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# A new family of maximum scattered linear sets in $\mathrm{PG}(1, q^6)$*

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## Abstract

We generalize the example of linear set presented by the last two authors in "Vertex properties of maximum scattered linear sets of $\mathrm{PG}(1, q^n)$" (2019) to a more general family, proving that such linear sets are maximum scattered when $q$ is odd and, apart from a special case, they are new. This solves an open problem posed in "Vertex properties of maximum scattered linear sets of $\mathrm{PG}(1, q^n)$" (2019). As a consequence of Sheekey's results in "A new family of linear maximum rank distance codes" (2016), this family yields to new MRD-codes with parameters $(6, 6, q; 5)$.

*Keywords: Scattered linear set, MRD-code, linearized polynomial.*

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# 1 Introduction

Let $\Lambda = \mathrm{PG}(V, \mathbb{F}_{q^n}) = \mathrm{PG}(1, q^n)$, where $V$ is a vector space of dimension 2 over $\mathbb{F}_{q^n}$. If $U$ is a $k$-dimensional $\mathbb{F}_q$-subspace of $V$, then the $\mathbb{F}_q$*-linear set* $L_U$ is defined as

$$L_U = \{\langle \mathbf{u} \rangle_{\mathbb{F}_{q^n}} : \mathbf{u} \in U \setminus \{\mathbf{0}\}\},$$

and we say that $L_U$ has *rank* $k$. Two linear sets $L_U$ and $L_W$ of $\mathrm{PG}(1, q^n)$ are said to be $\mathrm{P\Gamma L}$*-equivalent* if there is an element $\phi$ in $\mathrm{P\Gamma L}(2, q^n)$ such that $L_U^\phi = L_W$. It may happen that two $\mathbb{F}_q$-linear sets $L_U$ and $L_W$ of $\mathrm{PG}(1, q^n)$ are $\mathrm{P\Gamma L}$-equivalent even if the $\mathbb{F}_q$-vector subspaces $U$ and $W$ are not in the same orbit of $\Gamma\mathrm{L}(2, q^n)$ (see [5, 12] for further details). In this paper we focus on *maximum scattered* $\mathbb{F}_q$-linear sets of $\mathrm{PG}(1, q^n)$, that is, $\mathbb{F}_q$-linear sets of rank $n$ in $\mathrm{PG}(1, q^n)$ of size $(q^n - 1)/(q - 1)$.

If $\langle (0, 1) \rangle_{\mathbb{F}_{q^n}}$ is not contained in the linear set $L_U$ of rank $n$ of $\mathrm{PG}(1, q^n)$ (which we can always assume after a suitable projectivity), then $U = U_f := \{(x, f(x)) : x \in \mathbb{F}_{q^n}\}$ for some linearized polynomial (or $q$*-polynomial*) $f(x) = \sum_{i=0}^{n-1} a_i x^{q^i} \in \mathbb{F}_{q^n}[x]$. In this case we will denote the associated linear set by $L_f$. If $L_f$ is scattered, then $f(x)$ is called a *scattered* $q$-polynomial; see [24].

The first examples of scattered linear sets were found by Blokhuis and Lavrauw in [3] and by Lunardon and Polverino in [18] (recently generalized by Sheekey in [24]). Apart from these, very few examples are known, see Section 3.

In [24, Section 5], Sheekey established a connection between maximum scattered linear sets of $\mathrm{PG}(1, q^n)$ and MRD-codes, which are interesting because of their applications to random linear network coding and cryptography. We point out his construction in the last section. By the results of [1] and [2], it seems that examples of maximum scattered linear sets are rare.

In this paper we will prove that any

$$f_h(x) = h^{q-1} x^q - h^{q^2-1} x^{q^2} + x^{q^4} + x^{q^5}, \quad h \in \mathbb{F}_{q^6}, \quad h^{q^3+1} = -1, \quad q \text{ odd} \tag{1.1}$$

is a scattered $q$-polynomial. This will be done by considering two cases:

**Case 1:** $h \in \mathbb{F}_q$, that is, $f_h(x) = x^q - x^{q^2} + x^{q^4} + x^{q^5}$; the condition $h^{q^3+1} = -1$ implies $q \equiv 1 \pmod 4$.

**Case 2:** $h \notin \mathbb{F}_q$. In this case $h \neq \pm\sqrt{-1}$, otherwise $h \in \mathbb{F}_{q^2}$ and then we have $h^{q+1} = 1$, a contradiction to $h^{q^3+1} = -1$.

Note that in Case 1, this example coincides with the one introduced in [27], where it has been proved that $f_h$ is scattered for $q \equiv 1 \pmod 4$ and $q \leq 29$. In Corollary 3.11 we will prove that the linear set $\mathcal{L}_h$ associated with $f_h(x)$ is new, apart from the case of $q$ a power of 5 and $h \in \mathbb{F}_q$. This solves an open problem posed in [27].

Finally, in Section 4 we prove that the $\mathbb{F}_q$-linear MRD-codes with parameters $(6, 6, q; 5)$ arising from linear sets $\mathcal{L}_h$ are not equivalent to any previously known MRD-code, apart from the case $h \in \mathbb{F}_q$ and $q$ a power of 5; see Theorem 4.1.

# 2 $\mathcal{L}_h$ is scattered

A $q$*-polynomial* (or *linearized polynomial*) over $\mathbb{F}_{q^n}$ is a polynomial of the form

$$f(x) = \sum_{i=0}^{t} a_i x^{q^i},$$

where $a_i \in \mathbb{F}_{q^n}$ and $t$ is a positive integer. We will work with linearized polynomials of degree less than or equal to $q^{n-1}$. For such a kind of polynomial, the *Dickson matrix*[1] $M(f)$ is defined as

$$M(f) := \begin{pmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ a_{n-1}^q & a_0^q & \cdots & a_{n-2}^q \\ \vdots & \vdots & \ddots & \vdots \\ a_1^{q^{n-1}} & a_2^{q^{n-1}} & \cdots & a_0^{q^{n-1}} \end{pmatrix} \in \mathbb{F}_{q^n}^{n\times n},$$

where $a_i = 0$ for $i > t$.

Recently, different results regarding the number of roots of linearized polynomials have been presented, see [4, 9, 22, 23, 26]. In order to prove that a certain polynomial is scattered, we make use of the following result; see [4, Corollary 3.5].

**Theorem 2.1.** *Consider the q-polynomial* $f(x) = \sum_{i=0}^{n-1} a_i x^{q^i}$ *over* $\mathbb{F}_{q^n}$ *and, with* $m$ *as a variable, consider the matrix*

$$M(m) := \begin{pmatrix} m & a_1 & \cdots & a_{n-1} \\ a_{n-1}^q & m^q & \cdots & a_{n-2}^q \\ \vdots & \vdots & \ddots & \vdots \\ a_1^{q^{n-1}} & a_2^{q^{n-1}} & \cdots & m^{q^{n-1}} \end{pmatrix}.$$

*The determinant of the* $(n-i)\times(n-i)$ *matrix obtained by* $M(m)$ *after removing the first* $i$ *columns and the last* $i$ *rows of* $M(m)$ *is a polynomial* $M_{n-i}(m) \in \mathbb{F}_{q^n}[m]$. *Then the polynomial* $f(x)$ *is scattered if and only if* $M_0(m)$ *and* $M_1(m)$ *have no common roots.*

## 2.1 Case 1

Let

$$f(x) = x^q - x^{q^2} + x^{q^4} + x^{q^5} \in \mathbb{F}_{q^6}[x].$$

By Theorem 2.1, $f(x)$ is scattered if and only if for each $m \in \mathbb{F}_{q^6}$ the determinants of the following two matrices do not vanish at the same time

$$M_5(m) = \begin{pmatrix} 1 & -1 & 0 & 1 & 1 \\ m^q & 1 & -1 & 0 & 1 \\ 1 & m^{q^2} & 1 & -1 & 0 \\ 1 & 1 & m^{q^3} & 1 & -1 \\ 0 & 1 & 1 & m^{q^4} & 1 \end{pmatrix},$$

$$M_6(m) = \begin{pmatrix} m & 1 & -1 & 0 & 1 & 1 \\ 1 & m^q & 1 & -1 & 0 & 1 \\ 1 & 1 & m^{q^2} & 1 & -1 & 0 \\ 0 & 1 & 1 & m^{q^3} & 1 & -1 \\ -1 & 0 & 1 & 1 & m^{q^4} & 1 \\ 1 & -1 & 0 & 1 & 1 & m^{q^5} \end{pmatrix}.$$

[1] This is sometimes called *autocirculant matrix*.

**Theorem 2.2.** *The polynomial $f(x)$ is scattered if and only if $q \equiv 1 \pmod 4$.*

*Proof.* If $q$ is even, then for $m = 0$ the matrix $M_6(0)$ has rank two and $f(x)$ is not scattered.

Suppose now $q \equiv 3 \pmod 4$. Then let $\overline{m} \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$ such that $\overline{m}^2 = -4$. So $\overline{m} = \overline{m}^{q^2} = \overline{m}^{q^4} = -\overline{m}^q = -\overline{m}^{q^3} = -\overline{m}^{q^5}$ and, by direct checking,

$$\det(M_5(\overline{m})) = (\overline{m}^2 + 4)^2 = 0, \quad \det(M_6(\overline{m})) = -(\overline{m}^2 + 4)^3 = 0$$

and $f(x)$ is not scattered.

