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On the largest subsets avoiding the diameter of $(0, \pm 1)$-vectors

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Abstract

Let $L_{mkl} \subset \mathbb{R}^{m+k+l}$ be the set of vectors which have m of entries -1, k of entries 0, and l of entries 1. In this paper, we investigate the largest subset of L_{mkl} whose diameter is smaller than that of L_{mkl}. The largest subsets for $m = 1$, $l = 2$, and any k will be classified. From this result, we can classify the largest 4-distance sets containing the Euclidean representation of the Johnson scheme $J(9, 4)$. This was an open problem in Bannai, Sato, and Shigezumi (2012).

Keywords: The Erdős–Ko–Rado theorem, s-distance set, diameter graph, independent set, extremal set theory.

Math. Subj. Class.: 05D05, 05C69

1 Introduction

The famous theorem in Erdős–Ko–Rado [8] stated that for $n \geq 2k$ and a family $\mathfrak{A}$ of k-element subsets of $I_n = \{1, \ldots, n\}$, if any two distinct $A, B \in \mathfrak{A}$ satisfy $A \cap B \neq \emptyset$, then

$$|\mathfrak{A}| \leq \binom{n-1}{k-1}.$$

For $n > 2k$, the set $\{A \subset I_n \mid |A| = k, 1 \in A\}$ is the unique family achieving equality, up to permutations on I_n. For $n = 2k$, the largest set is any family which contains only one of A or $I_n \setminus A$ for any k-element $A \subset I_n$. This result plays a central role in extremal set theory, and similar or analogous theorems are proved for various objects [2, 5, 9].

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We can naturally interpret $A \subset I_n$ as $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ by the manner $x_i = 1$ if $i \in A$, $x_i = 0$ if $i \notin A$. By this identification, the Erdős–Ko–Rado Theorem can be rewritten that for $n \geq 2k$ and a subset X of $L_k = \{x \in \mathbb{R}^n \mid x_i \in \{0,1\}, \sum x_i = k\}$ if any distinct $x, y \in X$ satisfy $d(x,y) < D(L_k) = \sqrt{2k}$, then

$$|X| \leq \binom{n-1}{k-1},$$

where $d(,)$ is the Euclidean distance, and $D(L_k)$ is the diameter of L_k. We would like to consider the following problem to generalize the Erdős–Ko–Rado Theorem.

Problem 1.1. Let $L_{mkl} \subset \mathbb{R}^{m+k+l}$ be the set of vectors which have m of entries -1, k of entries 0, and l of entries 1. Classify the largest $X \subset L_{mkl}$ with $D(X) < D(L_{mkl})$.

It is almost obvious for the cases $m = l$ (Proposition 2.1) and $m + k \leq l$ (Proposition 2.2). In this paper, we solve the first non-trivial case $m = 1$, $l = 2$ and any k (Theorem 2.5). Using the largest sets for the case $(m,k,l) = (1,6,2)$, we can classify the largest 4-distance sets containing the Euclidean representation of the Johnson scheme $J(9,4)$. This was an open problem in [1].

We will give a brief survey on related results. Let $\mathfrak{L}_{nm}$ be the set of $(0,\pm1)$-vectors in $\mathbb{R}^n$ which have m non-zero coordinates. For a fixed set D of integers, let $V(n,m,D)$ be the family of subsets $V = \{v_1, \ldots, v_k\}$ of $\mathfrak{L}_{nm}$ such that $(v_i, v_j) \in D$ for any $i \neq j$. There are several results relating to the largest sets in $V(n,m,D)$ for some (n,m,D) [4, 6, 7]. Since $X \subset \mathfrak{L}_{nm}$ is on a sphere, if $|D| = s$ holds, then $|X| \leq \binom{n+s-1}{s} + \binom{d+s-2}{s-1}$ [3]. The case $D = \{d\}$ is investigated in [4]. For non-negative integers $d < m$, $t \geq 2$, and $n > n_0(m)$ (see [4] about $n_0(m)$), if $X \in V(n,m,\{d, d+1, \ldots, d+t-1\})$, then $|X| \leq \binom{n-d}{t}/\binom{m-d}{t}$ [6]. This equality can be attained whenever a Steiner system $S(n-d, m-d, t)$ (equivalently t-$(n-d, m-d, 1)$ design) exists . We also have if $X \in V(n,m,\{-(t-1), -(t-2), \ldots, t-1\})$, then $|X| \leq 2^{t-1}(m-t+1)\binom{n}{t}/\binom{m}{t}$ [7]. When $m = t+1$, this equality can be attained whenever a Steiner system $S(n,m,m-1)$ exists.

2 Largest subsets avoiding the diameter of L_{mkl}

Let L_{mkl} denote the finite set in $\mathbb{R}^n = \mathbb{R}^{m+k+l}$, which consists of all vectors whose number of entries -1, 0, 1 is equal to m, k, l, respectively. For two subsets X, Y of L_{mkl}, X is *isomorphic* to Y if there exists a permutation $\sigma \in S_n$ such that $X = \{(y_{\sigma(1)}, \ldots, y_{\sigma(n)}) \mid (y_1, \ldots, y_n) \in Y\}$. The *diameter* $D(X)$ of $X \subset \mathbb{R}^n$ is defined to be

$$D(X) = \max\{d(x,y) \mid x, y \in X\},$$

where $d(,)$ is the Euclidean distance. Let M_{mkl} denote the largest possible number of cardinalities of $X \subset L_{mkl}$ such that $D(X) < D(L_{mkl})$. The *diameter graph* of $X \subset \mathbb{R}^n$ is defined to be the graph (X,E), where $E = \{(x,y) \mid d(x,y) = D(X)\}$. The problem of determining M_{mkl} is equivalent to determining the independence number of the diameter graph of L_{mkl}. Note that $M_{mkl} = M_{lkm}$ because we have $L_{mkl} = -L_{lkm} = \{-x \mid x \in L_{lkm}\}$. Thus we may assume $m \leq l$. In this section, we determine M_{mkl}, and classify the largest sets for several cases of m, k, l.

First we determine M_{mkl} for the cases $m = l$ and $m + k \leq l$.

Proposition 2.1. *Assume* $m = l$*. Then we have*

$$M_{mkl} = \frac{1}{2}\binom{n}{m}\binom{k+m}{m} = \frac{1}{2}|L_{mkl}|,$$

and the largest sets contain only one of x *or* $-x$ *for any* $x \in L_{mkl}$*.*

Proof. For any $x \in L_{mkl}$, we have $\{y \mid d(x, y) = D(L_{mkl})\} = \{-x\}$. Therefore the diameter graph of L_{mkl} is the set of independent edges. The proposition can be easily proved from this fact. □

For $X \subset L_{mkl}$, we use the notation

$$N_i(X, j) = \{(x_1, \ldots, x_n) \in X \mid x_i = j\}, \qquad \text{and} \qquad n_i(X, j) = |N_i(X, j)|.$$

Proposition 2.2. *Assume* $m + k \leq l$*. Then we have*

$$M_{mkl} = \binom{n-1}{m+k-1}\binom{m+k}{m}.$$

For $m + k > l$*, the largest set is* $N_1(L_{mkl}, -1) \cup N_1(L_{mkl}, 0)$*, up to isomorphism. For* $m + k = l$*, then the largest sets contain only one of* $\{(x_1, \ldots, x_n) \in L_{mkl} \mid x_i = 1, \forall i \in J\}$ *or* $\{(x_1, \ldots, x_n) \in L_{mkl} \mid x_i = 1, \forall i \in I_n \setminus J\}$ *for any* $J \subset I_n$ *of order* l*.*

Proof. A finite subset X of L_{mkl} satisfies $D(X) < D(L_{mkl})$ if and only if $\{i \mid x_i = -1, 0\} \cup \{i \mid y_i = -1, 0\}$ is not empty for any distinct $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in X$. We can therefore apply the Erdős–Ko–Rado Theorem [8] to determine the positions of entries -1 or 0. The number of possible positions of -1, 0 is $\binom{n-1}{m+k-1}$. After fixing the position, -1, 0 can be placed in $\binom{m+k}{k}$ ways. This determines M_{mkl}. The largest sets are classified from the optimal sets of the Erdős–Ko–Rado Theorem. □

The remaining part of this section is devoted to proving

$$M_{1k2} = \mathfrak{M}_k = \binom{k+3}{3} + 2,$$

and determining the classification of the largest sets. Note that $D(L_{1k2}) = \sqrt{10}$ and if $X \subset L_{1k2}$ satisfies $D(X) < D(L_{1k2})$, then $D(X) \leq \sqrt{8}$. The following two lemmas are used later.

Lemma 2.3. *Let* $X \subset L_{1k2}$ *with* $D(X) < D(L_{1k2})$*. Suppose* $k \geq 4$*, and* $|X| \geq \mathfrak{M}_k$*. Then there exists* $i \in \{1, \ldots, n\}$ *such that* $n_i(X, 0) \geq \mathfrak{M}_{k-1}$*.*

Proof. This lemma is immediate because the average of $n_i(X, 0)$ is

$$\frac{1}{n}\sum_{i=1}^{n} n_i(X, 0) = \frac{k|X|}{k+3} \geq \frac{k\mathfrak{M}_k}{k+3} = \mathfrak{M}_{k-1} - \frac{6}{k+3} > \mathfrak{M}_{k-1} - 1. \qquad \square$$

Lemma 2.4. *Let* $G = (V, E)$ *be a connected simple graph, and* E' *a matching in* G*. Assume that* G *has an independent set* I *of size* $|V| - |E'|$*. Then for* $z \in I$ *if* $x \in V$ *satisfies* $(x, y) \in E'$ *for some* y *adjacent to* z*, then* $x \in I$*.*

Proof. Since the cardinality of I is $|V| - |E'|$, only one of x or y is an element of I for any $(x, y) \in E'$. By assumption, $y \notin I$, and hence $x \in I$. □

The subsets $S_k(i)$, $T_k(i)$, $U_k(i)$ of L_{1k2} are defined by

$$\begin{aligned} S_k(i) &= \{(x_1, \ldots, x_n) \in L_{1k2} \mid x_1 = \cdots = x_{i-1} = 0, x_i = -1\}, \\ T_k(i) &= \{(x_1, \ldots, x_n) \in L_{1k2} \mid x_1 = \cdots = x_{i-1} = 0, x_i = 1\}, \\ U_k(i) &= \left\{(x_1, \ldots, x_n) \in L_{1k2} \mid \begin{array}{l} x_1 = 1, x_l = -1, x_j = 1, \\ \exists l \in \{2, \ldots, i\}, \exists j \in \{l+1, \ldots, n\} \end{array} \right\} \end{aligned}$$

for $i = 2, \ldots, k+2$. We define $S_k(1) = N_1(L_{1k2}, -1)$, and $T_k(1) = N_1(L_{1k2}, 1)$. The following are candidates of the largest subsets avoiding the largest distance $\sqrt{10}$.

$$X_k = T_k(k+1) \cup (\bigcup_{i=1}^{k+1} S_k(i)) \text{ for } k \geq 1,$$

$$Y_1 = T_1(1), \qquad Y_k = T_k(k) \cup (\bigcup_{i=1}^{k-1} S_k(i)) \text{ for } k \geq 2,$$

$$Z_2 = T_2(1), \qquad Z_k = T_k(k-1) \cup (\bigcup_{i=1}^{k-2} S_k(i)) \text{ for } k \geq 3.$$

Note that $|X_k| = |Y_k| = |Z_k| = \mathfrak{M}_k$, and they can be inductively constructed by

$$\begin{aligned} X_k &= \{(0, x) \mid x \in X_{k-1}\} \cup N_1(L_{1k2}, -1), \\ Y_k &= \{(0, x) \mid x \in Y_{k-1}\} \cup N_1(L_{1k2}, -1), \\ Z_k &= \{(0, x) \mid x \in Z_{k-1}\} \cup N_1(L_{1k2}, -1). \end{aligned}$$

We also use the following notation.

$$\begin{aligned} X'_k &= X_k \setminus S_k(1) = \{(0, x) \mid x \in X_{k-1}\} \quad (k \geq 2), \\ Y'_k &= Y_k \setminus S_k(1) = \{(0, x) \mid x \in Y_{k-1}\} \quad (k \geq 2), \\ Z'_k &= Z_k \setminus S_k(1) = \{(0, x) \mid x \in Z_{k-1}\} \quad (k \geq 3). \end{aligned}$$

Theorem 2.5. *Let $X \subset L_{1k2}$ with $D(X) < D(L_{1k2})$. Then we have*

$$|X| \leq \mathfrak{M}_k.$$

If equality holds, then

(1) *for $k = 1$, $X = X_1$, or Y_1,*

(2) *for $k \geq 2$, $X = X_k$, Y_k, or Z_k,*

up to isomorphism.

This theorem will be proved by induction. We first prove the inductive step.

Lemma 2.6. *Let $k \geq 2$. Assume that the statement in Theorem 2.5 holds for some $k-1$. Let $X \subset L_{1k2}$ with $D(X) < D(L_{1k2})$, such that $n_i(X, 0) = \mathfrak{M}_{k-1}$ for some i. Then we have $|X| \leq \mathfrak{M}_k$. If equality holds, then $X = X_k$, Y_k, or Z_k, up to isomorphism.*

Proof. Without loss of generality, $n_1(X, 0) = \mathfrak{M}_{k-1}$, and hence X contains X'_k, Y'_k, or Z'_k for $k \geq 3$, and X'_1, or Y'_1 for $k = 2$.

(i) Suppose $X'_k \subset X$ for $k \geq 2$. The set of other candidates of elements of X is $S_k(1) \cup U_k(k)$. The diameter graph G of $S_k(1) \cup U_k(k)$ is a bipartite graph of the partite sets $S_k(1)$ and $U_k(k)$. Since the three elements

$$(-1, 0, \dots, 0, 0, 1, 1), (-1, 0, \dots, 0, 1, 0, 1), (-1, 0, \dots, 0, 1, 1, 0) \in S_k(1)$$

are isolated vertices in G, they may be contained in X. Let G' be the subgraph of G formed by removing the three isolated vertices. A perfect matching of G' is given as follows.

Matching (i)

$S_k(1)$ $(-1, x_2, \dots, x_{k+3})$	$U_k(k)$ $(1, y_2, \dots, y_{k+3})$
$x_i = 1, x_j = 1$ $(2 \leq i \leq k, i < j < n)$	$y_i = -1, y_{j+1} = 1$
$x_i = 1, x_n = 1$ $(2 \leq i \leq k)$	$y_i = -1, y_{i+1} = 1$

By this matching, we can show

$$|X| \leq \mathfrak{M}_{k-1} + |S_k(1)| = \mathfrak{M}_k.$$

We will classify the sets attaining this bound. First assume that $x \in X$ for some $x \in S_k(1)$ with $x_2 = 1$. By Lemma 2.4, X must contain any $x \in S_k(1)$ with $x_2 = 1$. In particular, $(-1, 1, 1, 0, \dots, 0) \in X$. Using Lemma 2.4 again, X must contain $x \in S_k(1)$ with $x_3 = 1$. By a similar manner, X must contain any $x \in S_k(1)$. Therefore $X = X_k$.

Assume X does not contain any $x \in S_k(1)$ with $x_2 = 1$, namely $n_2(X, 1) = 0$. By assumption, we have

$$|X| = n_2(X, -1) + n_2(X, 0) \leq \binom{k+2}{2} + \mathfrak{M}_{k-1} = \mathfrak{M}_k.$$

If $|X| = \mathfrak{M}_k$, then we have $n_2(X, -1) = \binom{k+2}{2}$ and $n_2(X, 0) = \mathfrak{M}_{k-1}$. This implies that X is isomorphic to X_k, Y_k, or Z_k.

(ii) Suppose $Y'_k \subset X$ for $k \geq 2$. The set of other candidates of elements of X is the union of $S_k(1)$, $U_k(k-1)$, and

$$\mathcal{S}_1 = \{(x_1, \dots, x_{k+3}) \in L_{1k2} \mid x_1 = 1, x_k = 1, x_j = -1, k < j\}$$

for $k \geq 3$, and $S_2(1) \cup \mathcal{S}_1$ for $k = 2$. The diameter graph G of $S_k(1) \cup U_k(k-1) \cup \mathcal{S}_1$ is a bipartite graph of the partite sets $S_k(1)$ and $U_k(k-1) \cup \mathcal{S}_1$. Since the three elements

$$(-1, 0, \dots, 0, 1, 1, 0, 0), (-1, 0, \dots, 0, 1, 0, 1, 0), (-1, 0, \dots, 0, 1, 0, 0, 1) \in S_k(1)$$

are isolated vertices in G, they may be contained in X. Let G' be the subgraph of G formed by removing the three isolated vertices. A perfect matching of G' is given as follows.

Matching (ii)

$S_k(1)$ $(-1, x_2, \dots, x_{k+3})$	$U_k(k-1)$ $(1, y_2, \dots, y_{k+3})$
$x_i = 1, x_j = 1$ $(2 \leq i \leq k-1, i < j < n)$	$y_i = -1, y_{j+1} = 1$
$x_i = 1, x_n = 1$ $(2 \leq i \leq k-1)$	$y_i = -1, y_{i+1} = 1$

$S_k(1)$	$\mathcal{S}_1$
$(-1,0,\ldots,0,1,1,0)$	$(1,0,\ldots,0,1,-1,0,0)$
$(-1,0,\ldots,0,0,1,1)$	$(1,0,\ldots,0,1,0,-1,0)$
$(-1,0,\ldots,0,1,0,1)$	$(1,0,\ldots,0,1,0,0,-1)$

By this maching, we can show $|X| \leq \mathfrak{M}_k$.

We will classify the sets attaining this bound. For $k = 2$, the maximum indepdent sets of G' is $\{(-1,0,0,1,1),(-1,0,1,0,1),(-1,0,1,1,0)\} \subset S_2(1)$ or $\mathcal{S}_1$. This implies that $X = Y_2$ or Z_2. For $k \geq 3$, we assume that $x \in X$ for some $x \in S_k(1)$ with $x_2 = 1$. By Lemma 2.4, X must contain any $x \in S_k(1)$. Therefore $X = Y_k$. If X does not contain any $x \in S_k(1)$ with $x_2 = 1$, namely $n_2(X,1) = 0$. It can be proved that X is isomorphic to X_k, Y_k, or Z_k.

(iii) Suppose $k \geq 3$, and $Z'_k \subset X$. The set of other candidates of elements of X is the union of $S_k(1)$, $U_k(k-2)$, and

$$\mathcal{S}_2 = \{(x_1,\ldots,x_{k+3}) \in L_{1k2} \mid x_1 = 1, x_{k-1} = 1, x_j = -1, k < j\}$$

for $k \geq 4$, and $S_3(1) \cup \mathcal{S}_2$ for $k = 3$. The diameter graph G of $S_k(1) \cup U_k(k-2) \cup \mathcal{S}_2$ is a bipartite graph of the partite sets $S_k(1)$ and $U_k(k-2) \cup \mathcal{S}_2$. Since the four vectors

$$(-1,0,\ldots,0,1,1,0,0,0),(-1,0,\ldots,0,1,0,1,0,0),$$
$$(-1,0,\ldots,0,1,0,0,1,0),(-1,0,\ldots,0,1,0,0,0,1) \in S_k(1)$$

are isolated vertices in G, they may be contained in X. Let G' be the subgraph of G formed by removing the four isolated vertices. A maximum matching of G' is given as follows.

Matching (iii)

$S_k(1)$ $(-1,x_2,\ldots,x_{k+3})$	$U_k(k-2)$ $(1,y_2,\ldots,y_{k+3})$
$x_i = 1, x_j = 1$ $(2 \leq i \leq k-2, i < j < n)$	$y_i = -1, y_{j+1} = 1$
$x_i = 1, x_n = 1$ $(2 \leq i \leq k-2)$	$y_i = -1, y_{i+1} = 1$

$S_k(1)$	$\mathcal{S}_2$
$(-1,0,\ldots,0,1,1,0,0)$	$(1,0,\ldots,0,1,-1,0,0,0)$
$(-1,0,\ldots,0,0,1,1,0)$	$(1,0,\ldots,0,1,0,-1,0,0)$
$(-1,0,\ldots,0,0,0,1,1)$	$(1,0,\ldots,0,1,0,0,-1,0)$
$(-1,0,\ldots,0,1,0,0,1)$	$(1,0,\ldots,0,1,0,0,0,-1)$

Note that the two vectors

$$(-1,0,\ldots,0,1,0,1,0),(-1,0,\ldots,0,0,1,0,1) \in S_k(1) \tag{2.1}$$

are unmatched in this matching. By this matching, we can show $|X| \leq \mathfrak{M}_k$.

We will classify the sets attaining this bound. If $|X| = \mathfrak{M}_k$, then the two vectors in (2.1) must be contained in X. Therefore X does not contain any element of $\mathcal{S}_2$, and contains an element of $S_k(1)$ which matches some element of $\mathcal{S}_2$. For $k = 3$, X therefore contains $S_k(1)$, and $X = Z_3$. For $k \geq 4$, we assume that $x \in X$ for some $x \in S_k(1)$ with $x_2 = 1$. By Lemma 2.4, X must contain any $x \in S_k(1)$. Therefore $X = Z_k$. If X does not contain any $x \in S_k(1)$ with $x_2 = 1$, namely $n_2(X,1) = 0$. Therefore X is isomorphic to X_k, Y_k, or Z_k. □

Matchings (i)–(iii) and the notation $\mathcal{S}_1, \mathcal{S}_2$ defined in the proof of Lemma 2.6 are used again later. The base case in the induction is the case $k = 3$. We will prove the cases $k = 1, 2, 3$ in order.

Proposition 2.7. *Let $X \subset L_{112}$ with $D(X) < D(L_{112})$. Then we have*

$$|X| \leq \mathfrak{M}_1 = 6.$$

If equality holds, then $X = X_1$, or Y_1, up to isomorphism.

Proof. Since the diameter graph G of L_{112} is isomorphic to $C_4 \cup C_4 \cup C_4$, where C_4 is the 4-cycle, the bound $|X| \leq 6$ clearly holds. Considering the permutation of coordinates, G has the automorphism group S_4. Since the stabilizer of X_1 in S_4 is of order 6, the orbit of X_4 has length 4. Similarly the orbit of Y_1 has length 4. Since the number of maximum independent sets of G is $2^3 = 8$, this proposition follows. □

For $k = 2$, we also classify $(\mathfrak{M}_2 - 1)$-point sets X with $D(X) < D(L_{122})$ in order to prove the case $k = 3$.

Proposition 2.8. *Let $X \subset L_{122}$ with $D(X) < D(L_{122})$. Then we have*

$$|X| \leq \mathfrak{M}_2 = 12.$$

If $|X| = 12$, then $X = X_2$, Y_2, or Z_2, up to isomorphism. If $|X| = 11$, then X is

$$V_2 = X_2' \cup \{(-1,0,0,1,1), (-1,0,1,0,1), (-1,0,1,1,0), (-1,1,1,0,0), (1,-1,1,0,0)\},$$

$$W_2 = Y_2' \cup \{(-1,1,1,0,0), (-1,1,0,1,0), (-1,1,0,0,1), (-1,0,0,1,1), (1,1,-1,0,0)\},$$

or the set obtained by removing a point from X_2, Y_2, or Z_2, up to isomorphism.

Proof. First suppose $n_i(X, 0) = 6$ for some i. Then we have $|X| \leq 12$, and X with $|X| = 12$ is X_2, Y_2, or Z_2 by Lemma 2.6. In order to find X with $|X| = 11$, we consider 5-point independent sets in the diameter graph of $S_2(1) \cup U_2(2)$ or $S_2(1) \cup U_2(1) \cup \mathcal{S}_1$. If X is not isomorphic to a subset of X_2, Y_2, or Z_2, then $X = V_2$ from $S_2(1) \cup U_2(2)$, and $X = W_2$ from $S_2(1) \cup U_2(1) \cup \mathcal{S}_1$.

Suppose $n_i(X, 0) \leq 5$ for any i. If $|X| \geq 11$, then the average of $n_i(X, 0)$ is greater than 4. Without loss of generality, we may assume $n_1(X, 0) = 5$. Since the diameter graph of L_{112} is $C_4 \cup C_4 \cup C_4$, we can show that X contains a 5-point subset of X_2' or Y_2'.

(i) Suppose X contains a 5-point subset of X_2'. By considering the automorphism group of X_2', we may assume X contains the 5-point subset obtained by removing $(0, -1, 0, 1, 1)$ or $(0, 0, -1, 1, 1)$. First assume that X contains the 5-point subset obtained by removing $(0, -1, 0, 1, 1)$. Since other candidates of elements of X are still in $S_2(1) \cup U_2(2)$, we have $|X| \leq 11$, and if $|X| = 11$, then X is isomorphic to a subset of X_2, Y_2, or Z_2. Assume that X contains the 5-point subset obtained by removing $(0, 0, -1, 1, 1)$. The set of other candidates of elements of X is $S_2(1) \cup U_2(2) \cup \{(1, 0, 1, -1, 0), (1, 0, 1, 0, -1)\}$. If X does not contain both $(1, 0, 1, -1, 0)$ and $(1, 0, 1, 0, -1)$, then $|X| \leq 11$, and X

attaining this bound is isomorphic to a subset of X_2, Y_2, or Z_2. To make a new set, X may contain $(1,0,1,-1,0)$. The two vectors $(-1,1,0,1,0), (-1,0,0,1,1) \in S_2(1)$, which are at distance $\sqrt{10}$ from $(1,0,1,-1,0)$, are not contained in X. The set P_1 consisting of the two isolated vertices

$$(-1,0,1,0,1), (-1,0,1,1,0) \in S_2(1)$$

and 6 points

$$(-1,1,1,0,0), (-1,1,0,0,1), (1,-1,1,0,0),\\ (1,-1,0,1,0), (1,-1,0,0,1), (1,0,1,0,-1)$$

has the unique maximum 6-point independent set

$$\left\{ \begin{array}{l} (-1,0,1,0,1), (-1,0,1,1,0), (1,-1,1,0,0), \\ (1,-1,0,1,0), (1,-1,0,0,1), (1,0,1,-1,0) \end{array} \right\},$$

which gives X isomorphic to Y_2, and $n_2(X,0) = 6$. If X contains a 5-point independent set in P_1 and is not isomorphic to a subset of Y_2, then X contains the 5-point independent set

$$\{(-1,0,1,0,1), (-1,0,1,1,0), (-1,1,1,0,0), (1,-1,1,0,0), (1,0,1,0,-1)\}.$$

Then X is isomorphic to W_2 and $n_2(X,0) = 6$.

(ii) Suppose X contains a 5-point subset of Y_2'. By considering the automorphism group of Y_2', we may assume X contains the 5-point subset obtained by removing $(0,1,-1,0,1)$. The set of other candidates of elements of X is $S_2(1) \cup \mathcal{S}_1 \cup \{(1,0,1,0,-1)\}$. To make a new set, X may contain $(1,0,1,0,-1)$. The two vectors $(-1,1,0,0,1), (-1,0,0,1,1) \in S_2(1)$, which are at distance $\sqrt{10}$ from $(1,0,1,0,-1)$, are not contained in X. The set consisting of the two isolated vertices

$$(-1,1,1,0,0), (-1,1,0,1,0) \in S_2(1)$$

and 5 points

$$(-1,0,1,1,0), (-1,0,1,0,1), (1,1,-1,0,0), (1,1,0,-1,0), (1,1,0,0,-1)$$

has the unique maximum 5-point independent set

$$\{(-1,1,1,0,0), (-1,1,0,1,0), (1,1,-1,0,0), (1,1,0,-1,0), (1,1,0,0,-1)\},$$

which gives X is isomorphic to a subset of Z_2. □

Proposition 2.9. *Let $X \subset L_{132}$ with $D(X) < D(L_{132})$. Then we have*

$$|X| \le \mathfrak{M}_3 = 22.$$

If equality holds, then $X = X_3$, Y_3, or Z_3, up to isomorphism.

Proof. If $n_i(X,0) = 12$ for some i, then we have $|X| \le 22$, and the set attaining this bound is X_3, Y_3, or Z_3 by Lemma 2.6.

Suppose $n_i(X, 0) \leq 11$ for any i. If $|X| > 22$, then the average of $n_i(X, 0)$ is greater than 11, which gives a contradiction. Therefore $|X| \leq 22$, and if $|X| = 22$, then the average of $n_i(X, 0)$ is 11, and $n_i(X, 0) = 11$ for any i. By Proposition 2.8, X may contain

$$V_3' = \{(0, v) \in L_{132} \mid v \in V_2\},$$

$$W_3' = \{(0, w) \in L_{132} \mid w \in W_2\},$$

or an 11-point set obtained by removing a point from X_3', Y_3', or Z_3'.

(i) Suppose X contains an 11-point subset of X_3'. By considering the automorphism group of X_3', X may contain the set in X_3' obtained by removing $(0, -1, 0, 0, 1, 1)$, $(0, -1, 1, 1, 0, 0)$, $(0, 0, -1, 0, 1, 1)$, or $(0, 0, 0, -1, 1, 1)$. If X contains the set X_3' with $(0, -1, 0, 0, 1, 1)$, $(0, -1, 1, 1, 0, 0)$, or $(0, 0, -1, 0, 1, 1)$ removed, then the set of other candidates of X is still $S_3(1) \cup U_3(3)$, and $|X| < 22$. Suppose X contains the set X_3' with $(0, 0, 0, -1, 1, 1)$ removed. Then new candidates of vectors of X are only $(1, 0, 0, 1, -1, 0)$ and $(1, 0, 0, 1, 0, -1)$, and X may contain $(1, 0, 0, 1, -1, 0)$. The three vectors $(-1, 1, 0, 0, 1, 0)$, $(-1, 0, 1, 0, 1, 0)$, and $(-1, 0, 0, 0, 1, 1)$, which are at distance $\sqrt{10}$ from $(1, 0, 0, 1, -1, 0)$, are not contained in X. Therefore by $|X| = 22$, the other new candidate $(1, 0, 0, 1, 0, -1)$, and two isolated vectors $(-1, 0, 0, 1, 0, 1)$, and $(-1, 0, 0, 1, 1, 0)$ must be contained in X. Moreover a 7-point independent set must be obtained from Matching (i). Since $(-1, 1, 0, 0, 1, 0)$ and $(-1, 0, 1, 0, 1, 0)$ are not contained in X, by Lemma 2.4, $(1, -1, 0, 0, 0, 1)$ and $(1, 0, -1, 0, 0, 1)$ must be contained in X, and consequently any element of $U_2(2)$ is contained in X. This implies $n_2(X, 1) = 0$, and X is isomorphic to X_3, Y_3, or Z_3.

(ii) Suppose X contains an 11-point subset of Y_3'. By considering the automorphism group of Y_3', X may contain the set in Y_3' obtained by removing $(0, -1, 0, 0, 1, 1)$, $(0, -1, 1, 1, 0, 0)$, or $(0, 0, 1, -1, 0, 1)$. If X contains the set Y_3' with $(0, -1, 0, 0, 1, 1)$, or $(0, -1, 1, 1, 0, 0)$ removed, then the set of other candidates of X is still $S_3(1) \cup U_3(2) \cup \mathcal{S}_1$, and $|X| < 22$. Suppose X contains the set Y_3' with $(0, 0, 1, -1, 0, 1)$ removed. Then a new candidate of an element of X is only $(1, 0, 0, 1, 0, -1)$, and X may contain $(1, 0, 0, 1, 0, -1)$. The three vectors $(-1, 1, 0, 0, 0, 1)$, $(-1, 0, 1, 0, 0, 1)$, and $(-1, 0, 0, 0, 1, 1)$, which are at distance $\sqrt{10}$ from $(1, 0, 0, 1, 0, -1)$, are not contained in X. By considering Matching (ii), we can show $|X| < 22$.

(iii) Suppose X contains an 11-point subset of Z_3'. By considering the automorphism group of Z_3', X may contain the set in Z_3' obtained by removing $(0, 1, -1, 0, 0, 1)$. Then a new candidate of an element of X is only $(1, 0, 1, 0, 0, -1)$, and X may contain $(1, 0, 1, 0, 0, -1)$. The three vectors $(-1, 1, 0, 0, 0, 1)$, $(-1, 0, 0, 1, 0, 1)$, and $(-1, 0, 0, 0, 1, 1)$, which are at distance $\sqrt{10}$ from $(1, 0, 1, 0, 0, -1)$, are not contained in X. By considering Matching (iii), we can show $|X| < 22$.

(iv) Suppose X contains V_3'. The set of other candidates of X is $S_3(1) \cup U_3(3) \setminus \{(1, -1, 1, 0, 0, 0)\}$, and the maximum independent set is of order at most 10 by Matching (i). Thus $|X| < 22$.

(v) Suppose X contains W_3'. The set of other candidates of X is $S_3(1) \cup U_3(2) \cup \mathcal{S}_1 \setminus \{(1, -1, 0, 1, 0, 0)\}$, and the maximum independent set is of order at most 10 by Matching (ii). Thus $|X| < 22$.

Therefore this proposition follows. □

Finally we prove Theorem 2.5.

Proof of Theorem 2.5. By Propositions 2.7–2.9, the statement holds for $k = 1, 2, 3$. By the inductive hypothesis and Lemma 2.3, if $|X| \geq \mathfrak{M}_k$, then there exists $i \in \{1, \dots, n\}$ such that $n_i(X, 0) = \mathfrak{M}_{k-1}$ for $k \geq 4$. By Lemma 2.6, this theorem holds for any k. □

3 Classification of the largest 4-distance sets which contain $\tilde{J}(n, 4)$

A finite set X in $\mathbb{R}^d$ is called an s-distance set if the set of Euclidean distances of two distinct vectors in X has size s. The Johnson graph $J(n, m) = (V, E)$, where

$$V = \{\{i_1, \dots, i_m\} \mid 1 \leq i_1 < \dots < i_m \leq n, i_j \in \mathbb{Z}\},$$
$$E = \{(v, u) \mid |v \cap u| = m - 1, v, u \in V\},$$

is represented into $\mathbb{R}^{n-1}$ as the m-distance set $\tilde{J}(n, m) = L_{0,n-m,m}$. Indeed $\tilde{J}(n, m) \subset \mathbb{R}^n$, but the summation of all entries of any $x \in \tilde{J}(n, m)$ is m, and $\tilde{J}(n, m)$ is on a hyperplane isometric to $\mathbb{R}^{n-1}$. Bannai, Sato, and Shigezumi [1] investigated m-distance sets containing $\tilde{J}(n, m)$. In their paper, for $m \leq 5$ and any n, the largest m-distance sets containing $\tilde{J}(n, m)$ are classified except for $(n, m) = (9, 4)$. In this section, the case $(n, m) = (9, 4)$ will be classified.

The set of Euclidean distances of two distinct points of $\tilde{J}(9, 4)$ is $\{\sqrt{2}, \sqrt{4}, \sqrt{6}, \sqrt{8}\}$. The set of vectors which can be added to $\tilde{J}(9, 4)$ while maintaining 4-distance is the union of the following sets [1].

$$X^{(i)} = \left(\left(\frac{2}{3}\right)^7, \left(-\frac{1}{3}\right)^2\right)^P, \qquad X^{(ii)} = \left(\left(\frac{2}{3}\right)^8, -\frac{4}{3}\right)^P,$$
$$X^{(iii)} = \left(\frac{4}{3}, \left(\frac{1}{3}\right)^8\right)^P, \qquad X^{(iv)} = \left(\left(\frac{4}{3}\right)^2, \left(\frac{1}{3}\right)^6, -\frac{2}{3}\right)^P,$$

where the exponents inside indicate the number of occurrences of the corresponding numbers, and the exponent P outside indicates that we should take every permutation. They conjectured that $\tilde{J}(9, 4) \cup X^{(i)} \cup X^{(iii)} \cup \{(-4/3, (2/3)^8)\} \cup X^{(iv)'}$ is largest, where $(-4/3, (2/3)^8) \in X^{(ii)}$, and

$$X^{(iv)'} = \left\{(x_1, \dots, x_9) \in X^{(iv)} \mid x_i = -\frac{2}{3}, x_{j_1} = \frac{4}{3}, x_{j_2} = \frac{4}{3}, i < j_1, j_2\right\}$$
$$\cup \left\{\left(\left(\frac{1}{3}\right)^6, \frac{4}{3}, -\frac{2}{3}, \frac{4}{3}\right), \left(\left(\frac{1}{3}\right)^6, \left(\frac{4}{3}\right)^2, -\frac{2}{3}\right)\right\}.$$

Actually $X^{(iv)'}$ is isometric to X_6 in Section 2 by replacing $-2/3$, $1/3$, $4/3$ to -1, 0, 1, respectively. Let $X^{(iv)''}$ (*resp.* $X^{(iv)'''}$) be the set obtained from Y_6 (*resp.* Z_6) by the same manner. Using Theorem 2.5, we can classify the largest 4-distance sets containing $\tilde{J}(9, 4)$.

Theorem 3.1. *Let $X \subset \{(x_1, \dots, x_9) \in \mathbb{R}^9 \mid x_1 + \dots + x_9 = 1\}$ be a 4-distance set which contains $\tilde{J}(9, 4)$. Then we have*

$$|X| \leq 258.$$

If equality holds, then X is one of the following, up to permutations of coordinates.

(1) $\tilde{J}(9,4) \cup X^{(i)} \cup X^{(iii)} \cup \{(-4/3, (2/3)^8)\} \cup X^{(iv)'}$,
(2) $\tilde{J}(9,4) \cup X^{(i)} \cup X^{(iii)} \cup \{(-4/3, (2/3)^8)\} \cup X^{(iv)''}$,
(3) $\tilde{J}(9,4) \cup X^{(i)} \cup X^{(iii)} \cup \{(-4/3, (2/3)^8)\} \cup X^{(iv)'''}$.

Proof. For any $x \in X^{(i)} \cup X^{(iii)}$, $y \in \cup_{j=1}^4 X^{(j)}$, the Euclidean distance of x, y is in $\{\sqrt{2}, \sqrt{4}, \sqrt{6}, \sqrt{8}\}$, and hence X may contain $X^{(i)} \cup X^{(iii)}$. The set $X^{(iv)}$ is isometric to L_{162} by replacing $-2/3$, $1/3$, $4/3$ to -1, 0, 1, respectively. Therefore the largest subsets of $X^{(iv)}$ with distances $\{\sqrt{2}, \sqrt{4}, \sqrt{6}, \sqrt{8}\}$ are $X^{(iv)'}$, $X^{(iv)''}$, and $X^{(iv)'''}$, up to permutations of coordinates. If X does not contain any element of $X^{(ii)}$, then

$$|X| \leq |\tilde{J}(9,4) \cup X^{(i)} \cup X^{(iii)}| + |X^{(iv)'}| = 257.$$

If X contains $x \in X^{(ii)}$ with $x_i = -4/3$, then X cannot contain $y \in X^{(iv)}$ with $y_i = 4/3$. By re-ordering the vectors, we may assume that the set

$$X^{(ii)}(t) = \{x \in X^{(ii)} \mid x_i = -4/3, \exists i \in \{1, \ldots, t\}\}$$

is in X for some t. Clearly, from the definition of $X^{(ii)}(t)$, this set must have size t. For $t = 7, 8, 9$, X contains at most one element of $X^{(iv)}$, and hence

$$|X| \leq |\tilde{J}(9,4) \cup X^{(i)} \cup X^{(iii)}| + t + 1 \leq 181.$$

If the set $X^{(ii)}(t)$ is in X for $1 \leq t \leq 6$, then consider the set of vectors in $X \cap X^{(iv)}$ in which the entry $1/3$ occurs in all of the first t positions. The final $9 - t$ entries of one of these vectors forms a vector from $L_{1,6-t,2}$; no two vectors in this set can be at the maximum distance. Thus the size of

$$|\{x \in X \cap X^{(iv)} \mid x_i = 1/3, \forall i \in \{1, \ldots, t\}\}|$$

is bounded by $\mathfrak{M}_{6-t}$. It is clear that

$$|\{x \in X \cap X^{(iv)} \mid x_i = -2/3, x_{j_1} = 4/3, x_{j_2} = 4/3, \\ \exists i \in \{1, \ldots, t\}, \exists j_1, j_2 \in \{t+1, \ldots, 9\}\}|$$

is bounded by $t\binom{9-t}{2}$. Thus, for $1 \leq t \leq 6$, we have

$$\begin{aligned} |X| &\leq |\tilde{J}(9,4) \cup X^{(i)} \cup X^{(iii)}| + t + \mathfrak{M}_{6-t} + t\binom{9-t}{2} \\ &= \frac{t^3}{3} - \frac{9t^2}{2} + \frac{31t}{6} + 257 \leq 258, \end{aligned}$$

and equality holds only if $t = 1$. The sets attaining this bound are only the three sets in the statement. □

4 Remarks on other M_{mkl}

Actually it is hard to determine M_{mkl} for other (m, k, l) by a similar manner in Section 2. Fix m, l, where $m < l$. By Proposition 2.2, if $k \leq l - m$, then $M_{mkl} = \binom{n-1}{m+k-1}\binom{m+k}{m}$.

In general there are many largest sets for $k = l - m$. For $k > l - m$, we can inductively construct a large set $X_k \subset L_{mkl}$ satisfying $D(X_k) < D(L_{mkl})$ as follows

$$X_k = \{(0, x') \mid x' \in X_{k-1}\} \cup \{(x_1, \dots, x_n) \in L_{mkl} \mid x_1 = -1\},$$

where X_{l-m} is a largest set for $k = l - m$. Therefore we have

$$M_{mkl} \geq \mathfrak{M}_{mkl} := \binom{m+l-1}{m-1}\binom{k+m+l}{m+l} + \binom{m+l-1}{m}.$$

We can generalize Lemma 2.3 as follows.

Lemma 4.1. *Let $X \subset L_{mkl}$ with $D(X) \leq D(L_{mkl})$. Suppose $k \geq m\binom{m+l}{m} - m - l + 1$, and $|X| \geq \mathfrak{M}_{mkl}$. Then there exists $i \in \{1, \dots, n\}$ such that $n_i(X, 0) \geq \mathfrak{M}_{m,k-1,l}$.*

Proof. This lemma is immediate because the average of $n_i(X, 0)$ is

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n} n_i(X,0) &= \frac{k|X|}{m+k+l} \geq \frac{k\mathfrak{M}_{mkl}}{m+k+l} \\ &= \mathfrak{M}_{m,k-1,l} - \frac{m+l}{m+k+l}\binom{m+k+l}{l} > \mathfrak{M}_{k-1} - 1. \quad \square\end{aligned}$$

In the manner of Section 2, it is hard to classify M_{mkl} for $m - l + 1 \leq k \leq m\binom{m+l}{m} - m - l$. Moreover it seems to be difficult to give matchings, like Matching (i) or (ii), of many possibilities of X_k. We need another idea to determine other M_{mkl}.

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The distinguishing index of the Cartesian product of countable graphs

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Abstract

The *distinguishing index* $D'(G)$ of a graph G is the least cardinal d such that G has an edge colouring with d colours that is preserved only by the trivial automorphism.

We derive some bounds for this parameter for infinite graphs. In particular, we investigate the distinguishing index of the Cartesian product of countable graphs.

Finally, we prove that $D'(K_2^{\aleph_0}) = 2$, where $K_2^{\aleph_0}$ is the infinite dimensional hypercube.

Keywords: Distinguishing index, automorphism, infinite graph, edge colouring, infinite dimensional hypercube.

Math. Subj. Class.: 05C25, 05C80, 03E10

1 Introduction

Albertson and Collins [1] introduced the *(vertex-)distinguishing number* $D(G)$ of a graph G as the least cardinal d such that G has a labelling with d labels that is only preserved by the trivial automorphism. This concept has spawned numerous papers, mostly on finite graphs. But countable infinite graphs have also been investigated with respect to the distinguishing number; see [12], [13], and [14]. For graphs of higher cardinality, see [8]. The corresponding notion for endomorphisms instead of automorphisms is investigated in [5].

Let us consider now any *edge colouring* of a graph G; it is merely a function $f : E(G) \to C$ which labels each edge of G with a *colour* from some set C. Given a graph

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G with an edge colouring f, we say that a graph automorphism $\varphi : V(G) \to V(G)$ of G *preserves the edge colouring* f if $f(xy) = f(\varphi(x)\varphi(y))$ for every edge $xy \in E(G)$; if, on the other hand, there is an edge xy such that $f(xy) \neq f(\varphi(x)\varphi(y))$, then we say that φ is *broken* by xy. It is easy to see that there is, for every connected graph $G \neq K_2$, an edge colouring of G which is preserved only by the *trivial automorphism* of G, i.e., only by the identity $\mathrm{id}_G : V(G) \to V(G)$: Merely choose different colours for different edges. The *distinguishing index* $D'(G)$ of a graph G is the least cardinal d such that G has an edge colouring with d colours that is only preserved by the trivial automorphism. Obviously for K_2 the distinguishing index is not defined and it is the only such connected graph.

For finite graphs this concept is investigated by Kalinowski and Pilśniak in [9] and by Pilśniak in [11]. In [2], the following general upper bound was proved.

Theorem 1.1. *Let G be a connected, infinite graph such that the degree of every vertex of G is not greater than Δ. Then $D'(G) \leq \Delta$.*

A graph G is said to be *prime* with respect to the Cartesian product if whenever $G \cong G_1 \Box G_2$, then either G_1 or G_2 is the graph consisting of a single vertex. It is well known (see [6]) that if G is connected, then G has a unique prime factorization, *i.e.*,

$$G \cong G_1 \Box G_2 \Box \cdots \Box G_t$$

such that for $1 \leq i \leq t, G_i$ is prime. Two graphs G and H are called *relatively prime* if K_1 is the only common factor of G and H. About forty-five years ago Imrich and Miller independently proved the following theorem – see Thm. 6.10 in [6].

Theorem 1.2. *If G is connected and $G = G_1 \Box G_2 \Box \cdots \Box G_r$ is its prime decomposition, then every automorphism of G is generated by the automorphisms of the factors and the transpositions of isomorphic factors.*

A basic fact, which is a reformulation of the above theorem for $r = 2$ and which is used frequently in this paper, is:

If φ is an automorphism of the Cartesian product $G_1 \Box G_2$ of two connected relatively prime graphs, then there are automorphisms φ_i of G_i, $i = 1, 2$, such that $\varphi(v_1, v_2) = (\varphi_1(v_1), \varphi_2(v_2))$ for all $(v_1, v_2) \in V(G_1 \Box G_2)$.

In this case we write $\varphi = (\varphi_1, \varphi_2)$ for short and we note that φ is non-trivial if and only if at least one of φ_1 and φ_2 is non-trivial.

An *asymmetric* graph has only one automorphism, the trivial automorphism. We now state an easy corollary of these properties and definitions for product graphs with distinguishing index 1.

Proposition 1.3. *Let G be the Cartesian product of two graphs G_1 and G_2. Then*

$$D'(G_1 \Box G_2) = 1$$

if and only if G_1 and G_2 are relatively prime and both are asymmetric graphs.

The aim of this paper is to present new results for the distinguishing index of the Cartesian product of infinite graphs. Most graphs in this document are countable, i.e., finite or denumerable; numbers used are either finite or $\aleph_0$.

Subgraphs of the Cartesian product $G_1 \Box G_2$ of the form $G_1 \Box \{v\}$ (for any $v \in V(G_2)$) are isomorphic to G_1 and are called G_1*-layers* of $G_1 \Box G_2$. The G_2*-layers* of $G_1 \Box G_2$ are defined similarly.

The distinguishing index of the Cartesian product of finite graphs is investigated in [4] where the authors prove, amongst others, a result which will be useful in the next section and which we now record as

Theorem 1.4. *Let G be a connected finite graph and $k \geq 2$. Then $D'(G^k) = 2$ with the only exception: $D'(K_2^2) = 3$.*

2 The distinguishing index of the Cartesian product

First we consider the Cartesian product of two denumerable graphs with infinite edge sets.

Lemma 2.1. *Let G_1 and G_2 be two connected relatively prime denumerable graphs. Then $D'(G_1 \Box G_2) \leq 2$.*

Proof. We start by labelling the edges of G_1 with $e_1, e_2, \ldots$ and those of G_2 with $f_1, f_2, \ldots$ This is possible since both edge sets have to be denumerable. Note that these labellings effectively *order* the edges of these graphs. We can now easily describe the required edge distinguishing colouring in colours 1 and 2:
Colour the first (in terms of the above ordering) k edges of the k'th layer of G_1 and the first k edges of the k'th layer of G_2 with 1; colour all other edges with 2. Recall that every edge in $G_1 \Box G_2$ lies in a G_1-layer or a G_2-layer; hence this process colours indeed all edges of $G_1 \Box G_2$. Using the labels, this means that the edges corresponding to the edges $\{e_1, e_2, \ldots, e_k\}$ of G_1 in the k'th G_1-layer and the edges corresponding to the edges $\{f_1, f_2, \ldots, f_k\}$ of G_2 in the k'th G_2-layer, for all $k = 1, 2, \ldots$, are coloured 1 and all other edges are coloured 2.

Now consider, if possible, any non-trivial automorphism $\varphi = (\varphi_1, \varphi_2)$ of $G_1 \Box G_2$ which preserves the above edge colouring of $G_1 \Box G_2$. Since every two different G_1-layers have different numbers of edges coloured with 1, the automorphism φ_2 of G_2 must be trivial. Similarly, φ_1 must be trivial. Hence φ is the trivial automorphism, proving that for every non-trivial automorphism φ of $G_1 \Box G_2$ there is an edge e of $G_1 \Box G_2$ for which e and $\varphi(e)$ are coloured differently. $\Box$

The same result was obtained for the distinguishing number of two connected relatively prime denumerable graphs by Imrich and Klavžar in [7]. Recently it was shown by Estaji, Imrich, Kalinowski, Pilśniak and Tucker in [3] that the condition that the two graphs are relatively prime can be omitted.

Note that Lemma 2.1 assures us that $D'(G_1 \Box G_2)$ is at most two irrespective of the values of $D'(G_1)$ and $D'(G_2)$. Next we consider the case in which both G_1 and G_2 of orders being any cardinals and with finite values for the distinguishing index.

Lemma 2.2. *Suppose G_1 and G_2 are connected relatively prime graphs with finite distinguishing indexes. If $D'(G_i) \leq k_i$, $i = 1, 2$, then $D'(G_1 \Box G_2) \leq max\{k_1, k_2\}$.*

Proof. Since $D'(G_i) \leq k_i$, $i = 1, 2$, there are, with $k = \max\{k_1, k_2\}$, edge colourings f_1 of G_1 and f_2 of G_2 using the colours $1, 2, \ldots, k$ which are distinguishing colourings of G_1 and G_2 respectively. In order to prove now that $D'(G_1 \Box G_2) \leq k$, we again use the notion of a "first" layer through a labelling of the vertices (which here is not explicitly chosen or named). Hence consider the function $f : E(G_1 \Box G_2) \to \{1, 2, \ldots, k\}$ defined by
1) $f((v_1, w)(v_2, w)) = f_1(v_1 v_2)$ for edges of the first G_1-layer and

2) $f((v, w_1)(v, w_2)) = f_2(w_1 w_2)$ for edges of the first G_2-layer and
3) $f(e) = 1$ for all remaining edges.

Consider any non-trivial automorphism $\alpha = (\alpha_1, \alpha_2)$ of $G_1 \square G_2$ with α_1 a non-trivial automorphism of G_1 or α_2 a non-trivial automorphism of G_2. Assume that the first is true (for G_1): Then, since f_1 is a distinguishing colouring of the first G_1-layer, there is an edge e of G_1 such that $f_1(e) \neq f_1(\alpha_1(e))$. Now, if α_2 does not move the first layer, then this edge (considered as an edge of $G_1 \square G_2$) is an edge of the required kind in the first G_1-layer. On the other hand, if α_2 does move the first layer to another layer, we can remark, since $f_1(e) \neq f_1(\alpha_1(e))$, that at least one of $f_1(e)$ and $f_1(\alpha_1(e))$ is different from 1 so that this edge is moved by α_2 to an edge in another layer which has colour 1 by 3) above.

A similarly argument holds if the second is true (for G_2) – merely interchange the roles of G_1 and G_2 (and their colourings and automorphisms) in the above argument.

Hence we are assured that all non-trivial automorphisms of $G_1 \square G_2$ are broken by the colouring f. $\square$

Observe, that $D'(G_1 \square G_2)$ can be arbitrary large, for instance if G_1 is isomorphic to P_3 and G_2 is isomorphic to an infinite ray with many (but finitely many) leaves adjacent to its first vertex.

In our next result we prove that if G_1 satisfies $D'(G_1) = \aleph_0$ and the graph G_2 is finite (so that, in particular $D'(G_2)$ is finite), then $D'(G_1 \square G_2) = \aleph_0$.

Lemma 2.3. *Suppose G_1 and G_2 are connected relatively prime graphs with $D'(G_1) = \aleph_0$ and G_2 is finite. Then $D'(G_1 \square G_2) = \aleph_0$.*

Proof. Suppose, for a proof by contradiction, that $D'(G_1 \square G_2)$ is finite. Since G_2 is a finite graph, there are finite values for $||G_2||$, the number of edges of G_2, and $D'(G_2)$ too. Hence we can choose a positive integer k such that each of these three numbers is at most k.

Since $D'(G_1 \square G_2) \leq k$, there is a k-distinguishing edge colouring f of the edges of $G_1 \square G_2$. Furthermore, since $D'(G_1) = \aleph_0$, there exists, for every positive integer t, a non-trivial automorphism α_t of G_1 which needs at least $t+1$ colours to break it. So if $t \geq k$, the colouring by f of any layer of G_1 induces a colouring on G_1 which cannot be broken by the automorphism α_t of G_1. Since there are infinitely many such automorphisms, we may assume without loss of generality that $\alpha_s \neq \alpha_t$ when $s \neq t$.

Now consider non-trivial automorphisms of $G_1 \square G_2$ of the form $\alpha = (\alpha_t, \mathrm{id}_{G_2})$ (for some $t \geq k$). For each such t, and each edge vw of G_1 (which we can consider as an edge of any G_1-layer of $G_1 \square G_2$), we have that $f(vw) = f(\alpha_t(v)\alpha_t(w))$, i.e., these automorphisms of $G_1 \square G_2$ are not broken by edges in layers of G_1.

The automorphisms α of the above form should therefore be broken by edges of layers of G_2. But this means that, for each $t \geq k$, for at least one edge xy of the G_2-layer determined by a vertex $v \in V(G_1)$, we have that $f(xy)$ in this layer is different from $f(\alpha_t(x)\alpha_t(y))$ in the G_2-layer determined by $\alpha_t(v) \in V(G_1)$. Since there are infinitely many G_2-layers, this requires infinitely many different colourings of G_2. However, there are at most $k^{||G_2||}$ different colourings of G_2-layers. Hence the colouring f cannot break all the infinitely many automorphisms described above. $\square$

As a consequence of the above three lemmas we immediately obtain the following characterisation.

Theorem 2.4. *If G_1 and G_2 are connected relatively prime countable graphs, then $D'(G_1 \Box G_2)$ is infinite if and only if for some $i \in \{1,2\}$ we have that $D'(G_i)$ is infinite while for $j \neq i$ we have that G_j is finite.* □

Now we consider a graph which is the Cartesian power G^k of a denumerable graph G. For a finite graph G, the distinguishing number of the Cartesian power of G is considered in [4]. Here we prove a result for graphs G with a finite number of prime factors (counted with their multiplicities). We begin with a result for prime graphs.

Lemma 2.5. *Let $k \geq 2$ be an integer. If a connected denumerable graph G is prime with respect to the Cartesian product, then $D'(G^k) = 2$.*

Proof. If $k = 2$, the proof is similar to the proof of Lemma 2.1. Indeed, denote $G^2 = G_1 \Box G_2$, where G_1, G_2 are isomorphic to G. Using an analogous proof technique but colouring distinct even numbers of edges of each G_1-layer with red and distinct odd numbers of edges of each G_2-layer with red will also take care of the additional automorphisms generated by the isomorphism between G_1 and G_2.

Now we show that $D'(G \Box H) = 2$ if $D'(H) = 2$ and G is prime. In particular, if we consider $H = G^{k-1}$ then we obtain the thesis by induction. Namely, let f be a distinguishing colouring of H with two colours. We define a colouring of $G \Box H$ as follows: One H-layer is given the colouring f, hence all automorphisms of this H-layer are broken. We colour another H-layer completely blue and all remaining H-layers we colour with distinct numbers of red edges different from the number of red edges in f. Hence all automorphisms of G are broken. If G' isomorphic with G is a factor of H, then we have additional automorphisms, generated by interchanging of G and G'. To break them, we colour each G-layer red. Then every G'-layer contained in a blue H-layer is completely blue, so it cannot be interchanged with G. In this way we break all nontrivial automorphisms of $G \Box H$ with two colours if $D'(H) = 2$ and G is prime. □

The above proof is analogous to the proof of a similar result in [7]. Observe that $D'(G \Box H) = 2$ if $D'(H) = 2$ and G is prime, also if G is finite.

Theorem 2.6. *Let $k \geq 2$ be an integer and G be a connected denumerable graph with the prime factor decomposition $G = G_1 \Box ... \Box G_r$, where $G_1, ..., G_r$ are not necessarily distinct. Then $D'(G^k) = 2$.*

Proof. If G is prime, the claim follows from Lemma 2.5. If G is not prime, we consider the prime factorization $G = G_1 \Box ... \Box G_r$ and apply Lemma 2.5 to every infinite factor (G has at least one infinite prime factor). Moreover, we can use Theorem 1.4 for every finite factor. The result then follows from Lemma 2.2 unless $G = K_2 \Box H$ and $k = 2$, where H is an infinite graph relatively prime with K_2. But we already know that $D'(H^2) = 2$ due to the above arguments, so let f be a distinguishing colouring of H^2 with two colours. We then define a colouring of G^2 in terms of its four H^2-layers as follows: One H^2-layer is given the colouring f, hence all automorphisms of this H^2-layer are broken. The three remaining H^2-layers are coloured with distinct numbers of red edges (while all remaining edges are blue), hence all automorphisms of G^2 are broken. □

We say the G has infinite diameter if there are vertices of arbitrarily large distance. Such a situation occurs in particular in any weak Cartesian product G of infinitely many non-trivial factors (finite or infinite). Hence the above theorem immediately implies the following.

Corollary 2.7. *Let $k \geq 2$ be an integer and let G be a connected denumerable graph with finite diameter. Then $D'(G^k) = 2$.*

3 The distinguishing index of the infinite hypercube

The situation is quite different when we have infinitely many factors in the Cartesian power – consider for example the *infinite dimensional hypercube* $K_2^{\aleph_0}$. This (uncountable) graph has vertices represented by (denumerable) sequences of 0's and 1's and two vertices are adjacent whenever their binary sequences differ in exactly one entry. This graph also has uncountably many connected components, each a countable graph, which are pairwise isomorphic. The automorphism group of $K_2^{\aleph_0}$ is well described (see [10]). Using this information, we are now ready to prove

Theorem 3.1. *Let $K_2^{\aleph_0}$ be the infinite dimensional hypercube. Then $D'(K_2^{\aleph_0}) = 2$.*

Proof. We first construct an asymmetric spanning tree and then show how it can be used to prove the existence of an asymmetric spanning subgraph in every component of $K_2^{\aleph_0}$; these subgraphs will be constructed in such a way that different components have non-isomorphic subgraphs. Towards the end of the proof, we shall show how they can be exploited to break all non-trivial automorphisms of the hypercube $K_2^{\aleph_0}$.

It is convenient to describe the required asymmetric subgraphs by first handling the connected component C^0 in which all sequences have only finitely many 1's (and therefore an infinite tail of 0's). First we build an asymmetric tree T, which is a spanning subgraph of C^0, as follows:

Take $(0,0,0,0,\ldots)$ and let it be the central vertex. Then add $(1,0,0,0,\ldots)$, and the edge between it and the central vertex, to form the first branch of the tree. Next take $(0,1,0,0,0,\ldots)$ and $(1,1,0,0,0,\ldots)$ and the path between them and the central vertex to form the second branch of the tree. The i'th branch of this tree will therefore be the path on the central vertex and $(0^{i-1},1,0,0,0,\ldots)$, $(0^{i-2},1,1,0,0,0,\ldots)$, $\ldots$ and will have length 2^{i-1}. All these binary sequences have 1 on the i'th entry and if we restricted them to the first $i-1$ entries, then we obtain the binary-reflected Gray code list with $i-1$ bits. It can be generated recursively from the list for $i-2$ bits by reflecting the list (i.e. listing the entries in reverse order), concatenating the original list with the reversed list, prefixing the entries in the original list with 0, and then prefixing the entries in the reflected list with 1. In particular, the last vertex of the i'th branch has the code $(1,0^{i-2},1,0,0,0,\ldots)$, and the last but one has the code $(1,0^{i-3},1,1,0,0,0,\ldots)$.

Note that all branches of T are of different length, which ensures us that T is asymmetric, and note that it is a spanning tree of the component C^0. So it means that we can easily distinguish the weak Cartesian product of $\aleph_0$ copies of K_2 by two colors: Namely we colour all the edges of T with one colour and the remaining edges with the second colour.

Now we would like to distinguish the Cartesian product of $\aleph_0$ copies of K_2 by two colours. Consider any sequence $\mathbf{x} = (x_1, x_2, \ldots)$ of 0's and 1's and suppose it is in the connected component C of the hypercube $K_2^{\aleph_0}$. Since C is isomorphic to C^0, we can find a copy of T, say T^C, in C. Now we use $\mathbf{x}$ and T^C to create a spanning subgraph $T_{\mathbf{x}}^C$ of C by adding edges to T^C as follows:

For every positive integer i we add the edge of $K_2^{\aleph_0}$ between the endvertex of the i'th branch and the last but one vertex of the $(i+1)$'th branch of T^C to this tree if and only if

$x_i = 1$. We remark that this edge is indeed in $K_2^{\aleph_0}$ since the binary sequences representing these vertices in C^0 differ in exactly one entry, namely the $(i+1)$'th entry, and therefore the same is true in the isomorphic copy T^C of T. Note also that the choice of the added edges ensures us that $T_{\mathbf{x}}$ is not isomorphic to $T_{\mathbf{x}'}$ whenever $\mathbf{x} \neq \mathbf{x}'$. Since there are uncountably many sequences $\mathbf{x}$, we thus have uncountably many pairwise non-isomorphic subgraphs all of which are asymmetric.

Finally we prove, using these subgraphs of the components of $K_2^{\aleph_0}$, that the infinite hypercube is 2-distinguishable. Consider the following colouring f of the edges of $K_2^{\aleph_0}$: Colour, for each component C of $K_2^{\aleph_0}$ and some fixed choice of a vertex $\mathbf{x}$ of C, all the edges of the spanning subgraph $T_{\mathbf{x}}^C$ with 1; colour all the other edges of $K_2^{\aleph_0}$ with 2. Then consider any automorphism α of $K_2^{\aleph_0}$. Since isomorphisms, and thus α, preserve connectivity, α has to take every component C of $K_2^{\aleph_0}$ to a component C' of $K_2^{\aleph_0}$. But, if $C \neq C'$, then the asymmetric spanning subgraphs $T_{\mathbf{x}}^C$ and $T_{\mathbf{x}'}^{C'}$ of C and C' are not isomorphic (because $\mathbf{x} \neq \mathbf{x}'$), hence the colouring f breaks α. □

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Classification of convex polyhedra by their rotational orbit Euler characteristic

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Abstract

Let $\mathcal{P}$ be a polyhedron whose boundary consists of flat polygonal faces on some compact surface $S(\mathcal{P})$ (not necessarily homeomorphic to the sphere S^2). Let $vo_R(\mathcal{P}), eo_R(\mathcal{P})$, $fo_R(\mathcal{P})$ be the numbers of rotational orbits of vertices, edges and faces, respectively, determined by the group $G = G_R(P)$ of all the rotations of the Euclidean space E^3 preserving $\mathcal{P}$. We define the *rotational orbit Euler characteristic* of $\mathcal{P}$ as the number $Eo_R(\mathcal{P}) = vo_R(\mathcal{P}) - eo_R(\mathcal{P}) + fo_R(\mathcal{P})$.

Using the Burnside lemma we obtain the lower and the upper bound for $Eo_R(\mathcal{P})$ in terms of the genus of the surface $S(P)$. We prove that $Eo_R \in \{2, 1, 0, -1\}$ for any convex polyhedron $\mathcal{P}$. In the non-convex case Eo_R may be arbitrarily large or small.

Keywords: Polyhedron, rotational orbit, Euler characteristic.

Math. Subj. Class.: 52B05, 52B10

1 Introduction

CONTEXT: Euler (1752) discovered the famous relation $v - e + f = 2$ between the numbers of vertices v, edges e and faces f of any convex polyhedron. This *Euler polyhedron formula* was implicitly stated in the formulas of Descartes (1630) $p = 2f + 2v - 4, p = 2e$, where p is the number of "plane angles" – corners of faces determined by pairs of adjacent edges ([5], p.469).

The number $\chi = v - e + f$ can be defined for any *map* (a graph cellularly embedded into a compact surface S) and is called its *Euler characteristic*. It is related to the *genus* g of the surface S as follows: $\chi = 2 - 2g$ ([5], p. 473.) and it may be used for the

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classification of surfaces by two parameters: one is χ and the other is orientability (or non-orientability) of the surface. In [4] we introduced the concept of *Euler orbit characteristic* $Eo = vo - eo + fo$, where vo, eo, fo denote the number of orbits of vertices, edges and faces, respectively, determined by the group of all rotations and reflections of the Euclidean space E^3, preserving the polyhedron $\mathcal{P}$, and we used it for the classification of the 92 Johnson solids ([4], p.258). In this paper we introduce a similar concept, called *rotational orbit Euler characteristic*, and we use it for the classification of convex polyhedra.

Definition 1.1. The rotational Euler orbit characteristic of the polyhedron $\mathcal{P}$ is defined as the number $Eo_R = vo_R - eo_R + fo_R$ where vo_R, eo_R, fo_R are the numbers of rotational orbits of the vertices, edges and faces, respectively, of P (these orbits are determined by the group $G_R(\mathcal{P})$ of all the rotations of the Euclidean space E^3 preserving $\mathcal{P}$).

Proposition 1.2. $Eo_R = 1$ *for all the Platonic solids and all the* n*-prisms and* n*-antiprisms, while* $1 \leq Eo_R \leq 2$ *for all the Archimedean solids.*

Proof. In the Table 1 the number of rotational orbits of Platonic and Archimedean solids are given. These values can be easily found for each solid directly or deduced from the symmetry-type graphs of Platonic and Archimedean solids [3]. The 5 Platonic solids have just one rotational orbit of vertices, edges and faces. The 13 Archimedean solids have at most two rotational orbits of vertices and at most three rotational orbits of edges and faces. The n-prisms and the n-antiprisms have just one rotational orbit of vertices and two rotational orbits of edges and faces. □

class	solid $\mathcal{P}$	vertex pattern	vo_R	eo_R	fo_R	Eo_R
I.	tetrahedron	(3.3.3)	1	1	1	1
I.	octahedron	(3.3.3.3)	1	1	1	1
I.	cube	(4.4.4)	1	1	1	1
I.	icosahedron	(3.3.3.3.3)	1	1	1	1
I.	dodecahedron	(5.5.5)	1	1	1	1
II.	cuboctahedron	(3.4.3.4)	1	1	2	2
II.	icosidodecahedron	(3.5.3.5)	1	1	2	2
III.	truncated tetrahedron	(3.6.6)	1	1	2	2
III.	truncated cube	(3.8.8)	1	1	2	2
III.	truncated octahedron	(4.6.6)	1	1	2	2
III.	truncated dodecahedron	(3.10.10)	1	1	2	2
III.	truncated icosahedron	(5.6.6)	1	1	2	2
IV.	rhombicuboctahedron	(3.4.4.4)	1	2	3	2
IV.	rhombicosidodecahedron	(3.4.5.4)	1	2	3	2
V.	truncated cuboctahedron	(4.6.8)	2	3	3	2
V.	truncated icosidodecahedron	(4.6.10)	2	3	3	2
VI.	snub cube	(3.3.3.3.4)	1	3	3	1
VI.	snub dodecahedron	(3.3.3.3.5)	1	3	3	1
VII.	n-prism	(4.4.n)	1	2	2	1
VIII.	n-antiprism	(3.3.n)	1	2	2	1

Table 1: Values of vo_R, eo_R, fo_R for Platonic and Archimedean solids and for the infinite families of n-prisms and n-antiprisms.

MOTIVATION: Similar bounds on Eo_R exist for the Johnson solids (i.e. convex polyhedra with regular polygonal faces and at least two orbits of vertices [2]). The direct motivation for writing this paper came from the empirical observation that the values of Eo_R for the 92 Johnson solids are in a small range between -1 and 2. This was discovered during the process of constructing a table of 16 parameters of the Johnson solids presented in [4], while the range for Eo for the same solids turned out to be bigger: $0 \le Eo \le 5$.

COMPARISON OF Eo AND Eo_R: The two characteristics behave very differently on the set of all convex polyhedra: the main result of the paper (Theorem 2.1) states that the relation $-1 \le Eo_R \le 2$ holds for all convex polyhedra, while for Eo there is no fixed upper bound, we can obtain only the following estimate: $Eo = vo - eo + fo \le vo + fo \le vo_R + fo_R = \le (vo_R - eo_R + fo_R) + eo_R = Eo_R + eo_R \le 2 + eo_R$.

Definition 1.3. Let $G_R(\mathcal{P}) = \{R_1, R_2, ..., R_{n-1}, R_n = Id\}$ be the group of rotational symmetries of the polyhedron $\mathcal{P}$. The poles of the rotation R_i are the points in which the axis of the rotation R_i intersects the surface $S(\mathcal{P})$. Let $v_p(R_i)$, $e_p(R_i)$, $f_p(R_i)$ denote the numbers of poles of R_i in the vertices, edge centers and face centers of $\mathcal{P}$, respectively. The number $E_p(R_i) = v_p(R_i) - e_p(R_i) + f_p(R_i)$ is called the Euler polar characteristics of the rotation R_i.

Lemma 1.4. *Let n_i denote the number of poles of any non-trivial rotation R_i of the polyhedral map $\mathcal{P}$ on the surface S of genus g. Then*

$$
\begin{gathered}
n_i \le 2(g+1), \\
n_i \in \{0, 2, 4, \ldots, 2(g+1)\}, \\
v_p(R_i) + e_p(R_i) + f_p(R_i) = n_i, \\
0 \le e_p(R_i) \le n_i, \\
E_p(R_i) = n_i - 2e_p(R_i), \\
-2(g+1) \le -n_i \le E_p(R_i) \le n_i \le 2(g+1).
\end{gathered}
$$

If the order of the rotation R_i is greater than 2 (i.e. $R_i^n = id$ and $n > 2$), then $e_p(R_i) = 0$ and $E_p(R_i) = n_i$.

Proof. Any line intersecting $\mathcal{P}$ has at most $2g$ intersecting points with S. If P is a convex polyhedron then any nontrivial rotation R_i has exactly two poles, hence $n_i = 2$. If $n_i > 2$ then each segment of the rotational axis r_i not lying in the interior of P contributes two poles (hence n_i is an even number!) and at least one new handle. Thus it "increases" the genus of S for 1 (since it is well known that the genus counts the numbers of "handles" of a surface), therefore it must be $n_i \le 2(g+1)$. The poles can be only in vertices, edge centers or face centers, hence $v_p(R_i) + e_p(R_i) + f_p(R_i) = n_i$ and $0 \le e_p(R_i) \le n_i$. Obviously $E_p(R_i) = v_p(R_i) - e_p(R_i) + f_p(R_i) = n_i - 2e_p(R_i)$. Therefore the upper bound for $E_p(R_i)$ is n_i and the lower bound is $-n_i$. □

Corollary 1.5. *If $\mathcal{P}$ is a convex polyhedron, then $E_p(R_i) = 2 - 2e_p(R_i) \in \{2, 0, -2\}$. If $e_p = 0$ then $E_p = 2$, if $e_p = 1$ then $E_p = 1$, and if $e_p = 2$ then $E_p = -2$. If the order of the rotation R_i is greater than 2, then $E_p(R_i) = 2$.*

Proof. Every convex polyhedron is homeomorphic (by a radial projection from any point of its interior) to a sphere, which has genus $g = 0$. Now $n_i = 2$ and the formulas follow from the Lemma 1.4. □

The next tool we need (in order to prove the main result, Theorem 2.1) is the Burnside lemma, a standard tool for calculating the number of orbits.

Lemma 1.6. *(Burnside lemma) Let a group G act on some set Q. Let $|G| = n$ denote the number of elements of G and let $|Fix(g)|$ denote the number of elements a of the set Q, preserved by the given element g of the group: $g(a) = a$. Then the number of orbits Qo of the set Q is given by the formula*

$$Qo = \frac{1}{|G|} \sum_{g \in G} |Fix(g)|.$$

If the group of rotational symmetries of a convex polyhedron is the cyclical group C_n then the exact value for the Euler orbit characteristic $Eo_R(\mathcal{P})$ can be obtained by a straightforward application of the Burnside lemma. This is a generalization of the similar result for the spherical polyhedra ([4], p.253).

Proposition 1.7. *Let $\mathcal{P}$ be a convex polyhedron. If $G_R(P) = C_n$ where C_n is generated by a rotation R_1 (thus $R_1^n = I$), then $Eo_R(\mathcal{P}) \in \{0, 1, 2\}$.*

Proof. The identity transformation fixes all vertices, edges and faces while the other $n-1$ rotations fix only the poles. Hence we get by the Burnside lemma and using the Euler formula $v - e + f = 2$ (valid for any convex polyhedron) the following formulas for the numbers of rotational orbits:

$$vo_R = \frac{1}{n}(v + (n-1)v_p),$$

$$eo_R = \frac{1}{n}(e + (n-1)e_p),$$

$$fo_R = \frac{1}{n}(f + (n-1)f_p),$$

$$Eo_R = \frac{1}{n}(2 + (n-1)E_p(R_1)),$$

and using $E_p(R_1) = 2 - 2e_p(R_1)$ we see: if $e_p(R_1) = 0$ then $E_p(R_1) = 2$ and $Eo_R = 2$; if $e_p(R_1) = 1$ then $n = 2$, $E_p(R_1) = 0$ and $Eo_R = 1$; if $e_p(R_1) = 2$ then $n = 2$, $E_p(R_1) = -2$ and $Eo_R = 0$. Thus, if $n > 2$ then $Eo_R = 2$. □

2 The main result

Theorem 2.1. *Let $\mathcal{P}$ be a polyhedron with faces on the surface S of genus g. Then*

$$Eo_R(\mathcal{P}) = \frac{1}{n}(\chi(\mathcal{P}) + \sum_{i=1}^{n-1} E_p(R_i)),$$

and we get the following bounds on $Eo_R(\mathcal{P})$:

$$\frac{1}{n}(\chi(\mathcal{P}) - \sum_{i=1}^{n-1} 2(g+1)) \le Eo_R(\mathcal{P}) \le \frac{1}{n}(\chi(\mathcal{P}) + \sum_{i=1}^{n-1} 2(g+1)).$$

If $\mathcal{P}$ is a convex polyhedron, then

$$-1 \le Eo_R(\mathcal{P}) \le 2.$$

Proof. Let n be the number of elements in the group $G_R(P)$.

The identity transformation fixes each vertex, edge or face.

Every rotation R_i fixes $v_p(R_i)$ vertices, $e_p(R_i)$ edges and $f_p(R_i)$ faces.

Therefore, by the Burnside lemma:

$$vo_R = \frac{1}{n}(v + v_p(R_1) + \cdots v_p(R_{n-1})),$$

$$eo_R = \frac{1}{n}(e + e_p(R_1) + \cdots e_p(R_{n-1})),$$

$$fo_R = \frac{1}{n}(f + f_p(R_1) + \cdots f_p(R_{n-1})),$$

$$Eo_R = \frac{1}{n}(\chi + E_p(R_1) + \cdots E_p(R_{n-1})).$$

Using $-2(g+1) \leq E_p(R_i) \leq 2(g+1)$ (proved in Lemma 1.4) we get

$$\frac{1}{n}(\chi - \sum_{i=1}^{n-1} 2(g+1)) \leq Eo_R \leq \frac{1}{n}(\chi + \sum_{i=1}^{n-1} 2(g+1)).$$

If $\mathcal{P}$ is a convex polyhedron, then $g = 0$, $\chi = 2$, hence

$$Eo_R \leq \frac{1}{n}(2 + (n-1)2) = 2,$$

$$Eo_R \geq \frac{1}{n}(2 + (n-1)(-2)) = \frac{1}{n}(4 + n(-2)) \geq -2 + \frac{4}{n} \geq -1,$$

because $\frac{4}{n} > 0$ and Eo_R must be an integer. □

Thus there are 4 classes C_2, C_1, C_0, C_{-1} of convex polyhedra, whose Eo_R are $2, 1, 0, -1$, respectively.

Is the lower bound $Eo_R = -1$ actually obtained, and (if it is so) for which convex polyhedra? And is there any simple description of these four classes?

Proposition 2.2. *Let a, b, c be the numbers of rotations R_i in the group $G_R(\mathcal{P})$ of a convex polyhedron for which $E_p(R_i)$ equals 2, 0 and -2, respectively, and let n be the number of elements in $G_R(\mathcal{P})$. Then*

$$Eo_R(\mathcal{P}) = \frac{1}{n}(2 + a \cdot 2 + c(-2)) = \frac{2}{n}(1 + a - c).$$

Thus the number $1 + a - c$ is an integer multiple of $\frac{n}{2}$. The numbers a and c can be (for each of the 4 possible values of Eo_R) expressed by b, n and Eo_R.

Proof. This formula follows immediately from the Burnside lemma. Also, it is clear that

$$a + b + c + 1 = n,$$

hence

$$a + c = n - b - 1.$$

The equation $\frac{2}{n}(1+a-c) = Eo_R$ implies $2(1+a-c) = Eo_R \cdot n$ and

$$c - a = 1 - \frac{Eo_R \cdot n}{2}.$$

Then $(a+c)+(c-a) = 2c = n-b-1+1-\frac{Eo_R \cdot n}{2} = \frac{(2-Eo_R)\cdot n}{2} - b$ and

$$c = \frac{(2-Eo_R)n - 2b}{4}.$$

Similarly, $2a = n-b-1-1+\frac{Eo_R \cdot n}{2}$, hence

$$a = \frac{n \cdot (2+Eo_R) - (2b+4)}{4}.$$

For example, if $Eo_R = -1$ and $b = 0$ then $a = \frac{n}{4} - 1$ and $c = \frac{3n}{4}$. In that case n must be divisible by 4. □

Example 2.3. To find such a solid with 4 symmetries we have to look for one having three rotations with poles in edge centers! The lower bound $Eo_R = -1$ is really obtained for the Johnson solid J84 (Snub Disphenoid, see Figure 1), where $vo_R = 2, eo_R = 6, fo_R = 3$, hence $Eo_R = 2 - 6 + 3 = -1$. Here the number of symmetries is 4 (the identity transformation and 3 rotations of order two with axes going through edge centers), $b = 0, a = 0$ and $c = 3$. Thus this lower bound -1 is sharp.

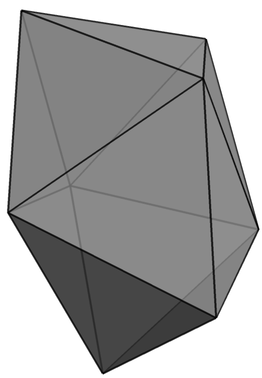

Figure 1: The Johnson solid J84, also known as the Snub Disphenoid.

Remark 2.4. A rotational axis of a non-convex polyhedron may have more than 2 ”poles”. As a consequence, there is no upper or lower bound for Eo_R in the non-convex case.

3 Classification of convex polyhedra

As an immediate consequence of the formulas in Proposition 3 we get:

Corollary 3.1. *The four classes C_2, C_1, C_0, C_{-1} of convex polyhedra (whose Eo_R are $2, 1, 0, -1$, respectively) can be characterized as follows:*

C_2: $a - c = n - 1$

C_1: $a - c = -1 + \frac{n}{2}$

C_0: $a - c = -1$

C_{-1}: $a - c = -1 - \frac{n}{2}$, *all poles in edge centers,* $a = 0$.

Corollary 3.2. *If $a \geq n/2$ then $Eo_R \in \{1, 2\}$.*

Proof. The relation $a \geq n/2$ is a sufficient condition for $a - c > 0$ (since there is also the identity transformation in the group $G_R(\mathcal{P})$) that holds only for polyhedra from C_2 and C_1. □

Corollary 3.3. *Let q be the number of all rotations $R_i \in G_R(\mathcal{P})$ with the property that R_i has the same rotational axis as some k-fold rotation for any $k > 2$. If $q \geq n/2$, where n is the order of the group G_R, then $Eo_R(\mathcal{P}) \in \{1, 2\}$.*

Proof. No rotation with such an axis can have any of its two poles in an edge center, hence $a \geq q \geq n/2$, therefore $Eo_R(\mathcal{P}) \in \{1, 2\}$. □

Now we can classify convex polyhedra with respect to their rotational symmetry groups and their rotational orbit Euler characteristic.

The only possible rotational groups of the Euclidean space E^3 are the rotational groups of 1) the n-gonal pyramid, 2) the n-gonal dipyramid or prism, 3) the regular tetrahedron, 4) the cube or the regular octahedron, 5) the regular dodecahedron or the regular icosahedron ([1], p.34).

Theorem 3.4. *Convex polyhedra with at least one rotational symmetry can be classified by their G_R and by their Eo_R into 13 classes (in Table 2 the impossible cases are marked with $\emptyset$):*

	C_2	C_1	C_0	C_{-1}
C_n cyclical group				$\emptyset$
D_n dihedral group				
T tetrahedron group			$\emptyset$	$\emptyset$
O octahedron group			$\emptyset$	$\emptyset$
D dodecahedron group			$\emptyset$	$\emptyset$

Table 2: Classification of convex polyhedra by G_R and Eo_R.

Proof. If $G_R = C_n$ then $Eo_R \neq -1$ (by Proposition 2). If $G_R \in \{T, O, D\}$ then $q \geq n/2$ (Table 3) hence $\mathcal{P}$ cannot be in C_0 or C_{-1} (by Corollary 3.3). □

	2	3	4	5	n	q
T tetrahedron	3	4			3 + 4.2 + 1 = 12	9
O cube	6	4	3		6 + 4.2 + 3.3 + 1 = 24	17
D dodecahedron	15	10		6	15 + 10.2+12.4 + 1= 60	44

Table 3: Numbers of the 2-,3-,4-,5-fold axes in the solids $\mathcal{P}$ with the rotational groups T, O or D, orders n of the groups T, O, D and the numbers q of all rotations $R_i \in G_R(\mathcal{P})$ with the property that R_i has the same rotational axis as some k-fold rotation for any $k > 2$.

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Lifting symmetric pictures to polyhedral scenes

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Abstract

Scene analysis is concerned with the reconstruction of d-dimensional objects, such as polyhedral surfaces, from $(d-1)$-dimensional pictures (i.e., projections of the objects onto a hyperplane). In this paper we study the impact of symmetry on the lifting properties of pictures. We first use methods from group representation theory to show that the lifting matrix of a symmetric picture can be transformed into a block-diagonalized form. Using this result we then derive new symmetry-extended counting conditions for a picture with a non-trivial symmetry group in an arbitrary dimension to be minimally flat (i.e., 'non-liftable'). These conditions imply very simply stated restrictions on the number of those structural components of the picture that are fixed by the various symmetry operations of the picture. We then also transfer lifting results for symmetric pictures from Euclidean $(d-1)$-space to Euclidean d-space via the technique of coning. Finally, we offer some conjectures regarding sufficient conditions for a picture realized generically for a symmetry group to be minimally flat.

Keywords: Incidence structure, picture, polyhedral scene, lifting, symmetry, coning.

Math. Subj. Class.: 68T45, 20C99, 52C25

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1 Introduction

An important and well studied problem in Artificial Intelligence, Computer Vision and Robotics is to find efficient methods for determining whether a straight line drawing in the Euclidean plane (also known as a '2-picture') corresponds to the projection of a 3-dimensional polyhedral surface (also known as a polyhedral '3-scene'). Applications of these results include image understanding, monocular vision and automatic reconstructions of 3-dimensional polyhedral objects or environments.

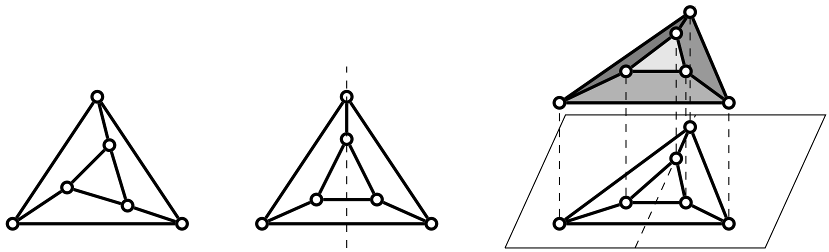
In the Computer Vision community, this problem was first studied by Mackworth and Huffman [12, 7]. Using 'labeling schemes' and 'reciprocal diagrams', they obtained necessary conditions for the realizability of 2-pictures as polyhedral 3-scenes. However, the geometric method of the 'reciprocal diagram' has already been used by J. C. Maxwell and L. Cremona in the 19th century as a graphical tool to analyze the statics of trusses and mechanical structures [5, 13]. This work provides a beautiful connection between the field of polyhedral scene analysis and the field of static (or equivalently, infinitesimal) rigidity of frameworks [27, 29]. For further connections between these fields and other areas of discrete geometry, such as parallel redrawings of configurations (which is the dual interpretation of liftings of pictures), Minkowski decomposability of polytopes, and projective point-line configurations, see [21, 33, 34] for example. While reciprocal diagrams provide a powerful tool to check for inconsistencies in pictures, they do not provide *sufficient* conditions for the realizability of pictures as polyhedral scenes.

In [22, 23, 24] Sugihara used linear programming methods to establish both a necessary and sufficient condition for a general picture to represent a polyhedron (see also [25]). Various other necessary and sufficient conditions were subsequently obtained by Crapo, Whiteley, et al. using a variety of different methods ranging from projective geometry and Grassmann-Cayley algebra to invariant theory (see e.g. [3, 4, 14, 26, 30, 31, 32]).

A fundamental tool for analyzing a given picture is the lifting matrix, whose rank, row dependencies and column dependencies completely determine the liftability properties of the picture (see e.g. [3, 31, 33, 34]). In particular, this matrix yields some simple necessary counting conditions for a $(d-1)$-dimensional picture to be 'flat' (i.e., non-liftable to a d-dimensional polyhedral scene) in terms of the number of vertices, faces and incidences of the underlying combinatorial incidence structure. Following a conjecture of Sugihara [23], Whiteley showed in [31] that these counts are also sufficient for 'generic' pictures (with the same underlying incidence structure) to be flat.

In this paper, we study the impact of symmetry on the lifting properties of $(d-1)$-dimensional pictures. This has important practical applications since symmetry is ubiquitous in both man-made and natural structures. Moreover, there has recently been a surge of interest in studying the impact of symmetry on the static or infinitesimal rigidity properties of structures (see e.g. [2, 6, 10, 15, 18, 19]), and hence it is natural to apply similar group-theoretic methods to the lifting analysis of symmetric pictures.

In Section 4 we first show that the lifting matrix of a symmetric $(d-1)$-picture can be transformed into a block-diagonalized form using methods from group representation theory. This is a fundamental result, since the block-decomposition of the lifting matrix can be used to break up the lifting analysis of a symmetric picture into a number of independent subproblems, one for each block of the lifting matrix. In fact, the analogous result for the rigidity matrix of a symmetric framework (see [10, 15]) is basic to most of the recent results regarding the rigidity analysis of symmetric structures (see e.g. [2, 6, 18]). Similarly to [15], the block-decomposition of the lifting matrix is obtained by showing that



(a) (b) (c)

Figure 1: A (minimally) flat 2-picture (where all four interior regions are faces) (a) which becomes sharp if realized with reflectional symmetry (b). A non-trivial (and sharp) lifting of the picture in (b) is shown in (c).

it intertwines two particular matrix representations of the given symmetry group. For the lifting matrix, one of these representations is associated with the incidences of the picture and the other one is associated with the vertices and faces of the picture (see Theorem 4.1).

In Section 5 we then use these results, together with some methods from character theory, to derive new necessary counting conditions for a symmetric picture to be 'minimally flat' (i.e., flat with the property that the removal of any incidence leads to a picture which does have a non-trivial lifting). Such pictures may be thought of as the basic building blocks for symmetric flat pictures, as we may (symmetrically) add further incidence constraints to a minimally flat picture to obtain classes of (over-constrained) flat pictures. We then follow the approach in [2] to derive a complete list of the necessary conditions for 2-dimensional pictures to be minimally flat, as these are the most important structures for practical applications. Similar counts for higher-dimensional pictures can easily be obtained for any symmetry group using Corollary 5.5. A well established tool in rigidity theory for transferring results from an Euclidean space to the next higher dimension (as well as to other types of metrics) is the technique of 'coning' (see e.g. [20, 28]). In the end of Section 5 we show that coning can also be used to transfer lifting results for pictures from $(d-1)$-space to d-space.

Finally, in Section 6 we offer some conjectures regarding combinatorial characterizations of minimally flat pictures which are as generic as possible subject to the given symmetry constraints. Moreover, we briefly discuss the question of whether a picture which is generic modulo symmetry has a 'sharp' lifting, i.e. a lifting where any two faces sharing a vertex lie in different hyperplanes.

2 Pictures, liftings, and scenes

A *(polyhedral) incidence structure* S is an abstract set of vertices V, an abstract set of faces F, and a set of incidences $I \subseteq V \times F$.

A $(d-1)$-*picture* is an incidence structure S together with a corresponding location map $r : V \to \mathbb{R}^{d-1}$, $r_i = (x_i, y_i, \dots, w_i)^T$, and is denoted by $S(r)$.

A d-*scene* $S(p, P)$ is an incidence structure $S = (V, F; I)$ together with a pair of location maps, $p : V \to \mathbb{R}^d$, $p_i = (x_i, \dots, w_i, z_i)^T$, and $P : F \to \mathbb{R}^d$, $P^j = (A^j, \dots, C^j, D^j)^T$, such that for each $(i, j) \in I$ we have $A^j x_i + \dots + C^j w_i + z_i + D^j = 0$. (We assume that no hyperplane is vertical, i.e., is parallel to the vector $(0, \ldots, 0, 1)^T$.)

A *lifting* of a $(d-1)$-picture $S(r)$ is a d-scene $S(p, P)$, with the vertical projection $\Pi(p) = r$. That is, if $p_i = (x_i, \ldots, w_i, z_i)^T$, then $r_i = (x_i, \ldots, w_i)^T = \Pi(p_i)$.

A lifting $S(p, P)$ is *trivial* if all the faces lie in the same plane. Further, $S(p, P)$ is *folded* (or *non-trivial*) if some pair of faces have different planes, and is *sharp* if each pair of faces sharing a vertex have distinct planes. A picture is called *sharp* if it has a sharp lifting. Moreover, a picture which has no non-trivial lifting is called *flat* (or *trivial*). A picture with a non-trivial lifting is called *foldable*.

The *lifting matrix* for a picture $S(r)$ is the $|I| \times (|V| + d|F|)$ coefficient matrix $M(S, r)$ of the system of equations for liftings of a picture $S(r)$: For each $(i, j) \in I$, we have the equation $A^j x_i + B^j y_i + \ldots + C^j w_i + z_i + D^j = 0$, where the variables are ordered as $[\ldots, z_i, \ldots; \ldots, A^j, B^j, \ldots, D^j, \ldots]$. That is the row corresponding to $(i, j) \in I$ is:

$$(i,j)\ \underbrace{\boxed{0\ldots0 \quad \overset{i}{1} \quad 0\ldots0}}_{|V|}\ \underbrace{\boxed{0\ldots0 \quad \overbrace{r_i \quad 1}^{j} \quad 0\ldots0}}_{d|F|}$$

Theorem 2.1 (Picture Theorem)**.** *[31, 33] A generic picture of an incidence structure $S = (V, F; I)$ with at least two faces has a sharp lifting, unique up to lifting equivalence, if and only if $|I| = |V| + d|F| - (d+1)$ and $|I'| \leq |V'| + d|F'| - (d+1)$ for all subsets I' of incidences with at least two faces.*

A generic picture of S has independent rows in the lifting matrix if and only if for all non-empty subsets I' of incidences, we have $|I'| \leq |V'| + d|F'| - d$.

Note that it follows from the Picture Theorem that a generic picture of an incidence structure $S = (V, F; I)$ is *minimally flat*, i.e. flat with independent rows in the lifting matrix, if and only if $|I| = |V| + d|F| - d$ and $|I'| \leq |V'| + d|F'| - d$ for all non-empty subsets I' of incidences.

3 Symmetric incidence structures and pictures

An *automorphism* of an incidence structure $S = (V, F; I)$ is a pair $\alpha = (\pi, \sigma)$, where π is a permutation of V and σ is a permutation of F such that $(v, f) \in I$ if and only if $(\pi(v), \sigma(f)) \in I$ for all $v \in V$ and $f \in F$. For simplicity, we will write $\alpha(v)$ for $\pi(v)$ and $\alpha(f)$ for $\sigma(f)$.

The automorphisms of S form a group under composition, denoted $\mathrm{Aut}(S)$. An *action* of a group Γ on S is a group homomorphism $\theta : \Gamma \to \mathrm{Aut}(S)$. The incidence structure S is called Γ-*symmetric* (with respect to θ) if there is such an action. For simplicity, if θ is clear from the context, we will sometimes denote the automorphism $\theta(\gamma)$ simply by γ.

Let Γ be an abstract group, and let S be a Γ-symmetric incidence structure (with respect to θ). Further, suppose there exists a group representation $\tau : \Gamma \to O(\mathbb{R}^{d-1})$. Then we say that a picture $S(r)$ is Γ-*symmetric* (with respect to θ and τ) if

$$\tau(\gamma)(r_i) = r_{\theta(\gamma)(i)} \text{ for all } i \in V \text{ and all } \gamma \in \Gamma. \tag{3.1}$$

In this case we also say that $\tau(\Gamma) = \{\tau(\gamma) |\, \gamma \in \Gamma\}$ is a *symmetry group* of $S(r)$, and each element of $\tau(\Gamma)$ is called a *symmetry operation* of $S(r)$. Throughout this paper, we will use the Schoenflies notation for symmetry operations and symmetry groups, as this is one of the standard notations in the literature on symmetric structures [2, 6, 10, 15].

