}_{R}$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\mid\stackrel{56}{[+]}\stackrel{78}{(+)}\mid\mid\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}$
18	$\hat{u}^{c3\dagger}_{R}$	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\mid\stackrel{56}{[+]}\stackrel{78}{(+)}\mid\mid\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	1	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}$
19	$\hat{d}^{c3\dagger}_{R}$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\mid\stackrel{56}{(-)}\stackrel{78}{[-]}\mid\mid\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$
20	$\hat{d}^{c3\dagger}_{R}$	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\mid\stackrel{56}{(-)}\stackrel{78}{[-]}\mid\mid\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$

Continued on next page

i		$a\hat{b}^\dagger_i$	$\Gamma^{(3+1)}$	S^{12}	τ^{13}	τ^{23}	τ^{33}	τ^{38}	τ^{4}	Y	Q
		(Anti)octet, $\Gamma^{(7+1)} = (-1)\,1\,,\ \Gamma^{(6)} = (1)\,-1$ of (anti)quarks and (anti)leptons									
21	$\hat{d}^{c3\dagger}_L$	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	-1	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$
22	$\hat{d}^{c3\dagger}_L$	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	-1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$
23	$\hat{u}^{c3\dagger}_L$	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	-1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$
24	$\hat{u}^{c3\dagger}_L$	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	-1	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$
25	$\hat{\nu}^{\dagger}_R$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	0	0
26	$\hat{\nu}^{\dagger}_R$	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	1	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	0	0
27	$\hat{e}^{\dagger}_R$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	0	$-\frac{1}{2}$	−1	−1
28	$\hat{e}^{\dagger}_R$	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	0	$-\frac{1}{2}$	−1	−1
29	$\hat{e}^{\dagger}_L$	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	-1	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	−1
30	$\hat{e}^{\dagger}_L$	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	-1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	−1
31	$\hat{\nu}^{\dagger}_L$	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	-1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	0
32	$\hat{\nu}^{\dagger}_L$	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	-1	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	0
33	$\hat{d}^{\bar{c}1\dagger}_L$	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	-1	$\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$
34	$\hat{d}^{\bar{c}1\dagger}_L$	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$
35	$\hat{u}^{\bar{c}1\dagger}_L$	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	-1	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{2}{3}$	$-\frac{2}{3}$
36	$\hat{u}^{\bar{c}1\dagger}_L$	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	-1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{2}{3}$	$-\frac{2}{3}$
37	$\hat{d}^{\bar{c}1\dagger}_R$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	1	$\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$
38	$\hat{d}^{\bar{c}1\dagger}_R$	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	1	$-\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$
39	$\hat{u}^{\bar{c}1\dagger}_R$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{2}{3}$
40	$\hat{u}^{\bar{c}1\dagger}_R$	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{2}{3}$
41	$\hat{d}^{\bar{c}2\dagger}_L$	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	-1	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$
42	$\hat{d}^{\bar{c}2\dagger}_L$	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$
43	$\hat{u}^{\bar{c}2\dagger}_L$	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	-1	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{2}{3}$	$-\frac{2}{3}$
44	$\hat{u}^{\bar{c}2\dagger}_L$	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	-1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{2}{3}$	$-\frac{2}{3}$
45	$\hat{d}^{\bar{c}2\dagger}_R$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	1	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$
46	$\hat{d}^{\bar{c}2\dagger}_R$	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	1	$-\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$
47	$\hat{u}^{\bar{c}2\dagger}_R$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{2}{3}$
48	$\hat{u}^{\bar{c}2\dagger}_R$	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{2}{3}$
49	$\hat{d}^{\bar{c}3\dagger}_L$	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$	-1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$
50	$\hat{d}^{\bar{c}3\dagger}_L$	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$
51	$\hat{u}^{\bar{c}3\dagger}_L$	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$	-1	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{2}{3}$	$-\frac{2}{3}$
52	$\hat{u}^{\bar{c}3\dagger}_L$	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$	-1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{2}{3}$	$-\frac{2}{3}$
53	$\hat{d}^{\bar{c}3\dagger}_R$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$
54	$\hat{d}^{\bar{c}3\dagger}_R$	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$	1	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$
55	$\hat{u}^{\bar{c}3\dagger}_R$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{2}{3}$
56	$\hat{u}^{\bar{c}3\dagger}_R$	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$	1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{2}{3}$
57	$\hat{e}^{\dagger}_L$	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$	-1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$	1	1
58	$\hat{e}^{\dagger}_L$	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$	1	1
59	$\hat{\nu}^{\dagger}_L$	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$	-1	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	0	$\frac{1}{2}$	0	0
60	$\hat{\nu}^{\dagger}_L$	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$	-1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	0	$\frac{1}{2}$	0	0
61	$\hat{\nu}^{\dagger}_R$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0
62	$\hat{\nu}^{\dagger}_R$	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$	1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0
63	$\hat{e}^{\dagger}_R$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	1
64	$\hat{e}^{\dagger}_R$	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$	1	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	1