Assume $q \equiv 1 \pmod 4$ and suppose that $f(x)$ is not scattered. Then there exists $m_0 \in \mathbb{F}_{q^6}$ such that

$$(\det(M_5(m_0)))^{q^s} = 0, \quad (\det(M_6(m_0)))^{q^t} = 0, \quad s, t = 0, 1, 2, 3, 4, 5. \tag{2.1}$$

Consider

$$P_1 = \det\begin{pmatrix} 1 & -1 & 0 & 1 & 1 \\ Y & 1 & -1 & 0 & 1 \\ 1 & Z & 1 & -1 & 0 \\ 1 & 1 & U & 1 & -1 \\ 0 & 1 & 1 & V & 1 \end{pmatrix}, \quad P_2 = \det\begin{pmatrix} X & 1 & -1 & 0 & 1 & 1 \\ 1 & Y & 1 & -1 & 0 & 1 \\ 1 & 1 & Z & 1 & -1 & 0 \\ 0 & 1 & 1 & U & 1 & -1 \\ -1 & 0 & 1 & 1 & V & 1 \\ 1 & -1 & 0 & 1 & 1 & W \end{pmatrix}. \tag{2.2}$$

Therefore,

$$X = m_0,\ Y = m_0^q,\ \ldots,\ W = m_0^{q^5} \tag{2.3}$$

is a root of $P_1 =: P_1^{(0)}$, $P_2 =: P_2^{(0)}$ and of the polynomials inductively defined by

$$P_i^{(j)}(X, Y, Z, U, V, W) = P_i^{(j-1)}(Y, Z, U, V, W, X), \quad j = 1, 2, 3, 4, 5, \quad i = 1, 2,$$

which arise from Equation 2.1. These polynomials satisfy

$$\left(P_i^{(j-1)}(m_0, m_0^q, m_0^{q^2}, m_0^{q^3}, m_0^{q^4}, m^{q^5})\right)^q = P_i^{(j)}(m_0, m_0^q, m_0^{q^2}, m_0^{q^3}, m_0^{q^4}, m^{q^5}).$$

One obtains a set $S$ of twelve equations in $X, Y, Z, U, V, W$ having a nonempty zero set. The following arguments are based on the fact that taking the resultant $R$ of two polynomials in $S$ with respect to any variable, the equations $S \cup \{R\}$ admit the same solutions.

We have

$$P_1 = YZUV - YZU - 2YZ + 2YU + 4Y - ZUV + 2ZV - 2UV + 4V + 16 = 0. \tag{2.4}$$

Consider the following resultants:

$$\begin{aligned} Q_1 := Res_V(P_1^{(3)}, P_1) = {} & 2(XY^2ZU - XY^2ZW + XY^2UW + 2XY^2W \\ & - 2XYZU + 2XYZW - 2XYUW + 8XYW + 8XY - 8XW + 16X \\ & - Y^2ZUW - 2Y^2ZU + 2YZUW - 8YZU - 8YZ + 8YU - 8YW \\ & + 8ZU - 16Z + 16U - 16W), \end{aligned}$$

$$\begin{aligned} Q_2 := Res_V(P_1^{(4)}, P_1) = {} & XYZW - XYZ - XYW + 2XZ \\ & - 2XW - 2YZ + 2YW + 4Z + 4W + 16, \end{aligned}$$

$$\begin{aligned} Q_3 := Res_V(P_1^{(5)}, P_1) = {} & XYZU - XYZ - 2XY + 2XZ \\ & + 4X - YZU + 2YU - 2ZU + 4U + 16. \end{aligned}$$

They all must be zero, as well as

$$Res_W(Res_U(Q_1,Q_3),Q_2)=8(YZ-4)(Y^2+4)(X-Z)(XZ+4)(XY-4). \quad (2.5)$$

We distinguish a number of cases.

1. Suppose that $Y^2=-4$. Since $q\equiv 1 \pmod 4$, $X=Y=Z=U=V=W$. So
$$P_1=X^4-2X^3+8X+16$$
and the resultant between $X^2+4$ and $P_1$ with respect to $X$ is $2^{27}\neq 0$ and then (2.3) is not a root of $P_1$, a contradiction.
2. Condition $YZ=4$ is clearly equivalent to $XY=4$. This means that $Y=U=W=4/X$, $Z=V=X$. Therefore, by (2.4) we get $X^2+4=0$ and we proceed as above.
3. Case $XZ=-4$. In this case $Z=-4/X$, $U=-4/Y$, $V=-4/Z=X$, $W=Y$, $X=Z$ and therefore $X^2=-4$ and we can proceed as above.
4. Condition $X=Z$ implies $X\in\mathbb{F}_{q^2}$ and so $X=Z=V$ and $Y=U=W$. By substituting in $P_1$ and $P_2$,
$$X^3Y^3+3X^3Y-6X^2Y^2-12X^2+3XY^3+24XY-12Y^2-64=0,$$
$$X^2Y^2-X^2Y+2X^2-XY^2-4XY+4X+2Y^2+4Y+16=0.$$
Eliminating $Y$ from these two equations one gets
$$8(X^2+4)^6=0,$$
and so $X^2+4=0$. We proceed as in the previous cases.

This proves that such $m_0\in\mathbb{F}_{q^6}$ does not exist and the assertion follows. □

## 2.2 Case 2

We apply the same methods as in Section 2.1. In the following preparatory lemmas (and in the rest of the paper) $q$ is a power of an arbitrary prime $p$.

**Lemma 2.3.** *Let $h\in\mathbb{F}_{q^6}$ be such that $h^{q^3+1}=-1$, $h^4\neq 1$. Then*

1. *$h^q\neq -h$;*
2. *$h^{q^2+1}\neq 1$;*
3. *$h^{q^2+1}\neq \pm h^q$, if $q$ is odd;*
4. *$h^{4q^2+4}+14h^{2q^2+2q+2}+h^{4q}=0$ implies $p=2$ and $h^{q^2-q+1}=1$ or $q=3^{2s}$, $s\in\mathbb{N}^*$, $h^{q^2-q+1}=\pm\sqrt{-1}$.*

*Proof.* The first three are easy computations. Consider now

$$h^{4q^2+4}+14h^{2q^2+2q+2}+h^{4q}=0.$$

For $p=2$ the equation above implies $h^{q^2-q+1}=1$.

Assume now $p \neq 2$. Since $h \neq 0$, it is equivalent to

$$(h^{q^2-q+1})^4 + 14(h^{q^2-q+1})^2 + 1 = 0,$$

that is $(h^{q^2-q+1})^2 = -7 \pm 4\sqrt{3} = (\sqrt{-3} \pm 2\sqrt{-1})^2$. Let $z = -7 \pm 4\sqrt{3}$. Note that $h^{q^2-q+1} = \pm\sqrt{z}$ belongs to $\mathbb{F}_{q^2}$. We distinguish two cases.

- $\sqrt{z} \in \mathbb{F}_q$. Then

  $$-1 = h^{q^3+1} = (h^{q^2-q+1})^{q+1} = \left(\pm\sqrt{z}\right)^{q+1} = z = -7 \pm 4\sqrt{3},$$

  a contradiction if $p \neq 3$. Also, $z = -1$, $q$ is an even power of 3, and $h^{q^2-q+1} = \pm\sqrt{-1}$.
- $\sqrt{z} \notin \mathbb{F}_q$. Then

  $$-1 = h^{q^3+1} = (h^{q^2-q+1})^{q+1} = \left(\pm\sqrt{z}\right)^{q+1} = -z = 7 \mp 4\sqrt{3},$$

  a contradiction if $p \neq 2$. □

**Lemma 2.4.** *Let $h \in \mathbb{F}_{q^6}$ be such that $h^{q^3+1} = -1$, $h^4 \neq 1$. If a root $\sigma$ of the polynomial*

$$h^{q+1}T^{q+1} + (h^{q^2+q+2} + h^{2q^2+2})T^q + (h^{2q^2+2} - h^{q^2+1})T + h^{q^2+2q+1} + h^{2q^2+q+1} - h^{2q} - h^{q^2+q} \in \mathbb{F}_{q^6}[T]$$

*belongs to $\mathbb{F}_{q^6}$, then one of the following cases occurs:*

- *$p = 2$, $h^{q^2-q+1} = 1$; or*
- *$q = 3^{2s}$, $s > 0$, $h^{q^2-q+1} = \pm\sqrt{-1}$; or*
- *$\sigma = \pm(h^{q^2} + h^q)$; or*
- *$h \in \mathbb{F}_q$.*

