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ON THE SUBMULTIPLICATIVITY AND SUBADDITIVITY OF THE CONE SPECTRAL RADIUS

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MARKO KANDIĆ AND ALJOŠA PEPERKO

Abstract. Let S be a cone equipped with an associative product $*$ and let $p : S \to [0, \infty)$ be a monote $*$-submultiplicative seminorm. We introduce the notion of the cone spectral radius r_p associated to p and $*$. We prove that under certain conditions the inequalities

$$r_p(a * b) \le r_p(a)r_p(b) \;\text{ and }\; r_p(a + b) \le r_p(a) + r_p(b)$$

hold. Our results apply to several radii appearing in the literature including the spectral radius of positive operators, the spectral radius in max algebra, the Bonsall's cone spectral radius and the essential cone spectral radius. We also obtain new results in the setting of general Banach algebras.

1. Introduction

It is well known, that the inequalities

$$r(AB) \le r(A)r(B) \;\text{ and }\; r(A + B) \le r(A) + r(B) \tag{1.1}$$

(for the spectral radius r) do not hold for arbitrary bounded operators A and B on the Banach space X. It is also well known that the inequalities in (1.1) hold if the operators A and B commute. One of the goals of this paper is to find other conditions under which these inequalities hold. These conditions depend on the existence of a certain partial order and on the monotonicity and submultiplicativity of the operator norm. The main tool in our proofs is the well known Gelfand-Beurling's formula

$$r(A) = \lim_{n\to\infty} \|A^n\|^{1/n}.$$

Moreover, our techniques extend also to several other radii appearing in the literature. A simultaneous proof of this fact is enabled by an introduction of the cone spectral radius, which generalizes the above mentioned radii.

One of these radii is the spectral radius in max algebra. The algebraic system max algebra (and its isomorphic versions) is an attractive way of describing a class of non-linear problems appearing for instance in machine-scheduling, information technology, discrete event-dynamic systems, computational biology, combinatorial optimisation,... (see e.g. [8], [9], [3], [30], [17], [4], [7], [5]). Max algebra's usefulness arises from a fact that these non-linear problems become linear when decribed in the max algebra language. In this theory the role of the spectral radius (see e.g.

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[4], [8]) is played by the maximum cycle geometric mean of a non-negative square matrix (see e.g. [18, p. 366], [5, p. 130],[8], [15], [16], [14], [12]).

The max algebra is the set of non-negative numbers with sum $a \oplus b = \max\{a, b\}$ and the standard product ab, where $a, b \geq 0$. The $n \times n$ matrix $A = [a_{ij}]$ is called *non-negative*, if $a_{ij} \geq 0$ for all $i, j = 1, \ldots, n$. The operations between non-negative matrices and vectors are defined by analogy with the usual linear algebra. For instance, the product of non-negative $n \times n$ matrices A and B in the max algebra, denoted by $A \otimes B$, is defined by

$$[A \otimes B]_{ij} = \max_{k=1,\ldots,n} a_{ik} b_{kj}.$$

By $A_\otimes^k$ we denote the k-th max power of A. The usual associative and distributive laws hold in this algebra.

The weighted directed graph $\mathcal{D}(A)$ associated with A has a vertex set $\{1, 2, \ldots, n\}$ and edges (i, j) from a vertex i to a vertex j with weight a_{ij} if and only if $a_{ij} > 0$. A path of length k is a sequence of edges $(i_1, i_2), (i_2, i_3), \ldots, (i_k, i_{k+1})$. A circuit of length k is a path with $i_{k+1} = i_1$, where $i_1, i_2, \ldots, i_k$ are distinct. Associated with this circuit is the *circuit geometric mean* known as $(a_{i_1 i_2} a_{i_2 i_3} \ldots a_{i_k i_1})^{1/k}$. The maximum circuit geometric mean in $\mathcal{D}(A)$ is denoted by $\mu(A)$. Note that circuits (i_1, i_1) of length 1 (loops) are included here and that we also consider empty circuits, i.e., circuits that consist of only one vertex and have length 0. For empty circuits, the associated circuit geometric mean is zero.