Table 7.3. The left handed ($\Gamma^{(13,1)} = -1$), multiplet of creation operators of fermions — the members of one fundamental representation of the $SO(13,1)$ group, manifesting the subgroup $SO(7,1)$ of the colour charged quarks and antiquarks and the colourless leptons and antileptons — is presented in the massless basis as products of nilpotents and projectors. The multiplet contains the left handed ($\Gamma^{(3+1)} = -1$ weak ($SU(2)_I$) charged ($\tau^{13} = \pm\frac{1}{2}$, ($\vec{\tau}^1 = \frac{1}{2}(S^{58} - S^{67}, S^{57} + S^{68}, S^{56} - S^{78}$)) and $SU(2)_{II}$ chargeless ($\tau^{23} = 0$, $\vec{\tau}^2 = \frac{1}{2}(S^{58} + S^{67}, S^{57} - S^{68}, S^{56} + S^{78}$)) quarks and leptons and the right handed ($\Gamma^{(3+1)} = 1$), weak ($SU(2)_I$) chargeless and $SU(2)_{II}$ charged ($\tau^{23} = \pm\frac{1}{2}$) quarks and leptons, both with the spin S^{12} up and down ($\pm\frac{1}{2}$, respectively). The creation operators of quarks distinguish from those of leptons only in the $SU(3) \times U(1)$ part: Quarks are triplets of three colours ($(\tau^{33}, \tau^{38}) = [(\frac{1}{2}, \frac{1}{2\sqrt{3}}), (-\frac{1}{2}, \frac{1}{2\sqrt{3}}), (0, -\frac{1}{\sqrt{3}})]$, ($\vec{\tau}^3 = \frac{1}{2}(S^{9\,12} - S^{10\,11}, S^{9\,11} + S^{10\,12}, S^{9\,10} - S^{11\,12}, S^{9\,14} - S^{10\,13}, S^{9\,13} + S^{10\,14}, S^{11\,14} - S^{12\,13}, S^{11\,13} + S^{12\,14}, \frac{1}{\sqrt{3}}(S^{9\,10} + S^{11\,12} - 2S^{13\,14})$)), carrying the "fermion charge" ($\tau^4 = \frac{1}{6}, = -\frac{1}{3}(S^{9\,10} + S^{11\,12} + S^{13\,14})$. The colourless leptons carry the "fermion charge" ($\tau^4 = -\frac{1}{2}$). In the same multiplet of creation operators the left handed weak ($SU(2)_I$) chargeless and $SU(2)_{II}$ charged antiquarks and antileptons and the right handed weak ($SU(2)_I$) charged and $SU(2)_{II}$ chargeless antiquarks and antileptons are included. Antiquarks distinguish from antileptons again only in the $SU(3) \times U(1)$ part: Anti-quarks are antitriplets, carrying the "fermion charge" ($\tau^4 = -\frac{1}{6}$). The anti-colourless antileptons carry the "fermion" charge ($\tau^4 = \frac{1}{2}$). $Y = (\tau^{23} + \tau^4)$ is the hyper charge, the electromagnetic charge is $Q = (\tau^{13} + Y)$. The creation operators of opposite charges (antifermion creation operators) are reachable from the particle ones besides by S^{ab} also by the application of the discrete symmetry operator $\mathbb{C}_{\mathcal{N}}\, \mathcal{P}_{\mathcal{N}}$, presented in Refs. [65,66]. The reader can find this Weyl representation also in Refs. [4,9,71,72] and in the references therein.