*Proof.* First, note that $\sigma = 0$ would imply $h^q(h^q+h)^q(h^{q^2+1}-1) = 0$ which is impossible by Lemma 2.3. Therefore $\sigma \neq 0$ and $\sigma^{q^i} = \frac{\ell_i(X)}{m_i(X)}$, where

$$\begin{aligned}
\ell_1(X) &= -(h^{q^2+1} - 1)(h^{q^2+1}X + h^{2q} + h^{q^2+q}) \\
m_1(X) &= h(h^q X + h^{q^2+q+1} + h^{2q^2+1}) \\
\ell_2(X) &= -(h^q + h)(2h^{q^2+q+1}X + h^{2q^2+q+2} + h^{3q^2+2} + h^{3q} + h^{q^2+2q}) \\
m_2(X) &= h^{q+1}(h^{2q^2+2}X + h^{2q}X + 2h^{q^2+2q+1} + 2h^{2q^2+q+1}) \\
\ell_3(X) &= (h^q + h)^q(3h^{2q^2+q+2}X + h^{3q}X + h^{3q^2+q+3} + h^{4q^2+3} + 3h^{q^2+3q+1} \\
&\quad + 3h^{2q^2+2q+1}) \\
m_3(X) &= h^{q^2+q}(h^{3q^2+3}X + 3h^{q^2+2q+1}X + 3h^{2q^2+2q+2} + 3h^{3q^2+q+2} + h^{4q} + h^{q^2+3q})
\end{aligned}$$

$$
\begin{aligned}
\ell_4(X) &= (h^{q^2+1}-1)(h^{4q^2+4}X+6h^{2q^2+2q+2}X+h^{4q}X+4h^{3q^2+2q+3}+4h^{4q^2+q+3}\\
&\quad +4h^{q^2+4q+1}+4h^{2q^2+3q+1})\\
m_4(X) &= h^{q^2}(4h^{3q^2+q+3}X+4h^{q^2+3q+1}X+h^{4q^2+q+4}+h^{5q^2+4}+6h^{2q^2+3q+2}\\
&\quad +6h^{3q^2+2q+2}+h^{5q}+h^{q^2+4q})\\
\ell_5(X) &= -(h^q+h)(h^{5q^2+5}X+10h^{3q^2+2q+3}X+5h^{q^2+4q+1}X+5h^{4q^2+2q+4}\\
&\quad +5h^{5q^2+q+4}+10h^{2q^2+4q+2}+10h^{3q^2+3q+2}+h^{6q}+h^{q^2+5q})\\
m_5(X) &= 5h^{4q^2+q+4}X+10h^{2q^2+3q+2}X+h^{5q}X+h^{5q^2+q+5}+h^{6q^2+5}\\
&\quad +10h^{3q^2+3q+3}+10h^{4q^2+2q+3}+5h^{q^2+5q+1}+5h^{2q^2+4q+1}\\
\ell_6(X) &= (h^q+h)^q(6h^{5q^2+q+5}X+20h^{q^3+3q+3}X+6Xh^{q^2+5q+1}+h^{6q^2+q+6}\\
&\quad +h^{7q^2+6}+15h^{4q^2+3q+4}+15h^{5q^2+2q+4}+15h^{2q^2+5q+2}\\
&\quad +15h^{3q^2+4q+2}+h^{7q}+h^{q^2+6q})\\
m_6(X) &= h^{6q^2+6}X+15h^{4q^2+2q+4}X+15h^{2q^2+4q+2}X+h^{q^6}X+6h^{5q^2+2q+5}\\
&\quad +6h^{6q^2+q+5}+20h^{3q^2+4q+3}+20h^{4q^2+3q+3}+6h^{q^2+6q+1}+6h^{2q^2+5q+1}.
\end{aligned}
$$

Since $\sigma^{q^6}=\sigma$, in particular

$$(h^{2q^2+2}+h^{2q})(h^{4q^2+4}+14h^{2q^2+2q+2}+h^{4q})(h^{q^2}-h^q)(\sigma+h^q+h^{q^2})(\sigma-h^q-h^{q^2})=0.$$

The claim follows from Lemma 2.3. □

**Lemma 2.5.** *Let* $h\in\mathbb{F}_{q^6}$ *be such that* $h^{q^3+1}=-1$, $h^4=1$. *If a root* $\sigma$ *of the polynomial*

$$h^{q+1}T^{q^2+1}+(h^q+h)^{q+1}\in\mathbb{F}_{q^6}[T]$$

*belongs to* $\mathbb{F}_{q^6}$, *then*

$$\sigma=\pm(h^{q^2}+h^q).$$

*Proof.* If $\sigma=0$, then $h^q+h=0$, a contradiction to Lemma 2.3. So we can suppose $\sigma\neq 0$. Then

$$
\begin{aligned}
\sigma^{q^2} &= -\frac{(h^{q-1}+1)^{q+1}}{\sigma}\\
\sigma^{q^4} &= (h^{q-1}+1)^{q^3+q^2-q-1}\sigma\\
\sigma^{q^6} &= -\frac{(h^{q-1}+1)^{q^5+q^4-q^3-q^2+q+1}}{\sigma}=\frac{(h^q+h)^{2q}}{\sigma}.
\end{aligned}
$$

So, $\sigma=\pm(h^{q^2}+h^q)$. □

Let $h\in\mathbb{F}_{q^6}$ be such that $h^{q^3+1}=-1$, $h^4\neq 1$. By Theorem 2.1 the polynomial

$$f_h(x)=h^{q-1}x^q-(h^{q^2-1})x^{q^2}+x^{q^4}+x^{q^5}$$

is scattered if and only if for each $m \in \mathbb{F}_{q^6}$ the determinant of the following two matrices do not vanish at the same time

$$M_6(m) = \begin{pmatrix} m & h^{q-1} & -h^{q^2-1} & 0 & 1 & 1 \\ 1 & m^q & h^{q^2-q} & h^{-q-1} & 0 & 1 \\ 1 & 1 & m^{q^2} & -h^{-q^2-1} & h^{-q^2-q} & 0 \\ 0 & 1 & 1 & m^{q^3} & h^{1-q} & -h^{1-q^2} \\ h^{q+1} & 0 & 1 & 1 & m^{q^4} & h^{q-q^2} \\ -h^{q^2+1} & h^{q^2+q} & 0 & 1 & 1 & m^{q^5} \end{pmatrix}, \tag{2.6}$$

$$M_5(m) = \begin{pmatrix} h^{q-1} & -h^{q^2-1} & 0 & 1 & 1 \\ m^q & h^{q^2-q} & h^{-q-1} & 0 & 1 \\ 1 & m^{q^2} & -h^{-q^2-1} & h^{-q^2-q} & 0 \\ 1 & 1 & m^{q^3} & h^{1-q} & -h^{1-q^2} \\ 0 & 1 & 1 & m^{q^4} & h^{q-q^2} \end{pmatrix}. \tag{2.7}$$

**Theorem 2.6.** *Let $h \in \mathbb{F}_{q^6}$, $q = 2^s$, be such that $h^{q^3+1} = 1$. Then the polynomial $f_h(x) = h^{q-1}x^q - (h^{q^2-1})x^{q^2} + x^{q^4} + x^{q^5}$ is not scattered.*

*Proof.* Consider $\overline{m} = h^{q^2} + h^q$. So,

$$\overline{m}^q = \frac{1}{h} + h^{q^2}, \qquad \overline{m}^{q^2} = \frac{1}{h^q} + \frac{1}{h}, \qquad \overline{m}^{q^3} = \frac{1}{h^{q^2}} + \frac{1}{h^q},$$
$$\overline{m}^{q^4} = h + \frac{1}{h^{q^2}}, \qquad \overline{m}^{q^5} = h^q + h.$$

By direct checking, in this case, both $\det(M_6(\overline{m})) = \det(M_5(\overline{m})) = 0$ and therefore $f_h(x)$ is not scattered. □

**Theorem 2.7.** *Let $h \in \mathbb{F}_{q^6}$, $q = p^s$, $p > 2$, be such that $h^{q^3+1} = -1$ and $h \notin \mathbb{F}_q$. Then the polynomial $f_h(x) = h^{q-1}x^q - (h^{q^2-1})x^{q^2} + x^{q^4} + x^{q^5}$ is scattered.*

*Proof.* First we note that $h^4 \neq 1$ since $q$ is odd, $h \notin \mathbb{F}_q$, and $h^{q^3+1} = -1$. Suppose that $f(x)$ is not scattered. Then $\det(M_6(m_0)) = \det(M_5(m_0)) = 0$ for some $m_0 \in \mathbb{F}_{q^6}$. Consider

$$X = m_0, \quad Y = m_0^q, \quad Z = m_0^{q^2}, \quad U = m_0^{q^3}, \quad V = m_0^{q^4}, \quad W = m_0^{q^5}.$$

With a procedure similar to the one in the proof of Theorem 2.2, we will compute resultants starting from the polynomials associated with $\det(M_6(m_0))$, $\det(M_5(m_0))^{q^3}$, and $\det(M_5(m_0))^{q^5}$.