4 Block-decomposing the lifting matrix of a symmetric picture

In this section we will show that by changing the canonical bases for $\mathbb{R}^{|I|}$ and $\mathbb{R}^{|V|+d|F|}$ to appropriate symmetry-adapted bases, the lifting matrix of a symmetric picture can be transformed into a block-decomposed form.

Let Γ be an abstract group, and let $S = (V, F; I)$ be a Γ-symmetric incidence structure (with respect to $\theta : \Gamma \to \mathrm{Aut}(S)$). Further, let $S(r)$ be a Γ-symmetric $(d-1)$-picture with respect to the action θ and the homomorphism $\tau : \Gamma \to O(\mathbb{R}^{d-1})$. We fix an ordering of the vertices in V, the faces in F and the incidences in I.

We let $P_V : \Gamma \to GL(\mathbb{R}^{|V|})$ be the linear representation of Γ defined by $P_V(\gamma) = [\delta_{\theta(\gamma)(j)}]_{i,j}$, where δ denotes the Kronecker delta symbol. That is, $P_V(\gamma)$ is the permutation matrix of the permutation of V induced by $\theta(\gamma)$. Similarly, we let $P_F : \Gamma \to GL(\mathbb{R}^{|F|})$ be the linear representation of Γ defined by $P_F(\gamma) = [\delta_{\theta(\gamma)(j)}]_{i,j}$. That is, $P_F(\gamma)$ is the permutation matrix of the permutation of F induced by $\theta(\gamma)$. Moreover, note that for each $\gamma \in \Gamma$, the automorphism $\theta(\gamma)$ of S clearly also induces a permutation of the incidences I of S. So, analogously to P_V and P_F, we let $P_I : \Gamma \to GL(\mathbb{R}^{|I|})$ be the linear representation of Γ which consists of the permutation matrices of the permutations of I induced by θ. We call P_I the *internal representation* of Γ (with respect to θ and τ).

The *external representation* of Γ (with respect to θ and τ) is the linear representation

$$P_V \oplus (\hat{\tau} \otimes P_F) : \Gamma \to GL(\mathbb{R}^{|V|} \oplus \mathbb{R}^{d|F|}),$$

where $\hat{\tau} : \Gamma \to O(\mathbb{R}^d)$ is the *augmented representation of* τ, defined by $\hat{\tau}(\gamma) = \begin{pmatrix} \tau(\gamma) & 0 \\ 0 & 1 \end{pmatrix}$.

Recall that given two linear representations, $\rho_1 : \Gamma \to GL(X)$ and $\rho_2 : \Gamma \to GL(Y)$ with representation spaces X and Y, a linear map $T : X \to Y$ is said to be a Γ-linear map of ρ_1 and ρ_2 if $T \circ \rho_1(\gamma) = \rho_2(\gamma) \circ T$ for all $\gamma \in \Gamma$. The vector space of all Γ-linear maps of ρ_1 and ρ_2 is denoted by $\mathrm{Hom}_\Gamma(\rho_1, \rho_2)$.

Theorem 4.1. *Let $S = (V, F; I)$ be a Γ-symmetric incidence structure (with respect to θ), let $\tau : \Gamma \to O(\mathbb{R}^{d-1})$ be a homomorphism, and let $S(r)$ be a Γ-symmetric $(d-1)$-picture (with respect to θ and τ). Then $M(S, r) \in \mathrm{Hom}_\Gamma(P_V \oplus (\hat{\tau} \otimes P_F), P_I)$.*

Proof: Suppose we have $M(S, r)c = b$. Then we need to show that for all $\gamma \in \Gamma$, we have $M(S, r)(P_V \oplus (\hat{\tau} \otimes P_F))(\gamma)c = P_I(\gamma)b$.

Fix $\gamma \in \Gamma$, and let $\theta(\gamma) = (\pi, \sigma)$. The automorphism of I induced by $\theta(\gamma)$ we denote by α.

Let $(i, j) \in I$. Further, let $\pi(i) = k$ and $\sigma(j) = l$. Thus, $\alpha((i, j)) = (k, l)$. Note that $P_I(\gamma)b$ is an $|I| \times 1$ column vector which is indexed by the incidences in I. By the definition of $P_I(\gamma)$, for the (k, l)-th component, $(P_I(\gamma)b)_{(k,l)}$, of $P_I(\gamma)b$ we have $(P_I(\gamma)b)_{(k,l)} = (b)_{(i,j)}$. So we need to show that $(M(S, r)(P_V \oplus (\hat{\tau} \otimes P_F))(\gamma)c)_{(k,l)} = (b)_{(i,j)}$.

Note that $(M(S, r)c)_{(k,l)} = (b)_{(k,l)} = z_k + A^l x_k + B^l y_k + \ldots + C^l w_k + D^l$. By the definition of $P_V \oplus (\hat{\tau} \otimes P_F)(\gamma)$, $(M(S, r)(P_V \oplus (\hat{\tau} \otimes P_F))(\gamma)c)_{(k,l)}$ is equal to

$$z_i + (x_k, y_k, \ldots, w_k, 1) \begin{pmatrix} \tau(\gamma) & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A^j \\ B^j \\ \vdots \\ D^j \end{pmatrix}.$$

By symmetry (recall that $\pi(i) = k$, and hence $\tau(\gamma)(r_i) = r_k$), this is equal to

$$z_i + (x_i, y_i, \dots, w_i, 1) \begin{pmatrix} \tau(\gamma) & 0 \\ 0 & 1 \end{pmatrix}^T \begin{pmatrix} \tau(\gamma) & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A^j \\ B^j \\ \vdots \\ D^j \end{pmatrix}.$$

Since $\tau(\gamma)$ (and hence also $\hat{\tau}(\gamma)$) is an orthogonal matrix, this is equal to

$$z_i + x_i A^j + \dots + w_i C^j + D^j = (b)_{(i,j)}.$$

This completes the proof. □

Since $M(S,r) \in \mathrm{Hom}_\Gamma(P_V \oplus (\hat{\tau} \otimes P_F), P_I)$, there are non-singular matrices A and B such that $B^T M(S,r) A$ is block-diagonalized, by Schur's lemma (see [8] e.g.). If $\rho_0, \dots, \rho_n$ are the irreducible representations of Γ, then for an appropriate choice of symmetry-adapted coordinate systems, the lifting matrix takes on the following block form

$$B^T M(S,r) A := \widetilde{M}(S,r) = \begin{pmatrix} \widetilde{M}_0(S,r) & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \widetilde{M}_n(S,r) \end{pmatrix}, \tag{4.1}$$

where the submatrix block $\widetilde{M}_i(S,r)$ corresponds to the irreducible representation ρ_i of Γ.

More precisely, the symmetry-adapted coordinate systems can be obtained as follows. Recall that every linear representation of Γ can be written uniquely, up to equivalency of the direct summands, as a direct sum of the irreducible linear representations of Γ. So we have

$$P_V \oplus (\hat{\tau} \otimes P_F) = \lambda_0\rho_0 \oplus \dots \oplus \lambda_n\rho_n, \text{ where } \lambda_0, \dots, \lambda_n \in \mathbb{N} \cup \{0\}. \tag{4.2}$$

For each $t = 0, \dots, n$, there exist λ_t subspaces $\left(V^{(\rho_t)}\right)_1, \dots, \left(V^{(\rho_t)}\right)_{\lambda_t}$ of the $\mathbb{R}$-vector space $\mathbb{R}^{|V|+d|F|}$ which correspond to the λ_t direct summands in (4.2), so that

$$\mathbb{R}^{|V|+d|F|} = V^{(\rho_0)} \oplus \dots \oplus V^{(\rho_n)}, \tag{4.3}$$

where

$$V^{(\rho_t)} = \left(V^{(\rho_t)}\right)_1 \oplus \dots \oplus \left(V^{(\rho_t)}\right)_{\lambda_t}. \tag{4.4}$$

Similarly, for the internal representation P_I of Γ, we have the direct sum decomposition

$$P_I = \mu_0\rho_0 \oplus \dots \oplus \mu_n\rho_n, \text{ where } \mu_0, \dots, \mu_n \in \mathbb{N} \cup \{0\}. \tag{4.5}$$

For each $t = 0, \dots, n$, there exist μ_t subspaces $\left(W^{(\rho_t)}\right)_1, \dots, \left(W^{(\rho_t)}\right)_{\mu_t}$ of the $\mathbb{R}$-vector space $\mathbb{R}^{|I|}$ which correspond to the μ_t direct summands in (4.5), so that

$$\mathbb{R}^{|I|} = W^{(\rho_0)} \oplus \dots \oplus W^{(\rho_n)}, \tag{4.6}$$

where

$$W^{(\rho_t)} = \left(W^{(I_t)}\right)_1 \oplus \dots \oplus \left(W^{(\rho_t)}\right)_{\mu_t}. \tag{4.7}$$

If we choose bases $\left(A^{(\rho_t)}\right)_1, \dots, \left(A^{(\rho_t)}\right)_{\lambda_t}$ for the subspaces in (4.4) and we also choose bases $\left(B^{(\rho_t)}\right)_1, \dots, \left(B^{(\rho_t)}\right)_{\mu_t}$ for the subspaces in (4.7), then $\bigcup_{t=0}^{n} \bigcup_{i=1}^{\lambda_t} \left(A^{(\rho_t)}\right)_i$ and $\bigcup_{t=0}^{n} \bigcup_{i=1}^{\mu_t} \left(B^{(\rho_t)}\right)_i$ are symmetry-adapted bases with respect to which the lifting matrix is block-decomposed as shown in (4.1).

Definition 4.2. A vector $c \in \mathbb{R}^{|V|+d|F|}$ is *symmetric with respect to the irreducible linear representation* ρ_t of Γ if $c \in V^{(\rho_t)}$. Similarly, we say that a vector $b \in \mathbb{R}^{|I|}$ is *symmetric with respect to* ρ_t if $b \in W^{(\rho_t)}$.

Note that the kernel of the block matrix $\widetilde{M}_t(S,r)$ is isomorphic to the space of all liftings of the Γ-symmetric picture $S(r)$ which are symmetric with respect to ρ_t.

5 Symmetry-extended counting rules for the foldability of pictures

Recall from the Picture Theorem (Theorem 2.1) that if $S(r)$ is minimally flat, then it satisfies $|I| = |V| + d|F| - d$. Clearly, if $|I| < |V| + d|F| - d$, then $S(r)$ has a non-trivial lifting, and if $|I| > |V| + d|F| - d$, then the lifting matrix $M(S,r)$ has a non-trivial row dependence.

In Section 5.1, we will use the results of the previous section to derive a symmetry-extended version of this formula, which will provide further necessary counting conditions for a symmetric picture in an arbitrary dimension to be minimally flat. As we will see, these conditions can be stated in a very simple way in terms of the numbers of structural components of the picture that are fixed by the various symmetry operations. In Section 5.2 we will then derive a complete list of the necessary counting conditions for symmetric pictures in the plane to be minimally flat. Finally, in Section 5.3 we consider the transfer of lifting results for pictures from $(d-1)$-space to d-space via coning.

5.1 Symmetry-extended counting rules

Recall that if $\rho : \Gamma \to GL(X)$ is a linear representation of a group Γ with representation space X then a subspace U of X is said to be *ρ-invariant* if $\rho(\gamma)(U) \subseteq U$ for all $\gamma \in \Gamma$.

Proposition 5.1. *Let $S(r)$ be a picture which is Γ-symmetric with respect to $\theta : \Gamma \to \mathrm{Aut}(S)$ and $\tau : \Gamma \to \mathbb{R}^{d-1}$. Then the space $\mathcal{T}(S,r)$ of trivial liftings of $S(r)$ is a $P_V \oplus (\hat{\tau} \otimes P_F)$-invariant subspace of $\mathbb{R}^{|V|} \oplus \mathbb{R}^{d|F|}$.*

Proof: Let t be any element of $\mathcal{T}(S,r)$. Then t is an element of the kernel of the lifting matrix $M(S,r)$ of the form $(\ldots, z_i, \ldots, \ldots, A^j, B^j, \ldots, D^j, \ldots)^T$, where $A^j = A^k$, $B^j = B^k$, …, $D^j = D^k$ for all $1 \leq j,k \leq |F|$. It follows from Theorem 4.1 that $(P_V \oplus (\hat{\tau} \otimes P_F))(t)$ is also an element of the kernel of $M(S,r)$, and it follows immediately from the definition of $P_V \oplus (\hat{\tau} \otimes P_F)$ that $(P_V \oplus (\hat{\tau} \otimes P_F))(t)$ is of the form $(\ldots, z'_i, \ldots, \ldots, A'^j, B'^j, \ldots, D'^j, \ldots)^T$, where $A'^j = A'^k$, $B'^j = B'^k$, …, $D'^j = D'^k$ for all $1 \leq j,k \leq |F|$. This gives the result. □

We denote by $(P_V \oplus (\hat{\tau} \otimes P_F))^{(\mathcal{T})}$ the subrepresentation of $P_V \oplus (\hat{\tau} \otimes P_F)$ with representation space $\mathcal{T}(S,r)$. Recall that the *character* of a representation $\rho : \Gamma \to GL(X)$ is the row vector $\chi(\rho)$ whose i-th component is the trace of $\rho(\gamma_i)$, for some fixed ordering $\gamma_1, \ldots, \gamma_{|\Gamma|}$ of the elements of Γ.

Theorem 5.2 (Symmetry-extended counting rule)**.** *Let $S(r)$ be a $(d-1)$-picture which is Γ-symmetric with respect to θ and τ. If $S(r)$ is minimally flat, then we have*

$$\chi(P_I) = \chi(P_V \oplus (\hat{\tau} \otimes P_F)) - \chi((P_V \oplus (\hat{\tau} \otimes P_F))^{(\mathcal{T})}). \qquad (5.1)$$

Proof: By Maschke's Theorem (see [8] e.g.), $\mathcal{T}(S,r)$ has a $(P_V \oplus (\hat{\tau} \otimes P_F))$-invariant complement Q in $\mathbb{R}^{|V|+d|F|}$. We may therefore form the subrepresentation $(P_V \oplus (\hat{\tau} \otimes$

$P_F))^{(Q)}$ of $P_V \oplus (\hat{\tau} \otimes P_F)$ with representation space Q. Since (S, r) is minimally flat, the restriction of the linear map represented by the lifting matrix $M(S, r)$ to Q is an isomorphism onto $\mathbb{R}^{|I|}$. Moreover, since $M(S, r)$ is Γ-linear with respect to the representations $P_V \oplus (\hat{\tau} \otimes P_F)$ and P_I, this restriction is Γ-linear for the representations $(P_V \oplus (\hat{\tau} \otimes P_F))^{(Q)}$ and P_I. Hence $(P_V \oplus (\hat{\tau} \otimes P_F))^{(Q)}$ and P_I are isomorphic representations of Γ. It follows that

$$\chi(P_I) = \chi((P_V \oplus (\hat{\tau} \otimes P_F))^{(Q)}) = \chi(P_V \oplus (\hat{\tau} \otimes P_F)) - \chi((P_V \oplus (\hat{\tau} \otimes P_F))^{(\mathcal{T})}).$$

□

Suppose for a Γ-symmetric picture $S(r)$ we have $\chi(P_I) \neq \chi(P_V \oplus (\hat{\tau} \otimes P_F)) - \chi((P_V \oplus (\hat{\tau} \otimes P_F))^{(\mathcal{T})})$. Then, by Theorem 5.2, we my conclude that $S(r)$ is not minimally flat, that is, $S(r)$ either has a non-trivial lifting, or a row-dependence in the lifting matrix $M(S, r)$, or both. Further information about the non-trivial liftings of $S(r)$ and the row dependencies of $M(S, r)$ may be obtained as follows.

It is a well-known fact from group representation theory that if Γ is a finite group with the irreducible linear representations $\rho_0, \dots, \rho_n$, and $H : \Gamma \to GL(X)$ is any linear representation of Γ with $H = \alpha_0 \rho_0 \oplus \dots \oplus \alpha_n \rho_n$, where $\alpha_0, \dots, \alpha_n \in \mathbb{N} \cup \{0\}$, then for the character of H we have $\chi(H) = \alpha_0 \chi(\rho_0) \oplus \dots \oplus \alpha_n \chi(\rho_n)$. Thus, by (4.5) we have $\chi(P_I) = \mu_0 \chi(\rho_0) \oplus \dots \oplus \mu_n \chi(\rho_n)$. Similarly, the character of $(P_V \oplus (\hat{\tau} \otimes P_F))^{(Q)}$ can be written as $\chi((P_V \oplus (\hat{\tau} \otimes P_F))^{(Q)}) = \kappa_0 \chi(\rho_0) \oplus \dots \oplus \kappa_n \chi(\rho_n)$ for some $\kappa_0, \dots, \kappa_n \in \mathbb{N} \cup \{0\}$. Since $\chi(P_I) \neq \chi((P_V \oplus (\hat{\tau} \otimes P_F))^{(Q)})$, we have $\kappa_t \neq \mu_t$ for some $t \in \{0, \dots, n\}$.

For $t = 0, \dots, n$, the dimension of $W^{(\rho_t)}$ (recall (4.7)) is equal to the dimension of the representation ρ_t multiplied by μ_t. Similarly, the dimension of $V^{(\rho_t)}$ (recall (4.4)) minus the dimension of the space of trivial liftings of $S(r)$ which are symmetric with respect to ρ_t (recall Def. 4.2) is equal to the dimension of ρ_t multiplied by κ_t. Thus, if $\kappa_t > \mu_t$, then, by (4.1), there exists a non-trivial lifting of $S(r)$ (i.e., a non-trivial element in the kernel of $M(S, r)$) which is symmetric with respect to ρ_t, and if $\kappa_t < \mu_t$, then there exists a non-trivial row dependence of $M(S, r)$ (i.e., a non-zero element in the co-kernel of $M(S, r)$) which is symmetric with respect to ρ_t.

Before we illustrate this symmetry-adapted counting rule by means of an example, we show how the characters in (5.1) can be computed in a very simple way. We need the following definitions.

Let S be an incidence structure and let $\theta : \Gamma \to \mathrm{Aut}(S)$ be a group action on S. A vertex v of S is said to be *fixed* by $\gamma \in \Gamma$ (with respect to θ) if $\gamma v = v$. Similarly, a face $f = \{v_1, \dots, v_m\}$ of S is said to be *fixed* by $\gamma \in \Gamma$ (with respect to θ) if $\gamma f = f$, i.e., if $\gamma(\{v_1, \dots, v_m\}) = \{v_1, \dots, v_m\}$. Finally, an incidence (i, j) of S is said to be *fixed* by $\gamma \in \Gamma$ (with respect to θ) if $\gamma((i, j)) = (i, j)$ The sets of vertices, faces, and incidences of a Γ-symmetric incidence structure S which are fixed by $\gamma \in \Gamma$ are denoted by V_γ, F_γ, and I_γ, respectively.

Remark 5.3. Note that if a $(d-1)$-picture $S(r)$ is Γ-symmetric (with respect to θ and τ) and a vertex i is fixed by some $\gamma \in \Gamma$, then r_i must occupy a special geometric position in $\mathbb{R}^{d-1}$. For example, if $\tau(\gamma)$ is a reflection in the plane, then r_i must lie in the corresponding mirror line. Similarly, if $\tau(\gamma)$ is a rotation in the plane, then r_i must lie at the center of rotation (i.e., the origin). Similar geometric restrictions are imposed on any faces (or incidences) of $S(r)$ that are fixed by an element $\gamma \in \Gamma$.

Proposition 5.4. *Let* $\Gamma = \{\gamma_1, \dots, \gamma_{|\Gamma|}\}$ *be an abstract group and let* $S(r)$ *be a* $(d-1)$*-picture which is* Γ*-symmetric with respect to* θ *and* τ*. Then we have*

(i) $\chi(P_I) = (|I_{\gamma_1}|, \dots, |I_{\gamma_{|\Gamma|}}|)$*;*

(ii) $\chi(P_V \oplus (\hat{\tau} \otimes P_F)) = (|V_{\gamma_1}| + tr(\hat{\tau}(\gamma_1))\,|F_{\gamma_1}|, \dots, |V_{\gamma_{|\Gamma|}}| + tr(\hat{\tau}(\gamma_{|\Gamma|}))\,|F_{\gamma_{|\Gamma|}}|)$*;*

(iii) $\chi((P_V \oplus (\hat{\tau} \otimes P_F))^{(\mathcal{T})}) = \chi(\hat{\tau})$.

Proof: (i) Note that $tr(P_I(\gamma)) = |I_\gamma|$ for each $\gamma \in \Gamma$.

(ii) Similarly, we have $tr(P_V(\gamma)) = |V_\gamma|$, $tr(P_F(\gamma)) = |F_\gamma|$, and $tr((\hat{\tau} \otimes P_F)(\gamma)) = tr(\hat{\tau}(\gamma))tr(P_F(\gamma))$ for each $\gamma \in \Gamma$.

(iii) A basis for the space $\mathcal{T}(S, r)$ of trivial liftings of $S(r)$ is given by the d vectors $\tau_1 = (x_1, \dots, x_{|V|}, -e_1, \dots, -e_1)^T$, $\tau_2 = (y_1, \dots, y_{|V|}, -e_2, \dots, -e_2)^T$, ..., $\tau_{d-1} = (w_1, \dots, w_{|V|}, -e_{d-1}, \dots, -e_{d-1})^T$ and $\tau_d = (1, \dots, 1, -e_d, \dots, -e_d)^T$, where e_i denotes the i-th canonical basis vector of $\mathbb{R}^d$. An elementary calculation shows that for every $\gamma \in \Gamma$, we have

$$(P_V \oplus (\hat{\tau} \otimes P_F))^{(\mathcal{T})}(\gamma)\tau_j = (\hat{\tau}(\gamma))_{1j}\tau_1 + \cdots + (\hat{\tau}(\gamma))_{dj}\tau_d.$$

This gives the result. □

By Proposition 5.4, the symmetry-extended counting rule in (5.1) can be simplified to

$$\chi(P_I) = \chi(P_V \oplus (\hat{\tau} \otimes P_F)) - \chi(\hat{\tau}), \tag{5.2}$$

and each of these characters can be computed very easily. (The calculations of the characters $\chi(P_I)$, $\chi(P_V \oplus (\hat{\tau} \otimes P_F))$, and $\chi(\hat{\tau})$ for pictures in dimension 2 are shown in Table 2.) Moreover, note that this vector equation gives one equation for each element of the group Γ. This leads to the following very useful corollary of Theorem 5.2, which allows us to detect non-trivial liftings in symmetric pictures by simply counting the number of structural components of the picture that are 'fixed' by a given symmetry operation of the picture.

Corollary 5.5. *Let* (S, r) *be a* $(d-1)$*-picture which is* Γ*-symmetric with respect to* θ *and* τ*. If* (S, r) *is minimally flat, then for each* $\gamma \in \Gamma$*,*

$$|I_\gamma| = |V_\gamma| + tr(\hat{\tau}(\gamma))\,(|F_\gamma| - 1). \tag{5.3}$$

Proof: This follows immediately from Theorem 5.2 and Proposition 5.4. □

The following example illustrates how to apply the symmetry-extended counting rule to a picture in dimension 2.

Example 5.6. Consider the 2-picture with the reflectional symmetry group $\mathcal{C}_s = \{id, s\}$ in Figure 2. For this picture, we have $|V| = 6$, $|F| = 4$, $|I| = 15$, and hence $|I| = |V| + 3|F| - 3 = 15$. So, for generic positions of the vertices, we obtain flat pictures. However, using our symmetry-extended counting rule we can easily show that the mirror symmetry shown in Figure 2 induces a non-trivial lifting. The group $\mathcal{C}_s$ has two non-equivalent irreducible representations ρ_0 and ρ_1, each of which is of dimension 1 (see Table 1).

Since $tr(\hat{\tau}(id)) = 3$, $tr(\hat{\tau}(s)) = 1$, $|V| + 3|F| = 18$, and $|V_s| + |F_s| = 2 + 2 = 4$, we have

$$\chi(P_V \oplus (\hat{\tau} \otimes P_F)) - \chi((P_V \oplus (\hat{\tau} \otimes P_F))^{(\mathcal{T})}) = (18, 4) - (3, 1) = (15, 3) = 9\rho_0 + 6\rho_1.$$

$\mathcal{C}_s$	id	s
ρ_0	1	1
ρ_1	1	-1

Table 1: The characters of the irreducible representations of the group $\mathcal{C}_s$.

Further, since $|I| = 15$ and $|I_s| = 1$, we have

$$\chi(P_I) = (15, 1) = 8\rho_0 + 7\rho_1.$$

Thus, we may conclude that the picture in Figure 2 has a non-trivial lifting which is symmetric with respect to ρ_0 (i.e., the lifting preserves the mirror symmetry) and a row dependence which is symmetric with respect to ρ_1.

Note that for this particular example, we could also have used the results in [2] to see that the corresponding bar-joint framework has a self-stress, and then deduce the existence of a (sharp) lifting via the technique of the reciprocal diagram [13, 12].

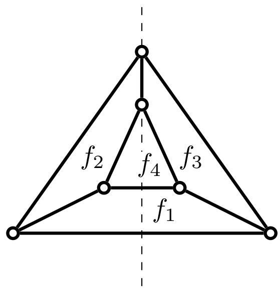


Figure 2: A 2-picture with mirror symmetry which has a symmetry-induced non-trivial lifting (see also Figure 1(c)).

5.2 When is a symmetric 2-picture minimally flat?

The possible non-trivial symmetry operations of a picture in dimension 2 are reflections in lines through the origin (denoted by s), and rotations about the origin by an angle of $\frac{2\pi}{n}$, where $n \geq 2$ (denoted by C_n). Therefore, the possible symmetry groups in the plane are the identity group $\mathcal{C}_1$, the rotational groups $\mathcal{C}_n$, $n \geq 2$ (generated by a single rotation C_n), the reflection group $\mathcal{C}_s$ (generated by a single reflection s), and the dihedral groups $\mathcal{C}_{nv}$, $n \geq 2$.

In Table 2, we show the calculations of characters for the counting condition in (5.1) (or equivalently, (5.2)) for 2-pictures. In this table, $|V_n|$, $|F_n|$, and $|I_n|$ denote the numbers of vertices, faces, and incidences that are fixed by an n-fold rotation C_n, $n \geq 2$, respectively. Similarly, $|V_s|$, $|F_s|$, and $|I_s|$ denote the numbers of vertices, faces, and incidences that are fixed by a reflection s, respectively (recall also the notation introduced in Subsection 5.1).

From equation (5.3) and Table 2, we obtain the following necessary conditions for a Γ-symmetric 2-picture (with respect to θ and τ) to be minimally flat. If $\chi(P_I) = \chi(P_V \oplus (\hat{\tau} \otimes P_F)) - \chi(\hat{\tau})$, then

	id	$C_{n>2}$	C_2	s
$\chi(P_I)$	$\lvert I\rvert$	$\lvert I_n\rvert$	$\lvert I_2\rvert$	$\lvert I_s\rvert$
$\chi(P_V \oplus (\hat{\tau} \otimes P_F))$	$\lvert V\rvert + 3\lvert F\rvert$	$\lvert V_n\rvert + (1+2\cos\frac{2\pi}{n})\lvert F_n\rvert$	$\lvert V_2\rvert - \lvert F_2\rvert$	$\lvert V_s\rvert + \lvert F_s\rvert$
$\chi(\hat{\tau})$	3	$1+2\cos\frac{2\pi}{n}$	-1	1

Table 2: Calculations of characters for the symmetry-extended counting rule for minimally flat pictures in dimension 2.

id: $$|I| = |V| + 3|F| - 3 \tag{5.4}$$

C_2: $$|I_2| = |V_2| - |F_2| + 1 \tag{5.5}$$

s: $$|I_s| = |V_s| + |F_s| - 1 \tag{5.6}$$

$C_{n>2}$: $$|I_n| = |V_n| + (|F_n| - 1)(1 + 2\cos\frac{2\pi}{n}) \tag{5.7}$$

where a given equation applies when the corresponding symmetry operation is present in $\tau(\Gamma)$. Some observations on minimally flat 2-pictures, arising from this set of equations are:

1. Every symmetry group contains the identity id, and hence we must always have the standard count $|I| = |V| + 3|F| - 3$.
2. It follows from (5.5) that the presence of a half-turn C_2 imposes limitations on the placements of vertices and faces. Note that if $|V_2| = 0$ or $|F_2| = 0$, then $|I_2| = 0$. Thus, we must have $|F_2| > 0$. Also, if $|V_2| = 0$, then we must have $|F_2| = 1$. If $|V_2| = 1$ then by $|I_2| \leq |F_2|$ either $|I_2| = |F_2| = 1$ or $|I_2| = 0, |F_2| = 2$ holds.
3. Similarly, by (5.6), presence of a mirror line implies that if $|V_s| = 0$, then $|F_s| = 1$, and if $|F_s| = 0$, then $|V_s| = 1$.
4. By (5.7), presence of a rotation of higher order $C_{n>2}$ gives rise to the following conditions.

   If $n = 3$, then the equation in (5.7) becomes $|I_3| = |V_3|$.

   If $n = 4$, then the equation in (5.7) becomes $|I_4| = |V_4| + |F_4| - 1$. Therefore, if $|V_4| = 0$, then $|F_4| = 1$ and, if $|V_4| = 1$ then $|I_4| = |F_4| \leq 1$ by $|I_4| \leq |I_2|$.

   If $n = 6$, then the equation in (5.7) becomes $|I_6| = |V_6| + 2|F_6| - 2$. Therefore, $|F_6| > 0$ (for otherwise, $|V_6| = 2$ and the location map of the picture would be non-injective). Further, if $|V_6| = 0$, then $|F_6| = 1$ and, if $|V_6| = 1$ then $|I_6| = |F_6| = 1$ holds by $|I_6| \leq |I_2|$.

   Finally, if C_n is present, where $n \notin \{2, 3, 4, 6\}$, we must have $|F_n| = 1$ and $|I_n| = |V_n|$.

Similarly, we may obtain lists of necessary conditions for symmetric pictures in 3- or higher-dimensional space to be minimally flat (using Corollary 5.5).

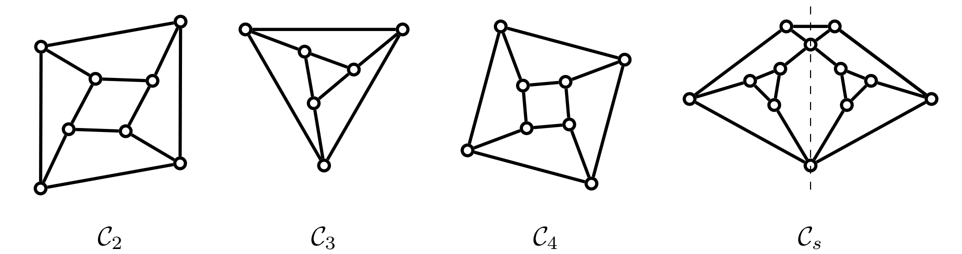


Figure 3: Some symmetric minimally flat 2-pictures (where all interior regions are faces).

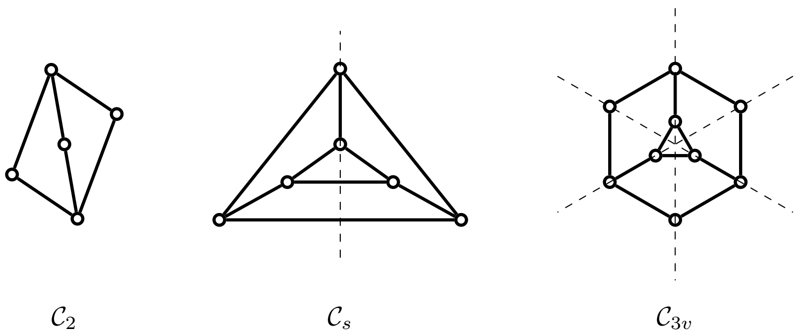


Figure 4: Some symmetric 2-pictures with a (sharp) symmetry-induced lifting (where all interior regions are faces).

5.3 Coning $(d-1)$-pictures

In the following we show that the technique of 'coning' can be used to construct (minimally) flat Γ-symmetric d-pictures from (minimally) flat Γ-symmetric $(d-1)$-pictures. Let $S = (V, F; I)$ be an incidence structure and let $S(r)$ be a $(d-1)$-picture for $d \geq 2$. The *coned incidence structure* $\tilde{S} = (\tilde{V}, \tilde{F}; \tilde{I})$ is obtained by adding a new vertex v to V, replacing each face $f \in F$ by $f \cup \{v\}$, and adding the incidences $(v, \tilde{f})$, $\tilde{f} \in \tilde{F}$. A realization of $\tilde{S}$ as a d-picture $\tilde{S}(\tilde{r})$ is called a *coned picture* of $S(r)$. An example of a coned 2-picture is shown in Figure 5.

Let Γ be a group, and let $S(r)$ be a Γ-symmetric $(d-1)$-picture (with respect to θ and τ) with n vertices. Then $S(r)$ is said to be Γ-*generic* if the set of coordinates of the image of r are algebraically independent over $\mathbb{Q}_\Gamma$, where $\mathbb{Q}_\Gamma$ denotes the field generated by $\mathbb{Q}$ and the entries of the matrices in $\tau(\Gamma)$. In other words, $S(r)$ is Γ-generic if there does not exist a polynomial $h(x_1, \ldots, x_{(d-1)n})$ with coefficients in $\mathbb{Q}_\Gamma$ such that $h((r_1)_1, \ldots, (r_n)_{d-1}) = 0$. (Note that if Γ is the trivial group, then $\mathbb{Q}_\Gamma = \mathbb{Q}$. In this case, a Γ-generic picture is simply called *generic*.) Clearly, the set of all Γ-generic realizations of S is a dense (but not open) subset of all Γ-symmetric realizations of S. Moreover, all Γ-generic realizations of S share the same lifting properties. We say that S is Γ-*generically (minimally) flat* in $\mathbb{R}^{d-1}$ if all Γ-generic realizations of S in $\mathbb{R}^{d-1}$ are (minimally) flat.

For a Γ-symmetric $(d-1)$-picture $S(r)$ (with respect to $\theta : \Gamma \to \mathrm{Aut}(S)$ and $\tau : \Gamma \to O(\mathbb{R}^{d-1})$), we let $\tilde{\tau} : \Gamma \to O(\mathbb{R}^d)$ be the augmented representation, i.e., $\tilde{\tau}(\gamma) =$

$\begin{pmatrix} \tau(\gamma) & 0 \\ 0 & 1 \end{pmatrix}$. Moreover, for the coned incidence structure $\tilde{S} = (\tilde{V}, \tilde{F}; \tilde{I})$ (with cone vertex v), we define $\tilde{\theta} : \Gamma \to \mathrm{Aut}(\tilde{S})$ as follows: $\tilde{\theta}(\gamma)|_V = \theta(\gamma)$, $\tilde{\theta}(\gamma)(v) = v$, and $\tilde{\theta}(\gamma)(f \cup \{v\}) = (\theta(\gamma)(f)) \cup \{v\}$ for all $f \in F$ and $\gamma \in \Gamma$.

Theorem 5.7. *Let $S = (V, F; I)$ be a Γ-symmetric incidence structure (with respect to θ), and let $S(r)$ be a $(d-1)$-picture which is Γ-symmetric with respect to θ and τ. Then the following hold:*

(i) $S(r)$ has a non-trivial lifting if and only if the coned d-picture $\tilde{S}(\tilde{r})$, with the cone vertex at the point $(0, \ldots, 0, \alpha) \in \mathbb{R}^d$, for some non-zero $\alpha \in \mathbb{R}$, has a non-trivial lifting.

(ii) S is Γ-generically (minimally) flat (with respect to θ and τ) in $\mathbb{R}^{d-1}$ if and only if $\tilde{S}$ is Γ-generically (minimally) flat (with respect to $\tilde{\theta}$ and $\tilde{\tau}$) in $\mathbb{R}^d$.

Proof: (i) Let $r_1, \ldots, r_{|V|}$ be the vertices of the picture $S(r)$. Embed $S(r)$ into the space $\mathbb{R}^d$ via $\tilde{r}_i = (r_i, 0)$ for $i = 1, \ldots, |V|$. Then form the coned picture $\tilde{S}(\tilde{r})$, with the cone vertex $\tilde{r}_0 = (0, \ldots, 0, \alpha) \in \mathbb{R}^d$, $\alpha \neq 0$.

The lifting matrix of $S(r)$ is of the form

$$M(S, r) = \begin{array}{c} \\ \\ (i, f_j) \\ \\ \end{array} \begin{array}{c} \qquad\qquad i \qquad\qquad\qquad\qquad\qquad f_j \\ \left[\begin{array}{ccc|ccc} & \vdots & & & \vdots & \\ 0 \ldots 0 & 1 & 0 \ldots 0 & 0 \ldots 0 & [x_i, y_i, \ldots, w_i, 1] & 0 \ldots 0 \\ & \vdots & & & \vdots & \end{array}\right] \end{array}.$$

The lifting matrix of the coned picture $\tilde{S}(\tilde{r})$ (with the cone vertex fixed) is of the form

$$M'(\tilde{S}, \tilde{r}) = \begin{array}{c} \qquad\qquad\qquad i \qquad\qquad\qquad\qquad\qquad f_j \\ \begin{array}{c} \\ (i, f_j) \\ \\ \\ (0, f_j) \\ \\ \end{array} \left[\begin{array}{ccc|ccc} & \vdots & & & \vdots & \\ 0 \ldots 0 & 1 & 0 \ldots 0 & 0 \ldots 0 & [x_i, y_i, \ldots, w_i, 0, 1] & 0 \ldots 0 \\ & \vdots & & & \vdots & \\ \hline & \vdots & & & \vdots & \\ 0 \ldots 0 & 0 & 0 \ldots 0 & 0 \ldots 0 & [0, 0, \ldots, 0, \alpha, 1] & 0 \ldots 0 \\ & \vdots & & & \vdots & \end{array}\right] \end{array}.$$

Note that we added $|F|$ rows (one for each incidence of the form $(0, f_j)$, $j = 1, \ldots, |F|$, where 0 is the cone vertex) and $|F|$ columns. Furthermore, for each added column (under face j) we have a 0 in each row, except in the one added row corresponding to the incidence $(0, f_j)$, where the entry is equal to α. Thus we have increased the rank by $|F|$, and preserved the dimension of the kernel. Now, if we add the missing column for the cone vertex, then we obtain the lifting matrix of the coned picture $\tilde{S}(\tilde{r})$:

$$M(\tilde{S},\tilde{r}) = \begin{array}{c} \\ (i,f_j) \\ \\ \\ (0,f_j) \\ \\ \end{array} \overset{\begin{array}{ccc} 0 & \quad i \quad & \qquad f_j \end{array}}{\left[\begin{array}{c|ccc|ccc} 0 & & \vdots & & & \vdots & \\ \vdots & 0\dots0 & 1 & 0\dots0 & 0\dots0 & [x_i,y_i,\dots,w_i,0,1] & 0\dots0 \\ 0 & & \vdots & & & \vdots & \\ \hline 1 & & \vdots & & & \vdots & \\ \vdots & 0\dots0 & 0 & 0\dots0 & 0\dots0 & [0,0,\dots,0,\alpha,1] & 0\dots0 \\ 1 & & \vdots & & & \vdots & \end{array}\right]}$$

This added a 1-dimensional space of liftings (namely the space $\{\lambda\tau_d|\, \lambda\in\mathbb{R}\}$ of all 'vertical' translations of the picture (recall the proof of Prop. 5.4)), but did not add any non-trivial liftings. The rank of the matrix has not changed, nor has the space of row-dependencies. This proves (i).

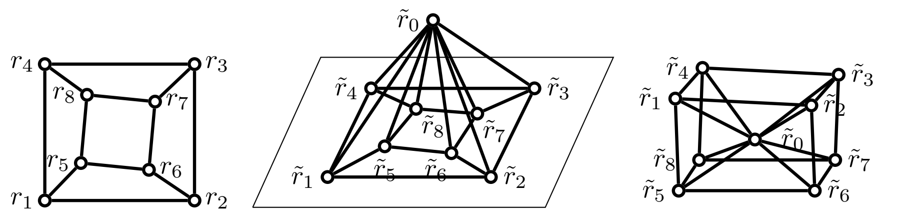
(ii) Note that if there exists some (minimally) flat Γ-symmetric realization of an incidence structure S, then clearly all Γ-generic realizations of S are also (minimally) flat. Therefore, by (i), it suffices to show that $S(r)$ is Γ-symmetric with respect to θ and τ if and only if the coned picture $\tilde{S}(\tilde{r})$ (with the cone vertex at the point $(0,\dots,0,\alpha)\in\mathbb{R}^d$, $\alpha\neq 0$) is Γ-symmetric with respect to $\tilde{\theta}$ and $\tilde{\tau}$.

Let $r_1,\dots,r_{|V|}$ be the vertices of the picture $S(r)$, and let $\tilde{r}_0,\tilde{r}_1,\dots,\tilde{r}_{|V|}$ be the vertices of the picture $\tilde{S}(\tilde{r})$, i.e., $\tilde{r}_0=(0,0,\dots,0,\alpha)$, and $\tilde{r}_i=(r_i,0)$ for $i=1,\dots,|V|$.

For all $\gamma\in\Gamma$, we clearly have $\tilde{\tau}(\gamma)\tilde{r}_0=\tilde{r}_0=\tilde{r}_{\tilde{\theta}(\gamma)(0)}$. Furthermore, for $i\neq 0$, $S(r)$ is Γ-symmetric with respect to θ and τ if and only if

$$\tilde{\tau}(\gamma)\tilde{r}_i=(\tau(\gamma)r_i,0)=(r_{\theta(\gamma)(i)},0)=(r_{\tilde{\theta}(\gamma)(i)},0)=\tilde{r}_{\tilde{\theta}(\gamma)(i)}.$$

This gives the result. □



(a) (b) (c)

Figure 5: A $\mathcal{C}_4$-symmetric 2-picture $S(r)$ (where all five interior regions are faces) (a) and a $\mathcal{C}_4$-symmetric coned picture $\tilde{S}(\tilde{r})$ in 3-space with cone vertex $\tilde{r}_0=(0,\dots,0,\alpha)$ and $\tilde{r}_i=(r_i,0)$ for $i=1,\dots,8$ (b). A $\mathcal{C}_4$-generic realization of $\tilde{S}$ is shown in (c). Note that $\tilde{S}$ also has five faces, namely the ones corresponding to the 'interior cells' of the cube in (c) except for the 'top cell' $\{\tilde{r}_0,\tilde{r}_1,\tilde{r}_2,\tilde{r}_3,\tilde{r}_4\}$.

6 Further work

In the previous sections we have derived new necessary conditions for a symmetric picture to be minimally flat. It is now natural to ask whether these conditions, together with the standard non-symmetric counts, are also sufficient for a picture which is realized generically for the given symmetry group to be minimally flat. We conjecture that this is in fact the case, for all symmetry groups in all dimensions.

Conjecture 6.1. *A Γ-generic $(d-1)$-picture $S(r)$ is minimally flat if and only if*

(i) $|I| = |V| + d|F| - d$ *and* $|I'| \leq |V'| + d|F'| - d$ *for all nonempty subsets* I' *of incidences;*

(ii) S *satisfies the conditions for* Γ *in the symmetry extended counting rule (Corollary 5.5);*

(iii) *For every subset* I' *of* I *which induces a* Γ'*-symmetric incidence structure* S' *with* $|I'| = |V'| + d|F'| - d$ *(where* $\Gamma' \subseteq \Gamma$*),* S' *satisfies the conditions for* Γ' *in the symmetry extended counting rule.*

Note that if every face of the incidence structure S is incident to exactly four vertices (i.e., if S induces a 4-uniform hypergraph), then the count $|I| = |V| + 3|F| - 3$ in condition (i) for $d = 3$ simplifies to $|F| = |V| - 3$. Thus, a natural approach to prove this conjecture for $d = 3$ is to first focus on incidence structures which induce 4-uniform hypergraphs and to develop a symmetry-adapted version of the recently established recursive construction of 4-uniform $(1, 3)$-tight hypergraphs [9]. For each of the symmetric hypergraph operations in this construction scheme, we then need to check that it preserves the full rank of the lifting matrix. Finally, one could then try to extend the result to general incidence structures with the same symmetry. Using this approach, Conjecture 6.1 has recently been verified for 2-pictures with three-fold rotational symmetry [11]. All the other cases, however, remain open, and we invite the reader to join in these explorations.

Note that a similar approach based on symmetry-adapted recursive Henneberg-type graph constructions was successfully used in [16, 17] to establish symmetry-adapted versions of Laman's theorem for various groups. These results provide combinatorial characterizations of symmetry-generic isostatic (i.e. minimally infinitesimally rigid) graphs in the plane. However, the recursive construction of (non-symmetric) $(1, 3)$-tight hypergraphs developed in [9] is more complex than the recursive Henneberg construction of (non-symmetric) Laman graphs, and hence Conjecture 6.1 presents us with new challenges of both combinatorial and geometric nature.

For practical applications of scene analysis, it is of particular interest to develop methods which allow us to determine whether there exist a (unique) *sharp* lifting for a given picture, rather than just a non-trivial lifting. It is therefore natural to ask whether our results can be extended to provide necessary and/or sufficient conditions for a symmetric picture to be sharp, rather than just foldable.

A combinatorial characterization of those pictures which have a unique sharp lifting if realized generically in $(d-1)$-space (without symmetry) is given by the Picture Theorem (recall Theorem 2.1). This result is essentially a corollary of the combinatorial characterization of generic independent pictures (i.e., pictures whose lifting matrices have independent rows) given in [31]. Therefore, in order to obtain a complete symmetry-adpated version of the Picture Theorem we need to first obtain a combinatorial characterization of those pictures which are independent if realized generically with respect to the given symmetry group. Note that in the non-symmetric situation, any generic independent picture is a substructure of a minimally flat picture. In general, however, this is no longer the case for symmetric pictures. For example, it is easy to construct a $\mathcal{C}_3$-generic picture in the plane which satisfies $|V_3| = 1$ and $|I_3| = |F_3| > 1$, so that it is not contained in a minimally flat $\mathcal{C}_3$-generic picture (by Corollary 5.5), but whose lifting matrix has independent rows. It follows that a combinatorial characterization of Γ-generic minimally flat pictures would in general not provide us with a combinatorial characterization of Γ-generic independent pictures. However, once a characterization of Γ-generic independent pictures has been established for a group Γ (again using a symmetric recursive construction scheme, e.g.), then we expect that the proof idea in [31] can be extended to obtain a characterization of those pictures which have a unique sharp lifting if realized generically with respect to Γ. For some initial results for $\mathcal{C}_3$-generic pictures in the plane, we refer the reader to [11].

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On the Erdős-Sós Conjecture for graphs on $n = k + 4$ vertices*

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Abstract

The Erdős-Sós Conjecture states that if G is a simple graph of order n with average degree more than $k - 2$, then G contains every tree of order k. In this paper, we prove that Erdős-Sós Conjecture is true for $n = k + 4$.

Keywords: Erdős-Sós Conjecture, tree, maximum degree.

Math. Subj. Class.: 05C05, 05C35

1 Introduction

The graphs considered in this paper are finite, undirected, and simple (no loops or multiple edges). Let $G = (V(G), E(G))$ be a graph of order n, where $V(G)$ is the vertex set and $E(G)$ is the edge set with size $e(G)$. The *degree* of $v \in V(G)$, the number of edges incident to v, is denoted $d_G(v)$ and the set of neighbors of v is denoted $N(v)$. If u and v in $V(G)$ are adjacent, we say that u *hits* v or v *hits* u. If u and v are not adjacent, we say that u *misses* v or v *misses* u. If $S \subseteq V(G)$, the induced subgraph of G by S is denoted by $G[S]$. Denote by $D(G)$ the diameter of G. In addition, $\delta(G)$, $\Delta(G)$ and $avedeg(G) = \frac{2e(H)}{|V(H)|}$ are denoted by the minimum, maximum and average degree in $V(G)$, respectively. Let T be a tree on k vertices. If there exists an injection $g : V(T) \to V(G)$ such that $g(u)g(v) \in E(G)$ if $uv \in E(T)$ for $u, v \in V(T)$, we call g an *embedding* of T into G and G contains a copy of T as a subgraph, denoted by $T \subseteq G$. In addition, assume that $T' \subseteq T$ is a subtree of T

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and g' is an embedding of T' into G. If there exists an embedding $g : V(T) \to V(G)$ such that $g(v) = g'(v)$ for all $v \in V(T')$, we say that g' is $T-$*extensible.*

In 1959, Erdős and Gallai [6] proved the following theorem.

Theorem 1.1. *Let G be a graph with $avedeg(G) > k - 2$. Then G contains a path of order k.*

Based on the above result, Later Erdős and Sós proposed the following well known conjecture (for example, see [5]).

Conjecture 1.2. *Let G be a graph with $avedeg(G) > k - 2$. Then G contains every tree on k vertices as a subgraph.*

Various specific cases of Conjecture 1.2 have already been proven. For example, Brandt and Dobson [2] proved the conjecture for graphs having girth at least 5. Balasubramanian and Dobson [1] proved this conjecture for graphs without any copy of $K_{2,s}$, $s < \frac{k}{12} + 1$. Li, Liu and Wang [15] proved the conjecture for graphs whose complement has girth at least 5. Dobson [3] improved this to graphs whose complements do not contain $K_{2,4}$. More results on this conjecture can be referred to [7, 8, 9] and [11, 12]. On the other hand, in 2003, Mclennan [10] proved the following theorem.

Theorem 1.3. *Let G be a graph with $avedeg(G) > k - 2$. Then G contains every tree of order k whose diameter does not excess 4 as a subgraph.*

In 2010, Eaton and Tiner [4] proved the the following two theorems.

Theorem 1.4. *[4] Let G be a graph with $avedeg(G) > k - 2$. If $\delta(G) \geq k - 4$, then G contains every tree of order k as a subgraph.*

Theorem 1.5. *[4] Let G be a graph with $avedeg(G) > k - 2$. If $k \leq 8$, then G contains every tree of order k as a subgraph.*

In 1984, Zhou [17] proved that Conjecture 1.2 holds for $k = n$. Later, Slater, Teo and Yap [13] and Woźniak [16] proved that Conjecture 1.2 holds for $k = n - 1$ and $k = n - 2$, respectively.

Theorem 1.6. *[16] Let G be a graph of order n with $avedeg(G) > k - 2$. If $k = n - 2$, then G contains every tree of order k as a subgraph.*

Recently, Tiner [14] proved that Conjecture 1.2 holds for $k = n - 3$.

Theorem 1.7. *[14] Let G be a graph of order n with $avedeg(G) > k - 2$. If $k \geq n - 3$, then G contains every tree of order k as a subgraph.*

In this paper, we establish the following:

Theorem 1.8. *Let G be a graph of order n with $avedeg(G) > k - 2$. If $k \geq n - 4$, then G contains every tree of order k as a subgraph.*

2 Proof of Theorem 1.8

Let T be any tree of order k. If $k \geq n - 3$, or $k \leq 8$ or the diameter of T is at most 4, the assertion holds by Theorems 1.3, 1.5 and 1.7. We only consider $k = n - 4 \geq 9$, $D(T) \geq 5$ and prove the assertion by the induction. Clearly the assertion holds for $n = 6$. Hence assume Theorem 1.8 holds for all of the graphs of order fewer than n and let G be a graph of order n. If there exists a vertex v with $d_G(v) < \lfloor \frac{k}{2} \rfloor$, then $avedeg(G - v) > k - 2$ and the assertion holds by Theorems 1.7. Furthermore, by Theorem 1.4, without loss of generality, there exists a vertex z in $V(G)$ such that $\lfloor \frac{k}{2} \rfloor \leq d_G(z) = \delta(G) \leq k-5$. Without loss of generality, we can assume that $e(G) = 1 + \lfloor \frac{1}{2}(k-2)(k+4) \rfloor$. Let T be any tree of order k with a longest path $P = a_0a_1 \ldots a_{r-1}a_r$ and $N_T(a_1) \setminus \{a_2\} = \{b_1, \ldots, b_s\}$ and $N_T(a_{r-1}) \setminus \{a_{r-2}\} = \{c_1, \ldots, c_t\}$. Since $avedeg(G) > k - 2$, we can consider the following cases: $\Delta(G) = k+3, k+2, k+1, k, k-1$.

2.1 $\Delta(G) = k + 3$

Let $u \in V(G)$ be such vertex that $d_G(u) = k + 3$ and let $G' = G - \{u, z\}$ and $T' = T - \{a_1, b_1, \ldots, b_s\}$. Then $e(G') \geq e(G) - (k+3) - (k-5) + 1 > \frac{1}{2}(k+4)(k-2) - (k+3) - (k-5) + 1 = \frac{1}{2}(k^2 - 2k - 2)$. So *avedeg*$(G') > (k^2 - 2k - 2)/(k+2) > k - 4$ and $| V(T') | \leq k - 2$. By the induction hypothesis, $T' \subseteq G'$. Let f' be an embedding of T' into G'. Then let $f = f'$ in T' and $f(a_1) = u$, $X = V(G) \setminus f'(V(T'))$. Since $d_G(u) = k + 3$, u hits at least s vertices in X. Hence f can be extended to an embedding of T into G or we can say that f is $T-$extensible.
Remark: For the remainder of this paper we shall always let f' be an embedding of T' into G' and when we do not define the value of f on any vertex of T', we always let $f = f'$ on those vertices.

2.2 $\Delta(G) = k + 2$

Let $u \in V(G)$ be such vertex that $d_G(u) = k + 2$. Then there exists only one vertex $x \in V(G) \setminus \{u\}$ not adjacent to u. We consider two subcases: $d_G(x) \leq k - 2$ and $d_G(x) \geq k - 1$.

2.2.1 $d_G(x) \leq k - 2$

Let $G' = G - \{u, x\}$ and $T' = T - \{a_1, b_1, \ldots, b_s\}$. Then $e(G') \geq e(G) - (k+2) - (k-2) > \frac{1}{2}(k^2 - 2k - 8)$. So *avedeg*$(G') > (k^2 - 2k - 8)/(k+2) = k - 4$ and $| V(T') | \leq k - 2$. By the induction hypothesis, $T' \subseteq G'$. Then let $f(a_1) = u$ and $X = V(G) \setminus f'(V(T'))$. Since $d_G(u) = k + 2$, u hits at least s vertices in X, f is $T-$extensible.

2.2.2 $d_G(x) \geq k - 1$

Since $x \neq z$, we consider the following two cases.

(A). x misses z. Let $G' = G - \{u, z, x\}$ and $T' = T - \{a_1, b_1, \ldots, b_s, a_r\}$. Then $e(G') \geq e(G) - (k+2) - (k-5) - (k+1) + 1 > \frac{1}{2}(k^2 - 4k - 2)$. Hence *avedeg*$(G') > (k^2 - 4k - 2)/(k+1) > k - 5$ and $| V(T') | \leq k - 3$. By the induction hypothesis, we have $T' \subseteq G'$. Since x misses z, u and $d_G(x) \geq k - 1$, x misses at most two vertices of G'. If x hits $f'(a_2)$, let $f(a_1) = x$ and $f(a_r) = u$. Since $d_G(x) \geq k - 1$ and u hits all vertices of T', f is $T-$extensible. Hence we assume that x misses $f'(a_2)$. If x hits $f'(a_{r-1})$, let

$f(a_r) = x$ and $f(a_1) = u$. Then f is $T-$extensible. If x misses $f'(a_2)$ and $f'(a_{r-1})$, then x hits all of $V(G') \setminus \{f'(a_2), f'(a_{r-1})\}$, because $D(T) \geq 5$, a_2 and a_{r-1} are not adjacent. Then let $f(a_{r-1}) = x$, $f(a_1) = u$, which implies that f is $T-$extensible.

(B). x hits z. We consider the following two subcases.

(B.1). $d_G(x) > k-1$. Let $G' = G - \{u, z, x\}$, $T' = T - \{a_1, b_1, \dots, b_s, a_r\}$. Since x misses u and $d_G(x) > k-1$, x misses at most two vertices of G', the assertion can be proven by the similar method of (A).

(B.2). $d_G(x) = k-1$. Then x misses 3 vertices of $V(G) \setminus \{u\}$, says y_1, y_2, y_3.

(a). There exists one vertex y_i with $1 \leq i \leq 3$ such that $d_G(y_i) \geq k+1$. Let $G' = G - \{u, z, y_i, x\}$ and $T' = T - \{a_1, b_1, \dots, b_s, a_{r-1}, c_1, \dots, c_t\}$. Then $e(G') \geq e(G) - (k+2) - (k-5) - (k+2) - (k-1) + 3 + 1 > \frac{1}{2}(k^2 - 6k + 4)$, which implies *avedeg*$(G') > (k^2 - 6k + 4)/k > k - 6$ and $\mid V(T') \mid \leq k - 4$. Hence by the induction hypothesis, $T' \subseteq G'$. Note that y_i misses at most one vertex of G'. If y_i misses $f'(a_2)$, let $f(a_1) = u, f(a_{r-1}) = y_i$; if y_i misses $f'(a_{r-2})$, let $f(a_{r-1}) = u$,$f(a_1) = y_i$. Thus f is $T-$extensible.

(b). There exists one vertex y_i with $1 \leq i \leq 3$ such that $d_G(y_i) = k$ and y_i misses z. Then the proof is similar to (a) and omitted.

(c). There exists one vertex y_i with $1 \leq i \leq 3$ such that $d_G(y_i) \leq k-2$. Let $G' = G - \{u, y_i, x\}$ and $T' = T - \{a_1, b_1, \dots, b_s, a_r\}$. Then $e(G') \geq e(G) - (k+2) - (k-2) - (k-1) + 1 > \frac{1}{2}(k^2 - 4k - 4)$, which implies *avedeg*$(G') > (k^2 - 4k - 4)/(k+1) > k - 5$ and $\mid V(T') \mid \leq k - 3$. Hence by the induction hypothesis, $T' \subseteq G'$. Similarly as in case (A), there exists an embedding from T into G.

(d). $d_G(y_i) = k$, y_i hits z or $d_G(y_i) = k-1$ for $i \in \{1, 2, 3\}$.

(d.1). $d_T(a_1) + d_T(a_{r-1}) \geq 5$. Let $G' = G - \{u, z, y_1, y_2, x\}$ and $T' = T - \{a_1, b_1, \dots, b_s, a_{r-1}, c_1, \dots, c_t\}$. Then $e(G') \geq e(G) - (k+2) - (k-5) - (k-1) - (k-1) - (k-1) + 3 > \frac{1}{2}(k^2 - 8k + 10)$ which implies *avedeg*$(G') > (k^2 - 8k + 10)/(k-1) > k - 7$ and $\mid V(T') \mid \leq k - 5$. Hence by the induction hypothesis, $T' \subseteq G'$. Moreover, x misses only one vertex of G'. If x misses $f'(a_2)$, let $f(a_1) = u, f(a_{r-1}) = x$; if x misses $f'(a_{r-2})$, let $f(a_{r-1}) = u, f(a_1) = x$. In both situations, f is $T-$extensible.

(d.2). $d_T(a_1) = d_T(a_{r-1}) = 2$. Let $G' = G - \{u, z\}$ and $T' = T - \{a_0, a_1\}$. Then $e(G') \geq e(G) - (k+2) - (k-5) + 1 > \frac{1}{2}(k^2 - 2k)$, which implies *avedeg*$(G') > (k^2 - 2k)/(k+2) > k - 4$ and $\mid V(T') \mid \leq k - 2$. By the induction hypothesis, $T' \subseteq G'$. Moreover, u hits all vertices of $V(G) \setminus \{x\}$ and z hits x. Let $f(a_1) = u$ or z and $f(a_0) = z$ or u. Then f is $T-$extensible.

2.3 $\Delta(G) = k + 1$

Let $u \in V(G)$ be such vertex that $d_G(u) = k+1$ with u missing vertices x_1 and x_2. Without loss of the generality, we can assume $d_G(x_1) \geq d_G(x_2)$ and $d_T(a_1) \geq d_T(a_{r-1})$.

2.3.1 $d_T(a_1) + d_T(a_{r-1}) \geq 5$

We consider the following two cases: (A) and (B).

(A). x_1 misses x_2.

(A.1) $d_G(x_1) + d_G(x_2) \leq 2k - 3$. Let $G' = G - \{u, x_1, x_2\}$ and $T' = T - \{a_1, b_1, \dots, b_s\}$. Then $e(G') \geq e(G) - (k+1) - (2k-3) > \frac{1}{2}(k^2 - 4k - 4)$, which implies *avedeg*$(G') > (k^2 - 4k - 4)/(k+1) > k - 5$ and $\mid V(T') \mid \leq k - 3$. Hence by the induction hypothesis, $T' \subseteq G'$. Let $f(a_1) = u$. It is easy to see that f is $T-$extensible.

(A.2). $d_G(x_1) + d_G(x_2) \geq 2k - 2$.

(a). $d_G(x_1) = k-1$ Then $d_G(x_2) = k-1$ and x_1 misses $\{u, x_2, y_1, y_2\}$. If $y_1, y_2 \neq z$, let $G' = G - \{u, z, x_1, x_2, y_1\}$ and $T' = T - \{a_1, b_1, \ldots, b_s\, a_{r-1}, c_1, \ldots, c_t\}$. Then $e(G') \geq e(G) - (k+1) - (k-5) - (2k-2) - (k+1) + 3 > \frac{1}{2}(k^2 - 8k + 8)$, which implies *avedeg*$(G') > (k^2 - 8k + 8)/(k-1) > k - 7$ and $\mid V(T') \mid \leq k - 5$. Hence by the induction hypothesis, $T' \subseteq G'$. Note that x_1 misses only one vertex of G'. If x_1 misses $f'(a_2)$, let $f(a_1) = u$ and $f(a_{r-1}) = x_1$; if x_1 misses $f'(a_{r-2})$, let $f(a_{r-1}) = u$ and $f(a_1) = x_1$. In both situations, f is $T-$extensible. Now assume that $y_1 = z$ or $y_2 = z$. Let $G' = G - \{u, x_1, x_2, y_1, y_2\}$ and $T' = T - \{a_1, b_1, \ldots, b_s, a_{r-1}, c_1, \ldots, c_t\}$. Then $e(G') \geq e(G) - (k+1) - (k-5) - (2k-2) - (k+1) + 2 + 1 > \frac{1}{2}(k^2 - 8k + 8)$, which implies *avedeg*$(G') > (k^2 - 8k + 8)/(k-1) > k - 7$ and $\mid V(T') \mid \leq k - 5$. Let $f(a_{r-1}) = u$ and $f(a_1) = x_1$. Then f is $T-$extensible.

(b). $d_G(x_1) \geq k$. Let $G' = G - \{u, z, x_1, x_2\}$ and $T' = T - \{a_1, b_1, \ldots, b_s, a_{r-1}, c_1, \ldots, c_t\}$. Then $e(G') \geq e(G) - (k+1) - (k-5) - (2k+2) + 1 + 2 > \frac{1}{2}(k^2 - 6k + 2)$, which implies *avedeg*$(G') > (k^2 - 6k + 2)/k > k - 6$ and $\mid V(T') \mid \leq k - 4$. Hence by the induction hypothesis, $T' \subseteq G'$. Note that x_1 misses at most one vertex of G'. If x_1 misses $f'(a_2)$, let $f(a_1) = u$ and $f(a_{r-1}) = x_1$; if x_1 misses $f'(a_{r-2})$, let $f(a_{r-1}) = u$ and $f(a_1) = x_1$. In both situations, f is $T-$extensible.

(B). x_1 hits x_2.

(B.1). $d_G(x_1) + d_G(x_2) \leq 2k - 2$. The proof is similar to (A.1) and omitted.

(B.2). $d_G(x_1) + d_G(x_2) \geq 2k - 1$. The proof is similar to (A.2) with (a)$d_G(x_1) = k$,$d_G(x_2) = k - 1$ or k, (b)$d_G(x_1) = k + 1$.

2.3.2 $d_T(a_1) = d_T(a_{r-1}) = 2$.

We consider the following four cases.

(A). There exists a vertex $v \neq u$ of degree at most k such that it hits both x_1 and x_2. Let $G' = G - \{u, v\}$ and $T' = T - \{a_0, a_1\}$. Then $e(G') \geq e(G) - (k+1) - k + 1 > \frac{1}{2}(k^2 - 2k - 8)$, which implies *avedeg*$(G') > (k^2 - 2k - 8)/(k+2) = k - 4$ and $\mid V(T') \mid \leq k - 2$. Hence by the induction hypothesis, $T' \subseteq G'$. If $f'(a_2)$ hits u, let $f(a_1) = u$. If $f'(a_2)$ misses u, then $f'(a_2) = x_1$ or x_2 and let $f(a_1) = v, f(a_0) = u$. Thus f is $T-$extensible.

(B). There exists a vertex $v \neq u$ of degree at least $k+1$ such that it hits both x_1 and x_2. Then $d_G(v) = k + 1$ and v misses y_1 and y_2. Since the case $z \in \{x_1, x_2, y_1, y_2\}$ is much easier, we may suppose $z \neq x_1, x_2, y_1, y_2$. Let $G' = G - \{u, v, z\} - x_1x_2 - y_1y_2$ and $T' = T - \{a_0, a_1, a_r\}$. Then $e(G') \geq e(G) - 2(k+1) - (k-5) + 1 - 2 > \frac{1}{2}(k^2 - 4k - 4)$, which implies *avedeg*$(G') > (k^2 - 4k - 4)/(k+1) > k - 5$ and $\mid V(T') \mid \leq k - 3$. Hence by the induction hypothesis, $T' \subseteq G'$. If $f'(a_2) = x_1$ or x_2, and $f'(a_{r-1}) = y_1$ or y_2, then let $f(a_1) = v$ and $f(a_r) = u$. If $f'(a_2) = x_1$ and $f'(a_{r-1}) = x_2$, then let $f(a_1) = v, f(a_{r-1}) = u$, because u hits all the neighbours of $f'(a_{r-1})$. If $f'(a_2) = y_1$,$f'(a_{r-1}) = y_2$, then let $f(a_1) = u$ and $f(a_{r-1}) = v$. For the rest situations, it is easy to find an embedding from T into G.

(C). There is no vertex in $V(G) \setminus \{u\}$ hitting both x_1 and x_2, and x_1 misses x_2. Then $d_G(x_1) + d_G(x_2) \leq k + 1$. Let $G' = G - \{u, x_1, x_2\}$ and $T' = T - \{a_0, a_1\}$. Then $e(G') \geq e(G) - (k+1) - (k+1) > \frac{1}{2}(k^2 - 2k - 12)$, Since $k \geq 9$, *avedeg*$(G') > (k^2 - 2k - 12)/(k+1) > k - 4$ and $\mid V(T') \mid \leq k - 2$. By theorem 1.7, $T' \subseteq G'$. Let $f(a_1) = u$. Then f is $T-$extensible.

(D). There is no vertex in $V(G) \setminus \{u\}$ hitting both x_1 and x_2, and x_1 hits x_2. Then $d_G(x_1) + d_G(x_2) \le k+3$. If $d_G(x_1) + d_G(x_2) \le k+2$, the assertion follows from (C). Hence assume that $d_G(x_1) + d_G(x_2) = k+3$. If $z \ne x_1, x_2$, then z has to hit x_1 or x_2, say that z hits x_1. Let $G' = G - \{u, z\} - x_1x_2$ and $T' = T - \{a_0, a_1\}$. Then $e(G') \ge e(G) - (k+1) - (k-5) + 1 - 1 > \frac{1}{2}(k^2 - 2k)$, which implies *avedeg*$(G') > (k^2-2k)/(k+2) > k-4$ and $|\ V(T')\ | \le k-2$. Hence by the induction hypothesis, $T' \subseteq G'$. If $f'(a_2)$ hits u, let $f(a_1) = u$; if $f'(a_2) = x_1$, let $f(a_1) = z$ and $f(a_0) = u$. If $f'(a_2) = x_2$ and if there is a vertex w in T' such that $f'(w) = x_1$, let $f(w) = u$, $f(a_1) = x_1$ and $f(a_0) = z$, because u hits all neighbours of $f'(w)$ in G'; if $f'(a_2) = x_2$ and there does not exist any vertex w in T' such that $f'(w) = x_1$, let $f(a_1) = x_1$, and $f(a_0) = z$. In all situations, f is $T-$extensible. If $z = x_1$ or x_2, then let $G' = G - \{u, z\}$ and $T' = T - \{a_0, a_1\}$. Similarly, we can find an embedding from T into G.

2.4 $\Delta(G) = k$

Let $u \in V(G)$ be a vertex of degree $d_G(u) = k$ and misses three vertices x_1, x_2, x_3. Denote by $S = \{x_1, x_2, x_3\}$.

2.4.1 $G[S]$ contains no edges.

Let $G' = G - \{u\}$ and $T' = T - \{a_0\}$. Then $e(G') \ge e(G) - k > \frac{1}{2}(k^2 - 8)$, which implies *avedeg*$(G') > (k^2-8)/(k+3) > k-3$ and $|\ V(T')\ | \le k-1$. By the induction hypothesis, $T' \subseteq G'$. If $f'(a_1)$ hits u, let $f(a_0) = u$; if $f'(a_1) = x_i$, $1 \le i \le 3$, let $f(a_1) = u$. Since u hits all neighbours of $f'(a_1)$ in G', f is $T-$extensible.

2.4.2 $G[S]$ contains exactly one edge.

Without loss of the generality, x_1 hits x_2, $d_G(x_1) \ge d_G(x_2)$, and $d_T(a_1) \ge d_T(a_{r-1})$. We consider two cases.

(A). $d_T(a_1) + d_T(a_{r-1}) \ge 5$.

(A.1). $d_G(x_2) \ge k-1$. If $x_3 \ne z$, let $G' = G - \{u, z, x_3\} - x_1x_2$ and $T' = T - \{a_1, b_1, \ldots, b_s\}$. Then $e(G') \ge e(G) - k - (k-5) - k - 1 > \frac{1}{2}(k^2 - 4k)$, which implies *avedeg*$(G') > (k^2-4k)/(k+1) > k-5$ and $|\ V(T')\ | \le k-3$. By the induction hypothesis, $T' \subseteq G'$. If $f'(a_2)$ hits u, then let $f(a_1) = u$; if $f'(a_2) = x_1$ and $x_2 \notin f'(V(T'))$, then let $f(a_1) = x_2$; if $f'(a_2) = x_1$ and $x_2 \in f'(V(T'))$ and $f'(w) = x_2$, then let $f(w) = u$, $f(a_2) = x_1$, and $f(a_1) = x_2$. Hence f is $T-$extensible. On the other hand, if $x_3 = z$, let $G' = G - \{u, z\} - \{x_1x_2\}$ and $T' = T - \{a_1, b_1, \ldots, b_s\}$. Similarly, we can prove that the assertion holds.

(A.2). $d_G(x_3) \ge k-1$. By (A.1), we can assume that $d_G(x_2) \le k-2$. If $z \ne x_1, x_2$, let $G' = G - \{u, z, x_1, x_2, x_3\}$ and $T' = T - \{a_1, b_1, \ldots, b_s, a_{r-1}, c_1, \ldots, c_t\}$. Then $e(G') \ge e(G) - k - (k-5) - (k-2) - k - k + 2 + 1 > \frac{1}{2}(k^2 - 8k + 12)$, which implies *avedeg*$(G') > (k^2 - 8k + 12)/(k-1) > k-7$ and $|\ V(T')\ | \le k-5$. Hence by the induction hypothesis, $T' \subseteq G'$. Moreover, x_3 misses at most one vertex of $V(G')$. If x_3 misses $f'(a_2)$, let $f(a_1) = u$ and $f(a_{r-1}) = x_3$; if x_3 hits $f'(a_2)$, let $f(a_{r-1}) = u$ and $f(a_1) = x_3$. then f is $T-$extensible. On the other hand, if $x_1 = z$ or $x_2 = z$, let $G' = G - \{u, x_1, x_2, x_3\}$ and $T' = T - \{a_1, b_1, \ldots, b_s, a_{r-1}, c_1, \ldots, c_t\}$. Using the same above argument, we can prove the assertion.

(A.3). $d_G(x_1) = k$,$d_G(x_2) \leq k-2$ and $d_G(x_3) \leq k-2$. If $z \neq x_2, x_3$, let $G' = G - \{u, z, x_1, x_2, x_3\}$ and $T' = T - \{a_1, b_1, \ldots, b_s, a_{r-1}, c_1, \ldots, c_t\}$. Hence $e(G') \geq e(G) - k - (k-5) - (k-2) - k - (k-2) + 2 > \frac{1}{2}(k^2 - 8k + 10)$, which implies *avedeg*$(G') > (k^2 - 8k + 10)/(k-1) > k - 7$ and $\mid V(T') \mid \leq k-5$. By the induction hypothesis, $T' \subseteq G'$. Note that x_1 misses at most one vertex in $V(G')$. If x_1 misses $f'(a_2)$, let $f(a_1) = u$ and $f(a_{r-1}) = x_1$; if x_1 hits $f'(a_2)$, let $f(a_{r-1}) = u$ and $f(a_1) = x_1$. Hence f is $T-$extensible. On the other hand, if $x_2 = z$ or $x_3 = z$, let $G' = G - \{u, x_1, x_2, x_3\}$ and $T' = T - \{a_1, b_1, \ldots, b_s, a_{r-1}, c_1, \ldots, c_t\}$. By the same above argument, we can prove the assertion.

(A.4). $d_G(x_1) \leq k-1$,$d_G(x_2) \leq k-2$ and $d_G(x_3) \leq k-2$. Then there exists a vertex u' in $V(G) \setminus \{x_1, x_2, x_3, u\}$ with degree at least $k-1$. Otherwise, by $\delta(G) \leq k-5$, we have *avedeg*$(G) \leq \frac{k+(k-1)(k-2)+(k-1)+2(k-2)+(k-5)}{k+4} \leq k-2$, which is a contradiction. Let $G' = G - \{u, u'\} - \{x_1 x_2\}$ and $T' = T - \{a_1, b_1, \ldots, b_s\}$. Then $e(G') \geq e(G) - k - k + 1 - 1 > \frac{1}{2}(k^2 - 2k - 8)$, which implies *avedeg*$(G') > (k^2 - 2k - 8)/(k+2) = k-4$ and $\mid V(T') \mid \leq k-2$. By the induction hypothesis, $T' \subseteq G'$. If $f'(a_2)$ hits u, let $f(a_1) = u$; if $f'(a_2)$ misses u, let $f(a_2) = u$ and $f(a_1) = u'$. Then f is $T-$extensible.

(B). $d_T(a_1) = 2$ and $d_T(a_{r-1}) = 2$. If there exists a vertex w that hits both x_1 and x_3, let $G' = G - \{u, w\} - x_1 x_2$ and $T' = T - \{a_0, a_1\}$. Then $e(G') \geq e(G) - 2k + 1 - 1 > \frac{1}{2}(k^2 - 2k - 8)$, which implies *avedeg*$(G') > (k^2 - 2k + 8)/(k+2) = k-4$ and $\mid V(T') \mid \leq k-2$. By the induction hypothesis, $T' \subseteq G'$. If $f'(a_2) = x_1$ or x_3, let $f(a_1) = w$ and $f(a_0) = u$; if $f'(a_2) = x_2$ and $x_1 \notin f'(V(T'))$, let $f(a_1) = x_1$ and $f(a_0) = w$; if $f'(a_2) = x_2$ and $x_1 \in f'(V(T'))$,$f'(v) = x_1$, let $f(v) = u, f(a_1) = x_1$ and $f(a_0) = w$. In the above situations, f is $T-$extensible. On the other hand, if there is no vertex hits both x_1 and x_3, or x_2 and x_3. then $d_G(x_1) + d_G(x_3) \leq k$, $d_G(x_2) + d_G(x_3) \leq k$. Since $d_G(x_i) \geq \lfloor \frac{k}{2} \rfloor$ and $k \geq 9$, $d_G(x_i) \leq k-2$. Hence, similarly as in (A.4), there exists a vertex u' in $V(G) \setminus \{x_1, x_2, x_3, u\}$ with degree at least $k-1$, and an embedding of T into G.

2.4.3 $G[S]$ contains exactly two edges

Without loss of the generality, assume that x_1 hits both x_2 and x_3. We consider the following two cases.

(A). $d_T(a_1) = 2$. Let $G' = G - \{u, x_1\}$ and $T' = T - \{a_0, a_1\}$. Then $e(G') \geq e(G) - 2k > \frac{1}{2}(k^2 - 2k - 8)$, which implies *avedeg*$(G') > (k^2 - 2k - 8)/(k+2) > k-4$ and $\mid V(T') \mid \leq k-2$. By the induction hypothesis, $T' \subseteq G'$. If $f'(a_2) = x_2$ or x_3 (say x_2), let $f(a_1) = x_1$; Moreover, if $x_3 \notin f'(V(T'))$, let $f(a_0) = x_3$; if $x_3 \in f'(V(T'))$ and $f'(v) = x_3$, let $f(v) = u, f(a_1) = x_1$, and $f(a_0) = x_3$. Hence, f is T-extensible. If $f'(a_2) \neq x_2, x_3$, then it is easy to find an embedding from T to G.

(B). $d_T(a_1) \geq 3$.

(a). $d_G(x_1) \geq k-1$. If $z \neq x_2, x_3$, let $G' = G - \{u, z, x_1\}$ and $T' = T - \{a_1, b_1, \ldots, b_s\}$. Then $e(G') \geq e(G) - k - (k-5) - k + 1 > \frac{1}{2}(k^2 - 4k + 4)$, which implies *avedeg*$(G') > (k^2 - 4k + 4)/(k+1) > k-5$ and $\mid V(T') \mid \leq k-3$. By the induction hypothesis, $T' \subseteq G'$. If $f'(a_2) = x_2$ or x_3 (say x_2), let $f(a_1) = x_1$. Moreover, if $x_3 \notin f'(V(T'))$, let $f(b_1) = x_3$; if $x_3 \in f'(V(T'))$ and $f'(v) = x_3$, let $f(v) = u, f(a_1) = x_1$ and $f(b_1) = x_3$. Because u hits all neighbours of $f'(v)$ and $d_G(x_1) \geq k-1$, f is $T-$extensible. If $f'(a_2) \neq x_2, x_3$, it is easy to find an embedding from T to G. On the other hand, if $z = x_2$ or x_3 (say x_2), let $G' = G - \{u, x_1, x_2\}$, by the same argument above, the assertion holds.

(b). $d_G(x_1) \leq k-2$, $d_G(x_2) = k$ or $d_G(x_3) = k$ (say $d_G(x_2) = k$). Then there exists a vertex $y \in V(G) \setminus \{u, x_1, x_2, x_3\}$ such that x_2 misses y. So x_2 misses u, x_3 and y and u misses x_3. By Case 2.4.2, we can assume y hits x_3. Further, by (a), we can assume $d_G(y) \leq k-2$. If $z \neq x_1, y$, let $G' = G - \{u, z, x_2, x_3, y\}$ and $T' = T - \{a_1, b_1, \ldots, b_s, a_{r-1}, c_1, \ldots, c_t\}$. Then $e(G') \geq e(G) - k - (k-5) - k - k - (k-2) + 3 > \frac{1}{2}(k^2 - 8k + 12)$, which implies *avedeg*$(G') > (k^2 - 8k + 12)/(k-1) > k-7$ and $|\ V(T')\ | \leq k-5$. By the induction hypothesis, $T' \subseteq G'$. Further, if $f'(a_2) = x_1$, let $f(a_1) = x_2$ and $f(a_{r-1}) = u$; if $f'(a_{r-2}) = x_1$, let $f(a_{r-1}) = x_2$ and $f(a_1) = u$. Hence f is $T-$extensible. On the other hand, if $z = y$, let $G' = G - \{u, x_2, x_3, y\}$ and $T' = T - \{a_1, b_1, \ldots, b_s, a_{r-1}, c_1, \ldots, c_t\}$; if $z = x_1$, let $G' = G - \{u, z, x_2, x_3, y\}$ and $T' = T - \{a_1, b_1, \ldots, b_s, a_{r-1}, c_1, \ldots, c_t\}$. Then by the same argument, it is easy to prove that the assertion holds.

(c). $d_G(x_1) \leq k-2$, $d_G(x_2) = k-1$ and $d_G(x_3) = k-1$. Let $G' = G - \{u, x_2, x_3\}$ and $T' = T - \{a_1, b_1, \ldots, b_s\}$. Then $e(G') \geq e(G) - k - (k-1) - (k-1) > \frac{1}{2}(k^2 - 4k - 4)$, which implies *avedeg*$(G') > (k^2 - 4k - 4)/(k+1) > k-5$ and $|\ V(T')\ | \leq k-3$. By the induction hypothesis, $T' \subseteq G'$. If $f'(a_2) = x_1$, let $f(a_1) = x_2$, which f is $T-$extensible. If $f'(a_2) \neq x_1$, it is easy to find an embedding from T to G.

(d). $d_G(x_1) \leq k-2$, and $d_G(x_2) \leq k-2$ or $d_G(x_3) \leq k-2$ (say $d_G(x_2) \leq k-2$), hence $d_G(x_3) \leq k-1$ by (b). Then there exists a vertex $u' \in V(G) \setminus \{x_1, x_2, x_3, u\}$ of degree at least $k-1$, otherwise $2e(G) \leq (k-1)(k-2) + (k-5) + k + 2(k-2) + (k-1) \leq (k+4)(k-2)$ which is impossible. Let $G' = G - \{u, u', x_1\}$ and $T' = T - \{a_1, b_1, \ldots, b_s\}$. Then $e(G') \geq e(G) - 2k - (k-2) + 1 > \frac{1}{2}(k^2 - 4k - 2)$, which implies *avedeg*$(G') > (k^2 - 4k - 2)/(k+1) > k-5$ and $|\ V(T')\ | \leq k-3$. By the induction hypothesis, $T' \subseteq G'$. Hence if $f'(a_2)$ hits u, let $f(a_1) = u$; if $f'(a_2) = x_2$ or x_3 (say x_2), let $f(a_2) = u$ and $f(a_1) = u'$ since u hits all the neighbours of $f'(a_2)$. Then f is $T-$extensible.

2.4.4 $G[S]$ contains exactly three edges

The following two cases are considered.

(A). $d_T(a_1) = 2$. If there exists an $1 \leq i \leq 3$ (say $i = 1$) such that $d_G(x_1) \leq k-1$, let $G' = G - \{u, x_1\} - x_2x_3$ and $T' = T - \{a_0, a_1\}$. Then $e(G') \geq e(G) - k - (k-1) - 1 > \frac{1}{2}(k^2 - 2k - 8)$, which implies *avedeg*$(G') > (k^2 - 2k - 8)/(k+2) = k-4$ and $|\ V(T')\ | \leq k-2$. By the induction hypothesis, $T' \subseteq G'$. If $f'(a_2) = x_2$ or x_3 (say x_2), let $f(a_1) = x_1$. Moreover, if $x_3 \notin f'(V(T'))$, let $f(a_0) = x_3$; and if $x_3 \in f'(V(T'))$ and $f'(v) = x_3$, let $f(v) = u, f(a_1) = x_1, f(a_0) = x_3$. Hence f is $T-$extensible. On the other hand, if $d_G(x_1) = d_G(x_2) = d_G(x_3) = k$, let $G' = G - \{u, x_1\}$ and $T' = T - \{a_0, a_1\}$. Then $e(G') \geq e(G) - 2k > \frac{1}{2}(k^2 - 2k - 8)$, which implies *avedeg*$(G') > (k^2 - 2k - 8)/(k+2) = k-4$ and $|\ V(T')\ | \leq k-2$. By the induction hypothesis,$T' \subseteq G'$. If $f'(a_2) = x_2$ or x_3, let $f(a_1) = x_1$; if $f'(a_2) \neq x_2, x_3$, let $f(a_1) = u$. Hence f is T-extensible.

(B). $d_T(a_1) \geq 3$. If there exists an $1 \leq i \leq 3$ (say $i = 1$) such that $d_G(x_1) \geq k-1$, Let $G' = G - \{u, z, x_1\} - x_2x_3$. By the same argument as Case 2.4.3.(B).(a), the assertion holds. The rest is similar as Case 2.4.3.(B).(d).

2.5 $\Delta(G) = k-1$

Since $\Delta(G) = k-1$ and $\delta(G) \le k-5$, there exist at least four vertices of degree $k-1$. Otherwise $2e(G) \le 3(k-1)+k(k-2)+(k-5) = (k-2)(k+4)$, which is a contradiction. Let u_i be vertex of $d_G(u_i) = k-1$ missing four vertices of $S_i = \{x_{i1}, x_{i2}, x_{i3}, x_{i4}\}$ for $i = 1,2,3,4$. If there exists a vertex u_i with $1 \le i \le 4$ such that $G[S_i]$ contains at most one edge. Let $G' = G - \{u_i\} - E(G[S_i])$ and $T' = T - \{a_0\}$. Then $e(G') \ge e(G) - (k-1) - 1 > \frac{1}{2}(k^2-8)$, which implies *avedeg*$(G') > (k^2-8)/(k+3) > k-3$ and $\mid V(T') \mid \le k-1$. By the induction hypothesis, $T' \subseteq G'$. If u_i hits $f'(a_1)$, let $f(a_0) = u_i$, and if u_i misses $f'(a_1)$, let $f(a_1) = u_i$. Then f is $T-$extensible. Hence we assume that $G[S_i]$ contains at least two edges for $i = 1,2,3,4$.

2.5.1 $d_T(a_1) \ge 3, d_T(a_{r-1}) \ge 2$

We consider the number of the edges in $G[u_1, u_2, u_3, u_4]$.

(A). $G[u_1, u_2, u_3, u_4]$ contains at least one edge, say u_1 hits u_2. If $z \notin S_1 = \{x_{11}, x_{12}, x_{13}, x_{14}\}$, let $G' = G - \{u_1, u_2, z\} - E(G[S_1])$ and $T' = T - \{a_1, b_1, \ldots, b_s\}$. Then $e(G') \ge e(G) - 2(k-1) - (k-5) + 1 - 6 > \frac{1}{2}(k^2-4k-4)$, which implies *avedeg*$(G') > (k^2-4k-4)/(k+1) > k-5$ and $\mid V(T') \mid \le k-3$. By the induction hypothesis, $T' \subseteq G'$. Hence if u_1 hits $f'(a_2)$, let $f(a_1) = u_1$; and if u_1 misses $f'(a_2)$, let $f(a_2) = u_1$ and $f(a_1) = u_2$. Since u_1 hits all the neighbours of $f'(a_2)$ in G', f is $T-$extensible. On the other hand, if $z \in S_1 = \{x_{11}, x_{12}, x_{13}, x_{14}\}$, say $z = x_{11}$. Let $G' = G - \{u_1, u_2, z\} - E(G[x_{12}, x_{13}, x_{13}])$. By the same argument, the assertion holds.

(B). $G[u_1, u_2, u_3, u_4]$ contains no edges.

(B.1). If there exist two vertices, say u_1 and u_2, in $\{u_1, u_2, u_3, u_4\}$ such that u_1 misses y_1 and u_2 misses y_2, where $y_1 \ne y_2$ and $y_1, y_2 \notin \{u_1, \ldots, u_4\}$. Let $G' = G - \{u_1, u_2, u_3, u_4\}$ and $T' = T - \{a_1, b_1, \ldots, b_s, a_{r-1}, c_1, \ldots, c_t\}$. Then $e(G') \ge e(G) - 4(k-1) > \frac{1}{2}(k^2-6k)$, which implies *avedeg*$(G') > (k^2-6k)/k = k-6$ and $\mid V(T') \mid \le k-4$. By the induction hypothesis, $T' \subseteq G'$. Hence if $f'(a_2) = y_1$, let $f(a_1) = u_2$ and $f(a_{r-1}) = u_1$; if $f'(a_2) = y_2$, let $f(a_1) = u_1$ and $f(a_{r-1}) = u_2$. Moreover, if $f'(a_{r-2}) = y_1$, let $f(a_1) = u_1$ and $f(a_{r-1}) = u_2$; and if $f'(a_{r-2}) = y_2$, let $f(a_1) = u_2$ and $f(a_{r-1}) = u_1$. Therefore, f is $T-$extensible.

(B.2). There exist a vertex $y \notin \{u_1, \ldots, u_4\}$ such that y misses $u_1, \ldots, u_4$. Then $G[u_1, \ldots, u_4, y]$ contains no edges.

(a). $d_T(a_{r-1}) = 2$. Then there exists a vertex w hits $\{u_1, u_2, u_3, u_4\}$ and y. Let $G' = G - \{u_1, w\}$ and $T' = T - \{a_{r-1}, a_r\}$. Then $e(G') \ge e(G) - 2(k-1) + 1 > \frac{1}{2}(k^2-2k-2)$, which implies *avedeg*$(G') > (k^2-2k-2)/(k+2) > k-4$ and $\mid V(T') \mid \le k-2$. By the induction hypothesis, $T' \subseteq G'$. Hence if $f'(a_{r-2}) = u_2, u_3, u_4$ or y, let $f(a_{r-1}) = w$ and $f(a_r) = u_1$; and if $f'(a_{r-2}) \ne u_2, u_3, u_4, y$, let $f(a_{r-1}) = u_1$ and $f(a_r) = w$. Therefore f is $T-$extensible.

(b). $d_T(a_{r-1}) \ge 3$. If $z \ne y$, let $G' = G - \{u_1, u_2, u_3, u_4, y, z\}$ and $T' = T - \{a_1, b_1, \ldots, b_s, a_{r-1}, c_1, \ldots, c_t\}$. Then $e(G') \ge e(G) - 4(k-1) - (k-1) - (k-5) + 4 > \frac{1}{2}(k^2-10k+20)$, which implies *avedeg*$(G') > (k^2-10k+20)/(k-2) > k-8$ and $\mid V(T') \mid \le k-6$. By the induction hypothesis, $T' \subseteq G'$. Let $f(a_1) = u_1$ and $f(a_{r-1}) = u_2$. Then f is $T-$extensible. On the other hand, if $z = y$, let $G' = G - \{u_1, u_2, u_3, u_4, z\}$ and $T' = T - \{a_1, b_1, \ldots, b_s, a_{r-1}, c_1, \ldots, c_t\}$. By the same argument, the assertion holds.

2.5.2 $d_T(a_1) = 2, d_T(a_{r-1}) = 2$.

We will discuss the following four cases: $(A), (B), (C)$ and (D).

(A). There exists a $1 \le i \le 4$, say $i = 1$, such that $G[S_1]$ contains two or three edges. If u_1 hits one vertex, say u_2, of three vertices u_2, u_3, u_4. Let $G' = G - \{u_1, u_2\} - E(G[S_1])$ and $T' = T - \{a_0, a_1\}$. Then $e(G') \ge e(G) - 2(k-1) + 1 - 3 > \frac{1}{2}(k^2 - 2k - 8)$, which implies *avedeg*$(G') > (k^2 - 2k - 8)/(k+2) = k - 4$ and $| V(T') | \le k - 2$. By the induction hypothesis, $T' \subseteq G'$. Hence if u_1 hits $f'(a_2)$, let $f(a_1) = u_1$; and if u_1 misses $f'(a_2)$, let $f(a_2) = u_1$ and $f(a_1) = u_2$. Since u_1 hits all the neighbours of $f'(a_2)$ in G', f is $T-$extensible. Therefore, we assume that u_1 misses u_j for $j = 2, 3, 4$. Then u_1 misses $x_{11} = u_2, x_{12} = u_3, x_{13} = u_4, x_{14}$ and $G[u_2, u_3, u_4, x_{14}]$ contains two or three edges.

(A.1). x_{14} hits one vertex, say u_2, of three vertices u_2, u_3, u_4. Let $G' = G - \{u_1, u_2, u_3, u_4\}$ and $T' = T - \{a_0, a_1, a_{r-1}, a_r\}$. Then $e(G') \ge e(G) - 4(k-1) > \frac{1}{2}(k^2 - 6k)$, which implies *avedeg*$(G') > (k^2 - 6k)/k = k - 6$ and $| V(T') | \le k - 4$. By the induction hypothesis, $T' \subseteq G'$. Since $G[u_2, u_3, u_4, x_{14}]$ contains two or three edges, there exists a vertex, say u_3, of two vertices u_3,u_4 misses at most one vertex, say y_1, in $V(G) \setminus \{u_1, u_2, u_4, x_{14}\}$. Hence if $f'(a_2) = x_{14}$ or y_1, and $f'(a_{r-2}) = y_1$ or x_{14}, let $f(a_1) = u_2$ or u_1 and $f(a_{r-1}) = u_1$ or u_2, then f is $T-$extensible. For the rest cases, it is easy to find an embedding from T to G.

(A.2). x_{14} misses three vertices u_2, u_3, u_4. Then $G[u_2, u_3, u_4]$ contains two or three edges. We can assume that u_2 hits u_3 and u_4. If u_3 misses u_4, u_3 misses at most one vertex, says y_1, in $V(G) \setminus \{u_1, u_2, u_4, x_{14}\}$. Then let $G' = G - \{u_1, x_{14}, u_3, u_4\}$ and $T' = T - \{a_0, a_1, a_{r-1}, a_r\}$. By the similar argument as Case (A.1), the assertion holds. Hence we can assume that u_3 hits u_4 and u_3 misses z_1, z_2, u_1, x_{14}. Let $G' = G - \{u_1, x_{14}, u_3, u_4\} - \{z_1 z_2\}$ and $T' = T - \{a_0, a_1, a_{r-1}, a_r\}$. Then $e(G') \ge e(G) - 4(k-1) + 1 - 1 > \frac{1}{2}(k^2 - 6k)$, which implies *avedeg*$(G') > (k^2 - 6k)/k = k - 6$ and $| V(T') | \le k - 4$. By the induction hypothesis, $T' \subseteq G'$. Hence if $f'(a_2) = z_1$ or z_2, and $f'(a_{r-2}) = z_2$ or z_1, let $f(a_2) = u_3$, $f(a_1) = u_4$, $f(a_{r-1}) = u_1$. Therefore f is $T-$ extensible. If $f'(a_2) = z_1$ or z_2, and $f'(a_{r-2}) = u_2$, let $f(a_1) = u_1$, $f(a_{r-1}) = u_4$. Therefore f is $T-$ extensible. For the rest cases, it is easy to find an embedding from T to G.

(B). There exists a $1 \le i \le 4$, say $i = 1$, such that $G[S_1]$ contains exactly four edges.