Table 7.3 represents in the *spin-charge-family* theory the creation operators for observed *quarks and leptons and antiquarks and antileptons* for a particular family, Table (7.4). Hermitian conjugation of the creation operators of Table 7.3 generates the corresponding annihilation operators, fulfilling together with the creation operators anticommutation relations for fermions of Eq. (7.23).

The condensate of two right handed neutrinos with the family quantum numbers of the upper four families, causing the break of the starting symmetry $SO(13,1)$ into $SO(7,1) \times SU(3) \times U(1)$, is presented in Table 7.5.

7.6 APPENDIX: Expressions for scalar fields in term of $\omega_{s's''s}$ and $\tilde{\omega}_{abs}$

The scalar fields, responsible for masses of the family members and of the heavy bosons [6,7] after gaining nonzero vacuum expectation values and triggering the electroweak break, are presented in the second line of Eq. (7.27). These scalar fields are included in the covariant derivatives as $-\frac{1}{2}\, S^{s's"}\omega_{s's"s} - \frac{1}{2}\tilde{S}^{ab}\tilde{\omega}_{abs}$, $s \in (7,8)$, $(a,b), \in (0,\ldots,3), (5,\ldots,8)$.

One can express the scalar fields carrying the quantum numbers of the subgroups of the family groups, expressed in terms of $\tilde{\omega}_{abs}$ (they contribute to mass matrices of quarks and leptons and to masses of the heavy bosons), if taking into account Eqs. (7.55, 7.56, 7.58),

$$
\begin{aligned}
\sum_{a,b} -\frac{1}{2}\tilde{S}^{ab}\,\tilde{\omega}_{abs} &= -(\vec{\tilde{\tau}}^{\tilde{1}}\,\vec{\tilde{A}}^{\tilde{1}}_s + \vec{\tilde{N}}_{\tilde{L}}\,\vec{\tilde{A}}^{\tilde{N}_{\tilde{L}}}_s + \vec{\tilde{\tau}}^{\tilde{2}}\,\vec{\tilde{A}}^{\tilde{2}}_s + \vec{\tilde{N}}_{\tilde{R}}\,\vec{\tilde{A}}^{\tilde{N}_{\tilde{R}}}_s)\,,\\
\vec{\tilde{A}}^{\tilde{1}}_s &= (\tilde{\omega}_{58s} - \tilde{\omega}_{67s},\, \tilde{\omega}_{57s} + \tilde{\omega}_{68s},\, \tilde{\omega}_{56s} - \tilde{\omega}_{78s})\,,\\
\vec{\tilde{A}}^{\tilde{N}_{\tilde{L}}}_s &= (\tilde{\omega}_{23s} + i\,\tilde{\omega}_{01s},\, \tilde{\omega}_{31s} + i\,\tilde{\omega}_{02s},\, \tilde{\omega}_{12s} + i\,\tilde{\omega}_{03s})\,,\\
\vec{\tilde{A}}^{\tilde{2}}_s &= (\tilde{\omega}_{58s} + \tilde{\omega}_{67s},\, \tilde{\omega}_{57s} - \tilde{\omega}_{68s},\, \tilde{\omega}_{56s} + \tilde{\omega}_{78s})\,,\\
\vec{\tilde{A}}^{\tilde{N}_{\tilde{R}}}_s &= (\tilde{\omega}_{23s} - i\,\tilde{\omega}_{01s},\, \tilde{\omega}_{31s} - i\,\tilde{\omega}_{02s},\, \tilde{\omega}_{12s} - i\,\tilde{\omega}_{03s})\,,\\
&\quad (s \in (7,8))\,.
\end{aligned}
\tag{7.60}
$$