Eliminating $W$ using $\det(M_5(m_0))^{q^3} = 0$ and $U$ using $\det(M_5(m_0))^{q^5} = 0$, one gets from $\det(M_6(m_0)) = 0$

$$h^{q^2+2q+1}\varphi_1(X,Y)\varphi_2(X,Y,Z,V)\varphi_3(X,Y,Z,V) = 0,$$

where

$$\begin{aligned}
\varphi_1(X,Y) &= h^{q+1}XY + h^{2q^2+2}X - h^{q^2+1}X + h^{q^2+q+2}Y + h^{2q^2+2}Y \\
&\quad + h^{q^2+2q+1} + h^{2q^2+q+1} - h^{2q} - h^{q^2+q};\\
\varphi_2(X,Y,Z,V) &= h^{q^2+q+2}XYZV - h^{q^2+q+2}XYZ - h^2XY - h^{q+1}XY \\
&\quad - h^{2q^2+q+1}XZV - h^{2q^2+2}XV - h^{2q^2+q+1}XV - h^{q^2+2q+3}YZ \\
&\quad - h^{2q^2+q+3}YZ - h^{q^2+q+2}Y - h^{2q^2+2}Y - h^{q^2+2q+1}Y \\
&\quad - h^{2q^2+q+1}Y - h^{q^2+2q+1}ZV - h^{2q^2+q+1}ZV - h^{2q^2+q+1}V \\
&\quad - h^{3q^2+1}V - h^{2q^2+2q}V - h^{3q^2+q}V + h^{2q^2+q+3} + h^{3q^2+3} \\
&\quad + h^{2q^2+2q+2} + h^{3q^2+q+2} - 2h^{q^2+q+2} - 2h^{2q^2+2} - 2h^{q^2+2q+1} \\
&\quad - 2h^{2q^2+q+1} + h^{q+1} + h^{q^2+1} + h^{2q} + h^{q^2+q};\\
\varphi_3(X,Y,Z,V) &= h^{q^2+q+2}XYZV + h^{q^2+q+2}XYZ - h^2XY - h^{q+1}XY \\
&\quad + h^{2q^2+q+1}XZV - h^{2q^2+2}XV - h^{2q^2+q+1}XV - h^{q^2+2q+3}YZ \\
&\quad - h^{2q^2+q+3}YZ + h^{q^2+q+2}Y + h^{2q^2+2}Y + h^{q^2+2q+1}Y \\
&\quad + h^{2q^2+q+1}Y - h^{q^2+2q+1}ZV - h^{2q^2+q+1}ZV + h^{2q^2+q+1}V \\
&\quad + h^{3q^2+1}V + h^{2q^2+2q}V + h^{3q^2+q}V + h^{2q^2+q+3} + h^{3q^2+3} \\
&\quad + h^{2q^2+2q+2} + h^{3q^2+q+2} - 2h^{q^2+q+2} - 2h^{2q^2+2} - 2h^{q^2+2q+1} \\
&\quad - 2h^{2q^2+q+1} + h^{q+1} + h^{q^2+1} + h^{2q} + h^{q^2+q}.
\end{aligned}$$

- If $\varphi_1(X,Y)=0$, then by Lemma 2.4 either $q=3^{2s}$ and $h^{q^2-q+1}=\pm\sqrt{-1}$, or $X=\pm(h^{q^2}+h^q)$.

  In this last case,

$$\begin{aligned}
&Y=\pm(-h^{-1}+h^{q^2}), \quad Z=\pm(-h^{-q}-h^{-1}), \quad U=\pm\left(-h^{-q^2}-h^{-q}\right)\\
&V=\pm(h-h^{-q^2}), \quad W=\pm(h^q+h).
\end{aligned} \tag{2.8}$$

  By substituting in $\det(M_5(m_0))$ one obtains

$$4(h+h^q)^{q+1}(h^{q^2+1}-1)(h^{q^2+1}-h^q)=0$$

  and

$$4(h+h^q)^{q+1}(h^{q^2+1}-1)(h^{q^2+1}+h^q)=0,$$

  respectively. Both are not possible due to Lemma 2.3.

  Consider now the case $q=3^{2s}$, $h^{q^2-q+1}=\pm\sqrt{-1}$ and $X\neq\pm(h^{q^2}+h^q)$. So, using $\varphi_1(X,Y)=0$ and $h^{q^2-q+1}=\pm\sqrt{-1}$,

$$\begin{aligned}
&\det(M_5(m_0))=0 \Longrightarrow\\
&\quad h^{q^2+2q+1}(h^{q^2}+h^q)(h^q+h)(h^{q^2+1}-1)(h^{q^2+q}+h^q)^3(h^{q^2+q}-h^q)^3\cdot\\
&\qquad \cdot(h^{2q^2+2}-h^{q^2+1}+h^{2q})(X+h^q+h^{q^2})^2(X-h^q-h^{q^2})^2=0.
\end{aligned}$$

  By Lemma 2.3 we get

$$h^{2q^2+2}-h^{q^2+1}+h^{2q}=0,$$

  which yields to a contradiction.

- If $\varphi_2(X,Y,Z,V)=0$ and $\varphi_1(X,Y)\neq 0$, eliminating $V$ in $\det(M_5(m_0))=0$ one gets

$$2h^{3q^2+2q+1}(h^{q+2}YZ-h^{q^2+2}-h^{q^2+q+1}+h^q+h)\cdot$$
$$\cdot(hXY+h^{q^2+q+1}+h^{2q^2+1}-h^{q^2}-h^q)\cdot$$
$$\cdot(h^{q+1}XZ+h^{q+1}+h^{q^2+1}+h^{2q}+h^{q^2+q})\cdot$$
$$\cdot(h^{q+2}YZ+hY+h^qY-h^{q^2+q+1}Z+h^qZ-h^{q^2+2}-h^{q^2+q+1}+h^q+h)=0.$$

  - If $h^{q+2}YZ-h^{q^2+2}-h^{q^2+q+1}+h^q+h=0$ then, from

$$Z=\frac{h^{q^2+2}+h^{q^2+q+1}-h^q-h}{h^{q+2}Y},$$

    $\det(M_5)=0$ gives

$$(h^q+h)^{q+1}(hY-h^{q^2+1}+1)(hY+h^{q^2+1}-1)=0.$$

    So, (2.8) holds and as in the case $\varphi_1(X,Y)=0$ a contradiction arises.
  - If $hXY+h^{q^2+q+1}+h^{2q^2+1}-h^{q^2}-h^q=0$ then, from

$$Y=\frac{-h^{q^2+q+1}-h^{2q^2+1}+h^{q^2}+h^q}{hX},$$

    the equation $\det(M_5(m_0))=0$ yields

$$(h^q+h)(h^{q^2+1}-1)(X-h^{q^2}-h^q)(X+h^{q^2}+h^q)=0.$$

    So, (2.8) holds and as in the case $\varphi_1(X,Y)=0$, a contradiction.
  - If $h^{q+1}XZ+h^{q+1}+h^{q^2+1}+h^{2q}+h^{q^2+q}=0$ then by Lemma 2.5

$$(X-h^{q^2}-h^q)(X+h^{q^2}+h^q)=0,$$

    again a contradiction as before.
  - If $h^{q+2}YZ+hY+h^qY-h^{q^2+q+1}Z+h^qZ-h^{q^2+2}-h^{q^2+q+1}+h^q+h=0$ then

$$Z=-\frac{(h^q+h)Y-h^{q^2+2}-h^{q^2+q+1}+h^q+h}{h^{q+2}Y-h^{q^2+q+1}+h^q}.$$

    So, substituting $U=Z^q, V=Z^{q^2}, W=Z^{q^3}, X=Z^{q^4}$ in $\det(M_5(m_0))=0$ we get

$$(h-1)^{q+1}(h+1)^{q+1}(h^q+h)^{q+1}(h^{q^2+1}-1)\cdot$$
$$\cdot(hY-h^{q^2+1}+1)^2(hY+h^{q^2+1}-1)^2=0.$$

    By Lemma 2.3, $(hY-h^{q^2+1}+1)(hY+h^{q^2+1}-1)=0$. Since $Y=\pm(h^{q^2}-1/h)$ then (2.8) holds and a contradiction arises as in the case $\varphi_1(X,Y)=0$.

- If $\varphi_3(X,Y,Z,V)=0$ and $\varphi_1(X,Y)\neq 0$, eliminating $U$ from $\det(M_5(m_0))=0=\det(M_5(m_0))^{q^5}$ and then eliminating $V$ using $\varphi_3(X,Y,Z,V)=0$ one gets

$$2h^{3q^2+q+1}(h^q+h)^q(h^{q+2}YZ-h^{q^2+2}-h^{q^2+q+1}+h^q+h)^2\cdot \\ \cdot(hXY+h^{q^2+q+1}+h^{2q^2+1}-h^{q^2}-h^q)\cdot \\ \cdot(h^{q+1}XZ+h^{q+1}+h^{q^2+1}+h^{2q}+h^{q^2+q})=0.$$

A contradiction follows as in the case $\varphi_2(X,Y,Z,V)=0$ and $\varphi_1(X,Y)\neq 0$. $\square$

## 3 The equivalence issue

We will deal with the linear sets $\mathcal{L}_h = L_{f_h}$ associated with the polynomials defined in (1.1). Note that when $h\in\mathbb{F}_q$, such a linear set coincide with the one introduced in [27, Section 5].

### 3.1 Preliminary results

We start by listing the non-equivalent (under the action of $\Gamma\mathrm{L}(2,q^6)$) maximum scattered subspaces of $\mathbb{F}_{q^6}^2$, i.e. subspaces defining maximum scattered linear sets.

**Example 3.1.**

1. $U^1 := \{(x,x^q) : x\in\mathbb{F}_{q^6}\}$, defining the linear set of pseudoregulus type, see [3, 11];
2. $U^2_\delta := \{(x,\delta x^q + x^{q^5}) : x\in\mathbb{F}_{q^6}\}$, $\mathrm{N}_{q^6/q}(\delta)\notin\{0,1\}$, defining the linear set of LP-type, see [16, 18, 20, 24];
3. $U^3_\delta := \{(x,x^q+\delta x^{q^4}) : x\in\mathbb{F}_{q^6}\}$, $\mathrm{N}_{q^6/q^3}(\delta)\notin\{0,1\}$, satisfying further conditions on $\delta$ and $q$, see [6, Theorems 7.1 and 7.2] and [23] [2];
4. $U^4_\delta := \{(x,x^q+x^{q^3}+\delta x^{q^5}) : x\in\mathbb{F}_{q^6}\}$, $q$ odd and $\delta^2+\delta=1$, see [10, 21].