(B.1). There exists a vertex, say x_{11}, of degree 3 in $G[S_1]$ and $| E(G[S_1]) | \le 5$. Then x_{11} hits x_{12}, x_{13} and x_{14}. Let $G' = G - \{u_1, x_{11}\} - E(G[x_{12}, x_{13}, x_{14}])$ and $T' = T - \{a_0, a_1\}$. Then $e(G') \ge e(G) - 2(k-1) - 2 > \frac{1}{2}(k^2 - 2k - 8)$, which implies *avedeg*$(G') > (k^2 - 2k - 8)/(k+2) = k - 4$ and $| V(T') | \le k - 2$. By the induction hypothesis, $T' \subseteq G'$. Hence if u_1 hits $f'(a_2)$, let $f(a_1) = u_1$, which implies that f is $T-$extensible. If u_1 misses $f'(a_2)$ and $f'(a_2) = x_{12}$, let $f(a_1) = x_{11}$. Moreover, if x_{13} or $x_{14} \notin f'(V(T'))$, then let $f(a_0) = x_{13}$ or x_{14}. Then f is $T-$extensible. If x_{13} and $x_{14} \in f'(V(T'))$, $f'(w) = x_{13}$ or x_{14}, let $f(w) = u_1, f(a_0) = x_{13}$ or x_{14}. Then f is $T-$extensible. For the rest cases, it is easy to find an embedding from T to G.

(B.2). The degree of every vertex in $G[S_1]$ is two. We assume that x_{11} hits x_{12}, x_{12} hits x_{13}, x_{13} hits x_{14}, x_{14} hits x_{11}.

(a). u_1 hits all vertices of $\{u_2, u_3, u_4\}$.

(a.1). There exists a vertex u_i, say u_2, in $\{u_2, u_3, u_4\}$ which misses x_{11}, x_{12}, x_{13} and x_{14}. Let $G' = G - \{u_1, u_2, x_{11}, x_{12}\} - x_{13}x_{14}$ and $T' = T - \{a_0, a_1, a_{r-1}, a_r\}$. Then $e(G') \ge e(G) - 4(k-1) + 1 > \frac{1}{2}(k^2 - 6k + 2)$, which implies *avedeg*$(G') > (k^2 - 6k + 2)/k > k - 6$ and $| V(T') | \le k - 4$. By the induction hypothesis, $T' \subseteq G'$. If $f'(a_2) = x_{13}, f'(a_{r-2}) = x_{14}$, let $f(a_1) = x_{12}, f(a_0) = x_{11}, f(a_{r-2}) = u_1, f(a_{r-1}) =$

u_2. Since u_1 hits all the neighbours of $f'(a_{r-2})$ in G', f is $T-$extensible. For the rest cases, similarly, it is easy to find an embedding from T to G.

(a.2). There exists a vertex, say u_2, in $\{u_2, u_3, u_4\}$ such that it hits at least two vertices of $\{x_{11}, x_{12}, x_{13}, x_{14}\}$, say u_2 hits x_{11} and x_{13}, or u_2 hits x_{11} and x_{12}.

If u_2 hits x_{11} and x_{13}, let $G' = G - \{u_1, u_2\} - x_{11}x_{12} - x_{12}x_{13} - x_{13}x_{14}$ and $T' = T - \{a_0, a_1\}$. Then $e(G') \geq e(G) - 2(k-1) + 1 - 3 > \frac{1}{2}(k^2 - 2k - 8)$, which implies *avedeg*$(G') > k - 4$ and $\mid V(T') \mid \leq k - 2$. By the induction hypothesis, $T' \subseteq G'$. Hence if $f'(a_2) = x_{11}$ or x_{13}, let $f(a_1) = u_2$; if $f'(a_2) = x_{12}$, let $f(a_2) = u_1$ and $f(a_1) = u_2$; if $f'(a_2) = x_{14}$ and $x_{13} \notin f'(V(T'))$, let $f(a_1) = x_{13}$ and $f(a_0) = u_2$; if $f'(a_2) = x_{14}$ and $x_{13} \in f'(V(T'))$, let $f(v) = u_1, f(a_1) = x_{13}, f(a_0) = u_2$, because there is a vertex v, $f'(v) = x_{13}$ and u_1 hits all the neighbours of $f'(v)$ in G'. Therefore f is $T-$extensible.

If u_2 hits x_{11} and x_{12}, let $G' = G - \{u_1, u_2\} - x_{12}x_{13} - x_{13}x_{14} - x_{11}x_{14}$ and $T' = T - \{a_0, a_1\}$. Then $e(G') \geq e(G) - 2(k-1) + 1 - 3 > \frac{1}{2}(k^2 - 2k - 8)$, which implies *avedeg*$(G') > k - 4$ and $\mid V(T') \mid \leq k - 2$. By the induction hypothesis, $T' \subseteq G'$. Hence if $f'(a_2) = x_{11}$ or x_{12}, let $f(a_1) = u_2$; if $f'(a_2) = x_{13}$ or x_{14}, let $f(a_2) = u_1, f(a_1) = u_2$, because u_1 hits all the neighbours of $f'(a_2)$ in G'. Therefore f is $T-$extensible.

(a.3). u_i hits exactly one vertex of $\{x_{11}, x_{12}, x_{13}, x_{14}\}$ for $i = 2, 3, 4$.

(i). There exist two vertices of $\{u_2, u_3, u_4\}$ such that they hit the same vertex in $\{x_{11}, x_{12}, x_{13}, x_{14}\}$, says both u_2 and u_3 hit x_{14}.

If u_2 and u_3 misses the same vertices, say, $\{x_{11}, x_{12}, x_{13}, y\}$, then u_2 hits u_3. Further, if $G[x_{11}, x_{12}, x_{13}, y]$ contains at most three edges or has a vertex of degree 3, the assertion follows from Case 2.5.2.(A) or Case 2.5.2.(B.1). Therefore we can assume that y hits both x_{11} and x_{13}. Let $G' = G - \{u_2, u_3, x_{11}, x_{12}\} - x_{13}y$ and $T' = T - \{a_0, a_1, a_{r-1}, a_r\}$. The assertion follows from Case 2.5.2. (B.2).(a.1).

If u_2 misses $\{x_{11}, x_{12}, x_{13}, y_1\}$ and u_3 misses $\{x_{11}, x_{12}, x_{13}, y_2\}$ with $y_1 \neq y_2$, let $G' = G - \{u_1, u_2, u_3, x_{14}\} - x_{11}x_{12} - x_{12}x_{13}$ and $T' = T - \{a_0, a_1, a_{r-1}, a_r\}$. Then $e(G') \geq e(G) - 4(k-1) + 4 - 2 > \frac{1}{2}(k^2 - 6k + 4)$, which implies *avedeg*$(G') > k - 6$ and $\mid V(T') \mid \leq k - 4$. By the induction hypothesis, $T' \subseteq G'$. Hence if $f'(a_2) = x_{11}$ or x_{13}, let $f(a_1) = x_{14}$,$f(a_0) = u_3$ or u_2 or let $f(a_2) = u_1$,$f(a_1) = u_3$ or u_2. If $f'(a_2) = x_{12}$, let $f(a_2) = u_1$,$f(a_1) = u_3$ or u_2. If $f'(a_2) = y_1$ or y_2, let $f(a_1) = u_3$ or u_2. Since there is a choice which uses distinct vertices of $\{u_1, u_2, u_3, x_{14}\}$ for any two vertices of $\{x_{11}, x_{12}, x_{13}, y_1, y_2\}$, we can find an embedding from T to G. (For example, if $f'(a_2) = x_{11}$,$f'(a_{r-2}) = x_{13}$, let $f(a_1) = x_{14}$,$f(a_0) = u_3$,$f(a_{r-2}) = u_1$,$f(a_{r-1}) = u_2$.)

(ii). $\{u_2, u_3, u_4\}$ hits the different vertices of $\{x_{11}, x_{12}, x_{13}, x_{14}\}$. Without loss of generality, we assume that u_2 hits x_{11} and u_3 hits x_{13}, u_2 misses y_1 and u_3 misses y_2. Let $G' = G - \{u_1, u_2, u_3, x_{13}\} - x_{11}x_{12} - x_{11}x_{14}$ and $T' = T - \{a_0, a_1, a_{r-1}, a_r\}$. Then $e(G') \geq e(G) - 4(k-1) + 3 + 0 - 2 > \frac{1}{2}(k^2 - 6k + 2)$, which implies *avedeg*$(G') > k - 6$ and $\mid V(T') \mid \leq k - 4$. By the induction hypothesis, $T' \subseteq G'$. Hence if $f'(a_2) = x_{12}$ or x_{14}, let $f(a_1) = x_{13}$ and $f(a_0) = u_3$, or let $f(a_2) = u_1$ and $f(a_1) = u_2$, if $f'(a_2) = y_1$ or y_2, let $f(a_1) = u_1$, if $f'(a_2) = x_{11}$, let $f(a_1) = u_2$, Therefore f is $T-$extensible. For the rest cases, by the same argument, it is easy to find an embedding from T to G.

(b). u_1 hits one or two vertices of $\{u_2, u_3, u_4\}$. Without loss of the generality, we assume that u_1 hits u_2 and u_1 misses u_4. Then $u_4 \in \{x_{11}, x_{12}, x_{13}, x_{14}\}$, say $u_4 = x_{14}$, u_4 misses u_1,x_{12},z_1,z_2.

If $u_2 \neq z_1, z_2$, then u_2 hits u_4. Let $G' = G - \{u_1, u_2, u_4, x_{12}\} - z_1z_2$ and $T' =$

$T - \{a_0, a_1, a_{r-1}, a_r\}$. Then $e(G') \geq e(G) - 4(k-1) + 2 - 1 > \frac{1}{2}(k^2 - 6k)$, which implies *avedeg*$(G') > k - 6$ and $| V(T') | \leq k - 4$. By the induction hypothesis, $T' \subseteq G'$. Hence if $f'(a_2) = x_{11}$ and $f'(a_{r-2}) = x_{13}$, let $f(a_1) = u_4, f(a_{r-2}) = u_1$ and $f(a_{r-1}) = u_2$, if $f'(a_2) = z_1$ and $f'(a_{r-2}) = z_2$, let $f(a_1) = u_1, f(a_{r-2}) = u_4$ and $f(a_{r-1}) = u_2$. Therefore f is T-extensible. For the rest cases, it is easy to find an embedding from T to G. If $u_2 = z_1$ or z_2, say $u_2 = z_1$, let $G' = G - \{u_1, u_2, u_4, x_{12}\}$ and $T' = T - \{a_0, a_1, a_{r-1}, a_r\}$. This situation is much easier than the above case.

(c). u_1 misses all the vertices of $\{u_2, u_3, u_4\}$. Without loss of generality, we assume $u_2 = x_{11}$, $u_3 = x_{12}$, $u_4 = x_{13}$. Let u_2 miss $\{u_1, x_{13}, y_1, y_2\}$. If $G[u_1, x_{13}, y_1, y_2]$ contains two, or three edges, or a vertex of degree 3, the assertion follows from Case 2.5.2 (A). and Case 2.5.2 (B.1). Hence we assume that u_1 hits y_1, y_1 hits $u_4 = x_{13}$, u_4 hits y_2 and y_2 hits u_1. Hence the assertion follows from Case 2.5.2. (B.2). (a) and Case 2.5.2. (B.2).(b).

(C). There exists a $1 \leq i \leq 4$, say $i = 1$, such that $G[x_{11}, x_{12}, x_{13}, x_{14}]$ contains five edges. Then we assume that x_{11} hits x_{12}, x_{13} and x_{14}. Let $G' = G - \{u_1, x_{11}\} - E(G[x_{12}, x_{13}, x_{14}])$ and $T' = T - \{a_0, a_1\}$. The assertion follows from the proof of Case 2.5.2 (B.1).

(D). There exists a $1 \leq i \leq 4$, say $i = 1$, such that $G[x_{11}, x_{12}, x_{13}, x_{14}]$ contains six edges. If $d_G(x_{11}) \leq k - 2$, similar as Case 2.5.2 (B.1), we can prove the assertion. So we can assume $d_G(x_{11}) = d_G(x_{12}) = d_G(x_{13}) = d_G(x_{14}) = k - 1$, we can also assume if $d_G(x) = k - 1$, and x misses y then $d_G(y) = k - 1$, furthermore we can assume x hits all of the vertices whose degree is less than $k - 1$. Let $G' = G - \{u_1, z\}$, z hits all of $\{x_1, x_2, x_3, x_4\}$, $T' = T - \{a_0, a_1\}$. So $e(G') \geq e(G) - (k-1) - (k-5) + 1 > \frac{1}{2}(k^2 - 2k + 6)$. *avedeg*$(G') > (k^2 - 2k + 6)/(k + 2) > k - 4$ and $| V(T') | \leq k - 2$. By the induction assumption, $T' \subseteq G'$. If $f'(a_2)$ hits u_1, then $f(a_1) = u_1, f(a_0) = z$. If $f'(a_2)$ misses u_1, then $f(a_0) = u_1, f(a_1) = z$. f is T-extensible.

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The JLS model with ARMA/GARCH errors

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Abstract

Prior to crashes, the stock index price time series is characterised by the Log-Periodic Power Law (LPPL) equation of the Johansen–Ledoit–Sornette (JLS) model, which leads to a critical point indicating a change to a new market regime. In this paper, we describe the hierarchical diamond lattice, upon which the JLS model is derived, using the diamond lattice operation D_i and derive the recursion for the coefficients of the growth function in a diamond lattice rooted at the main root vertex r_m. Further, to verify the adequacy of the JLS model, we analyse the model's residuals and propose its generalization, using the ARMA/GARCH error model. We determine the ARMA/GARCH orders using the extended autocorrelation function (EACF) method and compare the results with those of the Akaike and Bayesian Information Criteria. Using the data for 33 major world stock indices we show, that proposed generalization of the JLS model in general performs better in predicting the market regime changes and has also the ability to recognise false bubble identification, indicated by the JLS model.

Keywords: Graph operations, hierarchical diamond lattice, JLS model, financial bubbles and crashes, ARMA/GARCH errors.

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1 Introduction

Financial bubbles and crashes are fascinating events for academics and practitioners alike, and such occurrences are especially interesting in the field of econophysics. These events are extremely important because of their usually strong impacts on not only financial markets but also the global economy. Unfortunately, there is still no general agreement in the literature on what defines a financial bubble or crash. A financial bubble may be recognised as a long-term positive deviation of a financial asset's market price from its fundamental value. A crash may be defined as a sudden dramatic decline of market price over a short time period. Understanding the behaviour of financial markets and the relationship between financial bubbles and crashes may help to minimise the damage of the speculative bubbles that end up with crashes. Consequently, identifying a financial bubble and predicting its end has become very important issue in financial markets behaviour research.

The Johansen–Ledoit–Sornette (JLS) model [14] was developed to describe the dynamics of financial bubbles and crashes. The model states that, prior to crashes, the mean function of a stock index price time series is characterised by a Log-Periodic Power Law (LPPL) equation that leads to a critical point indicating the change to a new market regime, either a large crash or a change in the average growth rate. The model assumes the presence of two types of agents in the market, namely a group of agents with rational expectations and a group of irrational agents with herding behaviours, and these agents potentially lead to the development of speculative bubbles. These agents are organised into a network in which each exists in only one of two possible states (buy or sell), while their trading actions depend on the opinions of other agents and on external influences. If the tendency of irrational agents to imitate their neighbours increases up to a certain critical point, then a large proportion of agents will be in the same state (sell) at the same time, thus causing a crash. In the JLS model, bubbles are characterised by faster-than-exponential price growth due to herding behaviours and imitation of irrational agents during the bubble period. The key parameter of the model is the critical time t_c, which is interpreted as the moment at which the bubble ends and the transition to another market regime begins.

Numerous empirical results have been reported by several authors on this subject. The JLS model has been used in various types of markets, such as the bubbles of stock market indices [14, 16, 9, 44, 12], the anti-bubbles in different financial markets [15, 32, 40], exchange rate bubbles [20], the oil bubble [31], real estate bubbles [42, 43], corporate bond spread bubbles [3], credit default swap (CDS) spread bubbles [36], and the repo market size [38]. Most of the published research papers on the JLS model have focused on the existence of log-periodic fluctuations by fitting LPPL equation to the data. Although some papers have included several statistical methods for the detection of log-periodicity [14, 29, 6, 39, 41, 2], only a few papers have focused on the JLS model residuals [8, 19, 12].

The aim of this paper is to propose the JLS model generalization, based on the analysis of the JLS model residuals. Specifically, we investigate the presence of ARMA/GARCH patterns in the JLS model residuals, wherein we also compare our results with the log-periodic-AR(1)-GARCH(1,1) specification, proposed in [8]. According to ARMA/GARCH model determination, we examine the adequacy of the JLS model. In doing so, we compare the critical time parameter estimates, calculated with the JLS model, versus those calculated with generalized JLS model. We explore whether the generalized JLS model improves the JLS model estimates of the critical time parameters. To assure the generality of the results, we perform an analysis on large number of data samples.

The rest of the paper is organised as follows. Section 2 describes derivation of the

JLS model, together with the q-state Potts model and growth functions on hierarchical diamond lattice. In this section we also present details of the JLS model estimation, optimisation and verification. Our proposed generalization of the JLS model, methodology for the ARMA/GARCH model determination and parameter estimation are described in section 3. The data, empirical results of our analysis and main contributions of this paper are reported in section 4. Section 5 concludes the paper.

2 The Johansen–Ledoit–Sornette model

2.1 Motivation

Financial markets consist of numerous interacting traders, that differ in size from small individuals to large institutional traders, such as pension funds. Moreover, all traders worldwide are organised inside a social network (family, friends, etc), within which they locally influence each other. The structure of financial markets resembles to hierarchical systems with traders on all different levels of the market.

To develop the Johansen–Ledoit–Sornette (JLS) model, a model of rational imitation, Johansen *et al.* [14] used hierarchical diamond lattice representation for the structure of financial markets. In the case of hierarchical diamond lattice discussed by Berker and Ostlund [1], the lattice is generated in an iterative manner as shown in Figure 1. This is quite realistic model of complicated network of interactions between traders. The model is derived using the q-state Potts model on hierarchical diamond lattice defined in [5], where free energy exhibits log-periodic oscillations as the critical point is approached. For more details, see [13].

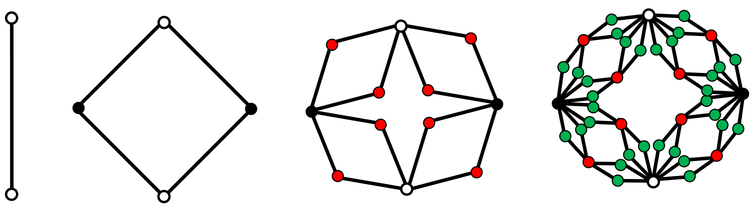

Figure 1: First few steps of building a hierarchical diamond lattice.

2.2 The q-state Potts model on hierarchical diamond lattice

Let G be a graph and consider a set $\{1, 2, \dots, q\}$ of q elements, called *spins*. A *state* of a graph G is an assignment of a single spin to each vertex of the graph. Denote by $V(G) = \{v_1, \dots, v_n\}$ vertex set of G and by $E(G)$ edge set of G. Then the state of G is a function $\sigma : V(G) \to \{1, 2, \dots, q\}$. Let $S(G)$ denote the set of states of G.

The *interaction energy* may be thought of simply as weights on the edges of the graph G. Denote by $J_e = J_{i,j} = J_{v_i,v_j}$ interaction energy on an edge $e = \{v_i, v_j\}$. Then the *Hamiltonian* is

$$h(\sigma) = - \sum_{\{i,j\} \in E(G)} J_{i,j} \delta(\sigma_i, \sigma_j),$$

where σ is a state of graph G, σ_i is the spin at vertex v_i, δ is Kronecker delta function and each edge $\{i, j\}$ is assigned an interaction energy $J_{i,j}$.

Let the spins be positioned on a hierarchical diamond lattice constructed by the iterative process shown in Figure 1. Denote by G_n graph on the nth level of hierarchical diamond lattice. Then the q-state Potts model *partition function* on the nth level is defined as

$$Z_n\left(G_n\right) = \sum_{\sigma \in S(G_n)} e^{-\beta h(\sigma)}.$$

The Potts model partition function is the sum of all possible states of an exponential function of the Hamiltonian.

There exists a connection between the Potts model, which is useful to study phase transitions and critical phenomena in physics, and the graph theory, for example the Tutte polynomial [35] or Chromatic polynomial [28]. The graph theory is mathematical area, useful to describe and study the relations between participants in networks, such as physical, biological, social or economic networks. For some basic concepts used in models of economic networks, see [17].

Figure 1 depicts the first few steps of the *diamond lattice operation*, denoted by Di, where the original graph is a single edge. The graphs in the Figure are $G_0 = K_2, G_1 = Di(G_0), G_2 = Di(G_1), G_3 = Di(G_2)$. The original graph is planar, so is each next graph in the sequence. Di can be applied on any map (graph embedded on a surface). Note that Di is a composite operation. In [7, 24] several operations on maps are considered. One, Pa, *parallelization* replaces each edge by a pair of parallel edges and another one $Su1$, *one-dimensional subdivision* subdivides each edge of the original map. In this way $Di(M) = Su1(Pa(M))$. Note that Di may be regarded as an operation on map or due to its simplicity also as an operation on the underlying graph. A theory of representations of graphs and maps has been laid down by Pisanski and Žitnik [26], where such operations were considered. Repeated operations were used in other contexts, see for instance [25]. Operations on maps have been studied in connection with symmetry in several papers, see for instance [23, 22, 4, 10].

2.3 Growth function in rooted diamond lattice

Recall the definition of growth function in rooted graphs [25]. Let G be a connected, finite or locally finite (infinite graph with finite vertex degrees) graph. Let $V(G)$ be the set of vertices of G and let $r \in V(G)$ be the so-called *root* of graph G. The distance $d(u, v)$ between $u, v \in V(G)$ is defined as number of edges on the shortest path between u and v. Further, we define the (*spherical*) *growth sequence* as $\{\delta(G, r, n) | n = 0, 1, 2, \dots\}$, where $\delta(G, r, n)$ denote the number of vertices at distance n from root r in graph G. Then the (*spherical*) *growth function* of graph G rooted at r can be written as follows:

$$f(G, r, x) = \sum_{n=0}^{\infty} \delta(G, r, n) x^n,$$

i.e. the generating function for $\delta(G, r, n)$ of graph G at root r. Here we limit to the case when the root is single vertex, *vertex root*, although we can extend the definition by allowing that a root is any induced subgraph of graph G.

We start the construction of hierarchical diamond lattice with a graph $G_0 = K_2$. Its growth function is $f(G_0, r, x) = 1 + x$ and is independent of the selected root r. Using the diamond lattice operation on graph G_0 we get graph $G_1 = Di(G_0)$, for which

$f(G_1, r, x) = 1 + 2x + x^2$ is also independent of r. If we use diamond lattice operation at least two times, we have to calculate each of the growth functions separately depending on the root. Here we first limit ourselves to the case of selecting one of the two vertices at the top and bottom in each graph G_n, called *main root vertex*, denoted by r_m (white vertices of graph G_n on Figure 1). Note that since each diamond lattice G_n can be mirrored horizontally (vertically for $n > 0$), horizontally (vertically) symmetrically selected roots produce an identical growth function. In the next subsection we examine the cases of selecting vertex root different from r_m in G_n for $n = 1, 2, 3, 4$. The results for all possible growth functions of a graph G_n for $n > 4$ will be published elsewhere.

Let us present the growth function of graph G_n rooted at r_m for $n = 2$ and $n = 3$:

$$\begin{aligned} f(G_2, r_m, x) &= 1 + 4x + 2x^2 + 4x^3 + x^4 \\ f(G_3, r_m, x) &= 1 + 8x + 4x^2 + 8x^3 + 2x^4 + 8x^5 + 4x^6 + 8x^7 + x^8. \end{aligned}$$

If we denote the growth function of graph G_n rooted at r_m by $g_n(x)$, we can write the system of equations that determine the growth functions $f(G_n, r_m, x)$ as follows:

$$\begin{aligned} g_0(x) &= (1 + x) \\ g_1(x) &= (1 + x)(2g_0(x) - (1 + x)) \\ g_2(x) &= (1 + x^2)(2g_1(x) - (1 + x^2)) \\ g_3(x) &= (1 + x^4)(2g_2(x) - (1 + x^4)) \\ &\dots \\ g_n(x) &= (1 + x^{2^{n-1}})(2g_{n-1}(x) - (1 + x^{2^{n-1}})). \end{aligned} \tag{2.1}$$

Moreover, if we define the function

$$f_0(x) = 1 \quad \text{and} \quad f_n(x) = \prod_{i=1}^{n} (1 + x^{2^i}) \quad \text{for} \quad n > 0, \tag{2.2}$$

we can write the recursion of the growth function $f(G_n, r_m, x)$ also as:

$$g_n(x) = g_{n-1}(x^2) + 2^n x f_{n-1}(x).$$

To derive the recursion for the list of coefficients of g_n (i.e. for the growth sequence of graph G_n rooted at r_m), we first define some operations on lists. Let $w = \{w_1, w_2, \dots, w_m\}$ and $\bar{w} = \{\bar{w}_1, \bar{w}_2, \dots, \bar{w}_n\}$ be two lists of coefficients. Let $a^{(k)} = \{a, a, \dots, a\}$ denote the list of k repetitions of the value a. Then we define the following operations on the lists:

$$\begin{aligned} t(w, \bar{w}) &= w + \bar{w} = \{w_1, w_2, \dots, w_m, \bar{w}_1, \bar{w}_2, \dots, \bar{w}_n\} \\ m(w, k) &= w \cdot k = \{w_1 k, w_2 k, \dots, w_m k\} \\ v(w, a) &= w * a^{(m-1)} = \{w_1, a, w_2, a, \dots, w_{m-1}, a, w_m\} \\ b(w) &= w[-1] = \{w_2, w_3, \dots, w_m\} \\ c(w) &= w[-1, m] = \{w_2, w_3, \dots, w_{m-1}\}. \end{aligned}$$

Let a_n denote the list of coefficients of g_n. Then we derive the following rule for generating the list of coefficients for g_n from the list of coefficients for g_{n-1} and g_0:

$$\begin{aligned} a_0 &= \{1, 1\} \\ a_n &= t(b(a_0), t(m(t(b(a_{n-1}), c(a_{n-1})), 2), b(a_0))). \end{aligned} \tag{2.3}$$

Alternatively, the recursion in (2.3) can be written as follows:

$$a_n \quad = \quad v(a_{n-1}, 2^n). \tag{2.4}$$

The first four lists a_n of coefficients for g_n are:

$$\begin{aligned} a_0 &= \{1,1\} \\ a_1 &= \{1,2,1\} \\ a_2 &= \{1,4,2,4,1\} \\ a_3 &= \{1,8,4,8,2,8,4,8,1\} \end{aligned}$$

At each step we get a list a_n of coefficients for g_n of length $2^n + 1$.

2.4 Different growth functions of G_n for $n = 1, 2, 3, 4$

In the previous subsection we examined the growth function in diamond lattice G_n rooted at r_m. Here we examine the number of different growth functions in G_n and calculate the growth functions separately depending on the root r different from r_m in G_n for $n = 1, 2, 3, 4$ (the graph G_0 has only two main root vertices).

For a graph G and $v \in V(G)$, the *degree* of a vertex v is denoted by $d_G(v)$ and the *degree set* of a graph G (i.e. the set of (distinct) degrees of the vertices of G) by $D(G)$, where we write the degree set in ascending order. Then, we define the set $VD(G) = \{p_{k_i}(G) | k_i \in D(G), i = 1, 2, \ldots, |D(G)|\}$, where $p_k(G)$ denotes the number of vertices of a degree k in G. Let A be the automorphism group of G that partitions the vertex set $V(G)$ into orbits:

$$V(G) = [v_1] \cup [v_2] \cup \cdots \cup [v_s],$$

where the number of orbits of G is $|Orb(G)| = s$.

If u and v belong to the same orbit of G_n, then $f(G_n, u, x) = f(G_n, v, x)$. We provide in Table 1 a number of different growth functions in G_n for $n = 1, 2, 3, 4$. Additionally, we provide the number of vertices and edges, degree set and set of number of vertices with the same degree.

n	$\lvert V(G_n)\rvert$	$\lvert E(G_n)\rvert$	$D(G_n)$	$VD(G_n)$	$\lvert Orb(G_n)\rvert$
1	4	4	$\{2\}$	$\{4\}$	1
2	12	16	$\{2,4\}$	$\{8,4\}$	2
3	44	64	$\{2,4,8\}$	$\{32,8,4\}$	3
4	172	256	$\{2,4,8,16\}$	$\{128,32,8,4\}$	5

Table 1: Number of vertices and edges, ascending ordered degree set, set of number of vertices with the same degree and number of orbits in diamond lattice G_n for $n = 1, 2, 3, 4$.

It is not hard to see that

$$\begin{aligned} |V(G_n)| &= \frac{2}{3}(2 + 4^n) \\ |E(G_n)| &= 4^n \\ D(G_n) &= \left\{2^i | i = 1, 2, \ldots, n\right\} \\ VD(G_n) &= \left\{2^{2(n-i)+1} | i = 1, 2, \ldots, n-1\right\} \cup \left\{2^2\right\}. \end{aligned}$$

If we are given a root r in G_n this defines a root r in the corresponding $Di^k(G_n)$. It is not hard to see that, if $d_{G_n}(r) = 2^i$ for $i \in \{1, 2, \ldots, n\}$, then $d_{Di(G_n)}(r) = d_{G_{n+1}}(r) = 2^{i+1}$.

We now consider the cases of all possible growth functions of a graph G_n $(1 \leq n \leq 4)$ when the root r is a single vertex. First we examine root r of graph G_n which produces an identical growth function as main root vertex r_m. If $r \in V(G_n)$, $r \neq r_m$ and $d_{G_n}(r) = d_{G_n}(r_m)$, then $f(G_n, r, x) = g_n(x)$ and $p_{d_{G_n}(r_m)}(G_n) = 4$. That means only four vertices in graph G_n $(n > 1)$ produce an identical growth function $g_n(x)$ (black and white vertices on Figure 1). Next, for $n = 2, 3, 4$ the growth functions of a rooted graph G_n at r, where $d_{G_2}(r) = 2$, $d_{G_3}(r) = 4$, $d_{G_4}(r) = 8$ (red vertices on Figure 1) are:

$$\begin{aligned}
f(G_2, r, x) &= 1 + 2x + 5x^2 + 2x^3 + 2x^4 \\
f(G_3, r, x) &= 1 + 4x + 2x^2 + 12x^3 + 5x^4 + 8x^5 + 2x^6 + 8x^7 + 2x^8 \\
f(G_4, r, x) &= 1 + 8x + 4x^2 + 8x^3 + 2x^4 + 24x^5 + 12x^6 + 24x^7 + 5x^8 + 16x^9 \\
&\quad + 8x^{10} + 16x^{11} + 2x^{12} + 16x^{13} + 8x^{14} + 16x^{15} + 2x^{16}.
\end{aligned}$$

Moreover, using the polynomial $p(x) = 1 + 3x + 2x^2 + 2x^3$ and function $f_n(x)$ defined in (2.2), we can write the following system of equations that determine the growth functions of a rooted graph G_n at r, where $d_{G_n}(r) = 2^{n-1}$:

$$\begin{aligned}
f(G_3, r, x) &= f(G_2, r, x^2) + 2^2 x f_0(x) p(x^2) \\
f(G_4, r, x) &= f(G_3, r, x^2) + 2^3 x f_1(x) p(x^4) \\
&\cdots \\
f(G_n, r, x) &= f(G_{n-1}, r, x^2) + 2^{n-1} x f_{n-3}(x) p(x^{2^{n-2}}).
\end{aligned}$$

For $n = 3, 4$, the growth functions of a rooted graph G_n at r, where $d_{G_3}(r) = 2$ and $d_{G_4}(r) = 4$ (green vertices on Figure 1) are:

$$\begin{aligned}
f(G_3, r, x) &= 1 + 2x + 9x^2 + 4x^3 + 10x^4 + 3x^5 + 8x^6 + 3x^7 + 4x^8 \\
f(G_4, r, x) &= 1 + 4x + 2x^2 + 20x^3 + 9x^4 + 16x^5 + 4x^6 + 24x^7 + 10x^8 + 16x^9 \\
&\quad + 3x^{10} + 16x^{11} + 8x^{12} + 16x^{13} + 3x^{14} + 16x^{15} + 4x^{16}, \\
&= f(G_3, r, x^2) + 4xq(x^2),
\end{aligned}$$

where $q(x) = 1 + 5x + 4x^2 + 6x^3 + 4x^4 + 4x^5 + 4x^6 + 4x^7$.

Finally, we examine the root r of graph G_4 with $d_{G_4}(r) = 2$, which belongs to one of two orbits. We denote the root in the first orbit by r_1 $(ur_1 \in E(G_4)$ and $d_{G_4}(u) = 16)$ and in the second orbit by r_2 $(ur_2 \in E(G_4)$ and $d_{G_4}(u) = 8)$. Then the growth functions $f(G_n, r_i, x)$ for $n = 4$ and $i = 1, 2$ are:

$$\begin{aligned}
f(G_4, r_1, x) &= 1 + 2x + 17x^2 + 8x^3 + 18x^4 + 5x^5 + 16x^6 + 7x^7 + 20x^8 + 5x^9 \\
&\quad + 16x^{10} + 6x^{11} + 16x^{12} + 6x^{13} + 16x^{14} + 5x^{15} + 8x^{16} \\
f(G_4, r_2, x) &= 1 + 2x + 9x^2 + 4x^3 + 16x^4 + 7x^5 + 24x^6 + 9x^7 + 20x^8 + 6x^9 \\
&\quad + 16x^{10} + 5x^{11} + 16x^{12} + 5x^{13} + 16x^{14} + 8x^{15} + 8x^{16}.
\end{aligned}$$

2.5 Derivation of the JLS model

The structure of financial markets, given by hierarchical diamond lattice, can be described as follows. Start with a two linked traders. Secondly, replace this link by a diamond, where the original traders occupy two diametrically opposite vertices, and the two other vertices are occupied by two new traders. Thirdly, for each of these four links, replace it by a diamond in the same way. If we repeat this process, we get a hierarchical diamond lattice and after n iterations we have $\frac{2}{3}\left(2+4^{n}\right)$ traders and 4^{n} links among them.

Johansen *et al.* [14] constructed so-called JLS model, a non-linear model suitable for the prediction of crash time in both microscopic and macroscopic modelling. This model characterises the occurrence of a crash by the crash hazard rate $h\left(t\right)$. All traders are organised into a network and they locally influence each other through this model. The model assumes that agents tend to imitate the opinions of their nearest neighbours. The imitation process is described by the *crash hazard rate* $h\left(t\right)$ with a power law, i.e. $dh/dt = Ch^{\delta}$, where C is a positive constant and $\delta > 1$.

In the JLS model [14] we consider a network of traders, where each trader is indexed by an integer number $i = 1, \ldots, N$ and $N\left(i\right)$ denotes the set of traders who are directly connected to trader i in the network (hierarchical diamond lattice). For simplicity we consider a special case of the Potts model for $q = 2$. We assume that trader i can be in only one of two possible states at time t: the buy ($s_i = +1$) or the sell ($s_i = -1$) decision. Then the *state* of trader i is determined by the following Markov process:

$$s_i = \operatorname{sign}\left(K \sum_{j \in N(i)} s_j + \sigma\epsilon_i + G\right),$$

where the sign function, $\operatorname{sign}\left(x\right)$, is equal to $+1$ (-1) if $x > 0$ $(x < 0)$, K is positive constant, $\epsilon_i \sim N\left(0,1\right)$ is an independent and identically distributed random variable and term G is a measure of some external influence, which tends to favor state $+1$ (-1) if $G > 0$ $(G < 0)$. In this model, K governs the tendency of imitation among traders, while σ governs their idiosyncratic behaviour. If we define the average state of the market as $M = (1/I)\sum_{i=1}^{I} s_i$, the *susceptibility* of the system is defined as $\chi = \left.\frac{dE[M]}{dG}\right|_{G=0}$ and measures the sensitivity of the average state of the system to a small external influence.

Further the JLS model assumes that the crash hazard rate behaves in a similar way as the susceptibility in the neighbourhood of a critical point. By considering a hierarchical diamond lattice for the financial market, the dynamics of the crash hazard rate can be described as follows:

$$h\left(t\right) \approx B_0\left(t_c - t\right)^{m-1} + C_0\left(t_c - t\right)^{m-1}\cos\left[\omega \ln\left(t_c - t\right) - \psi\right],$$

where B_0, C_0 and ω are real number.

The dynamics of the price is described as $dp = \mu\left(t\right)p\left(t\right)dt - \kappa p\left(t\right)dj$, where $p\left(t\right)$ is the price, $\mu\left(t\right)$ is time-varying drift and j is a jump process, such that $dj = 0$ before crash and $dj = 1$ after the crash occurs at critical time t_c. The parameter κ determines the loss amplitude associated with the occurrence of a crash.

One assumption of this model is that the price $p(t)$ follows a martingale process, i.e. $E_t\left[p\left(t'\right)\right] = p\left(t\right), \quad \forall t' > t$, where $E_t\left[\cdot\right]$ represents the conditional expectation on all information available up to time t. Then we have $E_t\left[dp\right] = 0$. The dynamics of the jumps is governed by a crash hazard rate $h\left(t\right)$ and $E_t\left[dj\right] = h\left(t\right)dt$. Furthermore, the drift $\mu\left(t\right)$ is chosen so that the martingale condition is satisfied, which yields $\mu\left(t\right) = \kappa h\left(t\right)$.

Then the simplest form of a log-price dynamics up to the end of the financial bubble can be written as follows:

$$\ln\left(p_t\right) = A + B\left(t_c - t\right)^{m} + C\left(t_c - t\right)^{m}\cos\left(\omega \ln\left(t_c - t\right) - \phi\right) + u_t, \tag{2.5}$$

where p_t is the price of a stock index or some other specific asset at time t, $A = \ln(p_{t_c}) > 0$ is the logarithm of the price at the critical time t_c and $B < 0$ for a growing bubble before the crash. The critical time t_c is the end of a bubble and indicates a change to a new market regime, which could be a large crash or a change in the price growth rate. Note that t_c is the most probable time for the crash, but there also exists a nonzero probability that the bubble ends without a crash. If $C \neq 0$, then the presence of log-periodic behaviour is indicated. The exponent m lies between 0 and 1 to ensure a finite price, even at t_c. Parameter ω is the frequency of oscillations during the bubble period, while ϕ is a phase parameter and lies between 0 and 2π. The error term u_t has a zero mean and some standard deviation.

2.6 Estimation (optimisation) of the JLS model

The JLS model in equation (2.5) is described by three linear parameters, A, B and C, and four nonlinear parameters, t_c, m, ω and ϕ. For simplicity, we denote the 7-dimensional vector of these parameters by $\theta = [A, B, t_c, m, C, \omega, \phi]$. We need to find the vector θ that best fits the observed log-price time series $\{\ln(p_t)\}_{t=t_1}^{t_n}$ within the estimation time period $[t_1, t_n]$, where $t_n < t_c$. Although different measures can be used, the most common approach is to minimise the sum of the squared residuals:

$$S(\theta) = \sum_{t=t_1}^{t_n} u_t^2(\theta), \tag{2.6}$$

where u_t is the error term in the JLS model.

The minimisation of such an objective function $S(\theta)$ is not an easy task due to the presence of many local minima. Therefore, we first reduce the number of parameters by expressing three linear parameters as a function of the remaining four nonlinear parameters, as proposed in [14]. If we rewrite the equation (2.5) as $\ln(p_t) \approx A + Bf_t + Cg_t$, where $f_t = (t_c - t)^m$ and $g_t = (t_c - t)^m \cos(\omega \ln(t_c - t) - \phi)$, then we obtain the estimates of linear parameters A, B and C by using the ordinary least squares (OLS) method:

$$\sum_{t=t_1}^{t_n} \begin{pmatrix} \ln(p_t) \\ f_t \ln(p_t) \\ g_t \ln(p_t) \end{pmatrix} = \sum_{t=t_1}^{t_n} \begin{pmatrix} 1 & f_t & g_t \\ f_t & f_t^2 & f_t g_t \\ g_t & f_t g_t & g_t^2 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix}. \tag{2.7}$$

If we rewrite the system of equations (2.7) in matrix form as $\mathbf{X}^{\mathrm{T}}\mathbf{y} = (\mathbf{X}^{\mathrm{T}}\mathbf{X})\beta$, where

$$\mathbf{X} = \begin{pmatrix} 1 & f_{t_1} & g_{t_1} \\ \vdots & \vdots & \vdots \\ 1 & f_{t_n} & g_{t_n} \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} \ln(p_{t_1}) \\ \vdots \\ \ln(p_{t_n}) \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} A \\ B \\ C \end{pmatrix},$$

then the well-known solution is $\hat{\beta} = (\mathbf{X}^{\mathrm{T}}\mathbf{X})^{-1}\mathbf{X}^{\mathrm{T}}\mathbf{y}$.

By reducing the number of parameters from seven to four we decreased the complexity of the optimisation problem. However, we still need to find the global minimum in a 4-dimensional space of the objective function:

$$S_1(\theta_1) = \min_{A,B,C} S(\theta), \tag{2.8}$$

where $\theta_1 = [t_c, m, \omega, \phi]$ denotes the vector of nonlinear parameters and $S(\theta)$ is given in equation (2.6).

Many different optimisation algorithms have been proposed to estimate the JLS model [30, 14, 11, 27, 8, 18]. In this paper, we use the Differential Evolution (DE) algorithm, which is a simple and efficient population-based search heuristic developed by Storn and Price [33]. We employ the `DEoptim` function in the `R` package **DEoptim** (see [21] for more details) together with the following intervals for each parameter's optimisation:

$$t_c \in [t_n + 1, 2t_n - t_1 + 1] \quad , \quad m \in \left[10^{-5}, 1 - 10^{-5}\right] \tag{2.9}$$

$$\omega \in \left[10^{-5}, 40\right] \quad , \quad \phi \in \left[10^{-5}, 2\pi - 10^{-5}\right] \tag{2.10}$$

where t_1 and t_n are first and last data point in the estimation time period. These intervals are similar to those used in [37].

If the parameter m is too small, then we obtain a bubble with a sudden acceleration at the end, while too large an m corresponds to an almost linear non-accelerating bubble. Similarly, if the frequency ω of the oscillations is too small, then the log-periodic oscillations will be too slow, while for too large a value of ω, the oscillations are too fast. Therefore, after the estimation of the JLS model, we accept all results that satisfy the following four additional constraints:

$$B < 0, \qquad 0.1 \leq m \leq 0.8, \qquad 4 \leq \omega \leq 15 \qquad \text{and} \qquad t_c \leq 2t_n - t_1. \tag{2.11}$$

Similar bounds for m and ω were used in [29]. It is also reasonable to eliminate the results, for which the upper bound of the search space for t_c under (2.9) is attained. Therefore we use the last constraint under (2.11). For the sake of brevity, we refer to conditions (2.11) as the *LPPL conditions*.

2.7 Verification of the JLS model

Important step in building a model is determination of its quality. If the model specification is appropriate, then the residuals should behave like true stochastic components. If this component is white noise, then the residuals should behave like independent (normal) random variables, with a zero mean and some standard deviation.

To investigate the stationarity of the residuals u_t in the JLS model (2.5), we perform Phillips–Perron (PP) and Augmented Dickey–Fuller (ADF) unit root tests, where the null hypothesis is the presence of a unit root. Additionally, the Kwiatkowski–Phillips–Schmidt–Shin (KPSS) test is employed for testing the null hypothesis, which posits that the observable time series is stationary. We perform the Shapiro–Wilk test on the residuals as a formal test of normality.

We are also interested in searching for possible dependencies in the JLS model residuals. We apply a runs test to verify the independence of the residuals. Using the autocorrelation function (ACF) of the residuals and squared residuals, we can observe the presence of linear and nonlinear dependence in the residuals. However, for a mixed autoregressive moving average (ARMA) model, it is usually difficult to identify the ARMA orders p and q from these plots. Tsay and Tiao [34] proposed an extended autocorrelation function (EACF) method for model identification that is able to identify mixed $\mathrm{ARMA}(p, q)$ models, as well as pure $\mathrm{AR}(p)$ and $\mathrm{MA}(q)$ models. Details about the EACF method and our proposed algorithm for ARMA order determination can be found in supplementary file (section 1).

We first apply the EACF method on the JLS model residuals to select the appropriate ARMA model. Then we compare this result with two of the information criteria, namely Akaike's Information Criterion (AIC), given by $\mathrm{AIC} = \log\left(\hat{\sigma}^2\right) + \frac{2k}{n}$, and Schwarz Bayesian Information Criterion (BIC), given by $\mathrm{BIC} = \log\left(\hat{\sigma}^2\right) + \frac{k}{n}\log\left(n\right)$, where k is the number of estimated parameters, $\hat{\sigma}^2$ is the estimated error variance of the fitted model, and n is the number of observations. We estimate the parameters of all $\mathrm{ARMA}(p,q)$ models with $0 \leq p, q \leq 5$ and select the ARMA orders by minimising AIC and BIC, respectively. For this purpose, we employ an `arima` function in the `R` package **TSA**.

3 The JLS model with ARMA/GARCH errors

Gazola *et al.* [8] proposed a log-periodic-AR(1)-GARCH(1,1) model to capture the structure of the JLS model residuals. According to the analysis of residuals described in subsection 2.7, our empirical results on different stock indices show similar results, namely, that the residuals are strongly autocorrelated and in many cases also heteroscedastic. Furthermore, by using the EACF method we also found that an ARMA model is sometimes more appropriate than a pure AR model. Consequently, to capture the behaviour of the error term u_t in (2.5), we propose the following $\mathrm{ARMA}(p,q)/\mathrm{GARCH}(P,Q)$ error model:

$$u_t = \sum_{i=1}^{p} \rho_i u_{t-i} + \eta_t + \sum_{j=1}^{q} \theta_j \eta_{t-j}, \tag{3.1}$$

$$\eta_t = \sigma_t \epsilon_t, \tag{3.2}$$

$$\sigma_t^2 = \alpha_0 + \sum_{k=1}^{P} \alpha_k \eta_{t-k}^2 + \sum_{l=1}^{Q} \beta_l \sigma_{t-l}^2, \tag{3.3}$$

where ϵ_t is independent and identically distributed process with a zero mean and unit variance. If $q = 0$ in equation (3.1), we have a pure $\mathrm{AR}(p)$ process and if $p = 0$ we have a pure $\mathrm{MA}(q)$ process. We also include the conditional variance σ_t^2, which evolves according to a $\mathrm{GARCH}(P,Q)$ process described by equation (3.3). If $P = Q = 0$ in equation (3.3), we have a constant variance and obtain a pure $\mathrm{ARMA}(p,q)$ error model.

3.1 Estimation of the JLS model with ARMA/GARCH errors

Let us denote by $\mathbf{\Theta} = [A, B, t_c, m, C, \omega, \phi, \rho_i, \theta_j]$ the vector of the parameters of the JLS model with $\mathrm{ARMA}(p,q)$ errors and by $\mathbf{\Theta}_1 = [A, B, t_c, m, C, \omega, \phi, \rho_i, \theta_j, \alpha_k, \beta_l]$, the vector of the parameters of the JLS model with $\mathrm{ARMA}(p,q)/\mathrm{GARCH}(P,Q)$ errors, where $i = 1, \ldots, p$, $j = 1, \ldots, q$, $k = 0, \ldots, P$, and $l = 1, \ldots, Q$. In our empirical analysis, we use the following three-step procedure for model identification and estimation of the JLS model with ARMA/GARCH errors:

1. Estimate the JLS model (2.5), as described in subsection 2.6 and verify that the LPPL conditions under (2.11) are satisfied.
2. Identify the ARMA orders with EACF method (compare with AIC and BIC) as described in subsection 2.7 and estimate the JLS model with $\mathrm{ARMA}(p,q)$ errors. Using the conditional maximum likelihood method, under the normality assumption for ϵ_t in equation (3.2), the estimates of parameters $\mathbf{\Theta}$ are obtained by the maximization of the log-likelihood function:

$$\mathcal{L}\left(\mathbf{\Theta}\right) = -\frac{n-p}{2}\log(2\pi) - \frac{n-p}{2}\log(\sigma^2) - \sum_{t=p+1}^{n} \frac{\eta_t^2}{2\sigma^2}, \tag{3.4}$$

where η_t^2 and $\sigma^2 = \sigma_t^2$ are obtained with equation (3.2), and n is the length of the residuals series.

3. Perform the Engle's Lagrange Multiplier (LM) test (for lags 10, 12 and 20) to verify the presence of autoregressive conditional heteroscedasticity (i.e. ARCH effect) in the residuals from the estimated model in the previous step. If necessary, identify the GARCH orders with EACF method (compare with AIC and BIC) on the squared and absolute residuals. The maximum likelihood estimation for the JLS model with $\mathrm{ARMA}(p,q)/\mathrm{GARCH}(P,Q)$ errors can be carried out by maximizing the log-likelihood function:

$$\mathcal{L}\left(\mathbf{\Theta}_1\right) = -\frac{n-p}{2}\log(2\pi) - \frac{1}{2}\sum_{t=p+1}^{n}\log(\sigma_t^2) - \frac{1}{2}\sum_{t=p+1}^{n}\frac{\eta_t^2}{\sigma_t^2}, \tag{3.5}$$

where η_t^2 and σ_t^2 are obtained with equation (3.2), and n is the length of the residuals series.

For simplicity, if there is an ARCH effect in the residuals, we incorporate a simple $\mathrm{GARCH}(1,1)$ model. That means, we are not interested in particular GARCH order, but only in the existence of GARCH structure in the residuals. Therefore, we propose new algorithm for determination if there exist an ARCH effect; see section 1 in supplementary file. Some additional comments on described three-step procedure can be found in supplementary file (section 2).

3.2 Verification of the JLS model with ARMA/GARCH errors

If our proposed model specification is appropriate, than the standardized residuals $\hat{\epsilon}_t = \hat{\eta}_t/\hat{\sigma}_t$ are approximately independent and identically distributed. Therefore, we proceed with the analysis of the standardized residuals from the fitted JLS model with ARMA errors (the model estimated in step 2 of three-step procedure described in previous subsection), if the results in step 3 show that there is no need to incorporate a GARCH process. Otherwise we perform the analysis of the standardized residuals from the fitted JLS model with ARMA/GARCH errors. The standardized residuals are investigated using the same set of tests as described in subsection 2.7.

4 Empirical results

4.1 The data

The dataset consists of daily closing prices for 33 major stock indices worldwide, namely 10 American, 12 European and 11 Asian/Pacific indices. Detailed information about the dataset can be found in supplementary file (section 3). The data were downloaded from Yahoo! finance to the 28th July 2014. According to the available data, we selected stock indices that represent trading activities on main stock exchanges in each geographic region to cover. We did so to cover adequate volume of trading activity in order to assure the generality of our results.

4.2 The JLS model estimation

We apply the procedure as described in section 3 to estimate the JLS model parameters. For all stock indices in the dataset, we generate a set of time windows, each consisting of

500 successive trading days (which is approximately two calendar years). Note, the JLS model is usually estimated on 2-3 years large time window. Each set is obtained by rolling such time windows over the whole dataset with increment of 1 day.

To summarize, the results of the JLS model application to rolling time windows for which the LPPL conditions, as specified in (2.11), are satisfied show, that fraction of such time windows varies between 15.14% and 2.39%. We note that such variation among stock indices reflects also different starting points of the dataset and therefore different bubbles representation across stock indices. To increase the possibility that the selected samples (for further analysis) resemble the bubble periods, we set the minimum number of successive time windows (that satisfy LPPL conditions) to 50. For more details, see section 4 in supplementary file.

4.3 The JLS model residual analysis

In this subsection we investigate the JLS model residuals u_t in (2.5) and estimate the JLS model with ARMA/GARCH errors. We perform analysis of 30 stock indices (121 selected samples). For three indices no sample is selected. Detailed information about the selected samples can be found in supplementary file (section 5).

To summarize, results shows that PP, ADF and KPSS tests indicate the stationarity for most cases at the 5% level. For each stock index, we examine any possible dependence in the JLS model residuals by performing a runs test on all time windows of selected samples. For all cases we get p-values of < 0.05, therefore we can reject the null hypothesis of independence at the 5% level. The Shapiro–Wilk test rejects the normality assumption in most cases at the 5% level. For more details, see section 6 in supplementary file.

To see how Gazola *et al.* [8] proposed model specification holds in the case of our selected data, we first investigate the presence of ARMA orders, different from $\mathrm{ARMA}(1,0)$ in selected samples. According to the EACF method the results indicate that the fraction of $\mathrm{ARMA}(1,0)$ models varies significantly across the stock indices. We found that, contrary to findings in [8], for only four stock indices the EACF method suggest only the $\mathrm{ARMA}(1,0)$ model, while for two indices it suggests only ARMA models with $p, q > 0$. In most cases the BIC method confirms the results of EACF method, while in general the selected ARMA orders with AIC method are greater compared to those selected by BIC and EACF methods. For more details, see section 7 in supplementary file.

Next, we investigate the existence of GARCH structure in the residuals. We use the following strategy to decide, whether there is no need to incorporate GARCH process: if at least one EACF result on squared or absolute residuals suggest that $P = Q = 0$ and LM test for at least two lags (10, 12 or 20) confirms that results, or LM test results cannot reject the null hypothesis at the 5% level for all three lags, then the JLS model with only ARMA errors is selected. The results show that, contrary to findings in [8], only for about third of all analysed stock indices the incorporation of GARCH is always necessary and also that the biggest proportion of such indices is in Asian/Pacific region. Here we note that the results comparison between different geographic regions is limited due to different historical data availability. For more details, see section 7 in supplementary file.

As a last step, we proceed with the analysis of the standardized residuals from our proposed generalized JLS (GJLS) model. The Shapiro–Wilk test on standardized residuals of the GJLS model shows quite similar result as JLS model residual analysis. There exists only one index for which the fraction of rejected normality assumption is smaller than 85%. Similarly, in most cases, the PP, ADF and KPSS indicate the stationarity of the standardized residuals at the 5% level. The key difference between analysis of the JLS and GJLS model residuals is obtained using a runs test. For 16 indices, performing this test yields fraction smaller than 5% of the rejected null hypothesis at the 5% level. Note that for almost all indices, where this fraction is larger than 5%, using the GJLS model we identify the false JLS model bubble identification. Detailed explanation of the false JLS model bubble identification can be found in supplementary file (section 9).

4.4 The JLS model versus GJLS model

In this subsection we compare the results of the JLS and GJLS model. In doing so, we focus on the parameter of our key interest, critical time t_c. For all 121 selected samples, we compute two location parameters for the distribution of t_c, namely the mean and the median, and 25-75% quantile interval for t_c. For some explanatory comments on the estimation methodology; see section 8 in supplementary file.

Our main results show that GJLS model in general performs better in predicting the actual local or global market peak, denoted by $t_{c,obs}$, which is consistent with findings in [8], that the GJLS model outperforms the JLS model in predicting t_c. For the American stock indices the mean or median estimate of the GJLS model is 26 times closer to $t_{c,obs}$, while the JLS model only 14 times. Similar results are also obtained for the European and Asian/Pacific stock indices; see section 9 in supplementary file.

Estimated 25-75% quantile intervals (QI) for t_c (calculated with the JLS or GJLS model) cover the $t_{c,obs}$ for more than half (sub)samples for stock indices in all three geographic regions. The results differ, however, when comparing the JLS and GJLS models; in the case of the American stock indices the coverage for the JLS model is 14 times, while for the GJLS 18 times. There are also 6 cases for each model, where the estimated QI misses the $t_{c,obs}$ for less than 11 trading days. For the European stock indices the QI coverage is 17 (25) times for the JLS (GJLS) model and the number of estimates outside the interval by less or equal to 2 weeks is 11 (12). For the Asian/Pacific stock indices the JLS (GJLS) model QI covers the $t_{c,obs}$ 5 (10) times, with 3 estimates outside the intervals for less or equal to 2 weeks for each model. For more details, see section 9 in supplementary file.

Comparing the mean and median estimates across the JLS and GJLS models for the American stock indices, the former performs better in 24 cases, while the later in 14 cases. Similar outcome is observed also for the European stock indices. For Asian/Pacific indices the mean and median perform almost equally good, but we note that for this region we have considerably lower number of (sub)samples. It therefore makes sense to take into account also the choice of the point estimate measure. For more details, see section 9 in supplementary file.

5 Conclusions

In this paper we consider the Johansen–Ledoit–Sornette (JLS) model, which describes the behaviour of stock index prices during a bubble regime and also predicts the most probable time for a change in the regime. We introduce the diamond lattice operation D_i which is a composite operation of parallelization and one-dimensional subdivision, to describe the construction of the hierarchical diamond lattice used in JLS model derivation. It turns out that the operation Di has interesting properties that we are investigating. The results will be published elsewhere.

The idea of our JLS model generalization is motivated by the behaviour of the JLS model residuals. The results of our analysis reveal the presence of a strong autocorrelation in JLS model residuals and heteroscedasticity in many cases. To incorporate these two properties into the model, we propose an $\mathrm{ARMA}(p, q)/\mathrm{GARCH}(P, Q)$ error model and a methodology for model identification, parameter estimation and its verification.

As the first part of our analysis we investigate the behaviour of the JLS model residuals. To assure the generality of the results, we perform an analysis over a large size of samples for 33 stock indices from three broader world geographic regions. Our results suggest that there is no general rule for determination of ARMA/GARCH specification of the JLS model as the proportion of ARMA models, excluding $\mathrm{ARMA}(1, 0)$ and proportions of ARMA models without GARCH specification varies significantly across the samples and also within each geographic region. We take this result as a reminder for careful ARMA/GARCH order determination, when analysing particular time period for stock indices.

In the second part of the analysis we show that specified JLS model generalization outperforms the JLS model estimates of critical time t_c. Using the mean value and the median of t_c parameter estimates, the results show smaller deviations from the actual crash dates for the GJLS model. Moreover, we also show that 25-75% quantile intervals more often cover the t_c parameter estimates of the GJLS model than for the JLS model. Further more, with GJLS model we are able to detect false alarms, meaning that there is no actual bubble period, which is otherwise identified with JLS model.

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The existence of square integer Heffter arrays

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Abstract

An integer Heffter array $H(m, n; s, t)$ is an $m \times n$ partially filled matrix with entries from the set $\{\pm 1, \pm 2, \ldots, \pm ms\}$ such that *i)* each row contains s filled cells and each column contains t filled cells, *ii)* every row and column sums to 0 (in $\mathbb{Z}$), and *iii)* no two entries agree in absolute value. Heffter arrays are useful for embedding the complete graph K_{2ms+1} on an orientable surface in such a way that each edge lies between a face bounded by an s-cycle and a face bounded by a t-cycle. In 2015, Archdeacon, Dinitz, Donovan and Yazıcı constructed square (i.e. $m = n$) integer Heffter arrays for many congruence classes. In this paper we construct square integer Heffter arrays for all the cases not found in that paper, completely solving the existence problem for square integer Heffter arrays.

Keywords: Heffter array, biembedding.

Math. Subj. Class.: 05B20, 05C10

1 Introduction

We begin with the general definition of Heffter arrays [1]. A Heffter array $H(m, n; s, t)$ is an $m \times n$ matrix with nonzero entries from $\mathbb{Z}_{2ms+1}$ such that

1. each row contains s filled cells and each column contains t filled cells,
2. the elements in every row and column sum to 0 in $\mathbb{Z}_{2ms+1}$, and
3. for every $x \in \mathbb{Z}_{2ms+1} \setminus \{0\}$, either x or $-x$ appears in the array.

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The notion of a Heffter array $H(m, n; s, t)$ was first defined by Archdeacon in [1]. It is shown there that a Heffter array with a pair of special row and column orderings can be used to construct an embedding of the complete graph K_{2ms+1} on a surface. This connection is given in the following theorem.

Theorem 1.1 ([1]). *Given a Heffter array $H(m, n; s, t)$ with compatible orderings ω_r of the symbols in the rows of the array and ω_c on the symbols in the columns of the array, then there exists an embedding of K_{2ms+1} on an orientable surface such that every edge is on a face of size s and a face of size t. Moreover, if ω_r and ω_c are both simple, then all faces are simple cycles.*

We refer the reader to [1] for the definition of a simple ordering and the definition of compatible orderings. We will not concern ourselves with the ordering problem in this paper and will concentrate on the construction of Heffter arrays. In [4] the ordering problem is addressed in more detail in the case when $m = t = 3$ and $n = s$.

A Heffter array is called an *integer* Heffter array if Condition 2 in the definition of Heffter array above is strengthened so that the elements in every row and every column sum to zero in $\mathbb{Z}$. In [2], Archdeacon et al. study the case where the Heffter array has no empty cells. They show that there is an *integer* $H(m, n; n, m)$ if and only if $m, n \geqslant 3$ and $mn \equiv 0, 3 \pmod 4$ and in general that there is an $H(m, n; n, m)$ for all $m, n \geqslant 3$.

In this paper we will concentrate on constructing *square integer* Heffter arrays with empty cells. If the Heffter array is square, then $m = n$ and necessarily $s = t$. So for the remainder of this paper define a *square integer Heffter array* $H(n; k)$ to be an $n \times n$ partially filled array of nonzero integers satisfying the following:

1. each row and each column contains k filled cells,
2. the symbols in every row and every column sum to 0 in $\mathbb{Z}$, and
3. for every element $x \in \{1, 2, \dots, nk\}$ either x or $-x$ appears in the array.

In [3] the authors study the case of square integer Heffter arrays $H(n; k)$. The following theorem is from that paper.

Theorem 1.2 ([3]). *If an $H(n; k)$ exists, then necessarily*

$$3 \leqslant k \leqslant n \quad \textit{and} \quad nk \equiv 0, 3 \pmod 4.$$

Furthermore, this condition is sufficient except possibly when $n \equiv 0$ or $3 \pmod 4$ and $k \equiv 1 \pmod 4$.

It should be noted that [3] also contains partial results when $n \equiv 0$ or $3 \pmod 4$ and $k \equiv 1 \pmod 4$. In this paper we will solve those cases completely. Our main result is given in the following theorem.

Theorem 1.3. *There exists an integer $H(n; k)$ if and only if*

$$3 \leqslant k \leqslant n \quad \textit{and} \quad nk \equiv 0, 3 \pmod 4.$$

We will prove this theorem by first constructing an $H(n; 5)$ where all the filled cells are contained on 5 diagonals. Then we will add s disjoint $H(n; 4)$ to construct $H(n; 5+4s) = H(n; k)$ where $k \equiv 1 \pmod 4$. We begin in Section 2 by giving a general construction for $H(n; 4)$ where all of the filled cells are contained in 4 diagonals. In Section 3 we discuss the case when $n \equiv 3 \pmod 4$ and in Section 4 we discuss the case when $n \equiv 0 \pmod 4$.

2 $H(n;4)$ using two sets of consecutive diagonals

An important concept in the prior work on Heffter arrays has been the notion of a shiftable Heffter array. A *shiftable* Heffter array $H_s(n;k)$ is defined to be a Heffter array $H(n;k)$ where every row and every column contain equal numbers of positive and negative entries. Let A be a shiftable array and x a nonnegative integer. If x is added to each positive element and $-x$ is added to each negative element, then all of the row and column sums remain unchanged. Let $A \pm x$ denote the array where x is added to all the *positive* entries in A and $-x$ is added to all the *negative* entries.

If A is an integer array, define the *support* of A as the set containing the absolute value of the elements contained in A. So if A is shiftable with support S and x a nonnegative integer, then $A \pm x$ has the same row and column sums as A and has support $S + x$. In the case of a shiftable Heffter array $H_s(n;k)$, the array $H_s(n;k) \pm x$ will have row and column sums equal to zero and support $S = \{1 + x, 2 + x, \ldots, nk + x\}$.

In this section we describe an easy construction of a shiftable $H(n;4)$ where all of the filled cells are contained in two pairs of adjacent diagonals. If H is an $n \times n$ array with rows and columns labeled $1, \ldots, n$, for $i = 1, \ldots, n$ define the i-th diagonal D_i to be the set of cells $D_i = \{(i,1), (i+1,2), \ldots, (i-1,n)\}$ where all arithmetic is performed in $\mathbb{Z}_n$ (using the reduced residues $\{1, 2, \ldots, n\}$). We say that the diagonals D_i and D_{i+1} are *consecutive* diagonals. We should note that in [3] there is a construction of a shiftable $H(n;4)$ for all $n \geqslant 4$ that uses 4 consecutive diagonals.

All the constructions in this paper are based on filling in the cells of a fixed collection of diagonals. To aid in these constructions we define the following procedure for filling a sequence of cells on a diagonal. It is termed $diag$ and it has six parameters.

In an $n \times n$ array A the procedure $diag(r, c, s, \Delta_1, \Delta_2, \ell)$ installs the entries

$$A[r + i\Delta_1, c + i\Delta_1] = s + i\Delta_2 \text{ for } i = 0, 1, \ldots, \ell - 1.$$

Here all arithmetic on the row and column indices is performed modulo n, where the set of reduced residues is $\{1, 2, \ldots, n\}$. The following summarizes the parameters used in the $diag$ procedure:

- r denotes the starting row,
- c denotes the starting column,
- s denotes the starting symbol,
- Δ_1 denotes how much the row and column are changed at each step,
- Δ_2 denotes how much the symbol is changed at each step, and
- ℓ is the length of the chain.

The following example shows the use of the above definition and is also an example of the construction which will be described in Theorem 2.2.

Example 2.1. A shiftable $H(11;4)$ where the filled cells are contained in two sets of consecutive diagonals.

The Heffter array $H(11;4)$ below is constructed via the following procedures:

$$\begin{aligned}&diag(4,1,1,1,2,11);\\&diag(5,1,-2,1,-2,11);\\&diag(4,7,-23,1,-2,11);\\&diag(5,7,24,1,2,11).\end{aligned}$$

		38	−39				−16	17		
			40	−41				−18	19	
				42	−43				−20	21
1					44	−23				−22
−2	3					24	−25			
	−4	5					26	−27		
		−6	7					28	−29	
			−8	9					30	−31
−33				−10	11					32
34	−35				−12	13				
	36	−37				−14	15			

We point out a few properties of the Heffter array in Example 2.1 which will be useful in the proof of the main theorem of this section. First we note that all of the filled cells are in the two pairs of consecutive diagonals $\{D_4, D_5\}$ and $\{D_9, D_{10}\}$ and that the sum of the symbols in the columns of one of the pairs of diagonals is $+1$ while the other adds to -1. Hence every column adds to 0. The rows are similar except for row 4. In this row the sum of the symbols in D_4 and D_5 is -21 while the sum of the symbols in D_9 and D_{10} is $+21$. So all the row sums are 0. It is also apparent that each row and each column contain two positive values and two negative values making this a shiftable array. Finally it is clear that the support of D_4 and D_5 is $\{1, 2, \ldots, 22\}$ while the support of D_9 and D_{10} is $\{23, 24, \ldots, 44\}$. We have thus shown that this is indeed a shiftable integer $H(11;4)$ where all the filled cells are in the two pairs of consecutive diagonals. The following is the main theorem of this section.

Theorem 2.2. *For every $n \geqslant 4$ and any two disjoint pairs of consecutive diagonals, there exists a shiftable integer Heffter array $H(n;4)$ with filled cells contained in the four diagonals.*

Proof. Assume that the two pairs of consecutive diagonals are $\{D_a, D_{a+1}\}$ and $\{D_b, D_{b+1}\}$ with $b > a+1$. We define the square H using the $diag$ procedures as in Example 2.1 above. So let H be constructed from

$$\begin{aligned}&diag(a,1,1,1,2,n),\\&diag(a+1,1,-2,1,-2,n),\\&diag(a,n+a-b+1,-2n-1,1,-2,n), \text{ and}\\&diag(a+1,n+a-b+1,2n+2,1,2,n).\end{aligned}$$

Clearly diagonal D_a is filled from the procedure $diag(a,1,1,1,2,n)$ while diagonal D_{a+1} is filled from $diag(a+1,1,-2,1,-2,n)$. We next note that a cell (i,j) gets filled from the procedure $diag(a, n + a - b + 1, -2n - 1, 1, -2, n)$ if and only if $j - i = (n + a - b + 1) - a = n - b + 1 = 1 - b \pmod{n}$. So cell $(b, 1)$ is filled from this procedure. Since $\ell = n$ in this procedure we have that every cell in D_b is filled. Similarly every cell in D_{b+1} is filled from the procedure $diag(a + 1, n + a - b + 1, 2n + 2, 1, 2, n)$.

Considering the column sums, we see that in each column the sum of the cells in D_a and D_{a+1} is -1, while the sum of the cells in D_b and D_{b+1} is $+1$. Hence the sum of the symbols in each column is 0. Similarly, if $r \neq a$, then the sum of the cells in row r in D_a and D_{a+1} is $+1$, while the sum of the cells in row r in D_b and D_{b+1} is -1. So the sum of the symbols in each row $r \neq a$ is 0. Now consider row a. The symbols from D_a,D_{a+1}, D_b and D_{b+1} are $1, -2n, -2n - 1$ and $4n$, respectively, and so the symbols in this row also add to 0.

We next check the support of H. We see that the support of $diag(a, 1, 1, 1, 2, n)$ is $\{1, 3, \ldots, 2n - 1\}$, while the support of $diag(a + 1, 1, -2, 1, -2, n)$ is $\{2, 4, \ldots, 2n\}$ so together they cover the symbols $\{1, 2, \ldots, 2n\}$. Further, we have that the support of $diag(a, n + a - b + 1, -2n - 1, 1, -2, n)$ is $\{2n + 1, 2n + 3, \ldots, 4n - 1\}$, while the support of $diag(a + 1, n + a - b + 1, 2n + 2, 1, 2, n)$ is $\{2n + 2, 2n + 4, \ldots, 4n\}$, so these two diagonals cover the symbols $\{2n + 1, 2n + 2, \ldots, 4n\}$. Hence the support of H is the required $\{1, 2, \ldots, 4n\}$. Finally it is clear from the construction that each row and each column contains two positive numbers and two negative numbers. Thus we have shown that H is indeed a shiftable integer $H(n; 4)$, as desired. □

3 H(n;k) when $n \equiv 3 \pmod{4}$ and $k \equiv 1 \pmod{4}$

In this section we first give a direct construction for $H(n; 5)$ with $n \equiv 3 \pmod{4}$ where all of the filled cells are on exactly 5 diagonals. We then use Theorem 2.2 repeatedly to construct $H(n; k)$ for all $n \equiv 3 \pmod{4}$ with $n \geqslant 7$, and all $k \equiv 1 \pmod{4}$ with $5 \leqslant k \leqslant n - 2$.

We begin with an example of the main construction of this section. Hopefully, the reader can see the type of patterns which will exist in the general case.

Example 3.1. An $H(11, 5)$.

10	53				24	−33				−54
−36	−9	44				32	−31			
	−45	−8	52				30	−29		
		−37	−7	43				28	−27	
			−46	−6	51				26	−25
−21				−38	−11	47				23
12	−13				−42	4	39			
	14	−15				−50	3	48		
		16	−17				−41	2	40	
			18	−19				−49	−5	55
35				20	−22				−34	1

Theorem 3.2. *There exists an $H(n, 5)$ for all $n \equiv 3 \pmod{4}$ with $n \geqslant 7$.*

Proof. Let $h = (n + 1)/2$ and $q = (n - 3)/4$. We construct an $n \times n$ array H using the following procedures. The procedures are labeled A to N.

A $diag(h+1, h+1, h-2, 1, -1, h-3)$;
B $diag(3, 3, -(n-3), 1, 1, h-3)$;

C $diag(2, 3, 4n, 2, -1, q)$;
D $diag(3, 2, -(4n+1), 2, -1, q)$;
E $diag(3, 4, 5n-3, 2, -1, q)$;
F $diag(4, 3, -(3n+4), 2, -1, q)$;

G $diag(h+1, h, -(4n-q), 2, 1, q)$;
H $diag(h+2, h+1, -(5n-q-3), 2, 1, q)$;
I $diag(h, h+1, 4n+q+1, 2, 1, q)$;
J $diag(h+1, h+2, 3n+q+4, 2, 1, q)$;

K $diag(h+1, 2, -(n+2), 1, -2, h-2)$;
L $diag(h+1, 1, n+1, 1, 2, h-1)$;
M $diag(1, h+1, -3n, 1, 2, h-1)$;
N $diag(2, h+1, 3n-1, 1, -2, h-2)$;

We also fill the following cells in an *ad hoc* manner.

$$\begin{array}{rrr} H[1,1]=n-1; & H[1,2]=5n-2; & H[1,h]=2n+2; \\ H[1,n]=-(5n-1); & H[2,1]=-(3n+3); & H[2,2]=-(n-2); \\ H[h,1]=-(2n-1); & H[h,h]=-n; & H[h,n]=2n+1; \\ H[n-1,n-1]=-(h-1); & H[n-1,n]=5n; & H[n,1]=3n+2; \\ H[n,h]=-2n; & H[n,n-1]=-(3n+1); & H[n,n]=1. \end{array}$$

We now prove that the array constructed by the description above is indeed an integer $H(n;5)$. To aid in the proof we give a schematic picture of where each of the diagonal procedures fills cells (see Figure 1). The first cell in each of these procedures is shaded and we have placed an X in the *ad hoc* cells. (In this picture we used $n=15$, so $h=8$ and $q=3$.)

We first check that the rows all add to 0 (in the integers).

Row 1: There are four *ad hoc* values plus the first value in diagonal M. The sum is $(n-1)+(5n-2)+(2n+2)-3n-(5n-1)=0$.

Row 2: The sum is $-(3n+3)-(n-2)+4n+(3n-1)-(3n-2)=0$.

Rows 3 to $h-1$: First notice that in all of these rows the sum of the N and the M diagonal cells is $+1$ so we must show that the sum of the three cells in the three center diagonals is -1. There are two cases depending on whether the row r is odd or even. If r is odd, then write $r=3+2k$ where $0 \leqslant k \leqslant q-1$. Notice that from the D, B and E diagonal cells we get the following sum: $-(4n+1)-k-(n-3)+$

X	X						X	M						X
X	X	C						N	M					
	D	B	E						N	M				
		F	B	C						N	M			
			D	B	E						N	M		
				F	B	C						N	M	
					D	B	E						N	M
X						F	X	I						X
L	K						G	A	J					
	L	K						H	A	I				
		L	K						G	A	J			
			L	K						H	A	I		
				L	K						G	A	J	
					L	K						H	X	X
X						L	X						X	X

Figure 1: Construction in Theorem 3.2 for $n = 15$.

$2k + (5n - 3) - k = -1$ as desired. If r is even, then write $r = 4 + 2k$ where $0 \leqslant k \leqslant q - 2$. From the F, B and C diagonal cells we get the following sum: $-(3n + 4) - k - (n - 3) + 1 + 2k + 4n - 1 - k = -1$, as desired.

Row h: There are three *ad hoc* values plus the last of the F diagonal as well as the first of the I diagonal. We get the row sum: $-(2n - 1) - n + (2n + 1) - (3n + 4) - (q - 1) + 4n + q + 1 = 0$.

Rows $h+1$ to $n-2$: Note that in all of these rows the sum of the L and the K diagonal cells is -1 so we must show that the sum of the three cells in the three center diagonals is $+1$. There are again two cases depending on whether the row r is odd or even. If r is odd, noting that h is even, we write $r = (h + 1) + 2k$ where $0 \leqslant k \leqslant q - 1$. Now, from the G, A and J diagonal cells we get the following sum: $-(4n - q) + k + (h - 2) - 2k + (3n + q + 4) + k = -n + 2q + h + 2 = 1$. If r is even, write $r = (h+2) + 2k$ where $0 \leqslant k \leqslant q - 2$. From the H, A and I diagonal cells we get the following sum: $-(5n - q - 3) + k + (h - 3) - 2k + (4n + q + 2) + k = -n + 2q + h + 2 = 1$.

Row $n - 1$: We add the values in diagonals L, K and H with two *ad hoc* values to get: $(n + 1) + 2(h - 3) - (n + 2) - 2(h - 3) - (5n - q - 3) + (q - 1) - (h - 1) + 5n = -h + 2q + 2 = 0$.

Row n: The sum is $(3n+2) - 2n - (3n+1) + 1 + (n+1) + 2(h-2) = -n + 2h - 1 = 0$.

So all rows add to zero. Next we check that the columns also all add to zero.

Column 1: There are four *ad hoc* values plus the first value in diagonal L. The sum is $(n - 1) - (3n + 3) - (2n - 1) + (n + 1) + (3n + 2) = 0$.

Column 2: The sum is $5n - 2 - (n - 2) - (4n + 1) - (n + 2) + (n + 1) + 2 = 0$.

Columns 3 to $h-1$: Note that in all of these columns the sum of the L and the K diagonal cells is $+1$ so we must show that the sum of the three cells in the three center diagonals is -1. There are two cases depending on whether the column c is odd or even. If c is odd, then write $c = 3+2k$ where $0 \leqslant k \leqslant q-1$. From the C, B and F diagonal cells we get the following sum: $4n - k - (n-3) + 2k - (3n+4) - k = -1$. If c is even, then write $c = 4+2k$ where $0 \leqslant k \leqslant q-2$. From the E, B and D diagonal cells we get the following sum: $(5n-3) - k - (n-4) + 2k - (4n+1) - 1 - k = -1$, as desired.

Column h: There are three *ad hoc* values plus the last of the E diagonal as well as the first of the G diagonal. We get $(2n+2) - n - 2n + (5n-3) - (q-1) - (4n-q) = 0$.

Columns $h+1$ to $n-2$: In all of these columns the sum of the M and the N diagonal cells is -1, so we must show that the sum of the three cells in the three center diagonals is $+1$. There are again two cases depending on whether the column c is odd or even. If c is odd, noting that h is even, we write $c = (h+1) + 2k$ where $0 \leqslant k \leqslant q-1$. Now, from the I, A and H diagonal cells we get the following sum: $(4n+q+1) + k + (h-2) - 2k - (5n-q-3) + k = -n + 2q + h + 2 = 1$. If c is even, write $c = (h+2) + 2k$ where $0 \leqslant k \leqslant q-2$. From the J, A and G diagonal cells we get the following sum: $(3n+q+4) + k + (h-3) - 2k - (4n-q) + 1 + k = -n + 2q + h + 2 = 1$.

Column $n-1$: We add the values in diagonals M, N and J with two *ad hoc* values to get: $(-3n) + 2(h-3) + (3n-1) - 2(h-3) + (3n+q+4) + q - 1 - (h-1) - (3n+1) = 2q - h + 2 = 0$.

Column n: The sum is $-5n + 1 - 3n + 2(h-2) + 2n + 1 + 5n + 1 = -n + 2h - 1 = 0$.

So we have shown that all column sums are zero. Next we consider the support of H. We do this by looking at the elements used in each of the diagonals as well as the *ad hoc* symbols. We will write $[u, v](\text{W})$ if the elements in diagonal W consist of the integers in the closed interval $[u, v]$ and we give the *ad hoc* symbols individually. Note that we write all the numbers in terms of the value q (where $4q + 3 = n$). The support of H is:

$$\begin{gathered}\{1, [2, 2q](\text{A}), 2q+1, [2q+2, 4q](\text{B}), 4q+1, 4q+2, 4q+3, [4q+4, 8q+4](\text{K} \cup \text{L}),\\ 8q+5, 8q+6, 8q+7, 8q+8, [8q+9, 12q+9](\text{M} \cup \text{N}), 12q+10, 12q+11, 12q+12,\\ [12q+13, 13q+12](\text{F}), [13q+13, 14q+12](\text{J}), [14q+13, 15q+12](\text{G}),\\ [15q+13, 16q+12](\text{C}), [16q+13, 17q+12](\text{D}), [17q+13, 18q+12](\text{I}),\\ [18q+13, 19q+12](\text{H}), [19q+13, 20q+12](\text{E}), 20q+13, 20q+14, 20q+15\}\\ \hfill = [1, 20q+15] = [1, 5n].\end{gathered}$$

We have shown that H is indeed an integer Heffter array $H(n; 5)$. □

We are now ready to prove the main theorem of this section. Let $k = 5 + 4s$. To construct an $H(n; k)$ we start with the $H(n; 5)$ constructed in Theorem 3.2 and add s disjoint $H(n; 4)$ (with the symbols shifted accordingly), as constructed in Theorem 2.2. The details are given in the following theorem.

Theorem 3.3. *There exists an integer Heffter array $H(n; k)$ for all $n \equiv 3 \pmod 4$ and $k \equiv 1 \pmod 4$ with $n \geqslant 7$ and $5 \leqslant k \leqslant n-2$.*

Proof. Again let $h = (n+1)/2$, noting that h is necessarily even, and let $k = 5+4s$. When $s = 0$ we are done by Theorem 3.2. So we assume that $s \geqslant 1$, and hence that $4 \leqslant 4s \leqslant n-7$. Begin with $H = H(n;5)$ constructed in Theorem 3.2. We place s (shifted) $H(n;4)$ constructed in Theorem 2.2 in $4s$ empty diagonals of H. These empty diagonals will come in pairs of consecutive diagonals. Specifically, for each $0 \leqslant t \leqslant s-1$ place $H_t = H(n;4) \pm (5n+4nt)$ on the 4 diagonals $D_{3+2t}, D_{4+2t}, D_{h+2+2t}$, and D_{h+3+2t}.

A few things need to be checked. The filled diagonals in H are D_1, D_2, D_h, D_{h+1}, and D_n. The diagonals that get filled with the H_t's are $D_3, D_4, \ldots, D_{1+2s}, D_{2+2s}$ and $D_{h+2}, D_{h+3}, \ldots, D_{h+2s}, D_{h+2s+1}$. Since $4s \leqslant n-7$, then $2s+2 \leqslant h-2$ and also $h+2s+1 \leqslant n-2$. So the filled diagonals in $H, H_1, H_2, \ldots, H_s$ are all disjoint.

The row and column sums in H as well as in each H_t, $0 \leqslant t \leqslant s-1$ is zero, hence the resulting array has row and column sum zero. Finally, note that the support of H is $[1, 5n]$ and for each H_t the support is $[5n+4nt+1, 5n+4nt+4n] = [5n+4nt+1, 9n+4nt]$. So the support in the final array is

$$[1,5n] \cup \bigcup_{t=0}^{s-1} [5n+4nt+1, 9n+4nt]$$
$$= [1,5n] \cup [5n+1, 9n] \cup [9n+1, 13n] \cup \cdots \cup [5n+4n(s-1)+1, 9n+4n(s-1)].$$

Since $9n + 4n(s-1) = n(5+4s) = nk$, the support is $[1, nk]$, completing the proof. □

4 H(n;k) when $n \equiv 0 \pmod 4$ and $k \equiv 1 \pmod 4$

This section follows the same structure as Section 3. We first give a direct construction for $H(n;5)$ with $n \equiv 0 \pmod 4$ where all of the filled cells are on exactly 5 diagonals. We then use Theorem 2.2 repeatedly to construct $H(n;k)$ for all $n \equiv 0 \pmod 4$ with $n \geqslant 8$, and all $k \equiv 1 \pmod 4$ with $5 \leqslant k \leqslant n-3$. We again begin with an example of the main construction of this section.

Example 4.1. An $H(16,5)$.

46	−14	62						−49	−45						
	−63	−13	75						44	−43					
		−50	−12	61						42	−41				
			−64	−11	74						40	−39			
				−51	−10	60						38	−37		
					−65	−9	73						36	−35	
						−52	−8	59						34	−33
−30							−66	77	67						−48
15	−16							−58	6	53					
	17	−18							−72	5	68				
		19	−20							−57	4	54			
			21	−22							−71	3	69		
				23	−24							−56	2	55	
					25	−26							−70	−7	78
−32						27	80							−47	−28
1	76						−79	−29							31

Theorem 4.2. *There exists an* $H(n, 5)$ *for all* $n \equiv 0 \pmod 4$ *with* $n \geqslant 8$.

Proof. Let $h = n/2$ and $q = n/4$. We construct an $n \times n$ array H using the following procedures. The procedures are labeled A to N.

A $\quad diag(h+1, h+2, h-2, 1, -1, h-3)$;
B $\quad diag(1, 2, -(n-2), 1, 1, h-1)$;

C $\quad diag(1, 3, 4n-2, 2, -1, q)$;
D $\quad diag(2, 2, -(4n-1), 2, -1, q)$;
E $\quad diag(2, 4, 5n-5, 2, -1, q-1)$;
F $\quad diag(3, 3, -(3n+2), 2, -1, q-1)$;

G $\quad diag(h+1, h+1, -(4n-q-2), 2, 1, q-1)$;
H $\quad diag(h+2, h+2, -(5n-q-4), 2, 1, q-1)$;
I $\quad diag(h, h+2, 4n+q-1, 2, 1, q-1)$;
J $\quad diag(h+1, h+3, 3n+q+1, 2, 1, q-1)$;

K $\quad diag(h+1, 1, n-1, 1, 2, h-1)$;
L $\quad diag(h+1, 2, -n, 1, -2, h-2)$;
M $\quad diag(2, h+2, 3n-4, 1, -2, h-2)$;
N $\quad diag(1, h+2, -(3n-3), 1, 2, h-1)$;

We also fill the following cells in an *ad hoc* manner.

$$H[1,1] = 3n-2; \qquad H[1,h+1] = -(3n+1);$$
$$H[h,1] = -(2n-2); \qquad H[h,h+1] = 5n-3;$$
$$H[h,n] = -3n; \qquad H[n-2,n-1] = -(h-1);$$
$$H[n-2,n] = 5n-2; \qquad H[n-1,1] = -2n;$$
$$H[n-1,h] = 5n; \qquad H[n-1,n-1] = -(3n-1);$$
$$H[n-1,n] = -(2n-4); \qquad H[n,1] = 1;$$
$$H[n,2] = 5n-4; \qquad H[n,h] = -(5n-1);$$
$$H[n,h+1] = -(2n-3); \qquad H[n,n] = 2n-1.$$

We now prove that the array constructed by the description above is indeed an integer $H(n; 5)$. To aid in the proof we again give a schematic picture of where each of the diagonal procedures fills cells (see Figure 2). The first cell in each of these procedures is shaded and we have placed an X in the *ad hoc* cells. (In this picture we used $n = 16$, so $h = 8$ and $q = 4$.)

We first check that the rows all add to zero.

Row 1: There are two *ad hoc* values plus the first value in diagonals B, C and N. The sum is $(3n-2) - (n-2) + (4n-2) - (3n+1) - (3n-3) = 0$.

X	B	C						X	N						
	D	B	E						M	N					
		F	B	C						M	N				
			D	B	E						M	N			
				F	B	C						M	N		
					D	B	E						M	N	
						F	B	C						M	N
X							D	X	I						X
K	L							G	A	J					
	K	L							H	A	I				
		K	L							G	A	J			
			K	L							H	A	I		
				K	L							G	A	J	
					K	L							H	X	X
X						K	X							X	X
X	X						X	X							X

Figure 2: Construction in Theorem 4.2 for $n = 16$.

Rows 2 to $h - 1$: In all of these rows the sum of the N and the M diagonal cells is $+1$ so we must show that the sum of the three cells in the three center diagonals is -1. There are two cases depending on whether the row r is odd or even. If r is even, then write $r = 2 + 2k$ where $0 \leqslant k \leqslant q - 2$. From the D, B and E diagonal cells we get the following sum: $-(4n-1) - k - (n-3) + 2k + (5n-5) - k = -1$. If r is odd, then write $r = 3 + 2k$ where $0 \leqslant k \leqslant q - 2$. Notice that from the F, B and C diagonal cells we get the following sum: $-(3n+2) - k - (n-4) + 2k + (4n-3) - k = -1$ as desired.

Row h: The sum is: $-(2n-2) - (4n-1) - (q-1) + (5n-3) + (4n+q-1) - 3n = 0$.

Row $h+1$ to $n-3$: Note that in all of these rows the sum of the L and the K diagonal cells is -1 so we must show that the sum of the three cells in the three center diagonals is $+1$. There are again two cases depending on whether the row r is odd or even. If r is odd, noting that h is even, we write $r = (h+1) + 2k$ where $0 \leqslant k \leqslant q - 2$. Now, from the G, A and J diagonal cells we get the following sum: $-(4n - q - 2) + k + (h-2) - 2k + (3n+q+1) + k = -n + 2q + h + 1 = 1$, as desired. If r is even, write $r = (h+2) + 2k$ where $0 \leqslant k \leqslant q - 3$. From the H, A and I diagonal cells we get the following sum: $-(5n-q-4) + k + (h-3) - 2k + (4n+q) + k = -n + 2q + h + 1 = 1$.

Row $n-2$: The sum is: $(n-1) + 2(h-3) - n - 2(h-3) - (5n-q-4) + (q-2) - h + 1 + (5n-2) = 2q - h = 0$.

Row $n-1$: The sum is: $-2n + (n-1) + 2(h-2) + 5n - (3n-1) - (2n-4) = -n + 2h = 0$.

Row n: The sum is: $1 + (5n-4) - (5n-1) - (2n-3) + (2n-1) = 0$.

So all rows add to zero. Next we check that the columns also all add to zero.

Column 1: There are four *ad hoc* values plus the first value in diagonal K. The sum is $(3n-2) - (2n-2) + (n-1) - 2n + 1 = 0$.

Column 2: The sum is: $-(n-2)-(4n-1)-n+(n+1)+(5n-4)=0$.

Columns 3 to $h-1$: Note that in all of these columns the sum of the L and the K diagonal cells is $+1$, so we must show that the sum of the three cells in the three center diagonals is -1. There are two cases depending on whether the column c is odd or even. If c is odd, then write $c=3+2k$ where $0\leqslant k\leqslant q-2$. From the C, B and F diagonal cells we get the following sum: $(4n-2)-k-(n-3)+2k-(3n+2)-k=-1$. If c is even, then write $c=4+2k$ where $0\leqslant k\leqslant q-3$. From the E, B and D diagonal cells we get the following sum: $(5n-5)-k-(n-4)+2k-4n-k=-1$, as desired.

Column h: There are two *ad hoc* values plus the last of the E, B and D diagonals. We get $5n-(5n-1)+(5n-5)-(q-2)-(n-2)+(h-2)-(4n-1)-(q-1)=0$.

Column $h+1$: The sum is: $-(3n+1)+(4n-2)-(q-1)+(5n-3)-(4n-q-2)-(2n-3)=0$.

Columns $h+2$ to $n-2$: In all of these columns the sum of the M and the N diagonal cells is -1, so we must show that the sum of the three cells in the three center diagonals is $+1$. There are again two cases depending on whether the column c is odd or even. If c is even, noting that h is even, write $c=(h+2)+2k$ where $0\leqslant k\leqslant q-2$. From the I, A and H diagonal cells we get the following sum: $(4n+q-1)+k+(h-2)-2k-(5n-q-4)+k=-n+h+2q+1=1$. If c is odd, we write $c=(h+3)+2k$ where $0\leqslant k\leqslant q-3$. Now, from the J, A and G diagonal cells we get the following sum: $(3n+q+1)+k+(h-3)-2k-(4n-q-3)+k=-n+2q+h+1=1$.

Column $n-1$: The sum is: $-(3n-3)+2(h-3)+(3n-4)-2(h-3)+(3n+q+1)+(q-2)-(h-1)-(3n-1)=2q-h=0$.

Column n: The sum is: $-(3n-3)+2(h-2)-3n+(5n-2)-(2n-4)+(2n-1)=-n+2h=0$.

So we have shown that all column sums are zero. Next we consider the support of H. We do this by looking at the elements used in each of the diagonals as well as the *ad hoc* symbols used. We again write $[u,v](\text{W})$ if the elements in diagonal W consist of the integers in the closed interval $[u,v]$ and we give the *ad hoc* symbols individually. Note that we write all the numbers in terms of the value q (where $4q=n$). The support of H is:

$$\begin{gathered}\{1,[2,2q-2](\text{A}),2q-1,[2q,4q-2](\text{B}),[4q-1,8q-5](\text{K}\cup\text{L}),8q-4,8q-3,\\ 8q-2,8q-1,8q,[8q+1,12q-3](\text{M}\cup\text{N}),12q-2,12q-1,12q,12q+1,\\ [12q+2,13q](\text{F}),[13q+1,14q-1](\text{J}),[14q,15q-2](\text{G}),[15q-1,16q-2](\text{C}),\\ [16q-1,17q-2](\text{D}),[17q-1,18q-3](\text{I}),[18q-2,19q-4](\text{H}),[19q-3,20q-5](\text{E}),\\ 20q-4,20q-3,20q-2,20q-1,20q\}=[1,20q]=[1,5n].\end{gathered}$$

We have shown that the array H is indeed an integer Heffter array $H(n;5)$. □

We now present the main theorem of this section.

Theorem 4.3. *There exists an integer Heffter array* $H(n;k)$ *for all* $n\equiv 0 \pmod 4$ *and* $k\equiv 1 \pmod 4$ *with* $5\leqslant k\leqslant n-3$.

Proof. The proof is very similar to that of Theorem 3.3. As above, let $h = n/2$, where h is necessarily even, and let $k = 5 + 4s$. When $s = 0$ we are done by Theorem 4.2. So we assume that $s \geqslant 1$, and hence that $4 \leqslant 4s \leqslant n - 8$. Begin with $H = H(n; 5)$ constructed in Theorem 4.2. We place s (shifted) $H(n; 4)$ constructed in Theorem 2.2 in $4s$ empty diagonals of the $H(n; 5)$. These empty diagonals will again come in pairs of consecutive diagonals. Specifically, for each $0 \leqslant t \leqslant s - 1$ place $H_t = H(n; 4) \pm (5n + 4nt)$ on the four diagonals $D_{2+2t}, D_{3+2t}, D_{h+2+2t}$, and D_{h+3+2t}.

A few things need to be checked. The filled diagonals in H are $D_1, D_h, D_{h+1}, D_{n-1}$, and D_n. The diagonals that get filled with the H_t's are $D_2, D_3, \ldots, D_{2s}, D_{1+2s}$ and $D_{h+2}, D_{h+3}, \ldots, D_{h+2s}, D_{h+2s+1}$. Since $4s \leqslant n - 8$, then $2s + 1 \leqslant h - 3$ and also $h + 2s + 1 \leqslant n - 3$. So the filled diagonals in $H, H_1, H_2, \ldots, H_s$ are all disjoint.

The row and column sums in H as well as in each H_t, $0 \leqslant t \leqslant s - 1$ is zero, hence the resulting array has row and column sum zero. Finally, note that the support of H is $[1, 5n]$ and for each H_t the support is $[5n + 4nt + 1, 5n + 4nt + 4n] = [5n + 4nt + 1, 9n + 4nt]$. So the support in the final array is

$$[1, 5n] \cup \bigcup_{t=0}^{s-1} [5n + 4nt + 1, 9n + 4nt]$$

$$= [1, 5n] \cup [5n + 1, 9n] \cup [9n + 1, 13n] \cup \cdots \cup [5n + 4n(s - 1) + 1, 9n + 4n(s - 1)].$$

Since $9n + 4n(s - 1) = n(5 + 4s) = nk$, the support is $[1, nk]$, completing the proof. □

5 Conclusion

In the paper [3], it was proven that the necessary conditions for the existence of an integer $H(n; k)$ are that $n \geqslant k$ and $nk \equiv 0, 3 \pmod 4$. Furthermore, this condition was proved to be sufficient except possibly when $n \equiv 0$ or $3 \pmod 4$ and $k \equiv 1 \pmod 4$. In Section 3 we proved that $H(n; k)$ exist when $n \equiv 3 \pmod 4$ and $k \equiv 1 \pmod 4$ and in Section 4 we proved that $H(n; k)$ exist when $n \equiv 0 \pmod 4$ and $k \equiv 1 \pmod 4$. From this we have the main result of this paper.