Scalars, expressed in terms of ω_{abc} (contributing as well to the mass matrices of quarks and leptons and to masses of the heavy bosons) follow, if using Eqs. (7.53,

					$\tilde{\tau}^{13}$	$\tilde{\tau}^{23}$	$\tilde{N}^3_L$	$\tilde{N}^3_R$	$\tilde{\tau}^4$
I	$\hat{u}^{c1\dagger}_{R\,1}$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\mid\stackrel{56}{[+]}\stackrel{78}{(+)}\parallel\stackrel{9\,10}{(+)}\;\stackrel{11\,12}{(-)}\;\stackrel{13\,14}{(-)}$	$\hat{\nu}^{\dagger}_{R\,1}$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\mid\stackrel{56}{[+]}\stackrel{78}{(+)}\parallel\stackrel{9\,10}{(+)}\;\stackrel{11\,12}{[+]}\;\stackrel{13\,14}{[+]}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	$-\frac{1}{2}$
I	$\hat{u}^{c1\dagger}_{R\,2}$	$\stackrel{03}{[+i]}\stackrel{12}{(+)}\mid\stackrel{56}{[+]}\stackrel{78}{(+)}\parallel\stackrel{9\,10}{(+)}\;\stackrel{11\,12}{(-)}\;\stackrel{13\,14}{(-)}$	$\hat{\nu}^{\dagger}_{R\,2}$	$\stackrel{03}{[+i]}\stackrel{12}{(+)}\mid\stackrel{56}{[+]}\stackrel{78}{(+)}\parallel\stackrel{9\,10}{(+)}\;\stackrel{11\,12}{[+]}\;\stackrel{13\,14}{[+]}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	$-\frac{1}{2}$
I	$\hat{u}^{c1\dagger}_{R\,3}$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\mid\stackrel{56}{(+)}\stackrel{78}{[+]}\parallel\stackrel{9\,10}{(+)}\;\stackrel{11\,12}{(-)}\;\stackrel{13\,14}{(-)}$	$\hat{\nu}^{\dagger}_{R\,3}$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\mid\stackrel{56}{(+)}\stackrel{78}{[+]}\parallel\stackrel{9\,10}{(+)}\;\stackrel{11\,12}{[+]}\;\stackrel{13\,14}{[+]}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	$-\frac{1}{2}$
I	$\hat{u}^{c1\dagger}_{R\,4}$	$\stackrel{03}{[+i]}\stackrel{12}{(+)}\mid\stackrel{56}{(+)}\stackrel{78}{[+]}\parallel\stackrel{9\,10}{(+)}\;\stackrel{11\,12}{(-)}\;\stackrel{13\,14}{(-)}$	$\hat{\nu}^{\dagger}_{R\,4}$	$\stackrel{03}{[+i]}\stackrel{12}{(+)}\mid\stackrel{56}{(+)}\stackrel{78}{[+]}\parallel\stackrel{9\,10}{(+)}\;\stackrel{11\,12}{[+]}\;\stackrel{13\,14}{[+]}$	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$-\frac{1}{2}$
II	$\hat{u}^{c1\dagger}_{R\,5}$	$\stackrel{03}{[+i]}\stackrel{12}{[+]}\mid\stackrel{56}{[+]}\stackrel{78}{[+]}\parallel\stackrel{9\,10}{(+)}\;\stackrel{11\,12}{(-)}\;\stackrel{13\,14}{(-)}$	$\hat{\nu}^{\dagger}_{R\,5}$	$\stackrel{03}{[+i]}\stackrel{12}{[+]}\mid\stackrel{56}{[+]}\stackrel{78}{[+]}\parallel\stackrel{9\,10}{(+)}\;\stackrel{11\,12}{[+]}\;\stackrel{13\,14}{[+]}$	0	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$
II	$\hat{u}^{c1\dagger}_{R\,6}$	$\stackrel{03}{(+i)}\stackrel{12}{(+)}\mid\stackrel{56}{[+]}\stackrel{78}{[+]}\parallel\stackrel{9\,10}{(+)}\;\stackrel{11\,12}{(-)}\;\stackrel{13\,14}{(-)}$	$\hat{\nu}^{\dagger}_{R\,6}$	$\stackrel{03}{(+i)}\stackrel{12}{(+)}\mid\stackrel{56}{[+]}\stackrel{78}{[+]}\parallel\stackrel{9\,10}{(+)}\;\stackrel{11\,12}{[+]}\;\stackrel{13\,14}{[+]}$	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$
II	$\hat{u}^{c1\dagger}_{R\,7}$	$\stackrel{03}{[+i]}\stackrel{12}{[+]}\mid\stackrel{56}{(+)}\stackrel{78}{(+)}\parallel\stackrel{9\,10}{(+)}\;\stackrel{11\,12}{(-)}\;\stackrel{13\,14}{(-)}$	$\hat{\nu}^{\dagger}_{R\,7}$	$\stackrel{03}{[+i]}\stackrel{12}{[+]}\mid\stackrel{56}{(+)}\stackrel{78}{(+)}\parallel\stackrel{9\,10}{(+)}\;\stackrel{11\,12}{[+]}\;\stackrel{13\,14}{[+]}$	0	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$
II	$\hat{u}^{c1\dagger}_{R\,8}$	$\stackrel{03}{(+i)}\stackrel{12}{(+)}\mid\stackrel{56}{(+)}\stackrel{78}{(+)}\parallel\stackrel{9\,10}{(+)}\;\stackrel{11\,12}{(-)}\;\stackrel{13\,14}{(-)}$	$\hat{\nu}^{\dagger}_{R\,8}$	$\stackrel{03}{(+i)}\stackrel{12}{(+)}\mid\stackrel{56}{(+)}\stackrel{78}{(+)}\parallel\stackrel{9\,10}{(+)}\;\stackrel{11\,12}{[+]}\;\stackrel{13\,14}{[+]}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{3}$