In order to simplify the notation, we will denote by $L^1$ and $L^i_\delta$ the $\mathbb{F}_q$-linear set defined by $U^1$ and $U^i_\delta$, respectively. We will also use the following notation:

$$\mathcal{U}_h := U_{h^{q-1}x^q-h^{q^2-1}x^{q^2}+x^{q^4}+x^{q^5}}.$$

**Remark 3.2.** Consider the non-degenerate symmetric bilinear form of $\mathbb{F}_{q^6}$ over $\mathbb{F}_q$ defined by

$$\langle x,y\rangle = \mathrm{Tr}_{q^6/q}(xy),$$

for each $x,y\in\mathbb{F}_{q^6}$. Then the *adjoint* $\hat{f}$ of the linearized polynomial $f(x)=\sum_{i=0}^5 a_ix^{q^i}\in\tilde{\mathcal{L}}_{6,q}$ with respect to the bilinear form $\langle\ ,\ \rangle$ is

$$\hat{f}(x)=\sum_{i=0}^{5} a_i^{q^{6-i}}x^{q^{6-i}},$$

i.e.

$$\mathrm{Tr}_{q^6/q}(xf(y))=\mathrm{Tr}_{q^6/q}(y\hat{f}(x)),$$

for any $x,y\in\mathbb{F}_{q^6}$.

[2] Here $q>2$, otherwise it is not scattered.

In [10, Propositions 3.1, 4.1 and 5.5] the following result has been proved.

**Lemma 3.3.** *Let $L_f$ be one of the maximum scattered of $\mathrm{PG}(1,q^6)$ listed before. Then a linear set $L_U$ of $\mathrm{PG}(1,q^6)$ is $\mathrm{P\Gamma L}$-equivalent to $L_f$ if and only if $U$ is $\Gamma\mathrm{L}$-equivalent either to $U_f$ or to $U_{\hat{f}}$ Furthermore, $L_U$ is $\mathrm{P\Gamma L}$-equivalent to $L^3_\delta$ if and only if $U$ is $\Gamma\mathrm{L}$-equivalent to $U^3_\delta$.*

We will work in the following framework. Let $x_0,\dots,x_5$ be the homogeneous coordinates of $\mathrm{PG}(5,q^6)$ and let

$$\Sigma = \{\langle (x, x^q, \dots, x^{q^5})\rangle_{\mathbb{F}_{q^6}} : x \in \mathbb{F}_{q^6}\}$$

be a fixed canonical subgeometry of $\mathrm{PG}(5,q^6)$. The collineation $\hat{\sigma}$ of $\mathrm{PG}(5,q^6)$ defined by $\langle (x_0,\dots,x_5)\rangle_{\mathbb{F}_{q^6}}^{\hat{\sigma}} = \langle (x_5^q, x_0^q, \dots, x_4^q)\rangle_{\mathbb{F}_{q^6}}$ fixes precisely the points of $\Sigma$. Note that if $\sigma$ is a collineation of $\mathrm{PG}(5,q^6)$ such that $\mathrm{Fix}(\sigma)=\Sigma$, then $\sigma=\hat{\sigma}^s$, with $s\in\{1,5\}$.

Let $\Gamma$ be a subspace of $\mathrm{PG}(5,q^6)$ of dimension $k\geq 0$ such that $\Gamma\cap\Sigma=\emptyset$, and $\dim(\Gamma\cap\Gamma^\sigma)\geq k-2$. Let $r$ be the least positive integer satisfying the condition

$$\dim(\Gamma\cap\Gamma^\sigma\cap\Gamma^{\sigma^2}\cap\dots\cap\Gamma^{\sigma^r}) > k-2r. \tag{3.1}$$

Then we will call the integer $r$ the *intersection number of* $\Gamma$ w.r.t. $\sigma$ and we will denote it by $\mathrm{intn}_\sigma(\Gamma)$; see [27].

Note that if $\hat{\sigma}$ is as above, then $\mathrm{intn}_{\hat{\sigma}}(\Gamma)=\mathrm{intn}_{\hat{\sigma}^5}(\Gamma)$ for any $\Gamma$.

As a consequence of the results of [11, 27] we have the following result.

**Result 3.4.** *Let $L$ be a scattered linear set of $\Lambda=\mathrm{PG}(1,q^6)$ which can be realized in $\mathrm{PG}(5,q^6)$ as the projection of $\Sigma=\mathrm{Fix}(\sigma)$ from $\Gamma\simeq\mathrm{PG}(3,q^6)$ over $\Lambda$. If $\mathrm{intn}_\sigma(\Gamma)\neq 1,2$, then $L$ is not equivalent to any linear set neither of pseudoregulus type nor of LP-type.*

### 3.2 $\mathcal{L}_h$ is new in most of the cases

The linear set $\mathcal{L}_h$ can be obtained by projecting the canonical subgeometry

$$\Sigma = \{\langle (x, x^q, x^{q^2}, x^{q^3}, x^{q^4}, x^{q^5})\rangle_{\mathbb{F}_{q^6}} : x \in \mathbb{F}^*_{q^6}\}$$

from

$$\Gamma\colon \begin{cases} x_0 = 0 \\ h^{q-1}x_1 - h^{q^2-1}x_2 + x_4 + x_5 = 0 \end{cases}$$

to

$$\Lambda\colon \begin{cases} x_1 = 0 \\ x_2 = 0 \\ x_3 = 0 \\ x_4 = 0. \end{cases}$$

Then

$$\Gamma^{\hat{\sigma}}\colon \begin{cases} x_1 = 0 \\ h^{q^2-q}x_2 + h^{-q-1}x_3 + x_5 + x_0 = 0 \end{cases}$$

and

$$\Gamma^{\hat{\sigma}^2}: \begin{cases} x_2 = 0 \\ -h^{-1-q^2}x_3 + h^{-q^2-q}x_4 + x_0 + x_1 = 0. \end{cases}$$

Therefore,

$$\Gamma \cap \Gamma^{\hat{\sigma}}: \begin{cases} x_0 = 0 \\ x_1 = 0 \\ -h^{q^2-1}x_2 + x_4 + x_5 = 0 \\ h^{q^2-q}x_2 + h^{-q-1}x_3 + x_5 = 0 \end{cases}$$

and

$$\Gamma \cap \Gamma^{\hat{\sigma}} \cap \Gamma^{\hat{\sigma}^2}: \begin{cases} x_0 = 0 \\ x_1 = 0 \\ x_2 = 0 \\ x_4 + x_5 = 0 \\ h^{-q-1}x_3 + x_5 = 0 \\ -h^{-q^2-1}x_3 + h^{-q^2-q}x_4 = 0. \end{cases}$$

Hence, $\dim_{\mathbb{F}_{q^6}}(\Gamma\cap\Gamma^{\hat{\sigma}}) = 1$ and $\dim_{\mathbb{F}_{q^6}}(\Gamma\cap\Gamma^{\hat{\sigma}}\cap\Gamma^{\hat{\sigma}^2}) = -1$, since $q$ is odd and $h^{q^3+1} \neq 1$. So, $\mathrm{intn}_{\sigma}(\Gamma) = 3$ and hence, by Result 3.4 it follows that $\mathcal{L}_h$ is not equivalent neither to $L^1$ nor to $L^2_\delta$.

Generalizing [27, Propositions 5.4 and 5.5] we have the following two propositions.

**Proposition 3.5.** *The linear set* $\mathcal{L}_h$ *is not* $\mathrm{P\Gamma L}$*-equivalent to* $L^3_\delta$.

*Proof.* By Lemma 3.3, we have to check whether $\mathcal{U}_h$ and $U^3_\delta$ are $\Gamma$L-equivalent, with $\mathrm{N}_{q^6/q^3}(\delta) \notin \{0,1\}$. Suppose that there exist $\rho \in \mathrm{Aut}(\mathbb{F}_{q^6})$ and an invertible matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ such that for each $x \in \mathbb{F}_{q^6}$ there exists $z \in \mathbb{F}_{q^6}$ satisfying

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x^\rho \\ h^{\rho(q-1)}x^{\rho q} - h^{\rho(q^2-1)}x^{\rho q^2} + x^{\rho q^4} + x^{\rho q^5} \end{pmatrix} = \begin{pmatrix} z \\ z^q + \delta z^{q^4} \end{pmatrix}.$$

Equivalently, for each $x \in \mathbb{F}_{q^6}$ we have[3]

$$\begin{aligned} cx^\rho + d(h^{q-1}x^{\rho q} - h^{q^2-1}x^{\rho q^2} + x^{\rho q^4} + x^{\rho q^5}) = \\ a^q x^{\rho q} + b^q(h^{q^2-q}x^{\rho q^2} + h^{-q-1}x^{\rho q^3} + x^{\rho q^5} + x^\rho) \\ + \delta[a^{q^4}x^{\rho q^4} + b^{q^4}(h^{-q^2+q}x^{\rho q^5} - h^{q+1}x^\rho + x^{\rho q^2} + x^{\rho q^3})]. \end{aligned}$$

This is a polynomial identity in $x^\rho$ and hence we have the following relations:

$$\begin{cases} c = b^q + \delta h^{q+1}b^{q^4} \\ dh^{q-1} = a^q \\ -dh^{q^2-1} = h^{q^2-q}b^q + \delta b^{q^4} \\ 0 = h^{-1-q}b^q + \delta b^{q^4} \\ d = \delta a^{q^4} \\ d = b^q + \delta h^{q-q^2}b^{q^4}. \end{cases} \tag{3.2}$$

[3]We may replace $h^\rho$ by $h$, since $h^{q^3+1} = -1$ if and only if $(h^\rho)^{q^3+1} = -1$.