Theorem 5.1. *There exists an integer Heffter array $H(n; k)$ if and only if*

$$3 \leqslant k \leqslant n \quad \textit{and} \quad nk \equiv 0, 3 \pmod 4.$$

In future work we will consider the case when the Heffter array $H(n; k)$ is not an integer Heffter array. In this case the only necessary condition is that $3 \leqslant k \leqslant n$.

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3-pyramidal Steiner triple systems*

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Abstract

A design is said to be f-pyramidal when it has an automorphism group which fixes f points and acts sharply transitively on all the others. The problem of establishing the set of values of v for which there exists an f-pyramidal Steiner triple system of order v has been deeply investigated in the case $f = 1$ but it remains open for a special class of values of v. The same problem for the next possible f, which is $f = 3$, is here completely solved: there exists a 3-pyramidal Steiner triple system of order v if and only if $v \equiv 7, 9, 15 \pmod{24}$ or $v \equiv 3, 19 \pmod{48}$.

Keywords: Steiner triple system, group action, difference family, Skolem sequence, Langford sequence.

Math. Subj. Class.: 51E10, 20B25, 05B07, 05B10

1 Introduction

A Steiner triple system of order v, briefly STS(v), is a pair $(V, \mathcal{B})$ where V is a set of v points and $\mathcal{B}$ is a set of 3-subsets (*blocks* or *triples*) of V with the property that any two distinct points are contained in exactly one block. Apart from the trivial case $v = 0$ in

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which both V and $\mathcal{B}$ are empty, it is well known that a $\mathrm{STS}(v)$ exists if and only if $v \equiv 1$ or 3 (mod 6). For general background on STSs we refer to [7].

Steiner triple systems having an automorphism with a prescribed property or an automorphism group with a prescribed action have drawn much attention since a long time. It was proved by Peltesohn [15] that a $\mathrm{STS}(v)$ with an automorphism cyclically permuting all points, briefly a *cyclic* $\mathrm{STS}(v)$, exists for any possible v but $v \neq 9$. The existence question for a $\mathrm{STS}(v)$ with an involutory automorphism fixing exactly one point, briefly a *reverse* $\mathrm{STS}(v)$, has been settled by means of three different contributions of Doyen [9], Rosa [18] and Teirlinck [21]; it exists if and only if $v \equiv 1, 3, 9, 19$ (mod 24). In [16] Phelps and Rosa proved that there exists a $\mathrm{STS}(v)$ with an automorphism cyclically permuting all but one point, briefly a 1-*rotational* $\mathrm{STS}(v)$, if and only if $v \equiv 3$ or 9 (mod 24).

Note that a cyclic or 1-rotational STS may be viewed as a STS with a cyclic automorphism group acting sharply transitively on all points or all but one point, respectively. Thus one may ask, more generally, for a STS with an automorphism group G having the same kind of action without the request that G be cyclic.

Speaking of a regular $\mathrm{STS}(v)$ we mean a $\mathrm{STS}(v)$ for which there is at least one group G acting sharply transitively on the points. Also, speaking of a 1-rotational $\mathrm{STS}(v)$ we mean a $\mathrm{STS}(v)$ with at least one automorphism group acting regularly on all but one points.

The only STS(9), which is the point-line design associated with the affine plane over $\mathbb{Z}_3$, is clearly regular under $\mathbb{Z}_3^2$. Thus, in view of Peltesohn's result, there exists a regular $\mathrm{STS}(v)$ for any admissible v.

The problem of determining the set of values of v for which there exists a 1-rotational $\mathrm{STS}(v)$ (under some group) has been deeply investigated in [2, 4]. Such a STS is necessarily reverse so that v must be congruent to 1, 3, 9, or 19 (mod 24). For the cases $v \equiv 3$ or 9 (mod 24) the existence clearly follows from the result by Phelps and Rosa. In the case $v \equiv 19$ (mod 24) we do not have existence only when $v = 6PQ + 1$ with $P = 1$ or a product of pairwise distinct primes congruent to 5 (mod 12) and with Q a product of an odd number of pairwise distinct primes congruent to 11 (mod 12). The most difficult case is $v \equiv 1$ (mod 24) where the existence remains still uncertain only when all the following conditions are simultaneously satisfied: $v = (p^3 - p)n + 1 \equiv 1$ (mod 96) with p a prime; $n \not\equiv 0$ (mod 4); the odd part of $v - 1$ is square-free and without prime factors $\equiv 1$ (mod 6).

Of course one might consider the more specific problems of determining all groups G for which there exists a STS which is regular under G and all groups G for which there exists a STS which is 1-rotational under G. These problems appear at this moment quite hard. The same problems can be relaxed by asking for which v there is a $\mathrm{STS}(v)$ which is regular (1-rotational) under a group belonging to an assigned general class. For instance, the first author proved that there exists a $\mathrm{STS}(v)$ which is 1-rotational under some abelian group if and only if either $v \equiv 3, 9$ (mod 24) or $v \equiv 1, 19$ (mod 72). Also, Mishima [14] proved that there exists a $\mathrm{STS}(v)$ which is 1-rotational under a dicyclic group if and only if $v \equiv 9$ (mod 24).

We have quoted only the results which are closer to the problem that we are going to study in this paper. Indeed the literature on Steiner triple systems and their automorphism groups is quite wide. For example, results concerning the *full* automorphism group of a STS have been obtained by Mendelsohn [13] and Lovegrove [12].

Now we want to consider the problem of determining the set of values of v for which there exists a $\mathrm{STS}(v)$ with an automorphism group fixing f points and acting sharply transitively on the other $v - f$ points. Such a STS will be called f-*pyramidal*. Of course the cases $f = 0$ and $f = 1$ correspond, respectively, to the *regular* and 1-*rotational* STSs discussed above. It is natural to study the next possible case that is $f = 3$. This is because, as we are going to see in the next lemma, the fixed points of an f-pyramidal STS(v) form a subsystem of order f so that, for $f \neq 0$, we have $f \equiv 1$ or 3 (mod 6); also, if $f \neq v$, then $f < v/2$.

Lemma 1.1. *A necessary condition for the existence of an f-pyramidal* $\mathrm{STS}(v)$ *is that* $f = 0$ *or* $f \equiv 1, 3 \pmod 6$, *and* $f = v$ *or* $f < v/2$.

Proof. Let $(V, \mathcal{B})$ be an f-pyramidal STS(v) under the action of a group G. Assume, w.l.o.g., that G is additive and let $F = \{\infty_1, \ldots, \infty_f\}$ be the set of points fixed by G. Obviously G has order $v - f$, the set of points V can be identified with $F \cup G$, and the action of G on V can be identified with the addition on the right with the assumption that $\infty_i + g = \infty_i$ for each $\infty_i \in F$ and each $g \in G$. If a block $B \in \mathcal{B}$ contains two distinct fixed points, say ∞_i and ∞_j, then $B + g = B$ for every $g \in G$ otherwise $\mathcal{B}$ would have two distinct blocks, B and $B + g$, passing through the two points ∞_i and ∞_j. So, the third vertex of B is also fixed by G. It easily follows that all blocks of $\mathcal{B}$ contained in F form a STS(f) so that we have $f = 0$ or $f \equiv 1, 3$ (mod 6), and $f = v$ or $f < v/2$. □

The main result of this paper is a complete solution to the existence problem for a 3-pyramidal STS(v).

Theorem 1.2. *There exists a* 3*-pyramidal* $\mathrm{STS}(v)$ *if and only if* $v \equiv 7,\ 9,\ 15 \pmod{24}$ *or* $v \equiv 3,\ 19 \pmod{48}$.

The "if part" of this theorem will be proved in Section 3 which therefore will give non-existence results: for $v \equiv 1,\ 13,\ 21$ (mod 24) or $v \equiv 27,\ 43$ (mod 48) there is no 3-pyramidal STS(v). The "only if part" will be proved in Section 4 where we will give an explicit construction of a 3-pyramidal STS(v) whenever $v \equiv 7, 9, 15$ (mod 24) or $v \equiv 3, 19$ (mod 48).

First, in the next section, we have to translate our problem into algebraic terms: any f-pyramidal STS is completely equivalent to a suitable *difference family*.

2 Difference families and pyramidal STSs

As a natural generalization of the concept of a *relative difference set* [17], the first author introduced [3] *difference families* in a group G relative to a subgroup of G or, even more generally [5], relative to a *partial spread* of G. By a partial spread of a group G one means a set Σ of subgroups of G whose mutual intersections are all trivial. One omits the attribute "partial" in the special case that the subgroups of Σ cover all G. Let Σ be a partial spread of an additively written group G, let $\mathcal{F}$ be a set of k-subsets G, and let $\Delta\mathcal{F}$ be the list of all possible differences $x - y$ with (x, y) an ordered pair of distinct elements of a member of $\mathcal{F}$. One says that $\mathcal{F}$ is a $(G, \Sigma, k, 1)$-difference family (DF) if every group element appears 0 or 1 times in $\Delta\mathcal{F}$ according to whether it belongs or does not belong to some member of Σ, respectively. We say that Σ is of *type* $\{d_1^{e_1}, \ldots, d_n^{e_n}\}$ if this is the multiset (written in "exponential" notation) of the orders of all subgroups belonging to Σ and we speak of a $(G, \{d_1^{e_1}, \ldots, d_n^{e_n}\}, k, 1)$-DF.

It is obvious that any STS(v) is v-pyramidal under the trivial group. The following theorem explains how to construct an f-pyramidal STS(v) with $f < v/2$. It generalizes Theorem 1.1 in [3] which corresponds to the case $f = 1$.

Theorem 2.1. *There exists an f-pyramidal* $\mathrm{STS}(v)$ *with* $f < v/2$ *if and only if there exists a* $(G, \{2^f, 3^e\}, 3, 1)$*-DF for a suitable group G of order $v - f$ with exactly f involutions, and a suitable integer e.*

Proof. ($\Longrightarrow$). Let $(V, \mathcal{B})$ be an f-pyramidal STS(v) under an additive group G. We can assume that $V = F \cup G$ with F and the action of G on V defined as in Lemma 1.1. For $1 \leq i \leq f$, let $B_i = \{\infty_i, 0, x_i\}$ be the block of $\mathcal{B}$ containing the points ∞_i and 0. We have $B_i - x_i = \{\infty_i, -x_i, 0\}$ so that both B_i and $B_i - x_i$ contain the points ∞_i and 0. It necessarily follows that $B_i - x_i = B_i$, hence $-x_i = x_i$ which means that x_i is an involution. Conversely, if x is an involution of G and $B = \{0, x, y\}$ is the block through 0 and x, then $B + x = \{x, 0, y + x\}$ would also contain 0 and x so that $B + x = B$. This means that $y + x = y$ and this is possible only if $y \in F$. Hence $y = \infty_i$ and $x = x_i$ for a suitable i. We conclude that $\{x_1, ..., x_f\}$ is the set of all involutions of G.

Let $\mathcal{F}$ be a complete system of representatives for the G-orbits on the blocks of $\mathcal{B}$ with trivial G-stabilizer. Reasoning as in the "if part" of Theorem 2.2 in [4], one can see that $\mathcal{F}$ is a $(G, \Sigma, 3, 1)$-DF where Σ is the partial spread of G consisting of all 2-subgroups $\{0, x_i\}$ $(i = 1, \dots, f)$ of G and all 3-subgroups of G belonging to $\mathcal{B}$.

($\Longleftarrow$). Now assume that $f \equiv 1$ or 3 (mod 6) and that $\mathcal{F}$ is a $(G, \Sigma, 3, 1)$-DF with G a group of order $v - f$ having exactly f involutions and with Σ a partial spread of G of type $\{2^f, 3^e\}$.

Take an f-set $F = \{\infty_1, \dots, \infty_f\}$ disjoint with G and let $(F, \mathcal{B}_\infty)$ be any STS(f) (which exists because we assumed that $f \equiv 1$ or 3 (mod 6)). For $i = 2, 3$, let Σ_i be the set of subgroups of order i belonging to Σ. Set $\Sigma_2 = \{S_i \mid 1 \leq i \leq f\}$ and $\Sigma_2^+ = \{S_i \cup \{\infty_i\} \mid 1 \leq i \leq f\}$. Then, as in the "only if part" of Theorem 2.2 in [4], one can see that

$$\Sigma_2^+ \cup \Sigma_3 \cup \mathcal{F} \cup \mathcal{B}_\infty$$

is a complete system of representatives for the block-orbits of a 3-pyramidal STS(v) under the action of G on $F \cup G$ defined as in the proof of Lemma 1.1. □

Remark 2.2. Considering that "cyclic STS" means "0-pyramidal STS under the cyclic group", as a very special case of the above theorem we have the well known fact that any cyclic STS$(6n+1)$ is equivalent to a $(\mathbb{Z}_{6n+1}, \{1\}, 3, 1)$-DF and that any cyclic STS$(6n+3)$ is equivalent to a $(\mathbb{Z}_{6n+3}, \{3\}, 3, 1)$-DF.

Example 2.3. The empty-set clearly is a $(\mathbb{Z}_2^n, \{2^{2^n-1}\}, 3, 1)$-DF since every non-zero element of $\mathbb{Z}_2^n$ is an involution. It is not difficult to see that one of the associated $(2^n - 1)$-pyramidal STS$(2^{n+1} - 1)$ is the point-line design of the n-dimensional projective geometry over $\mathbb{Z}_2$.

Let $\mathbb{D}_{2n}$ be the dihedral group of order $2n$, namely the group with defining relations $\mathbb{D}_{2n} = \langle x, y \mid y^2 = x^n = 1;\ yx = x^{-1}y\rangle$. We give here an example of a STS$(3f)$ which is f-pyramidal under $\mathbb{D}_{2f}$.

Example 2.4. Let $f \equiv 1$ or 3 (mod 6) but $f \neq 9$. Let $\phi : \mathbb{Z}_f \longrightarrow \mathbb{D}_{2f}$ be the group monomorphism defined by $\phi(i) = x^i$ for each $i \in \mathbb{Z}_f$. The hypotesis on f guarantees, in view of Peltesohn's result, that there exists a cyclic STS(f). Thus, by Remark 2.2, there exists a $(\mathbb{Z}_f, \{1\}, 3, 1)$-DF or a $(\mathbb{Z}_f, \{3\}, 3, 1)$-DF $\mathcal{F}$ according to whether $f \equiv 1$ or 3 (mod 6), respectively. It is then obvious that $\{\phi(B) \mid B \in \mathcal{F}\}$ is a $(\mathbb{D}_{2f}, \{2^f, 3^e\}, 3, 1)$-DF with $e = 0$ or 1, respectively.

Thus there exists an f-pyramidal STS$(3f)$ under the dihedral group $\mathbb{D}_{2f}$ for any $f \equiv 1$ or $3 \pmod 6$ but $f \neq 9$.

If, in the above example, we put $f = 3$, we obtain a representation of the affine plane of order 3 as a 3-pyramidal STS(9) under $\mathbb{D}_6$. In this case the difference family $\mathcal{F}$ is empty because it is relative to a spread which is not partial; its subgroups $\{1, y\}$, $\{1, xy\}$, $\{1, x^2y\}$ and $\{1, x, x^2\}$ cover indeed all elements of $\mathbb{D}_6$. Following the instructions of the "only if part" of Theorem 2.1 the blocks of the STS(9) are:

$$\begin{array}{ccc}
\{\infty_1, \infty_2, \infty_3\}, & \{1, x, x^2\}, & \{y, xy, x^2y\}, \\
\{\infty_1, 1, y\}, & \{\infty_1, x, x^2y\}, & \{\infty_1, x^2, xy\}, \\
\{\infty_2, 1, xy\}, & \{\infty_2, x, y\}, & \{\infty_2, x^2, x^2y\}, \\
\{\infty_3, 1, x^2y\}, & \{\infty_3, x, xy\}, & \{\infty_3, x^2, y\}.
\end{array}$$

In Section 4 we will make use of $\mathbb{D}_6$ again, for the construction of a 3-pyramidal STS$(24n + 9)$ under $\mathbb{D}_6 \times \mathbb{Z}_{4n+1}$.

3 The "if part"

In this section we determine the values of v for which a 3-pyramidal STS(v) cannot exist, we namely prove the "if part" of the main result Theorem 1.2. For this, we need two lemmas about elementary group theory.

Lemma 3.1. *If G is a group of order $24n + 18$, then G has a subgroup of index 2.*

Proof. It is well known that a group of order twice an odd number has a subgroup of index 2. See, for example, [19, Exercise 262]. □

The next lemma makes use of the so-called "Burnside normal p-complement theorem" which is here recalled (see [19, Theorem 6.17]).

Theorem 3.2. *Let P be a Sylow p-subgroup of a finite group G. If $C_G(P) = N_G(P)$, then P has a normal complement in G.*

Lemma 3.3. *If G is a group of order $16n + 8$ containing exactly 3 involutions, then G has a subgroup of index 2 containing exactly one involution.*

Proof. Let j_1, j_2, and j_3 be the three involutions of G and let $H = \langle j_1, j_2, j_3\rangle$ be the group they generate. We point out that H is a normal subgroup of G since it is generated by all the elements of order 2. Now, let P be a 2−Sylow subgroup of G. As P has order 8 and G contains exactly three involutions, by taking into account the classification of the groups of order 8, we have three possibilities: namely P is either isomorphic to the group $\mathbb{Z}_4 \times \mathbb{Z}_2$ or P contains exactly one involution, i.e., it is $P \simeq \mathbb{Z}_8$ or $P \simeq Q_8$ (the quaternion group of order 8).

First of all we prove that it is necessarily $P \simeq \mathbb{Z}_4 \times \mathbb{Z}_2$. By contradiction, assume that P contains exactly one involution. It is known that, in general, any two involutions of G generate a dihedral group; also, a dihedral group of order $2h$ contains at least h involutions. Therefore, either $\langle j_1, j_2\rangle \simeq \mathbb{D}_4 \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\langle j_1, j_2\rangle \simeq \mathbb{D}_6$. In both cases, $\langle j_1, j_2\rangle$ has three involutions, hence $j_3 \in \langle j_1, j_2\rangle$ and $H = \langle j_1, j_2\rangle$. Since P contains just one involution, it is necessarily $H = \mathbb{D}_6$, otherwise P should contain a subgroup isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Let T be the subgroup of H of order 3. As usual, we denote by $N_G(T)$ and $C_G(T)$ the normalizer and the centralizer of T in G, respectively. By the N/C-theorem, the quotient group $N_G(T)/C_G(T)$ is isomorphic to a subgroup of $Aut(T) \simeq \mathbb{Z}_2$. Since T is the unique subgroup of H of order 3, T is characteristic in H and then it is normal in G, that is $N_G(T) = G$. Therefore, either $C_G(T) = G$ or $C_G(T)$ is a subgroup of G of index 2. In both cases, $C_G(T)$ is a normal subgroup of G of even order. The three involutions of G are pairwise conjugate, hence, they are contained in any normal subgroup of G of even order and then in $C_G(T)$. It then follows that T is central in H contradicting the fact that H is dihedral.

We conclude that $P \simeq \mathbb{Z}_4 \times \mathbb{Z}_2$; in particular, P is abelian and contains a subgroup $Q \simeq \mathbb{Z}_4$; also, $P \supseteq H \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$. To prove the assertion, we need to show that G has a normal subgroup O of order $2n+1$. In fact, the semidirect product of O and Q will be a subgroup of G of index 2 containing exactly one involution.

Recall that $|G : C_G(j_i)|$ is the size of $cl(j_i)$, the conjugacy class of j_i in G, which cannot exceed the total number of involutions in G hence, $|G : C_G(j_i)| \leq 3$ for any $i = 1, 2, 3$. On the other hand, P is abelian hence, $P \leq C_G(j_i)$ for any $i = 1, 2, 3$. It then follows that either $|G : C_G(j_i)| = 1$ for any $i = 1, 2, 3$ or $|G : C_G(j_i)| = 3$ for any $i = 1, 2, 3$.

We first deal with the former case in which all involutions of G are central. Since G/H has order $2d$, d odd, then it has a subgroup of index 2 (see for example [19, Exercise 262]). In other words, there exists a normal subgroup N of G of index 2 which contains H. Since H is a central 2-Sylow subgroup of N, by Theorem 3.2, H has a normal complement O in N; in particular, O has order $2n+1$. Now, suppose the existence of another subgroup K of N of order $2n+1$. Then $K \cap O$ is normal in K and the quotient $Q_1 = K/(K \cap O)$ should be isomorphic to $Q_2 = (K+O)/O$. However, Q_1 has odd order, while Q_2 has even order. Therefore, O is the only subgroup of N of order $2n+1$, hence it is normal in G.

We finally consider the case $|G : C_G(j_i)| = 3$ for any $i = 1, 2, 3$. By the N/C-theorem, $G/C_G(H)$ is isomorphic to a subgroup of $Aut(H) \simeq \mathbb{D}_6$. Since $P \leq C_G(H) \leq C_G(j_i)$, we have that $C_G(H) = C_G(j_i)$ for any $i = 1, 2, 3$. Now note that $C_G(H)$ satisfies the same assumption as G and all involutions are central in $C_G(H)$. Therefore, we can proceed as in the previous case to show that there is a normal subgroup Ω of $C_G(H)$ of order $\frac{2n+1}{3}$. As before, it is not difficult to check that Ω is the only subgroup of $C_G(H)$ of order $\frac{2n+1}{3}$, therefore it is normal in G.

Set now $\overline{G} = G/\Omega$ and $\overline{P} = C_G(H)/\Omega$. Note that $\overline{P} \simeq P$, $\overline{P}$ is normal in $\overline{G}$ (i.e., $N_{\overline{G}}(\overline{P}) = \overline{G}$) and $\overline{G}/\overline{P}$ has order 3. Also, since $\overline{P}$ is abelian, then $\overline{P} \leq C_{\overline{G}}(\overline{P})$ hence, $|\overline{G} : C_{\overline{G}}(\overline{P})| = 1$ or 3. Considering that, by the N/C-theorem, $\overline{G}/C_{\overline{G}}(\overline{P})$ is isomorphic to a subgroup of $Aut(\overline{P})$, and that $Aut(\overline{P})$ has order 8, we then have that $|\overline{G} : C_{\overline{G}}(\overline{P})| = 1$. This means that $\overline{P}$ is central in $\overline{G}$. Hence, $\overline{G}$ is the direct product of $\overline{P}$ by a normal subgroup $\overline{O}$ of order 3 which is the quotient by Ω of a normal subgroup O of G of order $2n+1$. □

Theorem 3.4. *There is no* 3*-pyramidal* $\mathrm{STS}(v)$ *in each of the following cases:*

(i) $v \equiv 1 \pmod{24}$*;*

(ii) $v \equiv 13 \pmod{24}$*;*

(iii) $v \equiv 21 \pmod{24}$*;*

(iv) $v \equiv 27 \pmod{48}$*;*

(v) $v \equiv 43 \pmod{48}$.

Proof. Cases (i)-(ii). Suppose that there exists a STS(v) with $v = 24n+1$ or $v = 24n+13$ which is 3-pyramidal under a group G. Therefore G is a group of order $24n-2$ or $24n+10$ with exactly three subgroups of order 2. Now note that these subgroups are precisely the 2-Sylow subgroups of G since the order of G is divisible by 2 but not by 4. Hence $n_2(G)$, the number of 2-Sylow subgroups of G, is 3. By the third Sylow theorem $n_2(G)$ should be also a divisor of $\frac{|G|}{2}$. We conclude that 3 should be a divisor of $|G|$ which is clearly false.

Case (iii). Now assume that there exists a 3-pyramidal STS$(24n+21)$. Then there exists a $(G, \{2^3, 3^e\}, 3, 1)$-DF $\mathcal{F}$ for a suitable group G of order $24n + 18$ with exactly 3 involutions and a suitable $e \geq 0$.

By Lemma 3.1 there is a subgroup S of G of index 2. Note that an element of G has odd or even order according to whether it is in S or not, respectively. Thus, in particular, the three involutions of G are all contained in $G \setminus S$ while every subgroup of G of order 3 is contained in S. Thus the subset of $G \setminus S$ which is covered by $\Delta\mathcal{F}$ has size $|G \setminus S| - 3 = 12n + 6$.

If B is any block of $\mathcal{F}$ having some differences in $G \setminus S$, then it necessarily has two points lying in distinct cosets of S in G. Thus, up to translations, we have $B = \{0, s, t\}$ with $s \in S$ and $t \in G \setminus S$. It follows that $\Delta B \cap (G \setminus S) = \{\pm t, \pm(s-t)\}$, hence ΔB has exactly four elements in $G \setminus S$.

From the above two paragraphs we conclude that $12n + 6$ should be divisible by 4 which is clearly absurd.

Cases (iv)-(v). Assume that there exists a 3-pyramidal STS(v) with $v \equiv 27$ or $43 \pmod{48}$. Thus there exists a $(G, \{2^3, 3^e\}, 3, 1)$-DF $\mathcal{F}$ for a suitable group G of order $48n + 24$ or $48n + 40$ with exactly three involutions and where $e > 0$ in case (iv) or $e = 0$ in case (v). Note that in both cases we have $|G| \equiv 8 \pmod{16}$ so that, by Lemma 3.3, G has a subgroup S of index 2 containing exactly one involution. Thus the subset of $G \setminus S$ which is covered by $\Delta\mathcal{F}$ has size $|G \setminus S| - 2 = 24n + 10$ or $24n + 18$. Then, reasoning as in case (iii), $24n + 10$ or $24n + 18$ should be divisible by 4 which is absurd. □

4 The "only if part"

For proving the "only if part" of our main theorem, we have to give a direct construction for a 3-pyramidal STS(v) whenever v is admissible and not forbidden by Theorem 3.4, hence for any $v \equiv 7, 9, 15 \pmod{24}$ and for any $v \equiv 3, 19 \pmod{48}$. The most laboured constructions are those for the last two cases where we will use *extended Skolem sequences* and *extended Langford sequences*.

Given a pair (k, n) of positive integers with $1 \leq k \leq 2n + 1$, a *k-extended Skolem sequence* of order n can be viewed as a sequence $(s_1, \ldots, s_n)$ of n integers such that

$$\bigcup_{i=1}^{n} \{s_i, s_i + i\} = \{1, 2, \ldots, 2n + 1\} \setminus \{k\}.$$

The existence question for extended Skolem sequences was completely settled by C. Baker in [1].

Theorem 4.1 ([1]). *There exists a k-extended Skolem sequence of order n if and only if either k is odd and $n \equiv 0, 1 \pmod{4}$ or k is even and $n \equiv 2, 3 \pmod{4}$.*

A k-extended Langford sequence of order n and defect d can be viewed as a sequence of n integers $(\ell_1, ..., \ell_n)$ such that

$$\bigcup_{i=1}^{n}\{\ell_i, \ell_i + i + d - 1\} = \{1, 2, \ldots, 2n + 1\} \setminus \{k\}.$$

We need the following partial result about extended Langford sequences by V. Linek and S. Mor.

Theorem 4.2 ([11])**.** *There exists a k-extended Langford sequence of order n and defect d for any triple (k, n, d) with $n \geq 2d$, $n \equiv 2 \pmod 4$ and k even.*

For general background concerning Skolem sequences, their variants, and their applications, we refer to [20]. We also refer to the recent survey [10] which, however, fails to mention some of our work; for instance, extended Skolem sequences have been crucial in our mentioned work on 1-rotational STSs [2, 4] and also in two papers [6, 22] dealing, more generally, with 1-rotational k-cycle systems.

In the next lemma we combine Skolem sequences and Langford sequences to get the last ingredient that we need for proving the "only if part" of our main result. This lemma will be used, specifically, in the construction of a 3-pyramidal STS(v) with $v \equiv 3$ (mod 96).

Lemma 4.3. *There exists a $(\mathbb{Z}_{12n}, \{3, 4\}, 3, 1)$-DF for any even $n \geq 2$.*

Proof. We have to construct a set $\mathcal{F}_n$ of $2n - 1$ triples with elements in $\mathbb{Z}_{12n}$ whose differences cover $\mathbb{Z}_{12n} \setminus \{0, 3n, 4n, 6n, 8n, 9n\}$ exactly once. For the small cases $n \in \{2, 4, 6, 8, 12, 14, 20\}$ one can check that we can take $\mathcal{F}_n = \{B_1, \ldots, B_{2n-1}\}$ with $B_i = \{0, i, b_i\}$ and the b_is as in the following table:

n	$(b_1, b_2, \ldots, b_{2n-1})$
2	$(5, 9, 13)$
4	$(40, 37, 34, 30, 38, 29, 28)$
6	$(13, 16, 20, 19, 26, 28, 30, 39, 34, 37, 40)$
8	$(17, 20, 22, 25, 28, 33, 36, 34, 39, 47, 46, 52, 54, 45, 53)$
12	$(66, 63, 37, 64, 38, 62, 39, 58, 40, 59, 41, 67, 42, 68, 43, 69, 44, 70, 45, 71, 46, 57, 47)$
14	$(76, 74, 43, 70, 44, 77, 45, 73, 46, 68, 47, 69, 48, 78, 49, 79, 50, 80, 51, 81, 52, 82, 53, 83, 54, 67, 55)$
20	$(110, 103, 61, 108, 62, 102, 63, 100, 64, 105, 65, 106, 66, 107, 67, 98, 68, 99, 69, 111, 70, 112, 71, 113, 72, 114, 73, 115, 74, 116, 75, 117, 76, 118, 77, 119, 78, 97, 79)$

For all the other values of n, set $n = 2m$ and $\epsilon = r + (-1)^r$ where r is the remainder of the Euclidean division of m by 4.

By applying Theorem 4.1 one can see that there exists a $(2m + 1)$-extended Skolem sequence $(s_1, \ldots, s_{m+\epsilon})$ of order $m + \epsilon$. Also, by applying Theorem 4.2 one can see that there exists a $(2m-2\epsilon)$-extended Langford sequence $(\ell_1, \ldots, \ell_{3m-\epsilon-1})$ of order $3m-\epsilon-1$ and defect $m + \epsilon + 1$. Then one can see that the desired difference family is the one whose blocks are the following:

$$\{0,\ -i,\ s_i + 4m - 1\} \qquad \text{with } 1 \leq i \leq m + \epsilon;$$
$$\{0,\ -(m + \epsilon + i),\ \ell_i + 6m + 2\epsilon\} \qquad \text{with } 1 \leq i \leq 3m - \epsilon - 1.$$

□

Theorem 4.4. *There exists a* 3*-pyramidal* $\mathrm{STS}(v)$ *for any admissible value of* v *not forbidden by Theorem 3.4.*

Proof. Considering that the admissible values of v are those congruent to 1 or 3 (mod 6), we have to prove that there exists a 3-pyramidal STS(v) in the following five cases:

(i) $v \equiv 7 \pmod{24}$;

(ii) $v \equiv 15 \pmod{24}$;

(iii) $v \equiv 9 \pmod{24}$;

(iv) $v \equiv 3 \pmod{48}$;

(v) $v \equiv 19 \pmod{48}$.

Cases (i)-(ii). Assume that $v \equiv 7, 15$ (mod 24). Let $\mathcal{F}$ be an $(H, \{h\}, 3, 1)$-DF with $H = \mathbb{Z}_{6n+1}$ and $h = 1$ if $v = 24n + 7$, or $H = \mathbb{Z}_3 \times \mathbb{Z}_{2n+1}$ and $h = 3$ if $v = 24n + 15$. In the former case the existence of $\mathcal{F}$ is guaranteed by Peltesohn's result (see Remark 2.2) while in the latter it is enough to take $\mathcal{F} = \{\{(0, 0),(1, i),(1, -i)\} \mid 1 \leq i \leq n\}$.

The group $G := \mathbb{Z}_2 \times \mathbb{Z}_2 \times H$ has order $v - 3$ and exactly 3 involutions. Thus, by Theorem 2.1, it is enough to exhibit a $(G, \{2^3, 3^e\}, 3, 1)$-DF for some suitable e. Such a DF is, for instance, the one whose blocks are:

$$\{(0, 0)\} \times B \qquad \text{with } B \in \mathcal{F};$$
$$\{(0, 1, 0),(1, 0, h),(1, 1, -h)\} \qquad \text{with } h \in \overline{H}$$

where $\overline{H}$ is a subset of H such that $\{\{h, -h\} \mid h \in \overline{H}\}$ is the *patterned starter* of H, i.e., the set of all symmetric 2-subsets of H (see [8]).

Case (iii). Assume that $v \equiv 9$ (mod 24), say $v = 24n+9$. The group $G := \mathbb{D}_6 \times \mathbb{Z}_{4n+1}$ has order $v-3$ and clearly has exactly three involutions. Thus, by Theorem 2.1, it is enough to exhibit a $(G, \{2^3, 3^1\}, 3, 1)$-DF (of course here the DF will "use" multiplication on the first component and addition on the second). For this, we have to distinguish three cases according to whether n is odd or congruent to 0 or 2 (mod 4). The three cases are very similar; the blocks of the desired difference family can be taken as indicated in the table below.

Blocks	n odd	$n \equiv 0$ (mod 4)	$n \equiv 2$ (mod 4)
$\{(1, 0),(x, n),(x, 2n)\}$	Yes	Yes	Yes
$\{(y, 0),(1, -\frac{n+1}{2}),(1, \frac{3n+1}{2})\}$	Yes	No	No
$\{(1, 0),(x, -\frac{n}{2}),(x, \frac{3n}{2})\}$	No	Yes	Yes
$\{(y, 0),(1, i),(1, 2n + 1 - i)\}$	$1 \leq i \leq n$ $i \neq \frac{n+1}{2}$	$1 \leq i \leq n$	$1 \leq i \leq n$
$\{(y, 0),(x, i),(x^2, -i)\}$	$1 \leq i \leq n$	$1 \leq i \leq n$, $i \neq \frac{n}{4}$	$1 \leq i \leq n$, and $i = \frac{7n+2}{4}$
$\{(y, 0),(x, -i),(x^2, i)\}$	$n + 1 \leq i \leq 2n$	$n + 1 \leq i \leq 2n$ and $i = \frac{n}{4}$	$n + 1 \leq i \leq 2n$ and $i \neq \frac{7n+2}{4}$
$\{(1, 0),(x, i),(x, 2n - i)\}$	$1 \leq i \leq n - 1$	$1 \leq i \leq n - 1$ and $i \neq \frac{n}{2}$	$1 \leq i \leq n - 1$ and $i \neq \frac{n}{2}$

We note that the subgroup of order 3 which is not covered by the differences of the above families is, in any subcase, $\{(1, 0),(x, 0),(x^2, 0)\}$.

Case (iv). Assume that $v \equiv 3 \pmod{48}$, say $v = 48n + 3$. The group $G = \mathbb{Z}_4 \times \mathbb{Z}_{12n}$ has order $v - 3$ and it has exactly three involutions. Then, by Theorem 2.1, it is enough to exhibit a $(G, \{2^3, 3\}, 3, 1)$-DF.

Subcase (iv.1): n is odd. Take a $(2n + 1)$-extended Skolem sequence $(s_1, ..., s_{2n-1})$ of order $2n - 1$ (which exists by Theorem 4.1). Set $n = 2t + 1$ and check that the blocks of a $(G, \{2^3, 3\}, 3, 1)$-DF are the following:

$$\begin{array}{ll}
\{(0,0),(1,0),(3,6t+3)\}; & \\
\{(0,0),(1,3t+2),(1,-9t-5)\}; & \\
\{(0,0),(1,i),(3,12t+7-i)\} & \text{with } 1 \le i \le 6t+3 \text{ and } i \ne 3t+2; \\
\{(0,0),(1,6t+3+i),(3,6t+3-i)\} & \text{with } 1 \le i \le 6t+2; \\
\{(0,0),(0,i),(0,-s_i-4t-1)\} & \text{with } 1 \le i \le 4t+1.
\end{array}$$

Subcase (iv.2): n is even. Take a $(\mathbb{Z}_{12n}, \{3, 4\}, 3, 1)$-DF $\mathcal{F}$ using Lemma 4.3. One can see that a $(G, \{2^3, 3\}, 3, 1)$-DF is the one whose blocks are the following.

$$\begin{array}{ll}
\{(0,0),(1,0),(1,9n)\}; & \\
\{0\} \times B & \text{with } B \in \mathcal{F}; \\
\{(0,0),(1,i),(3,6n+1-i)\} & \text{with } 1 \le i \le 3n; \\
\{(0,0),(1,i),(3,6n-i)\} & \text{with } 3n+1 \le i \le 6n-1.
\end{array}$$

Case (v). Assume that $v \equiv 19 \pmod{48}$, say $v = 48n + 19$. The group $G = \mathbb{Z}_4 \times \mathbb{Z}_{12n+4}$ has order $v - 3$ and it has exactly three involutions. Then, by Theorem 2.1, it is enough to exhibit a $(G, \{2^3\}, 3, 1)$-DF. Let $(s_1, s_2, ..., s_{2n})$ be any $(n+1)$-extended Skolem sequence of order $2n$ (which exists by Theorem 4.1). Then the required difference family is the one whose blocks are the following:

$$\begin{array}{ll}
\{(0,0),(1,0),(1,3n+1)\}; & \\
\{(0,0),(0,i),(0,-s_i-2n)\} & \text{with } 1 \le i \le 2n; \\
\{(0,0),(1,i),(3,6n+2-i)\} & \text{with } 1 \le i \le 3n; \\
\{(0,0),(1,6n+3-i),(3,i)\} & \text{with } 1 \le i \le 3n+1.
\end{array}$$

□

5 Conclusion

Theorem 3.4 and Theorem 4.4 are the "if part" and the "only if part" of the main result Theorem 1.2 which therefore is now completely proved.

The existence of a 3-pyramidal STS(v) can be summarized in the following table where, in the third column, we put a group acting 3-pyramidally on a STS(v).

v	Existence	Group
$24n+1$	No	–
$24n+3$	Yes $\Longleftrightarrow n$ is even	$\mathbb{Z}_4 \times \mathbb{Z}_{6n}$
$24n+7$	Yes	$\mathbb{Z}_2^2 \times \mathbb{Z}_{6n+1}$
$24n+9$	Yes	$\mathbb{D}_6 \times \mathbb{Z}_{4n+1}$
$24n+13$	No	–
$24n+15$	Yes	$\mathbb{Z}_2^2 \times \mathbb{Z}_3 \times \mathbb{Z}_{2n+1}$
$24n+19$	Yes $\Longleftrightarrow n$ is even	$\mathbb{Z}_4 \times \mathbb{Z}_{6n+4}$
$24n+21$	No	–

Note that an abelian group of order $24n+6$ has only one involution so that there is no STS$(24n+9)$ which is 3-pyramidal under an abelian group. Thus we see, from the above table, that there exists a STS(v) which is 3-pyramidal under an abelian group if and only if $v \equiv 7, 15$ (mod 24) or $v \equiv 3, 19$ (mod 48).

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On some generalization of the Möbius configuration

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Abstract

The Möbius (8_4) configuration is generalized in a purely combinatorial approach. We consider $(2n_n)$ configurations $\mathfrak{M}_{(n,\varphi)}$ depending on a permutation φ in the symmetric group S_n. Classes of non-isomorphic configurations of this type are determined. The parametric characterization of $\mathfrak{M}_{(n,\varphi)}$ is given. The uniqueness of the decomposition of $\mathfrak{M}_{(n,\varphi)}$ into two mutually inscribed n-simplices is discussed. The automorphisms of $\mathfrak{M}_{(n,\varphi)}$ are characterized for $n \geq 3$.

Keywords: Möbius configuration, (8_4) *configurations, Möbius pair,* n*-simplex.*

Math. Subj. Class.: 51D20, 05B30, 51E30

1 Introduction

The Möbius (8_4) configuration is a certain configuration in a projective 3-dimensional space consisting of two mutually inscribed and circumscribed tetrahedra (cf. [7]). Each vertex of one tetrahedron lies on a face plane of the other tetrahedron and vice versa. Configurations with parameters (n_4) were studied in detail in [4], but this is not the case, since the Möbius (8_4) configuration is not a point-line structure. An important role of the theorem connected with the Möbius configuration (which says, roughly speaking, that the Möbius configuration "closes") in a projective 3-dimensional space was presented in [12]: it is equivalent to the commutativity of the ground division ring.

In this paper we deal with two n-simplices (simplices with n vertices, $n \geq 3$)[1] instead of two tetrahedra (4-simplices). The way how an n-simplex is inscribed into another we define by a permutation φ in the group S_n. The generalization of the Möbius configuration we obtain, is a $(2n_n)$-configuration and it will be referred to as a *Möbius pair of* n*-simplices*,

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[1] In geometry an n-simplex usually means a simplex having $n+1$ vertices. Our definition is slightly different.

or shortly a *Möbius n-pair*. Only a combinatorial scheme (an abstract incidence structure, see e.g. [2, 10]) of a Möbius n-pair is investigated and we do not discuss problems regarding embeddability into projective (or other) spaces. Although these problems have been partially solved in [5] (the case with $\varphi = \mathrm{id}$), they are interesting and still open in general.

As we know from [6], in a projective space, up to an isomorphism there are five (8_4) point-plane configurations with the property that at most two planes share two points, and dually at most two points are shared by two planes. These are precisely those configurations with two mutually circumscribed tetrahedra, and thus all of them are sometimes called the Möbius configurations. It is also known (cf. [10]), that these (8_4) configurations correspond to conjugacy classes of the permutation group S_4. We shall prove, that two Möbius n-pairs are isomorphic if and only if the permutations, that determine them, are conjugate. Another important impact of the permutation on the geometry of the Möbius n-pair is that the cycle structure of φ is associated with circuits in the incidence graph of the Möbius n-pair.

As we shall see, the decomposition of the points of the generalized Möbius configuration into two complementary and mutually inscribed simplices is, generally, a unique one. Exceptions appear "near" the classical case $n = 4$. Three of five (8_4) Möbius configurations contain at least two distinct pairs of complementary 4-simplices.

The next problem, which is considered in the paper, involves Möbius subpairs of a Möbius n-pair. We simply delete some number of points and blocks of one n-simplex and the same number of points and blocks of the second n-simplex with a hope to obtain a Möbius pair again. The conditions, under which we get a subpair in the Möbius n-pair, are determined.

In the last part we use most of the established properties to characterize the automorphism group of the Möbius n-pair for $n \geq 3$.

2 Definitions, parameters and basic properties

By a *configuration* we mean any point-block structure $\mathfrak{M} = \langle S, \mathcal{L}\rangle$, where the blocks are subsets of the set of points, i.e. $\mathcal{L} \subseteq 2^S$. The *rank of a point* is the number of blocks containing this point, and dually the *size of a block* is the number of points contained in this block. Let n be a natural number and X be a set. The family of all n-subsets of the set X will be denoted by $\wp_n(X)$. Let $n \geq 3$. We say that a configuration $\mathfrak{M}$ is an *n-simplex* iff $|\mathcal{L}| = n$, there is a subset $V \in \wp_n(S)$ such that for every $V' \in \wp_{n-1}(V)$ there is a unique block $L \in \mathcal{L}$ containing V', and the rank of each point $s \in S \setminus V$ is less than $n - 1$. Elements of V will be called *vertices* of the simplex, and blocks of the simplex are said to be its *faces*. We say that two configurations $\mathfrak{M}_1 = \langle S_1, \mathcal{L}_1\rangle$, $\mathfrak{M}_2 = \langle S_2, \mathcal{L}_2\rangle$ are *isomorphic* (and we write $\mathfrak{M}_1 \cong \mathfrak{M}_2$) iff there exists a bijective map $f\colon S_1 \longrightarrow S_2$ such that conditions $k \in \mathcal{L}_1$ and $f(k) \in \mathcal{L}_2$ are equivalent. In case $\mathfrak{M}_1 = \mathfrak{M}_2 = \mathfrak{M}$ the map f will be called an *automorphism of* $\mathfrak{M}$.

Let us consider two sets $A = \{a_1, \dots, a_n\}$ and $B = \{b_1, \dots, b_n\}$ such that $A \cap B = \emptyset$. Let $\varphi \in S_n$ be a permutation of the set $I = \{1, \dots, n\}$. Now we introduce the following sets:

$$\mathcal{L}_A := \{A' \cup \{b_i\} : A' \in \wp_{n-1}(A) \text{ and } a_i \notin A'\},$$
$$\mathcal{L}_B := \{B' \cup \{a_{\varphi(i)}\} : B' \in \wp_{n-1}(B) \text{ and } b_i \notin B'\}.$$

The configuration

$$\mathfrak{M}_{(n,\varphi)} := \langle A \cup B, \mathcal{L}_A \cup \mathcal{L}_B\rangle,$$

will be called a *Möbius n-pair*. The Möbius configurations can be identified with the Möbius 4-pairs, which Levi graphs are Figures 1, 2, 3, 4, 5. All of them are also presented in [10]. In particular, $\mathfrak{M}_{(4,\mathrm{id})}$ is the classical (8_4) Möbius configuration.

Let $\mathcal{M}$ be a Möbius n-pair. We write: A_i, B_i for blocks of $\mathcal{M}$ not containing a_i, b_i, respectively; *a-points*, *b-points*, *A-blocks*, *B-blocks* for points in A, B, and blocks in $\mathcal{L}_A$, $\mathcal{L}_B$, respectively. The configuration $\mathcal{M}$ reflects the main abstract properties of the classical Möbius configuration.

1. The a-points yield a simplex in $\mathcal{M}$: for any $(n-1)$-subset $A \setminus \{a_i\}$ of the a-points there is a unique block of $\mathcal{M}$, which contains this subset (A_i, a face of the simplex in question); the remaining points (b-points) yield another simplex.

2. The simplex with a-points and the simplex with b-points are mutually inscribed: on each face, A_i, of the first simplex there is a unique vertex (b_i) of the second one; on each face, B_i, of the second simplex there is a unique vertex ($a_{\varphi(i)}$) of the first simplex.

Thus, we can decompose $\mathcal{M}$ into two complementary substructures $S_A(\mathcal{M}) = \langle A, \mathcal{L}_A \rangle$ and $S_B(\mathcal{M}) = \langle B, \mathcal{L}_B \rangle$, which we call *simplices of* $\mathcal{M}$ (although, formally, a block of each of them is not a subset of its points; there is one extra point on each of its faces).

In the forthcoming part we will use the notion of the incidence graph (*the Levi graph*) $\mathcal{G}_{\mathcal{M}}$ associated with $\mathcal{M}$. Recall that a Levi graph is a bipartite graph with partition induced by points vs. blocks (cf. [9, 10]). Two of its vertices x, y are said to be *adjacent* (which is written $x \sim y$) if x is a point, y is a block (or vice versa) and $x \in y$ (or $y \in x$). Otherwise x is not adjacent to y, which we write $x \nsim y$. The *rank of a vertex* is the number of vertices adjacent to it. A vertex of $\mathcal{G}_{\mathcal{M}}$ will be called point-vertex, block-vertex, a,b-vertex, A,B-vertex, or simply a_i, b_i, A_i, B_i as it corresponds to the point or to the block of $\mathcal{M}$. The Levi graph associated with $S_A(\mathcal{M}), S_B(\mathcal{M})$ will be denoted by $\mathcal{G}_{S_A(\mathcal{M})}, \mathcal{G}_{S_B(\mathcal{M})}$, respectively.

Remark 2.1. Let $\mathcal{M}$ be a Möbius n-pair. The Levi graph $\mathcal{G}_{\mathcal{M}}$ has the following properties:

(i) for $X = A, B$, every point-vertex from $\mathcal{G}_{S_X(\mathcal{M})}$ is adjacent to all but one block-vertices from $\mathcal{G}_{S_X(\mathcal{M})}$, and vice versa,

(ii) for $X, Y = A, B$ and $X \neq Y$, every point-vertex from $\mathcal{G}_{S_X(\mathcal{M})}$ is adjacent to precisely one block-vertex from $\mathcal{G}_{S_Y(\mathcal{M})}$, and vice versa.

Immediately from the definition of $\mathfrak{M}_{(n,\varphi)}$, the number of its points coincides with the number of its blocks and equals $2n$, and the rank of every point coincides with the size of every block and equals n. Thus the structures we investigate are $(2n_n)$-configurations. A standard parametric question related to configurations is: what is the number of points that are contained in two distinct blocks, and dually: what is the number of blocks containing two distinct points.

Proposition 2.2. *Let k, l be two different blocks of the structure $\mathfrak{M}_{(n,\varphi)}$. Then $|k \cap l| \in \{0, 1, 2, n-2\}$. If both k, l are A-blocks, or both k, l are B-blocks then $|k \cap l| = n - 2$. Otherwise, $k = A_i$ and $l = B_j$ for some $i, j \in I$, and the following equivalences hold*

(i) $|A_i \cap B_j| = 0$ *iff* $\varphi(j) = i = j$,

(ii) $|A_i \cap B_j| = 1$ *iff* $\varphi(j) = i \neq j$ *or* $\varphi(j) \neq i = j$,

(iii) $|A_i \cap B_j| = 2$ *iff* $\varphi(j) \neq i \neq j$.

Proof. It is straightforward from the definition that if k, l are both A-blocks or B-blocks then $k \cap l$ has $n-2$ elements. Let $k = A_i \in \mathcal{L}_A$ and $l = B_j \in \mathcal{L}_B$ for some $i, j \in I$. Let $i \neq j$. If $\varphi(j) \neq i$ then $A_i \cap B_j = \{b_i, a_{\varphi(j)}\}$. Otherwise, for $\varphi(j) = i$, we get $A_i \cap B_j = \{b_i\}$. Let $i = j$. If $\varphi(i) \neq i$ we obtain $A_i \cap B_i = \{a_{\varphi(i)}\}$. In case $\varphi(i) = i$ it holds $A_i \cap B_i = \emptyset$. □

Each conjugacy class of S_n corresponds to exactly one decomposition of a permutation $\varphi \in S_n$ into cycles, up to a permutation of the elements of I. Now we describe how the cycle structure of φ is reflected in block paths of $\mathfrak{M}_{(n,\varphi)}$.

Fact 2.3. *A permutation φ contains a cycle of length $k \leq n$ iff there is a closed path of length $2k$ consisting of blocks of $\mathfrak{M}_{(n,\varphi)}$ such that every two consecutive blocks intersect in precisely one point of $\mathfrak{M}_{(n,\varphi)}$.*

Proof. Assume that φ contains the cycle $(i_1 i_2 \ldots i_k)$. Then $a_{i_{j+1}} \in A_{i_j} \cap B_{i_j}$ and $b_{i_{j+1}} \in B_{i_j}, A_{i_{j+1}}$ for each $j \leq k$. Thus, the closed path in question is the following: A_{i_1}, B_{i_1}, $A_{i_2}, B_{i_2}, \ldots, A_{i_k}, B_{i_k}$.

Now assume that there exists a closed path $l_1, l'_1, \ldots, l_k, l'_k$ of blocks of $\mathfrak{M}_{(n,\varphi)}$ such that every two consecutive blocks intersect in a point. By Proposition 2.2(ii) every two consecutive blocks of the path are $A_i \in \mathcal{L}_A$, $B_j \in \mathcal{L}_B$ with $\varphi(j) = i \neq j$ or $\varphi(j) \neq i = j$. Suppose $\varphi(j) \neq i = j$ holds for the first two blocks of our path, namely $l_1 = A_i$, $l'_1 = B_i$ and $\varphi(i) \neq i$ for some $i \in I$. To obtain $|l'_1 \cap l_2| = 1$ we must have $l_2 = A_j$ with $\varphi(i) = j$. Thus the next two blocks are $l_2 = A_{\varphi(i)}$, $l'_2 = B_{\varphi(i)}$ and $\varphi(\varphi(i)) \neq \varphi(i)$. In general we obtain $l_j = A_{\varphi^{j-1}(i)}$, $l'_j = B_{\varphi^{j-1}(i)}$ and $\varphi^{j-1}(i) \neq \varphi^{j-2}(i)$ for every $j = 2, \ldots, k$. To close the path we need $\varphi^k(i) = i$. Let us put $i = i_0$. Then the cycle $(i_0, i_1, \ldots, i_{k-1})$, where $i_j = \varphi^j(i)$ for $j = 0, \ldots, k-1$, is one of the cycles in the cycle decomposition of φ. □

As the configuration $\mathfrak{M}_{(n,\varphi)}$ is symmetric, it makes sense to consider the dual configuration $\mathfrak{M}^{\circ}_{(n,\varphi)}$.

Fact 2.4. *The configuration $\mathfrak{M}^{\circ}_{(n,\varphi)}$ is isomorphic to $\mathfrak{M}_{(n,\varphi)}$.*

Proof. It is easy to note that $\mathfrak{M}^{\circ}_{(n,\varphi)} \cong \mathfrak{M}_{(n,\varphi^{-1})}$. Consider $\alpha \in S_n$ such that $\alpha(1) = 1$ and $\alpha(m) = n - m + 2$ for $m \in I \setminus \{1\}$. Let $x \in \{a, b, A, B\}$, $i \in I$. Then $F : x_i \mapsto x_{\alpha(i)}$ is an isomorphism mapping $\mathfrak{M}_{(n,\varphi^{-1})}$ onto $\mathfrak{M}_{(n,\varphi)}$. □

The problem of two isomorphic Möbius n-pairs will be considered in general in the last section of the paper. Another parametric characterization is now a simple consequence of Proposition 2.2 and Fact 2.4.

Proposition 2.5. *Let x, y be two different points of $\mathfrak{M}_{(n,\varphi)}$. There exist* 0, 1, 2, *or* $n-2$ *blocks of $\mathfrak{M}_{(n,\varphi)}$ containing x and y.*

3 Hidden Möbius pairs

The goal of this section is to characterize $\mathcal{M} = \mathfrak{M}_{(n,\varphi)}$ that can be transformed into Möbius pair with simplices distinct from $S_A(\mathcal{M})$, $S_B(\mathcal{M})$ by a decomposition of the points or by a deletion of some points and blocks. Informally, we say that these Möbius pairs are *hidden* in $\mathcal{M}$.

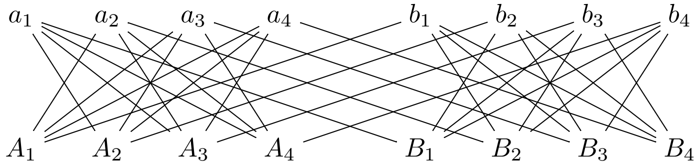

Figure 1: The Levi graph of $\mathfrak{M}_{(4,\mathrm{id})}$ (isomorphic to the hypercube graph Q_4).

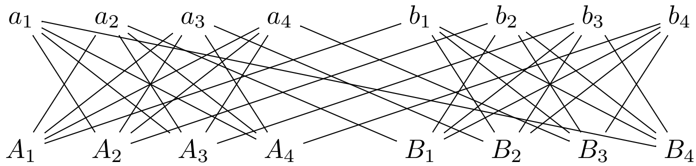

Figure 2: The Levi graph of $\mathfrak{M}_{(4,\varphi)}$ with $\varphi = (1234)$.

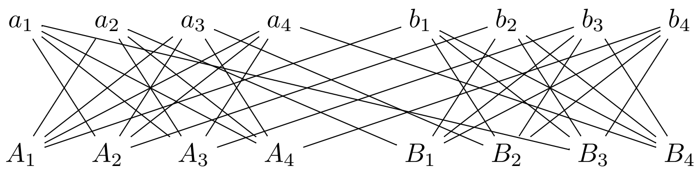

Figure 3: The Levi graph of $\mathfrak{M}_{(4,\varphi)}$ with $\varphi = (123)(4)$.

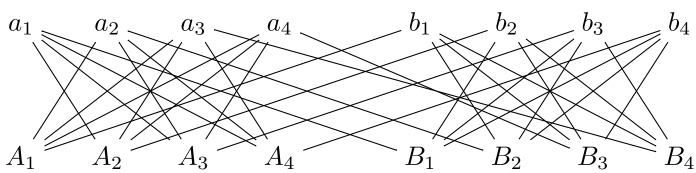

Figure 4: The Levi graph of $\mathfrak{M}_{(4,\varphi)}$ with $\varphi = (1)(2)(34)$.

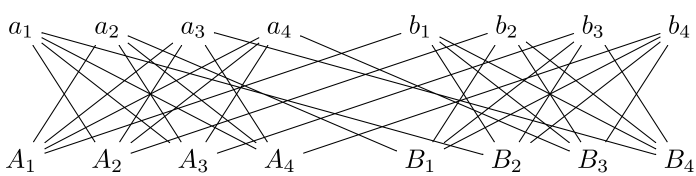

Figure 5: The Levi graph of $\mathfrak{M}_{(4,\varphi)}$ with $\varphi = (12)(34)$.

3.1 Möbius n-pairs with the special decompositions

Let us start with the following combinatorial observation:

Remark 3.1. The Möbius configuration $\mathcal{M} = \mathfrak{M}_{(4,\mathrm{id})}$ can be presented in 3 distinct ways as two mutually circumscribed simplices such that each of them is distinct from $S_A(\mathcal{M}), S_B(\mathcal{M})$.

One could say that there are four Möbius 4-pairs hidden in $\mathfrak{M}_{(4,\mathrm{id})}$. Let $n \geq 4$, $\mathcal{M} = \mathfrak{M}_{(n,\varphi)}$, and assume that it is possible to decompose the points of $\mathcal{M}$ into two complementary and mutually inscribed simplices $S_1(\mathcal{M})$, $S_2(\mathcal{M})$ such that $S_t(\mathcal{M}) \neq S_X(\mathcal{M})$ for each $t = 1, 2$, $X = A, B$. Such a decomposition will be called a *special decomposition*.

Lemma 3.2. *Let $S_1(\mathcal{M})$, $S_2(\mathcal{M})$ be two simplices, that arise from a special decomposition of $\mathcal{M}$.*

(i) *For each $i \in I$, and each $t = 1, 2$, it is impossible to have both B_i,b_i in $S_t(\mathcal{M})$, or both A_i,a_i in $S_t(\mathcal{M})$.*

(ii) *For each $t = 1, 2$, the blocks of $S_t(\mathcal{M})$ are two B-blocks and two A-blocks.*

Proof. The proof involves only $S_1(\mathcal{M})$, since the reasoning for $S_2(\mathcal{M})$ will be the same.

(i) Assume that $S_1(\mathcal{M})$ contains both of B_i, b_i. Then also some a_j is a point of $S_1(\mathcal{M})$ for $j \in I$. Consider the graph $\mathcal{G}_{\mathcal{M}}$. The vertices B_i, b_i are not adjacent, so from Remark 2.1(i) $a_j \sim B_i$ and $j = \varphi(i)$. The unique block-vertex not adjacent to a_j in $\mathcal{G}_{S_1(\mathcal{M})}$ is A_j or B_s for some $s \neq \varphi^{-1}(j)$.

Let A_j be this vertex, so from Remark 2.1(ii) $A_j \sim b_i$, and thus $j = i$. Consider in $\mathcal{G}_{S_1(\mathcal{M})}$ another vertex a_t or b_t with $t \neq i$. Since $\varphi(i) = i \neq t$, a contradiction arises: $b_t \nsim A_i$, and $a_t \nsim B_i$ (see the scheme presented in Figure 6).

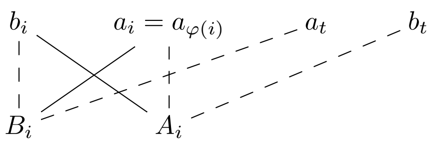

Figure 6: The fragment of $\mathcal{G}_{f(\mathcal{M})}$ containing B_i, b_i and A_i, a_i.

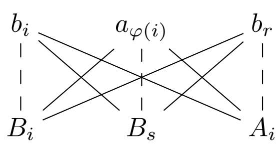

Figure 7: The fragment of $\mathcal{G}_{f(\mathcal{M})}$ containing B_i, b_i and $B_s, a_{\varphi(i)}$.

Assume that $s \neq \varphi^{-1}(j)$ and B_s is a unique vertex not adjacent to a_j in $\mathcal{G}_{S_1(\mathcal{M})}$. We get $s \neq i$, as far as $b_i \sim B_s$. Let us take another vertex: A_t or B_t. For $t \neq i$ there is no B-vertex adjacent to a_j, and $A_t \sim a_j, b_i$ if $t = i$. A vertex, which is not adjacent to A_i, is a_i or b_r with $r \neq i, s$. The vertex a_i is not adjacent to B_i since $\varphi(i) = j \neq i$, and thus a_i cannot be the vertex in question. Consequently, this vertex is $b_r \sim B_i, B_s$. Following the assumption $n \geq 4$, there exists another block in $S_1(\mathcal{M})$, that is different from B_i, B_s, A_i. We have two b-points in $S_1(\mathcal{M})$ so far, thus this block is a B-block. The B-vertex of $\mathcal{G}_{\mathcal{M}}$, that is associated with this block, must be adjacent to $a_{\varphi(i)}$. So this block is B_i, which is already one of the blocks in $S_1(\mathcal{M})$ (comp. with the scheme presented in Figure 7), a contradiction.

(ii) Let B_i be the unique B-block of $S_1(\mathcal{M})$ for some $i \in I$. Then the remaining blocks of $S_1(\mathcal{M})$ are A-blocks. In view of Lemma 3.2(i), there are $n-1$ b-vertices in $\mathcal{G}_{S_1(\mathcal{M})}$: every A-vertex is associated with the b-vertex, which is not adjacent to it. For $n \geq 4$ a contradiction with Remark 2.1(i) arises: every b-vertex is adjacent to precisely one of A-vertices, and thus it is not adjacent to at least two A-vertices in $\mathcal{G}_{S_1(\mathcal{M})}$.
Let $S_1(\mathcal{M})$ contain at least three B-blocks. Without loss of generality, assume B_1, B_2, B_3 are blocks of $S_1(\mathcal{M})$. From Lemma 3.2(i), b_1, b_2, b_3 are not in $S_1(\mathcal{M})$. Thus, from Remark 2.1(i), $S_1(\mathcal{M})$ contains $a_{i_1}, a_{i_2}, a_{i_3}$ such that $i_j \neq \varphi(j)$ for $j = 1, 2, 3$. Every block-vertex B_j must be adjacent to at least two of the point-vertices $a_{i_{j'}}$ with $j' \neq j$. On the other hand, it is adjacent to at most one of them, what follows from Remark 2.1(ii) applied to $\mathcal{G}_{\mathcal{M}}$. This contradiction actually completes the proof as other cases run dually. □

By Lemma 3.2 we prove a generalization of Remark 3.1.

Proposition 3.3. *Let $\mathcal{M} = \mathfrak{M}_{(n,\varphi)}$. The following conditions are equivalent*

(i) *there is a special decomposition of $\mathcal{M}$,*

(ii) *$n = 4$ and there is $X \subset I$ such that $|X| = 2$ and $\varphi(X) = X$.*

Proof. (i) ⇒ (ii): From Lemma 3.2(ii) we get $n = 4$, and two B-vertices and two A-vertices in $\mathcal{G}_{S_1(\mathcal{M})}$. Let (e.g.) B_1, B_2 be the B-vertices of $\mathcal{G}_{S_1(\mathcal{M})}$. In view of Remark 2.1(i), there are vertices x, y in $\mathcal{G}_{S_1(\mathcal{M})}$ such that $x \sim B_1, y \sim B_2$ and $x \nsim B_2, y \nsim B_1$. By Lemma 3.2(i), $x \neq b_2, y \neq b_2$, and thus $x = a_i, y = a_j$ where $\varphi(1) = i, \varphi(2) = j$. Then two A-vertices in $\mathcal{G}_{S_1(\mathcal{M})}$ are A_s, A_t with $s, t \neq i, j$. The remaining two point-vertices must be of the form $b_{s'}, b_{t'}$ with $s', t' \neq 1, 2$, since they must be adjacent to both of B_1, B_2. On the other hand, $b_{s'}, b_{t'}$ need to be adjacent to precisely one of A_s, A_t, so $\{s', t'\} = \{s, t\}$. Thus $s, t \neq 1, 2$, $\{1, 2\} = \{i, j\} = \{\varphi(1), \varphi(2)\}$, and $X = \{1, 2\}$ is the required set.

(ii) ⇒ (i): Assume, without loss of generality, $X = \{1, 2\}$ and consider $\mathcal{M} = \mathfrak{M}_{(4,\varphi)}$ with $\varphi(X) = X$. Take blocks B_1, B_2, A_3, A_4 and points $a_{\varphi(1)}, a_{\varphi(2)}, b_3, b_4$ of $\mathcal{M}$, and consider $\mathcal{G}_{\mathcal{M}}$. We have $B_1 \nsim a_{\varphi(2)}$, $B_2 \nsim a_{\varphi(1)}$, and $B_1, B_2 \sim b_3, b_4$. Similarly $A_3 \nsim b_4$, $A_4 \nsim b_3$, and $A_3, A_4 \sim a_{\varphi(1)}, a_{\varphi(2)}$, since $\varphi(1), \varphi(2) \in \{1, 2\}$. Thus the Levi graph we consider is a Levi graph of a 4-simplex. It is easy to verify that A_1, A_2, B_3, B_4 and b_1, b_2, a_3, a_4 form another 4-simplex. The two obtained simplices are mutually circumscribed. Indeed, B_1, b_2; B_2, b_1; A_3, a_4; A_4, a_3, and $A_1, a_{\varphi(1)}$ (or $A_1, a_{\varphi(2)}$); $A_2, a_{\varphi(2)}$ (or $A_2, a_{\varphi(1)}$); B_3, b_4; B_4, b_3 are all pairs of adjacent vertices representing blocks (points) of the first simplex and points (blocks) of the second simplex in each pair. In other words, we have found a special decomposition of $\mathcal{M}$. □

Due to Proposition 3.3 there is a correspondence between the special decompositions of $\mathfrak{M}_{(n,\varphi)}$ and 2-subsets of I preserved by φ. The correspondence is established up to complements, since the special decompositions arise only for $n = 4$, and thus if φ preserves a 2-subset of $\{1, 2, 3, 4\}$ then it preserves its complement as well. So, directly from Proposition 3.3 we get:

Corollary 3.4. *All (up to an isomorphism) Möbius n-pairs with a special decomposition are the following:*

1. $\mathfrak{M}_{(4,\mathrm{id})}$ *with* 3 *distinct special decompositions associated with* $X = \{1, 2\}, \{1, 3\}, \{1, 4\}$,
2. $\mathfrak{M}_{(4,(13)(24))}$ *with the special decomposition associated with* $X = \{1, 3\}$,
3. $\mathfrak{M}_{(4,(12)(3)(4))}$ *with the special decomposition associated with* $X = \{1, 2\}$.

3.2 Subpairs of Möbius n-pairs

Let $\mathcal{M} = \mathfrak{M}_{(n,\varphi)}$, $n \geq 4$, $k \geq 3$, $k < n$, and $\mathcal{M}'$ be a Möbius k-pair obtained from $\mathcal{M}$ by deleting $2(n-k)$ points and $2(n-k)$ blocks. We call $\mathcal{M}'$ a *k-subpair* of $\mathcal{M}$. The blocks of $\mathcal{M}'$ are *subblocks of* $\mathcal{M}$, that is every block of $\mathcal{M}'$ arises as a block of $\mathcal{M}$ with $n-k$ points removed. The subblocks of the A-blocks, the B-blocks are called the *A-subblocks*, the *B-subblocks*, respectively. Let $S_1(\mathcal{M}')$, $S_2(\mathcal{M}')$ be two simplices of $\mathcal{M}'$. For any $t = 1, 2$, $X = A, B$ we write $S_t(\mathcal{M}') \prec S_X(\mathcal{M})$ if all the points and the blocks of $S_t(\mathcal{M}')$ are points and subblocks of $S_X(\mathcal{M})$. Otherwise we write $S_i(\mathcal{M}') \not\prec S_X(\mathcal{M})$. For $Y \subset I$ by $\varphi \restriction_Y$ we mean the restriction of φ to the set Y.

In order to determine all Möbius n-pairs with k-subpairs we need to prove some auxiliary facts.

Lemma 3.5. *One of the following conditions holds*

(i) $S_1(\mathcal{M}') \prec S_A(\mathcal{M})$ *and* $S_2(\mathcal{M}') \prec S_B(\mathcal{M})$,

(ii) $S_2(\mathcal{M}') \prec S_A(\mathcal{M})$ *and* $S_1(\mathcal{M}') \prec S_B(\mathcal{M})$,

(iii) $S_1(\mathcal{M}') \not\prec S_A(\mathcal{M}), S_B(\mathcal{M})$ *and* $S_2(\mathcal{M}') \not\prec S_A(\mathcal{M}), S_B(\mathcal{M})$.

Moreover, if $\mathcal{M}'$ satisfies (iii) *then there is a special decomposition of $\mathcal{M}'$.*

Proof. Let $S_1(\mathcal{M}') \prec S_A(\mathcal{M})$ and $S_2(\mathcal{M}') \not\prec S_B(\mathcal{M})$. So there is an a-point or A-subblock in $S_2(\mathcal{M}')$. We consider only the case with an a-point, as the case with an A-subblock is symmetric. From Remark 2.1(ii) applied to $\mathcal{G}_{\mathcal{M}}$, and Remark 2.1(i) applied to $\mathcal{G}_{\mathcal{M}'}$, there are at most two B-subblocks in $S_2(\mathcal{M}')$. Since $k \geq 3$, there is at least one A-subblock in $S_2(\mathcal{M}')$. Note that a unique A-subblock, which does not contain an a-point of $S_1(\mathcal{M}')$, is a block of $S_1(\mathcal{M}')$. Thus all the points of $S_1(\mathcal{M})$ are in an A-subblock of $S_2(\mathcal{M}')$. This yields a contradiction with Remark 2.1(ii). The proof for each of the remaining cases (i.e. $S_2(\mathcal{M}') \prec S_A(\mathcal{M})$ and $S_1(\mathcal{M}') \not\prec S_B(\mathcal{M})$, $S_1(\mathcal{M}') \prec S_B(\mathcal{M})$ and $S_2(\mathcal{M}') \not\prec S_A(\mathcal{M})$, or $S_2(\mathcal{M}') \prec S_B(\mathcal{M})$ and $S_1(\mathcal{M}') \not\prec S_A(\mathcal{M})$) is analogous.

Let $\mathcal{M}'$ satisfy (iii). The steps of the proof of Lemma 3.2 can be repeated for simplices of $\mathcal{M}'$. As a result we get $k = 4$, and two A-subblocks and two B-subblocks in each of simplices of $\mathcal{M}'$. Let $Y \subset I$ be the set of subscripts of A-subblocks and B-subblocks in one of these simplices. From the reasoning analogous to the first part of the proof of Proposition 3.3 we get that Y is the set of all the subscripts used for labelling the points and the blocks of $\mathcal{M}'$, and there is a two-element set $X \subset Y$ such that $\varphi \restriction_Y (X) = X$. Therefore, in view of Proposition 3.3, there is a special decomposition of $\mathcal{M}'$. □

Lemma 3.6. *If the number of deleted B-blocks and the number of deleted A-blocks coincide (and equals $n - k$), then there is $X \subset I$ such that $|X| = n - k$ and $\varphi(X) = X$.*

Proof. Assume that $B_{i_1}, \dots, B_{i_{n-k}}$ and $A_{j_1}, \dots, A_{j_{n-k}}$ are removed blocks. Consider a vertex $a_{\varphi(i_s)}$ with $s = 1, \dots, n-k$ of $\mathcal{G}_{\mathcal{M}'}$, and assume $a_{\varphi(i_s)}$ is in $\mathcal{G}_{S_1(\mathcal{M}')}$ (the case with $a_{\varphi(i_s)}$ in $\mathcal{G}_{S_2(\mathcal{M}')}$ will be analogous). Note that $a_{\varphi(i_s)} \sim B_{i_s}$, and from Remark 2.1(ii) B_{i_s} is a unique B-vertex adjacent to $a_{\varphi(i_s)}$. According to Lemma 3.5 two cases arise: (i) or (iii) holds for $\mathcal{M}'$. Let $\mathcal{M}'$ satisfy (i) of Lemma 3.5. Then there is a B-vertex in $\mathcal{G}_{S_2(\mathcal{M}')}$ adjacent to $a_{\varphi(i_s)}$, a contradiction. If $\mathcal{M}'$ satisfies (iii) of Lemma 3.5 then there is a special decomposition of $\mathcal{M}'$. So, by Proposition 3.3, there is a B-vertex in $\mathcal{G}_{S_1(\mathcal{M}')}$ adjacent to $a_{\varphi(i_s)}$, a contradiction again. Therefore all $a_{\varphi(i_1)} \dots, a_{\varphi(i_{n-k})}$ are removed. Likewise we consider the pairs a_{j_s}, A_{j_s}, b_{j_s}, A_{j_s}, and b_{i_s}, B_{i_s}. Each of these reasonings leads us to contradiction. Consequently points $a_{j_1} \dots, a_{j_{n-k}}$, $b_{j_1} \dots, b_{j_{n-k}}$, and $b_{i_1} \dots, b_{i_{n-k}}$ are deleted as well. Hence

$$\{j_1, \dots, j_{n-k}\} = \{\varphi(i_1), \dots, \varphi(i_{n-k})\} \text{ and } \{i_1, \dots, i_{n-k}\} = \{j_1, \dots, j_{n-k}\}.$$

Finally we get $\{\varphi(i_1), \dots, \varphi(i_{n-k})\} = \{i_1, \dots, i_{n-k}\}$, and $X = \{i_1, \dots, i_{n-k}\}$ is the set from our claim. $\square$

Let us present a condition, which is sufficient and necessary to find a k-subpair in $\mathfrak{M}_{(n,\varphi)}$.

Proposition 3.7. *Let $\mathcal{M} = \mathfrak{M}_{(n,\varphi)}$. The following conditions are equivalent*

(i) *there is $\mathcal{M}'$, which is a k-subpair of $\mathcal{M}$,*

(ii) *there is $X \subset I$ such that $|X| = n-k$ and $\varphi(X) = X$.*

Furthermore, if (ii) *holds then $\mathcal{M}' \cong \mathfrak{M}_{(k,\varphi\restriction(I\setminus X))}$.*

Proof. (i) $\Rightarrow$ (ii): By Lemma 3.5 $\mathcal{M}'$ satisfies one of (i) – (iii) of Lemma 3.5. In cases (i) and (ii) of Lemma 3.5 the numbers of A-blocks and B-blocks deleted from $\mathcal{M}$ coincide and are equal to $n-k$. The claim follows directly from Lemma 3.6. If case (iii) of Lemma 3.5 holds, then there is a special decomposition of $\mathcal{M}'$, and we get our claim by Proposition 3.3.

(ii) $\Rightarrow$ (i): Without any loss of generality, let $X = \{1, \dots, n-k\}$. Recall that the rank of every vertex in $\mathcal{G}_{\mathcal{M}}$ is n. Observe the Levi graph obtained from $\mathcal{G}_{\mathcal{M}}$ by removing the vertices a_i, A_i and b_i, B_i for every $i \in X$, and all edges passing through these vertices. We denote this Levi graph by $\mathcal{H}$. Note that $a_{\varphi(i)}$ is not a vertex of $\mathcal{H}$, since $\varphi(i) \in X$. Let $j \notin X$ and take A_j. Clearly A_j is a vertex of $\mathcal{H}$. There are $n-k$ edges joining A_j with all a_i in $\mathcal{G}_{\mathcal{M}}$. Thus, the rank of A_j in $\mathcal{H}$ is $n-(n-k) = k$. Similarly we set ranks of the remaining vertices a_j, b_j, B_j of $\mathcal{H}$. All these ranks are k. From this and the construction of $\mathcal{H}$ we get that $\mathcal{H}$ is the Levi graph of two mutually circumscribed k-simplices, where the way they are inscribed one into another is induced by the action of φ on the set $I \setminus X$. Therefore $\mathcal{H} = \mathcal{G}_{\mathcal{M}'}$ for some $\mathcal{M}'$, which is a k-subpair of $\mathcal{M}$. $\square$

4 Isomorphisms and automorphisms

4.1 Isomorphic Möbius n-pairs

Recall that the Möbius (8_4) configurations (i.e. Möbius 4-pairs) correspond to conjugacy classes of the permutation group S_4. In this section we generalize this property to all Möbius n-pairs.

Let us start with a key lemma that gives an account on isomorphisms of configurations $\mathfrak{M}_{(n,\varphi)}$ with the unique decomposition into two n-simplices.

Lemma 4.1. *Let f be an isomorphism mapping $\mathfrak{M}_{(n,\varphi)}$ onto $\mathfrak{M}_{(n,\psi)}$. Assume that either $n = 4$ and both $\varphi, \psi \neq \mathrm{id}$ contain no cycle of length 2, or $n \geq 5$. There is $\alpha \in S_n$ such that $f(B_i) = B_{\alpha(i)}$ for each $i \in I$, or $f(B_i) = A_{\alpha(i)}$ for each $i \in I$.*

(i) *If $f(B_i) = B_{\alpha(i)}$ then $f(b_i) = b_{\alpha(i)}$, $f(A_i) = A_{\alpha(i)}$, $f(a_i) = a_{\alpha(i)}$ for each $i \in I$.*

(ii) *If $f(B_i) = A_{\alpha(i)}$ then $f(b_i) = a_{\alpha(i)}$, $f(A_i) = B_{\psi^{-1}(\alpha(i))}$, $f(a_i) = b_{\psi^{-1}(\alpha(i))}$ for each $i \in I$.*

Furthermore, $\alpha\varphi = \psi\alpha$ holds in both cases: (i) *and* (ii).

Proof. Let $\mathcal{M}_1 := \mathfrak{M}_{(n,\varphi)}$ and $\mathcal{M}_2 := \mathfrak{M}_{(n,\psi)}$. Let $i, j \in I$ and B_i be an arbitrary B-block of $\mathcal{M}_1$. Clearly, either $f(B_i) = B_j$ for some B-block B_j of $\mathcal{M}_2$, or $f(B_i) = A_j$ for some A-block A_j of $\mathcal{M}_2$.

Assume that $f(B_i) = B_j$. In view of Corollary 3.4, both $\mathcal{M}_1$, $\mathcal{M}_2$ are Möbius n-pairs without the special decompositions. Thus all the B-blocks of $\mathcal{M}_1$ are mapped onto B-blocks of $\mathcal{M}_2$. We introduce a map $\alpha \in S_n$ associated with f by the formula

$$\alpha \colon i \mapsto j \text{ iff } f(B_i) = B_j,$$

for all $i, j \in I$. Then $f(B_i) = B_{\alpha(i)}$. Let us analyze graphs $\mathcal{G}_{\mathcal{M}_1}$ and $\mathcal{G}_{\mathcal{M}_2}$: $f(b_i) = b_{\alpha(i)}$ as b_i, $b_{\alpha(i)}$ are unique b-vertices not adjacent to B_i, $B_{\alpha(i)}$ respectively in graphs $\mathcal{G}_{\mathcal{M}_1}$, $\mathcal{G}_{\mathcal{M}_2}$; $f(A_i) = A_{\alpha(i)}$ as A_i, $A_{\alpha(i)}$ are unique A-vertices adjacent to b_i, $b_{\alpha(i)}$ respectively in $\mathcal{G}_{\mathcal{M}_1}$, $\mathcal{G}_{\mathcal{M}_2}$; $f(a_i) = a_{\alpha(i)}$ as a_i, $a_{\alpha(i)}$ are unique a-vertices not adjacent to A_i, $A_{\alpha(i)}$ respectively in $\mathcal{G}_{\mathcal{M}_1}$, $\mathcal{G}_{\mathcal{M}_2}$. On the other hand, $f(a_{\varphi(i)}) = a_{\psi(\alpha(i))}$ as $a_{\varphi(i)}$, $a_{\psi(\alpha(i))}$ are unique a-vertices adjacent to B_i, $B_{\alpha(i)}$ respectively in $\mathcal{G}_{\mathcal{M}_1}$, $\mathcal{G}_{\mathcal{M}_2}$. So $a_{\alpha(\varphi(i))} = a_{\psi(\alpha(i))}$ and thus $\alpha\varphi = \psi\alpha$.

In case $f(B_i) = A_j$ the map $\alpha \in S_n$ is determined by the condition

$$\alpha \colon i \mapsto j \text{ iff } f(B_i) = A_j,$$

for all $i, j \in I$. Then we proceed in a similar way as in the former case, namely: $f(b_i) = a_{\alpha(i)}$ as b_i, $a_{\alpha(i)}$ are a unique b-vertex and a-vertex not adjacent to B_i, $A_{\alpha(i)}$ respectively in $\mathcal{G}_{\mathcal{M}_1}$, $\mathcal{G}_{\mathcal{M}_2}$; $f(A_i) = B_{\psi^{-1}(\alpha(i))}$ as A_i, $B_{\psi^{-1}(\alpha(i))}$ are a unique A-vertex and B-vertex adjacent to b_i, $a_{\alpha(i)}$ respectively in $\mathcal{G}_{\mathcal{M}_1}$, $\mathcal{G}_{\mathcal{M}_2}$; $f(a_i) = b_{\psi^{-1}(\alpha(i))}$ as a_i, $b_{\psi^{-1}(\alpha(i))}$ are a unique a-vertex and b-vertex not adjacent to A_i, $B_{\psi^{-1}(\alpha(i))}$ respectively in $\mathcal{G}_{\mathcal{M}_1}$, $\mathcal{G}_{\mathcal{M}_2}$. But also $f(a_{\varphi(i)}) = b_{\alpha(i)}$ as $a_{\varphi(i)}$, $b_{\alpha(i)}$ are a unique a-vertex and b-vertex adjacent to B_i, $A_{\alpha(i)}$ respectively in $\mathcal{G}_{\mathcal{M}_1}$, $\mathcal{G}_{\mathcal{M}_2}$. Hence $b_{\psi^{-1}(\alpha(\varphi(i)))} = b_{\alpha(i)}$, and consequently $\alpha\varphi = \psi\alpha$. □

We are ready to characterize two isomorphic Möbius n-pairs.

Theorem 4.2. *Let $n \geq 4$ and $\varphi, \psi, \alpha \in S_n$. The following conditions are equivalent:*

(i) $\varphi^\alpha = \psi$,

(ii) $\mathfrak{M}_{(n,\varphi)} \cong \mathfrak{M}_{(n,\psi)}$.

Proof. Let $\mathcal{M}_1 = \mathfrak{M}_{(n,\varphi)}$ and $\mathcal{M}_2 = \mathfrak{M}_{(n,\psi)}$.

(i) $\Rightarrow$ (ii): Let $i \in I$, a_i, b_i be points and A_i, B_i be blocks of $\mathcal{M}_1$. Consider a map f associated to the permutation α given by the formula

$$f(x_i) = x_{\alpha(i)} \text{ for } x \in \{a, b\},$$

which maps the points of $\mathcal{M}_1$ onto the points of $\mathcal{M}_2$. Then $f(A_i) = A_{\alpha(i)}$ and $f(B_i) = B_{\alpha(i)}$, as the conditions $a_i \notin A_i$, $b_i \notin B_i$ uniquely determine blocks A_i, B_i, respectively. Clearly, conditions $b_i \in A_i$ and $b_{\alpha(i)} \in A_{\alpha(i)}$ are equivalent. Note that $a_{\alpha(\varphi(i))} \in B_{\alpha(i)}$ is equivalent to $a_{\psi(\alpha(i))} \in B_{\alpha(i)}$ as well, since $\alpha\varphi = \psi\alpha$. Thus f is the required isomorphism.

(ii) ⇒ (i): We restrict ourselves to $n \geq 5$ since for $n = 4$ this fact is well known, as it was mentioned at the beginning of this section. Let f be an isomorphism mapping $\mathcal{M}_1$ onto $\mathcal{M}_2$. By Lemma 4.1, there is $\alpha \in S_n$ associated with f such that $\alpha\varphi = \psi\alpha$. □

According to Theorem 4.2, the number of non-isomorphic configurations $\mathfrak{M}_{(n,\varphi)}$ is equal to the number of partitions $p(n)$ of a positive integer n. There is the generating function, recursive formula, asymptotic formula, and direct formula for $p(n)$ (cf. [1]). The increase of n implies quick growth of $p(n)$: $p(5) = 7, p(6) = 11, \ldots, p(100) = 190569292, \ldots, p(1000) = 24061467864032622473692149727991$.

4.2 The automorphism group structure of a Möbius n-pair

For $n = 3$ the structure $\mathfrak{M}_{(n,\varphi)}$ consists of two mutually inscribed triangles. From [8] the automorphism group of $\mathfrak{M}_{(3,\varphi)}$ is isomorphic to $S_3 \ltimes C_2$. From the original paper of Möbius [7] the automorphism group of $\mathfrak{M}_{(4,\mathrm{id})}$ has order 192. The Möbius configuration is also a particular case of the Cox configuration. Recall the definition of the Cox configuration (comp. [3]). Let X be a set with n elements. The incidence structure

$$(\mathbf{Cx})_X = (\mathbf{Cx})_n = \left\langle \bigcup \{\wp_{2k+1}(X) : 0 \leq k \leq n\}, \{\wp_{2k}(X) \colon 0 \leq k \leq n\}, \subset \cup \supset \right\rangle$$

is the $(2^{n-1}{}_n)$ configuration, which is called the Cox configuration. Since the automorphism group of $(\mathbf{Cx})_n$ is established in [11] and $\mathfrak{M}_{(4,\mathrm{id})} = (\mathbf{Cx})_4$ (see Figure 8), we get the following:

Fact 4.3. *The automorphism group of* $\mathfrak{M}_{(4,\mathrm{id})}$ *is isomorphic to* $S_4 \ltimes C_2^3$.

It follows from Theorem 4.2 that the centralizer of φ in S_n consists of automorphisms of $\mathfrak{M}_{(n,\varphi)}$ for any n. Nevertheless, we will give a detailed characterization of automorphism group of $\mathfrak{M}_{(4,\varphi)}$ with $\varphi \neq \mathrm{id}$, and of $\mathfrak{M}_{(n,\varphi)}$ with $n \geq 5$.

Let $\mathcal{M} = \mathfrak{M}_{(n,\varphi)}$ and $1 \leq \nu_1 < \ldots < \nu_r$ be the lengths of the cycles which are contained in the cycle decomposition of $\varphi \in S_n$. Assume that there are m_t cycles of length ν_t, so $n = \sum_{t=1}^{r} m_t \nu_t$. In other words

$$\varphi = \varphi_1^{\nu_1}\varphi_2^{\nu_1} \ldots \varphi_{m_1}^{\nu_1}\varphi_1^{\nu_2}\varphi_2^{\nu_2} \ldots \varphi_{m_2}^{\nu_2} \ldots \varphi_1^{\nu_r}\varphi_2^{\nu_r} \ldots \varphi_{m_r}^{\nu_r},$$

where $\varphi_k^{\nu_t}$ is a cycle of length ν_t for $k \leq m_t, t \leq r$. In view of Theorem 4.2 we can assume, that each cycle consists of consecutive natural numbers. If we set $\mu_k^t := \sum_{i=1}^{t-1} m_i \nu_i + (k-1)\nu_t + 1$ then

$$\varphi_k^{\nu_t} : \mu_k^t \mapsto \mu_k^t + 1 \mapsto \mu_k^t + 2 \mapsto \ldots \mapsto \mu_k^t + (\nu_t - 1) \mapsto \mu_k^t,$$

and the effective domain of $\varphi_k^{\nu_t}$ is the set $X_k^{\nu_t} := \{\mu_k^t, \mu_k^t + 1, \ldots, \mu_k^t + (\nu_t - 1)\} \subseteq I$. Taking all the domains of all the cycles we obtain the family of pairwise disjoint sets $X_1^{\nu_1}, \ldots, X_{m_1}^{\nu_1}, X_1^{\nu_2}, \ldots, X_{m_2}^{\nu_2}, \ldots, X_1^{\nu_r}, \ldots, X_{m_r}^{\nu_r}$ that yields a covering of I. Thus for any cycle $\varphi_k^{\nu_t}$ we have $\varphi_k^{\nu_t}(X_k^{\nu_t}) = X_k^{\nu_t}$ and $\varphi_k^{\nu_t} \upharpoonright_{I \setminus X_k^{\nu_t}} = \mathrm{id}$.

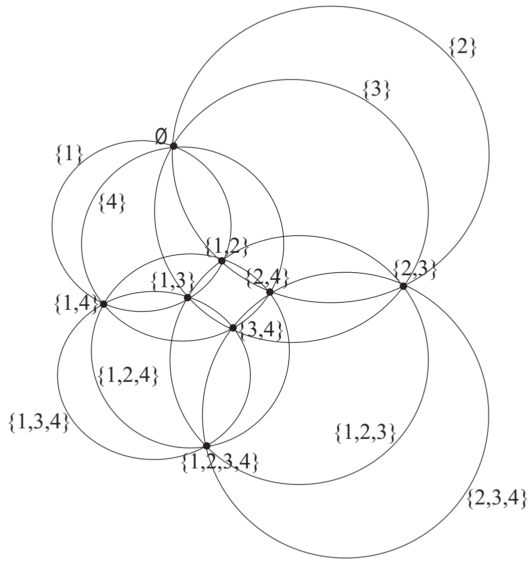

Figure 8: The Möbius configuration as $(\mathbf{Cx})_4$.

The points and the blocks of $\mathcal{M}$ can be identified with the sequences (t, k, i, ε) such that $t \leq r$, $k \leq m_t$, $i = 0, \dots, \nu_t - 1$, and $\varepsilon \in \{1, 2, -1, -2\}$ according to the formula:

$$(t, k, i, \varepsilon) = \begin{cases} a_{i+\mu_k^t} & \text{for } \varepsilon = 1, \\ b_{i+\mu_k^t} & \text{for } \varepsilon = -1, \\ A_{i+\mu_k^t} & \text{for } \varepsilon = 2, \\ B_{i+\mu_k^t} & \text{for } \varepsilon = -2. \end{cases} \tag{4.1}$$

Let $v_t = (v_1^t, \dots, v_{m_t}^t) \in C_{\nu_t}^{m_t}$, $\alpha_t \in S_{m_t}$, and $v = (v_1, \dots, v_r) \in \times_{t=1}^{r} C_{\nu_t}^{m_t}$, $\alpha = (\alpha_1, \dots, \alpha_r) \in \times_{t=1}^{r} S_{m_t}$. With the pair (v, α) we associate the map $f_{(v,\alpha)}$ as follows:

$$f_{(v,\alpha)}((t, k, i, \varepsilon)) = (t, \alpha_t(k), i + v_k^t \bmod \nu_t, \varepsilon). \tag{4.2}$$

In like manner we define the map $g_{(v,\alpha)}$ by:

$$g_{(v,\alpha)}((t, k, i, \varepsilon)) = \begin{cases} (t, \alpha_t(k), i + v_k^t - 1 \bmod \nu_t, -\varepsilon) & \text{for } \varepsilon = 1, 2, \\ (t, \alpha_t(k), i + v_k^t \bmod \nu_t, -\varepsilon) & \text{for } \varepsilon = -1, -2. \end{cases} \tag{4.3}$$

Lemma 4.4. *The map $f_{(v,\alpha)}$ is an automorphism of $\mathcal{M}$, which preserves each of simplices S_A, S_B.*

Proof. It follows directly from (4.2), that $f_{(v,\alpha)}$ maps S_A onto S_A, and $f_{(v,\alpha)}$ maps S_B onto S_B. Let $i \in X_k^{\nu_t}$ and $j \in I$.

Assume that $b_j \in B_i$. By (4.1), $B_i = (t, k, i_0, -2)$ for some $i_0 \in \{0, \dots, \nu_t - 1\}$, and $b_j = (t', k', j_0, -1)$ for some $t' \leq r$, $k' \leq m_{t'}$, $j_0 \in \{0, \dots, \nu_{t'} - 1\}$. Then $f(B_i) = (t, \alpha_t(k), i_0 + v_{\alpha_t(k)}^t \bmod \nu_t, -2)$ and $f(b_j) = (t', \alpha_{t'}(k'), j_0 + v_{\alpha_{t'}(k')}^{t'} \bmod \nu_{t'}, -2)$. Set $i' = (i_0 + v_{\alpha_t(k)}^t \bmod \nu_t) + \mu_{\alpha_t(k)}^t$ and $j' = (j_0 + v_{\alpha_{t'}(k')}^{t'} \bmod \nu_{t'}) + \mu_{\alpha_{t'}(k')}^{t'}$, so

$f(B_i) = B_{i'}$ and $f(b_j) = B_{j'}$. Recall that $b_j \in B_i$ iff $j \neq i$. If $j' \neq i'$ then: firstly $t' = t$, next $\alpha_t(k') = \alpha_t(k)$ and thus $k' = k$, and finally $j_0 = i_0$. It means that $j = i$, which yields a contradiction. Hence $f(b_j) \in f(B_i)$.

Let $a_j \in B_i$. Then $j = \varphi(i)$. We have $a_{\varphi(i)} = (t, k, i_0 + 1 \bmod \nu_t, 1)$, so $f(a_{\varphi(i)}) = (t, k, i_0 + 1 + v_k^t \bmod \nu_t, 1) = a_{\varphi(i')}$. Therefore $f(a_{\varphi(i)}) \in f(B_i)$.

The incidence (membership) relation is preserved by $f_{(v,\alpha)}$ in case $a_j \in A_i$ and in case $b_j \in A_i$ as well, that can be easily proved by similar reasoning. □

Let $\mathbf{v}_t = (\underbrace{v, \ldots, v}_{t})$ for all $t \leq r$, and $\mathbf{v} = (\mathbf{v}_1, \ldots, \mathbf{v}_r)$. Let us put $g_0 := g_{(\mathbf{0},\mathrm{id})}$.