Table 7.4. Eight families of creation operators of $\hat{u}^{c1\dagger}_R$ — the right handed u-quark with spin $\frac{1}{2}$ and the colour charge ($\tau^{33} = 1/2$, $\tau^{38} = 1/(2\sqrt{3})$), appearing in the first line of Table 7.3 — and of the colourless right handed neutrino $\hat{\nu}^{\dagger}_R$ — of spin $\frac{1}{2}$, appearing in the 25^{th} line of Table 7.3 — are presented in the left and in the right column, respectively. Table is taken from [9]. Families belong to two groups of four families, one (I) is a doublet with respect to ($\vec{\tilde{N}}_L$ and $\vec{\tilde{\tau}}^{(1)}$) and a singlet with respect to ($\vec{\tilde{N}}_R$ and $\vec{\tilde{\tau}}^{(2)}$), the other (II) is a singlet with respect to ($\vec{\tilde{N}}_L$ and $\vec{\tilde{\tau}}^{(1)}$) and a doublet with with respect to ($\vec{\tilde{N}}_R$ and $\vec{\tilde{\tau}}^{(2)}$), Eq. (7.52). All the families follow from the starting one by the application of the operators ($\tilde{N}^{\pm}_{R,L}$, $\tilde{\tau}^{(2,1)\pm}$), Eq. (7.59). The generators ($N^{\pm}_{R,L}$, $\tau^{(2,1)\pm}$) (Eq. (7.59)) transform $\hat{u}^{\dagger}_{1R}$ to all the members of one family of the same colour. The same generators transform equivalently the right handed neutrino $\hat{\nu}^{\dagger}_{1R}$ to all the colourless members of the same family.

state	S^{03}	S^{12}	τ^{13}	τ^{23}	τ^4	Y	Q	$\tilde{\tau}^{13}$	$\tilde{\tau}^{23}$	$\tilde{\tau}^4$	$\tilde{Y}$	$\tilde{Q}$	$\tilde{N}^3_L$	$\tilde{N}^3_R$
$(\|\nu^{VIII}_{1R}>_1 \|\nu^{VIII}_{2R}>_2)$	0	0	0	1	−1	0	0	0	1	−1	0	0	0	1
$(\|\nu^{VIII}_{1R}>_1 \|e^{VIII}_{2R}>_2)$	0	0	0	0	−1	−1	−1	0	1	−1	0	0	0	1
$(\|e^{VIII}_{1R}>_1 \|e^{VIII}_{2R}>_2)$	0	0	0	−1	−1	−2	−2	0	1	−1	0	0	0	1