From the second and the fifth equations, if $a \neq 0$ then $\delta h^{q-1} = a^{q-q^4}$ and $\mathrm{N}_{q^6/q^3}(\delta) = 1$, which is not possible and so $a = d = 0$ and $b, c \neq 0$. By the last equation, we would get $\mathrm{N}_{q^6/q^3}(\delta) = 1$, a contradiction. □

**Proposition 3.6.** *The linear set $\mathcal{L}_h$ is $\mathrm{P\Gamma L}$-equivalent to $L^4_\delta$ (with $\delta^2 + \delta = 1$) if and only if there exist $a, b, c, d \in \mathbb{F}_{q^6}$ and $\rho \in \mathrm{Aut}(\mathbb{F}_{q^6})$ such that $ad - bc \neq 0$ and either*

$$\begin{cases} c = b^q - \delta k^{q^2+1} b^{q^5} \\ a = -k^{q+1} b^{q^4} - \delta^q b^{q^2} \\ d = k^{-q+1} b^{q^3} + \delta b^{q^5} \\ b^{q^3} + (k^{q-1} + \delta k^{q+q^2}) b^{q^5} = 0 \\ k^{q^2-q} b^q + (1 + k^{q^2-q}) b^{q^3} + \delta k^{q^2-1} b^{q^5} = 0 \\ -\delta b^q + (k^{-q+1} + \delta^2 k^{1-q^2}) b^{q^3} + \delta b^{q^5} = 0 \end{cases} \tag{3.3}$$

*or*

$$\begin{cases} c = \delta b^q - k^{q^2+1} b^{q^5} \\ a = -\delta^q k^{q+1} b^{q^4} - b^{q^2} \\ d = k^{-q+1} b^{q^3} + b^{q^5} \\ \delta b^{q^3} + (k^{q-1} - \delta k^{q^2+q}) b^{q^5} = 0 \\ \delta k^{q^2-q} b^q + (k^{q^2-q} + 1) b^{q^3} + k^{q^2-1} b^{q^5} = 0 \\ \delta^2 b^q + (k^{-q+1} + \delta^2 k^{-q^2+1}) b^{q^3} + b^{q^5} = 0, \end{cases} \tag{3.4}$$

*where $k = h^\rho$.*

*Proof.* By Lemma 3.3 we have to check whether $\mathcal{U}_h$ is equivalent either to $U^4_\delta$ or to $(U^4_\delta)^\perp$. Suppose that there exist $\rho \in \mathrm{Aut}(\mathbb{F}_{q^6})$ and an invertible matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ such that for each $x \in \mathbb{F}_{q^6}$ there exists $z \in \mathbb{F}_{q^6}$ satisfying

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x^\rho \\ h^{\rho(q-1)} x^{\rho q} - h^{\rho(q^2-1)} x^{\rho q^2} + x^{\rho q^4} + x^{\rho q^5} \end{pmatrix} = \begin{pmatrix} z \\ z^q + z^{q^3} + \delta z^{q^5} \end{pmatrix}.$$

Equivalently, for each $x \in \mathbb{F}_{q^6}$ we have

$$\begin{aligned} c x^\rho + d(k^{q-1} x^{\rho q} - k^{q^2-1} x^{\rho q^2} + x^{\rho q^4} + x^{\rho q^5}) = \\ a^q x^{\rho q} + b^q (k^{q^2-q} x^{\rho q^2} + k^{-1-q} x^{\rho q^3} + x^{\rho q^5} + x^\rho) \\ + a^{q^3} x^{\rho q^3} + b^{q^3} (k^{-q+1} x^{\rho q^4} - k^{-q^2+1} x^{\rho q^5} + x^{\rho q} + x^{\rho q^2}) \\ + \delta [a^{q^5} x^{\rho q^5} + b^{q^5} (-k^{1+q^2} x^\rho + k^{q^2+q} x^{\rho q} + x^{\rho q^3} + x^{\rho q^4})]. \end{aligned}$$

This is a polynomial identity in $x^\rho$ which yields to the following equations

$$\begin{cases} c = b^q - \delta k^{q^2+1} b^{q^5} \\ d k^{q-1} = a^q + b^{q^3} + \delta k^{q+q^2} b^{q^5} \\ -d k^{q^2-1} = k^{q^2-q} b^q + b^{q^3} \\ 0 = k^{-q-1} b^q + a^{q^3} + \delta b^{q^5} \\ d = k^{-q+1} b^{q^3} + \delta b^{q^5} \\ d = b^q - k^{-q^2+1} b^{q^3} + \delta a^{q^5} \end{cases}$$

which can be written as (3.3).

Now, suppose that there exist $\rho \in \mathrm{Aut}(\mathbb{F}_{q^6})$ and an invertible matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ such that for each $x \in \mathbb{F}_{q^6}$ there exists $z \in \mathbb{F}_{q^6}$ satisfying

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x^\rho \\ h^{\rho(q-1)}x^{\rho q} - h^{\rho(q^2-1)}x^{\rho q^2} + x^{\rho q^4} + x^{\rho q^5} \end{pmatrix} = \begin{pmatrix} z \\ \delta z^q + z^{q^3} + z^{q^5} \end{pmatrix}.$$

Equivalently, for each $x \in \mathbb{F}_{q^6}$ we have

$$\begin{aligned} cx^\rho + d(k^{q-1}x^{\rho q} - k^{q^2-1}x^{\rho q^2} + x^{\rho q^4} + x^{\rho q^5}) = \\ \delta[a^q x^{\rho q} + b^q(k^{q^2-q}x^{\rho q^2} + k^{-1-q}x^{\rho q^3} + x^{\rho q^5} + x^\rho)] \\ + a^{q^3}x^{\rho q^3} + b^{q^3}(k^{-q+1}x^{\rho q^4} - k^{-q^2+1}x^{\rho q^5} + x^{\rho q} + x^{\rho q^2}) \\ + a^{q^5}x^{\rho q^5} + b^{q^5}(-k^{1+q^2}x^\rho + k^{q^2+q}x^{\rho q} + x^{\rho q^3} + x^{\rho q^4}). \end{aligned}$$

This is a polynomial identity in $x^\rho$ which yields to the following equations

$$\begin{cases} c = \delta b^q - k^{q^2+1}b^{q^5} \\ dk^{q-1} = \delta a^q + b^{q^3} + k^{q+q^2}b^{q^5} \\ -dk^{q^2-1} = \delta k^{q^2-q}b^q + b^{q^3} \\ 0 = \delta k^{-q-1}b^q + a^{q^3} + b^{q^5} \\ d = k^{-q+1}b^{q^3} + b^{q^5} \\ d = \delta b^q - k^{-q^2+1}b^{q^3} + a^{q^5} \end{cases}$$

which can be written as (3.4). □

We are now ready to prove that when $h \notin \mathbb{F}_{q^2}$, $\mathcal{L}_h$ is new.

**Proposition 3.7.** *If $h \notin \mathbb{F}_{q^2}$, then $\mathcal{L}_h$ is not* $\mathrm{P\Gamma L}$*-equivalent to $L^4_\delta$ (with $\delta^2 + \delta = 1$).*

*Proof.* By Proposition 3.6 we have to show that there are no $a, b, c$ and $d$ in $\mathbb{F}_{q^6}$ such that $ad - bc \neq 0$ and (3.3) or (3.4) are satisfied. Note that $b = 0$ in (3.3) and (3.4) yields $a = c = d = 0$, a contradiction. So, suppose $b \neq 0$. Since $h \notin \mathbb{F}_{q^2}$ then $k \notin \mathbb{F}_{q^2}$. We start by proving that the last three equations of (3.3), i.e.

$$\begin{cases} \mathrm{Eq}_1: b^{q^3} + (k^{q-1} + \delta k^{q+q^2})b^{q^5} = 0 \\ \mathrm{Eq}_2: k^{q^2-q}b^q + (1 + k^{q^2-q})b^{q^3} + \delta k^{q^2-1}b^{q^5} = 0 \\ \mathrm{Eq}_3: -\delta b^q + (k^{-q+1} + \delta^2 k^{1-q^2})b^{q^3} + \delta b^{q^5} = 0, \end{cases}$$

yield a contradiction. As in the above section, we will consider the $q$-th powers of $\mathrm{Eq}_1$, $\mathrm{Eq}_2$ and $\mathrm{Eq}_3$ replacing $b^{q^i}$, $k^{q^j}$, and $\delta^{q^\ell}$ (respectively) by $X_i$, $Y_j$, and $Z_\ell$ with $i, j \in \{0,1,2,3,4,5\}$ and $\ell \in \{0,1\}$. Consider the set $S$ of polynomials in the variables $X_i, Y_j$, and $Z_\ell$

$$S := \{\mathrm{Eq}_1^{q^\alpha}, \mathrm{Eq}_2^{q^\beta}, \mathrm{Eq}_3^{q^\gamma} : \alpha, \beta, \gamma \in \{0,1,2,3,4,5\}\}.$$