Lemma 4.5. *The map g_0 is an automorphism of $\mathcal{M}$, which interchanges simplices S_A, S_B.*

Proof. Immediately from (4.2), g_0 maps S_A onto S_B, and S_B onto S_A. We restrict our proof to the incidence relation involving the B-blocks, as the case with the A-blocks runs similarly. Let $i \in X_k^{\nu_t}$. From (4.1) B_i is represented by the sequence $(t, k, i_0, -2)$ for some $i_0 \in \{0, \ldots, \nu_t - 1\}$. The points that belongs to B_i are b_j with $j \in I \setminus \{i\}$ and $a_{\varphi(i)}$. Clearly, $g_0(b_j) = a_j \in A_i = g_0(B_i)$. We have $a_{\varphi(i)} = (t, k, i_0 + 1 \bmod \nu_t, 1)$ and thus $g_0(a_{\varphi(i)}) = (t, k, i_0, -1) = b_i$. Then finally $g_0(a_{\varphi(i)}) \in g_0(B_i)$. □

Since $g_{(v,\alpha)} = g_0 f_{(v,\alpha)}$, from Lemma 4.4 and Lemma 4.5 we infer that:

Corollary 4.6. *The map $g_{(v,\alpha)}$ is an automorphism of $\mathcal{M}$, which interchanges simplices S_A, S_B.*

We write $\mathcal{M}_k^{\nu_t}$ for the set of all the points and the blocks of $\mathcal{M}$ labelled by the elements of the set $X_k^{\nu_t}$, and $\mathcal{M}^{\nu_t} = \{\mathcal{M}_k^{\nu_t} : k \leq m_t\}$.

Lemma 4.7. *Let f be an automorphism of $\mathcal{M}$, which*

(1) maps the B-blocks onto the B-blocks, or

(2) maps the B-blocks onto the A-blocks.

There is $v \in \bigtimes_{t=1}^{r} C_{\nu_t}^{m_t}$ and $\alpha \in \bigtimes_{t=1}^{r} S_{m_t}$ such that

(i) $f = f_{(v,\alpha)}$ *in case* (1)*, or*

(ii) $f = g_{(v,\alpha)}$ *in case* (2)*.*

In particular, for each $k \leq m_t$ there is $k' \leq m_t$ such that $f(\mathcal{M}_k^{\nu_t}) = \mathcal{M}_{k'}^{\nu_t}$.

Proof. (i): Let $i \in X_k^{\nu_t}$. Assume that $f(B_i) = B_j$ for some $j \in I$. According to (4.1) there is $i_0 \in \{0, \ldots, \nu_t - 1\}$ such that $B_i = (t, k, i_0, -2)$, and $j_0 \in \{0, \ldots, \nu_{t'} - 1\}$ such that $B_j = (t', k', j_0, -2)$ for some $t' \leq r$, $k' \leq m_{t'}$. Then, by Lemma 4.1(ii) we get $f((t, k, i_0, \varepsilon)) = (t', k', j_0, \varepsilon)$ for each value of ε. The unique B-block containing $a_i = (t, k, i_0, 1)$ is $B_{\varphi^{-1}(i)} = (t, k, i_0 - 1 \bmod \nu_t, -2)$, and a unique B-block containing a_j is $B_{\varphi^{-1}(j)} = (t', k, j_0 - 1 \bmod \nu_{t'}, -2)$. Hence, f maps $(t, k, i_0 - 1 \bmod \nu_t, -2)$ onto $(t', k', j_0 - 1 \bmod \nu_{t'}, -2)$, and f maps $(t, k, i_0 - 1 \bmod \nu_t, \varepsilon)$ onto $(t', k', j_0 - 1 \bmod \nu_{t'}, \varepsilon)$ generally. By induction we get

$$f\colon (t, k, i_0 - u \bmod \nu_t, \varepsilon) \mapsto (t', k', j_0 - u \bmod \nu_{t'}, \varepsilon) \text{ for all } u = 0, \ldots, \nu_t - 1.$$

This characterizes the action of f on $\mathcal{M}_k^{\nu_t}$, in particular, $f(\mathcal{M}_k^{\nu_t}) \subseteq \mathcal{M}_{k'}^{\nu_{t'}}$. Conversely, f^{-1} maps B_j onto B_i. By the reasoning, analogous to this, which has been already done, we come to $f^{-1}(\mathcal{M}_{k'}^{\nu_{t'}}) \subseteq \mathcal{M}_k^{\nu_t}$. Consequently, $f(\mathcal{M}_k^{\nu_t}) = \mathcal{M}_{k'}^{\nu_{t'}}$, and therefore $t' = t$ since f is a bijection. It provides that f preserves the set $\mathcal{M}^{\nu_t}$. We define the map $\alpha \in S_{m_t}$ associated with $f \restriction_{\mathcal{M}^{\nu_t}}$ by the formula

$$\alpha\colon k \mapsto k' \text{ iff } f(\mathcal{M}_k^{\nu_t}) = \mathcal{M}_{k'}^{\nu_t},$$

for all $k, k' \leq m_t$. Set $v_k^t = j_0 - i_0 \bmod \nu_t$. Finally the formula for f is the following:

$$f\colon (t,k,i,\varepsilon) \mapsto (t, \alpha(k), i + v_k^t \bmod \nu_t, \varepsilon) \text{ for all } i = 0,\dots,\nu_t - 1.$$

(ii): Based on Lemma 4.5, $g_0 f$ is an automorphism of $\mathcal{M}$, which maps the B-blocks onto the B-blocks. Then, from Lemma 4.7(i), $g_0 f = f_{(v,\alpha)}$ for some $v \in \bigtimes_{t=1}^r C_{\nu_t}^{m_t}$ and $\alpha \in \bigtimes_{t=1}^r S_{m_t}$, and thus $f = g_0^{-1} f_{(v,\alpha)}$. Note that $g_0^{-1} = g_{(\mathbf{1},\mathrm{id})}$. Consequently, $f = g_{(\mathbf{1},\mathrm{id})} f_{(v,\alpha)} = g_0 f_{(v+\mathbf{1},\alpha)} = g_{(v+\mathbf{1},\alpha)}$. What is more, f preserves the set $\mathcal{M}^{\nu_t}$, that follows directly from (4.3). $\square$

Now we characterize automorphisms of $\mathfrak{M}_{(n,\varphi)}$, which can be uniquely decomposed into two mutually inscribed n-simplices.

Theorem 4.8. *Let $\mathcal{M} = \mathfrak{M}_{(n,\varphi)}$ and $1 \leq \nu_1 < \ldots < \nu_r$ be the lengths of the cycles in the cycle decomposition of $\varphi \in S_n$. Assume that either $n = 4$ and $\varphi \neq \mathrm{id}$ contains no cycle of length 2, or $n \geq 5$. Then $\mathrm{Aut}(\mathcal{M}) \cong \bigoplus_{i=1}^r \left(C_{2\nu_i}^{m_i} \rtimes S_{m_i}\right)$.*

Proof. Let F be an automorphism of $\mathcal{M}$. By Proposition 3.3, there is no special decomposition of $\mathcal{M}$. Thus, F either interchanges $S_A(\mathcal{M})$ with $S_B(\mathcal{M})$ or preserves each of them. According to Lemma 4.7 there is $v_0 \in \bigtimes_{t=1}^r C_{\nu_t}^{m_t}$ and $\alpha_0 \in \bigtimes_{t=1}^r S_{m_t}$ such that $F = f_{(v_0,\alpha_0)}$ or $F = g_{(v_0,\alpha_0)} = g_0 f_{(v_0,\alpha_0)}$. Furthermore, every $f_{(v,\alpha)}$, $g_{(v,\alpha)}$ with $v \in \bigtimes_{t=1}^r C_{\nu_t}^{m_t}$ and $\alpha \in \bigtimes_{t=1}^r S_{m_t}$ is an automorphism of $\mathcal{M}$ by Lemma 4.4 and Corollary 4.6. Since, by Lemma 4.7, F preserves each of the sets $\mathcal{M}^{\nu_t}$, we can restrict the proof to the one fixed set $\mathcal{M}^{\nu_t}$. Thus, we assume that $i = 0,\dots,\nu_t - 1$, $k \leq m_t$. For the simplicity of the notation, we will write $(\alpha(k), i + v_{\alpha(k)}, \varepsilon)$ instead of $(t, \alpha_t(k), i + v_{\alpha_t(k)}^t \bmod \nu_t, \varepsilon)$. Moreover, we identify $f_{(v,\alpha)}$ with $f_{(v,\alpha)} \restriction_{\mathcal{M}^{\nu_t}}$, and $g_{(v,\alpha)}$ with $g_{(v,\alpha)} \restriction_{\mathcal{M}^{\nu_t}}$, so we assume $v \in C_{\nu_t}^{m_t}$, $\alpha \in S_{m_t}$. Let $w \in C_{\nu_t}^{m_t}$, $\beta \in S_{m_t}$ and note that

$$f_{(w,\beta)} f_{(v,\alpha)}((k,i,\varepsilon)) = f_{(w,\beta)}((\alpha(k), i + v_k, \varepsilon)) = (\beta\alpha(k), i + v_k + w_{\alpha(k)}, \varepsilon).$$

Let $\phi^\alpha\colon S_{m_t} \longrightarrow \mathrm{Aut}(C_{\nu_t}^{m_t})$ be the map defined by

$$\phi_\alpha\colon (v_1,\dots,v_{m_t}) \mapsto (v_{\alpha(1)},\dots,v_{\alpha(m_t)}).$$

Then the formula for the composition of $f_{(v,\alpha)}$ and $f_{(w,\beta)}$ is

$$f_{(w,\beta)} f_{(v,\alpha)} = f_{(v+\phi_\alpha(w),\beta\alpha)}.$$

It is not difficult to check that g_0 and $f_{(v,\alpha)}$ commute. Note also that

$$g_0^z = \begin{cases} f_{(-\frac{\mathbf{z}}{\mathbf{2}},\mathrm{id})} & \text{if } z \text{ is even,} \\ g_{(-\frac{\mathbf{z-1}}{\mathbf{2}},\mathrm{id})} & \text{if } z \text{ is odd.} \end{cases}$$

Let $k' \leq m_{t'}$. We introduce the family of maps

$$g_{0_k}((k',i,\varepsilon)) = \begin{cases} g_0((k',i,\varepsilon)) & \text{if } k'=k, \\ (k',i,\varepsilon) & \text{otherwise.} \end{cases}$$

Then the following equalities hold

$$\begin{aligned} \{g_{0_k}^z : z = 0,\dots,2\nu_t - 1 \text{ and } z \text{ is even}\} &= \{f_{(v,\mathrm{id})} : v_{k'} = 0 \text{ for } k' \neq k\}, \\ \{g_{0_k}^z : z = 0,\dots,2\nu_t - 1 \text{ and } z \text{ is odd}\} &= \{g_{(v,\mathrm{id})} : v_{k'} = 0 \text{ for } k' \neq k\}. \end{aligned}$$

Therefore, for each $v \in C_{\nu_t}^{m_t}$ we have $f_{(v,\mathrm{id})} = g_{0_1}^{z_1} g_{0_2}^{z_2} \dots g_{0_{m_t}}^{z_{m_t}}$, where all numbers $z_k = 0,\dots,2\nu_t - 1$ are even. Likewise $g_{(v,\mathrm{id})} = g_{0_1}^{z_1} g_{0_2}^{z_2} \dots g_{0_{m_t}}^{z_{m_t}}$, where all numbers z_k are odd. Hence, for each $F \in \mathrm{Aut}(\mathcal{M})$ there is $v \in C_{\nu_t}^{m_t}$ and $\alpha \in S_{m_t}$ such that $F = f_{(v,\mathrm{id})} f_{(\mathbf{0},\alpha)}$ or $F = g_{(v,\mathrm{id})} f_{(\mathbf{0},\alpha)}$. To complete the proof it suffices to determine the remaining compositions:

$$\begin{aligned} f_{(v,\mathrm{id})} f_{(w,\mathrm{id})} &= f_{(w+v,\mathrm{id})}, \\ f_{(\mathbf{0},\alpha)} f_{(\mathbf{0},\beta)} &= f_{(\mathbf{0},\beta\alpha)}, \\ f_{(v,\mathrm{id})} f_{(\mathbf{0},\alpha)} &= f_{(\phi^\alpha(v),\alpha)}, \\ g_{(v,\mathrm{id})} f_{(\mathbf{0},\alpha)} = g_0 f_{(v,\mathrm{id})} f_{(\mathbf{0},\alpha)} &= g_0 f_{(\phi^\alpha(v),\alpha)} = g_{(\phi^\alpha(v),\alpha)}, \\ f_{(v,\mathrm{id})} g_{(w,\mathrm{id})} = f_{(v,\mathrm{id})} g_0 f_{(w,\mathrm{id})} &= g_0 f_{(v,\mathrm{id})} f_{(w,\mathrm{id})} = g_0 f_{(w+v,\mathrm{id})} = g_{(w+v,\mathrm{id})}, \\ g_{(v,\mathrm{id})} g_{(w,\mathrm{id})} = g_0 f_{(v,\mathrm{id})} g_0 f_{(w,\mathrm{id})} &= g_0^2 f_{(w+v,\mathrm{id})} = f_{(-\mathbf{1},\mathrm{id})} f_{(w+v,\mathrm{id})} = f_{(w+v-\mathbf{1},\mathrm{id})}. \end{aligned}$$

□

The Möbius n-pairs, which automorphism groups are not characterized by Theorem 4.8, admit a special decomposition. We say that an automorphism f of a Möbius n-pair $\mathcal{M}$ *yields a special decomposition* of $\mathcal{M}$ if f maps the pair $\{S_A, S_B\}$ onto a distinct pair of mutually inscribed simplices.

Theorem 4.9. *The automorphism group of* $\mathfrak{M}_{(4,\varphi)}$ *is isomorphic to*

(i) $(C_4 \oplus S_2) \rtimes C_2$ *if* $\varphi \in S_4$ *contains precisely one cycle of length* 2*,*

(ii) $(C_4^2 \rtimes S_2) \rtimes C_2$ *if* $\varphi \in S_4$ *contains two cycles of length* 2*.*

Proof. In view of Theorem 4.2, without loss of generality we can consider $\mathcal{M}_1 = \mathfrak{M}_{(4,\varphi_1)}$ with $\varphi_1 = (1)(2)(34)$ in case (i), and $\mathcal{M}_2 = \mathfrak{M}_{(4,\varphi_2)}$ with $\varphi_2 = (12)(34)$ in case (ii) (comp. Figures 4, 5). Let $F_s \in \mathrm{Aut}(\mathcal{M}_s)$ for $s = 1, 2$. By Corollary 3.4, there is the special decomposition of each of $\mathcal{M}_s$. Thus, F_s maps the pair $\{S_A, S_B\}$ onto $\{S_A, S_B\}$ or F_s yields the special decomposition of $\mathcal{M}_s$. In case F_s maps the pair $\{S_A, S_B\}$ onto $\{S_A, S_B\}$, by Lemma 4.7, there is $v_0 \in \{0\} \times \{0\} \times C_2$, $\alpha_0 \in S_2 \times \{\mathrm{id}\}$ for $\mathcal{M}_1$, or $v_0 \in C_2 \times C_2$, $\alpha_0 \in S_2$ for $\mathcal{M}_2$ such that $F_s = f_{(v_0,\alpha_0)}$ or $F_s = g_{(v_0,\alpha_0)} = g_0 f_{(v_0,\alpha_0)}$, respectively for $s = 1, 2$. By Lemma 4.4 and Corollary 4.6 all maps $F_s f_{(v,\alpha)}$, $F_s g_{(v,\alpha)}$, where $v \in \{0\} \times \{0\} \times C_2$ and $\alpha \in S_2 \times \{\mathrm{id}\}$ if $s = 1$, or $v \in C_2 \times C_2$, $\alpha \in S_2$ if $s = 2$, are automorphisms of $\mathcal{M}_s$ preserving the pair $\{S_A, S_B\}$. Based on the proof of Theorem 4.8, these maps form the group $C_4 \oplus S_2$ if $s = 1$, and the group $C_4^2 \rtimes S_2$ if $s = 2$.

Consider the following two transformations:

x	a_1	a_2	a_3	a_4	b_1	b_2	b_3	b_4
$\tilde{f}(x)$	b_1	b_2	a_4	a_3	a_1	a_2	b_3	b_4
$\hat{f}(x)$	a_2	a_1	b_3	b_4	b_1	b_2	a_3	a_4

The map $\tilde{f}$ is an automorphism, which yields a special decomposition of $\mathcal{M}_1$; and $\hat{f}$ is an automorphism, which yields a special decomposition of $\mathcal{M}_2$. Assume that F_s yields a special decomposition of $\mathcal{M}_s$. Then $F_1 = \tilde{f}F_1'$ and $F_2 = \hat{f}F_2'$, where F_s' is the automorphism of $\mathcal{M}_s$ given by (4.2) or (4.3).

Let us set the commutativity rules in the automorphism group of $\mathcal{M}_s$. By (4.1), the points of $\mathcal{M}_1$, $\mathcal{M}_2$ correspond to the sequences (t, i, k, ε) with $\varepsilon = 1, -1$. Using the convention introduced at the beginning of this paragraph we get $t = 1, 2$, $\nu_1 = 1$, $\nu_2 = 2$, $m_1 = 2$, $m_2 = 1$ and $X_1^1 = \{1\}$, $X_2^1 = \{2\}$, $X_1^2 = \{3, 4\}$ for $\mathcal{M}_1$; $t = 1$, $\nu_1 = 2$, $m_1 = 2$, and $X_1^2 = \{1, 2\}$, $X_2^2 = \{3, 4\}$ for $\mathcal{M}_2$. To avoid any misunderstanding, in case $\mathcal{M}_2$ we will write Y_1^2, Y_2^2 instead of X_1^2, X_2^2 respectively. Then $\tilde{f}$ maps the points of $\mathcal{M}_1$ by the formula:

$$\tilde{f}((t, k, i, \varepsilon)) = \begin{cases} (t, k, i, -\varepsilon) & \text{for } i + \mu_k^t \in X_1^1, X_2^1, \\ (t, k, i + 1 \bmod 2, \varepsilon) & \text{for } \varepsilon = 1,\ i + \mu_k^t \in X_1^2, \\ (t, k, i, \varepsilon) & \text{for } \varepsilon = -1,\ i + \mu_k^t \in X_1^2. \end{cases} \tag{4.4}$$

The map $\hat{f}$ can be defined on points of $\mathcal{M}_2$ as:

$$\hat{f}((t, k, i, \varepsilon)) = \begin{cases} (t, k, i + 1 \bmod 2, \varepsilon) & \text{for } \varepsilon = 1,\ i + \mu_k^t \in Y_1^2, \\ (t, k, i, \varepsilon) & \text{for } \varepsilon = -1,\ i + \mu_k^t \in Y_1^2, \\ (t, k, i, -\varepsilon) & \text{for } i + \mu_k^t \in Y_2^2. \end{cases} \tag{4.5}$$

Note, that $\tilde{f}^2 = \hat{f}^2 = \mathrm{id}$. Hence the cyclic group generated by $\tilde{f}$ and the cyclic group generated by $\hat{f}$ both coincide with C_2. All the formulas for compositions of $\tilde{f}$ with g_0, and $\tilde{f}$ with $f = f_{(v,\alpha)}$ can be calculated using (4.4) and (4.5) (it is rather technical and thus omitted) and then we get

$$\tilde{f}f = f\tilde{f} \text{ and } \tilde{f}g_0 = \tilde{f}g_{(\mathbf{0},\mathrm{id})} = g_{(\tau_{(0,0,1)}(\mathbf{0}),\mathrm{id})}\tilde{f},$$

where $\tau_{(0,0,1)}(v) = \tau_{(0,0,1)}((v_1^1, v_2^1, v_1^2)) = (v_1^1, v_2^1, v_1^2 + 1)$. Analogous calculation can be done for $\hat{f}$. If we set $\tau_{(1,1)}(v) = \tau_{(1,1)}((v_1^1, v_2^1)) = (v_1^1 + 1, v_2^1 + 1)$, then

$$\begin{aligned} \hat{f}g_0 = \hat{f}g_{(\mathbf{0},\mathrm{id})} &= g_{(\tau_{(1,1)}(\mathbf{0}),\mathrm{id})}\hat{f}, \\ \hat{f}f_{(v,\alpha)} &= f_{(v,\alpha)}\hat{f} && \text{if only} && \alpha = \mathrm{id}, \\ \hat{f}f_{(v,\alpha)} = g_0 f_{(v,\alpha)}\hat{f} &= g_{(v,\alpha)}\hat{f} && \text{provided that} && i + \mu_k^t \in Y_1^2 \text{ and } \alpha = (12), \\ \hat{f}f_{(v,\alpha)} = g_0^{-1} f_{(v,\alpha)}\hat{f} &= g_{(\tau_{(1,1)}(v),\alpha)}\hat{f} && \text{as long as} && i + \mu_k^t \in Y_2^2 \text{ and } \alpha = (12). \end{aligned}$$

□

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Domination game on paths and cycles

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Abstract

Domination game is a game on a simple graph played by two players, Dominator and Staller who are alternating in taking turns. In each turn a player chooses a vertex in such a way that at least one new vertex gets dominated by this move. The game ends when all vertices are dominated, and thus no legal move is possible. As the names of the players suggest, Dominator tries to finish the game as fast as possible, while Staller wants to prolong its end as long as she can. By γ_g (γ_g') we denote the total number of moves in the game when Dominator (resp. Staller) starts, and both players play according to their optimal strategies. In a manuscript from 2012, Kinnersley et al. determined γ_g and ${\gamma_g}'$ for paths and cycles, but have not yet published this very important result. In this paper we give an alternative proof for these formulas. Our approach also explicitly describes optimal strategies for both players.

Keywords: Domination game, game domination number, paths, cycles.

Math. Subj. Class.: 05C57, 91A43, 05C69

1 Introduction

The *Domination game* is a game played on a simple graph G by two players, *Dominator* and *Staller*. They are alternating in choosing vertices from G such that in every move at least one new, previously undominated, vertex gets dominated. We say that a vertex is dominated if it is either chosen or is a neighboor of a chosen vertex. The game ends when all vertices of G are dominated, that is when the set of chosen vertices forms a dominating set of G. As the names of the players suggest, Dominator wants to finish the game in the least possible number of moves, while Staller's goal is just the opposite: to play the game as long as she can. A game when Dominator makes the first move will be called a *D-game*, while *S-game* will denote a game when Staller starts. The *game domination number* $\gamma_g(G)$

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(*Staller-start game domination number* $\gamma_g'(G)$) denotes the total number of moves made in the D-game (resp. S-game) on G when both players play optimally.

Although the game was introduced not long ago by Brešar et al. in [4], it has already attracted several authors who have studied different aspects of the game, among which the most popular seems to be the so-called $3/5-$conjecture posed by Kinnersley et al. in [14]. A major breakthrough was made by Bujtás in [5] by introducing a powerful greedy method for proving upper bounds of the game domination number. For other papers with results regarding the mentioned conjecture see [2, 6, 10]. Moreover, the greedy method has already been applied in [7] to improve upper bounds on the domination number of a graph.

Other topics of the domination game that were studied include computational complexity of the game (see [1, 16]), metric properties with respect to the domination game (see [3]), the domination game played on disjoint union (see [9]), realizations of the game domination numbers (see [17]), and characterizations (see [15, 18]). Also, the game has motivated studies of new games such as the total version of domination game, introduced and studied in [11, 12], and the disjoint domination game introduced by Bujtás and Tuza in [8].

In this paper we will study the domination game on paths and cycles. Kinnersley et al. have in [13] already found formulas for the game domination number of these graphs. Since these results are fundamental for the theory of the domination game, it is rather unfortunate that they are not published yet. Moreover, their proof is analytical and does not offer us much of an insight into the optimal strategies of both players. The latter will implicitly follow from our proof in the next section. In the rest of this section we introduce some notation and results needed in the new proofs of formulas for paths and cycles.

For a graph G and a subset of vertices $S \subseteq V(G)$ we denote by $G|S$ a *partially dominated graph* where the vertices of S are considered to be already dominated, and thus not need to be dominated in the course of the game. Note that S can be an arbitrary subset of $V(G)$, and not only a union of closed neighborhoods of some vertices. From $G|S$ we get the *residual graph* if we remove from G all edges between already dominated vertices, and all vertices v that cannot be chosen anymore, more precisely $N[v] \subseteq S$.

Theorem 1.1 (Continuation Principle, [14])**.** *Let G be a graph and $A, B \subseteq V(G)$. If $B \subseteq A$, then $\gamma_g(G|A) \leq \gamma_g(G|B)$ and $\gamma_g'(G|A) \leq \gamma_g'(G|B)$.*

One of the corollaries of Continuation Principle is the fact that γ_g and γ_g' never differ for more than one.

Theorem 1.2 ([4, 14])**.** *For any graph G, $|\gamma_g(G) - \gamma_g'(G)| \leq 1$.*

We say that a partially dominated graph $G|S$ realizes a pair $(k, \ell) \in \mathbb{N}\times\mathbb{N}$ if $\gamma_g(G|S) = k$ and $\gamma_g'(G|S) = \ell$. In the light of Theorem 1.2 we can now classify graphs into three different families, each of them realizing one of the possible pairs $(k, k+1), (k, k)$ and $(k+1, k)$. By PLUS we denote the family of all graphs that realize $(k, k+1)$ for some positive k. Similarly we define EQUAL and MINUS. Lastly, we say that a graph is a *no-minus* graph if $\gamma_g(G|S) \leq \gamma_g'(G|S)$ for every $S \subseteq V(G)$.

Theorem 1.3 ([14])**.** *Forests are no-minus graphs.*

Some other families of no-minus graphs are given in [9], as well as the game domination numbers of the disjoint union of two no-minus graphs. If at least one of the components is an EQUAL graph the following holds.

Theorem 1.4 ([9]). *Let $G_1|S_1$ and $G_2|S_2$ be partially dominated no-minus graphs. If $G_1|S_1$ realizes (k,k) and $G_2|S_2$ realizes (ℓ,m) (where $m \in \{\ell,\ell+1\}$), then the disjoint union $(G_1 \cup G_2)|(S_1 \cup S_2)$ realizes $(k+\ell,k+m)$.*

In the second case, when both of the components are PLUS graphs we get the next result.

Theorem 1.5 ([9]). *Let $G_1|S_1$ and $G_2|S_2$ be partially dominated no-minus graphs such that $G_1|S_1$ realizes $(k,k+1)$ and $G_2|S_2$ realizes $(\ell,\ell+1)$. Then*

$$k+\ell \le \gamma_g((G_1 \cup G_2)|(S_1 \cup S_2)) \le k+\ell+1,$$
$$k+\ell+1 \le \gamma_g'((G_1 \cup G_2)|(S_1 \cup S_2)) \le k+\ell+2.$$

2 Paths and cycles

The following formulas for the game domination number of paths and cycles were proved in [13]:

$$\begin{aligned}
\gamma_g(P_n) = \gamma_g(C_n) &= \begin{cases} \lceil \frac{n}{2} \rceil - 1; & n \equiv 3 \pmod 4, \\ \lceil \frac{n}{2} \rceil; & \text{otherwise,} \end{cases} \\
\gamma_g'(P_n) &= \left\lceil \frac{n}{2} \right\rceil, \\
\gamma_g'(C_n) &= \begin{cases} \lceil \frac{n-1}{2} \rceil - 1; & n \equiv 2 \pmod 4, \\ \lceil \frac{n-1}{2} \rceil; & \text{otherwise.} \end{cases}
\end{aligned}$$

In the course of the game on a path or on a cycle, we come across two special partially dominated paths. Let P_n'' denote a partially dominated path of order $n+2$ with both pendant vertices being already dominated, see Fig. 1. By P_n' we denote a partially dominated path of order $n+1$ where exactly one of the leaves is dominated, see Fig. 1 again. Note that in both cases n denotes the number of undominated vertices.

Figure 1: Partially dominated paths P_n'' (left) and P_n' (right)

At first we prove the formulas for the game domination number of paths with both leaves dominated.

Lemma 2.1. *For every $n \ge 0$, we have*

$$\begin{aligned}
\gamma_g(P_n'') &= \begin{cases} \lceil \frac{n}{2} \rceil - 1; & n \equiv 3 \pmod 4, \\ \lceil \frac{n}{2} \rceil, & \text{otherwise,} \end{cases} \\
\gamma_g'(P_n'') &= \begin{cases} \lceil \frac{n}{2} \rceil + 1, & n \equiv 2 \pmod 4, \\ \lceil \frac{n}{2} \rceil; & \text{otherwise.} \end{cases}
\end{aligned}$$

Moreover, for every $i,j \ge 0$ such that $i+j=n$, $i_r = (i \bmod 4)$ and $j_r = (j \bmod 4)$, we *also have*

$$\gamma_g(P_i'' \cup P_j'') = \begin{cases} \gamma_g(P_i'') + \gamma_g(P_j''); & (i_r, j_r) \in \{0,1\} \times \{0,1,2,3\} \cup \\ & \qquad \{0,1,2,3\} \times \{0,1\}, \\ \gamma_g(P_i'') + \gamma_g(P_j'') + 1; & (i_r, j_r) \in \{2,3\} \times \{2,3\}, \end{cases}$$

$$\gamma_g'(P_i'' \cup P_j'') = \begin{cases} \gamma_g(P_i'') + \gamma_g(P_j''); & (i_r, j_r) \in \{0,1\} \times \{0,1\}, \\ \gamma_g(P_i'') + \gamma_g(P_j'') + 1; & (i_r, j_r) \in \{0,1\} \times \{2,3\} \cup \\ & \qquad \{2,3\} \times \{0,1\} \cup \{(2,2)\}, \\ \gamma_g(P_i'') + \gamma_g(P_j'') + 2; & (i_r, j_r) \in \{(2,3),(3,2),(3,3)\}. \end{cases}$$

Proof. We prove all four formulas simultaneously by induction on the number of undominated vertices n. It is easy to check by hand that all formulas hold for $n \le 8$. Let us now assume that $n \ge 9$.

Let us first consider a D-game on P_n''. Let u be one of the (dominated) leaves, v its (unique) neighbor and w the neighbor of v different from u. Since $N[u] \subseteq N[v] \subseteq N[w] \cup \{u\}$ holds, using Continuation Principle we get that $\gamma_g'(P_n''|(N[w] \cup \{u\})) \le \gamma_g'(P_n''|N[v]) \le \gamma_g'(P_n''|N[u])$, and thus we can assume that Dominator does not choose a leaf nor a leaf's neighbor in his first move on P_n''. Hence we get that the residual graph after the first move is the disjoint union $P_r'' \cup P_s''$ where $r, s \ge 0$ and $r + s = n - 3$. More precisely, we get the following:

$$\gamma_g(P_n'') = 1 + \min\{\gamma_g'(P_r'' \cup P_s'') \mid r + s = n - 3,\ r, s \ge 0\}\,.$$

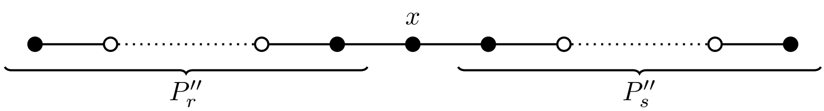

Figure 2: Partially dominated path after Dominator choosing x on P_n''

We now consider four cases regarding the value of $n \bmod 4$. By induction hypothesis we get that the following holds for every $k \ge 2$.

$$\begin{aligned} \gamma_g(P_{4k}'') &= 1 + \min\Big[\{\gamma_g'(P_{4\ell}'' \cup P_{4m+1}'') \mid \ell + m + 1 = k,\ \ell, m \ge 0\} \cup \\ &\qquad \{\gamma_g'(P_{4\ell+2}'' \cup P_{4m+3}'') \mid \ell + m + 2 = k,\ \ell, m \ge 0\}\Big] \\ &= 1 + \min\Big[\{\gamma_g(P_{4\ell}'') + \gamma_g(P_{4m+1}'') \mid \ell + m + 1 = k,\ \ell, m \ge 0\} \cup \\ &\qquad \{\gamma_g(P_{4\ell+2}'') + \gamma_g(P_{4m+3}'') + 2 \mid \ell + m + 2 = k,\ \ell, m \ge 0\}\Big] \\ &= 1 + \min\Big[\{2\ell + (2m+1) \mid \ell + m + 1 = k,\ \ell, m \ge 0\} \cup \\ &\qquad \{(2\ell+1) + (2m+1) + 2) \mid \ell + m + 2 = k,\ \ell, m \ge 0\}\Big] \end{aligned}$$

$$= \quad 1+\min\{2k-1,2k\}=2k,$$

$$\begin{aligned}\gamma_g(P''_{4k+1}) \quad = \quad & 1+\min\Big[\{\gamma_g'(P''_{4\ell}\cup P''_{4m+2}) \mid \ell+m+1=k,\ \ell,m\geq 0\}\cup \\ & \{\gamma_g'(P''_{4\ell+1}\cup P''_{4m+1}) \mid \ell+m+1=k,\ \ell,m\geq 0\}\cup \\ & \{\gamma_g'(P''_{4\ell+3}\cup P''_{4m+3}) \mid \ell+m+2=k,\ \ell,m\geq 0\}\Big] \\ = \quad & 1+\min\{2k,2k,2k\}=2k+1,\end{aligned}$$

$$\begin{aligned}\gamma_g(P''_{4k+2}) \quad = \quad & 1+\min\Big[\{\gamma_g'(P''_{4\ell}\cup P''_{4m+3}) \mid \ell+m+1=k,\ \ell,m\geq 0\}\cup \\ & \{\gamma_g'(P''_{4\ell+1}\cup P''_{4m+2}) \mid \ell+m+1=k,\ \ell,m\geq 0\}\Big] \\ = \quad & 1+\min\{2k,2k+1\}=2k+1,\end{aligned}$$

$$\begin{aligned}\gamma_g(P''_{4k+3}) \quad = \quad & 1+\min\Big[\{\gamma_g'(P''_{4\ell}\cup P''_{4m}) \mid \ell+m=k,\ \ell,m\geq 0\}\cup \\ & \{\gamma_g'(P''_{4\ell+1}\cup P''_{4m+3}) \mid \ell+m+1=k,\ \ell,m\geq 0\}\cup \\ & \{\gamma_g'(P''_{4\ell+2}\cup P''_{4m+2}) \mid \ell+m+1=k,\ \ell,m\geq 0\}\Big] \\ = \quad & 1+\min\{2k,2k+1,2k+1\}=2k+1\,.\end{aligned}$$

Similarly we compute the game domination number of P''_n of the S-game. Since the moves available are the same as those in the D-game, the set of possible residual graphs after Staller's first move is the same as above, that is $P''_{n-1}, P''_{n-2}, P''_{n-3}$ or $P''_r \cup P''_s$, where $r, s \geq 1$ and $r+s=n-3$. By Continuation Principle, Staller either plays on a dominated vertex (which is a leaf) or chooses such a vertex x that the residual graph of $P''_n|N[x]$ has two components, both of which are paths with both of their leaves already dominated. Continuation Principle assures us that choosing the neighbor of a leaf or the vertex on a distance two from a leaf is never better for Staller than choosing a leaf. Using induction hypothesis the following holds for every $k \geq 2$.

$$\begin{aligned}\gamma_g'(P''_{4k}) \quad = \quad & 1+\max\Big[\{\gamma_g(P''_{4(k-1)+3})\}\cup \\ & \{\gamma_g(P''_{4\ell}\cup P''_{4m+1}) \mid \ell+m+1=k,\ \ell,m\geq 0\}\cup \\ & \{\gamma_g(P''_{4\ell+2}\cup P''_{4m+3}) \mid \ell+m+2=k,\ \ell,m\geq 0\}\Big] \\ = \quad & 1+\max\Big[\{\gamma_g(P''_{4(k-1)+3})\}\cup \\ & \{\gamma_g(P''_{4\ell})+\gamma_g(P''_{4m+1}) \mid \ell+m+1=k,\ \ell,m\geq 0\}\cup \\ & \{\gamma_g(P''_{4\ell+2})+\gamma_g(P''_{4m+3})+1 \mid \ell+m+2=k,\ \ell,m\geq 0\}\Big]\end{aligned}$$

$$
\begin{aligned}
&= 1 + \max\Big[\{2k-1\} \cup \\
&\quad \{2\ell + (2m+1) \mid \ell + m + 1 = k,\ \ell, m \geq 0\} \cup \\
&\quad \{(2\ell+1) + (2m+1) + 1 \mid \ell + m + 2 = k,\ \ell, m \geq 0\}\Big] \\
&= 1 + \max\{2k-1, 2k-1, 2k-1\} = 2k,
\end{aligned}
$$

$$
\begin{aligned}
\gamma_g'(P''_{4k+1}) &= 1 + \max\Big[\{\gamma_g(P''_{4k})\} \cup \\
&\quad \{\gamma_g(P''_{4\ell} \cup P''_{4m+2}) \mid \ell + m + 1 = k,\ \ell, m \geq 0\} \cup \\
&\quad \{\gamma_g(P''_{4\ell+1} \cup P''_{4m+1}) \mid \ell + m + 1 = k,\ \ell, m \geq 0\} \cup \\
&\quad \{\gamma_g(P''_{4\ell+3} \cup P''_{4m+3}) \mid \ell + m + 2 = k,\ \ell, m \geq 0\}\Big] \\
&= 1 + \max\{2k, 2k-1, 2k, 2k-1\} = 2k+1,
\end{aligned}
$$

$$
\begin{aligned}
\gamma_g'(P''_{4k+2}) &= 1 + \max\Big[\{\gamma_g(P''_{4k+1})\} \cup \\
&\quad \{\gamma_g(P''_{4\ell} \cup P''_{4m+3}) \mid \ell + m + 1 = k,\ \ell, m \geq 0\} \cup \\
&\quad \{\gamma_g(P''_{4\ell+1} \cup P''_{4m+2}) \mid \ell + m + 1 = k,\ \ell, m \geq 0\}\Big] \\
&= 1 + \max\{2k+1, 2k-1, 2k\} = 2k+2,
\end{aligned}
$$

$$
\begin{aligned}
\gamma_g'(P''_{4k+3}) &= 1 + \max\Big[\{\gamma_g(P''_{4k+2})\} \cup \\
&\quad \{\gamma_g(P''_{4\ell} \cup P''_{4m}) \mid \ell + m = k,\ \ell, m \geq 0\} \cup \\
&\quad \{\gamma_g(P''_{4\ell+1} \cup P''_{4m+3}) \mid \ell + m + 1 = k,\ \ell, m \geq 0\} \cup \\
&\quad \{\gamma_g(P''_{4\ell+2} \cup P''_{4m+2}) \mid \ell + m + 1 = k,\ \ell, m \geq 0\}\Big] \\
&= 1 + \max\{2k+1, 2k, 2k, 2k+1\} = 2k+2\,.
\end{aligned}
$$

Next we show the formulas for the game domination number of the disjoint union $P''_i \cup P''_j$. From the first two formulas proven above we can quickly deduce that P''_k is an EQUAL if and only if $k \equiv 0 \pmod 4$ or $k \equiv 1 \pmod 4$. In the other two cases we get that P''_k is a PLUS. Since paths are no-minus graphs, by Theorem 1.4 it follows that formulas hold when at least one of the components is an EQUAL.

It remains to prove the case when both components are PLUS. Let us assume first that $i = 4\ell + 2$ and $j = 4m + 2$, where $\ell, m \geq 1$. We would like to prove that $P''_{4\ell+2} \cup P''_{4m+2}$ realizes the pair $(2\ell + 2m + 3, 2\ell + 2m + 3)$. By Theorem 1.5 it follows that we only need to prove the lower bound for γ_g and the upper bound for γ_g'. For the former we need to prove that Staller can ensure that on $P''_{4\ell+2} \cup P''_{4m+2}$ at least $2m + 2\ell + 3$ moves are played. Since we did not say anything about the relation between ℓ and m we can without loss of generality assume that Dominator plays his first move in $P''_{4\ell+2}$. Using Continuation Principle in the same way as above when proving the formula for $\gamma_g(P''_n)$, he has only two conceptually different options for his move so that the residual graph of $P''_{4\ell+2}$ after this move is either $P''_{4r} \cup P''_{4s+3}$ or $P''_{4r+1} \cup P''_{4s+2}$, where $r + s = \ell - 1$ and $r, s \geq 0$. If Staller then chooses a dominated leaf in the component with the odd number of undominated vertices, by induction hypothesis and by Theorem 1.4 we get the following.

$$\begin{aligned}
\gamma_g(P''_{4\ell+2} \cup P''_{4m+2}) &= 1 + \min \begin{pmatrix} \gamma'_g(P''_{4r} \cup P''_{4s+3} \cup P''_{4m+2}) \\ \gamma'_g(P''_{4r+1} \cup P''_{4s+2} \cup P''_{4m+2}) \end{pmatrix} \\
&\geq 2 + \min \begin{pmatrix} \gamma_g(P''_{4r} \cup P''_{4s+2} \cup P''_{4m+2}) \\ \gamma_g(P''_{4r} \cup P''_{4s+2} \cup P''_{4m+2}) \end{pmatrix} \\
&= 2 + (\gamma_g(P''_{4r}) + \gamma_g(P''_{4s+2}) + \gamma_g(P''_{4m+2}) + 1) \\
&= 2 + (2r + (2s+1) + (2m+1) + 1) \\
&= 2 + (2\ell + 2m + 1) \\
&= 2\ell + 2m + 3\,.
\end{aligned}$$

Proof of the upper bound in the S-game follows similar lines. Staller has three conceptually different options for her first move, and then Dominator plays in the component with an odd number of undominated vertices such that he chooses a vertex at a distance 2 from a dominated leaf. By induction hypothesis and Theorem 1.4 we then get the following for every $r, s \geq 0$, $r + s = \ell - 1$.

$$\begin{aligned}
\gamma'_g(P''_{4\ell+2} \cup P''_{4m+2}) &= 1 + \max \begin{pmatrix} \gamma_g(P''_{4\ell+1} \cup P''_{4m+2}) \\ \gamma_g(P''_{4r} \cup P''_{4s+3} \cup P''_{4m+2}) \\ \gamma_g(P''_{4r+1} \cup P''_{4s+2} \cup P''_{4m+2}) \end{pmatrix} \\
&\leq 2 + \max \begin{pmatrix} \gamma'_g(P''_{4(\ell-1)+2} \cup P''_{4m+2}) \\ \gamma'_g(P''_{4r} \cup P''_{4s} \cup P''_{4m+2}) \\ \gamma'_g(P''_{4(r-1)+2} \cup P''_{4s+2} \cup P''_{4m+2}) \end{pmatrix} \\
&= 2 + \max \begin{pmatrix} \gamma'_g(P''_{4(\ell-1)+2}) + \gamma_g(P''_{4m+2}) + 1 \\ \gamma'_g(P''_{4r}) + \gamma'_g(P''_{4s}) + \gamma_g(P''_{4m+2}) + 1 \\ \gamma'_g(P''_{4(r-1)+2}) + \gamma_g(P''_{4s+2}) + \gamma_g(P''_{4m+2}) + 2 \end{pmatrix} \\
&= 2 + \max \begin{pmatrix} (2(\ell-1)+1) + (2m+1) + 1 \\ 2r + 2s + (2m+1) + 1 \\ (2(r-1)+1) + (2s+1) + (2m+1) + 2 \end{pmatrix} \\
&= 2 + \max\{2\ell + 2m + 1, 2\ell + 2m, 2\ell + 2m + 1\} \\
&= 2\ell + 2m + 3\,.
\end{aligned}$$

Finally, we prove that the disjoint unions $P''_{4\ell+2} \cup P''_{4m+3}$ and $P''_{4\ell+3} \cup P''_{4m+3}$ both realize the pair $(2\ell + 2m + 3, 2\ell + 2m + 4)$ for every $\ell, m \geq 0$. By Theorem 1.5 it is enough to prove the lower bound in the S-game presenting Staller's strategy. If she chooses a dominated leaf in P''_{4m+3} in her first move, by the formulas proven above, we get that

$$\gamma'_g(P''_{4\ell+2} \cup P''_{4m+3}) \geq 1 + \gamma_g(P''_{4\ell+2} \cup P''_{4m+2}) = 1 + 2\ell + 2m + 3 = 2\ell + 2m + 4$$

and from here also

$$\gamma'_g(P''_{4\ell+3} \cup P''_{4m+3}) \geq 1 + \gamma_g(P''_{4\ell+3} \cup P''_{4m+2}) = 1 + 2\ell + 2m + 3 = 2\ell + 2m + 4\,.$$

This concludes the proof. □

A direct corollary of the above lemma are formulas for the game domination number on cycles. Since cycles are vertex-transitive graphs any first move on a cycle leads to the same residual graph – a path with both leaves dominated.

Theorem 2.2. *For every $n \geq 3$, we have*

$$\begin{aligned}\gamma_g(C_n) &= \begin{cases} \lceil \frac{n}{2} \rceil - 1; & n \equiv 3 \pmod 4, \\ \lceil \frac{n}{2} \rceil; & \text{otherwise}, \end{cases} \\ \gamma_g'(C_n) &= \begin{cases} \lceil \frac{n-1}{2} \rceil - 1; & n \equiv 2 \pmod 4, \\ \lceil \frac{n-1}{2} \rceil; & \text{otherwise}. \end{cases}\end{aligned}$$

Proof. By Lemma 2.1 it follows

$$\begin{aligned}\gamma_g(C_n) &= 1 + \gamma_g'(P_{n-3}'') \\ &= 1 + \begin{cases} \lceil \frac{n-3}{2} \rceil + 1; & n-3 \equiv 2 \pmod 4, \\ \lceil \frac{n-3}{2} \rceil; & \text{otherwise} \end{cases} \\ &= \begin{cases} \lceil \frac{n-1}{2} \rceil + 1; & n \equiv 1 \pmod 4, \\ \lceil \frac{n-1}{2} \rceil; & \text{otherwise} \end{cases} \\ &= \begin{cases} \lceil \frac{n}{2} \rceil - 1; & n \equiv 3 \pmod 4, \\ \lceil \frac{n}{2} \rceil; & \text{otherwise}. \end{cases}\end{aligned}$$

where we get the last equality by listing the values of both functions for $n = 4k, 4k+1, 4k+2, 4k+3$ and $k \geq 0$. Similarly, using Lemma 2.1 again we get the Staller-start game domination number. For every $n \geq 3$, we get that

$$\begin{aligned}\gamma_g'(C_n) &= 1 + \gamma_g(P_{n-3}'') \\ &= 1 + \begin{cases} \lceil \frac{n-3}{2} \rceil - 1; & n-3 \equiv 3 \pmod 4, \\ \lceil \frac{n-3}{2} \rceil; & \text{otherwise} \end{cases} \\ &= \begin{cases} \lceil \frac{n-1}{2} \rceil - 1; & n \equiv 2 \pmod 4, \\ \lceil \frac{n-1}{2} \rceil; & \text{otherwise}. \end{cases}\end{aligned}$$

□

The proof of Lemma 2.1 also gives us optimal strategies for both players. Choosing a vertex at a distance 2 from a dominated leaf is always optimal for Dominator, while playing on a dominated leaf in every move of the game is optimal for Staller. Since every residual graph of P_n' that occurs during the game has at least one dominated leaf, both players can play in the same way as on P_n''. Hence the following lemma directly follows.

Lemma 2.3. *For every $n, m \geq 0$, we have*

$$\gamma_g(P_n' \cup P_m') = \gamma_g(P_n'' \cup P_m') = \gamma_g(P_n'' \cup P_m'') \text{ and}$$
$$\gamma_g'(P_n' \cup P_m') = \gamma_g'(P_n'' \cup P_m') = \gamma_g'(P_n'' \cup P_m'') .$$

In particular, the last lemma says that $\gamma_g(P_n') = \gamma_g(P_n'')$ and $\gamma_g'(P_n') = \gamma_g'(P_n'')$ for every $n \geq 0$. Now everything is ready for the main theorem - the formulas for paths.

Theorem 2.4. *For every $n \geq 0$, we have*

$$\begin{aligned}\gamma_g(P_n) &= \begin{cases} \lceil \frac{n}{2} \rceil - 1; & n \equiv 3 \pmod 4, \\ \lceil \frac{n}{2} \rceil; & \text{otherwise}, \end{cases} \\ \gamma_g'(P_n) &= \left\lceil \frac{n}{2} \right\rceil .\end{aligned}$$

Proof. We can easily check by hand that the assertion holds for $n \leq 4$. Let us now assume that $n \geq 5$. In the first move Dominator can either dominate two vertices by choosing one of the leaves, or three vertices by choosing any other vertex. By using Continuation Principle we can assume that Dominator never chooses a leaf. More precisely, choosing the leaf's neighbor can never be worse for Dominator than choosing a leaf. Hence, in his first move exactly three vertices are dominated, and the residual graph after this move is the disjoint union $P_r' \cup P_s'$ where $r, s \geq 0$ and $r + s = n - 3$. We consider four cases regarding the value of $n \bmod 4$. In every case we first use the above argument about Dominator's first move, and then apply Lemma 2.3 and Lemma 2.1 to get the result for every $k \geq 1$.

$$\begin{aligned}\gamma_g(P_{4k}) &= 1 + \min \Big[\{\gamma_g'(P_{4\ell}' \cup P_{4m+1}') \mid \ell + m + 1 = k,\ \ell, m \geq 0\} \cup \\ &\qquad \{\gamma_g'(P_{4\ell+2}' \cup P_{4m+3}') \mid \ell + m + 2 = k,\ \ell, m \geq 0\} \Big] \\ &= 1 + \min \Big[\{\gamma_g'(P_{4\ell}'' \cup P_{4m+1}'') \mid \ell + m + 1 = k,\ \ell, m \geq 0\} \cup \\ &\qquad \{\gamma_g'(P_{4\ell+2}'' \cup P_{4m+3}'') \mid \ell + m + 2 = k,\ \ell, m \geq 0\} \Big] \\ &= 1 + \min \Big[\{2\ell + 2m + 1 \mid \ell + m + 1 = k,\ \ell, m \geq 0\} \cup \\ &\qquad \{2(\ell + m + 2) \mid \ell + m + 2 = k,\ \ell, m \geq 0\} \Big] \\ &= 1 + \min\{2k - 1, 2k\} = 2k,\end{aligned}$$

$$\begin{aligned}\gamma_g(P_{4k+1}) &= 1 + \min \Big[\{\gamma_g'(P_{4\ell}' \cup P_{4m+2}') \mid \ell + m + 1 = k,\ \ell, m \geq 0\} \cup \\ &\qquad \{\gamma_g'(P_{4\ell+1}' \cup P_{4m+1}') \mid \ell + m + 1 = k,\ \ell, m \geq 0\} \cup \\ &\qquad \{\gamma_g'(P_{4\ell+3}' \cup P_{4m+3}') \mid \ell + m + 2 = k,\ \ell, m \geq 0\} \Big] \\ &= 1 + \min\{2k, 2k, 2k\} = 2k + 1,\end{aligned}$$

$$\begin{aligned}\gamma_g(P_{4k+2}) &= 1 + \min \Big[\{\gamma_g'(P_{4\ell}' \cup P_{4m+3}') \mid \ell + m + 1 = k,\ \ell, m \geq 0\} \cup \\ &\qquad \{\gamma_g'(P_{4\ell+1}' \cup P_{4m+2}') \mid \ell + m + 1 = k,\ \ell, m \geq 0\} \Big] \\ &= 1 + \min\{2k, 2k + 1\} = 2k + 1,\end{aligned}$$

$$\begin{aligned}\gamma_g(P_{4k+3}) &= 1+\min\Big[\{\gamma_g'(P_{4\ell}'\cup P_{4m}') \mid \ell+m=k,\ \ell,m\geq 0\}\cup\\ &\quad \{\gamma_g'(P_{4\ell+1}'\cup P_{4m+3}') \mid \ell+m+1=k,\ \ell,m\geq 0\}\cup\\ &\quad \{\gamma_g'(P_{4\ell+2}'\cup P_{4m+2}') \mid \ell+m+1=k,\ \ell,m\geq 0\}\Big]\\ &= 1+\min\{2k,2k+1,2k+1\} = 2k+1\,.\end{aligned}$$

We can see that Dominator's moves are similar to those in a game when one or two leafs are dominated. Since this is not the case in the S-game, we have to be more careful when considering moves of Staller on P_n. In the first move she clearly can not select a dominated leaf and dominate only one new vertex. Hence she either plays on a leaf and dominates two new vertices, or plays on a vertex of degree two, and thus splits the path into two (smaller) partially dominated paths. By using Lemma 2.3 and Lemma 2.1 we get the following for every $k\geq 1$.

$$\begin{aligned}\gamma_g'(P_{4k}) &= 1+\max\Big[\{\gamma_g(P_{4(k-1)+2}')\}\cup\\ &\quad \{\gamma_g(P_{4\ell}'\cup P_{4m+1}') \mid \ell+m+1=k,\ \ell,m\geq 0\}\cup\\ &\quad \{\gamma_g(P_{4\ell+2}'\cup P_{4m+3}') \mid \ell+m+2=k,\ \ell,m\geq 0\}\Big]\\ &= 1+\max\Big[\{2k-1\}\cup\\ &\quad \{2\ell+2m+1 \mid \ell+m+1=k,\ \ell,m\geq 0\}\cup\\ &\quad \{2(\ell+m+1)+1 \mid \ell+m+2=k,\ \ell,m\geq 0\}\Big]\\ &= 1+\max\{2k-1,2k-1,2k-1\} = 2k,\end{aligned}$$

$$\begin{aligned}\gamma_g'(P_{4k+1}) &= 1+\max\Big[\{\gamma_g(P_{4(k-1)+3}')\}\cup\\ &\quad \{\gamma_g(P_{4\ell}'\cup P_{4m+2}') \mid \ell+m+1=k,\ \ell,m\geq 0\}\cup\\ &\quad \{\gamma_g(P_{4\ell+1}'\cup P_{4m+1}') \mid \ell+m+1=k,\ \ell,m\geq 0\}\cup\\ &\quad \{\gamma_g(P_{4\ell+3}'\cup P_{4m+3}') \mid \ell+m+2=k,\ \ell,m\geq 0\}\Big]\\ &= 1+\max\{2k-1,2k-1,2k,2k-1\} = 2k+1,\end{aligned}$$

$$\begin{aligned}\gamma_g'(P_{4k+2}) &= 1+\max\Big[\{\gamma_g(P_{4k}')\}\cup\\ &\quad \{\gamma_g(P_{4\ell}'\cup P_{4m+3}') \mid \ell+m+1=k,\ \ell,m\geq 0\}\cup\\ &\quad \{\gamma_g(P_{4\ell+1}'\cup P_{4m+2}') \mid \ell+m+1=k,\ \ell,m\geq 0\}\Big]\\ &= 1+\max\{2k,2k-1,2k\} = 2k+1,\end{aligned}$$

$$\begin{aligned}\gamma_g'(P_{4k+3}) &= 1+\max\Big[\{\gamma_g(P'_{4k+1})\}\cup \\ &\quad \{\gamma_g(P'_{4\ell}\cup P'_{4m}) \mid \ell+m=k,\ \ell,m\geq 0\}\cup \\ &\quad \{\gamma_g(P'_{4\ell+1}\cup P'_{4m+3}) \mid \ell+m+1=k,\ \ell,m\geq 0\}\cup \\ &\quad \{\gamma_g(P'_{4\ell+2}\cup P'_{4m+2}) \mid \ell+m+1=k,\ \ell,m\geq 0\}\Big] \\ &= 1+\max\{2k+1,2k,2k,2k+1\} = 2k+2\,.\end{aligned}$$

□

The last proof does not only give us the game domination number of paths, but also tells us what are the optimal moves of both players. We can see that choosing one of the leaves is an optimal first move of Staller on P_n when $n \not\equiv 1 \pmod 4$. In the case when $n \equiv 1 \pmod 4$ holds, Staller's optimal move is to play on a vertex v such that the residual graph obtained from $P_n|N[v]$ consists of exactly two components P'_r and P'_s, where $r+s=n-3$, $r \equiv 1 \pmod 4$ and $s \equiv 1 \pmod 4$.

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Affine primitive symmetric graphs of diameter two*

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Abstract

Let n be a positive integer, q be a prime power, and V be a vector space of dimension n over $\mathbb{F}_q$. Let $G := V \rtimes G_0$, where G_0 is an irreducible subgroup of $\mathrm{GL}(V)$ which is maximal by inclusion with respect to being intransitive on the set of nonzero vectors. We are interested in the class of all diameter two graphs Γ that admit such a group G as an arc-transitive, vertex-quasiprimitive subgroup of automorphisms. In particular, we consider those graphs for which G_0 is a subgroup of either $\Gamma\mathrm{L}(n, q)$ or $\Gamma\mathrm{Sp}(n, q)$ and is maximal in one of the Aschbacher classes $\mathcal{C}_i$, where $i \in \{2, 4, 5, 6, 7, 8\}$. We are able to determine all graphs Γ which arise from $G_0 \leq \Gamma\mathrm{L}(n, q)$ with $i \in \{2, 4, 8\}$, and from $G_0 \leq \Gamma\mathrm{Sp}(n, q)$ with $i \in \{2, 8\}$. For the remaining classes we give necessary conditions in order for Γ to have diameter two, and in some special subcases determine all G-symmetric diameter two graphs.

Keywords: Symmetric graphs, Cayley graphs, quasiprimitive permutation groups, linear groups.

Math. Subj. Class.: 05C25, 20B15, 20B25

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1 Introduction

A *symmetric* graph is one which admits a subgroup of automorphisms that acts transitively on its arc set; if G is such a subgroup, we say in particular that the graph is G-*symmetric*. We are interested in the family of all symmetric graphs with diameter two, a family which contains all symmetric strongly regular graphs. We consider those G-symmetric diameter two graphs where G is a primitive group of affine type, and where the point stabiliser G_0 is maximal in the general semilinear group or in the symplectic semisimilarity group. Our main result is Theorem 1.1. Those affine examples where G_0 is not contained in either of these groups were studied in [2].

Theorem 1.1. Let $V = \mathbb{F}_q^n$ for some prime power q and positive integer n, and let $G = V \rtimes G_0$, where G_0 is an irreducible subgroup of the general semilinear group $\Gamma\mathrm{L}(n,q)$ or the symplectic semisimilarity group $\Gamma\mathrm{Sp}(n,q)$, and G_0 is maximal by inclusion with respect to being intransitive on the set of nonzero vectors in V. If Γ is a connected graph with diameter two which admits G as a symmetric group of automorphisms, then Γ is isomorphic to a Cayley graph $\mathrm{Cay}(V,S)$ for some orbit S of G_0 satisfying $\langle S\rangle = V$ and $S = -S$, and one of the following holds:

1. (G_0, S) are as in Tables 1.0.1 and 1.0.2;
2. G_0 satisfies the conditions in Table 1.0.3;
3. G_0 belongs to the class $\mathcal{C}_9$.

Furthermore, all pairs (G_0, S) in Tables 1.0.1 and 1.0.2 yield G-symmetric diameter two graphs $\mathrm{Cay}(V,S)$.

Notation for Tables 1.0.1 and 1.0.2. The set X_s is as in (3.2) and W_β is as in (3.5) in Section 3.2, Y_s is as in (3.7) in Section 3.3, $c(v)$ is as in (3.9) in Section 3.4, S_0 is as in (2.4) in Section 2.2, and $S_\#$, $S_\square$ and $S_\boxtimes$ are as in (3.1) in Section 3.1. Cayley graphs are defined in Section 2.1. The graphs marked † did not appear in [2].

Table 1.0.1: Symmetric diameter two graphs from maximal subgroups of $\Gamma\mathrm{L}(n,q)$

	$G_0 \cap \mathrm{GL}(n,q)$	S	Conditions
1	$\mathrm{GL}(m,q) \wr \mathrm{Sym}(t)$, $mt = n$	X_s	$q^m > 2$ and $s \geq t/2$
2	$\mathrm{GL}(k,q) \otimes \mathrm{GL}(m,q)$, $km = n$	Y_s	$s \geq \frac{1}{2}\min\{k,m\}$
†3	$\mathrm{GL}\left(n,q^{1/r}\right) \circ Z_{q-1}$, $r > 2$ and $n > 2$	v^{G_0} as in (3.14)	$c(v) = r-1$ or $c(v) = r$
†4	$\mathrm{GL}\left(n,q^{1/r}\right) \circ Z_{q-1}$, $r = 2$ or $n = 2$	v^{G_0} as in (3.14)	$c(v) = 1$
5	$(Z_{q-1} \circ (Z_4 \circ Q_8)).\mathrm{Sp}(2,2)$, $n = 2$, q odd	v^{G_0}	$v \in V^\#$
6	$\mathrm{GL}(m,q) \wr_\otimes \mathrm{Sym}(2)$, $m^2 = n$	Y_s	$s \geq m/2$
†7	$\mathrm{GU}(n,q)$, $n \geq 2$	S_0, $S_\#$	
8	$\mathrm{GO}(n,q)$, $n = 3$ and $q = 3$	S_0	
9	$\mathrm{GO}(n,q)$, nq odd, $n > 3$ or $q > 3$	S_0, $S_\square$, or $S_\boxtimes$	
10	$\mathrm{GO}^+(n,q)$, n even, q odd, $n > 2$ or $q > 2$	S_0 or $S_\#$	
11	$\mathrm{GO}^-(n,q)$, n even, q odd, $n > 2$	S_0 or $S_\#$	

Table 1.0.2: Symmetric diameter two graphs from maximal subgroups of $\Gamma\mathrm{Sp}(n,q)$

	$G_0 \cap \mathrm{GL}\,(n,q)$	S	Conditions
1	$\mathrm{Sp}\,(m,q)^t\,.[q-1].\mathrm{Sym}\,(t)$, $mt=n$	X_s	$q^m>2$ and $s \geq t/2$
†2	$\mathrm{GL}\,(m,q)\,.[2]$, $2m=n$	$\bigcup_{\sigma\in\mathrm{Aut}(\mathbb{F}_q)} W_{\beta^\sigma}$	$q^m>2$ and $\beta \in \mathbb{F}_q$
†3	$(Z_{q-1}\circ Q_8).\mathrm{O}^-\,(2,2)$, $n=2$, q odd	v^{G_0}	$v \in V^\#$
4	$\mathrm{GO}^+(n,q)$, $n=2$ and $q=2$	S_0	
5	$\mathrm{GO}^+(n,q)$, q and n even, $n>2$ or $q>2$	S_0 or $S_\#$	
6	$\mathrm{GO}^-(n,q)$, q and n even, $n>2$	S_0 or $S_\#$	

Table 1.0.3: Restrictions for remaining cases

	$G_0 \cap \mathrm{GL}\,(n,q)$	Conditions	Restrictions
1	$\mathrm{GSp}\,(k,q)\otimes\mathrm{GO}^\epsilon(m,q)$, m odd, $q>3$		Proposition 3.14
2	$\mathrm{GL}\left(n,q^{1/r}\right)\circ Z_{q-1}$	$c(v)\neq r-1, r$	Proposition 3.16 (2), (3), (4)
3	$(Z_{q-1}\circ R).\mathrm{Sp}\,(2t,r)$, $n=r^t$	R Type 1, $t\geq 2$	Proposition 3.23 (1)
4	$(Z_{q-1}\circ R).\mathrm{Sp}\,(2t,2)$, $n=r^t$	R Type 2, $t\geq 2$	Proposition 3.23 (2)
5	$(Z_{q-1}\circ R).\mathrm{O}^-\,(2t,2)$, $n=r^t$	R Type 4, $t\geq 2$	Proposition 3.23 (3)
6	$\mathrm{GL}\,(m,q)\wr_\otimes \mathrm{Sym}\,(t)$, $m^t=n$	$t\geq 3$	Proposition 3.25
7	$\mathrm{GSp}\,(m,q)\wr_\otimes \mathrm{Sym}\,(t)$, $m^t=n$, q odd	$t\geq 3$	Proposition 3.26

The reduction to these cases is achieved as follows. It is shown in [1] that any symmetric diameter two graph has a normal quotient graph Γ which is G-symmetric for some group G and which satisfies one of the following:

(I) the graph Γ has at least one nontrivial G-normal quotient, and all nontrivial G-normal quotients of Γ are complete graphs (that is, every pair of distinct vertices are adjacent); or

(II) all G-normal quotients of Γ are trivial graphs (that is, consisting of a single vertex).

The context of our investigation is the following. It was shown that those that satisfy (II) fall into eight types according to the action of G [7]. One of these types is known as HA (see Subsection 2.1). In this case, the vertex set is a finite-dimensional vector space $V=\mathbb{F}_p^d$ over a prime field $\mathbb{F}_p$ and $G=V\rtimes G_0$, where V is identified with the group of translations on itself and G_0 is an irreducible subgroup of $\mathrm{GL}\,(d,p)$ which is intransitive on the set of nonzero vectors of V. The irreducible subgroups of $\mathrm{GL}\,(d,p)$ can be divided into eight classes $\mathcal{C}_i$, $i\in\{2,\ldots,9\}$, most of which can be described as preserving certain geometric configurations on V, such as direct sums or tensor decompositions [3]. Note that, if a diameter two graph Γ is G-symmetric, then the stabiliser G_v of a vertex v is not transitive on the remaining vertices since G_v leaves invariant the sets of vertices at distance 1, and distance 2, from v. Thus, in our situation, the group G_0 is intransitive on the set $V^\#$, where $V^\# := V\setminus\{0\}$, the set of nonzero vectors. In paper [2] we considered the graphs corresponding to the groups G_0 which are maximal in their respective classes $\mathcal{C}_i$, for $i\leq 8$, and which are intransitive on nonzero vectors. (We did not consider the last class

$\mathcal{C}_9$ since the groups in this class do not have a uniform geometric description.) Several classes were not considered because the maximal groups in these classes are transitive on $V^{\#}$, namely, the maximal groups are (a) symplectic groups preserving a nondegenerate alternating bilinear form on V, and (b) "extension field groups" preserving a structure on V of an n-dimensional vector space over $\mathbb{F}_q$, where $q^n = p^d$. The aim of this paper is to examine the cases not treated in [2], namely, G_0 preserves either an alternating form or an extension field structure on V, and:

(III) The group G_0 is irreducible and is maximal in $\mathrm{GL}\,(d,p)$ with respect to being intransitive on nonzero vectors.

All quasiprimitive groups of type HA are primitive; the condition of irreducibility of G_0 is necessary to guarantee that G_0 is maximal in G, and hence that G is primitive. In particular, since G_0 is intransitive on $V^{\#}$, G_0 does not contain $\mathrm{SL}\,(V)$ or $\mathrm{Sp}\,(V)$. The classification in [3] can be applied to the two groups $\Gamma\mathrm{L}(n,q)$ and $\mathrm{GSp}\,(d,p)$: the irreducible subgroups of $\Gamma\mathrm{L}(n,q)$ and of $\mathrm{GSp}\,(d,p)$ which do not contain $\mathrm{SL}\,(n,q)$ and $\mathrm{Sp}\,(d,p)$, respectively, are again organised into classes $\mathcal{C}_2$ to $\mathcal{C}_9$. Again we do not consider the $\mathcal{C}_9$-subgroups. Observe that of the maximal subgroups of $\Gamma\mathrm{L}(n,q)$ in classes $\mathcal{C}_2$ to $\mathcal{C}_8$, the only transitive ones are the $\mathcal{C}_3$-subgroups $\Gamma\mathrm{L}(m,q^{n/m})$ with n/m prime, and the $\mathcal{C}_8$-subgroup $\Gamma\mathrm{Sp}(n,q)$ of symplectic semisimilarities. We avoid these possibilities by choosing q maximal such that $q^n = p^d$. We then consider the two cases: (1) where $G_0 \leq \Gamma\mathrm{L}(n,q)$ and G_0 does not preserve an alternating form on $\mathbb{F}_q^n$, and (2) where $G_0 \leq \Gamma\mathrm{Sp}(n,q)$. Note that in this case it is possible for d/n to be not prime, and it follows from the maximality of q that G_0 is not contained in a proper $\mathcal{C}_3$-subgroup of $\Gamma\mathrm{L}(n,q)$ or $\Gamma\mathrm{Sp}(n,q)$, respectively. Since G_0 is irreducible and we are not considering $\mathcal{C}_9$-subgroups, we now have G_0 a maximal intransitive subgroup in the $\mathcal{C}_i$ (for $\Gamma\mathrm{L}(n,q)$ or $\Gamma\mathrm{Sp}(n,q)$) for some $i \in \{2,4,5,6,7,8\}$.

All such subgroups of $\Gamma\mathrm{L}(n,q)$ for which $n = d$ and $i \neq 5$ are considered in [2]; moreover, for some of these cases, the arguments were given in the general setting of $\mathcal{C}_i$-subgroups of $\Gamma\mathrm{L}(n,q)$, and so can be applied here. The cases requiring the most detailed arguments are those for subfield groups and, to a lesser extent, normalisers of symplectic-type r-groups ($\mathcal{C}_i$-groups with $i \in \{5,6\}$).

As in [2], for each family of groups G_0 we have two main tasks:

(i) to determine the G_0-orbits, and

(ii) to identify which of these orbits correspond to diameter two Cayley graphs.

In the instances where we are not able to achieve either of these, we obtain bounds on certain parameters to reduce the number of unresolved cases.

The rest of this paper is organised as follows: In Section 2 we give the relevant background on affine quasiprimitive permutation groups, semilinear transformations and semisimilarities. In Subsection 2.3 we present Aschbacher's classification of the subgroups of $\Gamma\mathrm{L}(n,q)$ and $\Gamma\mathrm{Sp}(n,q)$. Section 3 is devoted to the proof of Theorem 1.1, which we do by considering separately the maximal intransitive subgroups in each of the classes $\mathcal{C}_i$, where $i \in \{2,4,5,6,7,8\}$.

Notation. If A is a vector space, a finite field, or a group, $A^{\#}$ denotes the set of nonzero vectors, nonzero field elements, or non-identity group elements, respectively. The finite field of order q is denoted by $\mathbb{F}_q$. The notation used for the classical groups, some of which is nonstandard, is presented in Section 2. If Γ is a graph, $V(\Gamma)$ and $E(\Gamma)$ are, respectively, its vertex set and edge set.

2 Preliminaries

2.1 Cayley graphs and HA-type groups

The action of a group G on a set Ω is said to be *quasiprimitive of type HA* if G has a unique minimal normal subgroup N and N is elementary abelian and acts regularly on Ω. The group G is then a subgroup of the holomorph $N.\mathrm{Aut}\,(N)$ of N (hence the abbreviation HA, for *h*olomorph of an *a*belian group). It follows from [4, Lemma 16.3] that a graph Γ that admits G as a subgroup of automorphisms is isomorphic to a *Cayley graph* on N, that is, a graph with vertex set N and edge set $\{\{x, y\} \mid x - y \in S\}$ for some subset S of $N^{\#}$ with $S = -S$ and $0 \notin S$. (Since N is abelian we use additive notation, and in particular denote the identity by 0 and call it zero.) Such a graph is denoted by $\mathrm{Cay}(N, S)$. If, in addition, Γ is G-symmetric, then S must be an orbit of the point stabiliser G_0 of zero. Thus, in order for Γ to have diameter two, the group G_0 must be intransitive on the set of nonzero elements in N.

The result that is most relevant to our investigation is Lemma 2.1, which follows from the basic properties of Cayley graphs and quasiprimitive groups of type HA.

Lemma 2.1 ([7]). *Let Γ be a graph and $G \leq \mathrm{Aut}\,(\Gamma)$, where G acts quasiprimitively on $V(\Gamma)$ and is of type HA. Then $G \cong \mathbb{F}_p^d \rtimes G_0 \leq \mathrm{AGL}\,(d, p)$ and $\Gamma \cong \mathrm{Cay}(\mathbb{F}_p^d, S)$ for some finite field $\mathbb{F}_p$, where the vector space $\mathbb{F}_p^d$ is identified with its translation group and $G_0 \leq \mathrm{GL}\,(d, p)$ is irreducible. Moreover, Γ is G-symmetric with diameter 2 if and only if S is a G_0-orbit of nonzero vectors satisfying $-S = S$, $S \subsetneq V$ and $S \cup (S + S) = V$.*

The condition $-S = S$ implies that $|S + S| \leq |S|(|S| - 1) + 1$, and if S is a G_0-orbit then clearly $|S| \leq |G_0|$. It follows from Lemma 2.1 that if $\mathrm{Cay}(V, S)$ is G-symmetric with diameter two then

$$|V| \leq |S|^2 + 1 \leq |G_0|^2 + 1. \tag{2.1}$$

This fact will be frequently used in obtaining bounds for certain parameters.

In our situation $p^d = q^n$ and G_0 preserves on V the structure of an $\mathbb{F}_q$-space; we therefore regard V as $V = \mathbb{F}_q^n$, and G_0 as a subgroup of $\Gamma\mathrm{L}(n, q)$.

2.2 Semilinear transformations and semisimilarities

Throughout this subsection assume that q is an arbitrary prime power, V is a vector space with finite dimension n over $\mathbb{F}_q$, and $\mathcal{B} := \{v_1, \ldots, v_n\}$ is a fixed $\mathbb{F}_q$-basis of V.

The *general semilinear group* $\Gamma\mathrm{L}(n, q)$ consists of all invertible maps $h : V \to V$ for which there exists $\alpha(h) \in \mathbb{F}_q$, which depends only on h, satisfying

$$(\lambda u + v)^h = \lambda^{\alpha(h)} u^h + v^h \quad \text{for all } \lambda \in \mathbb{F}_q \text{ and } u, v \in V. \tag{2.2}$$

The group $\Gamma\mathrm{L}(n, q)$ is isomorphic to a semidirect product $\mathrm{GL}\,(n, q) \rtimes \mathrm{Aut}\,(\mathbb{F}_q)$ with the following action on V:

$$\left(\sum_{i=1}^{n} \lambda_i v_i\right)^{g\alpha} := \sum_{i=1}^{n} \lambda_i^{\alpha} v_i^{g} \quad \text{for all } g \in \mathrm{GL}\,(n, q),\ \alpha \in \mathrm{Aut}\,(\mathbb{F}_q),\ \text{and} \tag{2.3}$$
$$\lambda_1, \ldots, \lambda_n \in \mathbb{F}_q.$$

If V is endowed with a left-linear or quadratic form ϕ, then the elements of $\Gamma\mathrm{L}(n, q)$ that preserve ϕ up to a nonzero scalar factor or an $\mathbb{F}_q$-automorphism are called *semisimilarities* of ϕ. That is, h is a semisimilarity of ϕ if and only if for some $\lambda(h) \in \mathbb{F}_q^\#$ and some $\alpha'(h) \in \mathrm{Aut}(\mathbb{F}_q)$, both of which depend only on h,

$$\phi(u^h, v^h) = \lambda(h)\phi(u,v)^{\alpha'(h)} \quad \text{for all } u, v \in V$$

if ϕ is left-linear, and

$$\phi(v^h) = \lambda(h)\phi(v)^{\alpha'(h)} \quad \text{for all } v \in V$$

if ϕ is quadratic. It can be shown that $\alpha'(h)$ is the element $\alpha(h)$ in (2.2). The set of all semisimilarities of ϕ is a subgroup of $\Gamma\mathrm{L}(n,q)$ and is denoted by $\Gamma\mathrm{I}\,(n,q)$, where I is $\mathrm{Sp}, \mathrm{U}, \mathrm{O}, \mathrm{O}^+$, or O^-, if ϕ is symplectic (i.e., nondegenerate alternating bilinear), unitary (i.e., nondegenerate conjugate-symmetric sesquilinear), quadratic in odd dimension, quadratic of plus type, or quadratic of minus type, respectively.

The map $\alpha : \Gamma\mathrm{I}\,(n,q) \to \mathrm{Aut}\,(\mathbb{F}_q)$ defined by $h \mapsto \alpha(h)$ is a group homomorphism whose kernel $\mathrm{GI}(n,q)$ consists of all $g \in \mathrm{GL}\,(n,q)$ that preserve ϕ up to a nonzero scalar factor. The elements of $\mathrm{GI}(n,q)$ are called *similarities* of ϕ. Likewise, the map $g \mapsto \lambda(g)$ for any $g \in \Gamma\mathrm{I}\,(n,q)$ defines a homomorphism λ from $\mathrm{GI}(n,q)$ to the multiplicative group $\mathbb{F}_q^\#$. The kernel $\mathrm{I}(n,q)$ of λ consists of all ϕ-preserving elements in $\mathrm{GL}\,(n,q)$, which are called the *isometries* of ϕ. It should be emphasised that our notation for the similarity and isometry groups is non-standard, but follows for example [5]: the symbol $\mathrm{GI}(n,q)$ is sometimes used to denote the isometry group, whereas in the present paper this refers to the similarity group.

In Subsection 3.1 we determine the orbits in $V^\#$ of the groups $\Gamma\mathrm{I}\,(n,q)$. The following result, which gives the orbits of the isometry groups $\mathrm{I}(n,q)$, is useful:

Theorem 2.2 ([8, Propositions 3.11, 5.12, 6.8 and 7.10])**.** *Let $V = \mathbb{F}_q^n$ and ϕ a symplectic, unitary, or nondegenerate quadratic form on V. Then the orbits in $V^\#$ of the isometry group of (V, ϕ) are the sets S_λ for each $\lambda \in \mathrm{Im}\,(\overline{\phi})$, where*

$$S_\lambda := \{v \in V^\# \mid \overline{\phi}(v) = \lambda\} \tag{2.4}$$

and

$$\overline{\phi}(v) = \begin{cases} \phi(v,v) & \text{if } \phi \text{ is symplectic or unitary;} \\ \phi(v) & \text{if } \phi \text{ is quadratic.} \end{cases} \tag{2.5}$$

Observe that if ϕ is symplectic then $\phi(v,v) = 0$ for all nonzero vectors v, so it follows from Theorem 2.2 that $\mathrm{Sp}\,(n,q)$ is transitive on $V^\#$.

2.2.1 Some geometry

Let f be a left-linear form on V. A nonzero vector v is called *isotropic* if $f(v,v) = 0$; otherwise, it is *anisotropic*. If f is symplectic or unitary, then an isotropic vector is also called *singular*. If f is symmetric bilinear and Q is a quadratic form which polarises to f (that is, $f(u,v) = Q(u+v) - Q(u) - Q(v)$), then a singular vector is a nonzero vector v with $Q(v) = 0$. Hence, in general, all isotropic vectors are singular and vice versa, unless V is orthogonal and q is even; in this case all nonzero vectors are isotropic but not all are singular. A subspace U of V is *totally isotropic* if $f|_U \equiv 0$, and *totally singular* if all its nonzero vectors are singular. On the other hand, a subspace U is *anisotropic* if all of its nonzero vectors are anisotropic.

For any subspace U of V we define the subspace

$$U^\perp := \{v \in V \mid f(u,v) = 0\ \forall\, u \in U\}$$

and we write $V = U \perp W$ if $V = U \oplus W$ and $W \leq U^\perp$. Clearly a nonzero vector v is isotropic if and only if $v \in \langle v\rangle^\perp$, and the subspace U is totally isotropic if and only if $U \leq U^\perp$. A symplectic or unitary form f, or a quadratic form with associated bilinear form f, is *nondegenerate* (or *nonsingular*) if the radical $V^\perp$ of f is the zero subspace.

A *hyperbolic pair* in V is a pair $\{x, y\}$ of singular vectors such that $f(x, y) = 1$. The space V can be decomposed into an orthogonal direct sum of an anisotropic subspace and subspaces spanned by hyperbolic pairs, as stated in the following fundamental result on the geometry of formed spaces.

Theorem 2.3 ([6, Propositions 2.3.2, 2.4.1, 2.5.3]). *Let $V = \mathbb{F}_q^n$, and let f be a left-linear form on V which is symplectic, unitary, or a symmetric bilinear form associated with a nondegenerate quadratic form Q. Then*

$$V = \langle x_1, y_1\rangle \perp \ldots \perp \langle x_m, y_m\rangle \perp U$$

where $\{x_i, y_i\}$ is a hyperbolic pair for each i and U is an anisotropic subspace. Moreover:

1. *If f is symplectic then $U = 0$. Hence n is even and, up to equivalence, there is a unique symplectic geometry in dimension n over $\mathbb{F}_q$.*
2. *If f is unitary then $U = 0$ if n is even and* $\dim(U) = 1$ *if n is odd. Hence up to equivalence, there is a unique unitary geometry in dimension n over $\mathbb{F}_q$.*
3. *If f is symmetric bilinear with quadratic form Q and n is odd, then q is odd,* $\dim(U) = 1$*, and there are two isometry classes of quadratic forms in dimension n over $\mathbb{F}_q$, one a non-square multiple of the other. Hence all orthogonal geometries in dimension n over $\mathbb{F}_q$ are similar.*
4. *If f is symmetric bilinear with quadratic form Q and n is even, then $U = 0$ or* $\dim(U) = 2$. *For each n there are exactly two isometry classes of orthogonal geometries over $\mathbb{F}_q$, which are distinguished by* $\dim(U)$.

In Theorem 2.3 (4), the quadratic form Q and the corresponding geometry is said to be of *plus type* if $U = 0$, and of *minus type* if $\dim(U) = 2$.

2.2.2 Tensor products

Some of the subgroups listed in Aschbacher's classification arise as tensor products of classical groups. In order to describe the group action we define first the tensor product of forms. If $V = U \otimes W$, and if ϕ_U and ϕ_W are both bilinear or both unitary forms on U and W, respectively, then the form $\phi_U \otimes \phi_W$ on V is defined by

$$(\phi_U \otimes \phi_W)(u \otimes w, u' \otimes w') := \phi_U(u, u')\phi_W(w, w')$$

for all $u \otimes w$ and $u' \otimes w'$ in a tensor product basis of V, extended bilinearly if ϕ_U and ϕ_W are bilinear, and sesquilinearly if ϕ_U and ϕ_W are sesquilinear. If ϕ_U and ϕ_W are both bilinear then so is $\phi_U \otimes \phi_W$; moreover, $\phi_U \otimes \phi_W$ is alternating if at least one of ϕ_U and ϕ_W

Table 2.2.4: Tensor products of classical groups

$\mathrm{I}(U, \phi_U)$	$\mathrm{I}(W, \phi_W)$	$\mathrm{I}(U \otimes W, \phi_U \otimes \phi_W)$
Sp	O^{ϵ}	$\begin{cases} \mathrm{Sp} & \text{if the characteristic is odd;} \\ \mathrm{O}^{+} & \text{else} \end{cases}$
Sp	Sp	O^{+}
O^{ϵ_1}	O^{ϵ_2}	$\begin{cases} \mathrm{O}^{+} & \text{if } \epsilon_i = + \text{ for some } i, \text{ or } \epsilon_i = - \text{ for both } i; \\ \mathrm{O} & \text{if } \dim(U) \text{ and } \dim(W) \text{ are odd;} \\ \mathrm{O}^{-} & \text{else} \end{cases}$
U	U	U

is alternating, and $\phi_U \otimes \phi_W$ is symmetric if both ϕ_U and ϕ_W are symmetric. If ϕ_U and ϕ_W are both unitary then $\phi_U \otimes \phi_W$ is unitary. The tensor product $I(U, \phi_U) \otimes I(W, \phi_W)$ acts on V with the usual tensor product action — that is, for any $g \in I(U, \phi_U)$, $h \in I(W, \phi_W)$, $u \in U$ and $w \in W$,

$$(u \otimes w)^{(g,h)} := u^g \otimes w^h.$$

The types of forms $\phi_U \otimes \phi_W$ that arise according to the various possibilities for ϕ_U and ϕ_W, which are given in terms of the possible inclusions $I(U, \phi_U) \otimes I(W, \phi_W) \leq I(V, \phi_U \otimes \phi_W)$, are summarised in Table 2.2.4.

The tensor product of an arbitrary number of formed spaces can be defined similarly: If $V = U_1 \otimes \cdots \otimes U_t$ and ϕ_i is a nondegenerate form on U_i for each i, and either all ϕ_i are bilinear or all are sesquilinear, the form $\phi_1 \otimes \cdots \otimes \phi_t$ is given by

$$\left(\otimes_{i=1}^{t} \phi_i\right) \left(\otimes_{i=1}^{t} u_i, \otimes_{i=1}^{t} w_i\right) = \prod_{i=1}^{t} \phi(u_i, w_i)$$

as $\otimes_{i=1}^{t} u_i$ and $\otimes_{i=1}^{t} w_i$ vary over a tensor product basis of V, extended bilinearly if the ϕ are bilinear, and sesquilinearly if they are sesquilinear. Then $\otimes_{i=1}^{t} \phi_i$ is a nondegenerate bilinear (respectively, sesquilinear) form on V. If the spaces (U_i, ϕ_i) are all isometric, then we can extend the results of Table 2.2.4 to the following (see [6, 9]):

$$\otimes_{i=1}^{t} \mathrm{Sp}\,(m, q) < \begin{cases} \mathrm{Sp}\,(m^t, q) & \text{if } qt \text{ odd;} \\ \mathrm{O}^{+}\,(m^t, q) & \text{if } qt \text{ is even} \end{cases}$$

$$\otimes_{i=1}^{t} \mathrm{O}^{\epsilon}(m, q) < \begin{cases} \mathrm{O}\,(m^t, q) & \text{if } qm \text{ is odd;} \\ \mathrm{O}^{-}\,(m^t, q) & \text{if } \epsilon = - \text{ and } t \text{ is odd;} \\ \mathrm{O}^{+}\,(m^t, q) & \text{else} \end{cases}$$

$$\otimes_{i=1}^{t} \mathrm{U}\,(m, q) < \mathrm{U}\left(m^t, q\right)$$

2.3 Aschbacher's classification

The irreducible subgroups of semisimilarity and semilinear groups are classified by Aschbacher's Theorem [3]. In [6], Aschbacher's Theorem is used to identify those irreducible subgroups which are maximal. We present below the versions that correspond to $\Gamma\mathrm{L}(n, q)$

and to $\Gamma\mathrm{Sp}(n,q)$. Recall that G_0 does not contain either of the transitive groups $\mathrm{SL}\,(n,q)$ or $\mathrm{Sp}\,(n,q)$.

Theorem 2.4. *If M is a maximal irreducible subgroup of $\Gamma\mathrm{L}(n,q)$ that does not contain* $\mathrm{SL}\,(n,q)$*, then M is one of the following groups:*

($\mathcal{C}_2$) $(\mathrm{GL}\,(m,q) \wr \mathrm{Sym}\,(t)) \rtimes \mathrm{Aut}\,(\mathbb{F}_q)$, *where $mt = n$;*

($\mathcal{C}_3$) $\Gamma\mathrm{L}(m,q^r)$, *where r is prime and $mr = n$;*

($\mathcal{C}_4$) $(\mathrm{GL}\,(k,q) \otimes \mathrm{GL}\,(m,q)) \rtimes \mathrm{Aut}\,(\mathbb{F}_q)$, *where $km = n$ and $k \neq m$, and the action of τ is defined with respect to a tensor product basis of $\mathbb{F}_q^k \otimes \mathbb{F}_q^m$;*

($\mathcal{C}_5$) $\left(\mathrm{GL}\left(n,q^{1/r}\right) \circ Z_{q-1}\right) \rtimes \mathrm{Aut}\,(\mathbb{F}_q)$, *where $n \geq 2$, q is an rth power and r is prime;*

($\mathcal{C}_6$) $((Z_{q-1} \circ R).T) \rtimes \mathrm{Aut}\,(\mathbb{F}_q)$, *where $n = r^t$ with r prime, q is the smallest power of p such that $q \equiv 1 \pmod{r}$, and R and T are as given in Table* 2.3.5 *with R of type 1 or 2;*

($\mathcal{C}_7$) $(\mathrm{GL}\,(m,q) \wr_{\otimes} \mathrm{Sym}\,(t)) \rtimes \mathrm{Aut}\,(\mathbb{F}_q)$, *where $m^t = n$, $t \geq 2$, and the action of τ is defined with respect to a tensor product basis of $\otimes_{i=1}^t \mathbb{F}_q^m$;*

($\mathcal{C}_8$) $\Gamma\mathrm{O}\,(n,q)$ *or* $\Gamma\mathrm{O}^{\pm}(n,q)$ *with q odd,* $\Gamma\mathrm{Sp}(n,q)$*, or* $\Gamma\mathrm{U}\,(n,q)$*;*

($\mathcal{C}_9$) *the preimage of an almost simple group $H \leq \mathrm{P}\Gamma\mathrm{L}\,(n,q)$ satisfying the following conditions:*

(a) $T \leq H \leq \mathrm{Aut}\,(T)$ *for some nonabelian simple group T (i.e., H is almost simple).*

(b) The preimage of T in $\mathrm{GL}\,(n,q)$ *is absolutely irreducible and cannot be realised over a proper subfield of $\mathbb{F}_q$.*

In Theorem 2.5 the symbol $[o]$ denotes a group of order o. In case ($\mathcal{C}_2$) the group $[q-1]$ is generated by the map

$$\delta_\mu : x_i \mapsto \mu x_i,\ y_i \mapsto y_i$$

for all x_i and all y_i, $i \in \{1,\ldots,n/2\}$, where μ is a generator of the multiplicative group $\mathbb{F}_q^{\#}$ and $\{x_1,\ldots,x_{n/2},y_1,\ldots,y_{n/2}\}$ is a basis of $\mathbb{F}_q^n$, satisfying $\phi(x_i,x_j) = \phi(y_i,y_j) = \phi(x_i,y_j) = 0$ whenever $i \neq j$ and $\phi(x_i,y_i) = 1$ for all i. Such a basis is called a *symplectic basis*.

Theorem 2.5. *If M is a maximal irreducible subgroup of $\Gamma\mathrm{Sp}(n,q)$, then M is one of the following groups:*

($\mathcal{C}_2$) $\left((\mathrm{Sp}\,(m,q)^t.[q-1].\mathrm{Sym}\,(t))\right) \rtimes \mathrm{Aut}\,(\mathbb{F}_q)$, *where $m = n/t$; or*
$(\mathrm{GL}\,(m,q).[2]) \rtimes \mathrm{Aut}\,(\mathbb{F}_q)$, *where $m = n/2$;*

($\mathcal{C}_3$) $(\mathrm{Sp}\,(m,q^r).[q-1]) \rtimes \mathrm{Aut}\,(\mathbb{F}_q)$, *where r is prime and $m = n/r$; or*
$\Gamma\mathrm{U}\left(m,q^2\right)$, *where $m = n/2$ and q is odd;*

($\mathcal{C}_4$) $(\mathrm{GSp}\,(k,q) \times \mathrm{GO}^{\epsilon}(m,q)) \rtimes \mathrm{Aut}\,(\mathbb{F}_q)$, *where q is odd, $k \neq m$, $m \geq 3$, and* GO^{ϵ} *can be any of* GO, GO^+, *or* GO^-*;*

($\mathcal{C}_5$) $\left(\mathrm{GSp}\left(n, q^{1/r}\right) \circ Z_{q-1}\right) \rtimes \mathrm{Aut}\left(\mathbb{F}_q\right)$

($\mathcal{C}_6$) $\left(Z_{q-1} \circ R\right).\mathrm{O}^-\left(2t, 2\right)$, *where* $q \geq 3$ *and is prime, and* R *is of type 4 in Table* 2.3.5*;*

($\mathcal{C}_7$) $\left(\mathrm{GSp}\left(m, q\right) \wr_{\otimes} \mathrm{Sym}\left(t\right)\right) \rtimes \mathrm{Aut}\left(\mathbb{F}_q\right)$, *where* qt *is odd;*

($\mathcal{C}_8$) $\Gamma\mathrm{O}^{\pm}(n, q)$, *where* q *is even;*

($\mathcal{C}_9$) *the preimage of an almost simple group* $H \leq \mathrm{P\Gamma L}\left(n, q\right)$ *satisfying the following conditions:*

(a) $T \leq H \leq \mathrm{Aut}\left(T\right)$ *for some nonabelian simple group* T *(i.e.,* H *is almost simple).*

(b) The preimage of T *in* $\mathrm{GL}\left(n, q\right)$ *is symplectic, absolutely irreducible, and cannot be realised over a proper subfield of* $\mathbb{F}_q$*.*

Table 2.3.5: $\mathcal{C}_6$-subgroups

	r	R	T
Type 1	odd	$\underbrace{R_0 \circ \cdots \circ R_0}_{t}$, $R_0 := r_+^{1+2}$	$\mathrm{Sp}\left(2t, r\right)$
Type 2	2	$Z_4 \circ \underbrace{Q_8 \circ \cdots \circ Q_8}_{t}$	$\mathrm{Sp}\left(2t, 2\right)$
Type 4	2	$\underbrace{D_8 \circ \cdots \circ D_8}_{t-1} \circ Q_8$	$\mathrm{O}^-\left(2t, 2\right)$

3 Symmetric diameter two graphs from maximal subgroups of groups $\Gamma\mathrm{L}(n, q)$ and $\Gamma\mathrm{Sp}(n, q)$

In this section we prove Theorem 1.1. In view of the observations in Section 1, assume that the following hypothesis holds:

Hypothesis 3.1. Let $V = \mathbb{F}_p^d$ with p prime and $d \geq 2$, which is viewed as $\mathbb{F}_q^n$ with $q = p^{d/n}$ for some divisor n of d (possibly d/n composite or $n = d$). Let H be one of the subgroups below of $\mathrm{GL}\left(d, p\right)$:

1. $H = \Gamma\mathrm{L}(n, q) = \mathrm{GL}\left(n, q\right) \rtimes \langle\tau\rangle$, the general semilinear group on V, or

2. $H = \Gamma\mathrm{Sp}(n, q) = \mathrm{GSp}\left(n, q\right) \rtimes \langle\tau\rangle$, the group of symplectic semisimilarities of a symplectic form on V,

Let τ denote the Frobenius automorphism of $\mathbb{F}_q$ and $\mathcal{B}$ be a fixed $\mathbb{F}_q$-basis of V, with τ acting on V as in (2.3) with respect to $\mathcal{B}$ (with $g = 1$ and $\alpha = \tau$); for the case where $H = \Gamma\mathrm{Sp}(n, q)$ assume that $\mathcal{B}$ is a symplectic basis of V. Define $G = V \rtimes G_0 \leq V \rtimes H < \mathrm{AGL}\left(d, p\right)$ and $L = G_0 \cap \mathrm{GL}\left(n, q\right)$, where G_0 is a maximal $\mathcal{C}_i$-subgroup of H for some $i \in \{2, 4, 5, 6, 7, 8\}$ and G_0 does not contain $\mathrm{Sp}\left(n, q\right)$ or $\mathrm{SL}\left(n, q\right)$.

We note that the groups considered in [2] are the same as the subgroups L, as defined above, of $H = \Gamma\mathrm{L}(n, q)$.

All irreducible subgroups of $\mathrm{GL}\,(d, p)$ which are maximal with respect to being intransitive on $V^{\#}$ thus occur as subcases of the groups considered in Hypothesis 3.1 or belong to class $\mathcal{C}_9$. (Indeed, G_0 is maximal intransitive if $n = d$ or if d/n is prime.) For each Aschbacher class assume that $G_0 = M$ is of the form given in Theorem 2.4 or 2.5.

Since some of the other subgroups of $\Gamma\mathrm{Sp}(n, q)$ involve classical groups, we begin with class $\mathcal{C}_8$.