Table 7.5. The condensate of the two right handed neutrinos ν_R, with the quantum numbers of the VIII^{th} family, coupled to spin zero and belonging to a triplet with respect to the generators τ^{2i}, is presented, together with its two partners. The condensate carries $\tilde{\tau}^1 = 0$, $\tau^{23} = 1$, $\tau^4 = -1$ and $Q = 0 = Y$. The triplet carries $\tilde{\tau}^4 = -1$, $\tilde{\tau}^{23} = 1$ and $\tilde{N}^3_R = 1$, $\tilde{N}^3_L = 0$, $\tilde{Y} = 0$, $\tilde{Q} = 0$. The family quantum numbers of quarks and leptons are presented in Table 7.4.

7.54, 7.58)

$$\begin{aligned}
\sum_{s',s''} -\frac{1}{2} S^{s's''}\,\omega_{s's''s} &= -(g^{23}\,\tau^{23}\,A^{23}_s + g^{13}\,\tau^{13}\,A^{13}_s + g^4\,\tau^4\,A^4_s)\,,\\
g^{13}\,\tau^{13}\,A^{13}_s + g^{23}\,\tau^{23}\,A^{23}_s + g^4\,\tau^4\,A^4_s &= g^Q\,QA^Q_s + g^{Q'}\,Q'A^{Q'}_s + g^{Y'}\,Y'\,A^{Y'}_s\,,\\
A^4_s &= -(\omega_{9\,10\,s} + \omega_{11\,12\,s} + \omega_{13\,14\,s})\,,\\
A^{13}_s &= (\omega_{56s} - \omega_{78s})\,,\quad A^{23}_s = (\omega_{56s} + \omega_{78s})\,,\\
A^Q_s &= \sin\vartheta_1\,A^{13}_s + \cos\vartheta_1\,A^Y_s\,,\\
A^{Q'}_s &= \cos\vartheta_1\,A^{13}_s - \sin\vartheta_1\,A^Y_s\,,\\
A^{Y'}_s &= \cos\vartheta_2\,A^{23}_s - \sin\vartheta_2\,A^4_s\,,\\
&\quad (s \in (7,8))\,. \qquad (7.61)
\end{aligned}$$

Scalar fields from Eq. (7.60) interact with quarks and leptons and antiquarks and antileptons through the family quantum numbers, while those from Eq. (7.61) interact through the family members quantum numbers. In Eq. (7.61) the coupling constants are explicitly written in order to see the analogy with the gauge fields of the *standard model*.

Expressions for the vector gauge fields in terms of the spin connection fields and the vielbeins, which correspond to the colour charge ($\vec{A}^3_m$), the $SU(2)_{II}$ charge ($\vec{A}^2_m$), the weak $SU(2)_I$ charge ($\vec{A}^1_m$) and the $U(1)$ charge originating in $SO(6)$ ($\vec{A}^4_m$), can be found by taking into account Eqs. (7.53, 7.54). Equivalently one finds the vector gauge fields in the "tilde" sector, or one just uses the expressions from Eqs. (7.61, 7.60), if replacing the scalar index s with the vector index m.

The expression for $\sum_{tab} \gamma^t\, \frac{1}{2}\, \tilde{S}^{ab}\, \tilde{\omega}_{abt}$, used in Eq. (7.45) ($\tilde{S}^{ab}$ are the infinitesimal generators of either $\widetilde{SO}(3,1)$ or $\widetilde{SO}(4)$, while $\tilde{\omega}_{abt}$ belong to the corresponding gauge fields with $t = (9,\ldots,14)$), and obtained by using Eqs. (7.55 -