By eliminating from $S$ the variables $X_5$, $X_4$, $X_3$, and $X_2$ using $\mathrm{Eq}_1$, $\mathrm{Eq}_1^q$, $\mathrm{Eq}_1^{q^4}$, and $\mathrm{Eq}_1^{q^3}$ respectively we obtain

$$X_0Y_1(Z_1Y_0^2Y_2 - Z_1Y_0Y_2^2 - Z_1Y_0 + Z_1Y_2 - Z_0^2Z_2 - Z_2) = 0.$$

By the conditions on $b$ and $k$, $X_0Y_1 \neq 0$ and therefore

$$P := Z_1Y_0^2Y_2 - Z_1Y_0Y_2^2 - Z_1Y_0 + Z_1Y_2 - Z_0^2Z_2 - Z_2 = 0.$$

We eliminate $Z_1$ in $S$ using $P$, obtaining, w.r.t. $b$, $k$, and $\delta$,

$$bk^{q^2+1}(k - k^q)(k + k^q)(k^{q^2+1} - 1)(k^{q^2+1} + 1) = 0,$$

a contradiction to $k \notin \mathbb{F}_{q^2}$.

Consider now the last three equations of (3.4), i.e.

$$\begin{cases} \mathrm{Eq}_1\colon \delta b^{q^3} + (k^{q-1} - \delta k^{q^2+q})b^{q^5} = 0 \\ \mathrm{Eq}_2\colon \delta k^{q^2-q}b^q + (k^{q^2-q} + 1)b^{q^3} + k^{q^2-1}b^{q^5} = 0 \\ \mathrm{Eq}_3\colon \delta^2 b^q + (k^{-q+1} + \delta^2 k^{-q^2+1})b^{q^3} + b^{q^5} = 0. \end{cases}$$

As before, we will consider the $q$-th powers of $\mathrm{Eq}_1$, $\mathrm{Eq}_2$, and $\mathrm{Eq}_3$ replacing $b^{q^i}$, $k^{q^j}$, and $\delta^{q^\ell}$ (respectively) by $X_i$, $Y_j$, and $Z_\ell$ with $i, j \in \{0, 1, 2, 3, 4, 5\}$ and $\ell \in \{0, 1\}$. Consider the set $S$ of polynomials in the variables $X_i, Y_j$ and $Z_\ell$

$$S := \{\mathrm{Eq}_1^{q^\alpha}, \mathrm{Eq}_2^{q^\beta}, \mathrm{Eq}_3^{q^\gamma} : \alpha, \beta, \gamma \in \{0, 1, 2, 3, 4, 5\}\}.$$

We eliminate in $S$ the variables $X_5$, $X_4$, $X_3$, and $X_2$ using $\mathrm{Eq}_1$, $\mathrm{Eq}_1^q$, $\mathrm{Eq}_1^{q^4}$, and $\mathrm{Eq}_1^{q^3}$ respectively, and we get

$$Y_0X_0(Z_1Y_0^2Y_2^2 + 2Z_1Y_0Y_1^2Y_2 + 2Z_1Y_0Y_2 + Z_1Y_1^2 - Y_0^2Y_2^2 - Y_0Y_1^2Y_2 - Y_0Y_2 - Y_1^2) = 0.$$

Since $b \neq 0$ and $k \notin \mathbb{F}_{q^2}$, $X_0Y_0 \neq 0$ and therefore

$$P := Z_1Y_0^2Y_2^2 + 2Z_1Y_0Y_1^2Y_2 + 2Z_1Y_0Y_2 + Z_1Y_1^2 - Y_0^2Y_2^2 - Y_0Y_1^2Y_2 - Y_0Y_2 - Y_1^2 = 0.$$

Once again we consider the resultants of the polynomials in $S$ and $P$ w.r.t. $Z_1$ and we obtain

$$bk^{q^2+2q}(k - k^q)(k + k^q)(k^{q^2+1} - 1)(k^{q^2+1} + 1) = 0,$$

a contradiction to $k \notin \mathbb{F}_{q^2}$. □

As a consequence of the above considerations and Propositions 3.5 and 3.7, we have the following.

**Corollary 3.8.** *If $h \notin \mathbb{F}_{q^2}$, then $\mathcal{L}_h$ is not $\mathrm{P\Gamma L}$-equivalent to any known scattered linear set in $\mathrm{PG}(1, q^6)$.*

### 3.3 $\mathcal{L}_h$ may be defined by a trinomial

Suppose that $h \in \mathbb{F}_{q^2}$, then the condition on $h$ becomes $h^{q+1} = -1$. For such $h$ we can prove that the linear set $\mathcal{L}_h$ can be defined by the $q$-polynomial $(h^{-1} - 1)x^q + x^{q^3} + (h - 1)x^{q^5}$.

**Proposition 3.9.** *If $h \in \mathbb{F}_{q^2}$, then the linear set $\mathcal{L}_h$ is $\mathrm{P\Gamma L}$-equivalent to*

$$L_{\mathrm{tri}} := \{\langle (x, (h^{-1} - 1)x^q + x^{q^3} + (h - 1)x^{q^5}) \rangle_{\mathbb{F}_{q^6}} : x \in \mathbb{F}_{q^6}^*\}.$$

*Proof.* Let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}(2,q^6)$ with $a = -h + h^{-1}, b = 1, c = h^{-1} - 1 - h^3 + h^2$ and $d = h - h^2 - 1$. Straightforward computations show that the subspaces $\mathcal{U}_h$ and $U_{(h^{-1}-1)x^q + x^{q^3} + (h-1)x^{q^5}}$ are $\Gamma\mathrm{L}(2,q^6)$-equivalent under the action of the matrix $A$. Hence, the linear sets $\mathcal{L}_h$ and $L_{\mathrm{tri}}$ are $\mathrm{P\Gamma L}$-equivalent. □

The fact that $\mathcal{L}_h$ can also be defined by a trinomial will help us to completely close the equivalence issue for $\mathcal{L}_h$ when $h \in \mathbb{F}_{q^2}$. Indeed, we can prove the following:

**Proposition 3.10.** *If $h \in \mathbb{F}_{q^2}$, then the linear set $\mathcal{L}_h$ is* $\mathrm{P\Gamma L}$*-equivalent to some $L_\delta^4$ ($\delta^2 + \delta = 1$) if and only if $h \in \mathbb{F}_q$ and $q$ is a power of* 5.

*Proof.* Recall that by [27, Proposition 5.5] if $h \in \mathbb{F}_q$ and $q$ is a power of 5, then $\mathcal{L}_h$ is $\mathrm{P\Gamma L}$-equivalent to some $L_\delta^4$. As in the proof of Proposition 3.6, by Lemma 3.3 we have to check whether $U_{(h^{-1}-1)x^q + x^{q^3} + (h-1)x^{q^5}}$ is $\Gamma\mathrm{L}$-equivalent either to $U_\delta^4$ or to $(U_\delta^4)^\perp$. Suppose that there exist $\rho \in \mathrm{Aut}(\mathbb{F}_{q^6})$ and an invertible matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ such that for each $x \in \mathbb{F}_{q^6}$ there exists $z \in \mathbb{F}_{q^6}$ satisfying

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x^\rho \\ (h^{-\rho} - 1)x^{\rho q} + x^{\rho q^3} + (h^\rho - 1)x^{\rho q^5} \end{pmatrix} = \begin{pmatrix} z \\ z^q + z^{q^3} + \delta z^{q^5} \end{pmatrix}.$$

Let $k = h^\rho$, for which $k^{q+1} = -1$. As in Proposition 3.5, we obtain a polynomial identity, whence

$$\begin{cases} c = b^q(k^q - 1) + b^{q^3} + \delta b^{q^5}(k^{-q} - 1) \\ d(k^{-1} - 1) = a^q \\ 0 = b^q(k^{-q} - 1) + b^{q^3}(k^q - 1) + b^{q^5}\delta \\ d = a^{q^3} \\ 0 = b^q + b^{q^3}(k^{-q} - 1) + b^{q^5}(k^q - 1)\delta \\ d(k-1) = \delta a^{q^5}. \end{cases} \tag{3.5}$$

By subtracting the fifth equation from the third equation raised to $q^2$, we get

$$b^q = b^{q^5}(k^q - 1),$$

i.e. either $b = 0$ or $k^q - 1 = (b^q)^{q^4 - 1}$, whence we get either $b = 0$ or $\mathrm{N}_{q^6/q^2}(k^q - 1) = 1$.