3.1 Class $\mathcal{C}_8$

In this case the space V has a form ϕ, which is symplectic, unitary, or nondegenerate quadratic if $H = \Gamma\mathrm{L}(n, q)$, and is nondegenerate quadratic if $H = \Gamma\mathrm{Sp}(n, q)$ with q even. Since the symplectic group is transitive on $V^{\#}$, we consider only the unitary and orthogonal cases.

Throughout this section we shall use the following notation: for $\theta \in \{\square, \boxtimes, \#\}$ let

$$S_\theta := \bigcup_{\lambda \in \mathbb{F}_q^\theta} S_\lambda \qquad (3.1)$$

where the S_λ are as in (2.4). If q is a square (as in the unitary case), let $q_0 := \sqrt{q}$ and let $\mathbb{F}_{q_0}$ denote the subfield of $\mathbb{F}_q$ of index 2. Also let $\mathrm{Tr} : \mathbb{F}_q \to \mathbb{F}_{q_0}$ denote the trace map, that is, $\mathrm{Tr}(\alpha) = \alpha + \alpha^{q_0}$ for all $\alpha \in \mathbb{F}_q$.

We begin by describing the orbits of the similarity groups $\mathrm{GI}(n, q)$, where $\mathrm{I} \in \{\mathrm{U}, \mathrm{O}, \mathrm{O}^+, \mathrm{O}^-\}$.

Proposition 3.1. *Let $V = \mathbb{F}_q^n$, ϕ be a unitary or nondegenerate quadratic form on V, and $G_0 = \mathrm{GI}(n, q)$ with $\mathrm{I} \in \{\mathrm{U}, \mathrm{O}, \mathrm{O}^+, \mathrm{O}^-\}$, according to the type of ϕ. Let S_0 be as in* (2.4) *and $S_\square$, $S_\boxtimes$ and $S_\#$ be as in* (3.1).

1. *If ϕ is unitary, then the G_0-orbits in $V^{\#}$ are S_0 and $S_\#$.*
2. *If ϕ is nondegenerate quadratic, then the G_0-orbits in $V^{\#}$ are as follows:*
   - *(i) $S_\#$ if $n = 1$;*
   - *(ii) S_0 and $S_\#$ if n is even;*
   - *(iii) S_0, $S_\square$ and $S_\boxtimes$ if n is odd and $n \geq 3$.*

Proof. Statement 2 is precisely [2, Proposition 3.9], so we only need to prove statement 1. Assume that ϕ is unitary; hence q is a square and $q_0 = \sqrt{q}$. It follows from Theorem 2.2 that S_0 is a G_0-orbit (that is, provided that $S_0 \neq \varnothing$), so we only need to show that $S_\#$ is a G_0-orbit. Let $v \in S_\#$; clearly, $v^{G_0} \subseteq S_\#$. For any $u \in S_\#$ set $\alpha := f(u, u)f(v, v)^{-1}$. Then $\alpha \in \mathbb{F}_{q_0}^{\#}$, so $\alpha = \beta^{q_0+1}$ for some $\beta \in \mathbb{F}_q$. Hence $f(u, u) = \beta^{q_0+1} f(v, v) = f(\beta v, \beta v)$, so by Theorem 2.2 we have $u = (\beta v)^g$ for some $g \in \mathrm{U}\,(n, q)$. Then $u = v^{\beta g}$, where $\beta g \in \mathrm{GU}\,(n, q)$. Therefore $v^{G_0} = S_\#$, which proves statement 1. □

The orbits of the semisimilarity groups can be easily deduced from Proposition 3.1.

Proposition 3.2. *Let $V = \mathbb{F}_q^n$, ϕ be a unitary or nondegenerate quadratic form on V, and $G_0 = \Gamma\mathrm{I}(n, q)$ with $\mathrm{I} \in \{\mathrm{U}, \mathrm{O}, \mathrm{O}^+, \mathrm{O}^-\}$, according to the type of ϕ. Then for all cases, the G_0-orbits are exactly the same as the $\mathrm{GI}(n, q)$-orbits.*

Proof. This follows from Proposition 3.1 and the fact that the elements of $\Gamma\mathrm{I}(n,q)$ preserve the form up to an automorphism of $\mathbb{F}_q$. □

Hence, a direct consequence of Proposition 3.2 and [2, Proposition 3.12] is:

Proposition 3.3. *Let* Γ *be a graph and* $G \leq \mathrm{Aut}\,(\Gamma)$ *such that* G *satisfies Hypothesis* 3.1 *with* $G_0 = \Gamma\mathrm{O}\,(n,q)$ *or* $G_0 = \Gamma\mathrm{O}^{\epsilon}(n,q)$ *(*$\epsilon = \pm$*). Then* Γ *is* G*-symmetric with diameter* 2 *if and only if* $\Gamma \cong \mathrm{Cay}(V,S)$ *with* $V = \mathbb{F}_q^n$ *and the conditions listed in one of the lines 8–11 of Table* 1.0.1 *or lines 4–6 of Table* 1.0.2 *hold.*

We now consider the unitary case. Note that Theorem 2.3 implies that the space V contains a hyperbolic pair, which implies that there is some $v \in V$ which is nonsingular. The following are two easy but useful results which are analogous to Lemma 3.13 and Corollary 3.14 in [2].

Lemma 3.4. *Let* $V = \mathbb{F}_q^n$, ϕ *a unitary form on* V*, and* $\overline{\phi}$ *as in* (2.5)*. Then* $\mathrm{Im}\,(\overline{\phi}) = \mathbb{F}_{q_0}$*, the subfield of index* 2 *in* $\mathbb{F}_q$*.*

Proof. Recall that $f(v,v)^{\sqrt{q}} = f(v,v)$ for any $v \in V$, so $\mathrm{Im}\,(\overline{\phi}) \leq \mathbb{F}_{q_0}$. By the preceding remarks V contains a nonsingular vector, say u. So $f(\alpha u, \alpha u) = \alpha^{\sqrt{q}+1} f(u,u) = \eta(\alpha) f(u,u)$ for any $\alpha \in \mathbb{F}_q$, where $\eta : \mathbb{F}_q \to \mathbb{F}_{q_0}$ is the norm map. Since η is surjective so is $\overline{\phi}$, and the result follows. □

If $\phi(v,v) \neq 0$, then $\langle v\rangle^{\perp}$ is nondegenerate and $V = \langle v\rangle \perp \langle v\rangle^{\perp}$. On the other hand, if $\phi(v,v) = 0$ then $\langle v\rangle \leq \langle v\rangle^{\perp}$. By the remarks in [6, pp. 17–18], the form ϕ induces a nondegenerate unitary form ϕ_U on the space $U := \langle v\rangle^{\perp}/\langle v\rangle$, defined by $\phi_U(x + \langle v\rangle, y + \langle v\rangle) := \phi(x,y)$ for all $x, y \in \langle v\rangle^{\perp}$. It follows from [6, Propositions 2.1.6 and 2.4.1] that all maximal totally isotropic subspaces of V have the same dimension, which, in all cases, is at most $n/2$, so in particular $v^{\perp}$ contains a nonsingular vector whenever $n \geq 3$.

Corollary 3.5. *Let* $V = \mathbb{F}_q^n$, ϕ *a unitary form on* V, $\overline{\phi}$ *as in* (2.5)*, and* $v \in V^{\#}$*. Then* $\mathrm{Im}\,(\overline{\phi}|_{\langle v\rangle^{\perp}}) = \mathbb{F}_{q_0}$ *if* v *is nonsingular and* $n \geq 2$*, or if* v *is singular and* $n \geq 3$*.*

Proof. This follows immediately from Lemma 3.4 applied to $\langle v\rangle^{\perp}$, and the remarks above. □

Proposition 3.6. *Let* Γ *be a graph and* $G \leq \mathrm{Aut}\,(\Gamma)$ *such that* G *satisfies Hypothesis* 3.1 *with* $G_0 = \Gamma\mathrm{U}\,(n,q)$*. Then* Γ *is* G*-symmetric with diameter* 2 *if and only if* $n \geq 2$ *and* $\Gamma \cong \mathrm{Cay}(V,S)$*, where* $V = \mathbb{F}_q^n$ *and* $S \in \{S_0, S_{\#}\}$*, with* S_0 *and* $S_{\#}$ *as in* (2.4) *and* (3.1)*, respectively.*

Proof. By Lemma 2.1 and Proposition 3.1 we only need to prove that $\mathrm{Cay}(V,S)$ has diameter 2 if and only if $n \geq 2$. If $n = 1$ then V is anisotropic, so $\mathrm{GU}\,(n,q)$ is transitive on $V^{\#}$ by Proposition 3.1 (1) and $\mathrm{Cay}(V,S)$ is a complete graph. If $n \geq 2$ then $V^{\#} \setminus S_0 = S_{\#}$ and $V^{\#} \setminus S_{\#} = S_0$ by Proposition 3.1.

Claim 1: $S_{\#} \subseteq S_0 + S_0$. Let $v \in S_{\#}$. Then by Corollary 3.5 there exists $u \in \langle v\rangle^{\perp}$ with $\overline{\phi}(u) = -\overline{\phi}(v)$. Set $w := \beta(u+v)$, where $\beta := \alpha\overline{\phi}(v)^{-1}$ and $\alpha \in \mathbb{F}_q$ such that $\mathrm{Tr}(\alpha) = \overline{\phi}(v)$. Then $w, v - w \in S_0$, so $v \in S_0 + S_0$ and therefore $S_{\#} \subseteq S_0 + S_0$.

Claim 2: $S_0 \subseteq S_\mu + S_\mu$ *for any* $\mu \in (\mathrm{Im}\,(\overline{\phi}))^{\#}$. Let $v \in S_0$. Suppose first that $n \geq 3$. Then by Corollary 3.5, for any $\mu \in (\mathrm{Im}\,(\overline{\phi}))^{\#}$ there exists $w \in S_\mu \cap \langle v\rangle^{\perp}$. It is easy to verify that $\overline{\phi}(v-w) = \overline{\phi}(w)$, so $v - w \in S_\mu$ and $v \in S_\mu + S_\mu$. Therefore $S_0 \subseteq S_\mu + S_\mu$.

If $n = 2$ then $\langle v\rangle^\perp = \langle v\rangle$ for any $v \in S_0$. We show that there exists $u \in S_0$ such that $\phi(u, v) = 1$. Indeed, take $x \in V \setminus \langle v\rangle$. Then $\phi(v, x) \neq 0$. If $x \in S_0$ define $u' := x$; if $x \notin S_0$ let $u' := \alpha v + \phi(v, x)x$ where $\alpha \in \mathbb{F}_q$ with $\mathrm{Tr}(\alpha) = -\overline{\phi}(x)$. Then in both cases $u' \in S_0$ and $\phi(u', v) \neq 0$, and we take u to be the suitable scalar multiple of u' such that $\phi(u, v) = 1$. Let $w := \beta u + \gamma v$, where $\beta, \gamma \in \mathbb{F}_q$ with $\mathrm{Tr}(\beta) = 0$ and $\mathrm{Tr}(\beta^{q_0}\gamma) = \mu$. Then $w, v - w \in S_\mu$, and thus $v \in S_\mu + S_\mu$. Therefore $S_0 \subseteq S_\mu + S_\mu$.

It follows from Claims 1 and 2, respectively, that $\mathrm{Cay}(V, S_0)$ and $\mathrm{Cay}(V, S_\#)$ both have diameter 2. This completes the proof. □

3.2 Class $\mathcal{C}_2$

In this case $V = \oplus_{i=1}^t U_i$, where $U_i = \mathbb{F}_q^m$ for each i, $mt = n$ and $t \geq 2$. Assume that $\mathcal{B} = \bigcup_{i=1}^t \mathcal{B}_i$, where $\mathcal{B}_i$ is a basis for U_i for each i. We write the elements of V as t-tuples over $\mathbb{F}_q^m$; under this identification the τ-action is equivalent to the natural componentwise action.

Assume first that $H = \Gamma\mathrm{L}(n, q)$. It turns out that the G_0-orbits in $V^\#$ are the same as the L-orbits, and thus the graphs that we obtain are precisely those in [2, Proposition 3.2].

Lemma 3.7. *Let G_0 be as in case $(\mathcal{C}_2)$ of Theorem 2.4. Then the G_0-orbits in $V^\#$ are the sets X_s for each $s \in \{1, \dots, t\}$, where*

$$X_s := \{(u_1, \dots, u_t) \in V^\# \mid \textit{exactly } s \textit{ coordinates nonzero}\}. \tag{3.2}$$

Proof. Let $v \in X_s$. Clearly $v^{G_0} \subseteq X_s$; since $v^L = X_s$ by [2, Lemma 3.1] it follows that $v^{G_0} = X_s$. □

Proposition 3.8. *Let Γ be a graph and $G \leq \mathrm{Aut}(\Gamma)$ such that G satisfies Hypothesis 3.1, with $H = \Gamma\mathrm{L}(n, q)$ and G_0 as in case $(\mathcal{C}_2)$ of Theorem 2.4. Then Γ is G-symmetric with diameter 2 if and only if $\Gamma \cong \mathrm{Cay}(V, X_s)$, where X_s is as in (3.2), such that $q^m > 2$ and $s \geq t/2$.*

Proof. This follows immediately from Lemma 3.7 and [2, Proposition 3.2]. □

We now consider the case where $H = \Gamma\mathrm{Sp}(n, q)$ with $n \geq 4$. By Theorem 2.5 there are two types of $\mathcal{C}_2$-subgroups, corresponding to two kinds of decompositions. We refer to these subcases as $(\mathcal{C}_2.1)$ and $(\mathcal{C}_2.2)$.

$(\mathcal{C}_2.1)$ The dimension m of the subspaces U_i is even, U_i is a symplectic space for each i, the subspaces U_i are pairwise orthogonal, and

$$\begin{aligned} G_0 &= \{(g_1, \dots, g_t)\pi\sigma \mid \pi \in \mathrm{Sym}\,(t)\,,\, \sigma \in \langle\tau\rangle,\, g_i \in \mathrm{GSp}\,(m, q)\,, \lambda(g_i) = \lambda(g_1)\} \\ &\cong (\mathrm{Sp}\,(m, q)^t\,.[q-1].\mathrm{Sym}\,(t)) \rtimes \langle\tau\rangle, \end{aligned} \tag{3.3}$$

where $\lambda : \mathrm{GSp}\,(n, q) \to \mathbb{F}_q^\#$ is as defined in Subsection 2.2.

$(\mathcal{C}_2.2)$ The dimension $m = n/2$ so that $t = 2$, both subspaces U_i are totally singular with dimension $n/2$, q is odd if $n = 4$, and

$$\begin{aligned} G_0 &= \left\{\left(g, g^{-\top}\right)\pi\sigma \,\middle|\, \pi \in \mathrm{Sym}\,(t)\,,\, \sigma \in \langle\tau\rangle,\, g \in \mathrm{GL}\,(m, q)\right\} \\ &\cong (\mathrm{GL}\,(m, q)\,.[2]) \rtimes \langle\tau\rangle, \end{aligned} \tag{3.4}$$

where $g^\top$ denotes the transpose of g, and $g^{-\top} = (g^\top)^{-1}$.

Lemma 3.9. *For each $s \in \{1, \dots, t\}$ let X_s be as in* (3.2)*. The G_0-orbits in $V^\#$ are*

1. *the sets X_s for each $s \in \{1, \dots, t\}$ if case $(\mathcal{C}_2.1)$ holds and G_0 is as in* (3.3)*;*
2. *the sets X_1 and $\bigcup_{\sigma \in \langle \tau \rangle} W_{\beta^\sigma}$ for all $\beta \in \mathbb{F}_q$, if case $(\mathcal{C}_2.2)$ holds and G_0 is as in* (3.4)*, where*
$$W_\beta := (w_1, x_\beta)^L, \tag{3.5}$$
$L = G_0 \cap \mathrm{GL}\,(n, q) \cong \mathrm{GL}\,(m, q)\,.[2]$, $w_1 := (1, 0, \dots, 0) \in \mathbb{F}_q^m$, and $x_\beta \in (\mathbb{F}_q^m)^\#$ with first component β.

Proof. The proof of part (1) is similar to that of [2, Lemma 3.1] and uses the transitivity of $\mathrm{Sp}\,(m, q)$ on $U_i^\#$, so we only need to prove part (2). Assume that case $(\mathcal{C}_2.2)$ holds. Then $L = K.\mathrm{Sym}\,(2)$, where $K := \left\{ \left(g, g^{-\top}\right) \,\middle|\, g \in \mathrm{GL}\,(m, q) \right\}$. It is easy to see that $U_1 \oplus \{\mathbf{0}\}$ and $\{\mathbf{0}\} \oplus U_2$ are K-orbits, so $X_1 = (U_1 \otimes \{\mathbf{0}\}) \cup (\{\mathbf{0}\} \oplus U_2)$ is a G_0-orbit. Let $(u, v) \in X_2$, and for any $\beta \in \mathbb{F}_q$ define

$$w_\beta := \begin{cases} (\beta, 0, \dots, 0) & \text{if } \beta \neq 0, \\ (0, 1, 0, \dots, 0) & \text{if } \beta = 0. \end{cases} \tag{3.6}$$

Since $w_1 \in u^{\mathrm{GL}(m,q)}$ we can assume that $u = w_1$. Suppose that $v = (\beta, v_2, \dots, v_m)$.

Claim 1: $(w_1, y) \in (w_1, v)^K$ if and only if $y = (\beta, y_2, \dots, y_m)$ for some $y_2, \dots, y_m \in \mathbb{F}_q$. Indeed, $(w_1, y) \in (w_1, v)^K$ if and only if $y = v^{h^{-\top}}$ for some $h \in \mathrm{Stab}_{\mathrm{GL}(m,q)}(w_1)$. Now $w_1^h = w_1$ if and only if the matrix of $h^{-\top}$ has the form

$$\left(\begin{array}{c|c} 1 & C \\ \hline 0 & \\ \vdots & D \\ 0 & \end{array}\right)$$

where C is a $1 \times (m-1)$ matrix over $\mathbb{F}_q$ and $D \in \mathrm{GL}\,(m-1, q)$. Clearly, the orbit of v under the subgroup $\left\{ h^{-\top} \,\middle|\, h \in \mathrm{Stab}_{\mathrm{GL}(m,q)}(w_1) \right\}$ is the set of all nonzero vectors in $\mathbb{F}_q^m$ with first component β. Therefore Claim 1 holds.

Claim 2: $(w_1, v)^L = (w_1, v)^K$. By Claim 1 we can assume that $v = w_\beta$. If $\beta \neq 0$ let

$$g := \left(\begin{array}{c|ccc} \beta & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & I_{m-1} & \\ 0 & & & \end{array}\right).$$

If $\beta = 0$ let $g := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ if $m = 2$, and

$$g := \left(\begin{array}{cc|c} 0 & 1 & \\ 1 & 0 & 0 \\ \hline & & \\ \multicolumn{2}{c|}{0} & I_{m-2} \\ & & \end{array}\right)$$

if $m > 2$. Then $g \in \mathrm{GL}\,(m, q)$ for all cases, and $w_1^g = w_1^{g^\top} = v$. Hence $(w_1^g, v^{g^{-\top}}) = (v, w_1)$, so that $(v, w_1) \in (w_1, v)^K$. Therefore $(w_1, v)^L = (w_1, v)^K \cup (v, w_1)^K = (w_1, v)^K$, which proves Claim 2.

It follows from Claims 1 and 2 that each set W_β is an L-orbit (and moreover $W_\beta = W_{\beta'}$ if and only if $\beta = \beta'$). It follows from the definition of the τ-action on $V^\#$ that $(w_1, v)^{G_0} = \bigcup_{\sigma\in\langle\tau\rangle} W_{\beta^\sigma}$. This completes the proof of part (2). □

Proposition 3.10. *Let* Γ *be a graph and* $G \leq \mathrm{Aut}(\Gamma)$ *such that* G *satisfies Hypothesis* 3.1 *with* $H = \Gamma\mathrm{Sp}(n, q)$ *and* $i = 2$. *Then* Γ *is* G*-symmetric with diameter* 2 *if and only if* $\Gamma \cong \mathrm{Cay}(V, S)$, *where*

1. *if case* $(\mathcal{C}_2.1)$ *holds, then* $q^m > 2$, G_0 *is as in* (3.3), $S = X_s$, *and* $s \geq t/2$;
2. *if case* $(\mathcal{C}_2.2)$ *holds with* $q^m = 2$, *then* G_0 *is as in* (3.4), *and* $S = W_\beta$ *for any* $\beta \in \mathbb{F}_q$;
3. *if case* $(\mathcal{C}_2.2)$ *holds with* $q^m > 2$, *then* G_0 *is as in* (3.4), *and* $S = X_1$ *or* $S = \bigcup_{\sigma\in\langle\tau\rangle} W_{\beta^\sigma}$ *for some* $\beta \in \mathbb{F}_q$;

with X_s *as in* (3.2) *and* W_β *as* (3.5).

Proof. The graph of (1) is precisely that of Proposition 3.8, and the fact that it is G-symmetric follows from Lemma 2.1. So assume that case $(\mathcal{C}_2.2)$ holds. By Lemma 2.1 we only need to show that $V = S \cup (S + S)$ unless $S = X_1$ and $q = 2$. It follows from Proposition 3.8 (with $t = 2$) that $\mathrm{Cay}(V, X_1)$ has diameter 2 (with G quasiprimitive) if and only if $q^m > 2$, which proves part of statement (3). Thus we may assume that $S = \bigcup_{\sigma\in\langle\tau\rangle} W_{\beta^\sigma}$ for some $\beta \in \mathbb{F}_q$. It remains to prove that $V = S \cup (S + S)$.

Let w_β be as in (3.6) and $\gamma \in \mathbb{F}_q$, with $\gamma \neq \beta$. Define

$$g_0 := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } h_0 := \begin{pmatrix} 0 & -1 \\ -1 & \gamma_0 \end{pmatrix},$$

where $\gamma_0 := 1 - \beta^{-1}\gamma$ if $\beta \neq 0$ and $\gamma_0 := 0$ if $\beta = 0$. If $m = 2$ let $g := g_0$ and $h := h_0$; if $m \geq 3$ define g and h by

$$g := \begin{pmatrix} g_0 & 0 \\ 0 & I_{m-2} \end{pmatrix}$$

and

$$h := \left(\begin{array}{c|ccc} \multirow{2}{*}{h_0} & 0 & \cdots & 0 \\ & 1 & \cdots & 1 \\ \hline 0 & & I_{m-2} & \end{array}\right).$$

Then $g, h \in \mathrm{GL}(m, q)$ for all $m \geq 2$, and $w_1^g + w_1^h = w_1$. Recall that q is odd if $m = 2$, so we can take $x \in (\mathbb{F}_q^m)^\#$ where

$$x := \begin{cases} w_\beta & \text{if } \beta \neq 0; \\ (0, -\gamma/2) & \text{if } \beta = 0 \text{ and } m = 2; \\ (0, 0, 1, 0, \ldots, 0) & \text{if } \beta = 0 \text{ and } m \geq 3. \end{cases}$$

Then for all cases $y := x^{g^{-\top}} + x^{h^{-\top}}$ has first component γ. Hence, applying Lemma 3.9, we have $W_\gamma = (w_1, y)^L \subseteq W_\beta + W_\beta$ for any $\gamma \neq \beta$. Since also $\{\mathbf{0}\} \cup X_1 \subseteq W_\beta + W_\beta$, it follows that $V = W_\beta \cup (W_\beta + W_\beta)$. Therefore $V = S \cup (S + S)$, which completes the proof of parts (2) and (3). □

3.3 Class $\mathcal{C}_4$

In this case $V = U \otimes W = \mathbb{F}_q^k \otimes \mathbb{F}_q^m$ with $k, m \geq 2$, and $\mathcal{B}$ is a tensor product basis of V, that is,

$$\mathcal{B} = \{u_i \otimes w_j \mid 1 \leq i \leq k,\ 1 \leq j \leq m\},$$

where $\mathcal{B}_U := \{u_1, \dots, u_k\}$ and $\mathcal{B}_W := \{w_1, \dots, w_m\}$ are fixed bases of U and W, respectively. We choose τ to fix each of the vectors $u_i \otimes w_j$. Then for any simple vector $u \otimes w \in V$, we have $(u \otimes w)^\tau = u^\tau \otimes w^\tau$, and for any $v = \sum_{i=1}^r (a_i \otimes b_i) \in V$,

$$v^\tau = \sum_{i=1}^r a_i^\tau \otimes b_i^\tau.$$

Recall that $k \neq m$ in the description given in Theorems 2.4 and 2.5; however, all of the results in this section also hold for $k = m$, so we do not assume that k and m are distinct. In this way the results yield useful information for the $\mathcal{C}_7$ case.

A nonzero vector in V is said to be *simple* in the decomposition $U \otimes W$ if it can be written as $u \otimes w$ for some $u \in U$ and $w \in W$. The *tensor weight* $\mathrm{wt}(v)$ of $v \in V^\#$, with respect to this decomposition, is the least number s such that v can be written as the sum of s simple vectors in $U \otimes W$. It follows from [2, Lemma 3.3] that $\mathrm{wt}(v) \leq \min\{k, m\}$ for any $v \in V^\#$, and that for each $s \in \{1, \dots, \min\{k, m\}\}$ there is a vector $v \in V^\#$ with weight s.

The proof of the following observation is straightforward and is omitted.

Lemma 3.11. *For any $v \in V^\#$ and any $\sigma \in \mathrm{Aut}(\mathbb{F}_q)$,*

$$\mathrm{wt}(v^\sigma) = \mathrm{wt}(v).$$

Assume first that $H = \Gamma\mathrm{L}(n, q)$. As in the previous section, the G_0-orbits in $V^\#$ are the same as the L-orbits. This follows easily from Lemma 3.11 and the results in [2].

Lemma 3.12. *Let G_0 be as in case ($\mathcal{C}_4$) of Theorem 2.4. Then the G_0-orbits in $V^\#$ are the sets Y_s for each $s \in \{1, \dots, \min\{k, m\}\}$, where*

$$Y_s := \left\{v \in V^\# \;\middle|\; \mathrm{wt}(v) = s\right\}. \tag{3.7}$$

Proof. This is a consequence of Lemma 3.11 above, and of [2, Lemmas 3.3 and 3.4]. □

We then obtain the same graphs as those in [2, Proposition 3.5].

Proposition 3.13. *Let Γ be a graph and $G \leq \mathrm{Aut}(\Gamma)$ such that G satisfies Hypothesis* 3.1 *with G_0 as in case ($\mathcal{C}_4$) of Theorem* 2.4*, where k and m may be equal. Then Γ is G-symmetric with diameter* 2 *if and only if $\Gamma \cong \mathrm{Cay}(V, Y_s)$, where $s \geq \frac{1}{2}\min\{k, m\}$ and Y_s is as in* (3.7).

Proof. This follows immediately from Lemma 3.12 and [2, Propositon 3.5]. □

Now assume that $H = \Gamma\mathrm{Sp}(n, q)$. In this case k is even, $m \geq 3$, q is odd, and $\phi = \phi_U \otimes \phi_W$, where ϕ_U is a symplectic form on U and ϕ_W is a nondegenerate symmetric bilinear form on W. We can choose B_U and B_W appropriately so that B is a symplectic basis and hence we can again choose τ to fix each of the vectors $u_i \otimes w_j$. The G_0-orbits in this case are proper subsets of the sets Y_s in (3.7), and are in general rather difficult to describe, as are the L-orbits. For instance, if $v = \sum_{i=1}^{s} a_i \otimes b_i \in Y_s$, it is easy to see that

$$v^{G_0} = \left\{ \sum_{i=1}^{s} a_i' \otimes b_i' \;\middle|\; a_i' \in U^{\#},\ b_i' \in b_i^{\mathrm{GO}^{\epsilon}(m,q)} \right\}.$$

If $s = 1$ then the set Y_1 of simple vectors splits into the G_0-orbits Y_1^{θ}, where $\theta \in \{0, \#\}$ if m is even and $\theta \in \{0, \square, \boxtimes\}$ if m is odd, and

$$Y_1^{\theta} := \left\{ a \otimes b \mid a \in U^{\#},\ b \in S_{\theta} \right\}.$$

If $s > 1$ suppose that exactly r of the vectors b_i belong in $S_{\#}$ for some r, $0 \leq r \leq s$; if m is odd suppose further that exactly $r_{\square}$ belong in $S_{\square}$ and $r_{\boxtimes}$ in $S_{\boxtimes}$. If m is even then $v^{G_0} \subset Y_s^r$, where

$$Y_s^r := \left\{ \sum_{i=1}^{s} a_i' \otimes b_i' \in Y_s \;\middle|\; \text{exactly } r \text{ of the vectors } b_i' \text{ are in } S_{\#} \right\},$$

and if m is even then $v^{G_0} \subset Y_s^{r_{\square}, r_{\boxtimes}}$, where

$$Y_s^{r_{\square}, r_{\boxtimes}} := \left\{ \sum_{i=1}^{s} a_i' \otimes b_i' \in Y_s \;\middle|\; \text{exactly } r_{\theta} \text{ of the vectors } b_i' \text{ are in } S_{\theta} \text{ for } \theta \in \{\square, \boxtimes\} \right\}.$$

The sets Y_s^r and $Y_s^{r_{\square}, r_{\boxtimes}}$ above are, in general, not G_0-orbits. For instance, if $s = 2$, the weight-2 vectors $a_1 \otimes b_1 + a_2 \otimes b_2, a_1' \otimes b_1' + a_2' \otimes b_2' \in Y_2^0$ (or $Y_2^{0,0}$ if m is even), such that $b_1 \perp b_2$ and $b_1' \not\perp b_2'$, belong to different G_0-orbits.

The following is an easy consequence of the preceding discussion. However, as discussed, we do not have a good description of the G_0-orbits.

Proposition 3.14. *Let* Γ *be a graph and* $G \leq \mathrm{Aut}(\Gamma)$ *such that* G *satisfies Hypothesis* 3.1 *with* G_0 *as in case* ($\mathcal{C}_4$) *of Theorem* 2.5, *where* k *and* m *may be equal. If* Γ *is* G*-symmetric with diameter* 2, *then* $\Gamma \cong \mathrm{Cay}(V, S)$ *where* $S = v^{G_0}$ *for some* $v \in Y_s$, *where* Y_s *is as in* (3.7) *and* $s \geq \frac{1}{2}\min\{k, m\}$.

Proof. This follows immediately from the discussion above together with Proposition 3.13. □

3.4 Class $\mathcal{C}_5$

In this case $n \geq 2$, d/n is composite with a prime divisor r, and V has a fixed ordered basis

$$\mathcal{B} := (v_1, \ldots, v_n).$$

Let $q_0 := q^{1/r}$ and let $\mathbb{F}_{q_0}$ denote the subfield of $\mathbb{F}_q$ of index r. Let V_0 be the $\mathbb{F}_{q_0}$-span of $\mathcal{B}$. Then V_0 is a vector space over $\mathbb{F}_{q_0}$ that is contained in V, but V_0 is not an $\mathbb{F}_q$-subspace of V.

To any $v = \sum_{i=1}^{n} \alpha_i v_i \in V$ we can associate the $\mathbb{F}_{q_0}$-subspace D_v of $\mathbb{F}_q$, where

$$D_v := \langle \alpha_1, \ldots, \alpha_n \rangle_{\mathbb{F}_{q_0}}. \tag{3.8}$$

Set

$$c(v) := \dim_{\mathbb{F}_{q_0}}(D_v), \tag{3.9}$$

and note that $c(v) \le \min\{r, n\}$. For any $\lambda \in \mathbb{F}_q$ it is clear that $D_{\lambda v} = \lambda D_v$, so $c(\lambda v) = c(v)$, and it is also easy to show that $c(v^\sigma) = c(v)$ for any $\sigma \in \mathrm{Aut}(\mathbb{F}_q)$. Let

$$[D_v] := \{\lambda D_v \mid \lambda \in \mathbb{F}_q^\#\},$$

and observe that $D_u \in [D_{v^\sigma}]$ if and only if $D_u = \lambda D_{v^\sigma} = \left(\lambda^{\sigma^{-1}} D_v\right)^\sigma$ for some $\lambda \in \mathbb{F}_q^\#$. Hence $D_{u^{\sigma^{-1}}} = (D_u)^{\sigma^{-1}} = \lambda^{\sigma^{-1}} D_v$, so that $D_{u^{\sigma^{-1}}} \in [D_v]$. Thus $[D_{v^\sigma}] = [D_v]^\sigma$.

3.4.1 Case $H = \Gamma\mathrm{L}(n, q)$

By Theorem 2.4

$$G_0 = (\mathrm{GL}(n, q_0) \circ Z_{q-1}) \rtimes \langle \tau \rangle$$

and $L = \mathrm{GL}(n, q_0) \circ Z_{q-1}$.

Regard the field $\mathbb{F}_q$ as a vector space of dimension r over $\mathbb{F}_{q_0}$, and for any $a \in \{1, \dots, r\}$, define

$$\mathbb{K}(a) := \begin{cases} \mathbb{F}_q & \text{if } a = r, \\ \mathbb{F}_{q_0} & \text{otherwise.} \end{cases} \tag{3.10}$$

For $a \in \{1, \dots, r\}$ define

$$\eta(a) := \frac{\left[\begin{array}{c} r \\ a \end{array} \right]_{q_0}}{\left|\mathbb{F}_q^\# : \mathbb{K}(a)^\#\right|}, \tag{3.11}$$

where

$$\left[\begin{array}{c} r \\ a \end{array} \right]_{q_0} := \prod_{i=0}^{a-1} \frac{q_0^r - q_0^i}{q_0^a - q_0^i},$$

the number of a-dimensional subspaces of $\mathbb{F}_{q_0}^r$. In particular $\eta(r) = \eta(1) = 1$. Lemma 3.15 gives some elementary observations about $\mathbb{K}(a)$ and η, whose significance will be apparent in Corollary 3.19. The proof of Lemma 3.15 is straightforward and is omitted.

Lemma 3.15. *Let $\mathbb{F}_{q_0}$ be a proper nontrivial subfield of $\mathbb{F}_q$ with prime index r, and suppose that $\mathbb{F}_q$ is viewed as a vector space over $\mathbb{F}_{q_0}$ with dimension r. For any $a \in \{1, \dots, r\}$, let $\mathcal{D}$ denote the set of all $\mathbb{F}_{q_0}$-subspaces of $\mathbb{F}_q$ with dimension a, and let $\mathbb{K}(a)$ and $\eta(a)$ be as defined in* (3.10) *and* (3.11)*, respectively. Then the following hold:*

1. *For any $D \in \mathcal{D}$,*
$$\{\lambda \in \mathbb{F}_q \mid \lambda D = D\} = \mathbb{K}(a).$$
2. *For any $D \in \mathcal{D}$, the sets $[D] = \left\{\lambda D \mid \lambda \in \mathbb{F}_q^\#\right\}$ partition $\mathcal{D}$. Moreover, $|[D]| = \left|\mathbb{F}_q^\# : \mathbb{K}(a)^\#\right|$, and the number of distinct parts $[D]$ in $\mathcal{D}$ is $\eta(a)$.*

The main result for this case, which relies on the value of the parameter $c(v)$, is the following. It shows that examples do exist.

Proposition 3.16. *Let* Γ *be a graph and* $G \leq \mathrm{Aut}(\Gamma)$ *such that* G *satisfies Hypothesis* 3.1 *with* $H = \Gamma\mathrm{L}(n, q)$ *and* $i = 5$*. Then* Γ *is connected and* G*-symmetric if and only if* $\Gamma \cong \mathrm{Cay}(V, v^{G_0})$ *for some* $v \in V^{\#}$*. Moreover, if* D_v *and* $c(v)$ *are as in* (3.8) *and* (3.9)*, respectively, then the following hold.*

1. *If* $c(v) = r$ *or* $c(v) = r - 1$ *then* $\mathrm{diam}(\Gamma) = 2$.
2. *If* $c(v) = 1$ *then* $\mathrm{diam}(\Gamma) = \min\{n, r\}$*. In particular* $\mathrm{diam}(\Gamma) = 2$ *if and only if* $n = 2$ *or* $r = 2$.
3. *If* $2 \leq c(v) < \frac{1}{2}\min\{n, r\}$ *then* $\mathrm{diam}(\Gamma) > 2$.
4. *Let* η *be as defined in* (3.11)*,* s *be the largest divisor of* d/n *with* $s \leq \eta(c(v))$*, and*

$$k_1(q_0) := \begin{cases} 18s/17 & \text{if } q_0 = 2; \\ s - 5/4 & \text{if } q_0 > 2. \end{cases}$$

If $3 \leq n < r$ *and* $n/2 \leq c(v) < (r(n-2) + k_1(q_0))/(2n)$*, then* $\mathrm{diam}(\Gamma) > 2$.

The cases not covered by Proposition 3.16 are discussed briefly at the end of the section. The proof of Proposition 3.16 is given after Lemma 3.20, and relies on several intermediate results. We begin by describing the $\mathrm{GL}(n, q_0)$-orbits in terms of the subspaces D_v, which in turn leads to a description of the G_0-orbits in $V^{\#}$.

Lemma 3.17. *For any* $v \in V^{\#}$ *let* D_v *and* $c(v)$ *be as in* (3.8) *and* (3.9)*, respectively, and let* $\mathcal{U}$ *denote the set of all* $\mathbb{F}_{q_0}$*-independent* $c(v)$*-tuples in* V_0*. Then for any fixed* $\mathbb{F}_{q_0}$*-basis* $\{\beta_1, \ldots, \beta_{c(v)}\}$ *of* D_v*,*

$$\begin{aligned} v^{\mathrm{GL}(n,q_0)} &= \left\{ \sum_{i=1}^{c(v)} \beta_i u_i \;\middle|\; (u_1, \ldots, u_{c(v)}) \in \mathcal{U} \right\} \\ &= \left\{ u \in V^{\#} \;\middle|\; D_u = D_v \right\}. \end{aligned}$$

Proof. Suppose that $v = \sum_{i=1}^{n} \alpha_i v_i$. Define

$$U := \left\{ u \in V^{\#} \;\middle|\; D_u = D_v \right\} \tag{3.12}$$

and

$$W := \left\{ \sum_{i=1}^{c(v)} \beta_i u_i \;\middle|\; (u_1, \ldots, u_{c(v)}) \in \mathcal{U} \right\}. \tag{3.13}$$

Claim 1: $v^{\mathrm{GL}(n,q_0)} \subseteq U$. Let $g \in \mathrm{GL}(n, q_0)$ with matrix $[g_{jk}]$ with respect to $\mathcal{B}$. Then $v^g = \sum_{k=1}^{n} \alpha'_k v_k$, where $\alpha'_k = \sum_{j=1}^{n} \alpha_j g_{jk} \in D_v$ for each k. Hence $D_{v^g} \leq D_v$. Since v and g are arbitrary, we also have $D_v \leq D_{v^g}$. So $D_{v^g} = D_v$, and therefore $v^{\mathrm{GL}(n,q_0)} \subseteq U$.

Claim 2: $U \subseteq W$. Let $u = \sum_{j=1}^{n} \alpha'_j v_j \in U$. Writing $\alpha'_j = \sum_{i=1}^{c(v)} \beta_i \gamma_{ij}$ for each j, where all $\gamma_{ij} \in \mathbb{F}_{q_0}$, we get $u = \sum_{i=1}^{c(v)} \beta_i u_i$, with $u_i = \sum_{j=1}^{n} \gamma_{ij} v_j \in V_0$ for all i. It remains to show that the set $\mathbf{u} := \{u_1, \ldots, u_{c(v)}\}$ is $\mathbb{F}_{q_0}$-independent. Indeed, let $\{u'_1, \ldots, u'_b\}$ be a maximal $\mathbb{F}_{q_0}$-independent subset of $\mathbf{u}$, and extend this to an ordered $\mathbb{F}_{q_0}$-basis $\mathcal{B}' := (u'_1, \ldots, u'_d)$ of V_0. Then $u = \sum_{k=1}^{b} \beta'_k u'_k$ for some $\beta'_1, \ldots, \beta'_b \in \mathbb{F}_q$, and if $g \in \mathrm{GL}(n, q_0)$ is the change of basis matrix from $\mathcal{B}'$ to $\mathcal{B}$, then $u^g = \sum_{k=1}^{b} \beta_k' v_k$. So $D_u = D_{u^g}$ by Claim 1, and thus $b \leq c(v) = \dim_{\mathbb{F}_{q_0}}(D_u) = \dim_{\mathbb{F}_{q_0}}(D_{u^g}) \leq b$. Hence $b = c(v)$ and $\mathbf{u}$ is $\mathbb{F}_{q_0}$-independent. Therefore $U \subseteq W$.

Claim 3: $W \subseteq v^{\mathrm{GL}(n,q_0)}$. It is easy to see that W is contained in one orbit of $\mathrm{GL}(n, q_0)$, and it follows from Claims 1 and 2 that $v \in W$. So $W \subseteq v^{\mathrm{GL}(n,q_0)}$, as claimed.

Thus we have $v^{\mathrm{GL}(n,q_0)} = U = W$ by Claims 1 – 3. □

Proposition 3.18. *For any $v \in V^\#$ let D_v and $c(v)$ be as in* (3.8) *and* (3.9)*, respectively, and let $\mathcal{U}$ be the set of all $\mathbb{F}_{q_0}$-independent $c(v)$-tuples in V_0. Then for any fixed $\mathbb{F}_{q_0}$-basis $\{\beta_1, \ldots, \beta_{c(v)}\}$ of D_v we have*

$$\begin{aligned} v^L &= \left\{ \lambda \sum_{i=1}^{c(v)} \beta_i u_i \,\middle|\, (u_1, \ldots, u_{c(v)}) \in \mathcal{U},\, \lambda \in \mathbb{F}_q^\# \right\} \\ &= \left\{ u \in V^\# \mid D_u = \lambda D_v,\, \lambda \in \mathbb{F}_q^\# \right\} \end{aligned}$$

and

$$\begin{aligned} v^{G_0} &= \left\{ \lambda \sum_{i=1}^{c(v)} \beta_i^\sigma u_i \,\middle|\, (u_1, \ldots, u_{c(v)}) \in \mathcal{U},\, \lambda \in \mathbb{F}_q^\#,\, \sigma \in \langle \tau \rangle \right\} \\ &= \left\{ u \in V^\# \mid D_u = \lambda (D_v)^\sigma,\, \lambda \in \mathbb{F}_q^\#, \sigma \in \langle \tau \rangle \right\}. \end{aligned} \tag{3.14}$$

Proof. Let $U' := \{u \in V^\# \mid D_u = \lambda D_v \text{ for some } \lambda \in \mathbb{F}_q^\#\}$. Since $L = \mathrm{GL}(n, q_0) \circ Z_{q-1}$ and $D_{\lambda v} = \lambda D_v$ for any $\lambda \in \mathbb{F}_q$, it follows from Lemma 3.17 that $v^L = U'$. Thus

$$v^{G_0} = \bigcup_{\sigma \in \langle \tau \rangle} \left\{ u^\sigma \mid u \in v^L \right\} \subseteq W',$$

where $W' := \left\{ u \in V^\# \mid D_u = \lambda (D_v)^\sigma,\, \lambda \in \mathbb{F}_q^\#,\, \sigma \in \langle \tau \rangle \right\}$. For any $w \in W$ with $D_w = \mu (D_v)^\rho$ for $\mu \in \mathbb{F}_q^\#$ and $\rho \in \langle \tau \rangle$, we have $w \in (v^\rho)^L \subseteq v^{G_0}$. Therefore $v^{G_0} = W'$, and the rest follows from Lemma 3.17. □

Corollary 3.19. *Let $v \in V^\#$, and let $\mathbb{K}$, η, D_v and $c(v)$ be as defined in* (3.10)*,* (3.11)*,* (3.8) *and* (3.9)*, respectively.*

1. *For $a \in \{1, \ldots, \min\{n, r\}\}$, the number of orbits v^L with $c(v) = a$ is $\eta(a)$.*
2. $\left|v^L\right| = \begin{bmatrix} n \\ c(v) \end{bmatrix}_{q_0} \cdot \left|\mathrm{GL}(c(v), q_0)\right| \cdot \left|\mathbb{F}_q^\# : \mathbb{K}(c(v))^\#\right|$
3. $\left|v^{G_0}\right| = s\left|v^L\right|$ *for some divisor s of d/n with $s \leq \eta(c(v))$.*

Proof. It follows from Proposition 3.18 that the map $v^L \mapsto [D_v] := \{\lambda D_v \mid \lambda \in \mathbb{F}_q^\#\}$ is a one-to-one correspondence between the set of L-orbits and the set of classes $[D]$ of $\mathbb{F}_{q_0}$-subspaces of $\mathbb{F}_q$. Therefore, by Lemma 3.15 (2), there are exactly $\eta(a)$ orbits v^L with $c(v) = a$, which proves part (1). Also by Proposition 3.18, we have $|v^L| = |\mathcal{U}| \cdot |[D_v]|$, where $\mathcal{U}$ is the set of $\mathbb{F}_{q_0}$-independent $c(v)$-tuples in V_0. So

$$|\mathcal{U}| = \begin{bmatrix} n \\ c(v) \end{bmatrix}_{q_0} \left|\mathrm{GL}(c(v), q_0)\right|,$$

and by Lemma 3.15 (2), $|[D_v]| = |\mathbb{F}_q^\# : \mathbb{K}(c(v))^\#|$. This proves part (2). Since $L \lhd G_0$ we must have $\left|v^{G_0}\right| = s\left|v^L\right|$ for some s dividing $|G_0 : L| = |\mathrm{Aut}\,(\mathbb{F}_q)| = d/n$. Also $s \leq \eta(c(v))$ since $c(v^\sigma) = c(v)$, which proves part (3). □

Lemma 3.20. *Let* $\Gamma = \mathrm{Cay}(V, v^{G_0})$ *for some* $v \in V^\#$*, and let* $c(v)$ *be as in* (3.9)*. Let* $w \in V$.

1. *If* $w \in v^{G_0} + v^{G_0}$ *then* $c(w) \leq 2c(v)$.
2. *If* $D_w < D_v$ *then* $w \in v^{G_0} + v^{G_0}$.

Proof. Let $\mathcal{U}$ and $\mathcal{W}$ denote the sets of $\mathbb{F}_{q_0}$-independent $c(v)$- and $c(w)$-tuples, respectively, in V.

Suppose first that $w = x + y$ for some $x, y \in v^{G_0}$. Then by Proposition 3.18 we can write x and y as $x = \sum_{i=1}^{c(v)} \lambda\beta_i^\rho x_i$ and $y = \sum_{i=1}^{c(v)} \mu\beta_i^\sigma y_i$ for some scalars $\lambda, \mu \in \mathbb{F}_q^\#$, maps $\rho, \sigma \in \mathrm{Aut}\,(\mathbb{F}_q)$, and $c(v)$-tuples $\left(x_1, \ldots, x_{c(v)}\right), \left(y_1, \ldots, y_{c(v)}\right) \in \mathcal{U}$. Hence

$$D_w = D_{x+y} \subseteq \left\langle \lambda\beta_1^\rho, \ldots, \lambda\beta_{c(v)}^\rho, \mu\beta_1^\sigma, \ldots, \mu\beta_{c(v)}^\sigma \right\rangle_{\mathbb{F}_{q_0}},$$

and therefore $c(w) = c(x + y) \leq 2c(v)$. This proves part (1).

To prove part (2), observe that Lemma 3.17 implies that we can write v and w as $v = \sum_{i=1}^{c(v)} \gamma_i u_i$ and $w = \sum_{i=1}^{c(w)} \delta_i z_i$ for some $\left(u_1, \ldots, u_{c(v)}\right) \in \mathcal{U}$ and $\left(z_1, \ldots, z_{c(w)}\right) \in \mathcal{W}$, and for some fixed $\mathbb{F}_{q_0}$-bases $\{\gamma_i, \ldots, \gamma_{c(v)}\}$ and $\{\delta_1, \ldots, \delta_{c(w)}\}$ of D_v and D_w, respectively. Since $D_w < D_v$ then $c(w) < c(v)$, and we can extend $\{\delta_1, \ldots, \delta_{c(w)}\}$ to an $\mathbb{F}_{q_0}$-basis $\{\delta_1, \ldots, \delta_{c(v)}\}$ of D_v, and $\left(z_1, \ldots, z_{c(w)}\right)$ to $\left(z_1, \ldots, z_{c(v)}\right) \in \mathcal{U}$. Set $x := \sum_{i=1}^{c(v)} \delta_i z_i$ and $y := \sum_{i=1}^{c(v)} \delta_i y_i$, where $y_i := z_{i+1} - z_i$ if $1 \leq i \leq c(w) - 1$, $y_{c(w)} := z_1$, and $y_i := -z_i$ if $c(w) + 1 \leq i \leq c(v)$. Then $\left(y_1, \ldots, y_{c(v)}\right) \in \mathcal{U}$ and $D_x = D_y = D_v$, so by Lemma 3.17 we have $x, y \in v^{\mathrm{GL}(n,q_0)} \subseteq v^{G_0}$. Therefore $x + y \in v^{G_0} + v^{G_0}$. Now $D_w = D_{x+y}$, so applying Lemma 3.17 again we get $w \in (x + y)^{\mathrm{GL}(n,q_0)} \subseteq v^{G_0} + v^{G_0}$. Thus (2) holds. □

Proof of Proposition 3.16. Suppose that $r - 1 \leq c(v) \leq r$. Observe that $\eta(r-1) = \eta(r) = 1$, so for either value of $c(v)$ we have $v^L = \{u \in V \mid c(u) = c(v)\}$, which in turn implies that $v^{G_0} = v^L$. If $c(v) = r$ then $D_v = \mathbb{F}_q$, and clearly $D_w < D_v$ for any $w \in V^\# \setminus v^{G_0}$. So $w \in v^{G_0} + v^{G_0}$ by part (2) of Lemma 3.20, and thus $V^\# \setminus v^{G_0} \subseteq v^{G_0} + v^{G_0}$. Therefore $\mathrm{diam}(\Gamma) = 2$. Now suppose that $c(v) = r - 1$, and let $w \in V^\# \setminus v^{G_0}$. If $c(w) < r - 1$ then it follows from part (1) of Corollary 3.19 that $D_w < \lambda D_v = D_{\lambda v}$ for some $\lambda \in \mathbb{F}_q^\#$. Thus $w \in (\lambda v)^{G_0} + (\lambda v)^{G_0} = v^{G_0} + v^{G_0}$ by Lemma 3.17. If $c(w) = r$ let $x := \sum_{i=1}^{r-1} \alpha_i v_i$ and $y := \sum_{i=1}^{r-2} \beta_i v_i + \gamma v_r$, where $\{\alpha_1, \ldots, \alpha_{r-1}\}$ is an $\mathbb{F}_{q_0}$-basis of D_v, $\gamma \in \mathbb{F}_q^\# \setminus D_v$, and

$$\beta_i := \begin{cases} \alpha_{i+1} - \alpha_i & \text{if } 1 \leq i \leq r-3; \\ \alpha_1 - \alpha_{r-2} & \text{if } i = r-2. \end{cases}$$

Then $c(x) = c(y) = r - 1$ and $c(x+y) = r$, so $x, y \in v^{G_0}$ and $w \in (x+y)^{G_0} \subseteq v^{G_0} + v^{G_0}$. Therefore $V^\# \setminus v^{G_0} \subseteq v^{G_0} + v^{G_0}$, and again we have $\mathrm{diam}(\Gamma) = 2$. This completes the proof of part (1).

If $c(v) = 1$ then we get the special case $v^L = v^{G_0} = (\mathbb{F}_q V_0)^\#$. Let $\mathrm{d}ist_\Gamma(\mathbf{0}_V, w)$ denote the distance in Γ between the vertices $\mathbf{0}_V$ and w; we claim that $\mathrm{d}ist_\Gamma(\mathbf{0}_V, w) =$

$c(w)$ for any $w \in V$. Let $\ell(w) := dist_\Gamma(\mathbf{0}_V, w)$. Then $w \in Y$ by Proposition 3.18, where Y is as in (3.13), so w can be written as a sum of $c(w)$ elements of $(\mathbb{F}_q V_0)^\#$ and thus $\ell(w) \leq c(w)$. On the other hand $w = \sum_{i=1}^{\ell(w)} \lambda_i u_i$, where $\lambda_i \in \mathbb{F}_q^\#$ and $u_i \in V_0^\#$ for all i. Writing each u_i as $u_i = \sum_{j=1}^{n} \mu_{i,j} w_j$ where $\mu_{i,j} \in \mathbb{F}_{q_0}$ for all i, j, we get $w = \sum_{j=1}^{n} \lambda'_j w_j$ where $\lambda'_j = \sum_{j=1}^{\ell(w)} \lambda_i \mu_{i,j}$ for each j. Hence $D_w \leq \langle \lambda_1, \ldots, \lambda_{\ell(w)} \rangle_{\mathbb{F}_{q_0}}$, so that $c(w) \leq \ell(w)$. Therefore $\ell(w) = c(w)$, as claimed. It follows immediately that $\mathrm{diam}(\Gamma) = \min\{n, r\}$, and that $\mathrm{diam}(\Gamma) = 2$ if and only if $n = 2$ or $r = 2$. This proves (2).

Suppose that $\mathrm{diam}(\Gamma) = 2$. Then $c(w) \leq 2c(v)$ for any $w \in V^\#$ by part (1) of Lemma 3.20, and in particular $2c(v) \geq \min\{n, r\}$ since there clearly exists $u \in V^\#$ with $c(u) = \min\{n, r\}$. Hence $c(v) \leq \frac{1}{2}\min\{n, r\}$ implies that $\mathrm{diam}(\Gamma) > 2$, and part (3) holds.

Finally, let $a := c(v)$, $S := v^{G_0}$, and $\eta(a)$ as in (3.11). By Corollary 3.19 we have

$$|S| \leq \begin{bmatrix} n \\ a \end{bmatrix}_{q_0} |\mathrm{GL}(a, q_0)| \left|\mathbb{F}_q^\# : \mathbb{F}_{q_0}^\#\right| s,$$

where s is the largest divisor of d/n with $s \leq \eta(a)$. Hence

$$|S|^2 + 1 < q_0^{2an} \left|\mathbb{F}_q^\# : \mathbb{F}_{q_0}^\#\right|^2 s^2.$$

Observe that $s < q_0^{st}$ for all $s \geq 1$, where $t = \frac{9}{17}$ if $q_0 = 2$, and $t = \frac{1}{2}$ if $q_0 \geq 3$. Also, for $q_0 \geq 3$, we have $q_0 - 1 > q_0^{5/8}$, so that $\left|\mathbb{F}_q^\# : \mathbb{F}_{q_0}^\#\right| < q_0^{r-5/8}$. With these bounds we obtain

$$|S|^2 + 1 < q_0^{2(an+r)+k_1(q_0)},$$

where $k_1(q_0)$ is as defined in (4). It is easy to verify that if $a < (r(n-2) - k_1(q_0))/(2n)$ then $2(an + r) + k_1(q_0) < rn$, so $|S|^2 + 1 < |V|$, and thus $\mathrm{diam}(\Gamma) > 2$ by Lemma 2.1. This proves part (4). □

Remark 3.21. Some small cases covered by Proposition 3.16 are summarised in Table 3.4.6. The cases left unresolved by Proposition 3.16 are the following:

1. $5 \leq r \leq n, r/2 \leq c(v) \leq r - 2$;
2. $2 = n \leq r - 2, c(v) = 2$;
3. $3 \leq n < r, \max n/2, (r(n-2) - k_1(q_0))/(2n) \leq c(v) \leq r - 2$.

Let $a := c(v) < r$, $S = v^{G_0}$, and s as in Proposition 3.16 (4). Then $s \geq 1$, $\left|\mathbb{F}_q^\# : \mathbb{F}_{q_0}^\#\right| > q_0^{r-2}$ and

$$\begin{bmatrix} n \\ a \end{bmatrix}_{q_0} |\mathrm{GL}(a, q_0)| > q_0^{2a(n-1)},$$

so

$$\begin{aligned} |G_0|^2 + 1 &\geq \left(\begin{bmatrix} n \\ a \end{bmatrix}_{q_0} |\mathrm{GL}(a, q_0)| \left|\mathbb{F}_q^\# : \mathbb{F}_{q_0}^\#\right| s \right)^2 + 1 \\ &> q_0^{2a(n-1)+2(r-2)}. \end{aligned}$$

It is easy to show that if condition (1) or (2) holds then $2(a(n-1) + r - 2) > rn$, and thus $|G_0|^2 + 1 > |V|$. This, unfortunately, does not lead to any conclusion about $\mathrm{diam}(\Gamma)$.

Table 3.4.6: Γ as in Proposition 3.16 for small values of r and n

r	n	$c(v)$	Conclusion about $\Gamma = \mathrm{Cay}(V, v^{G_0})$
2	≥ 2	1	$\mathrm{diam}(\Gamma) = 2$ by Proposition 3.16 (2)
		2	$\mathrm{diam}(\Gamma) = 2$ by Proposition 3.16 (1)
3	2	1	$\mathrm{diam}(\Gamma) = 2$ by Proposition 3.16 (2)
		2	$\mathrm{diam}(\Gamma) = 2$ by Proposition 3.16 (1)
3	≥ 3	1	$\mathrm{diam}(\Gamma) = 3$ by Proposition 3.16 (2)
		2	$\mathrm{diam}(\Gamma) = 2$ by Proposition 3.16 (1)
		3	$\mathrm{diam}(\Gamma) = 2$ by Proposition 3.16 (2)
5	2	1	$\mathrm{diam}(\Gamma) = 2$ by Proposition 3.16 (2)
5	3	1	$\mathrm{diam}(\Gamma) = 3$ by Proposition 3.16 (2)
5	4	1	$\mathrm{diam}(\Gamma) = 4$ by Proposition 3.16 (2)
		4	$\mathrm{diam}(\Gamma) = 2$ by Proposition 3.16 (1)
5	≥ 5	1	$\mathrm{diam}(\Gamma) = 5$ by Proposition 3.16 (2)
		2	$\mathrm{diam}(\Gamma) > 2$ by Proposition 3.16 (3)
		4	$\mathrm{diam}(\Gamma) = 2$ by Proposition 3.16 (1)
		5	$\mathrm{diam}(\Gamma) = 2$ by Proposition 3.16 (1)

3.4.2 Case $H = \Gamma\mathrm{Sp}(n, q)$

By Theorem 2.5,

$$G_0 = (\mathrm{GSp}\,(n, q_0) \circ Z_{q-1}) \rtimes \langle\tau\rangle$$

and $L = \mathrm{GSp}\,(n, q_0) \circ Z_{q-1}$. The main result in this section is parallel to part (4) of Proposition 3.16.

Proposition 3.22. *Let* Γ *be a graph and* $G \leq \mathrm{Aut}\,(\Gamma)$ *such that* G *satisfies Hypothesis* 3.1 *with* $H = \Gamma\mathrm{Sp}(n, q)$ *and* $i = 5$*. Then* Γ *is connected and* G*-symmetric if and only if* $\Gamma \cong \mathrm{Cay}(V, v^{G_0})$ *for some* $v \in V^{\#}$*. Moreover, if* $s := |\tau| = |G_0 : L|$ *and* $c(v)$ *is as defined in* (3.9)*, and if*

$$t := \begin{cases} 9/17 & \text{if } q_0 = 2, \\ 1/2 & \text{if } q_0 > 2 \end{cases}$$

then the following hold:

1. *If* $c(v) < \frac{1}{2}\min\{n, r\}$ *then* $\mathrm{diam}(\Gamma) > 2$.
2. *If* $3 \leq n \leq r$, $c(v) \geq n/2$ *and* $r > (n^2 + n + 2st)/(n - 2)$*, then* $\mathrm{diam}(\Gamma) > 2$.

Proof. Assume that $c(v) < \frac{1}{2}\min\{n, r\}$. Let $S = v^{G_0}$, and let $\Gamma' = \mathrm{Cay}(V, v^{G'_0})$, such that G' satisfies Hypothesis 3.1 with $H = \Gamma\mathrm{L}(n, q)$ and $i = 5$. Then Γ is a subgraph of Γ', and hence $\mathrm{diam}(\Gamma) \geq \mathrm{diam}(\Gamma')$. If $c(v) = 1$ then $\mathrm{diam}(\Gamma') \geq \min\{n, r\} > 2$ by part (2) of Proposition 3.16, and if $c(v) \geq 2$ then $\mathrm{diam}(\Gamma') > 2$ by part (3) of Proposition 3.16. In both cases $\mathrm{diam}(\Gamma) > 2$. This proves statement (1).

We now prove statement (2). Observe that for any $\lambda \in \mathbb{F}_q^{\#}$ and $g \in \mathrm{GSp}\,(n, q_0)$, we have $\lambda v^g = v^{\lambda g} \in v^{\mathrm{GSp}(n,q_0)}$ if and only if $\lambda I_n \in Z_{q_0-1}$, the subgroup of scalar matrices in $\mathrm{GL}(n, q_0)$. Hence $v^L = \bigcup_{\lambda\in\mathbb{F}_q^\#} \lambda v^{\mathrm{GSp}(n,q_0)}$ can be written as a disjoint union $v^L = \bigcup_{\lambda\in T} \lambda v^{\mathrm{GSp}(n,q_0)}$, where T is a transversal of $\mathbb{F}_{q_0}^\#$ in $\mathbb{F}_q^\#$. Thus

$$|v^L| \leq |T||\mathrm{GSp}(n, q_0)| = (q_0^r - 1)|\mathrm{Sp}(n, q_0)|$$

and $|S| \leq s|v^L|$, where $s = |G_0 : L|$. We have

$$|\mathrm{Sp}(n, q_0)| = q_0^{n^2/4} \prod_{i=1}^{n/2} \left(q^{2i} - 1\right) < q_0^{(n^2+n)/2}.$$

Also, as in the proof of Proposition 3.16 (4), we have $s < q_0^{st}$ for any s, where $t = \frac{9}{17}$ if $q_0 = 2$, and $t = \frac{1}{2}$ if $q_0 \geq 3$. Hence

$$|S|^2 + 1 < s^2(q_0^r - 1)^2 q_0^{n^2+n} < q_0^{n^2+n+2r+2st}.$$

If $r > (n^2 + n + 2st)/(n - 2)$ then $rn > n^2 + n + 2r + 2st$, so $|V| > |S|^2 + 1$ and $\mathrm{diam}(\Gamma) > 2$ by Lemma 2.1. Therefore part (2) holds. □

3.5 Class $\mathcal{C}_6$

In this case $\dim(V) = r^t$ where r is a prime different from p, q is the smallest power of p such that $q \equiv 1 \pmod{|Z(R)|}$ for some R in Table 2.3.5, and

$$G_0 = (Z_{q-1} \circ R).T \rtimes \langle\tau\rangle,$$

with T as in Table 2.3.5. By Theorems 2.4 and 2.5, if $H = \Gamma\mathrm{L}(n, q)$ then R is of type 1 or 2, and if $H = \Gamma\mathrm{Sp}(n, q)$ with q odd then R is of type 4.

Proposition 3.23 is an extension of [2, Proposition 3.6], and is proved somewhat similarly.

Proposition 3.23. *Let V and G_0 be as above, and let $\Gamma := \mathrm{Cay}(V, S)$ for some G_0-orbit $S \subseteq V^\#$.*

1. *Suppose that r is odd, $q \equiv 1 \pmod r$, and R is Type 1. If $\mathrm{diam}(\Gamma) = 2$ then $1 \leq t \leq 3$, $r \leq r_0(t)$, and $q \leq q_0(r, t)$, where $r_0(t)$ and $q_0(r, t)$ are given in Table* 3.5.7.
2. *Suppose that $r = 2$, $t \geq 2$, $q \equiv 1 \pmod 4$, and R is Type 2. If $\mathrm{diam}(\Gamma) = 2$ then $2 \leq t \leq 6$ and $q \leq q_0(t)$, where $q_0(t)$ is given in Table* 3.5.8.
3. *Suppose that $r = 2$, $t \geq 2$, q is odd, and R is Type 4. If $\mathrm{diam}(\Gamma) = 2$ then $2 \leq t \leq 7$ and $q \leq q_0(t)$, where $q_0(t)$ is given in Table* 3.5.9.
4. *Suppose that $r = 2$, $t = 1$, q is odd, and R is Type 2 or 4. Then $\mathrm{diam}(\Gamma) = 2$ for any S.*

Proof. If $q = p^\ell$ and R is Type 1 or 2, then

$$|G_0| = \ell(q - 1)r^{2t}|\mathrm{Sp}(2t, r)| < \ell(q - 1)r^{2t^2+3t}.$$

Table 3.5.7: Bounds for r and q when R is Type 1

t	1	2	3
$r_0(t)$	11	3	3
$q_0(3,t)$	186619	73	11
$q_0(5,t)$	521	-	-
$q_0(7,t)$	71	-	-
$q_0(11,t)$	23	-	-

Table 3.5.8: Bounds for q when R is Type 2 and $t \geq 2$

t	2	3	4	5	6
$q_0(t)$	23029	569	73	17	5

Table 3.5.9: Bounds for q when R is Type 4 and $t \geq 2$

t	2	3	4	5	6	7
$q_0(t)$	1913	149	37	11	5	3

Suppose first that R is Type 1. In this case r is odd and $q = p^\ell \equiv 1 \pmod r$, so $\ell \leq r-1$, $q > r$, and

$$|G_0|^2 + 1 < \left((q-1)r^{2t^2+3t+1}\right)^2 + 1 < q^{4t^2+6t+4}.$$

It can be shown that $4t^2 + 6t + 4 < r^t$ for the following cases: $t \geq 5$ and $r \geq 3$, $t = 1$ and $r \geq 17$, $t = 2$ and $r \geq 7$, and $t \in \{3, 4\}$ and $r \geq 5$. Thus for all these cases $|G_0|^2 + 1 < |V|$. For all remaining pairs (r, t) define

$$\pi(q, r, t) := \left((r-1)(q-1)r^{2t}|\mathrm{Sp}\,(2t, r)\,|\right)^2 + 1 - q^{r^t}.$$

Then $|G_0|^2+1-|V| < \pi(q,r,t)$ and $\pi(q,r,t) < 0$ if $q > \left((r-1)r^{2t}|\mathrm{Sp}\,(2t,r)\,|\right)^{2/\left(r^t-2\right)}$. Getting the largest prime power $q = p^\ell \equiv 1 \pmod r$ less than or equal to this bound, with $\ell \leq r-1$ and $\pi(q,r,t) > 0$, gives the values $q_0(r,t)$ in Table 3.5.7, and for each t we take $r_0(t)$ to be the largest value of r for which there exist such q. In particular, $\pi(q,r,t) < 0$ for the following cases: $(r,t) = (13,1)$ and $q > 13$, $(r,t) = (5,2)$ and $q > 7$, $(r,t) = (3,4)$ and $q > 3$; for these cases there is no value of q less than or equal to the given bound that satisfies all the required conditions. This proves part (1).

Now suppose that R is Type 2 with $t \geq 2$. Then $r = 2$ and $q = p^\ell \equiv 1 \pmod 4$, so $\ell \leq 2$, $q > 4$, and

$$|G_0|^2 + 1 < \left((q-1)2^{2t^2+3t+1}\right)^2 + 1 < q^{2t^2+3t+3}.$$

We have $2t^2 + 3t + 3 < 2^t$ whenever $t \geq 7$, hence $|G_0|^2 + 1 \leq |V|$ for all such t. For $t \in \{1, \ldots, 6\}$ define

$$\pi(q,t) := \left(2(q-1)2^{2t}|\mathrm{Sp}\,(2t,2)\,|\right)^2 + 1 - q^{2^t},$$

and observe that $|G_0|^2+1-|V|<\pi(q,t)<0$ for all $q>\left(2^{2t+1}|\mathrm{Sp}\,(2t,2)\,|\right)^{1/\left(2^{t-1}-1\right)}$. The values of $q_0(t)$ in Table 3.5.8 are the largest prime powers $q=p^\ell\equiv 1 \pmod 4$ less than or equal to these bounds, with $\ell\le 2$ and satisfying $\pi(q,t)>0$. This proves (2).

For (3), suppose that R is Type 4 with $t\ge 2$. Then $r=2$ and $|Z(R)|=2$, so $\ell=1$ and $q=p$. Also $q\ge 3$, so $q^{3/2}>4$. We have

$$|G_0|=(q-1)2^{2t}\left|\mathrm{O}^-\,(2t,2)\right|<(q-1)2^{2t^2+t+2}$$

so

$$|G_0|^2+1<\left((q-1)2^{2t^2+t+2}\right)^2+1<q^2 4^{2t^2+t+2}<q^{3t^2+\frac{3}{2}t+5}.$$

We have $3t^2+\frac{3}{2}t+5<2^t$ (and hence $|G_0|^2+1<|V|$) for all $t\ge 8$. For $t\in\{2,\dots,7\}$ define

$$\pi(q,t):=\left((q-1)2^{2t}\left|\mathrm{O}^-\,(2t,2)\right|\right)^2+1-q^{2^t}.$$

Then $|G_0|^2+1-|V|<\pi(q,t)<0$ for all $q>\left(2^{2t}\left|\mathrm{O}^-\,(2t,2)\right|\right)^{1/\left(2^{t-1}-1\right)}$. As in the previous cases we take $q_0(t)$, $2\le t\le 7$, to be the largest prime q less than or equal to these bounds such that $\pi(q,t)>0$. This yields Table 3.5.9 and proves (3).

Statement 4 for the case where R is type 2 is precisely [2, Proposition 3.6 (2)]. For the case where R is type 4 define the matrices $a,c\in\mathrm{GL}\,(V)$ by

$$a:=\begin{pmatrix}0&1\\-1&0\end{pmatrix}\quad\text{and}\quad c:=\begin{pmatrix}\beta&\gamma\\\gamma&-\beta\end{pmatrix},$$

where $\beta,\gamma\in\mathbb{F}_q$ such that $\beta^2+\gamma^2=-1$. Then $\langle a,c\rangle$ is a representation of R in $\mathrm{GL}\,(2,q)$ (see [6, pp. 153-154]). Since R is irreducible on V, any R-orbit v^R in $V^\#$ contains a basis $\{v_1,v_2\}$ of V, and v^{G_0} contains $\langle v_1\rangle^\#\cup\langle v_2\rangle^\#$. Clearly $V^\#\subseteq\langle v_1\rangle^\#+\langle v_2\rangle^\#$. Therefore $V\subseteq v^{G_0}+v^{G_0}$, and thus $\mathrm{diam}(\Gamma)=2$. This proves (4), and completes the proof of the proposition. □

3.6 Class $\mathcal{C}_7$

In this case $V=\otimes_{i=1}^t U_i$ with $U_i=\mathbb{F}_q^m$ for all i, $m\ge 2$, $t\ge 2$, and $d=m^t$. Assume that $\mathcal{B}$ is a tensor product basis of V, with

$$\mathcal{B}:=\left\{\otimes_{i=1}^t u_{i,j}\,\middle|\,1\le j\le m\right\}.$$

As in the $\mathcal{C}_4$ case, it is not difficult to show that for any $v=\sum_{i=1}^r\left(\otimes_{j=1}^t v_{i,j}\right)\in V^\#$ we have

$$v^\tau=\sum_{i=1}^r\left(\otimes_{j=1}^t v_{i,j}^\tau\right),$$

where τ acts on each U_i with respect to the basis $\{u_{i,j}\mid 1\le j\le m\}$.

3.6.1 Case $H=\Gamma\mathrm{L}(n,\,q)$

By Theorem 2.4

$$G_0=(\mathrm{GL}\,(m,q)\wr_\otimes \mathrm{Sym}\,(t))\rtimes\langle\tau\rangle. \tag{3.15}$$

If $t=2$ then we obtain the examples in Proposition 3.13 with $k=m$. We state this in the next corollary, which is analogous to [2, Corollary 3.7].

Corollary 3.24. *Let* $V = \otimes_{i=1}^{t} \mathbb{F}_q^m$ *and let* G_0 *be as in* (3.15) *with* $m \geq 2$ *and* $t = 2$. *Then the* G_0*-orbits in* $V^{\#}$ *are the sets* Y_s *for each* $s \in \{1, \ldots, m\}$, *where* Y_s *is as defined in* (3.7). *Moreover, for any* G_0*-orbit* $S \subseteq V^{\#}$, *the graph* $\mathrm{Cay}(V, S)$ *has diameter* 2 *if and only if* $S = Y_s$ *for some* $s \geq m/2$.

Proof. This follows immediately from Lemma 3.12 and Proposition 3.13. □

Using Lemma 2.1, we get the following bounds which significantly reduce the cases that remain to be considered. It turns out that these are exactly the same as those in [2, Proposition 3.8]; we prove them here for subgroups of $\Gamma\mathrm{L}(n, q)$.

Proposition 3.25. *Let* Γ *be a graph and let* $G \leq \mathrm{Aut}\,(\Gamma)$, *such that* G *satisfies Hypothesis* 3.1 *with* G_0 *as in* (3.15), $m \geq 2$ *and* $t \geq 3$. *Then* Γ *is connected and* G*-symmetric if and only if* $\Gamma \cong \mathrm{Cay}(V, v^{G_0})$ *for some* $v \in V^{\#}$. *Moreover, if* $\mathrm{diam}(\Gamma) = 2$ *then either:*

1. $m = 2$ *and* $t \in \{3, 4, 5\}$; *or*
2. $t = 3$ *and* $m \in \{3, 4, 5\}$.

Proof. Recall that $(\alpha g_1) \otimes g_2 \otimes \cdots \otimes g_t = g_1 \otimes \cdots \otimes (\alpha g_i) \otimes \cdots \otimes g_t$ for all $g_1, \ldots, g_t \in \mathrm{GL}\,(m, q)$, so that

$$|G_0| \leq |\mathrm{GL}\,(m, q)\,|^t\, t!\, \ell(q-1)^{-(t-1)}.$$

Now

$$|\mathrm{GL}\,(m, q)\,| < q^{m(m-1)} q^{m-1}(q-1) = q^{m^2-1}(q-1),$$

$s \leq q^{s-1}$ for all $s \geq 2$ and $q \geq 2$, and $\ell < p^{\ell} = q$ for all $\ell \geq 1$ and $p \geq 2$, so that

$$|G_0|^2 + 1 < \left(q^{(m^2-1)t}(q-1)^t\right)^2 \left(q^{\frac{1}{2}t(t-1)}\right)^2 q^2(q-1)^{-2(t-1)} < q^{t^2+(2m^2-3)t+4}.$$

It can be shown that $t^2 + (2m^2 - 3)t + 4 < m^t$ whenever $t \geq 7$ and $m \geq 2$, and whenever $t \in \{3, 4, 5, 6\}$ and $m > m_0(t)$, where $m_0(t)$ is as given in Table 3.6.10. Hence $|G_0|^2 + 1 < |V|$ for all such pairs (m, t). Of the remaining pairs we can eliminate $(2, 6)$ and $(6, 3)$ by considering $\pi(q, m, t) := (t!)^2 q^{2t(m^2-1)+4} - q^{m^t}$; it can be shown that $\pi(q, 2, 6) < 0$ for all $q \geq 2$ and $\pi(q, 6, 3) < 0$ for all $q \geq 7$. For $q \in \{2, 3, 4, 5\}$ it can be checked that $36\,\ell^2\,|\mathrm{GL}\,(6, q)\,|^6 (q-1)^{-4} + 1 < q^{216}$. Therefore $|G_0|^2 + 1 < |V|$ if $(m, t) \in \{(2, 6), (6, 3)\}$, which completes the proof. □

Table 3.6.10: Values for $m_0(t)$

t	3	4	5	6
$m_0(t)$	6	2	2	2

3.6.2 Case $H = \Gamma\mathrm{Sp}(n,\ q)$

By Theorem 2.5, both q and t are odd and

$$G_0 = (\mathrm{GSp}\,(m, q) \wr_{\otimes} \mathrm{Sym}\,(t)) \rtimes \langle \tau \rangle. \tag{3.16}$$

Hence $q, t \geq 3$.

Proposition 3.26. *Let Γ be a graph and $G \leq \mathrm{Aut}(\Gamma)$ such that G satisfies Hypothesis* 3.1 *with G_0 as in* (3.16)*, $m \geq 2$ and $t \geq 3$. Then Γ is connected and G-symmetric if and only if $\Gamma \cong \mathrm{Cay}(V, v^{G_0})$ for some $v \in V^{\#}$. Moreover, if* $\mathrm{diam}(\Gamma) = 2$ *then either:*

1. *$m = 2$ and $t \in \{3, 5\}$; or*
2. *$t = 3$, $m = 4$, and $q = 9$.*

Proof. In this case $|G_0| \leq |\mathrm{GSp}\,(m,q)\,|^t\, t!\, \ell (q-1)^{-(t-1)}$, where

$$|\mathrm{GSp}\,(m,q)\,| = (q-1)\mathrm{Sp}\,(m,q) < (q-1)q^{\frac{1}{2}(m^2+m)}.$$

Also $s \leq k^{s/2}$ for all $k \geq 3$ and $s \geq 2$, so that $\ell \leq q$, $t! \leq q^{\frac{1}{4}(t-1)(t+2)}$, and

$$|G_0|^2 + 1 < (q-1)^{2t} q^{t(m^2+m)+\frac{1}{2}(t-1)(t+2)+1}(q-1)^{-2(t-1)} < q^{\frac{1}{2}t^2+\left(m^2+m+\frac{1}{2}\right)t+2}.$$

It can be shown that $\frac{1}{2}t^2 + \left(m^2+m+\frac{1}{2}\right)t + 2 < m^t$ whenever $t \geq 6$ and $m \geq 2$, $t = 3$ and $m \geq 5$, and $t = 5$ and $m \geq 3$. So $|G_0|^2 + 1 < |V|$ for all such pairs (m, t). Let $\pi(q, m, t) := (t!)^2 q^{t(m^2+m)+3} - q^{m^t}$. If $(m, t) = (4, 3)$ then for all $q \geq 37$ we get

$$|G_0|^2 + 1 - |V| < \pi(q, 4, 3) < 0.$$

For $3 \leq q \leq 31$, $q \neq 9$, we have $36\,\ell^2\,(q-1)^2 |\mathrm{Sp}\,(4,q)\,|^6 + 1 < q^{64}$. Therefore if $(m, t) = (4, 3)$ and $q \neq 9$ then $|G_0|^2 + 1 < |V|$, which completes the proof. □

3.7 Proof of Theorem 1.1

We now give the proof of Theorem 1.1.

Proof of Theorem 1.1. The first part follows immediately from Lemma 2.1, so we only need to show statements (1) – (3). Assume that G_0 does not belong in the Aschbacher class $\mathcal{C}_9$. Line 1 of Table 1.0.1 follows from Proposition 3.8, line 2 from Proposition 3.13, lines 3 and 4 from Proposition 3.16 (1) and (2), respectively. Line 5 follows from Proposition 3.23 (4), line 5 from Corollary 3.24, line 7 from Proposition 3.6, and lines 8 – 11 from Proposition 3.3. Line 1 of Table 1.0.2 follows from Proposition 3.10 (1), line 2 from Proposition 3.10 (2) and (3), and line 3 from Proposition 3.23 (4). Lines 4 – 6 follow from Proposition 3.3. This proves statement (1).

Statement (2) follows from the results given in the Restrictions column of Table 1.0.2. This completes the proof of Theorem 1.1. □

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On which groups can arise as the canonical group of a spherical latin bitrade

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Abstract

We address a question of Cavenagh and Wanless asking: which finite abelian groups arise as the canonical group of a spherical latin bitrade? We prove the existence of an infinite family of finite abelian groups that do not arise as canonical groups of spherical latin bitrades. Using a connection between abelian sandpile groups of digraphs underlying directed Eulerian spherical embeddings, we go on to provide several, general, families of finite abelian groups that do arise as canonical groups. These families include:

- any abelian group in which each component of the Smith Normal Form has composite order;
- any abelian group with Smith Normal Form $\mathbb{Z}_p^n \oplus \left(\bigoplus_{i=1}^k \mathbb{Z}_{pa_i}\right)$, where $1 \leq k$, $2 \leq a_1, a_2, \ldots, a_k, p$ and $n \leq 1 + 2\sum_{i=1}^k (a_i - 1)$; and
- with two exceptions and two potential exceptions any abelian group of rank two.

Keywords: Spherical latin bitrade, canonical group, abelian sand-pile group.

Math. Subj. Class.: 005C10, 05C25, 05B15, 05C20

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1 Introduction

Given two latin squares of the same order a latin trade describes the differences between them. Early motivation [13] for their study arose from considering the differences between the operation tables of a finite group and a latin square of the same order, that is: what is the 'distance' between a group and a latin square? The study of the topological and geometric properties of latin trades has lead to significant progress towards understanding such differences, see for example [1, 9, 11, 19, 20], also see [7] for a survey of earlier results.

Given a latin trade it may be the case that the constituent partial latin squares are not 'contained' (do not embed) in any group operation table, [9]. Hence, it is desirable to identify those that are. Connected latin bitrades of maximum size, equivalently spherical latin bitrades provide a family of latin bitrades for which the constituent partial latin squares do embed. We are interested in the 'minimal group' that such constituent partial latin squares embed in, and indeed what groups arise as such minimal groups.

1.1 Spherical latin bitrades

A *partial latin square* P is an $\ell \times m$ array, in which the cells either contain an element of a set S of symbols or are empty, such that each row and each column contains each of the symbols of S at most once. Without loss of generality we let $S = \{s_1, s_2, \ldots, s_n\}$ and index the rows and columns by the sets $R = \{r_1, r_2, \ldots, r_\ell\}$ and $C = \{c_1, c_2, \ldots, c_m\}$ respectively (we may assume that each symbol in S occurs at least once in the array and the rows of R and columns of C are all nonempty). As such a partial latin square P can be considered to be a subset of $R \times C \times S$ such that if (r_1, c_1, s_1) and (r_2, c_2, s_2) are distinct triples in P, then at most one of $r_1 = r_2$, $c_1 = c_2$ and $s_1 = s_2$ holds.

A *latin bitrade* is an ordered pair, (W, B) say, of non-empty partial latin squares such that for each triple $(r_i, c_j, s_k) \in W$ (respectively B) there exists unique $r_{i'} \neq r_i$, $c_{j'} \neq c_j$ and $s_{k'} \neq s_k$ such that

$$\big\{(r_{i,'} c_j, s_k), (r_i, c_{j'}, s_k), (r_i, c_j, s_{k'})\big\} \subset B \text{ (respectively } W).$$

Note that (W, B) is a latin bitrade if and only if (B, W) is also a latin bitrade. The *size* of such a latin bitrade is $|W|$ (equivalently $|B|$). A latin bitrade (W, B) for which there does not exist any latin bitrade (W', B') such that $W' \subsetneq W$ and $B' \subsetneq B$ is said to be *connected*.

Let (W, B) be a latin bitrade; for each row, r say, of (W, B) a permutation ρ_r of the symbols in row r can be defined by $\rho_r(s) = s'$ if and only if $(r, c, s) \in W$ and $(r, c, s') \in B$ for some c in C. A row r for which ρ_r is comprised of a single cycle is said to be *separated*. Similar definitions hold for separated columns and separated symbols. A latin bitrade in which each row, each column and each symbol is separated is called a *separated latin bitrade*. Suppose that (W, B) is a latin bitrade which is not separated. Then replacing each non-separated row x (respectively column, symbol) by new rows (respectively columns, symbol) for each of the cycles in ρ_x we obtain a separated latin bitrade. See the survey paper [7] for further details and discussion.

A connected latin bitrade (W, B) can be used to construct a face two-coloured triangulation $\mathcal{G}_{W,B}$ of a pseudo-surface Σ in which the vertex set is $R \sqcup C \sqcup S$ and there is an edge between a pair of vertices if and only if the vertices occur together in a triple of W (equivalently a triple of B). For each triple $(r, c, s) \in W$ a white triangular face with vertices r, c, s is constructed and for each $(r', c', s') \in B$ a black triangular face with vertices

r', c', s' is constructed. As (W, B) is a bitrade the graph underlying $\mathcal{G}_{W,B}$ is simple, and as (W, B) is connected $\mathcal{G}_{W,B}$ is also connected. The pseudo-surface Σ is a true surface if the rotation at each vertex is a full rotation; this occurs if and only if (W, B) is separated (in which case each row, column or symbol permutation corresponds to the rotation at the corresponding vertex). If Σ is not a surface, then replacing each pinch point of multiplicity t with t vertices, one on each of the sheets at the pinch point, corresponds to the above construction taking a non-separated bitrade to a separated one. As the triangulation $\mathcal{G}_{W,B}$ is face two-coloured and the underlying graph is vertex three-coloured it follows, see the proof of Theorem 10.1 in [14], that $\mathcal{G}_{W,B}$ is orientable.

The *genus* of a separated connected latin bitrade is the genus of the surface obtained in the above manner; in particular separated connected latin bitrades of genus zero are referred to as *spherical latin bitrades*. Note that for any connected latin bitrade of size ℓ we have that $|R| + |C| + |S| \leq \ell + 2$, with equality if and only if the bitrade is a spherical latin bitrade, see [1]. That is, spherical latin bitrades are the connected latin bitrades of minimal size (with respect to the sum of the number of rows, columns and symbols).

In [8] Cavenagh and Lisoněk prove the following result.

Theorem 1.1 (Cavenagh & Lisoněk, [8])**.** *Spherical latin bitrades are equivalent to spherical Eulerian triangulations whose underlying graphs are simple.*

Note that an Eulerian graph that has an embedding in the sphere is necessarily vertex three-colourable [16]. It is not hard to generalise Theorem 1.1 to surfaces of higher genus, however as face two-coloured triangulations of surfaces of higher genus may not be vertex three-colourable, an additional condition is required.

Corollary 1.2. *Separated connected latin bitrades of genus g are equivalent to vertex three-colourable Eulerian triangulations of genus g whose underlying graphs are simple.*

1.2 Embeddings of latin bitrades into abelian groups

Two partial latin squares are said to be *isotopic* if they are equal up to a relabelling of their sets of rows, columns and symbols. A partial latin square P, with row set R, column set C and symbol set S, is said to *embed in an abelian group* Γ if there exist injective maps $\phi_1 : R \to \Gamma$, $\phi_2 : C \to \Gamma$ and $\phi_3 : S \to \Gamma$ such that $\phi_1(r) + \phi_2(c) = \phi_3(s)$ for all $(r, c, s) \in P$. In other words P is isotopic to a partial latin square contained in the operation table of Γ. See Figure 1 for an example.

By defining $\phi|_R = \phi_1$, $\phi|_C = \phi_2$, and $\phi|_S = -\phi_3$ it follows, see [1], that P embeds in an abelian group Γ if and only if there exists a function $\phi : R \sqcup C \sqcup S \to \Gamma$ that is injective when restricted to each of R, C and S and is such that $\phi(r) + \phi(c) + \phi(s) = 0$ for all $(r, c, s) \in P$. The map ϕ is called an *embedding* of P. An abelian group Γ is said to be a *minimal abelian representation* of a partial latin square P if P embeds in Γ and the image of ϕ generates Γ for all embeddings ϕ of P in Γ.

Two partial latin squares are said to be *conjugate* if they are equal up to permutations of the roles of rows, columns and symbols. Two partial latin squares, say P and Q, for which a partial latin square isotopic to P is conjugate to a partial latin square isotopic to Q are said to be in the same *main class*. Note that if a partial latin square P has an embedding in an abelian group Γ, every partial latin square in the same main class as P also has an embedding in Γ.

a	b	c
c		a
	a	b

$+$	0	1	2	3
0	0	**1**	**2**	**3**
1	1	2	3	0
2	2	**3**	0	**1**
3	3	0	**1**	**2**

Figure 1: The partial latin square above left embeds into $\mathbb{Z}_4$ as illustrated by the bold faced entries in the operation table of $\mathbb{Z}_4$, above right.

As we are interested in embeddings (into abelian groups) of partial latin squares (and given that if a partial latin square P embeds in an abelian group Γ, so does any partial latin square isotopic to P) from here on we will assume that the row, column and symbol sets of a partial latin square are pairwise disjoint.

In [10] Cavenagh and Drápal asked the following questions "Can the individual partial latin squares of a connected separated latin bitrade be embedded into the operation table of an abelian group? If this is not true in general is it true for spherical latin bitrades?". The case of spherical latin bitrades was solved by Cavenagh and Wanless in [9] and independently by Drápal, Hämäläinen and Kala in [11]. Cavenagh and Wanless [9] also showed that separated connected latin bitrades of higher genus exist for which the constituent partial latin squares do not embed in any group. Hence our focus on spherical latin bitrades.

Let P be a partial latin square with row set R, column set C and symbol set S. Let $V = R \cup C \cup S$ and define an abelian group $\mathcal{A}_P$ with generating set V subject to the relations $\{r + c + s = 0 : (r, c, s) \in P\}$. Note that, if P and Q are two partial latin squares in the same main class, then $\mathcal{A}_P \cong \mathcal{A}_Q$. Also, note that two partial latin squares, P and Q, from different main classes may also satisfy $\mathcal{A}_P \cong \mathcal{A}_Q$ (see Figure 2 in [19]).

The group $\mathcal{A}_P$ has the 'universal' property that any minimal abelian representation of P is a quotient of $\mathcal{A}_P$, [12], also see [1]. Moreover $\mathcal{A}_P$ is of the form $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathcal{C}_P$, again see [1]. Drápal et al [11] and Cavenagh and Wanless [9] proved that $\mathcal{C}_W$ is finite when (W, B) is a spherical latin bitrade. So in this case $\mathcal{C}_W$ is the torsion subgroup of $\mathcal{A}_W$. Cavenagh and Wanless conjectured that $\mathcal{C}_W \cong \mathcal{C}_B$ (and hence $\mathcal{A}_W \cong \mathcal{A}_B$), [9], also see [6, 18]. This is indeed the case.

Theorem 1.3 (Blackburn & McCourt [1])**.** *Let (W, B) be a spherical latin bitrade, then $\mathcal{A}_W \cong \mathcal{A}_B \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathcal{C}$, where $\mathcal{C}$ is finite.*

The group $\mathcal{C}$ in Theorem 1.3 is referred to as the *canonical group* of the spherical latin bitrade (see [15, 19]).

In [9] Cavenagh and Wanless asked the following question.

Question 1. Which abelian groups arise as the canonical group of a spherical latin bitrade?[1]

It is this question that we address in this paper. For any cyclic group $\mathbb{Z}_n$ the existence of spherical latin bitrades whose canonical group is isomorphic to $\mathbb{Z}_n$ was established by Cavenagh and Wanless in [9]. They also noted that no spherical latin bitrade exists whose canonical group is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.

[1]Cavenagh and Wanless actually asked this for the finite torsion subgroup of $\mathcal{A}_W$ as Theorem 1.3 was not established at the time.

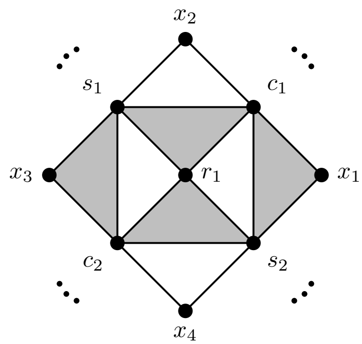


Figure 2: The vertex r_1 and nearby faces in $\mathcal{G}_{W,B}$.

Given a face 2-coloured triangulation of the sphere in which the underlying graph is not necessarily simple and leaving the definitions of $\mathcal{A}_W$ and $\mathcal{A}_B$ unchanged it is still the case that $\mathcal{A}_W \cong \mathcal{A}_B \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathcal{C}$ where $\mathcal{C}$ is finite [1]. In [19] the second author showed that given any finite abelian group Γ there exists a face 2-coloured triangulation of the sphere whose canonical group is isomorphic to Γ. However, unless Γ is a cyclic group the triangulations constructed have underlying graphs that are not simple.

In Section 2 we prove the existence of several, general, infinite families of abelian groups that arise as canonical groups of spherical latin bitrades. Before doing so, we first prove that there exist infinitely many abelian groups that do not arise as the canonical group of any spherical latin bitrade.

Theorem 1.4. *There does not exist a spherical latin bitrade whose canonical group is isomorphic to $\mathbb{Z}_2^k$ for any $k \geq 2$.*

Proof. In the following we will make repeated use of the fact that for $u, v, w, x, y, z \in \mathbb{Z}_2^k$, if $u + w = y$, $v + w = z$ and $v + x = y$, then $u + x = z$.

Let $k \geq 2$ and suppose that (W, B) is a spherical latin bitrade whose canonical group is isomorphic to $\mathbb{Z}_2^k$. So, by Theorem 1.3, both W and B embed in $\mathbb{Z}_2^k$.

Recall that we may assume that the row, column and symbol sets of W (and of B) are pairwise disjoint; denote them, respectively, by $R = \{r_1, r_2, \ldots, r_\ell\}$, $C = \{c_1, c_2, \ldots, c_m\}$ and $S = \{s_1, s_2, \ldots, s_n\}$. Let $\mathcal{G}_{W,B}$ be the related triangulation and G be the underlying graph of this triangulation. As $\mathcal{G}_{W,B}$ has a proper face 2-colouring, G is Eulerian, and, as (W, B) is a latin bitrade, the minimum degree of G is at least four. Moreover, $\mathcal{G}_{W,B}$ is a triangulation of the sphere, so, by Euler's formula, G contains at least six vertices of degree four.

As spherical latin bitrades in the same main class all have isomorphic canonical groups, without loss of generality, we may assume that the degree of r_1 is four, and (r_1, c_1, s_1), $(r_1, c_2, s_2) \in B$ and $(r_1, c_1, s_2), (r_1, c_2, s_1) \in W$ where $c_1 \neq c_2$ and $s_1 \neq s_2$. Hence, as (W, B) is a latin bitrade, there exist $x_1, x_2, x_3, x_4 \in R \setminus \{r_1\}$ such that (x_1, c_2, s_1), $(x_3, c_1, s_2) \in B$ and $(x_2, c_1, s_1), (x_4, c_2, s_2) \in W$ (see Figure 2 for an illustration of the corresponding faces).

As W embeds in $\mathbb{Z}_2^k$, $x_2 = x_4$ and, as B embeds in $\mathbb{Z}_2^k$, $x_1 = x_3$. Suppose that $x_1 = x_2$.