7.59), are

$$\sum_{abt} \gamma^t \frac{1}{2} \tilde{S}^{ab} \tilde{\omega}_{abt} = \sum_{+-tt'ab} \overset{tt'}{(\oplus)} \frac{1}{2} \tilde{S}^{ab} \tilde{\omega}_{ab \overset{tt'}{(\oplus)}} =$$
$$\sum_{+-tt'} \overset{tt'}{(\oplus)} \{ \tilde{\tau}^{2+} \tilde{A}^{2+}_{\overset{tt'}{(\oplus)}} + \tilde{\tau}^{2-} \tilde{A}^{2-}_{\overset{tt'}{(\oplus)}} + \tilde{\tau}^{23} \tilde{A}^{23}_{\overset{tt'}{(\oplus)}} +$$
$$\tilde{\tau}^{1+} \tilde{A}^{1+}_{\overset{tt'}{(\oplus)}} + \tilde{\tau}^{1-} \tilde{A}^{1-}_{\overset{tt'}{(\oplus)}} + \tilde{\tau}^{13} \tilde{A}^{13}_{\overset{tt'}{(\oplus)}} +$$
$$\tilde{N}^{+}_{R} \tilde{A}^{N_R+}_{\overset{tt'}{(\oplus)}} + \tilde{N}^{-}_{R} \tilde{A}^{N_R-}_{\overset{tt'}{(\oplus)}} + \tilde{N}^{3}_{R} \tilde{A}^{N_R 3}_{\overset{tt'}{(\oplus)}} +$$
$$\tilde{N}^{+}_{L} \tilde{A}^{N_L+}_{\overset{tt'}{(\oplus)}} + \tilde{N}^{-}_{L} \tilde{A}^{N_L-}_{\overset{tt'}{(\oplus)}} + \tilde{N}^{3}_{L} \tilde{A}^{N_L 3}_{\overset{tt'}{(\oplus)}} \},$$
$$\tilde{A}^{N_R \boxplus}_{\overset{tt'}{(\oplus)}} = (\tilde{\omega}_{23 \overset{tt'}{(\oplus)}} - i \tilde{\omega}_{01 \overset{tt'}{(\oplus)}}) \boxplus i (\tilde{\omega}_{31 \overset{tt'}{(\oplus)}} - i \tilde{\omega}_{02 \overset{tt'}{(\oplus)}}),$$
$$\tilde{A}^{N_R 3}_{\overset{tt'}{(\oplus)}} = (\tilde{\omega}_{12 \overset{tt'}{(\oplus)}} - i \tilde{\omega}_{03 \overset{tt'}{(\oplus)}}),$$
$$\tilde{A}^{N_L \boxplus}_{\overset{tt'}{(\oplus)}} = (\tilde{\omega}_{23 \overset{tt'}{(\oplus)}} + i \tilde{\omega}_{01 \overset{tt'}{(\oplus)}}) \boxplus i (\tilde{\omega}_{31 \overset{tt'}{(\oplus)}} + i \tilde{\omega}_{02 \overset{tt'}{(\oplus)}}),$$
$$\tilde{A}^{N_R 3}_{\overset{tt'}{(\oplus)}} = (\tilde{\omega}_{12 \overset{tt'}{(\oplus)}} + i \tilde{\omega}_{03 \overset{tt'}{(\oplus)}}). \quad (7.62)$$

The term $\sum_{tt't''} \gamma^t \frac{1}{2} S^{t't"} \omega_{t't"t}$ in Eq. (7.27) can be rewritten with respect to the generators $S^{t't"}$ and the corresponding gauge fields $\omega_{s's"t}$ as one colour octet scalar field and one $U(1)_{II}$ singlet scalar field (Eq. 7.54)

$$\gamma^t \frac{1}{2} S^{t"t'"} \omega_{t"t'"t} = \sum_{+,-} \sum_{(t\,t')} \overset{tt'}{(\oplus)} \{ \vec{\tau}^3 \cdot \vec{A}^3_{\overset{tt'}{(\oplus)}} + \tau^4 \cdot A^4_{\overset{tt'}{(\oplus)}} \},$$
$$(t\,t') \in ((9\,10), 11\,12), 13\,14)). \quad (7.63)$$

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