If $b \neq 0$, since $k - 1 \in \mathbb{F}_{q^2}$ and $\mathrm{N}_{q^6/q^2}(k-1) = (k-1)^3 = 1$, then

$$k^3 - 3k^2 + 3k - 2 = 0$$

and, since $\mathrm{N}_{q^6/q^2}(k^q - 1) = 1$ and $k^q = -1/k$,

$$2k^3 + 3k^2 + 3k + 1 = 0,$$

from which we get

$$9k^2 - 3k + 5 = 0. \tag{3.6}$$

- If $k \notin \mathbb{F}_q$ then $k$ and $k^q$ are the solutions of (3.6) and

$$-1 = k^{q+1} = \frac{5}{9},$$

which holds if and only if $q$ is a power of 7. By (3.6) it follows that $k \in \mathbb{F}_q$, a contradiction.

- If $k \in \mathbb{F}_q$, then $k^2 = -1$ and by (3.6) we have $k = -4/3$, which is possible if and only if $q$ is a power of 5.

Hence, if either $k \notin \mathbb{F}_q$ or $k \in \mathbb{F}_q$ with $q$ not a power of 5, we have that $b = 0$ and hence $c = 0$, $a \neq 0$ and $d \neq 0$.

By combining the second and the fourth equation of (3.5), we get $\mathrm{N}_{q^6/q^2}(k^{-1} - 1) = 1$ and, since $k^q = -1/k$, $\mathrm{N}_{q^6/q^2}(k^q + 1) = -1$. Arguing as above, we get a contradiction whenever $k \notin \mathbb{F}_q$ or $k \in \mathbb{F}_q$ with $q$ not a power of 5.

Now, suppose that there exist $\rho \in \mathrm{Aut}(\mathbb{F}_{q^6})$ and an invertible matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ such that for each $x \in \mathbb{F}_{q^6}$ there exists $z \in \mathbb{F}_{q^6}$ satisfying

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x^\rho \\ (h^{-\rho} - 1)x^{\rho q} + x^{\rho q^3} + (h^\rho - 1)x^{\rho q^5} \end{pmatrix} = \begin{pmatrix} z \\ \delta z^q + z^{q^3} + z^{q^5} \end{pmatrix}.$$

Let $k = h^\rho$. As before, we get the following equations

$$\begin{cases} c = \delta b^q (k^q - 1) + b^{q^3} + b^{q^5}(k^{-q} - 1) \\ d(k^{-1} - 1) = \delta a^q \\ 0 = \delta b^q (k^{-q} - 1) + b^{q^3}(k^q - 1) + b^{q^5} \\ d = a^{q^3} \\ 0 = \delta b^q + b^{q^3}(k^{-q} - 1) + b^{q^5}(k^q - 1) \\ d(k - 1) = a^{q^5}. \end{cases} \tag{3.7}$$

By subtracting the fifth equation from the third raised to $q^2$ of the above system we get

$$b^q = b^{q^3}(k^{-q} - 1).$$

If $b \neq 0$, then $\mathrm{N}_{q^6/q^2}(k^{-q} - 1) = 1$. Hence, arguing as above, we get that $b = 0$ and hence $c = 0$, $a, d \neq 0$. By combining the fourth equation with the second and the fifth equation of (3.7) we get $\mathrm{N}_{q^6/q^2}(k - 1) = 1$, which yields again to a contradiction when $k \notin \mathbb{F}_q$ or $k \in \mathbb{F}_q$ with $q$ not a power of 5. □

So, as a consequence of Corollary 3.8 and of the above proposition, we have the following result.

**Corollary 3.11.** *Apart from the case $h \in \mathbb{F}_q$ and $q$ a power of 5, the linear set $\mathcal{L}_h$ is not* $\mathrm{P\Gamma L}$*-equivalent to any known scattered linear set in* $\mathrm{PG}(1, q^6)$.

By Proposition 3.9, when $h \in \mathbb{F}_{q^2}$, $\mathcal{L}_h$ is a linear set of the family presented in [23, Section 7]. Also, we get an extension of [21, Table 1], where it is shown examples of scattered linear sets which could generalize the family presented in [10]. We do not know whether the linear set $\mathcal{L}_h$, for each $h \in \mathbb{F}_{q^6} \setminus \mathbb{F}_{q^2}$ with $h^{q^3+1} = -1$, may be defined by a trinomial or not.

## 4 New MRD-codes

Delsarte in [13] (see also [14]) introduced in 1978 rank metric codes as follows. A *rank metric code* (or *RM*-code for short) $\mathcal{C}$ is a subset of the set of $m \times n$ matrices $\mathbb{F}_q^{m \times n}$ over $\mathbb{F}_q$ equipped with the distance function

$$d(A, B) = \mathrm{rk}\,(A - B)$$

for $A,B\in\mathbb{F}_q^{m\times n}$. The *minimum distance* of $\mathcal{C}$ is

$$d=\min\{d(A,B):A,B\in\mathcal{C},\ A\neq B\}.$$

We will say that a rank metric code of $\mathbb{F}_q^{m\times n}$ with minimum distance $d$ has parameters $(m,n,q;d)$. When $\mathcal{C}$ is an $\mathbb{F}_q$-subspace of $\mathbb{F}_q^{m\times n}$, we say that $\mathcal{C}$ is $\mathbb{F}_q$-linear. In the same paper, Delsarte also showed that the parameters of these codes fulfill a Singleton-like bound, i.e.

$$|\mathcal{C}|\leq q^{\max\{m,n\}(\min\{m,n\}-d+1)}.$$

When the equality holds, we call $\mathcal{C}$ a *maximum rank distance* (*MRD* for short) code. We will consider only the case $m=n$ and we will use the following equivalence definition for codes of $\mathbb{F}_q^{m\times m}$. Two $\mathbb{F}_q$-linear RM-codes $\mathcal{C}$ and $\mathcal{C}'$ are *equivalent* if and only if there exist two invertible matrices $A,B\in\mathbb{F}_q^{m\times m}$ and a field automorphism $\sigma$ such that $\{AC^\sigma B: C\in\mathcal{C}\}=\mathcal{C}'$, or $\{AC^{T\sigma}B:C\in\mathcal{C}\}=\mathcal{C}'$, where $T$ denotes transposition. Also, the *left* and *right idealisers* of $\mathcal{C}$ are $L(\mathcal{C})=\{A\in\mathrm{GL}(m,q):A\mathcal{C}\subseteq\mathcal{C}\}$ and $R(\mathcal{C})=\{B\in\mathrm{GL}(m,q):\mathcal{C}B\subseteq\mathcal{C}\}$ [17, 19]. They are important invariants for linear rank metric codes, see also [15] for further invariants.

In [24, Section 5] Sheekey showed that scattered $\mathbb{F}_q$-linear sets of $\mathrm{PG}(1,q^n)$ of rank $n$ yield $\mathbb{F}_q$-linear MRD-codes with parameters $(n,n,q;n-1)$ with left idealiser isomorphic to $\mathbb{F}_{q^n}$; see [7, 8, 25] for further details on such kind of connections. We briefly recall here the construction from [24]. Let $U_f=\{(x,f(x)):x\in\mathbb{F}_{q^n}\}$ for some scattered $q$-polynomial $f(x)$. After fixing an $\mathbb{F}_q$-basis for $\mathbb{F}_{q^n}$ we can define an isomorphism between the rings $\mathrm{End}(\mathbb{F}_{q^n},\mathbb{F}_q)$ and $\mathbb{F}_q^{n\times n}$. In this way the set

$$\mathcal{C}_f:=\{x\mapsto af(x)+bx:a,b\in\mathbb{F}_{q^n}\}$$

corresponds to a set of $n\times n$ matrices over $\mathbb{F}_q$ forming an $\mathbb{F}_q$-linear MRD-code with parameters $(n,n,q;n-1)$. Also, since $\mathcal{C}_f$ is an $\mathbb{F}_{q^n}$-subspace of $\mathrm{End}(\mathbb{F}_{q^n},\mathbb{F}_q)$ its left idealiser $L(\mathcal{C}_f)$ is isomorphic to $\mathbb{F}_{q^n}$. For further details see [6, Section 6].

Let $\mathcal{C}_f$ and $\mathcal{C}_h$ be two MRD-codes arising from maximum scattered subspaces $U_f$ and $U_h$ of $\mathbb{F}_{q^n}\times\mathbb{F}_{q^n}$. In [24, Theorem 8] the author showed that there exist invertible matrices $A$, $B$ and $\sigma\in\mathrm{Aut}(\mathbb{F}_q)$ such that $A\mathcal{C}_f^\sigma B=\mathcal{C}_h$ if and only if $U_f$ and $U_h$ are $\Gamma\mathrm{L}(2,q^n)$-equivalent

Therefore, we have the following.

**Theorem 4.1.** *The $\mathbb{F}_q$-linear MRD-code $\mathcal{C}_{f_h}$ arising from the $\mathbb{F}_q$-subspace $\mathcal{U}_h$ has parameters $(6,6,q;5)$ and left idealiser isomorphic to $\mathbb{F}_{q^6}$, and is not equivalent to any previously known MRD-code, apart from the case $h\in\mathbb{F}_q$ and $q$ a power of* 5.

*Proof.* From [6, Section 6], the previously known $\mathbb{F}_q$-linear MRD-codes with parameters $(6,6,q;5)$ and with left idealiser isomorphic to $\mathbb{F}_{q^6}$ arise, up to equivalence, from one of the maximum scattered subspaces of $\mathbb{F}_{q^6}\times\mathbb{F}_{q^6}$ described in Section 3. From Corollaries 3.8 and 3.11 the result then follows. $\square$

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