Let

$$W' = \{(r_1, c_1, s_2), (r_1, c_2, s_1), (x_1, c_1, s_1), (x_1, c_2, s_2)\}$$

and

$$B' = \{(r_1, c_1, s_1), (r_1, c_2, s_2), (x_1, c_1, s_2), (x_1, c_2, s_1)\}.$$

Then (W', B') is a spherical latin bitrade such that $W' \subseteq W$ and $B' \subseteq B$. As (W, B) is connected, it must be the case that $W' = W$ and $B' = B$. However, the canonical group of (W', B') is $\mathbb{Z}_2$, a contradiction. So $x_1 \neq x_2$; in which case G contains a subgraph $H = (V, E)$ where $V = \{r_1, x_1, x_2, c_1, s_1, s_2\}$ and $E = \{r_1c_1, r_1s_1, r_1s_2, x_1c_1, x_1s_1, x_1s_2, x_2c_1, x_2s_1, x_2s_2\}$. However, H is isomorphic to $K_{3,3}$; which contradicts $\mathcal{G}_{W,B}$ being a spherical embedding. □

2 Existence results

2.1 Directed Eulerian spherical embeddings

Let D be a, not necessarily simple, digraph of order n with vertex set $V(D) = \{v_1, v_2, \ldots, v_n\}$. The *adjacency matrix* $A = [a_{ij}]$ of D is the $n \times n$ matrix where entry a_{ij} is the number of arcs from vertex v_i to vertex v_j. The *asymmetric Laplacian* of D is the $n \times n$ matrix $L(D) = B - A$ where B is the diagonal matrix in which entry b_{ii} is the out-degree of vertex v_i. The digraph D is said to be *Eulerian* if, for each $v \in V(D)$, the out-degree at v equals the in-degree at v. Hence, in an Eulerian digraph we will simply refer to the degree of a vertex v, i.e. $\deg v$.

Let D be a connected Eulerian digraph of order n with vertex set $V(D) = \{v_1, v_2, \ldots, v_n\}$. Fix an i, where $1 \leq i \leq n$ and define $L'(D, i)$ to be the matrix obtained by removing row and column i from $L(D)$. As D is connected and Eulerian, the group $\mathbb{Z}^{n-1}/L'(D, i)\mathbb{Z}^{n-1}$ is invariant of the choice of i, see [17, Lemma 4.12]. Hence, the *abelian sandpile group* of the connected Eulerian digraph D can be defined to be the group $\mathcal{S}(D) = \mathbb{Z}^{n-1}/\mathbb{Z}^{n-1}L'(D, n)$; moreover $\mathcal{S}(D) \cong \mathbb{Z}^{n-1}/\mathbb{Z}^{n-1}L'(D, i)$, for any $1 \leq i \leq n$.

Consider an embedding $\mathcal{D}$ of a connected Eulerian digraph D in an orientable surface S. If each face of the embedding corresponds to a directed cycle in D, equivalently the rotation at each vertex alternates between incoming and outgoing arcs, then the embedding is said to be a *directed Eulerian embedding*, see [2, 3]. If the embedding is in the sphere we call it a *directed Eulerian spherical embedding*.

Suppose that $\mathcal{G}$ is a face two-coloured triangulation of the sphere. By [16], the underlying digraph of $\mathcal{G}$ has a vertex three-colouring with colour classes R, C and S. Tutte [21] described a construction, from $\mathcal{G}$, of directed Eulerian spherical embeddings $D_I(\mathcal{G}) = D_I$ with vertex set I, where $I \in \{R, C, S\}$. We give a description of the construction from [19].

Let $\{I, I_1, I_2\} = \{R, C, S\}$. Consider a vertex $v_i \in I$. Then v_i has even degree, say d, the rotation at i is $(u_1, v_1, u_2, v_2, \ldots, u_{d/2}, v_{d/2})$, where, without loss of generality, $u_j \in I_1$ and $v_j \in I_2$ for all $1 \leq j \leq d/2$ and the edge e_j between u_j and v_j in the rotation is contained in a black face. Then in D_I there are $d/2$ outgoing arcs from vertex v_i, say a_j, $1 \leq j \leq d/2$, one for each black face, and the terminal vertex for arc a_j is the vertex in I contained in the white face containing edge e_j. Clearly, D_I inherits a spherical embedding from $\mathcal{G}$ in which the arc rotation at each vertex alternates between incoming and outgoing arcs, so D_I has a directed Eulerian spherical embedding. As the sphere is connected the graph underlying D_I is connected. Note that given any of D_R, D_C or D_S the original face two-coloured triangulation can be obtained by reversing the above construction:

Lemma 2.1 (Tutte, [21]). *Given a directed Eulerian spherical embedding D, there exists a face 2-coloured spherical triangulation $\mathcal{G}$ with a vertex 3-colouring given by the vertex sets R, C and S, such that for some $I \in \{R, C, S\}$, $D_I(\mathcal{G}) \cong D$.*

Tutte's Trinity Theorem [21] states that $|\mathcal{S}(D_R)| = |\mathcal{S}(D_C)| = |\mathcal{S}(D_S)|$. For a spherical latin bitrade (W, B) with corresponding face two-coloured triangulation $\mathcal{G}$, this result was strengthened implicitly in [1] and explicitly in [19] to $\mathcal{S}(D_R) \cong \mathcal{S}(D_C) \cong \mathcal{S}(D_S) \cong \mathcal{A}_W \cong \mathcal{A}_B$.

Given an arbitrary directed Eulerian spherical embedding applying the above construction in reverse yields a face two-coloured triangulation. However, the underlying graph is not necessarily simple. In order to make use of the above equivalences (between sandpile groups and canonical groups of spherical latin squares) we make use of the following result.

Proposition 2.2 (McCourt, [19]). *Suppose that $\mathcal{D}$ is a directed Eulerian spherical embedding with underlying digraph D. Further suppose that D is connected, has no loops, no cut vertices and its underling graph has no 2-edge-cuts. Then there exists a spherical latin bitrade whose canonical group is isomorphic to $\mathcal{S}(D)$.*

Hence, in order to construct a spherical latin bitrade with canonical group Γ it suffices to find a directed Eulerian spherical embedding satisfying the connectivity conditions of Proposition 2.2 whose abelian sandpile group is isomorphic to Γ.

2.2 Arbitrary rank

In this section we will construct families of canonical groups that have arbitrary rank. We will make repeated use of the following, elementary lemma.

Lemma 2.3. *Let $2 \le p, a$ and $0 \le x, y, \ell$. Further let $r = p(x+1) + a - x - 1$, $s = p(y+1) + a - y - 1$ and $t_{i,j} \in \mathbb{Z}$, for $1 \le i \le m$ and $1 \le j \le \ell$. Then the matrix*

$$L = \left[\begin{array}{cc|ccc|ccc|c|ccc}
p & -p+1 & 0 & \cdots & 0 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\
-p & r & -p & \cdots & -p & 0 & \cdots & 0 & x+1-a & 0 & \cdots & 0 \\
\hline
0 & -p+1 & & & & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\
\vdots & \vdots & & p\mathbb{I}_x & & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & -p+1 & & & & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\
\hline
0 & -1 & 0 & \cdots & 0 & & & & -p+1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & & p\mathbb{I}_y & & \vdots & \vdots & \ddots & \vdots \\
0 & -1 & 0 & \cdots & 0 & & & & -p+1 & 0 & \cdots & 0 \\
\hline
0 & y+1-a & 0 & \cdots & 0 & -p & \cdots & -p & s & t_{1,1} & \cdots & t_{1,\ell} \\
0 & -1 & 0 & \cdots & 0 & 0 & \cdots & 0 & -p+1 & t_{2,1} & \cdots & t_{2,\ell} \\
\hline
0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & t_{3,1} & \cdots & t_{3,\ell} \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & t_{m,1} & \cdots & t_{m,\ell}
\end{array}\right]$$

reduces (under row and column operations invertible over $\mathbb{Z}$) to

$$\left[\begin{array}{cc|ccc|c|ccc}
1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0\\
0 & ap & 0 & \cdots & 0 & 0 & 0 & \cdots & 0\\ \hline
0 & 0 & & & & 0 & 0 & \cdots & 0\\
\vdots & \vdots & & p\mathbb{I}_{x+y} & & \vdots & \vdots & \ddots & \vdots\\
0 & 0 & & & & 0 & 0 & \cdots & 0\\ \hline
0 & 0 & 0 & \cdots & 0 & p & t_{1,1} & \cdots & t_{1,\ell}\\
0 & 0 & 0 & \cdots & 0 & -p & t_{2,1} & \cdots & t_{2,\ell}\\ \hline
0 & 0 & 0 & \cdots & 0 & 0 & t_{3,1} & \cdots & t_{3,\ell}\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\
0 & 0 & 0 & \cdots & 0 & 0 & t_{m,1} & \cdots & t_{m,\ell}
\end{array}\right].$$

Proof. For $1 \leq i \leq x$ and $1 \leq j \leq y$ add Row $2+i$ and Row $2+x+j$ to Row $x+y+3$ of L. Subsequently, for $1 \leq i \leq x$, add Column $2+i$ to Column 2 and, for $1 \leq j \leq y$, Column $2+x+j$ to Column $3+x+y$. Next add Column 2 to Column 1.

Now add Column 1 to Column $3+x+y$ and $p-1$ copies of Column 1 to Column 2. Row 1 can now be used to clear all non-zeros from Column 1. Once this is completed it is easy to see that the remaining non-zeros in Column 2 can also be cleared. □

The proof of Lemma 2.4 is essentially a special case of the proof of Theorem 2.6, however, to aid the reader, we detail this simpler case before proving the general result.

Lemma 2.4. *Let $1 \leq k$ and let $2 \leq m, a_1, a_2, \ldots, a_k$. Then there exists a spherical latin bitrade whose canonical group is isomorphic to $\bigoplus_{i=1}^{k} \mathbb{Z}_{ma_i}$.*

Proof. We begin by defining a digraph $D_{m;a_1,a_2,\ldots,a_k}$ with vertex set $\{\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_k, \gamma_1, \gamma_2, \ldots, \gamma_k\}$ and

- for each $1 \leq i \leq k$:
  - $m-1$ arcs from α_i to γ_i and $m-1$ arcs from γ_i to α_i;
  - $a_i - 1$ arcs from α_{i-1} to γ_i and $a_i - 1$ arcs from γ_i to α_{i-1};
  - an arc from α_i to α_{i-1};
- for each $1 \leq i \leq k-1$: an arc from γ_i to γ_{i+1}; and
- an additional arc from α_0 to γ_1 and an additional arc from γ_k to α_k.

The digraph $D_{m;a_1,a_2,\ldots,a_k}$ has a directed Eulerian spherical embedding and satisfies the connectivity conditions of Proposition 2.2, as can be seen from Figure 3 (in this figure t arcs from u to v alternating with t arcs from v to u are represented by a bidirectional edge labelled t). Hence, there exists a spherical latin bitrade whose canonical group is isomorphic to $\mathcal{S}(D_{m;a_1,a_2,\ldots,a_k})$.

Suppose that we order the vertices of $D_{m;a_1,a_2,\ldots,a_k}$ by $\alpha_k, \gamma_k, \alpha_{k-1}, \gamma_{k-1}, \ldots, \alpha_2, \gamma_2, \alpha_1, \gamma_1, \alpha_0$, and construct the associated asymmetric Laplacian. Then, removing the row and column corresponding to α_0 yields the reduced asymmetric Laplacian $\mathcal{L}'(D_{m;a_1,a_2,\ldots,a_k})$.

Let $k \geq 1$ and $m, a_1, a_2, \ldots, a_{k+1} \geq 2$. Note that $\mathcal{L}'(D_{m;a_1}) = \begin{bmatrix} m & -m+1 \\ -m & m+a_1-1 \end{bmatrix}$ reduces to $\begin{bmatrix} 1 & 0 \\ 0 & ma_1 \end{bmatrix}$; so $\mathcal{S}(D_{m;a_1}) \cong \mathbb{Z}_{ma_1}$.

Assume that $\mathcal{S}(D_{m;a_1,a_2,\ldots,a_k})$ is isomorphic to $\bigoplus_{i=1}^{k} \mathbb{Z}_{ma_i}$. Setting $a_i - 1 = a_i'$ for $1 \leq i \leq k$, the reduced asymmetric Laplacian $\mathcal{L}_k' = \mathcal{L}'(D_{m;a_1,a_2,\ldots,a_k})$ is shown below.

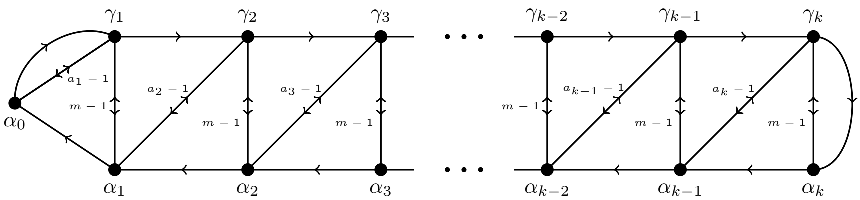


Figure 3: A directed Eulerian spherical embedding of $D_{m;a_1,a_2,\ldots,a_k}$.

$$\mathcal{L}'_k = \begin{bmatrix} m & -m+1 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ -m & m+a'_k & -a'_k & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -a'_k & m+a'_k & -m+1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & -m+1 & m+a'_{k-1} & -a'_{k-1} & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & -a'_{k-1} & m+a'_{k-1} & -m+1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & -1 & -m+1 & m+a'_{k-2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & m+a'_2 & -m+1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -m+1 & m+a'_1 \end{bmatrix}$$

Now, consider the digraph $D_{m;a_1,a_2,\ldots,a_{k+1}}$. Applying Lemma 2.3, with $p = m$ and $x = y = 0$, to rows $\alpha_{k+1}, \gamma_{k+1}, \alpha_k, \gamma_k$ we have that $\mathcal{L}'_{k+1} = \mathcal{L}'(D_{m;a_1,a_2,\ldots,a_{k+1}})$ reduces to

$$\left[\begin{array}{cc|ccc} 1 & 0 & 0 & \cdots & 0 \\ 0 & ma_{k+1} & 0 & \cdots & 0 \\ \hline 0 & 0 & & & \\ \vdots & \vdots & & \mathcal{L}'_k & \\ 0 & 0 & & & \end{array}\right].$$

It follows that $\mathcal{S}(D_{m;a_1,a_2,\ldots,a_{k+1}})$ is isomorphic to $\bigoplus_{i=1}^{k+1} \mathbb{Z}_{ma_i}$. □

It is now easy to establish the existence of spherical latin bitrades whose canonical groups can be expressed as the direct sum of components of composite order.

Theorem 2.5. *Suppose that Γ is a group isomorphic to a direct sum of cyclic groups of composite order; i.e. Γ is isomorphic to $\oplus_{i=1}^{k}\mathbb{Z}_{n_i}$, where each n_i is composite. Then there exists a spherical latin bitrade whose canonical group is isomorphic to Γ.*

Proof. Let $n_1, n_2, \ldots, n_k$ be composite integers and consider $\Gamma \cong \oplus_{i=1}^{k}\mathbb{Z}_{n_i}$. Recall that if $\gcd(n_u, n_v) = 1$, $u \neq v$, then $\oplus_{i=1}^{k}\mathbb{Z}_{n_i} \cong \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_{u-1}} \oplus \mathbb{Z}_{n_{u+1}} \oplus \cdots \oplus \mathbb{Z}_{n_{v-1}} \oplus \mathbb{Z}_{n_{v+1}} \oplus \cdots \oplus \mathbb{Z}_{n_k} \oplus \mathbb{Z}_{n_u n_v}$. Thus we may assume that $\gcd\{n_1, n_2, \ldots, n_k\} \neq 1$. Hence there exists a prime, p say, such that p divides $\gcd\{n_1, n_2, \ldots, n_k\}$. Note that, as n_i is composite for all $1 \leq i \leq k$, $p \neq n_i$. By setting $m = p$ and applying Lemma 2.4 the result follows. □

The next result addresses the existence of spherical latin bitrades for which the Smith Normal Form of their canonical groups contains components of prime order.

Theorem 2.6. *Let p be a prime and let $2 \leq a_1, a_2, \ldots, a_k$. Further let $n \leq 1 + 2\sum_{i=1}^{k}(a_i - 1)$. Then there exists a spherical latin bitrade whose canonical group is isomorphic to*

$$\mathbb{Z}_p^n \oplus \left(\bigoplus_{i=1}^{k} \mathbb{Z}_{pa_i} \right).$$

Proof. If $n = 0$, then this is Lemma 2.4. So for the remainder of the proof assume that $n \geq 1$. As $n \leq 1+2\sum_{i=1}^{k}(a_i-1)$ there exists a k', $0 \leq k' < k$, and t, $0 \leq t \leq 2(a_{k'+1}-1)$ such that

$$n = 1 + 2\sum_{i=1}^{k'}(a_i - 1) + t.$$

First construct the graph $D_{p;a_1,a_2,\ldots,a_k}$ (from the proof of Lemma 2.4). Next, add the following vertices,

- for each $1 \leq i \leq k'$: add vertices $\delta_{i,j}$ and $\epsilon_{i,j}$ for all $1 \leq j \leq a_i - 1$;
- for each $1 \leq j \leq \lceil t/2 \rceil$: add vertices $\delta_{k'+1,j}$;
- for each $1 \leq j \leq \lfloor t/2 \rfloor$: add vertices $\epsilon_{k'+1,j}$; and
- the vertex $\epsilon_{1,0}$.

Now add arcs to and replace arcs from $D_{p;a_1,a_2,\ldots,a_k}$ as illustrated in Figures 4 and 5.

Note that $D^n_{p;a_1,a_2,\ldots,a_k}$ has a directed spherical embedding, and that it satisfies the connectivity conditions of Proposition 2.2. Therefore, there exists a spherical latin bitrade whose canonical group is isomorphic to $\mathcal{S}(D^n_{p;a_1,a_2,\ldots,a_k})$.

For ease of notation, let

$$d_i = \begin{cases} a_i - 1 & \text{for } 1 \leq i \leq k' \\ \lceil t/2 \rceil & \text{for } i = k'+1 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad e_i = \begin{cases} a_i - 1 & \text{for } 1 \leq i \leq k' \\ \lfloor t/2 \rfloor & \text{for } i = k'+1 \\ 0 & \text{otherwise} \end{cases}.$$

Suppose that we order the vertices of $D^n_{p;a_1,a_2,\ldots,a_k}$ by

$$(\alpha_k, \gamma_k, \delta_{k,d_k}, \ldots, \delta_{k,1}, \epsilon_{k,e_k}, \ldots, \epsilon_{k,1}), \ldots, (\alpha_2, \gamma_2, \delta_{2,d_2}, \ldots, \delta_{2,1}, \epsilon_{2,e_2}, \ldots, \epsilon_{2,1}),$$

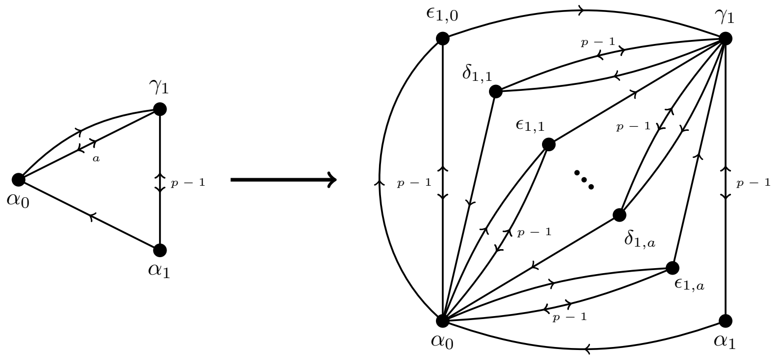
$$(\alpha_1, \gamma_1, \delta_{2,d_1}, \ldots, \delta_{1,1}, \epsilon_{1,e_1}, \ldots, \epsilon_{1,1}, \epsilon_{1,0}), \alpha_0$$

and construct the associated asymmetric Laplacian. Then, removing the row and column corresponding to α_0 yields the reduced asymmetric Laplacian $\mathcal{L}'(D^n_{p;a_1,a_2,\ldots,a_k})$.

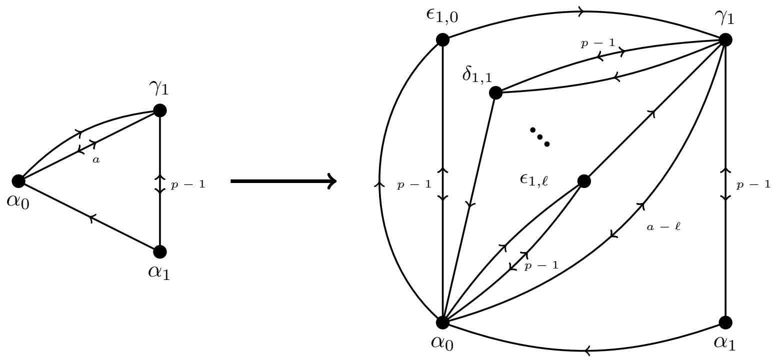
Let $k \geq 1$ and $p, a_1, a_2, \ldots, a_{k+1} \geq 2$ and let $1 \leq n \leq 1 + 2\sum_{i=1}^{k+1}(a_i - 1)$. Then, letting $x = d_1$, $y = e_1$ and $r = p(x+1) + a_1 - x - 1$,

$$\mathcal{L}'\left(D_{p;a_1}^{\min\{n,1+2(a_1-1)\}}\right) = \left[\begin{array}{cc|ccc|ccc|c} p & -p+1 & 0 & \cdots & 0 & 0 & \ldots & 0 & 0 \\ -p & r & -p & \ldots & -p & 0 & \ldots & 0 & 0 \\ \hline 0 & -p+1 & & & & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & p\mathbb{I}_{d_1} & & \vdots & \ddots & \vdots & \vdots \\ 0 & -p+1 & & & & 0 & \cdots & 0 & 0 \\ \hline 0 & -1 & 0 & \cdots & 0 & & & & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & & p\mathbb{I}_{e_1} & & 0 \\ 0 & -1 & 0 & \ldots & 0 & & & & \vdots \\ \hline 0 & -1 & 0 & \ldots & 0 & 0 & \cdots & 0 & p \end{array}\right].$$

Let $a = a_1 - 1$.
For $k' > 0$:



For $k' = 0$, if $t = 2\ell$:



For $k' = 0$, if $t = 2\ell + 1$:

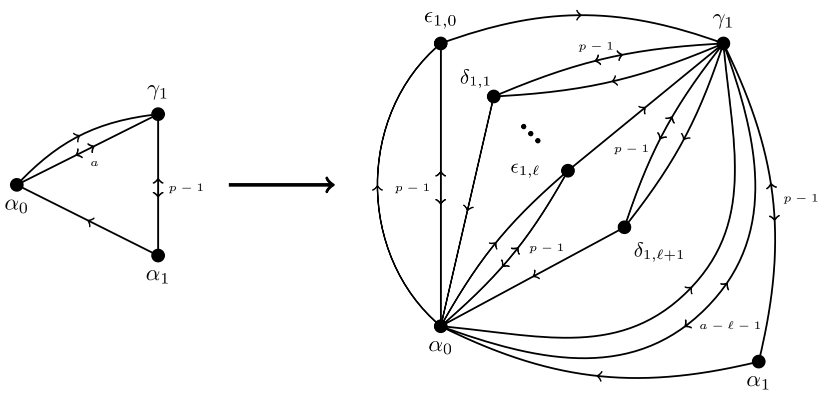
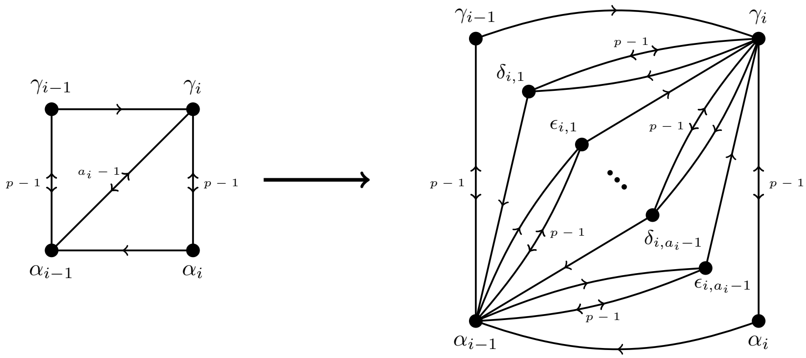
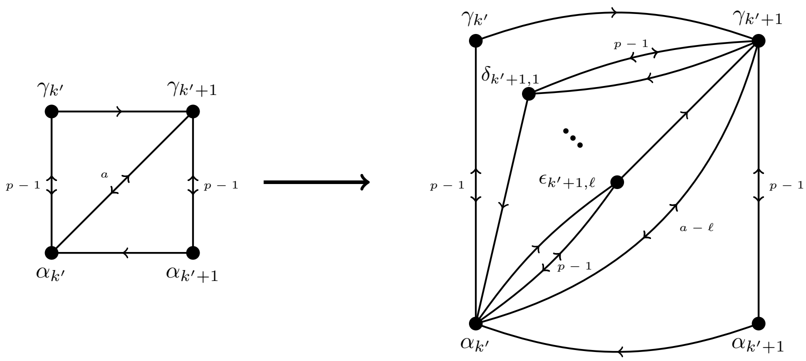


Figure 4: Constructing $D^n_{m;a_1,a_2,\ldots,a_k}$ for arcs incident with α_0.

For $1 < i \leq k'$:



Let $a = a_{k'+1} - 1$, then, if $t = 2\ell$:



Again let $a = a_{k'+1} - 1$, then, if $t = 2\ell + 1$:

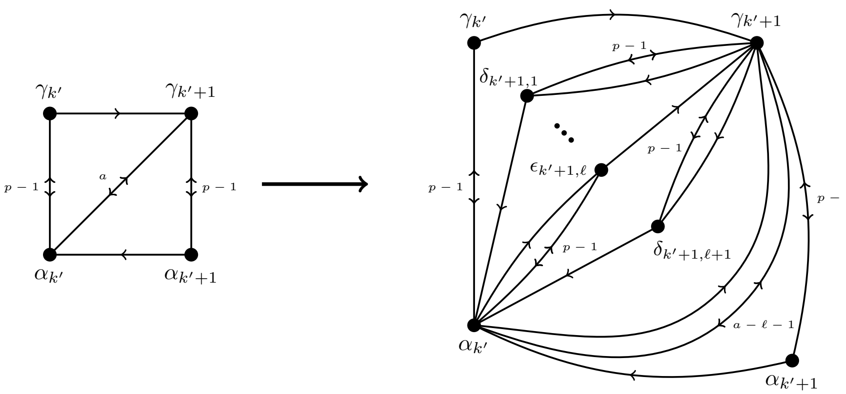


Figure 5: Constructing $D^n_{m;a_1,a_2,\ldots,a_k}$ for arcs not incident with α_0.

Which reduces, under a similar argument to that used to prove Lemma 2.3, to

$$\left[\begin{array}{cc|ccc} 1 & 0 & 0 & \cdots & 0 \\ 0 & pa_1 & 0 & \cdots & 0 \\ \hline 0 & 0 & & & \\ \vdots & \vdots & & p\mathbb{I}_{d_1+e_1+1} & \\ 0 & 0 & & & \end{array}\right].$$

Hence, $\mathcal{S}(D^{\min\{n,1+2(a_1-1)\}}_{p;a_1}) \cong \mathbb{Z}_p^{\min\{n,1+2(a_1-1)\}} \oplus \mathbb{Z}_{pa_1}$.

Assume that $\mathcal{S}\left(D^{\min\{n,1+2\sum_{i=1}^{k}(a_i-1)\}}_{p;a_1,a_2,\ldots,a_k}\right) \cong \mathbb{Z}_p^{\min\{n,1+2\sum_{i=1}^{k}(a_i-1)\}} \oplus \left(\bigoplus_{i=1}^{k} \mathbb{Z}_{pa_i}\right)$. Denote $\mathcal{L}'\left(D^{\min\{n,1+2\sum_{i=1}^{k}(a_i-1)\}}_{p;a_1,a_2,\ldots,a_k}\right)$ by $\mathcal{L}'_k = [\ell_{ij}]$. Let $x = d_{k+1}$, $y = e_{k+1}$, $p(x+1) + a_{k+1} = d_{k+1} - 1$ and $s = p(y+1) + a_{k+1} - y - 1$. Then the asymmetric Laplacian $\mathcal{L}'\left(D^n_{p;a_1,a_2,\ldots,a_{k+1}}\right)$ is

$$\left[\begin{array}{cc|ccc|ccc|c|ccc}
p & -p+1 & 0 & \cdots & 0 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\
-p & r & -p & \cdots & -p & 0 & \cdots & 0 & x+1-a_{k+1} & 0 & \cdots & 0 \\ \hline
0 & -p+1 & & & & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\
\vdots & \vdots & & p\mathbb{I}_x & & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & -p+1 & & & & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\ \hline
0 & -1 & 0 & \cdots & 0 & & & & -p+1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & & p\mathbb{I}_y & & \vdots & \vdots & \ddots & \vdots \\
0 & -1 & 0 & \cdots & 0 & & & & -p+1 & 0 & \cdots & 0 \\ \hline
0 & y+1-a_{k+1} & 0 & \cdots & 0 & -p & \ldots & -p & s & \ell_{1,2} & \cdots & \ell_{1,2k+n} \\
0 & -1 & 0 & \cdots & 0 & 0 & \ldots & 0 & -p+1 & \ell_{2,2} & \cdots & \ell_{2,2k+n} \\ \hline
0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \ell_{3,2} & \cdots & \ell_{3,2k+n} \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \ell_{2k+n,2} & \cdots & \ell_{2k+n,2k+n}
\end{array}\right].$$

Applying Lemma 2.3 to rows $\alpha_{k+1}, \gamma_{k+1}, \delta_{k+1,x}, \ldots, \delta_{k+1,1}, \epsilon_{k+1,y}, \ldots, \epsilon_{k+1,1}, \alpha_k$, γ_k of $\mathcal{L}_{k+1}(D^n_{p;a_1,a_2,\ldots,a_{k+1}})$ reduces it to

$$\left[\begin{array}{cc|ccc|ccc} 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & pa_{k+1} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \hline 0 & 0 & & & & 0 & \cdots & 0 \\ \vdots & \vdots & & p\mathbb{I}_{x+y} & & \vdots & \ddots & \vdots \\ 0 & 0 & & & & 0 & \cdots & 0 \\ \hline 0 & 0 & 0 & \cdots & 0 & & & \\ \vdots & \vdots & \vdots & \ddots & \vdots & & \mathcal{L}'_k & \\ 0 & 0 & 0 & \cdots & 0 & & & \end{array}\right].$$

Therefore $\mathcal{S}(D^n_{p;a_1,a_2,\ldots,a_{k+1}}) \cong \mathbb{Z}_p^n \oplus \left(\bigoplus_{i=1}^{k+1} \mathbb{Z}_{pa_i}\right)$. □

2.3 Canonical groups of rank two

In this section we will restrict our attention to canonical groups of rank two. We show that, with two exceptions and a further two possible exceptions, any finite abelian group of rank two is isomorphic to the canonical group of some spherical latin bitrade.

We will make use of the following elementary lemma.

Lemma 2.7. *Let $2 \le d$, $1 \le x$, $2 \le y$ and $t_{i,j} \in \mathbb{Z}$ for $1 \le i \le x$ and $1 \le j \le y$. Further let $M = [m_{ij}]$ be the $d-1$ by d matrix where*

$$m_{ij} = \begin{cases} 2 & \text{if } i = j \\ -1 & \text{if } j = i+1 \text{ or } j = i-1 \\ 0 & \text{otherwise} \end{cases} .$$

Then the $d+x-2$ by $d+y-2$ matrix

$$\left[\begin{array}{ccc|cc|ccc} & & & & & 0 & \cdots & 0 \\ & & M & & & 0 & \cdots & 0 \\ & & & & & \vdots & \ddots & \vdots \\ & & & & & 0 & \cdots & 0 \\ \hline 0 & \cdots & 0 & t_{1,1} & t_{1,2} & t_{1,3} & \cdots & t_{1,y} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & t_{x,1} & t_{x,2} & t_{x,3} & \cdots & t_{x,y} \end{array}\right]$$

reduces (under operations invertible over $\mathbb{Z}$) to

$$\left[\begin{array}{ccc|cc|ccc} & & & 0 & 0 & 0 & \cdots & 0 \\ & \mathbb{I}_{d-2} & & \vdots & \vdots & \vdots & \ddots & \vdots \\ & & & 0 & 0 & 0 & \cdots & 0 \\ \hline 0 & \cdots & 0 & d & 1-d & 0 & \cdots & 0 \\ \hline 0 & \cdots & 0 & t_{1,1} & t_{1,2} & t_{1,3} & \cdots & t_{1,y} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & t_{x,1} & t_{x,2} & t_{x,3} & \cdots & t_{x,y} \end{array}\right].$$

Proof. When $d = 2$, the result is trivial. Assume that the statement holds for $d = k$, and consider $L(k+1)$. Then $L(k+1)$ reduces to

$$\left[\begin{array}{ccc|cccccc} & & & 0 & 0 & 0 & 0 & \cdots & 0 \\ & \mathbb{I}_{k-2} & & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ & & & 0 & 0 & 0 & 0 & \cdots & 0 \\ \hline 0 & \cdots & 0 & k & 1-k & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & t_{1,1} & t_{1,2} & t_{1,3} & \cdots & t_{1,y} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & \cdots & 0 & 0 & t_{x,1} & t_{x,2} & t_{x,3} & \cdots & t_{x,y} \end{array}\right]$$

Adding $k-1$ copies of Row k to Row $k-1$ followed by adding one copy of the updated Row $k-1$ to Row k yields a 1 in entry $(k-1, k-1)$ and this is now the only non-zero in Column $k-1$. The result follows. □

Lemma 2.8. *Suppose that $1 \le a, b, c$. Then there exists a spherical latin bitrade whose canonical group is isomorphic to $\mathbb{Z}_{ab+bc+ac+1} \oplus \mathbb{Z}_{ab+bc+ac+1}$.*

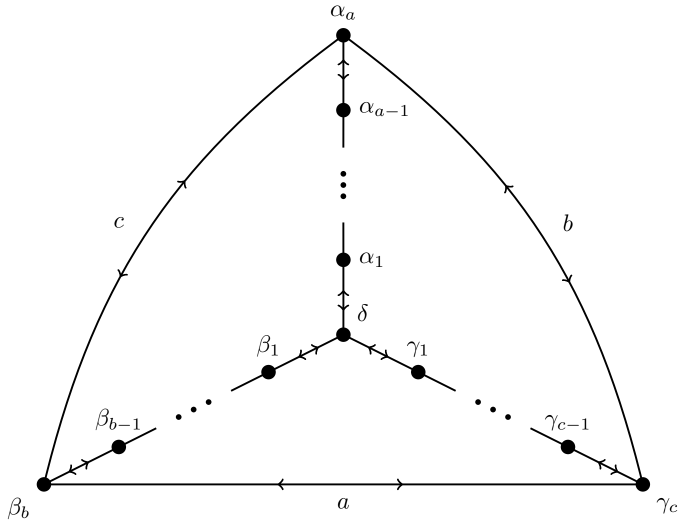


Figure 6: A directed Eulerian spherical embedding of $D_{a,b,c}$.

Proof. Without loss of generality we may assume that $1 \leq a \leq b \leq c$. Define $D_{a,b,c}$ to be the digraph of order $a+b+c+1$ with vertex set $\{\alpha_1, \alpha_2, \dots, \alpha_a, \beta_1, \beta_2, \dots, \beta_b, \gamma_1, \gamma_2, \dots, \gamma_c, \delta\}$ and

- for $1 \leq i \leq a-1$ an arc from α_i to α_{i+1} and an arc from α_{i+1} to α_i;
- for $1 \leq i \leq b-1$ an arc from β_i to β_{i+1} and an arc from β_{i+1} to β_i;
- for $1 \leq i \leq c-1$ an arc from γ_i to γ_{i+1} and an arc from γ_{i+1} to γ_i;
- for each $\iota \in \{\alpha, \beta, \gamma\}$ an arc from δ to ι_1 and from ι_1 to δ; and
- a arcs from β_b to γ_c and from γ_c to β_b; b arcs from α_a to γ_c and from γ_c to α_a; and c arcs from α_a to β_b and from β_b to α_a.

Note that $D_{a,b,c}$ has a directed Eulerian spherical embedding, see Figure 6, and that $D_{a,b,c}$ satisfies the connectivity conditions of Proposition 2.2. Hence, there exists a spherical latin bitrade whose canonical group is isomorphic to $\mathcal{S}(D_{a,b,c})$.

Suppose that we order the vertices of $D_{a,b,c}$ by

$$\gamma_1, \gamma_2, \dots, \gamma_{c-2}, \gamma_{c-1}, \gamma_c, \beta_1, \beta_2, \dots, \beta_{b-2}, \beta_{b-1}, \beta_b, \alpha_1, \alpha_2, \dots, \alpha_{a-2}, \alpha_{a-1}, \alpha_a, \delta.$$

Let $\mathcal{L}'(D_{a,b,c})$ be the reduced asymmetric Laplacian for $D_{a,b,c}$ obtained by removing the row and column corresponding to δ.

When $a=b=c=1$, $\mathcal{L}'(D_{1,1,1}) = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}$, which reduces to $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$.

Suppose that $2 \leq a, b, c$. Consider $\mathcal{L}'(D_{a,b,c})$, via three applications of Lemma 2.8 and setting $a+b+c+1=t$, this reduces to

$$\left[\begin{array}{c|cc|c|cc|c|cc}
 & 0 & 0 & 0\ \cdots\ 0 & 0 & 0 & 0\ \cdots\ 0 & 0 & 0 \\
\mathbb{I}_{c-2} & \vdots & \vdots & \vdots\ \ddots\ \vdots & \vdots & \vdots & \vdots\ \ddots\ \vdots & \vdots & \vdots \\
 & 0 & 0 & 0\ \cdots\ 0 & 0 & 0 & 0\ \cdots\ 0 & 0 & 0 \\
\hline
0\ \cdots\ 0 & c & 1-c & 0\ \cdots\ 0 & 0 & 0 & 0\ \cdots\ 0 & 0 & 0 \\
0\ \cdots\ 0 & -1 & t-c & 0\ \cdots\ 0 & 0 & -a & 0\ \cdots\ 0 & 0 & -b \\
\hline
0\ \cdots\ 0 & 0 & 0 & & 0 & 0 & 0\ \cdots\ 0 & 0 & 0 \\
\vdots\ \ddots\ \vdots & \vdots & \vdots & \mathbb{I}_{b-2} & \vdots & \vdots & \vdots\ \ddots\ \vdots & \vdots & \vdots \\
0\ \cdots\ 0 & 0 & 0 & & 0 & 0 & 0\ \cdots\ 0 & 0 & 0 \\
\hline
0\ \cdots\ 0 & 0 & 0 & 0\ \cdots\ 0 & b & 1-b & 0\ \cdots\ 0 & 0 & 0 \\
0\ \cdots\ 0 & 0 & -a & 0\ \cdots\ 0 & -1 & t-b & 0\ \cdots\ 0 & 0 & -c \\
\hline
0\ \cdots\ 0 & 0 & 0 & 0\ \cdots\ 0 & 0 & 0 & & 0 & 0 \\
\vdots\ \ddots\ \vdots & \vdots & \vdots & \vdots\ \ddots\ \vdots & \vdots & \vdots & \mathbb{I}_{a-2} & \vdots & \vdots \\
0\ \cdots\ 0 & 0 & 0 & 0\ \cdots\ 0 & 0 & 0 & & 0 & 0 \\
\hline
0\ \cdots\ 0 & 0 & 0 & 0\ \cdots\ 0 & 0 & 0 & 0\ \cdots\ 0 & a & 1-a \\
0\ \cdots\ 0 & 0 & -b & 0\ \cdots\ 0 & 0 & -c & 0\ \cdots\ 0 & -1 & t-a
\end{array}\right].$$

Computing the Smith Normal form of

$$\begin{bmatrix}
c & 1-c & 0 & 0 & 0 & 0 \\
-1 & t-c & 0 & -a & 0 & -b \\
0 & 0 & b & 1-b & 0 & 0 \\
0 & -a & -1 & t-b & 0 & -c \\
0 & 0 & 0 & 0 & a & 1-a \\
0 & -b & 0 & -c & -1 & t-a
\end{bmatrix}$$

we have that $\mathcal{S}(D_{a,b,c}) \cong \mathbb{Z}_{ab+bc+ac+1} \oplus \mathbb{Z}_{ab+bc+ac+1}$.

The cases where $1 = a < b \leq c$ and $1 = a = b < c$ follow similarly. □

Theorem 2.9. *For $n, m \geq 2$, with two exceptions and a further two possible exceptions, there exists a spherical latin bitrade whose canonical group is isomorphic to $\mathbb{Z}_n \oplus \mathbb{Z}_m$.*

The exceptions are as follows. There does not exist a spherical latin bitrade with canonical group isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ or $\mathbb{Z}_3 \oplus \mathbb{Z}_3$. There may or may not exist a spherical latin bitrade with canonical group isomorphic to $\mathbb{Z}_5 \oplus \mathbb{Z}_5$ or $\mathbb{Z}_r \oplus \mathbb{Z}_r$ for some r greater than 10^{11}.

Finally, if we assume the Generalised Riemann Hypothesis, then there exists a spherical latin bitrade with canonical group isomorphic to $\mathbb{Z}_r \oplus \mathbb{Z}_r$.

Proof. If n and m are coprime, then $\mathbb{Z}_n \oplus \mathbb{Z}_m \cong \mathbb{Z}_{nm}$ and the result follows from [9] (it also follows from Lemma 2.4 with $k = 1$). So assume that n and m are not coprime; that is, we are in the rank 2 case.

Suppose that $n \neq m$. If n and m are both composite, then the result follows from Theorem 2.5. So suppose that n is prime and m is composite. Then as n and m are not coprime $m = kn$ for some $k > 1$ and the result follows from Theorem 2.6.

So, suppose that $n = m$. If there exist $a, b, c \geq 1$ such that $ab + ac + bc + 1 = n$, then by Lemma 2.8 there exits a spherical latin bitrade whose canonical group is isomorphic to $\mathbb{Z}_n \oplus \mathbb{Z}_n$. In [4] Borwein and Choi proved that there are at most nineteen integers that are not of the form $ab + ac + bc + 1$ where $a, b, c \geq 1$. The first eighteen are: 2, 3, 5, 7, 11, 19, 23, 31, 43, 59, 71, 79, 103, 131, 191, 211, 331 and 463. The nineteenth is greater than

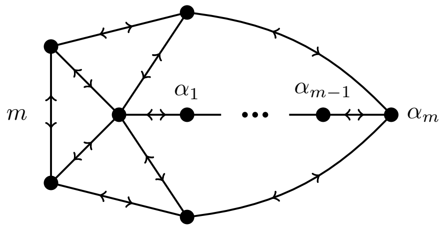


Figure 7: Directed Eulerian spherical embedding of a digraph with abelian sandpile group isomorphic to $\mathbb{Z}_{6m+5} \oplus \mathbb{Z}_{6m+5}$, when $m \in \{1, 3, 9, 11, 21, 31\}$.

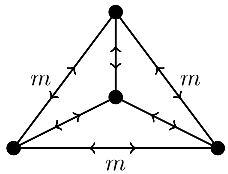


Figure 8: Directed Eulerian spherical embedding of a digraph with abelian sandpile group isomorphic to $\mathbb{Z}_{3m+1} \oplus \mathbb{Z}_{3m+1}$, when $m \in \{2, 6, 10, 14, 26, 34, 70, 110, 154\}$.

10^{11} and is not an exception if the Generalised Riemann Hypothesis is assumed. For $n \in \{7, 11, 19, 23, 31, 43, 59, 71, 79, 103, 131, 191, 211, 331, 463\}$ directed Eulerian spherical embeddings whose underlying digraphs satisfy the connectivity conditions of Proposition 2.2 and with abelian sandpile groups isomorphic to $\mathbb{Z}_n \oplus \mathbb{Z}_n$ are given in Figures 7 and 8.[2]

Cavenagh and Wanless noted in [9] that there does not exist a spherical latin bitrade with canonical group isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ (also see Theorem 1.4).

By [15, Theorems 5 and 6], the minimum order of the canonical group of a spherical Eulerian triangulation of order n is $(n-2)/2$. The computer program `plantri` [5] can be used to generate all spherical Eulerian triangulations of order up to 20. None of these triangulations have $\mathbb{Z}_3 \oplus \mathbb{Z}_3$ as their canonical group; hence this group is also an exception.[3]

□

2.4 Questions

We conclude with three questions for future consideration. The first two address the remaining cases to be considered in order to resolve Question 1.

Question 2. Let $p \neq 2$ be a prime, $n \geq 3$ if $p > 5$ and $n \geq 2$ if $p = 5$; does there exist a spherical latin bitrade with canonical group is isomorphic to $\mathbb{Z}_p^n$?

Question 3. Let p be a prime and let $2 \leq a_1, a_2, \ldots, a_k$. If $n > 1 + 2\sum_{i=1}^{k}(a_i - 1)$, does

[2] The families of indicated in Figures 7 and 8 generalise to give abelian sandpile groups isomorphic to $\mathbb{Z}_{6m+5} \oplus \mathbb{Z}_{6m+5}$, for all $m \geq 1$ and $\mathbb{Z}_{3m+1} \oplus \mathbb{Z}_{3m+1}$, for all $m \geq 1$, respectively. However, we do not require these more general results to prove Theorem 2.9.

[3] This proof that $\mathbb{Z}_3 \oplus \mathbb{Z}_3$ is an exception is due to an anonymous referee.

there exist a spherical latin bitrade with canonical group is isomorphic to

$$\mathbb{Z}_p^n \oplus \left(\bigoplus_{i=1}^{k} \mathbb{Z}_{pa_i} \right)?$$

Our final question arises naturally in response to the non-existence result Theorem 1.4. For a separated, connected latin bitrade (A, B) of genus greater than zero, the group $\mathcal{A}_W$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathcal{C}$, but the minimal abelian representation (if one exists) is now a quotient of $\mathcal{C}$, [1, Theorem 6]. Hence, we ask the following.

Question 4. Does there exist a family of separated, connected latin bitrades for which the minimum abelian representation of one (or both) of the partial latin squares is isomorphic to $\mathbb{Z}_2^k$ for arbitrary k? If so does such a family exist for a fixed genus?

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Pursuit-evasion in a two-dimensional domain

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Abstract

In a pursuit-evasion game, a team of pursuers attempt to capture an evader. The players alternate turns, move with equal speed, and have full information about the state of the game. We consider the most restrictive capture condition: a pursuer must become colocated with the evader to win the game. We prove two general results about this adversarial motion planning problem in geometric spaces. First, we show that one pursuer has a winning strategy in any compact $\text{CAT}(0)$ space. This complements a result of Alexander, Bishop and Ghrist, who provide a winning strategy for a game with positive capture radius. Second, we consider the game played in a compact domain in Euclidean two-space with piecewise analytic boundary and arbitrary Euler characteristic. We show that three pursuers always have a winning strategy by extending recent work of Bhadauria, Klein, Isler and Suri from polygonal environments to our more general setting.

Keywords: Pursuit-evasion, lion and man, $\text{CAT}(0)$ *space, motion planning.*

Math. Subj. Class.: 91A24, 49N75, 53A04

1 Introduction

A pursuit-evasion game in a domain $\mathcal{D}$ is played between a team of pursuers $p_1, p_2, \ldots, p_k$ and an evader e. The pursuers win if some p_i becomes colocated with the evader after a finite number of turns, meaning that the distance $d(p_i, e) = 0$. When this occurs, we say that p_i *captures* e. We consider the discrete time version of the game, which proceeds in turns. Initially, the pursuers choose their positions $p_1^0, p_2^0, \ldots, p_k^0$, and then the evader chooses his initial position e^0. In turn $t \geq 1$, each pursuer p_i moves from her current

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position p_i^{t-1} to a point $p_i^t \in B(p_i^{t-1}, 1) = \{x \in \mathcal{D} \mid d(p_i^{t-1}, x) \leq 1\}$. If the evader has been captured, then the game ends with the pursuers victorious. Otherwise, the evader moves from e^{t-1} to a point $e^t \in B(e^{t-1}, 1)$. The evader wins if he remains uncaptured forever. We consider the full-information (full-visibility) game in which each player knows the environment and the locations of all the other players. Furthermore, the pursuers may coordinate their movements.

Turn-based pursuit games in simply connected domains have been well-characterized: one pursuer is sufficient to capture the evader. Winning pursuer strategies have been found for environments in $\mathbb{E}^n$ [21, 16], and in simply connected polygons [14]. Taking a geometric viewpoint and using the weaker capture criterion $d(p, e) < \epsilon$ for some constant $\epsilon > 0$, Alexander, Bishop and Ghrist [3] proved that a single pursuer can capture an evader in any compact $\mathrm{CAT}(0)$ by heading directly towards the evader at maximum speed. We provide an alternate strategy for a compact $\mathrm{CAT}(0)$ space that achieves $d(p, e) = 0$ in a finite number of turns. Our winning pursuer strategy is a generalization of *lion's strategy*, which has been used successfully in $\mathbb{E}^n$ [21] and in simple polygons [14]. We defer the description of this strategy to Section 2.

Theorem 1.1. *A pursuer p using lion's strategy in a compact $\mathrm{CAT}(0)$ space $\mathcal{D}$, captures the evader e by achieving $d(p, e) = 0$ after at most $\mathrm{diam}(\mathcal{D})^2$ turns.*

Theorem 1.1 implies that a single pursuer can become colocated with an evader in a simply connected, compact domain $\mathcal{D} \subset \mathbb{E}^2$. In particular, this result holds for the polygonal setting in [14]. Notably, the general $\mathrm{CAT}(0)$ viewpoint leads to an improved capture time bound for polygons, compared to the $O(n \cdot \mathrm{diam}(\mathcal{D}))$ result in [14], where n is the number of vertices of polygon $\mathcal{D}$.

It is easy to construct compact domains that are evader win: removing one large open set from the middle of a simply connected domain tips the game in the evader's favor. Indeed, the evader can keep this large obstruction between himself and the pursuer, indefinitely. Such an open set is called an *obstacle* or *hole* in the environment. It is not hard to show that adding a second pursuer to this two-dimensional domain gives the game back to the pursuers. Adding multiple obstacles creates a distinct topology, and it is natural to wonder how many pursuers are needed to capture an evader in such an environment. The analogous question has been resolved for pursuit-evasion games in certain two-dimensional environments. Aigner and Fromme [1] proved that three pursuers are sufficient for pursuit-evasion on a planar graph. Bhadauria, Klein, Isler and Suri [8] showed that the analogous result holds in a two-dimensional polygonal environment with polygonal holes. We generalize the latter three-pursuer result to a broader class of geometric spaces.

Our pursuit game takes place in a compact and path-connected domain $\mathcal{D} \subset \mathbb{E}^2$. The set $\mathcal{D}$ contains a finite set of disjoint open obstacles $\mathcal{O} = \{O_1, O_2, \ldots, O_k\}$. The domain boundary is $\partial\mathcal{D} = \{\partial O_0, \partial O_1, \partial O_2, \ldots, \partial O_k\}$ where we define ∂O_0 to be the outer boundary of $\mathcal{D}$, for convenience. We place two conditions on the boundary. First, $\partial\mathcal{D}$ can be decomposed into a finite number of analytic curves $\gamma(t) = (x(t), y(t))$ for $0 \leq t \leq 1$, where each of $x(t), y(t)$ can locally be expanded as convergent power series. Second, we require that ∂D is a 1-manifold: for any $x \in \partial D$, there exists an $\epsilon > 0$, such that $B(x, \epsilon) \cap \partial D$ is homeomorphic to $\mathbb{E}^1$. In other words, we forbid self-intersections. For brevity, we say that a domain $\mathcal{D}$ satisfying these properties is *piecewise analytic*. We list three consequences of these conditions. First, the number of singular points on the boundary is finite. Second, the absolute value of the curvature at the nonsingular points of ∂D

is bounded above by some constant $\kappa_{\max} > 0$. Third, there is a minimum separation $d_{\min} > 0$ between boundary components: $d(O_i, O_j) > d_{\min}$ for all $0 \leq i < j \leq k$. During the game, the pursuers will guard a sequence of geodesics; crucially, we will see in Section 4 that each of these geodesics is also piecewise analytic. This brings us to our main result.

Theorem 1.2. *Three pursuers can capture an evader in a compact domain in $\mathbb{E}^2$ with piecewise analytic boundary. The number of turns required to capture the evader for a domain with k obstacles is $O(2k \cdot \mathrm{diam}(\mathcal{D}) + \mathrm{diam}(\mathcal{D})^2)$.*

At a high level, our winning three-pursuer strategy builds on those found in [1, 8], and we are indebted to those previous papers. However, our geometric and topological approach is entirely new. In particular, our arguments are grounded in a careful investigation of the convexity, curvature and homotopy classes of geodesic curves in our domain. Furthermore, Theorem 1.2 significantly extends the class of known three-pursuer-win domains.

Finally, we recently became aware of an unpublished technical report of Zhou et. al [22] that proves a similar result. Like our proof, their strategy adapts that of [8] to a more general setting. However, our underlying methodology is quite distinct: we use a homotopy based argument, while they use a geometric one. Furthermore, we devote more attention to the boundary of our region. In particular, we use analytic curves (rather than smooth curves) to avoid potential pathologies of geodesics. We provide more detail on finding guardable paths: we explain how to restrict ourselves to finding geodesics in closed sets, rather than in sets that are neither open nor closed (see Lemma 3.6 below). Finally, they use an endgame that requires two aggressive pursuers. We stick with lion's strategy due to its broad applicability to $\mathrm{CAT}(0)$ spaces.

1.1 Related Work

Pursuit-evasion games are a class of adversarial motion planning problems. Chung, Hollinger and Isler [11] provide an informative survey of pursuit games in mobile robotics. The interdisciplinary literature on pursuit-evasion games spans a range of settings and variations. Pursuit games have been studied in many environments, including graphs, in polygonal environments and in geometric spaces. Researchers have considered motion constraints such as speed differentials between the players, constraints on acceleration, and energy budgets. As for sensing models, the players may have full information about the positions of other players, or they may have incomplete or imperfect information. Typically, the capture condition requires achieving colocation, a proximity threshold, or sensory visibility (such as a non-obstructed view of the evader).

Pursuit games enjoy a wide range of applications, including intruder neutralization, search-and-rescue, and environmental monitoring of tagged wildlife. In these settings, modeling an adversarial evader gives worst-case feasibility and time bounds. For an overview of pursuit-evasion on graphs, see the monograph by Bonato and Nowakowski [9]. Kopparty and Ravishankar [16] give a nice an introduction to pursuit in the polygonal setting.

Research on pursuit-evasion spans nearly a century. In the 1930s, Rado posed the Lion and Man game in which a lion hunts the man in a circular arena. The players move with equal speeds, and the lion wins if it achieves colocation. At first blush, it seems that lion should be able to capture man, regardless of the man's evasive strategy. However, Besicovitch showed that when the game is played in continuous time, the man can follow a spiraling path so that lion can get arbitrarily close, but cannot achieve colocation [17]. However, when lion and man move in discrete time steps, our intuition prevails: lion does have a winning strategy [21].

The past decade has witnessed a renaissance of pursuit-evasion results in multiple disciplines. Prominent research efforts come from the robotics community, where pursuit-evasion in polygonal environments is a productive setting for exploring autonomous agents. Pursuit-evasion has also thrived in the graph theory community, where it is known as the game of Cops and Robbers. More recently, researchers have started exploring pursuit-evasion games in topological spaces. This is a natural evolution for the study of pursuit-evasion games. Indeed, determining the number of pursuers required to capture an evader in a given environment becomes a question about its topology since the various loops and holes of the environment provide escape routes for the evader.

The classic paper of Aigner and Fromme [1] initiated the study of multiple pursuers versus a single evader on a graph. In this turn-based game, agents can move to adjacent vertices, and the cops win if one of them becomes colocated with the robber. This paper introduced the *cop number* of a graph, which is the minimum number of pursuers (cops) needed to catch the evader (robber). Aigner and Fromme proved that the cop number of any planar graph is at most 3. This bound is tight, as the dodecahedron graph requires three cops. At a high level, their winning pursuer strategy proceeds as follows. Two cops guard distinct (u, v)-paths where u, v are vertices of the graph G. This restricts the robber movement to a subgraph of G. The third pursuer then guards another (u, v)-path, chosen so that (1) the robber's movement is further restricted, and (2) one of the other cops no longer needs to guard its path. This frees up that cop to continue the pursuit. This process repeats until the evader is caught.

More recently, an analogous result was proven by Bhadauria, Klein, Isler and Suri [8] for pursuit-evasion games in a two-dimensional polygonal environment with polygonal holes. In this turn-based game, an agent can move to any point within unit distance of its current location. Like Aigner and Fromme, they use colocation as their capture criterion. Bhadauria et al. prove that three pursuers are sufficient for pursuit-evasion in this setting, and that this bound is tight. The pursuer strategy is inspired by the Aigner and Fromme strategy for planar graphs: two pursuers guard paths that confine the evader while the third pursuer takes control of another path that further restricts the evader's movement. Of course, the details of the pursuit and the technical proofs are quite different from the graph case. Their proofs make heavy use of the polygonal nature of the environment, both to find the paths to guard and to guarantee that their pursuit finishes in finite time.

Just as the proofs of Bhadauria et al. were inspired by Aigner and Fromme, our proof of Theorem 1.2 is inspired by those for the polygonal environment. Bhadauria et al. actually give two different winning strategies for three pursuers. At a high level, these strategies progress in the same way, but the tactics for choosing paths and how to guard them are different. Herein, we adapt their *shortest path strategy* to our setting. Our more general geometric environment introduces a distinctive set of challenges to overcome. In particular, we do not have a finite set of polygonal vertices to use as a backbone for our guarded paths. Instead, we rely on homotopy classes to differentiate between paths to guard. Looking beyond the high-level structure of our pursuer strategy, the arguments (and their technical details) in this paper are wholly distinct from those found in [8], and our result applies to a much broader class of environments.

Finally, we note that our result follows in the footsteps of other recent explorations of pursuit-evasion games in general geometric and topological domains. Pursuit-evasion games in such spaces have further applications in robotics, where agents must navigate and coordinate in high dimensional configuration spaces. Alexander, Bishop and Ghrist helped to pioneer this subject, studying pursuit-evasion games with the capture condition $d(p, e) < \epsilon$ for some constant $\epsilon > 0$ (rather that colocation). In [3], Alexander, Bishop and Ghrist prove that a single pursuer can capture an evader in any compact $\mathrm{CAT}(0)$ space: the *simple pursuit* strategy of heading directly towards the evader is a winning strategy. In [5], these authors explore the simple pursuit strategy in unbounded $\mathrm{CAT}(K)$ spaces with positive curvature $K > 0$, developing connections between evader-win environments and the total curvature of the pursuer's trajectory. Finally, in [4], they consider pursuit games in unbounded Euclidean domains using multiple pursuers. They provide conditions on the initial configuration of the players that guarantee capture, generalizing (and amending) results of Sgall [21] and Kopparty and Ravishankar [16].

1.2 Preliminaries

We introduce some notation and review some concepts and results from algebraic topology [13]. We then prove three lemmas about convex paths in two-dimensional compact regions with piecewise analytic boundary.

A topological space is a set X along with a collection of subsets of X, called *open sets* that satisfy a sequence of axioms [7]. A map $f : X \to Y$ between two spaces is *continuous* when the inverse image of every open set in Y is open in X. A *path* $\Pi : [0, 1] \to \mathcal{D}$ is a continuous map from the interval $[0, 1]$ to $\mathcal{D}$, with initial point $\Pi(0)$ and terminal point $\Pi(1)$. The *length* $l(\Pi)$ of this path is its arc length in Euclidean space $\mathbb{E}^2$. A path Π is a *loop* when $\Pi(0) = \Pi(1)$. A *simple path* has no self-intersections, meaning that Π is injective. By abusing notation, we write $x \in \Pi$ when $x = \Pi(t)$ for some $t \in [0, 1]$. For $x, y \in \Pi$, we use $\Pi(x, y)$ to denote the subpath connecting these points. The space X is *path-connected* if there exists a path between any pair of points $x, y \in X$.

A *homotopy* of paths is a family of maps $f_t : [0, 1] \to X$, $t \in [0, 1]$, such that the associated map $F : [0, 1] \times [0, 1] \to X$ given by $F(s, t) = f_t(s)$ is continuous, and the endpoints $f_t(0) = x_0$ and $f_t(1) = x_1$ are independent of t. The paths f_0 and f_1 are called *homotopic*. The relation of homotopy on paths with fixed endpoints is an equivalence relation and we use $[f]$ to denote the homotopy class of the curve f under this relation. The set $[f]$ of loops in X at the basepoint x_0 forms a group under path composition, called the *fundamental group* of X at the basepoint x_0. The space X is *simply connected* when it is path-connected and its fundamental group is trivial. For example, a subspace X of $\mathbb{E}^2$ is simply connected if and only if it has the same homotopy type as a 2-disc.

We now turn to some geometric properties of paths in $\mathbb{E}^2$. The distance $d(x, y)$ between points $x, y \in X$ is the length of a shortest (x, y)-path in X. When restricting ourselves to $R \subset X$, we use $d_R(x, y)$ to denote the distance between these points in the subdomain. We will frequently consider a subdomain R enclosed by two simple (u, v)-paths Π_1, Π_2. We denote such a set as $R[\Pi_1, \Pi_2] \subset X$.

For a C^2 path $\gamma : [0, T] \to \mathbb{E}^2$, its curvature at $\gamma(t)$ is defined as $\kappa(t) = \pm \frac{||\gamma'(t) \times \gamma''(t)||}{||\gamma'(t)||^3}$, with the sign positive if the tangent turns counterclockwise, and negative if the tangent turns clockwise. The smoothness of a piecewise analytic curve $\gamma : [0, 1] \to \mathbb{E}^2$ ensures that its absolute curvature is bounded at its nonsingular points. If γ is piecewise C^2 and

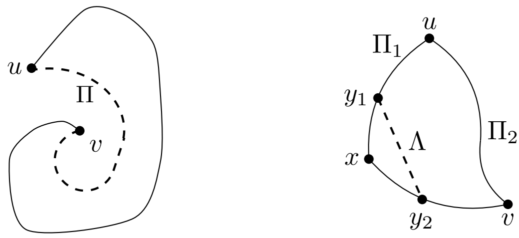


Figure 1: (a) A piecewise convex (u, v)-path Π. (b) Shortcutting a non-convex (u, v)-path Π_1.

continuous, with $t_0 < t_1 < \cdots < t_n$ as the preimages of the singular points, then its *total curvature* is

$$\kappa_{\text{total}}(\gamma) = \sum_{i=1}^{n} \int_{t_{i-1}}^{t_1} \kappa(t)dt + \sum_{i=1}^{n} \theta_i$$

where θ_i is the exterior angle at $\gamma(t_i)$, and $\theta_n = 0$ when $\gamma(t_0) \neq \gamma(t_n)$. This brings us to the celebrated Gauss-Bonnet Theorem which relates the total curvature of a closed curve with the Euler characteristic of its enclosed region. In our setting, the Euler characteristic equals $1 - k$, where k is the number of obstacles in the region R.

Theorem 1.3. *[Gauss-Bonnet Theorem, cf. [12]] Given a compact region $R \subset \mathbb{E}^2$ with boundary ∂R, we have*

$$\kappa_{\text{total}}(\partial R) = 2\pi\chi(R),$$

where $\chi(R)$ is the Euler characteristic of R.

We use the Gauss-Bonnet Theorem to understand the effect of obstacles on shortest paths. In particular, we will consider pairs of paths Π_1, Π_2 with shared endpoints u, v. These paths will be piecewise analytic (so that they have a finite number of singular points). Our goal is to prove that if the shortest (u, v)-path is not unique, then each shortest path must touch an obstacle in the given region. We begin with a definition of convexity, which we define for the broader family of piecewise C^2 smooth curves; an example is shown in Figure 1(a).

Definition 1.4. Let $\Pi : [0, 1] \to \mathbb{E}^2$ be a piecewise C^2 smooth curve in $\mathbb{E}^2$. Then Π is *convex* when the following holds for any point $x \in \Pi \backslash \{u, v\}$:
(a) If Π is C^2 smooth at x, then the curvature at x is nonpositive;
(b) If Π is not C^2 smooth at x, then the tangent line at x turns clockwise by an angle $0 \leq \theta \leq \pi$.

The definition for a *concave* curve is similar, but the curvature must be nonnegative and the tangent line must turn counterclockwise. Note that if the (u, v)-curve Π is convex, then the (v, u)-curve Γ given by $\Gamma(t) = \Pi(1 - t)$ is concave.

Lemma 1.5. *Let Π_1 and Π_2 be two piecewise analytic (u, v)-paths with $\Pi_1 \cap \Pi_2 = \{u, v\}$ and such that $R[\Pi_1, \Pi_2]$ lies to the left of Π_1. If the curve Π_1 is a shortest (u, v)-path in $R[\Pi_1, \Pi_2]$ and touches no obstacle inside $R[\Pi_1, \Pi_2]$, then Π_1 is convex in $R[\Pi_1, \Pi_2]$.*

Note that the curve Π_1 does not to need to be a straight line: see Figure 1(a) for an example. Similar convex bounding curves will arise as the pursuers remove obstacles from the evader's reach.

Proof. We prove the lemma by contradiction. Suppose that there exists an $x \in \Pi \backslash \{u, v\}$ where the convexity of Π_1 in $R[\Pi_1, \Pi_2]$ is violated. Either (a) Π_1 is C^2 smooth at x, but the curvature at x is positive, or (b) Π_1 is not C^2 smooth at x, and the tangent line turns counterclockwise by an angle $0 < \theta < \pi$ at x, creating a non-convex corner. Let d_0 denote the minimum distance between Π_1 and any obstacle $O \in R[\Pi_1, \Pi_2]$ with $\Pi_1 \cap \partial O = \emptyset$.

Suppose that the curvature at x is positive, see Figure 1(b). There must be a C^2 subpath Π_x between y_1 and y_2 of Π_1 around x with positive curvature. Using the lower bound d_0 on the separation between Π_1 and any obstacles inside $R[\Pi_1, \Pi_2]$ and Π_2, we may take y_1, y_2 to be close enough so that the line segment Λ connecting y_1 and y_2 lies inside $R[\Pi_1, \Pi_2]$ and does not encounter any obstacles. Replacing Π_x with Λ creates a path that is strictly shorter than Π_1, contradicting the minimality of Π_1.

Next suppose there is a non-convex corner at x. By an analogous argument to the previous case, we can create a short-cut Λ around x to make a shorter path than Π_1, a contradiction. □

Lemma 1.6. *Let Π_1, Π_2 be two (u, v)-paths with $\Pi_1 \cap \Pi_2 = \{u, v\}$. Suppose that Π_1 is a convex and piecewise analytic (u, v)-path in $R[\Pi_1, \Pi_2]$, and let Π_2 be a convex and piecewise analytic (v, u)-path in $R[\Pi_1, \Pi_2]$. Then Π_1 and Π_2 are both straight lines connecting u, v.*

Proof. Let $Q = Q[\Pi_1, \Pi_2]$ be the simply connected, closed region between Π_1 and Π_2 (so we ignore all obstacles in $\mathcal{D}$). The concatenation of $\Pi_1(u, v)$ and $\Pi_2(v, u)$ is a loop ∂Q bounding the simply connected region Q. By the Gauss-Bonnet Theorem 1.3, the total curvature along ∂Q equals $2\pi\chi(Q) = 2\pi$. We decompose the total curvature of ∂Q as the sum of total curvature of Π_1 and Π_2 respectively, and the exterior angles at u and v. Because of convexity, both Π_1 and Π_2 have total curvature no greater than 0. As for the two angles at u, v, neither can exceed π. Therefore the total curvature of the loop does not exceed 2π, and could only achieve 2π when $\kappa_{\text{total}}(\Pi_1) = \kappa_{\text{total}}(\Pi_2) = 0$. Therefore, Π_1 and Π_2 are both straight lines connecting u and v. □

Lemma 1.7. *Suppose Π_1, Π_2 are two shortest (u, v)-paths in the region $R = R[\Pi_1, \Pi_2]$, with $\Pi_1 \cap \Pi_2 = \{u, v\}$. Then each of Π_1, Π_2 touches at least one obstacle in R.*

Proof. Suppose that the conclusion is false. Without loss of generality, Π_1 does not touch any obstacles in R. By Lemma 1.5, the (u, v)-path Π_1 is convex in R. Let Q be the simply connected region obtained by removing the obstacles in R. We have $l(\Pi_1) = l(\Pi_2)$, so they are both shortest (u, v)-paths in the simply connected environment Q. Therefore Π_2 is also convex by Lemma 1.5, if parameterized as a path from v to u. By Lemma 1.6, Π_1 and Π_2 are both straight lines connecting u, v, which contradicts $\Pi_1 \cap \Pi_2 = \{u, v\}$. This proves that when there is more than one shortest (u, v)-path, each of these paths must touch an obstacle inside R. □

This concludes our topological and geometric preliminaries.

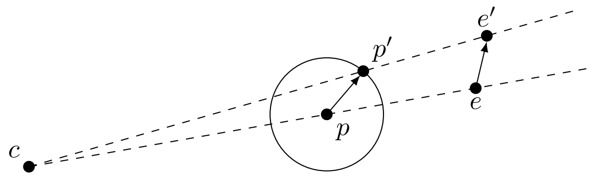


Figure 2: Lion's strategy in $\mathbb{E}^2$. On each move, the pursuer moves on the line segment connecting the center c to the evader, and increases her distance from c.

2 Lion's Strategy in a CAT(0) space

In this section, we describe a winning strategy for a single pursuer in a compact CAT(0) domain, and prove Theorem 1.1. Our strategy generalizes *lion's strategy* for pursuit in $\mathbb{E}^2$, introduced by Sgall [21]. This strategy was adapted for pursuit in polygonal regions by Isler, Kannan and Khanna [14]. Their adaptation relies heavily on the vertices of the polygon P and gives a capture time of $n \cdot \mathrm{diam}(P)^2$, where n is the number of vertices of P. We give a version of lion's strategy that succeeds in any compact CAT(0) domain $\mathcal{D}$ (including polygons) with capture time bounded by $\mathrm{diam}(\mathcal{D})^2$.

Sgall's version of lion's strategy proceeds as follows. Fix a point c as the center of our pursuit, see Figure 2. The pursuer starts at $p = c$ and the evader starts at some point e. On her first move, the pursuer moves directly towards e along the line ce. Considering a general round, the pursuer will be on the line segment between c and e prior to the evader move. After the evader moves to $e' \in B(e, 1)$, the pursuer looks at the circle centered at p with radius ϵ. If e is inside this circle, then the pursuer can capture the evader. Otherwise, this circle intersects the line segment ce' at two points. The pursuer moves to the point p' that is closer to e.

Lemma 2.1 (Sgall [21])**.** *A pursuer using lion's strategy in $\mathbb{E}^2$ re-establishes her location on the line segment between c and the evader. Furthermore, if the evader moves from e to e' and the pursuer moves from p to p' then $d(c,p')^2 \geq d(c,p)^2 + 1$.*

Before generalizing lion's strategy, we introduce of the basics of a CAT(0) geometry; see [10] for a thorough treatment. A complete metric space (X, d) is a *geodesic space* when there is a unique path $\Pi(x, y)$ whose length is the metric distance $d(x, y)$. This path $\Pi(x, y)$ is called a geodesic (or shortest path). A *triangle* $\triangle xyz$ between three points $x, y, z \in X$ is the triple of geodesics $\Pi(x, y), \Pi(y, z), \Pi(z, x)$. To each $\triangle xyz \in X$, we associate a *comparison triangle* $\triangle \tilde{x}\tilde{y}\tilde{z} \subset \mathbb{E}^2$ whose side lengths in $\mathbb{E}^2$ are the same as the lengths of the corresponding geodesics in X. The complete geodesic metric space (X, d) is CAT(0) when no triangle in X has a geodesic chord that is longer than the corresponding chord in the comparison triangle. In other words, pick any triangle $\triangle xyz$ and any points $u \in \Pi(x, y)$ and $v \in \Pi(y, z)$. Let $\tilde{u} \in \tilde{x}\tilde{y}$ and $\tilde{v} \in \tilde{y}\tilde{z}$ be the corresponding points, chosen distancewise on the edges $\tilde{x}\tilde{y}$ and $\tilde{y}\tilde{z}$. If the space X is CAT(0) then $d_X(u, v) \leq d_{\mathbb{E}^2}(\tilde{u}, \tilde{v})$. Colloquially, this is called the "no fat triangles" property, since it also implies that the sum of the angles of the triangle is not greater than π.

Our CAT(0) lion's strategy generalizes the *extended lion's strategy* for polygons of Isler et al. [14]. The pursuer starts at a fixed center point c and her goal is to stay on the shortest path $\Pi(c, e)$ at all times. In particular, assume that p_t is on the shortest path $\Pi(c, e_t)$ and that the evader moves from e_t to e_{t+1}. If $d(p_t, e_{t+1}) \leq 1$ then the purser responds by capturing the evader. Otherwise, the pursuer draws the unit circle C centered at p_t and moves to the point in $C \cap \Pi(c, e_{t+1})$ that is closest to e_{t+1}.

Lemma 2.2 (Lion's Strategy)**.** *A pursuer using lion's strategy in a* $\textsc{cat}(0)$ *space* (X, d) *re-establishes her location on the line segment between* c *and the evader. Furthermore, if the evader moves from* e *to* e' *and the pursuer moves from* p *to* p' *then* $d(c, p')^2 \geq d(c, p)^2 + 1$.

Proof. Suppose that $p \in \Pi(c, e)$ and then the evader moves to e'. Consider the $\textsc{cat}(0)$ triangle $\triangle cee'$ and its comparison triangle $\triangle \tilde{c}\tilde{e}\tilde{e}'$ in $\mathbb{E}^2$. Look at the corresponding $\mathbb{E}^2$ pursuit-evasion game with the pursuer at $\tilde{p} \in \tilde{c}\tilde{e}$. By Lemma 2.1, the pursuer can move to a point $\tilde{p}' \in \tilde{c}\tilde{e}'$ such that $d_{\mathbb{E}^2}(\tilde{c}, \tilde{p}')^2 \geq d_{\mathbb{E}^2}(\tilde{c}, \tilde{p})^2 + 1$. Since there are no fat triangles in X, we have $d_X(p, p') \leq d_{\mathbb{E}^2}(\tilde{p}, \tilde{p}')$ where $p' \in \Pi(c, e')$ is the point corresponding to $\tilde{p}'$. Therefore, in our original game, the pursuer can move to the point $p' \in \Pi(c, e')$, which satisfies $d_X(c, p')^2 \geq d_X(c, p)^2 + 1$. □

Finally, we prove Theorem 1.1: lion's strategy succeeds in a $\textsc{cat}(0)$ domain.

Proof of Theorem 1.1. Consider a pursuit-evasion game in the compact $\textsc{cat}(0)$ domain $\mathcal{D}$. Pick any $c \in \partial D$ as our center point. Using lion's strategy, the pursuer increases her distance from c with every step by Lemma 2.2, so she captures the evader after at most $\mathrm{diam}(\mathcal{D})^2$ rounds. □

3 Minimal Paths and Guarding

The key to our pursuit strategy is the ability of one pursuer to *guard* a shortest path, meaning that the evader cannot cross this path without being caught by a pursuer. When this shortest path splits the domain into two subdomains, the evader will be trapped in a smaller region. We refer to this region as the *evader territory*. In fact, we will be able to also guard a "second shortest path" when the shortest path is already guarded. The definitions and lemmas in this section are adaptations of the minimal path formulations introduced in [8] and further developed in [6]. Recall that we use $d(x, y)$ to denote the length of a shortest (x, y)-path in $\mathcal{D}$. In addition, we will use $\mathring{X}$ and $\overline{X}$ to denote the interior and the closure of a set X, respectively.

Definition 3.1. Let $X \subset \mathcal{D}$ be a path-connected region. The simple path $\Pi \subset X$ is *minimal in* X when for any $y_1, y_2 \in \Pi$ and any $z \in X$, we have $d_\Pi(y_1, y_2) \leq d_X(y_1, z) + d_X(z, y_2)$.

Definition 3.2. Let $Z \subset X$ and let Π be a minimal (u, v)-path in Z where $u, v \in \partial Z$. Then the *path projection with anchor* u is the function $\pi : Z \to \Pi$ defined as follows. If $d_Z(u, z) < d_Z(u, v) = d_\Pi(u, v)$, then $\pi(z)$ is the unique point $x \in \Pi$ with $d_\Pi(u, x) = d_Z(u, z)$. For the remaining $z \in Z \backslash \Pi$, we set $\pi(z) = v$.

We make a few observations. If $X = \mathcal{D}$, then a shortest (x, y)-path is always a minimal path in $\mathcal{D}$. In this case, we can define the path projection $\pi : \mathcal{D} \to \Pi$. When $X \subsetneq \mathcal{D}$, we might have $d_X(x_1, x_2) > d(x_1, x_2)$. In this case, a shortest path in X will be minimal in X, but it will not be minimal in $\mathcal{D}$. Next, we show that a path projection is non-expansive, meaning that distances cannot increase.

Lemma 3.3. *Let $\pi : Z \to \Pi$ be a path projection onto a minimal path in Z. Then $d_\Pi(\pi(z_1), \pi(z_2)) \leq d_Z(z_1, z_2)$ for all $z_1, z_2 \in Z$.*

Proof. The proof is a straight-forward argument using the triangle inequality. We consider the case $z_1, z_2 \in Z$ with $d(u, z_1) \leq d(u, z_2) \leq d(u, v)$. We have

$$\begin{aligned} d_Z(z_2, z_1) \geq d_Z(z_2, u) - d_Z(z_1, u) &= d_\Pi(\pi(z_2), u) - d_\Pi(\pi(z_1), u) \\ &= d_\Pi(\pi(z_2), \pi(z_1)). \end{aligned}$$

The other non-trivial cases are argued similarly. □

A single pursuer can turn a minimal path Π into an impassable boundary: once the pursuer has attained the position $p = \pi(e)$, the evader cannot cross Π without being captured in response. The proof of the following lemma is similar to the analogous result in [6], but we include this brief argument for completeness.

Lemma 3.4 (Guarding Lemma)**.** *Let $\pi : X \to \Pi$ be a path projection onto the minimal (u, v)-path $\Pi \subset X$. Consider a pursuit-evasion game between pursuer p and evader e in the environment X.*

(a) After $O(\mathrm{diam}(X))$ turns, the pursuer can attain $p^t = \pi(e^{t-1})$.

(b) Thereafter, the pursuer can re-establish $p^{s+1} = \pi(e^s)$ for all $s \geq t$.

(c) If the evader moves so that a shortest path from e^{s-1} to e^s intersects Π, then the pursuer can capture the evader at time $s + 1$.

Proof. To achieve (a), the pursuer moves as follows. First, p travels to u, reaching this point in $O(\mathrm{diam}(X))$ turns. Next, p traverses along Π until first achieving $d(u, p^i) \leq d(u, \pi(e^{i-1})) < d(u, p^i) + 1$. If $p^i = \pi(e^{i-1})$ then we are done. Otherwise, when the evader moves, we either have $d(u, p^i) - 1 < d(u, \pi(e^i)) \leq d(u, p^i) + 1$ or $d(u, p^i) + 1 < d(u, \pi(e^i)) < d(u, p^i) + 2$ by Lemma 3.3. In the former case, p can move to $\pi(e^i)$ in response, achieving her goal. In the latter case, p will increase her distance from u by one unit, re-establishing $d(u, p^{i+1}) \leq d(u, \pi(e^i)) < d(u, p^{i+1}) + 1$. This latter evader move can only be made $O(\mathrm{diam}(X))$ times, after which the pursuer acheives $p = \pi(e)$.

Next, suppose that $p^s = \pi(e^{s-1})$ and that $e^{s-1} \in X \backslash \Pi$. The pursuer can stay on the evader projection by induction since

$$d_\Pi(p^s, \pi(e^s)) = d_\Pi(\pi(e^{s-1}), \pi(e^s)) \leq d(e^{s-1}, e^s) \leq 1,$$

so (b) holds. As for (c), suppose that a shortest path from e^{s-1} to e^s includes the point $y \in \Pi$. Then

$$d(p^s, e^s) \leq d_\Pi(\pi(e^{s-1}), y) + d(y, e^s) \leq d(e^{s-1}, y) + d(y, e^s) = d(e^{s-1}, e^s) = 1.$$

Therefore the pursuer can capture the evader on her next move. □

The Guarding Lemma is the cornerstone of our pursuer strategy. When the pursuer moves as specified in the lemma, we say that she *guards* the path Π. In Section 4, our pursuers will repeatedly guard paths chosen to reduce the number of obstacles in the evader territory. Once the evader is trapped in a region that is obstacle-free, we have reached the endgame of the pursuit.

Lemma 3.5. *Suppose that the evader is located in a simply connected region R whose boundary consists of subcurves of the original boundary $\partial\mathcal{D}$ and two guarded paths Π_1 and Π_2. If the evader remains in R, then the third pursuer can capture him in finite time. If the evader tries to leave the region through Π_1 or Π_2, then he will be captured by the guarding pursuer.*

Proof. By Lemma 3.4, if the evader tries to leave this region, he will be caught by either p_1 or p_2. If the evader remains in this component, then Theorem 1.1 guarantees that pursuer p_3 captures the evader in a finite number of moves. □

The remainder of this section is devoted to identifying guardable paths that touch obstacles. Guarding such a path will neutralize the threat posed by the obstacle. First, we consider the case when p_1 guards the unique shortest (u, v)-path Π_1 that touches an obstacle O in the evader region. The objective of p_2 is to guard another (u, v)-path Π_2 of a different homotopy type. This path can be guarded even when Π_2 is longer than Π_1, provided that any path shorter than Π_2 also intersects Π_1.

Lemma 3.6. *Suppose that the evader territory $R = R[\Pi_1, \Delta]$ is bounded by the unique (u, v)-shortest path Π_1 and another boundary curve Δ. Furthermore, suppose that Π_1 touches an obstacle O and that Π_1 is guarded by p_1. Then we can find a (u, v)-path $\Pi_2 \subset R$ with the following properties: (a) $O \subset R[\Pi_1, \Pi_2]$, so that the homotopy type of Π_2 is different than that of Π_1; and (b) Π_2 is guardable by p_2, provided that Π_1 remains guarded by p_1.*

A naive attempt to find such a path is to pick some $x \in \Pi_1 \cap \bar{O}$ and find a shortest path that does not include the point x. However, $R\backslash\{x\}$ is not a closed set, which would complicate our argument. Furthermore, it could be that the next shortest path includes x without using this point as a shortcut around the obstacle O, as shown in Figure 3(b).[1] We handle both problems by removing a small and well-chosen open region A near x, rather than removing the point x. The delicate choice of A relies on two consequences of our piecewise analytic boundary: the finite upper bound $\kappa_{\max} > 0$ on the curvature and the minimum distance $d_{\min} > 0$ between boundary components.

Proof. First, suppose that $\Pi_1 \cap \partial O$ includes a continuous subcurve $C \subset \Pi_1$. Pick $x \in \mathring{C}$ and $\epsilon > 0$ so that $B(x, \epsilon) \cap R \subset \overline{O}$. Let $R' = R\backslash\mathring{B}(x, \epsilon)$, which effectively absorbs the obstacle into the boundary, see Figure 3 (a). The region R' is closed, so there is a well-defined shortest (u, v)-path $\Pi_2 \subset R'$. The path Π_2 is guardable in R', and therefore it is guardable by p_2 in R, provided that p_1 guards Π_1. Indeed, any shorter path in R must go through the point x, so Lemma 3.4 guarantees that an evader using such a path will be caught by p_1. Finally, we note that Π_1, Π_2 have distinct homotopy types because $O \subset R[\Pi_1, \Pi_2]$.

Next, we consider the case where $\Pi_1 \cap \bar{O}$ contains no continuous curves: we just focus on the first point $x \in \Pi_1 \cap \bar{O}$ encountered as we move from u to v. Locally around x, the path Π_1 and the boundary ∂O separate R into two external regions (outside of Π and inside ∂O) and two internal regions, see Figure 3 (b). The shortest path Π_1 does not self-intersect, so locally near x, this path consists of two curves meeting at x, creating an

[1] We note that this unusual circumstance is overlooked in [8], where it can occur during their minimal path strategy. This case can be easily handled in a manner analogous to our approach, but based on the visibility graph of their environment.

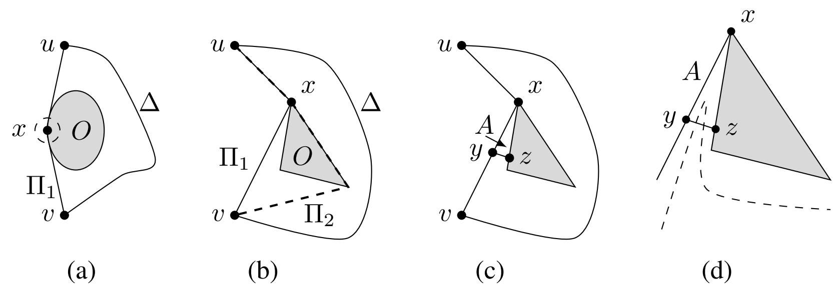


Figure 3: Finding the second shortest path. (a) When $\Pi_1 \cap \partial O$ contains a curve, we can remove a small open ball. (b) The shortest path Π_1 touches obstacle O at x. The second shortest path Π_2 goes around O, but includes the point x. (c) Finding Π_2 requires removing a small, open, triangular set A between Π_1 and O, and then finding the shortest (u, v)-path in the closed set $R \backslash A$. (d) Any path that crosses the line segment yz can be short-cut.

interior angle smaller than 2π. Therefore, at least one of the two interior angles made by Π_1 and the obstacle tangent line(s) at x is strictly less than π. This local region is where we will remove our triangular open set.

Without loss of generality, suppose that the subpath $\Pi_1(x, v)$ helps to bound this local region. Take points $y \in \Pi_1(x, v)$ and $z \in \partial O$ (traveling counterclockwise from x) such that $0 < d_{\Pi_1}(x, y) = d_{\partial O}(x, z) < d_{\min}/2$, and the angle $\angle yxz < \pi$. Let $A \subset B(x, d_{\min})$ be the closed region with endpoints (x, y, z), where the third curve is the unique shortest (y, z)-path Γ, see Figure 3 (c). The bound on the absolute curvature $\kappa_{\max}$ allows us to choose our y, z so that the region A is essentially triangular. Since $d_{\min}$ is the minimum distance between obstacles, A is obstacle-free, so Γ is a straight line segment.

We remove the relatively open set $A' = A \backslash \Gamma$ from our domain. We then find the shortest (u, v)-path Π_2 in the closed set $R = R \backslash A'$. We claim that p_2 can guard Π_2 in R, provided that p_1 guards Π_1. As in the previous case, the shorter paths that go through x are not available to the evader. Therefore, we must show that any path in R that visits A' is longer than Π_2. Such a path Λ must enter and leave A' through Γ, say at points a, b, see Figure 3 (d). However, the subpath $\Lambda(a, b)$ can be replaced with the unique shortest path $\Gamma(a, b)$ without changing the homotopy type, a contradiction. Once again, Π_1, Π_2 have distinct homotopy types because $O \subset R[\Pi_1, \Pi_2]$. □

We refer to the paths Π_1, Π_2 from Lemma 3.6 as a *guardable pair*. Provided that the shortest (u, v)-path Π_1 is guarded, the "second shortest (u, v)-path" Π_2 can also be guarded. The following corollary is a variation of the lemma.

Corollary 3.7. *Let Π_1, Π_2 be (u, v)-paths that are guarded by p_1, p_2, respectively. Suppose that for $i = 1, 2$, the path Π_i touches an obstacle O_i, where $O_1 \neq O_2$. Then we can find a path Π_3 with the following properties: (a) the homotopy type of Π_3 is different than the homotopy types of Π_1 and Π_2, and in particular, $O_i \in R[\Pi_i, \Pi_3]$ for $i = 1, 2$; and (b) Π_3 is guardable by p_3, provided that Π_1, Π_2 remain guarded by p_1, p_2.*

Proof. The proof is similar to the proof of Lemma 3.6. This time, we must remove an open set A near $x \in \Pi_1 \cap O_1$ and an open set B' near $y \in \Pi_2 \cap O_2$. We then find Π_3 in

$R \backslash (A' \cup B')$. □

This concludes our search for guardable paths that touch obstacles. The next section lays out the three-pursuer strategy for capturing the evader in a two-dimensional domain.

4 Shortest Path Strategy

In this section, we prove Theorem 1.2: three pursuers can capture an evader in a two-dimensional compact domain with piecewise analytic boundary. We adapt the the shortest path strategy of Bhaudaria et al. [8] to our more general setting. In particular, our guardable path lemmas from Section 3 supplant their use of polygon vertices to find successive paths. Their algorithm guarantees success by reducing the number of polygon vertices in the evader territory. Instead, we keep track of the threat level of obstacles to argue that the evader becomes trapped in a simply connected region.

Our pursuit proceeds in rounds. At the start of a round, at most two pursuers guard paths. The third pursuer moves to guard another path with the goal of eliminating obstacles from the evader territory. This third path will either be a shortest path, or it will create a guardable pair with the currently guarded path(s). Once this third path is guarded, the evader is trapped in a smaller region, which releases one of the other pursuers to continue the process. This continues until the evader is trapped in a simply connected region, where the free pursuer can capture the evader by Lemma 3.5.

We start by showing that the boundary of the evader territory is always piecewise analytic, after recalling two definitions. First, the endpoints of a line segment touching the boundary $\partial\mathcal{D}$ are called *switch points*. Second, a point x is an *accumulation point* (or limit point) of a set S when any open set containing x contains an infinite number of elements in S. We make use of the following result about the interaction of a geodesic with the boundary of the domain.

Theorem 4.1 (Albrecht and Berg [2]). *If M is a 2-dimensional analytic manifold with boundary embedded in $\mathbb{E}^2$, and γ is a geodesic in M, then the switch points on γ have no accumulation points.*

We restrict ourselves to analytic boundary, instead of smooth (C^2, or even C^∞) boundary, to avoid some potentially pathological behavior of geodesics. For example, Albrecht and Berg [2] construct a geodesic in C^∞ environment, that achieves a Cantor set of positive measure as the accumulation of switch points. This unusual geometry hampers our ability to confine the evader in a well-defined connected component. Theorem 4.1 ensures that our new evader territory will be bounded by piecewise analytic curves.

Lemma 4.2. *Let $\mathcal{D}$ be a compact domain with piecewise analytic boundary. If Π is a shortest path in $\mathcal{D}$, then Π is piecewise analytic. Furthermore, if $\mathcal{D}\backslash\Pi$ is disconnected, then it contains finitely many connected components, and the boundary of each connected component is piecewise analytic.*

Proof. Let $B \subset \partial(\Pi \backslash \partial\mathcal{D})$ be the set of switch points. We claim that B is finite. Otherwise, there must be an accumulation point of B since $l(\Pi)$ is finite, contradicting Theorem 4.1. Now we can use the finite set B as endpoints to partition Π so that each subcurve is either in the boundary $\partial\mathcal{D}$, or in the interior $\mathring{\mathcal{D}}$. Since any shortest path in the interior $\mathring{\mathcal{D}}$ must be a line segment, the path Π is piecewise analytic. For each connected component of $\mathcal{D}\backslash\Pi$, its boundary is a subset of $\Pi \cup \partial\mathcal{D}$, hence is piecewise analytic. □

In order to prove Theorem 1.2, we will show that our pursuit succeeds in finite time. To aid in this effort, we assign a threat level to each of the k obstacles in the original domain. These threat levels will reliably decrease during pursuit. An obstacle is in one of three states: dangerous, safe, or removed. A *removed* obstacle lies outside the evader territory. A *safe* obstacle lies in the evader territory and touches a currently guarded path. This obstacle is not a threat because the evader cannot circle around the object without being captured. The remaining obstacles are *dangerous*. Finally, we say that the evader territory is dangerous if it contains at least one dangerous obstacle.

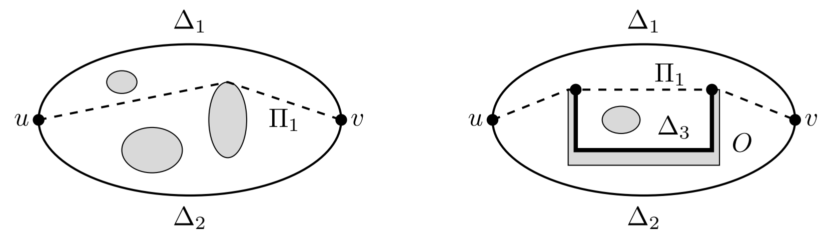
At the start of pursuit, all obstacles are dangerous. So long as there are still dangerous obstacles, a round consists of taking control of a guardable path. This effort succeeds in a finite number of moves by Lemma 3.4. We will show that after at most two rounds, either a dangerous obstacle transitions to safe/removed, or a safe obstacle transitions to removed. This is our notion of progress: after at most $2k$ rounds, the evader territory is not dangerous. From here forward, we focus on the transition of the threat levels of obstacles.

In general, our evader territory will be bounded by part of the domain boundary $\partial\mathcal{D}$ and by at most two guarded paths Π_1, Π_2. At the end of a round, the evader territory will be updated, bounded in part by updated paths Π_1', Π_2'. If these guarded paths intersect or share subpaths, then the evader is actually trapped in a smaller region by Lemma 3.4. When this is the case, we advance the endpoint(s) of our paths so that these are the only point(s) shared by our paths. This obviates the need to discuss degenerate cases.

The first round is an initialization round, so all obstacles might still be dangerous when this round completes. However, we will be able to neutralize at least one obstacle in the subsequent round. To kick off the first round, we pick points $u, v \in \partial\mathcal{D}$, chosen so that they divide the outer boundary into two curves Δ_1, Δ_2 of equal length, see Figure 4(a). Let Π_1 be a shortest (u, v)-path; if there are multiple shortest paths (in which case each touches an obstacle), then we pick one arbitrarily. Using Lemma 3.4, p_1 moves to guard Π_1. The round ends when p_1 has attained guarding position, trapping the evader in a subdomain that is bounded by Π_1 and one of Δ_1, Δ_2. The evader could be trapped in a smaller pocket region between Π_1 and a subcurve Δ_3 of an obstacle $O \subset \mathcal{D}$, see Figure 4(b). In the latter case, the obstacle O is marked as removed and we treat Δ_3 as the new outer boundary. After updating the evader territory R, any obstacle $O \not\subset R$ is marked as removed. Any obstacle $O \subset R$ that touches Π_1 or Π_2 is marked as safe.

For the remainder of the game, the evader territory is one of the following types.

- Type 0 region: A region containing no dangerous obstacles.
- Type 1 region: A dangerous three-sided region bounded by a (u, v)-shortest path Π_1, a (u, w)-shortest path Π_2 and a (v, w)-path $\Delta \subset \partial D$. No obstacle touches both Π_1, Π_2.
- Type $1'$ region: A dangerous two-sided region bounded by a (u, v)-shortest path Π_1 and a (u, v)-path $\Delta \subset \partial\mathcal{D}$. We treat this as a special case of the previous type, where Π_2 consists of the single point $w = u$. This point is on Π_1, so it is guarded by p_1.
- Type 2 region: A dangerous two-sided region bounded by (u, v)-paths Π_1, Π_2, each of which touches an obstacle in the evader territory. The path Π_1 is a shortest (u, v)-path in this region. The path Π_2 might also be a shortest (u, v)-path, or it could be a "second shortest path," meaning that it is a shortest (u, v)-path among the set of (u, v)-paths that are not homotopic to Π_1. No obstacle touches both Π_1, Π_2.



(a) (b)

Figure 4: The shortest (u, v)-path Π_1 guarded in round one. (a) The path Π_1 partitions the outer boundary to subcurves Δ_1, Δ_2 of equal length. (b) The evader may be trapped in a pocket between the path Π_1 and the boundary subcurve Δ_3 of obstacle O.

- Type 3 region: a dangerous 4-sided region bounded by a (u, v)-shortest path Π_1, a (w, x)-shortest path Π_2, a (v, w)-path Δ_1 from the boundary and a (u, x)-path Δ_2 from the boundary. These vertices are arranged so that they are ordered clockwise as u, v, w, x. No obstacle touches both Π_1, Π_2.

For example, after the initialization round, the evader territory is a type $1'$ region, bounded by a guarded path and part of the boundary $\partial\mathcal{D}$. Finally, we emphasize that Lemma 4.2 ensures that the boundary of the evader region is always piecewise analytic, since it consists of sub-curves of the piecewise analytic boundary along with one or more shortest paths.

We now describe the different types of rounds. In regions of type 1, $1'$ and 2, we will always transition at least one obstacle. At the end of such a round, the evader could now be trapped in a region of any type. Type 3 rounds are slightly different. Our primary goal is to trap the evader in a type 1 region, where we will surely make progress in the subsequent round. However, it is possible to transition an object via a type 3 move (just as in the initialization round). In this case, we make immediate progress, and the evader could then be trapped in a region of any type.

First we consider type 1 regions. This also handles type $1'$ regions as a special case. We use the following lemma to identify a point $x \in \Delta$ and a shortest (u, x)-path to guard during this round.

Lemma 4.3. *Let shortest paths $\Pi_0(u, v), \Pi_1(u, w)$ and boundary path $\Delta(v, w)$ bound a type 1 environment R. If R contains obstacles, then there exists a point $x \in \Delta$ such that there are multiple shortest (u, x)-paths in R, each of which touches at least one obstacle.*

Proof. Parameterize the boundary path as $\Delta : [0, 1] \to R$. We prove the lemma by contradiction.

Suppose that for every $t \in [0, 1]$, the shortest $(u, \Delta(t))$-path Π_t is unique. Denote its length by $l(\Pi_t) = d(u, \Delta(t))$. Define the function $n(t)$ to be the number of obstacles in the region R_t bounded by Π_0, Π_t and Δ. The function $n(t)$ is well-defined by the uniqueness of each Π_t. Furthermore, $n(0) = 0$, and $n(1) > 0$, so there must be a jump discontinuity somewhere in $[0, 1]$. Let $s = \inf\{t \in [0, 1] : n(t) > 0\}$.

Case 1: $n(s) > 0$ where $0 < s \leq 1$. (Recall that $n(0) = 0$, so $s > 0$.) Let Γ be a shortest $(u, \Delta(s))$-path that is of the same homotopy class as Π_0. The choice of s and

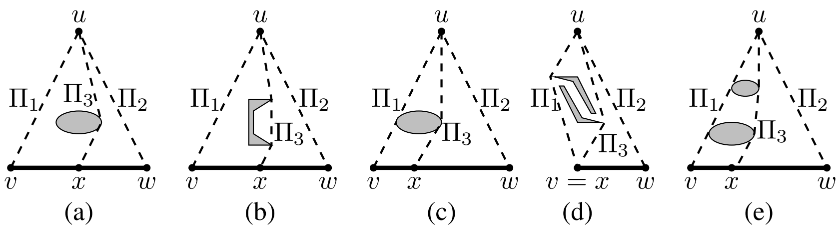


Figure 5: Representative examples of a type 1 move, where we transition to (a) a type 1 region, (b) a type 1 or type 1′ region, (c) a type 1 or a type 3 region, (d) a type 1 or type 2 region, (e) a type 1 or type 3 region.

the uniqueness of Π_s guarantee that $l(\Gamma) > l(\Pi_s)$. Also no obstacles are contained in the region bounded by Π_0, Γ and $\Delta(0, s)$. By the definition of Γ, for all $t \in [0, s)$, we have $\ell(\Gamma) \leq \ell(\Gamma_t) + \ell(\Delta(t, s))$. By the continuity of $l(\Pi_t) = d(u, \Delta(t))$ with respect to $t \in [0, s)$, we have $l(\Gamma) \leq l(\Pi_t) + l(\Delta(t, s))$, so

$$l(\Gamma) \leq \lim_{t \to s^-} \big(l(\Pi_t) + l(\Delta(t, s))\big) = l(\Pi_s) + 0 = l(\Pi_s).$$

This contradicts $l(\Gamma) > l(\Pi_s)$, so Π_s is not the unique shortest $(u, \Delta(s))$-path.

Case 2: $n(s) = 0$, where $0 \leq s < 1$. Let $\{s_i\}$ be an infinite sequence $s_i \to s^+$, such that $n(s_i) > 0$ for all i. There are finite number of obstacles, so by taking a subsequence if necessary, we can assume that the shortest paths $\{\Pi_{s_i}\}$ are of the same homotopy class. Let Γ be the shortest $(u, \Delta(s))$-path of this homotopy class. We have $l(\Gamma) \leq l(\Pi_{s_i}) + l(\Delta(s, s_i))$ for all i, and therefore

$$l(\Gamma) \leq \lim_{i \to \infty} \big(l(\Pi_{s_i}) + l(\Delta(s, s_i))\big) = l(\Pi_s),$$

where the limit holds by the continuity of distances in the region. However, this contradicts the uniqueness of Π_s which would require $l(\Pi_s) < l(\Gamma)$.

Thus we can conclude that are multiple shortest (u, x)-paths. By Lemma 1.7, each of these paths touches at least one obstacle. □

Having found the next path to guard, we now prove that we transition an object during a type 1 move.

Lemma 4.4. *Suppose that the evader is trapped in a type 1 (or type* $1'$*) region. Then the third pursuer can guard a path that transitions an obstacle state.*

Proof. By Lemma 4.3, there is some point $x \in \Delta$ with multiple shortest (u, x)-paths, each of which touches an obstacle. Let Π_3 be one of these shortest (u, x)-paths. If $x = v$ then we take a path $\Pi_3 \neq \Pi_1$. Similarly if $x = w$ we choose $\Pi_3 \neq \Pi_2$. When $x \notin \{v, w\}$, we can choose Π_3 arbitrarily from the collection of (u, x)-shortest paths. Pursuer p_3 moves to guard Π_3, which traps the evader in either $R[\Pi_1, \Pi_3]$ or $R[\Pi_3, \Pi_2]$. Any obstacles in the other region are marked as removed.

Without loss of generality, let $O \subset R[\Pi_1, \Pi_3]$ be an obstacle touched by Π_3, see Figure 5(a). Suppose that prior to p_3 guarding Π_3, the object O was dangerous. If $e \in R[\Pi_3, \Pi_2]$

then O transitions to removed. If $e \in R[\Pi_1, \Pi_3]$ then O transitions to safe. However, we may be in a more advantageous position, shown in Figure 5(b): the evader could be trapped in a pocket between obstacle O and path Π_3. In this case, the new evader territory is type $1'$ and the obstacle O is marked as removed, since it is now part of the outer boundary of the evader territory.

Next, suppose that O was already safe, touched by Π_1. If $e \in R[\Pi_2, \Pi_3]$ after p_3 guards Π_3, then O transitions from safe to removed. If the evader is trapped in a pocket region between O and Π_3, we proceed as in the previous case. Otherwise, we have $e \in R[\Pi_1, \Pi_3]$ and the obstacle O separates $R[\Pi_1, \Pi_3]$ into disjoint regions, as shown in Figure 5(c). The evader is trapped in one of these two subregions because both Π_1, Π_3 are guarded. Let Δ' be the subcurve of ∂O that bounds the effective evader territory. We update the evader territory appropriately, bounded by Δ' and subpaths of Π_1, Π_3, and perhaps part of Δ. The result is a region of type 1 or 3. The obstacle O is marked as removed: it is now part of the boundary. This reduces the number of safe obstacles.

When Π_3 touches multiple obstacles, each of them transitions to a lower threat level. Figures 5(d) and (e) show that we can also end up in a type 2 or 3 region, depending on the configuration of these obstacles and the location of the evader at the end of the round. □

Next, we consider a type 2 region. Such a region is bounded by (u, v)-paths Π_1, Π_2 that form a guardable pair, where Π_1, Π_2 touch safe obstacles O_1, O_2, respectively. Without loss of generality, Π_1 is a shortest (u, v)-path in the region, and Π_2 is either another shortest path, or a "second shortest path" as found in Lemma 3.6. (A type 1 move can lead to the first case. A type 2 move can lead to the second case, as we are about to see.)

Lemma 4.5. *Suppose that the evader is trapped in a type 2 region. Then the third pursuer can guard a path that transitions an obstacle state.*

Proof. Use Corollary 3.7 to find a guardable (u, v)-path Π_3 in $R[\Pi_1, \Pi_2]$ whose homotopy type is distinct from that of both Π_1, Π_2. Pursuer p_3 establishes a guarding position on Π_3. The evader is now trapped in either $R[\Pi_1, \Pi_3]$ or $R[\Pi_3, \Pi_2]$, so one of O_1, O_2 transitions from safe to removed. Furthermore, Π_3 must touch at least one obstacle in each of $R[\Pi_1, \Pi_3]$ or $R[\Pi_3, \Pi_2]$. Otherwise, Π_3 would be shorter than one of Π_1, Π_2, which contradicts the minimality of that path in $R[\Pi_1, \Pi_2]$. Depending on the configuration of the obstacles, we may be able to restrict the evader territory further. After doing so, the evader territory may be of any possible type, as shown in Figure 6 (a). □

This brings our discussion to a type 3 region, with p_1 guarding a shortest (u, v)-path Π_1 and p_2 guarding a shortest (w, x)-path Π_2. Our primary goal is to trap the evader in a type 1 region, but we might end up transitioning an obstacle instead. In the latter case, the new evader territory can be of any type, as explained below.

Lemma 4.6. *Suppose that the evader is trapped in a type 3 region. Then the third pursuer can guard a path so that either (a) the evader is trapped in a type 1 region, or (b) an obstacle transitions to a lower threat level.*

Proof. Let Π_3 be a (u, w)-minimal path. Pursuer p_3 moves to guard this path using Lemma 3.4. This traps the evader in a smaller region: without loss of generality, this region is bounded by Π_1, Δ_1, Π_3. If Π_3 does not touch any obstacles in this region, then the evader is now in a type 1 region. If Π_3 touches an obstacle O, then this obstacle transitions to

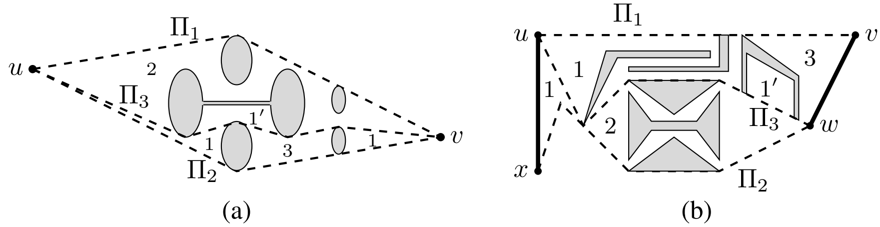


Figure 6: Examples of moves where the new guarded path Π_3 divides the region into five subregions, each identified by its type. (a) A type 2 move. (b) A type 3 move.

either safe or removed. The evader could be trapped in a region of any type, as shown in Figure 6 (b). □

We can now prove our main theorem: three pursuers can capture the evader in a two-dimensional compact domain with piecewise analytic boundary.

Proof of Theorem 1.2. The first round traps the evader in a type $1'$ region, or transitions an obstacle state. If we are in a region of type 1, $1'$ or 2 then we transition an obstacle state in the current round by Lemma 4.4 and Lemma 4.5. When we are in a type 3 region, Lemma 4.6 ensures that we either trap the evader in a type 1 region, or we transition an obstacle. With each path that we guard, the boundary of the updated evader territory is still piecewise analytic by Lemma 4.2. At the end of the round, we update the evader territory and our value for minimum obstacle separation since our new guarded path might be closer to an obstacle than the current value $d_{\min}$. (Note that the maximum boundary curvature $\kappa_{\max}$ never increases since all additions to the boundary are line segments.)

After at most $2k$ rounds, we have transitioned all k obstacles to either safe or removed. Once all obstacles have been transitioned, the evader is trapped in a simply connected type 0 region. Lemma 3.5 shows that the evader will then be caught. Each round completes in finite time, so the three pursuers win the game. The capture time upper bound of $O(2k \cdot \mathrm{diam}(\mathcal{D}) + \mathrm{diam}(\mathcal{D})^2)$ follows easily. The time required to guard any shortest path is $\mathrm{diam}(\mathcal{D})$ by Lemma 3.4 and lion's strategy completes in time $\mathrm{diam}(\mathcal{D})^2$ by Theorem 1.1. □

5 Conclusion

In this paper, we described a winning pursuer strategy for a single pursuer in a $\mathrm{CAT}(0)$ space for turn-based pursuit with capture criterion $d(p, e) = 0$. We then restricted our attention to compact domains in $\mathbb{E}^2$ with piecewise analytic boundary. We showed that three pursuers are sufficient to catch an evader in such environments. By adding a fourth pursuer for use in the final endgame, our strategy could be quickly adapted to a winning strategy in the continuous time version. However, a clever use of the two guarding pursuers during the endgame shows that three pursuers are actually sufficient: see [22] for details.

There are plenty of avenues for reseach in geometric pursuit-evasion. Pursuit-evasion results on polyhedral surfaces are an active area of current research [15, 18, 19]. For example, Klein and Suri [15] have proven that $4g + 4$ pursuers have a winning strategy on a polyhedral surface of genus g. Meanwhile, Schröder [20] has proven that at most $\lfloor 3g/2 \rfloor + 2$ pursuers are needed for a graph of genus g (meaning that such a graph can be drawn on a surface of genus g without edge crossings). It would be natural to consider this question for topological surfaces, or to start by trying to improve the bound for polyhedral surfaces. Likewise, there are a wealth of motion and sensory constraints to consider. Most of these variations of pursuit-evasion have a natural analog in a more general geometric setting.

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Spectrum, distance spectrum, and Wiener index of wreath products of complete graphs

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Abstract

We describe the adjacency matrix and the distance matrix of the wreath product of two complete graphs, and we give an explicit computation of their spectra. As an application, we deduce the spectrum of the transition matrix of the Lamplighter random walk over a complete base graph, with a complete color graph. Finally, an explicit computation of the Wiener index is given.

Keywords: Wreath product of complete graphs, adjacency matrix, distance matrix, spectrum, distance spectrum, Wiener index.

Math. Subj. Class.: 05C12, 05C50, 05C76, 05C81, 15A69

1 Introduction

The construction of new graphs starting from smaller factor graphs is a very natural and fruitful technique, largely developed in literature for its theoretical interest in several branches of Mathematics – Algebra, Combinatorics, Probability, Harmonic Analysis – but also for its practical applications. Among the standard products we find, for instance, the Cartesian product, the direct product, the strong product, the lexicographic product [22, 23, 30, 31]. More recently, the zig-zag product was introduced [29], in order to produce expanders of constant degree and arbitrary size; in [10, 14], some combinatorial and topological properties of such products, as well as connections with random walks, have been investigated.

It is worth mentioning that many of these constructions play an important role in Geometric Group Theory, since it turns out that, when applied to Cayley graphs of two finite groups, they provide the Cayley graph of an appropriate product of these groups (see [1],

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where this correspondence is shown for zig-zag products, or [15], for the case of wreath and generalized wreath products).

Spectral properties of graph products have been object of an intensive study in the last decades, both for their algebraic and combinatorial interest, and for applications to Probability, Computer Science, and Mathematical Chemistry. The spectrum of a graph is defined as the spectrum of its adjacency matrix; similarly, the distance spectrum of a graph is defined to be the spectrum of its distance matrix (see Section 2). The reader can refer, for instance, to the monograph [5] for an exhaustive treatment of spectra of graphs. We also want to mention the papers [24, 25, 34], where the distance spectrum of some graph compositions has been studied.

A related topic of research is the Wiener index, which is defined as the sum of the distances between all the unordered pairs of vertices of the graph. This index was introduced by Wiener [36] and, due to the wide range of applications, it is nowadays largely studied. In particular, it is one of the most frequently used topological indices in mathematical chemistry, as molecules can be represented by means of undirected graphs. For this reason, it has a strong correlation with many physical and chemical properties of molecular compounds, whose properties do not only depend on their chemical formula, but also on their molecular structure [13]. There exists a wide range of fields such as communication, facility location, cryptology, architecture where the Wiener index of a graph is of great interest. A large number of papers is devoted to the study of the Wiener index of graphs, sequences of graphs, products of graphs. In [12] the Wiener index of trees is investigated. In [16] the Wiener index and the related Hosoya polynomial are studied for a family of circulant graphs. See also the paper [9], where the Wiener index is studied on an increasing sequence of finite graphs, introduced in [6], and whose limit graphs have been studied in [7], which approximates the Sierpiński carpet fractal. In [17, 18] the study of Wiener index is developed for some graph compositions.

In the present paper, we focus our attention on a different kind of graph product known in literature, namely the wreath product of two graphs (see Definition 2.1). This construction is nowadays largely studied, and different generalizations have been introduced [15, 19]. Notice that this construction is interesting not only from an algebraic and combinatorial point of view, but also for its connection with Geometric group theory and Probability, via the notions of Lamplighter group and Lamplighter random walk (see, for instance, [3, 21, 32, 33, 37]). Notice that in a previous paper joint with D. D'Angeli [8], we introduced a matrix operation, called wreath product of matrices (recalled in Definition 2.2), which is a matrix analogue of the wreath product of graphs, since it provides the adjacency matrix of the wreath product of two graphs, when applied to the adjacency matrices of the factors (Theorem 2.3 below).

Let us denote by K_n the complete graph on n vertices. In this paper, the wreath product $K_n \wr K_m$ is studied. In Proposition 3.1, we describe in detail distances in $K_n \wr K_m$, and we deduce its diameter in Corollary 3.2. Moreover, in Proposition 3.4 we show that the graph $K_n \wr K_m$ is not distance-regular. After describing in detail the adjacency matrix of the wreath product $K_n \wr K_m$ of two complete graphs, we are able to explicitly compute its spectrum by using a reduction argument, allowing to reduce our computations to the study of the spectrum of smaller matrices, whose size is the cardinality of the vertex set of the first graph (Theorem 3.7); we then deduce the spectrum of the transition matrix of the Lamplighter random walk with base graph K_n and color graph K_m (Corollary 3.9). In Proposition 3.10, we provide the distance matrix of $K_n \wr K_m$, and its spectrum is determined in Theorem 3.11 by means of a second reduction argument. Finally, in Theorem 3.13, the Wiener index of the graph $K_n \wr K_m$ is computed.

Notice that the spectrum considered in the present paper concerns the "walk or switch" Lamplighter random walk. The analogous question for the so called "switch-walk-switch" Lamplighter random walk has been solved in [26, 27]. A common framework for such computations has been established in [20].

2 Preliminaries

Let $\mathcal{G} = (V, E)$ be a finite undirected graph, where V denotes the vertex set, and E is the edge set consisting of unordered pairs of type $\{u, v\}$, with $u, v \in V$. If $\{u, v\} \in E$, we say that the vertices u and v are adjacent in $\mathcal{G}$, and we use the notation $u \sim v$. A path in $\mathcal{G}$ is a sequence $u_0, u_1, \ldots, u_\ell$ of vertices such that $u_i \sim u_{i+1}$, for each $i = 0, \ldots, \ell - 1$. We say that such a path has length ℓ. The graph is connected if, for every $u, v \in V$, there exists a path $u_0, u_1, \ldots, u_\ell$ in $\mathcal{G}$ such that $u_0 = u$ and $u_\ell = v$. For a connected graph $\mathcal{G}$, we will denote by $d(u, v)$ the *geodesic distance* between the vertices u and v, that is, the length of a minimal path in $\mathcal{G}$ joining u and v. The *diameter* of $\mathcal{G}$ is then defined as $diam(\mathcal{G}) = \max_{u,v \in V}\{d(u, v)\}$.

Recall now that the *adjacency matrix* of an undirected graph $\mathcal{G} = (V, E)$ is the square matrix $A = (a_{u,v})_{u,v \in V}$, indexed by the vertices of $\mathcal{G}$, whose entry $a_{u,v}$ equals the number of edges connecting u and v. As the graph $\mathcal{G}$ is undirected, A is a symmetric matrix and so all its eigenvalues are real. The *spectrum* of $\mathcal{G}$ is then defined as the spectrum of its adjacency matrix. The *degree* of a vertex $u \in V$ is defined as $deg(u) = \sum_{v \in V} a_{u,v}$. In particular, we say that G is *regular* of degree d, or d-regular, if $deg(u) = d$, for each $u \in V$. In this case, the *normalized adjacency matrix* A' of $\mathcal{G}$ is obtained as $A' = \frac{1}{d}A$.

We recall now the definition of wreath product of graphs.

Definition 2.1. Let $\mathcal{G}_1 = (V_1, E_1)$ and $\mathcal{G}_2 = (V_2, E_2)$ be two finite graphs. The *wreath product* $\mathcal{G}_1 \wr \mathcal{G}_2$ is the graph with vertex set $V_2^{V_1} \times V_1 = \{(f, v) \mid f \colon V_1 \to V_2,\ v \in V_1\}$, where two vertices (f, v) and (f', v') are connected by an edge if:

(1) (*edges of type I*) either $v = v' =: \overline{v}$ and $f(w) = f'(w)$ for every $w \neq \overline{v}$, and $f(\overline{v}) \sim f'(\overline{v})$ in $\mathcal{G}_2$;

(2) (*edges of type II*) or $f(w) = f'(w)$, for every $w \in V_1$, and $v \sim v'$ in $\mathcal{G}_1$.

It follows from the definition that, if $\mathcal{G}_1$ is a regular graph on n_1 vertices with degree d_1 and $\mathcal{G}_2$ is regular graph on n_2 vertices with degree d_2, then the graph $\mathcal{G}_1 \wr \mathcal{G}_2$ is a $(d_1 + d_2)$-regular graph on $n_1 n_2^{n_1}$ vertices.

It is a classical fact (see, for instance, [37]) that the simple random walk on the graph $\mathcal{G}_1 \wr \mathcal{G}_2$ is the so called *Lamplighter random walk*, according to the following interpretation: suppose that at each vertex of $\mathcal{G}_1$ (the *base graph*) there is a lamp, whose possible states (or colors) are represented by the vertices of $\mathcal{G}_2$ (the *color graph*), so that the vertex (f, v) of $\mathcal{G}_1 \wr \mathcal{G}_2$ represents the configuration of the $|V_1|$ lamps at each vertex of $\mathcal{G}_1$ (for each vertex $u \in V_1$, the lamp at u is in the state $f(u) \in V_2$), together with the position v of a lamplighter walking on the graph $\mathcal{G}_1$. At each step, the lamplighter may either go to a neighbor of the current vertex v and leave all lamps unchanged (this situation corresponds to edges of type II in $\mathcal{G}_1 \wr \mathcal{G}_2$), or he may stay at the vertex $v \in \mathcal{G}_1$, but he changes the state of the lamp which is in v to a neighbor state in $\mathcal{G}_2$ (this situation corresponds to edges of type I in $\mathcal{G}_1 \wr \mathcal{G}_2$). For this reason, the wreath product $\mathcal{G}_1 \wr \mathcal{G}_2$ is also called the Lamplighter graph, or Lamplighter product, with base graph $\mathcal{G}_1$ and color graph $\mathcal{G}_2$. Also notice that the model described above is often called "walk or switch" Lamplighter random walk.

It is worth mentioning that the wreath product of graphs represents a graph analogue of the classical wreath product of groups [28], as it turns out that the wreath product of the Cayley graphs of two finite groups is the Cayley graph of the wreath product of the groups, with a suitable choice of the generating sets. In the paper [15], this correspondence is proven in the more general context of generalized wreath products of graphs, inspired by the construction introduced in [2] for permutation groups. Also notice that in [19] a different notion of generalized wreath product of graphs is presented.

In the paper [8], the following matrix construction involving wreath products is introduced. Let $\mathcal{M}_{m\times n}(\mathbb{C})$ denote the set of matrices with m rows and n columns over the complex field, and let I_n be the identity matrix of size n. We recall that the *Kronecker product* of two matrices $A = (a_{ij})_{i=1,\dots,m;j=1,\dots,n} \in \mathcal{M}_{m\times n}(\mathbb{C})$ and $B = (b_{hk})_{h=1,\dots,p;k=1,\dots,q} \in \mathcal{M}_{p\times q}(\mathbb{C})$ is defined to be the $mp \times nq$ matrix

$$A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{pmatrix}.$$

We denote by $A^{\otimes^n}$ the iterated Kronecker product $\underbrace{A \otimes \cdots \otimes A}_{n \text{ times}}$, and we put $A^{\otimes^0} = 1$.

Definition 2.2 ([8])**.** Let $A \in \mathcal{M}_{n\times n}(\mathbb{C})$ and $B \in \mathcal{M}_{m\times m}(\mathbb{C})$. For each $i = 1, \dots, n$, let $C_i = (c_{hk})_{h,k=1,\dots,n} \in \mathcal{M}_{n\times n}(\mathbb{C})$ be the matrix defined by

$$c_{hk} = \begin{cases} 1 & \text{if } h = k = i \\ 0 & \text{otherwise.} \end{cases}$$

The wreath product of A and B is the square matrix of size nm^n defined as

$$A \wr B = I_m^{\otimes^n} \otimes A + \sum_{i=1}^{n} I_m^{\otimes^{i-1}} \otimes B \otimes I_m^{\otimes^{n-i}} \otimes C_i.$$

In [8] the following theorem, which shows the correspondence between wreath products of matrices and wreath products of graphs, is proven.

Theorem 2.3. *Let A_1' be the normalized adjacency matrix of a d_1-regular graph $\mathcal{G}_1 = (V_1, E_1)$ and let A_2' be the normalized adjacency matrix of a d_2-regular graph $\mathcal{G}_2 = (V_2, E_2)$. Then the wreath product $\left(\frac{d_1}{d_1+d_2} A_1'\right) \wr \left(\frac{d_2}{d_1+d_2} A_2'\right)$ is the normalized adjacency matrix of the graph wreath product $\mathcal{G}_1 \wr \mathcal{G}_2$.*

For a finite connected graph $\mathcal{G} = (V, E)$, the *distance matrix* $D = (d_{u,v})_{u,v\in V}$ of $\mathcal{G}$ is, by definition, the square matrix indexed by the vertices of $\mathcal{G}$, such that $d_{u,v} = d(u, v)$. The matrix D is symmetric by definition, so that its spectrum is real. The spectrum of D is usually called the *distance spectrum* of the graph $\mathcal{G}$.

We complete this preliminary section by recalling the definition of Wiener index of a finite connected graph $\mathcal{G} = (V, E)$. The *Wiener index* of $\mathcal{G}$ is defined as the sum of the distances between all the unordered pairs of vertices, i.e.,

$$W(\mathcal{G}) = \frac{1}{2} \sum_{u,v\in V} d(u, v),$$

where $d(u, v)$ denotes the usual geodesic distance between u and v.

In Section 3, we will construct the adjacency matrix and the distance matrix of the graph $K_n \wr K_m$, and we will compute their spectra. Finally, we will provide an explicit computation of the Wiener index of $K_n \wr K_m$.

3 The wreath product $K_n \wr K_m$

From now on, we will focus our attention on the wreath product $K_n \wr K_m$, where $K_n = (V_n, E_n)$ is the complete graph on n vertices, that is, the graph on n vertices in which every pair of distinct vertices is connected by a unique edge. Notice that K_n is a regular graph of degree $n-1$, with diameter 1, where $d(u,v) = \begin{cases} 0 & \text{if } u = v \\ 1 & \text{if } u \neq v \end{cases}$ for each pair u, v of vertices.

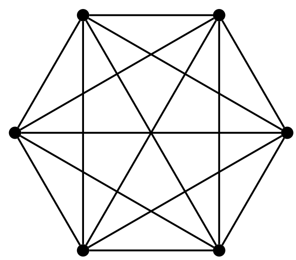

Figure 1: The complete graph K_6.

In particular, the adjacency matrix of K_n is given by $Ad_n = J_n - I_n$, where J_n denotes the uniform square matrix of size n, whose entries are all equal to 1. Moreover, it follows from Theorem 2.3 that the adjacency matrix of the graph $K_n \wr K_m$ is the matrix

$$Ad_n \wr Ad_m = I_m^{\otimes^n} \otimes Ad_n + \sum_{i=1}^{n} I_m^{\otimes^{i-1}} \otimes Ad_m \otimes I_m^{\otimes^{n-i}} \otimes C_i, \qquad (3.1)$$

with C_i as in Definition 2.2. Notice also that, by definition, $K_n \wr K_m$ is an $(n+m-2)$-regular graph on nm^n vertices. A vertex of $K_n \wr K_m$ will be usually denoted by $(y_1, \ldots, y_n)x_i$, where $y_j \in V_m$, for each $j = 1, \ldots, n$, and $x_i \in V_n$. In the lamplighter interpretation, we can think that the lamp placed at the j-th vertex x_j of K_n has color y_j, with $y_j \in V_m$, and the lamplighter is in position x_i.

Moreover, it follows from the definition of wreath product of graphs that two vertices $u = (y_1, \ldots, y_n)x_i$ and $v = (y'_1, \ldots, y'_n)x_k$ have distance 1 if either there exists a unique index $j \in \{1, \ldots, n\}$ such that $y_j \neq y'_j$ and $x_i = x_k$; or $y_j = y'_j$ for each j, and $x_i \neq x_k$ (observe that $x_i \sim x_k$ in K_n if and only if $x_i \neq x_k$, as the graph K_n is complete).

We are going to develop an explicit analysis of the variability of the distances between two vertices in the graph $K_n \wr K_m$. Let $u = (y_1, \ldots, y_n)x_i$ and $v = (y'_1, \ldots, y'_n)x_k$ be two vertices of $K_n \wr K_m$. Put $J = \{1, 2, \ldots, n\}$ and define the partition $J = J^0_{u,v} \sqcup J^1_{u,v}$ by

$$J^0_{u,v} = \{j \in J : y_j = y'_j\} \qquad J^1_{u,v} = \{j \in J : y_j \neq y'_j\}. \qquad (3.2)$$

Note that the cardinality $|J^1_{u,v}|$ can be interpreted as the Hamming distance between the "lamp strings" $(y_1, \ldots, y_n)$ and $(y'_1, \ldots, y'_n)$. The following proposition holds.

Proposition 3.1. *Let* $u = (y_1, \dots, y_n)x_i$ *and* $v = (y'_1, \dots, y'_n)x_k$ *be two vertices of* $K_n \wr K_m$ *and let* $J^0_{u,v}$ *and* $J^1_{u,v}$ *as in* (3.2). *Then*

$$d(u,v) = \begin{cases} 0 & \text{if } i = k \\ 1 & \text{if } i \neq k \end{cases} \quad \text{if } J^1_{u,v} = \emptyset;$$

$$d(u,v) = \begin{cases} 1 & \text{if } i = k = j_* \\ 3 & \text{if } i \neq j_* \neq k \\ 2 & \text{if } i = j_* \neq k; \text{ or } i \neq j_* = k \end{cases} \qquad \text{if } J^1_{u,v} = \{j_*\}.$$

More generally, for $2 \leq |J^1_{u,v}| \leq n$*:*

$$d(u,v) = \begin{cases} 2|J^1_{u,v}| + 1 & \text{if } k, i \in J^0_{u,v} \\ 2|J^1_{u,v}| & \text{if } i \in J^0_{u,v}, k \in J^1_{u,v}; \text{ or } i \in J^1_{u,v}, k \in J^0_{u,v} \\ 2|J^1_{u,v}| - 1 + \delta_{ik} & \text{if } i, k \in J^1_{u,v} \end{cases}$$

$$\text{with } \delta_{ik} = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{if } i \neq k. \end{cases}$$

Proof. First of all observe that, if $J^1_{u,v} = \emptyset$, we have $y_j = y'_j$ for each $j \in J$, so that u and v coincide if $i = k$, whereas they are adjacent, by an edge of type II in $K_n \wr K_m$, if $i \neq k$.

Suppose now $J^1_{u,v} = \{j_*\}$. In the first case, the vertices $u = (y_1, \dots, y_{j_*}, \dots, y_n)x_{j_*}$ and $v = (y_1, \dots, y'_{j_*}, \dots, y_n)x_{j_*}$, with $y_{j_*} \neq y'_{j_*}$, are adjacent in $K_n \wr K_m$. In the second case, when $i \neq j_* \neq k$, the path

$$(y_1, \dots, y_{j_*}, \dots, y_n)x_i \sim (y_1, \dots, y_{j_*}, \dots, y_n)x_{j_*} \sim (y_1, \dots, y'_{j_*}, \dots, y_n)x_{j_*} \sim \\ \sim (y_1, \dots, y'_{j_*}, \dots, y_n)x_k$$

is a path of minimal length joining u and v. In the third case, when $i = j_* \neq k$, the path

$$(y_1, \dots, y_{j_*}, \dots, y_n)x_{j_*} \sim (y_1, \dots, y'_{j_*}, \dots, y_n)x_{j_*} \sim (y_1, \dots, y'_{j_*}, \dots, y_n)x_k$$

is a path of minimal length joining u and v; the case $i \neq j_* = k$ is similar.

Now let $2 \leq |J^1_{u,v}| = h \leq n$, with $J^1_{u,v} = \{j_1, \dots, j_h\}$. In other words, the n-tuples $(y_1, \dots, y_n)$ and $(y'_1, \dots, y'_n)$ differ exactly in h places, indexed by the elements $j_1, \dots, j_h$ of $J^1_{u,v}$. In the first case, the path

$$(y_1, \dots, y_{j_1}, \dots, y_n)x_i \sim (y_1, \dots, y_{j_1}, \dots, y_n)x_{j_1} \sim (y_1, \dots, y'_{j_1}, \dots, y_n)x_{j_1} \sim \\ \sim (y_1, \dots, y'_{j_1}, \dots, y_n)x_{j_2} \sim \dots \sim (y_1, \dots, y'_{j_1}, \dots, y'_{j_{h-1}}, \dots, y_n)x_{j_h} \sim \\ \sim (y_1, \dots, y'_{j_1}, \dots, y'_{j_{h-1}}, \dots, y'_{j_h}, \dots, y_n)x_{j_h} \sim \\ \sim (y_1, \dots, y'_{j_1}, \dots, y'_{j_{h-1}}, \dots, y'_{j_h}, \dots, y_n)x_k$$

is a minimal path joining u and v, and it has length $2h + 1$. In the second case, when $i \in J^0_{u,v}, k \in J^1_{u,v}$, we can assume, without loss of generality, because K_n is complete, that $j_h = k$, so that the last step is not necessary, and a path of minimal length connecting u and v has length $2h$; a similar argument works in the case $i \in J^1_{u,v}, k \in J^0_{u,v}$. Finally, if $i \neq k$ and $i, k \in J^1_{u,v}$, we can assume that $j_1 = i$ and $j_h = k$. Now, a path of minimal length joining u and v is given by

$$\begin{gathered}(y_1,\dots,y_{j_1},\dots,y_n)x_{j_1}\sim(y_1,\dots,y'_{j_1},\dots,y_n)x_{j_1}\sim(y_1,\dots,y'_{j_1},\dots,y_n)x_{j_2}\sim\\ \sim(y_1,\dots,y'_{j_1},\dots,y'_{j_2},\dots,y_n)x_{j_2}\sim\cdots\sim\\ \sim(y_1,\dots,y'_{j_1},\dots,y'_{j_2}\dots,y'_{j_{h-1}},\dots,y_n)x_{j_{h-1}}\sim\\ \sim(y_1,\dots,y'_{j_1},\dots,y'_{j_2}\dots,y'_{j_{h-1}},\dots,y_n)x_{j_h}\sim\\ \sim(y_1,\dots,y'_{j_1},\dots,y'_{j_{h-1}},\dots,y'_{j_h},\dots,y_n)x_{j_h}\end{gathered}$$

and has length $2h-1$. In the special case $i=k$, we need one more step in order to reach the final vertex $x_k=x_i$, by means of an edge of type II in $K_n \wr K_m$. □

Corollary 3.2. *The diameter of the graph $K_n \wr K_m$ is $2n$.*

Proof. It follows from the proof of Proposition 3.1 that the maximal distance $d(u,v)$ between two vertices u and v of $K_n \wr K_m$ is equal to $2n$, and it is obtained when the vertices u,v have the form

$$u=(y_1,\dots,y_n)x_i \qquad v=(y'_1,\dots,y'_n)x_k,$$

with $y_j\neq y'_j$, for each $j=1,\dots,n$ and $x_i=x_k$. In fact, we get in this case

$$d(u,v)=2|J^1_{u,v}|-1+\delta_{ik}=2n-1+1=2n.$$

□

Now, for each $i=0,1,\dots,2n$, and every vertex u of $K_n \wr K_m$, we denote by $S_i(u)$ the sphere of radius i centered at u, that is:

$$S_i(u)=\{v\in V(K_n \wr K_m): d(u,v)=i\}.$$

Because of the complete symmetry of the graph, it is clear that the integer $s_i=|S_i(u)|$ does not depend on the particular choice of the vertex u. We recall below the classical definition of *distance-regular graph* (see, for instance, [4], or [5, Chapter 12] for some results about spectral properties of distance-regular graphs, also in connection with association scheme theory).

Definition 3.3. A connected graph $\mathcal{G}$ is said to be distance-regular if it is regular and, for any two vertices u,v at distance i, there are exactly c_i neighbors of v in $S_{i-1}(u)$ and b_i neighbors of v in $S_{i+1}(u)$.

If d is the diameter of $\mathcal{G}$, the sequence $\{b_0,b_1,\dots,b_{d-1};c_1,c_2,\dots,c_d\}$ is usually called the intersection array of $\mathcal{G}$; notice that the integers c_0 and b_d are undefined.

Proposition 3.4. *For every n,m, the wreath product $K_n \wr K_m$ is not distance-regular.*

Proof. Consider two vertices of type $u=(y_1,\dots,y_n)x_i$ and $v=(y_1,\dots,y_n)x_j$, with $j\neq i$, so that $d(u,v)=1$. Now, the neighbors of v having distance 2 from u are exactly the vertices of type $(y_1,\dots,y'_j,\dots,y_n)x_j$, with $y'_j\neq y_j$: the number of such vertices is $m-1$. On the other hand, consider the vertex $w=(y_1,\dots,y'_i,\dots,y_n)x_i$, with $y'_i\neq y_i$, so that we still have $d(u,w)=1$. It is clear that the neighbors of w having distance 2 from u are exactly the vertices of type $(y_1,\dots,y_i',\dots,y_n)x_j$, with $x_j \neq x_i$, and they are precisely $n-1$. This implies that the coefficient b_1 cannot be defined, and this is sufficient to conclude that $K_n \wr K_m$ is not distance-regular.

Also in the case $n = m$, the graph is not distance-regular. In order to show that, it suffices to consider the vertices $u = (y_1,\dots,y_{n-1},y_n)x_n$ and $v = (y_1',\dots,y_{n-1}',y_n)x_n$, with $y_j' \neq y_j$ for each $j = 1,\dots,n-1$, so that $d(u,v) = 2n-1$, according to Proposition 3.1. Now, the neighbors of v having distance $2n$ from u are exactly the vertices of type $(y_1',\dots,y_{n-1}',y_n')x_n$, with $y_n' \neq y_n$: the number of such vertices is $n-1$.

On the other hand, consider the vertices $w = (y_1,\dots,y_n)x_i$ and $z = (y_1',\dots,y_n')x_j$, with $y_j' \neq y_j$ for each $j = 1,\dots,n$ and $x_i \neq x_j$, so that we still have $d(w,z) = 2n-1$. In this case, the unique neighbor of z having distance $2n$ from w is the vertex $(y_1',\dots,y_n')x_i$. This implies that the coefficient b_{2n-1} cannot be defined, and this is sufficient to conclude that $K_n \wr K_n$ is not distance-regular. □

Example 3.5. Consider the graph $K_3 \wr K_2$ depicted in Figure 2, where the vertices of K_3 and K_2 are identified with the sets $\{0,1,2\}$ and $\{0,1\}$, respectively. The adjacency matrices of the graphs K_3 and K_2 are, respectively,

$$Ad_3 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \qquad \text{and} \qquad Ad_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

so that the matrix wreath product

$$\begin{aligned} Ad_3 \wr Ad_2 = I_2^{\otimes^3} \otimes Ad_3 + Ad_2 \otimes I_2 \otimes I_2 \otimes C_1 + \\ + I_2 \otimes Ad_2 \otimes I_2 \otimes C_2 + I_2 \otimes I_2 \otimes Ad_2 \otimes C_3 \end{aligned}$$

is the adjacency matrix of the graph $K_3 \wr K_2$. The graph $K_3 \wr K_2$ is regular of degree 3, and its diameter is 6.

3.1 Spectrum of the graph $K_n \wr K_m$

In this section, we will give an explicit description of the spectrum of the graph $K_n \wr K_m$ which is, by definition, the spectrum of its adjacency matrix $Ad_n \wr Ad_m$ described in Equation (3.1). In order to develop our analysis, we need to recall the definition of circulant matrix. A (complex) circulant matrix C of size m is a square matrix with m rows and m columns, of type

$$C = \begin{pmatrix} c_0 & c_1 & \cdots & \cdots & c_{m-1} \\ c_{m-1} & c_0 & c_1 & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & c_1 \\ c_1 & \cdots & \cdots & c_{m-1} & c_0 \end{pmatrix} \quad \text{with } c_i \in \mathbb{C}, \forall i = 0,\dots,m-1. \tag{3.3}$$

The reader can refer to [11] as an exhaustive monograph on circulant matrices.

The following theorem has been proven in [8], by using the spectral analysis developed in [35] for block circulant matrices.

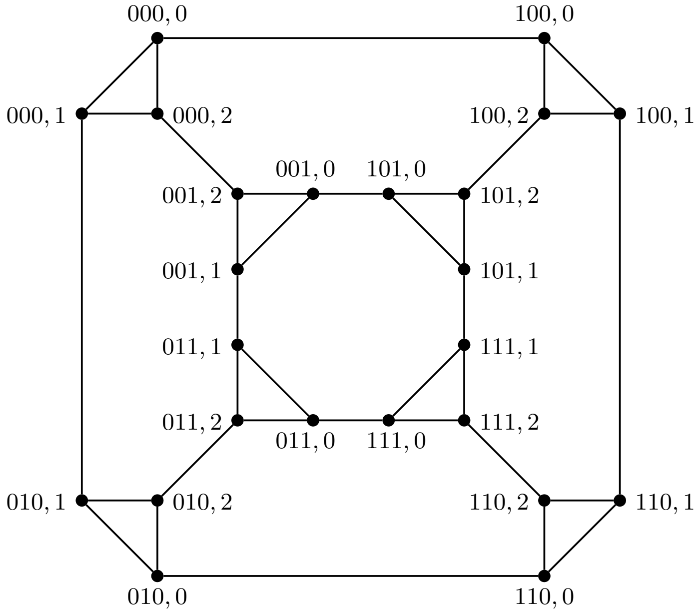

Figure 2: The graph $K_3 \wr K_2$.

Theorem 3.6. *Let A be a square matrix of size n, and let B be a circulant matrix of size m as in* (3.3)*. Then the spectrum Σ of the matrix $A \wr B$ is obtained by taking the union of the spectra $\Sigma_{i_1,\ldots,i_n}$ of the m^n matrices of size n given by*

$$\widetilde{M}^{i_1,i_2,\ldots,i_n} = A + \sum_{t=1}^{n} \sum_{i=0}^{m-1} c_i \rho^{i i_t} C_t,$$

where $i_j \in \{0, 1, \ldots, m-1\}$, for every $j = 1, \ldots, n$, and $\rho = \exp\left(\frac{2\pi i}{m}\right)$.

In particular, Theorem 3.6 can be applied in order to determine the spectrum of the adjacency matrix

$$Ad_n \wr Ad_m = I_m^{\otimes^n} \otimes Ad_n + \sum_{i=1}^{n} I_m^{\otimes^{i-1}} \otimes Ad_m \otimes I_m^{\otimes^{n-i}} \otimes C_i,$$

since the matrix Ad_m is a circulant matrix, with $c_0 = 0$ and $c_i = 1$, for each $i = 1, \ldots, m-1$. When listing eigenvalues and their multiplicities in the next theorem, and in the rest of the paper, we will write λ^h to say that the eigenvalue λ has multiplicity h; the multiplicity will be omitted when it is equal to 1. We obtain the following result.

Theorem 3.7. *The spectrum Σ of the graph $K_n \wr K_m$ is $\Sigma = \bigcup_{k=0}^{n} \Sigma_k^{\binom{n}{k}\cdot(m-1)^{n-k}}$, with*

$$\Sigma_0 = \left\{(-2)^{n-1};\ n-2\right\}$$

$$\Sigma_k = \left\{(m-2)^{k-1};\ (-2)^{n-k-1};\ \frac{m+n-4 \pm \sqrt{(m-n)^2+4km}}{2}\right\}, \quad k = 1, \ldots, n-1$$

and

$$\Sigma_n = \left\{(m-2)^{n-1};\ m+n-2\right\}.$$

Proof. By virtue of Theorem 3.6, the spectrum of $K_n \wr K_m$ is obtained by taking the union of the spectra $\Sigma_{i_1,\ldots,i_n}$ of the matrices

$$\widetilde{M}^{i_1,i_2,\ldots,i_n} = Ad_n + \sum_{t=1}^{n}\sum_{i=0}^{m-1} c_i \rho^{i i_t} C_t,$$

where $i_j \in \{0,1,\ldots,m-1\}$, for each $j = 1,\ldots,n$, and $\rho = \exp\left(\frac{2\pi i}{m}\right)$. Notice that $c_0 = 0$ and $c_i = 1$ for each $i = 1,\ldots,m-1$. Moreover, the following identity holds:

$$\sum_{i=1}^{m-1} \left(\rho^{i_t}\right)^i = \begin{cases} m-1 & \text{if } i_t = 0 \\ \frac{\left(\rho^{i_t}\right)^m - 1}{\rho^{i_t}-1} - 1 = -1 & \text{if } i_t \neq 0 \end{cases}$$

since ρ is an m-th root of unity. Therefore, the matrix $\widetilde{M}^{i_1,i_2,\ldots,i_n}$ can be rewritten as

$$\begin{aligned} \widetilde{M}^{i_1,i_2,\ldots,i_n} &= Ad_n + \sum_{t:i_t=0}(m-1)C_t - \sum_{t:i_t\neq 0} C_t \\ &= J_n - I_n + \sum_{t:i_t=0}(m-1)C_t - \left(\sum_{t:i_t\neq 0} C_t + \sum_{t:i_t=0} C_t\right) + \sum_{t:i_t=0} C_t \\ &= J_n - 2I_n + m\sum_{t:i_t=0} C_t. \end{aligned}$$

By using iterated conjugations with appropriate elementary permutation matrices, it can be shown that the spectrum of the matrix $\widetilde{M}^{i_1,i_2,\ldots,i_n}$ only depends on the number k of indices equal to 0 in the n-tuple $(i_1,i_2,\ldots,i_n)$, but it is independent of the particular position of such indices. As a consequence, for each $k = 0,1,\ldots,n$, we can reduce to investigate the spectrum of the matrix $\widetilde{M}^{0,\ldots,0,i_{k+1},\ldots,i_n}$, corresponding to the n-tuple $(\underbrace{0,\ldots,0}_{k \text{ times}}, i_{k+1},\ldots,i_n)$, with $i_j \neq 0$ for each $j = k+1,\ldots,n$. We have:

$$\widetilde{M}^{0,\ldots,0,i_{k+1},\ldots,i_n} = \begin{pmatrix} -1+m & 1 & \cdots & \cdots & \cdots & 1 \\ 1 & \ddots & 1 & & & \vdots \\ 1 & 1 & -1+m & 1 & \cdots & \vdots \\ \vdots & & & -1 & & 1 \\ \vdots & & & 1 & \ddots & \vdots \\ 1 & \cdots & \cdots & \cdots & 1 & -1 \end{pmatrix}.$$

Then we can write $\widetilde{M}^{0,\ldots,0,i_{k+1},\ldots,i_n} = J_n + Q$, where $Q = m\sum_{t=1}^{k} C_t - 2I_n$ is the diagonal matrix

$$Q = \begin{pmatrix} -2+m & & & & & \\ & \ddots & & & & \\ & & -2+m & & & \\ & & & -2 & & \\ & & & & \ddots & \\ & & & & & -2 \end{pmatrix}.$$

Now we have:

$$\begin{aligned} \det(\lambda I_n - \widetilde{M}^{0,\dots,0,i_{k+1},\dots,i_n}) &= \det\left(\lambda I_n - J_n - Q\right) \\ &= \det\left((\lambda I_n - Q)\left(I_n - (\lambda I_n - Q)^{-1} J_n\right)\right) \\ &= \det(\lambda I_n - Q) \cdot \det\left(I_n - (\lambda I_n - Q)^{-1} J_n\right). \end{aligned}$$

It is clear that

$$\det(\lambda I_n - Q) = (\lambda - (m-2))^k \cdot (\lambda + 2)^{n-k}. \tag{3.4}$$

Now it can be seen that the matrix $(\lambda I_n - Q)^{-1} J_n$ is the matrix of rank 1, whose first k rows are constant, with entries all equal to $\frac{1}{\lambda-(m-2)}$, whereas the remaining $n-k$ rows are constant, with entries all equal to $\frac{1}{\lambda+2}$. Therefore, $(\lambda I_n - Q)^{-1} J_n$ has $n-1$ eigenvalues equal to 0, and one eigenvalue equal to $\frac{k}{\lambda-(m-2)} + \frac{n-k}{\lambda+2}$. This implies that the matrix $I_n - (\lambda I_n - Q)^{-1} J_n$ has $n-1$ eigenvalues equal to 1, and one eigenvalue equal to $1 - \frac{k}{\lambda-(m-2)} - \frac{n-k}{\lambda+2}$, so that:

$$\det\left(I_n - (\lambda I_n - Q)^{-1} J_n\right) = 1 - \frac{k}{\lambda - (m-2)} - \frac{n-k}{\lambda+2}. \tag{3.5}$$

By gluing together (3.4) and (3.5), we obtain:

$$\begin{aligned} \det(\lambda I_n - \widetilde{M}^{0,\dots,0,i_{k+1},\dots,i_n}) &= (\lambda - (m-2))^{k-1} \cdot (\lambda+2)^{n-k-1} \cdot \\ &\cdot (\lambda^2 + (4-m-n)\lambda + mn + 4 - km - 2n - 2m). \end{aligned}$$

For the particular value $k = 0$, we get:

$$\det(\lambda I_n - \widetilde{M}^{i_1,\dots,i_n}) = (\lambda+2)^{n-1} \cdot (\lambda - (n-2));$$

for the particular value $k = n$, we have:

$$\det(\lambda I_n - \widetilde{M}^{0,\dots,0}) = (\lambda - (m-2))^{n-1} \cdot (\lambda - (m+n-2)).$$

The claim follows, if we observe that, for each $k = 0, 1, \dots, n$, the spectrum of Σ_k must be considered $\binom{n}{k} \cdot (m-1)^{n-k}$ times, corresponding to the number of n-tuples $(i_1, \dots, i_n)$ with k indices equal to 0, and the remaining indices varying in $\{1, \dots, m-1\}$. $\square$

Example 3.8. Consider the graph $K_3 \wr K_4$, so that $n = 3$ and $m = 4$. The spectrum of the matrix $Ad_3 \wr Ad_4$ consists of the following eigenvalues:

$$5;\ 2^{11};\ 1^{27};\ (-2)^{81};\ \left(\tfrac{3\pm\sqrt{17}}{2}\right)^{27};\ \left(\tfrac{3\pm\sqrt{33}}{2}\right)^{9}.$$

The corresponding matrices $\widetilde{M}^{i_1,i_2,i_3}$ of size 3, with $i_1, i_2, i_3 \in \{0, 1, 2, 3\}$, have eigenvalues:

(a) $(-2)^2; 1$, for $k = 0$. For instance, this is the case of the matrix

$$\widetilde{M}^{1,1,1} = J_3 - 2I_3 = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}.$$

(b) $-2; \frac{3\pm\sqrt{17}}{2}$, for $k = 1$. For instance, this is the case of the matrix

$$\widetilde{M}^{0,1,1} = J_3 - 2I_3 + 4C_1 = \begin{pmatrix} 3 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}.$$

(c) $2; \frac{3\pm\sqrt{33}}{2}$, for $k = 2$. For instance, this is the case of the matrix

$$\widetilde{M}^{0,0,1} = J_3 - 2I_3 + 4(C_1 + C_2) = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & -1 \end{pmatrix}.$$

(d) $2^2; 5$, for $k = 3$. This is the case of the matrix

$$\widetilde{M}^{0,0,0} = J_3 - 2I_3 + 4(C_1 + C_2 + C_3) = J_3 + 2I_3 = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}.$$

Corollary 3.9. *The spectrum Σ' of the transition matrix of the Lamplighter random walk with base graph K_n and color graph K_m is $\Sigma' = \bigcup_{k=0}^{n} \Sigma_k'^{\binom{n}{k}\cdot(m-1)^{n-k}}$, with*

$$\Sigma'_0 = \left\{ \left(-\frac{2}{m+n-2}\right)^{n-1};\ \frac{n-2}{m+n-2} \right\}$$

$$\Sigma'_k = \left\{ \left(\frac{m-2}{m+n-2}\right)^{k-1};\ \left(-\frac{2}{m+n-2}\right)^{n-k-1};\ \frac{m+n-4\pm\sqrt{(m-n)^2+4km}}{2(m+n-2)} \right\},$$

for $k = 1, \dots, n-1$, and

$$\Sigma'_n = \left\{ \left(\frac{m-2}{m+n-2}\right)^{n-1};\ 1 \right\}.$$

Proof. It suffices to take into account that the transition matrix of the Lamplighter random walk on the base graph K_n, with color graph K_m, is the normalized adjacency matrix of the graph $K_n \wr K_m$, which is a regular graph of degree $m + n - 2$. □

3.2 Distance spectrum and Wiener index of the graph $K_n \wr K_m$

The aim of this section is to describe the distance matrix of the graph $K_n \wr K_m$, together with its spectrum. Moreover, we will exhibit an explicit computation of the Wiener index of the graph.

Proposition 3.10. *The distance matrix of the graph $K_n \wr K_m$ is the matrix*

$$D = \sum_{(i_1,\dots,i_n)\in\{0,1\}^n} Ad_m^{i_1} \otimes Ad_m^{i_2} \otimes \cdots \otimes Ad_m^{i_n} \otimes A_{i_1,i_2,\dots,i_n}, \tag{3.6}$$

where we put $Ad_m^0 = I_m$, and the matrix $A_{i_1,i_2,\dots,i_n}$ is the square matrix of size n, indexed by the vertices of K_n, defined as follows. Let $\{i_1,\dots,i_n\} = I_0 \sqcup I_1$, with $I_0 = \{i_j : i_j = 0\}$ and $I_1 = \{i_j : i_j = 1\}$. Then, for any pair of vertices x_i and x_k of K_n:

(a) $A_{i_1,\dots,i_n} = Ad_n = J_n - I_n$ *if* $I_1 = \emptyset$;

(b) $$A_{i_1,\dots,i_n}(x_i,x_k) = \begin{cases} 1 & \text{if } i = k = j_* \\ 3 & \text{if } i \neq j_* \neq k \\ 2 & \text{if } i = j_* \neq k;\ \text{or } i \neq j_* = k \end{cases} \qquad \text{if } I_1 = \{i_{j_*}\};$$

(c) $$A_{i_1,\dots,i_n}(x_i,x_k) = \begin{cases} 2|I_1| + 1 & \text{if } i,k \in I_0 \\ 2|I_1| & \text{if } i \in I_0, k \in I_1;\ \text{or } i \in I_1, k \in I_0 \\ 2|I_1| - 1 + \delta_{ik} & \text{if } i,k \in I_1 \end{cases}$$

$$\text{if } 2 \leq |I_1| \leq n, \text{ where } \delta_{ik} = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{if } i \neq k. \end{cases}$$

Proof. Observe that, for each $j = 1,\dots,n$, the index $i_j \in \{0,1\}$ establishes whether the color of the lamp at the j-th vertex x_j of K_n is changed. More precisely, the index $i_j = 0$ produces the matrix $Ad_m^0 = I_m$ as j-th term of the Kronecker product, so that we are not changing the color of the lamp in that position; conversely, the index $i_j = 1$ provides the matrix Ad_m as j-th term of the Kronecker product, so that we are changing the color of the lamp in that position, with any other color, as K_m is the complete graph. Therefore, for any fixed n-tuple $(i_1,\dots,i_n) \in \{0,1\}^n$, the contribution $Ad_m^{i_1} \otimes Ad_m^{i_2} \otimes \cdots \otimes Ad_m^{i_n} \otimes A_{i_1,i_2,\dots,i_n}$ to D must take into account the distances between vertices u, v of $K_n \wr K_m$ corresponding to lamp configurations which differ exactly at the places indexed by I_1. Therefore, if the configurations of lamps corresponding to the vertices $u = (y_1,\dots,y_n)x_i$ and $v = (y'_1,\dots,y'_n)x_k$ of $K_n \wr K_m$ differ at exactly $|I_1|$ vertices of K_n, indexed by I_1, the last contribution in the Kronecker product is an $n \times n$ matrix, whose entry (x_i,x_k) must be equal to the distance $d(u,v)$. Then the claim follows from Proposition 3.1. $\square$

As in the case of the adjacency matrix $Ad_n \wr Ad_m$, the spectrum of the matrix D can be computed by using a reduction argument. In fact, the matrix D in (3.6) has the following block circulant structure

$$D = \begin{pmatrix} D_0 & D_1 & D_2 & \cdots & D_{m-1} \\ D_{m-1} & D_0 & D_1 & \ddots & \vdots \\ \vdots & D_{m-1} & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & D_1 \\ D_1 & D_2 & \cdots & D_{m-1} & D_0 \end{pmatrix} \tag{3.7}$$

with

$$D_0 = \sum_{(i_2,\dots,i_n)\in\{0,1\}^n} Ad_m^{i_2} \otimes \cdots \otimes Ad_m^{i_n} \otimes A_{0,i_2,\dots,i_n}$$
$$D_i = \sum_{(i_2,\dots,i_n)\in\{0,1\}^n} Ad_m^{i_2} \otimes \cdots \otimes Ad_m^{i_n} \otimes A_{1,i_2,\dots,i_n} \quad \text{for each } i = 1,\dots,m-1.$$

Then the spectral analysis of block circulant matrices developed in [35] ensures that the spectrum of D can be obtained by taking the union of the spectra of the following m matrices of size nm^{n-1}:

$$\begin{aligned} \widetilde{D}^{j_1} &= \sum_{h_1=0}^{m-1} \rho^{h_1 j_1} D_{h_1} \\ &= \sum_{(i_2,\dots,i_n)\in\{0,1\}^n} Ad_m^{i_2} \otimes \cdots \otimes Ad_m^{i_n} \otimes A_{0,i_2,\dots,i_n} + \\ &\quad + \sum_{h_1=1}^{m-1} \rho^{h_1 j_1} \sum_{(i_2,\dots,i_n)\in\{0,1\}^n} Ad_m^{i_2} \otimes \cdots \otimes Ad_m^{i_n} \otimes A_{1,i_2,\dots,i_n} \\ &= \sum_{(i_2,\dots,i_n)\in\{0,1\}^n} Ad_m^{i_2} \otimes \cdots \otimes Ad_m^{i_n} \otimes \left(A_{0,i_2,\dots,i_n} + \sum_{h_1=1}^{m-1} \rho^{h_1 j_1} A_{1,i_2,\dots,i_n} \right), \end{aligned}$$

with $j_1 \in \{0,1,\dots,m-1\}$. Observe that each of these matrices is still a block circulant matrix, with blocks of size nm^{n-2}, given by

$$D'_0 = \sum_{(i_3,\dots,i_n)\in\{0,1\}^n} Ad_m^{i_3} \otimes \cdots \otimes Ad_m^{i_n} \otimes \left(A_{0,0,i_3,\dots,i_n} + \sum_{h_1=1}^{m-1} \rho^{h_1 j_1} A_{1,0,i_3,\dots,i_n} \right),$$
$$D'_i = \sum_{(i_3,\dots,i_n)\in\{0,1\}^n} Ad_m^{i_3} \otimes \cdots \otimes Ad_m^{i_n} \otimes \left(A_{0,1,i_3,\dots,i_n} + \sum_{h_1=1}^{m-1} \rho^{h_1 j_1} A_{1,1,i_3,\dots,i_n} \right)$$

for $i = 1,\dots,m-1$. Therefore, the same argument can be repeated, so that the spectrum of D is obtained by taking the union of the spectra of the following m^2 matrices of size nm^{n-2}:

$$\begin{aligned} \widetilde{D}^{j_1,j_2} &= \sum_{h_2=0}^{m-1} \rho^{h_2 j_2} D'_{h_2} = \\ &= \sum_{(i_3,\dots,i_n)\in\{0,1\}^n} Ad_m^{i_3} \otimes \cdots \otimes Ad_m^{i_n} \otimes \left(A_{0,0,i_3,\dots,i_n} + \sum_{h_1=1}^{m-1} \rho^{h_1 j_1} A_{1,0,i_3,\dots,i_n} \right) + \\ &\quad + \sum_{h_2=1}^{m-1} \rho^{h_2 j_2} \sum_{(i_3,\dots,i_n)\in\{0,1\}^n} Ad_m^{i_3} \otimes \cdots \otimes Ad_m^{i_n} \otimes \\ &\qquad \otimes \left(A_{0,1,i_3,\dots,i_n} + \sum_{h_1=1}^{m-1} \rho^{h_1 j_1} A_{1,1,i_3,\dots,i_n} \right) \end{aligned}$$

$$= \sum_{(i_3,\dots,i_n)\in\{0,1\}^n} Ad_m^{i_3}\otimes\cdots\otimes Ad_m^{i_n}\otimes\Bigg(A_{0,0,i_3,\dots,i_n}+ + \sum_{h_1=1}^{m-1}\rho^{h_1j_1}A_{1,0,i_3,\dots,i_n} + \sum_{h_2=1}^{m-1}\rho^{h_2j_2}A_{0,1,i_3,\dots,i_n} + + \sum_{h_1=1}^{m-1}\rho^{h_1j_1}\sum_{h_2=1}^{m-1}\rho^{h_2j_2}A_{1,1,i_3,\dots,i_n}\Bigg)$$

with $(j_1, j_2) \in \{0,\dots,m-1\}^2$. This reduction argument can be iterated further, until we get blocks of size n. Once again, notice that $\sum_{h=1}^{m-1}\rho^{hj_s} = \begin{cases} -1 & \text{if } j_s \neq 0 \\ m-1 & \text{if } j_s = 0. \end{cases}$ We thus have proven the following theorem.

Theorem 3.11. *The distance spectrum Σ of the graph $K_n \wr K_m$ is obtained by taking the union of the spectra $\Sigma_{j_1,\dots,j_n}$ of the m^n matrices of size n:*

$$\widetilde{D}^{j_1,\dots,j_n} = \sum_{(i_1,\dots,i_n)\in\{0,1\}^n}\prod_{s=1}^{n}\left(\sum_{h_s=1}^{m-1}\rho^{h_sj_s}\right)^{i_s} A_{i_1,\dots,i_n}, \qquad (j_1,\dots,j_n)\in\{0,\dots,m-1\}^n,$$

where, if we put $I_0 = \{i_j : i_j = 0\}$ and $I_1 = \{i_j : i_j = 1\}$, we have:

$$A_{0,\dots,0} = Ad_n;$$

$$A_{i_1,\dots,i_n}(x_i,x_k) = \begin{cases} 1 & \text{if } i,k\in I_1 \\ 3 & \text{if } i,k \in I_0 \\ 2 & \text{if } i\in I_1, k\in I_0; \text{ or } i\in I_0, k\in I_1 \end{cases} \qquad \text{for } |I_1| = 1;$$

and

$$A_{i_1,\dots,i_n}(x_i,x_k) = \begin{cases} 2|I_1|+1 & \text{if } i,k\in I_0 \\ 2|I_1| & \text{if } i\in I_0, k\in I_1; i\in I_1, k\in I_0 \\ 2|I_1|-1+\delta_{ik} & \text{if } i,k\in I_1 \end{cases}$$

$$\text{for } 2\leq |I_1|\leq n, \text{ with } \delta_{ik} = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{if } i\neq k. \end{cases}$$

Example 3.12. Let us consider the explicit example $K_3 \wr K_3$. The distance matrix of this graph is

$$D = I_3\otimes I_3\otimes I_3\otimes A_{000} + I_3\otimes I_3\otimes Ad_3\otimes A_{001} + I_3\otimes Ad_3\otimes I_3\otimes A_{010}+ + I_3\otimes Ad_3\otimes Ad_3\otimes A_{011} + Ad_3\otimes I_3\otimes I_3\otimes A_{100} + Ad_3\otimes I_3\otimes Ad_3\otimes A_{101}+ + Ad_3\otimes Ad_3\otimes I_3\otimes A_{110} + Ad_3\otimes Ad_3\otimes Ad_3\otimes A_{111},$$

with

$$A_{000} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \qquad A_{001} = \begin{pmatrix} 3 & 3 & 2 \\ 3 & 3 & 2 \\ 2 & 2 & 1 \end{pmatrix}$$

$$A_{010} = \begin{pmatrix} 3 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & 2 & 3 \end{pmatrix} \qquad A_{011} = \begin{pmatrix} 5 & 4 & 4 \\ 4 & 4 & 3 \\ 4 & 3 & 4 \end{pmatrix}$$

$$A_{100} = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 3 & 3 \\ 2 & 3 & 3 \end{pmatrix} \qquad A_{101} = \begin{pmatrix} 4 & 4 & 3 \\ 4 & 5 & 4 \\ 3 & 4 & 4 \end{pmatrix}$$

$$A_{110} = \begin{pmatrix} 4 & 3 & 4 \\ 3 & 4 & 4 \\ 4 & 4 & 5 \end{pmatrix} \qquad A_{111} = \begin{pmatrix} 6 & 5 & 5 \\ 5 & 6 & 5 \\ 5 & 5 & 6 \end{pmatrix}.$$

The spectrum of D consists of the following eigenvalues:

$$312;\ 15^2;\ 0^{48};\ (-3)^{18}; (-24 \pm 3\sqrt{43})^6.$$

We have, for instance:

$$\widetilde{D}^{0,2,0} = A_{000} + 2A_{001} - A_{010} - 2A_{011} + 2A_{100} + 4A_{101} - 2A_{110} - 4A_{111} = \\ = \begin{pmatrix} -21 & -9 & -18 \\ -9 & -9 & -9 \\ -18 & -9 & -21 \end{pmatrix}$$

whose eigenvalues are -3 and $-24 \pm 3\sqrt{43}$.

Next, we pass to the computation of the Wiener index $W(K_n \wr K_m)$ of the graph $K_n \wr K_m$. It follows from the definition of the Wiener index that $W(K_n \wr K_m)$ is given by the sum of all the entries of D, divided by 2, due to the fact that each contribution $d(u,v)$ appears twice, as the matrix D is symmetric. Keeping in mind the block structure of the distance matrix D described in (3.7) and the fact that each block D_i, for $i = 0, \ldots, m-1$, appearing in (3.7) can be recursively regarded as a block circulant matrix, until one gets elementary blocks of size n represented by matrices of type $A_{i_1,\ldots,i_n}$, we obtain the following result.

Theorem 3.13. *The Wiener index of the graph $K_n \wr K_m$ is*

$$W(K_n \wr K_m) = \tfrac{nm^n}{2}(2m^n n^2 - nm^n - 2n^2 m^{n-1} + m^n + 2nm^{n-1} - m^{n-1} - m).$$

Proof. First of all, for every n-tuple $(i_1, \ldots, i_n) \in \{0,1\}^n$, put:

$$d_{i_1,\ldots,i_n} = \sum_{x_i, x_j \in V_n} A_{i_1,\ldots,i_n}(x_i, x_j).$$

Now observe that, by definition of the matrices $A_{i_1,\ldots,i_n}$, the sum $d_{i_1,\ldots,i_n}$ only depends on the cardinality of the sets $I_0 = \{i_j : i_j = 0\}$ and $I_1 = \{i_j : i_j = 1\}$, while it is independent from the particular position of the indices. Therefore, for every $k = 0, 1, \ldots, n$, it makes sense to define:

$$d_k = \sum_{x_i, x_j \in V_n} A_{\underbrace{1,\ldots,1}_{k \text{ times}},\underbrace{0,\ldots,0}_{n-k \text{ times}}}(x_i, x_j).$$

Moreover, by performing a direct computation which uses the explicit description of the matrices $A_{i_1,\ldots,i_n}$ given in Theorem 3.11, we are able to determine:

(a) $d_0 = n(n-1)$;

(b) $d_1 = 1 + 4(n-1) + 3(n-1)^2$;

(c) $d_k = (2k+1)(n-k)^2 + 4k^2(n-k) + 2k^2 + k(k-1)(2k-1)$ for $2 \leq k \leq n$.

Now we have to establish the number of contributions of type d_k to $W(K_n \wr K_m)$, for every k. First of all, a factor equal to $\binom{n}{k}$ appears, taking into account all the possible choices of k indices equal to 1. Moreover, a second factor given by $m^n(m-1)^k$ appears, since a fixed n-tuple $(\bar{i}_1, \ldots, \bar{i}_n)$ containing k indices equal to 1 (see Equation (3.6)) produces $m^n(m-1)^k$ blocks of size n, within the matrix D, which are equal to $A_{\bar{i}_1,\ldots,\bar{i}_n}$, due to the fact that, when we change the color of a lamp, we have $m-1$ possibilities for the choice of the new color. This implies that

$$W(K_n \wr K_m) = \frac{m^n}{2} \sum_{k=0}^{n} \binom{n}{k} (m-1)^k d_k. \tag{3.8}$$

By explicitly computing the sum in (3.8), we get the claim. □

Example 3.14. Consider the case of $K_3 \wr K_3$. Theorem 3.13 gives $W(K_3 \wr K_3) = 12636$, with $d_0 = 6$; $d_1 = 21$; $d_2 = 35$; $d_3 = 48$.

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Counting faces of graphical zonotopes

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Abstract

It is a classical fact that the number of vertices of the graphical zonotope Z_Γ is equal to the number of acyclic orientations of a graph Γ. We show that the f-polynomial of Z_Γ is obtained as the principal specialization of the q-analog of the chromatic symmetric function of Γ.

Keywords: Graphical zonotope, f-vector, graphical matroid, symmetric function.

Math. Subj. Class.: 05E05, 52B05, 16T05

1 Introduction

The f-polynomial of an n-dimensional polytope P is defined by $f(P,q) = \sum_{i=0}^{n} f_i(P)q^i$, where $f_i(P)$ is the number of i-dimensional faces of P. The f-polynomial $f(\mathcal{Z}_\Gamma, q)$ of the graphical zonotope $\mathcal{Z}_\Gamma$ is a combinatorial invariant of a finite, simple graph Γ. The vertices of $\mathcal{Z}_\Gamma$ are in one-to-one correspondence with regions of the graphical hyperplane arrangement $\mathcal{H}_\Gamma$, which are enumerated by acyclic orientations of Γ.

Stanley's chromatic symmetric function $\Psi(\Gamma) = \sum_{f proper} \mathbf{x}_f$ of a graph $\Gamma = (V, E)$, introduced in [7], is the enumerator function of proper colorings $f\colon V \to \mathbb{N}$, where $\mathbf{x}_f = x_{f(1)} \cdots x_{f(n)}$ and f is proper if there are no monochromatic edges. The chromatic polynomial $\chi(\Gamma, d)$ of the graph Γ, which counts proper colorings with a finite number of colors, appears as the principal specialization

$$\chi(\Gamma, d) = \mathbf{ps}(\Psi(\Gamma))(d) = \Psi(\Gamma) \mid_{x_1=\cdots=x_d=1, x_{d+1}=\cdots=0} .$$

The number of acyclic orientations of Γ is determined by the value of the chromatic polynomial $\chi(\Gamma, d)$ at $d = -1$, [6]

$$a(\Gamma) = (-1)^{|V|}\chi(\Gamma, -1). \tag{1.1}$$

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There is a q-analog of the chromatic symmetric function $\Psi_q(\Gamma)$ introduced in a wider context of the combinatorial Hopf algebra of simplicial complexes considered in [2]. It is a symmetric function over the field of rational functions in q. The principal specialization of $\Psi_q(\Gamma)$ is the q-analog of the chromatic polynomial $\chi_q(\Gamma, d)$.

The main result of this paper is the following generalization of formula (1.1):

Theorem 1.1. *Let $\Gamma = (V, E)$ be a simple connected graph and $\mathcal{Z}_\Gamma$ the corresponding graphical zonotope. Then the f-polynomial of $\mathcal{Z}_\Gamma$ is given by*

$$f(\mathcal{Z}_\Gamma, q) = (-1)^{|V|}\chi_{-q}(\Gamma, -1).$$

The cancellation-free formula for the antipode in the Hopf algebra of graphs, obtained by Humpert and Martin in [3], reflects the fact that $f(\mathcal{Z}_\Gamma, q)$ depends only on the graphical matroid $M(\Gamma)$ associated to Γ. For instance, for any tree T_n the graphical matroid is the uniform matroid $M(T_n) = U_n^n$ and the corresponding graphical zonotope is the cube $\mathcal{Z}_{T_n} = I^{n-1}$. Whitney's theorem from 1933 describes how two graphs with the same graphical matroid are related [9]. It can be used to find more interesting nonisomorphic graphs with the same f-polynomials of corresponding graphical zonotopes.

The paper is organized as follows. In Section 2, we review the basic facts about zonotopes. In Section 3, the q-analog of the chromatic symmetric function $\Psi_q(\Gamma)$ of a graph Γ is introduced. Theorem 1.1 is proved in Section 4. We present some examples and calculations in Section 5.

2 Zonotopes

A *zonotope* $\mathcal{Z} = \mathcal{Z}(v_1, \ldots, v_m)$ is a convex polytope determined by a collection of vectors $\{v_1, \ldots, v_m\}$ in $\mathbb{R}^n$ as the Minkowski sum of line segments

$$\mathcal{Z} = [-v_1, v_1] + \cdots + [-v_m, v_m].$$

It is a projection of the m-cube $[-1, 1]^m$ under the linear map $\mathbf{t} \mapsto A\mathbf{t}, \mathbf{t} \in [-1, 1]^m$, where $A = [v_1 \cdots v_m]$ is an $n \times m$-matrix whose columns are vectors $v_1, \ldots, v_m$. The zonotope $\mathcal{Z}$ is symmetric about the origin and all its faces are translations of zonotopes.

To a collection of vectors $\{v_1, \ldots, v_m\}$ is associated a central arrangement of hyperplanes $\mathcal{H} = \{H_{v_1}, \ldots, H_{v_m}\}$, where H_v denotes the hyperplane perpendicular to a vector $v \in \mathbb{R}^n$. The zonotope $\mathcal{Z}$ and the corresponding arrangement of hyperplanes $\mathcal{H}$ are closely related. In fact the associated fan $\mathcal{F}_\mathcal{H}$ of the arrangement $\mathcal{H}$ is the normal fan $\mathcal{N}(\mathcal{Z})$ of the zonotope $\mathcal{Z}$ (see [10, Theorem 7.16]). It follows that the face lattice of $\mathcal{F}_\mathcal{H}$ and the reverse face lattice of $\mathcal{Z}$ are isomorphic. In particular, vertices of $\mathcal{Z}$ correspond to regions of $\mathcal{H}$ and their total numbers coincide

$$f_0(\mathcal{Z}) = r(\mathcal{H}). \tag{2.1}$$

The faces of the zonotope $\mathcal{Z}$ are encoded by covectors of the oriented matroid $\mathcal{M}$ associated to the collection of vectors $\{v_1, \ldots, v_m\}$. The covectors are sign vectors

$$\mathcal{V}^* = \{\text{sign}(v) \in \{+, -, 0\}^m \mid v \in \mathbb{R}^n\},$$

where $\text{sign}(v)_i = \begin{cases} +, & \langle v, v_i\rangle > 0 \\ 0, & \langle v, v_i\rangle = 0 \\ -, & \langle v, v_i\rangle < 0 \end{cases}$, $i = 1, \ldots, m$. The face lattice of the zonotope $\mathcal{Z}$ is isomorphic to the lattice of covectors componentwise induced by $+, - < 0$ on $\mathcal{V}^*$.

A special class of zonotopes is determined by simple graphs. To a connected graph $\Gamma = (V, E)$, whose vertices are enumerated by integers $V = \{1, \ldots, n\}$, are associated the *graphical zonotope*

$$\mathcal{Z}_\Gamma = \mathcal{Z}(e_i - e_j \mid i < j, \{i, j\} \in E)$$

and the *graphical arrangement* in $\mathbb{R}^n$

$$\mathcal{H}_\Gamma = \{H_{e_i - e_j} \mid i < j, \{i, j\} \in E\}.$$

There is a bijective correspondence between regions of $\mathcal{H}_\Gamma$ and acyclic orientations of Γ, [8, Proposition 2.5], which by (2.1) implies

$$f_0(\mathcal{Z}_\Gamma) = r(\mathcal{H}_\Gamma) = a(\Gamma). \tag{2.2}$$

The arrangement $\mathcal{H}_\Gamma$ is refined by the braid arrangement $\mathcal{A}_{n-1}$ consisting of all hyperplanes $H_{e_i - e_j}, 1 \leq i < j \leq n$. Thus $\mathcal{Z}_\Gamma$ belongs to a wider class of convex polytopes called generalized permutohedra introduced in [4]. Since arrangements $\mathcal{H}_\Gamma$ and $\mathcal{A}_{n-1}$ are not essential we take their quotients by the line $l : x_1 = \cdots = x_n$ and without confusing retain the same notation. Consequently $\dim \mathcal{Z}_\Gamma = n - 1$.

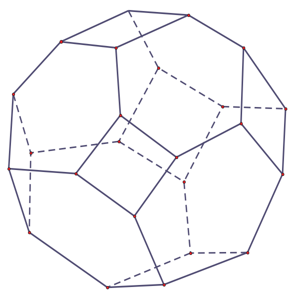
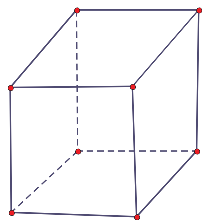

Figure 1: Permutohedron Pe^3 and cube I^3.

Example 2.1. (i) The permutohedron Pe^{n-1} is represented as the graphical zonotope $\mathcal{Z}_{K_n}$ corresponding to the complete graph K_n on n vertices (Figure 1).

(ii) The cube I^{n-1} is represented as the graphical zonotope $\mathcal{Z}_{T_n}$ corresponding to an arbitrary tree T_n on n vertices. This shows that the graph Γ is not determined by the combinatorial type of the zonotope $\mathcal{Z}_\Gamma$.

3 q-analog of chromatic symmetric function of graph

Stanley's chromatic symmetric function $\Psi(\Gamma)$ can be obtained in a purely algebraic way. A combinatorial Hopf algebra $\mathcal{H}$ is a graded, connected Hopf algebra equipped with the multiplicative linear functional $\zeta \colon \mathcal{H} \to \mathbf{k}$ to the ground field $\mathbf{k}$. For the theory of combinatorial Hopf algebras see [1]. Consider the combinatorial Hopf algebra of graphs $\mathcal{G}$ which is linearly generated over a field $\mathbf{k}$ by simple finite graphs with the product defined by disjoint union $\Gamma_1 \cdot \Gamma_2 = \Gamma_1 \sqcup \Gamma_2$ and the coproduct

$$\Delta(\Gamma) = \sum_{I \subset V} \Gamma\mid_I \otimes \Gamma\mid_{V \setminus I},$$

where $\Gamma\mid_I$ denotes the induced subgraph on $I\subset V$. The structure of $\mathcal{G}$ is completed by the character $\zeta\colon \mathcal{G}\to\mathbf{k}$ defined to be $\zeta(\Gamma)=1$ for Γ with no edges and $\zeta(\Gamma)=0$ otherwise. Then it turns out that $\Psi(\Gamma)$ is the image of the unique morphism of combinatorial Hopf algebras to symmetric functions $\Psi:\mathcal{G}\to Sym$, ([1, Example 4.5]).

An important part of the structure of the Hopf algebra $\mathcal{G}$ is the antipode $S\colon\mathcal{G}\to\mathcal{G}$. The cancellation-free formula for the antipode in terms of acyclic orientations of a graph Γ is obtained in [3]. We recall some basic definitions. Terminology comes from matroid theory. Given a graph $\Gamma=(V,E)$, for a collection of edges $F\subset E$ denote by $\Gamma_{V,F}$ the graph on V with the edge set F. A *flat* F of the graph Γ is a collection of its edges such that components of $\Gamma_{V,F}$ are induced subgraphs. The rank $\mathrm{rk}(F)$ is the size of spanning forests of $\Gamma_{V,F}$. We have that $|V|=\mathrm{rk}(F)+c(F)$, where $c(F)$ is the number of components of $\Gamma_{V,F}$. By contracting edges from a flat F we obtain the graph Γ/F. Finally, let $a(\Gamma)$ be the number of acyclic orientations of Γ. The formula of Humpert and Martin is as follows

$$S(\Gamma)=\sum_{F\in\mathcal{F}(\Gamma)}(-1)^{c(F)}a(\Gamma/F)\Gamma_{V,F}, \tag{3.1}$$

where the sum is over the set of flats $\mathcal{F}(\Gamma)$.

The following modification of the character ζ is considered in [2] in a wider context of the combinatorial Hopf algebra of simplicial complexes. Define $\zeta_q(\Gamma)=q^{\mathrm{rk}(\Gamma)}$, which determines the algebra morphism $\zeta_q\colon\mathcal{G}\to\mathbf{k}(q)$, where $\mathbf{k}(q)$ is the field of rational functions in q. This character produces the unique morphism $\Psi_q\colon\mathcal{G}\to QSym$ to quasisymmetric functions over $\mathbf{k}(q)$. The expansion of $\Psi_q(\Gamma)$ in the monomial basis of quasisymmetric functions is determined by the universal formula [1, Theorem 4.1]

$$\Psi_q(\Gamma)=\sum_{\alpha\models n}(\zeta_q)_\alpha(\Gamma)M_\alpha.$$

The sum above is over all compositions of the integer $n=|V|$ and the coefficient of the expansion corresponding to the composition $\alpha=(a_1,\ldots,a_k)\models n$ is given by

$$(\zeta_q)_\alpha(\Gamma)=\sum_{I_1\sqcup\ldots\sqcup I_k=V}q^{\mathrm{rk}(\Gamma|_{I_1})+\cdots+\mathrm{rk}(\Gamma|_{I_k})},$$

where the sum is over all set compositions of V of the type α. The coefficients $(\zeta_q)_\alpha(\Gamma)$ depend only on the partition corresponding to a composition α, so the function $\Psi_q(\Gamma)$ is actually symmetric and it can be expressed in the monomial basis of symmetric functions.

The invariant $\Psi_q(\Gamma)$ is more subtle than $\Psi(\Gamma)$. Obviously $\Psi_0(\Gamma)$ is the chromatic symmetric function of a graph Γ. It remains open to find two nonisomorphic graphs Γ_1 and Γ_2 with the same q-chromatic symmetric functions $\Psi_q(\Gamma_1)=\Psi_q(\Gamma_2)$. Let

$$\chi_q(\Gamma,d)=\mathbf{ps}(\Psi_q(\Gamma))(d)$$

be the q-analog of the chromatic polynomial $\chi(\Gamma,d)$. It is a consequence of a general fact for combinatorial Hopf algebras (see [1]) that

$$\chi_q(\Gamma,-1)=(\zeta_q\circ S)(\Gamma). \tag{3.2}$$

Example 3.1. Consider the graph Γ on four vertices with the edge set $E=\{12,13,23,34\}$. We find that

$$\Psi_q(\Gamma) = 24m_{1,1,1,1} + (8q+4)m_{2,1,1} + (2q^2+4q)m_{2,2} + (3q^2+q)m_{3,1} + q^3m_4.$$

By principal specialization and taking into account that

$$\mathbf{ps}(m_{\lambda_1^{i_1},\dots,\lambda_k^{i_k}})(d) = \frac{(i_1+\cdots+i_k)!}{i_1!\cdots i_k!}\binom{d}{i_1+\cdots+i_k},$$

we obtain

$$\chi_q(\Gamma,d) = d(d-1)^2(d-2) + qd(d-1)(4d-5) + 4q^2d(d-1) + q^3d,$$

which by Theorem 1.1 gives

$$f(\mathcal{Z}_\Gamma,q) = 12 + 18q + 8q^2 + q^3.$$

4 Proof of Theorem 1.1

Proof. By applying (3.2) and the formula for antipode (3.1) we obtain

$$(-1)^{|V|}\chi_{-q}(\Gamma,-1) = (-1)^{|V|}\sum_{F\in\mathcal{F}(\Gamma)}(-1)^{c(\Gamma)}a(\Gamma/F)(-q)^{\mathrm{rk}(F)}.$$

It follows that the statement of the theorem is equivalent to the following expression of the f-polynomial

$$f(\mathcal{Z}_\Gamma,q) = \sum_{F\in\mathcal{F}(\Gamma)} a(\Gamma/F)q^{\mathrm{rk}(F)}. \tag{4.1}$$

Therefore it should be shown that components of f-vectors are determined by

$$f_k(\mathcal{Z}_\Gamma) = \sum_{\substack{F\in\mathcal{F}(\Gamma)\\ \mathrm{rk}(F)=k}} a(\Gamma/F),\ \ 0\le k\le n-1. \tag{4.2}$$

By duality between the face lattice of $\mathcal{Z}_\Gamma$ and the face lattice of the fan $\mathcal{F}_{\mathcal{H}_\Gamma}$ we have

$$f_k(\mathcal{Z}_\Gamma) = f_{n-k-1}(\mathcal{F}_{\mathcal{H}_\Gamma}).$$

Let $L(\mathcal{H}_\Gamma)$ be the intersection lattice of the graphical arrangement $\mathcal{H}_\Gamma$. For a subspace $X\in L(\mathcal{H}_\Gamma)$ there is an arrangement of hyperplanes

$$\mathcal{H}_\Gamma^X = \{X\cap H \mid X\nsubseteq H, H\in\mathcal{H}_\Gamma\}$$

whose intersection lattice $L(\mathcal{H}_\Gamma^X)$ is isomorphic to the upper cone of X in $L(\mathcal{H}_\Gamma)$. Since $\mathcal{H}_\Gamma$ is central and essential we have

$$f_{n-k-1}(\mathcal{F}_{\mathcal{H}_\Gamma}) = \sum_{\substack{X\in L(\mathcal{H}_\Gamma)\\ \dim(X)=n-k-1}} r(\mathcal{H}_\Gamma^X), \tag{4.3}$$

where $r(\mathcal{H}_\Gamma^X)$ is the number of regions of the arrangement $\mathcal{H}_\Gamma^X$, see [8, Theorem 2.6].

The intersection lattice $L(\mathcal{H}_\Gamma)$ is isomorphic to the lattice of flats of the graphical matroid $M(\Gamma)$. By this isomorphism to a flat F of rank k corresponds the intersection subspace $X^F = \cap_{\{i,j\}\in F}H_{e_i-e_j}$ of dimension $n-k-1$. It is easy to see that arrangements $\mathcal{H}_\Gamma^{X^F}$ and $\mathcal{H}_{\Gamma/F}$ coincide, which by (2.2) and comparing formulas (4.2) and (4.3) proves theorem. $\square$

5 Examples

By applying Theorem 1.1 we obtain the following interpretation of identities elaborated in [2, Propositions 17, 19].

Example 5.1. (i) For the permutohedron $Pe^{n-1} = \mathcal{Z}_{K_n}$, the f-polynomial is given by

$$f(\mathcal{Z}_{K_n}, q) = A_n(q+1),$$

where $A_n(q) = \sum_{\pi \in S_n} q^{\mathrm{des}(\pi)}$ is the Euler polynomial. Recall that $\mathrm{des}(\pi)$ is the number of descents of a permutation $\pi \in S_n$. It recovers the fact that the h-polynomial of the permutohedron Pe^{n-1} is the Euler polynomial $A_n(q)$.

(ii) For the cube $I^{n-1} = \mathcal{Z}_{T_n}$, where T_n is a tree on n vertices, the f-polynomial is given by

$$f(\mathcal{Z}_{T_n}, q) = (q+2)^{n-1}.$$

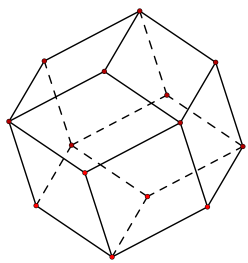

Figure 2: Rhombic dodecahedron $\mathcal{Z}_{C_4}$.

Proposition 5.2. *The f-polynomial of the graphical zonotope $\mathcal{Z}_{C_n}$ associated to the cycle graph C_n on n vertices is given by*

$$f(\mathcal{Z}_{C_n}, q) = q^n + q^{n-1} + (q+2)^n - 2(q+1)^n.$$

Proof. A flat $F \in \mathcal{F}(C_n)$ is determined by the complementary set of edges. If $\mathrm{rk}(F) = n-k, k > 1$ then the complementary set has k edges and $C_n/F = C_k$. Since $a(C_k) = 2^k - 2, k > 1$, by formula (4.2), we obtain

$$f_{n-k}(\mathcal{Z}_{C_n}) = (2^k - 2)\binom{n}{k}, \;\; 2 \leq k \leq n,$$

which leads to the required formula. □

Specially, for $n = 4$ the resulting zonotope is the rhombic dodecahedron (see Figure 2). We have

$$f(\mathcal{Z}_{C_4}, q) = 14 + 24q + 12q^2 + q^3.$$

Proposition 5.3. *Let $\Gamma = \Gamma_1 \vee_v \Gamma_2$ be the wedge of two connected graphs Γ_1 and Γ_2 at the common vertex v. Then*

$$f(\mathcal{Z}_{\Gamma}, q) = f(\mathcal{Z}_{\Gamma_1}, q) f(\mathcal{Z}_{\Gamma_2}, q).$$

Proof. The graphical matroids of involving graphs are related by $M(\Gamma) = M(\Gamma_1) \oplus M(\Gamma_2)$. For the sets of flats it holds $\mathcal{F}(\Gamma) = \{F_1 \cup F_2 \mid F_i \in \mathcal{F}(\Gamma_i), i = 1, 2\}$. For $F = F_1 \cup F_2$ we have $\Gamma/F = \Gamma_1/F_1 \vee_{[v]} \Gamma_2/F_2$, where $[v]$ is the component of the vertex v in $\Gamma_{V,F}$. Obviously $a(\Gamma/F) = a(\Gamma_1/F_1)a(\Gamma_2/F_2)$ and $\mathrm{rk}(F) = \mathrm{rk}(F_1) + \mathrm{rk}(F_2)$. The proposition follows from formula (4.1). □

The formula for cubes in Example 5.1 (ii) follows from Proposition 5.3 since any tree is a consecutive wedge of edges and $f(I^1, q) = q + 2$. It also allows us to restrict ourselves only to biconnected graphs. For a biconnected graph Γ with a disconnecting pair of vertices $\{u, v\}$ Whitney introduced the transformation called the *twist* around the pair $\{u, v\}$. This transformation does not have an affect on the graphical matroid $M(\Gamma)$ [9].

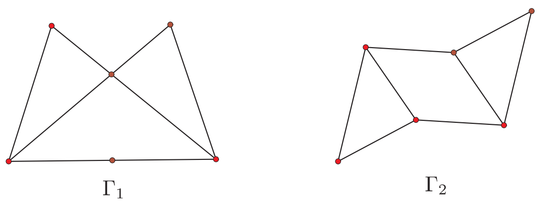

Figure 3: Biconnected graphs related by twist transformation.

Example 5.4. Figure 3 shows the pair of biconnected graphs on six vertices obtained one from another by the twist transformation. The corresponding zonotopes have the same f-polynomial

$$f(\mathcal{Z}_{\Gamma_1}, q) = f(\mathcal{Z}_{\Gamma_2}, q) = 126 + 348q + 358q^2 + 164q^3 + 30q^4 + q^5.$$

On the other hand their q-chromatic symmetric functions are different. One can check that corresponding coefficients by $m_{3,1^3}$ are different

$$[m_{3,1^3}]\Psi_q(\Gamma_1) = (11q^2 + 8q + 1) \cdot 3!,$$
$$[m_{3,1^3}]\Psi_q(\Gamma_2) = (10q^2 + 10q) \cdot 3!.$$

This shows that the q-analog of the chromatic symmetric function of a graph is not determined by the corresponding graphical matroid. By taking $q = 0$ we obtain that even the chromatic symmetric functions are different since $[m_{3,1^3}]\Psi(\Gamma_1) = 6$ and $[m_{3,1^3}]\Psi(\Gamma_2) = 0$.

Let us now consider Stanley's example of nonisomorphic graphs with the same chromatic symmetric functions, see [7]. We find that the f-polynomials of the corresponding graphical zonotopes differ for those graphs. From these examples we conclude that chromatic properties of a graph and the f-vector of the corresponding graphical zonotope are not related.

We have already noted that graphical zonotopes are generalized permutohedra. The h-polynomials of simple generalized permutohedra are determined in [5, Theorem 4.2]. The only simple graphical zonotopes are products of permutohedra [5, Proposition 5.2]. They are characterized by graphs whose biconnected components are complete subgraphs. Therefore Proposition 5.3 together with Example 5.1 (i) prove that the h-polynomial of a simple graphical zonotope is the product of Eulerian polynomials, the fact obtained in [5, Corollary 5.4]. Example 3.1 is of this sort and represents the hexagonal prism which is the product $\mathcal{Z}_{K_3} \times \mathcal{Z}_{K_2}$.

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