The maximum circuit geometric mean $\mu(A)$ was utilized in [15] and has been studied extensively since. There are many different characterizations of $\mu(A)$ (see e.g. [14], [18, p. 366], [5, p. 130], [12], [27], [28]). It is known that $\mu(A)$ is the largest max eigenvalue of A. Moreover, if A is irreducible, then $\mu(A)$ is the unique max eigenvalue and every max eigenvector is positive (see e.g. [4, Theorem 2] and [19, Theorem 1]). So $\mu(A)$ can be viewed as the max version of the spectral radius of a non-negative matrix A.

In [12, Lemma 4.1] the max version of Gelfand's formula was proved, i.e., for a non-negative $n \times n$ matrix A the equality

$$\mu(A) = \lim_{k \to \infty} \|A_\otimes^k\|^{1/k} \tag{1.2}$$

holds for an arbitrary vector norm $\|\cdot\|$ on $\mathbb{R}^{n \times n}$. For alternative proofs see [19], [26].

One of the goals of this paper is to provide unification and generalization of the notions of the spectral radius and the spectral radius in max algebra. This generalization includes the notions of the Bonsall's cone spectral radius and the essential cone spectral radius (see e.g. [20], [23], [24]) of non-linear maps.

The paper is organized as follows. In the second section we introduce the notion of the cone spectral radius r_p and prove several general results on its submultiplicativity and subadditivity. In section 3 we apply these results to the spectral radius and the spectral radius in max algebra and in section 4 to the Bonsall's cone spectral radius and the essential cone spectral radius. In the final section we state some related new results in the case of general Banach algebras.

2. Cone spectral radius

Let S be a semigroup with respect to (an associative) product $*$. By S^1 we denote the semigroup obtained by adjoining the neutral element e to the semigroup

S. For $a \in S$ and $n \in \mathbb{N}$ the power a_*^n is defined by

$$a_*^1 = a \text{ and } a_*^n = a_*^{n-1} * a.$$

Given a functional $p : S \to [0, \infty)$ and $a \in S$ we define

$$r_p(a) = \limsup_{n\to\infty} p(a_*^n)^{1/n}. \tag{2.1}$$

If the functional $q : S \to [0, \infty)$ is equivalent to p (i.e., there exists positive real numbers m, M such that $mp(a) \leq q(a) \leq Mp(a)$ for every $a \in S$), then $r_p(a) = r_q(a)$ for all $a \in S$. We say that $p : S \to [0, \infty)$ is $*$-submultiplicative if

$$p(a * b) \leq p(a)p(b) \tag{2.2}$$

for all $a, b \in S$. In this case it is not hard to see that we have $r_p(a) \leq p(a)$ and

$$r_p(a) = \lim_{n\to\infty} p(a_*^n)^{1/n} = \inf_{n\in\mathbb{N}} p(a_*^n)^{1/n} \tag{2.3}$$

for all $a \in S$. Also the equality $(b * a)_*^n = b * (a * b)_*^{n-1} * a$ implies

$$r_p(a * b) = r_p(b * a) \tag{2.4}$$

for all $a, b \in S$.

In our applications the semigroup S will also be a cone included in a certain vector space V and $p : S \to [0, \infty)$ a seminorm, i.e.,

$$p(a + b) \leq p(a) + p(b) \text{ and } p(\lambda a) = \lambda p(a)$$

for all $a, b \in S$ and every non-negative scalar λ.

For $a, b \in V$ we denote $a \leq b$ if $b - a \in S$. Thus $(V, \leq)$ is a partially ordered vector space and $a \geq 0$ if and only if $a \in S$ (we sometimes denote S by V^+ and call it the *positive cone* of V). In this setting we call the functional r_p a *cone spectral radius* associated to $*$ and p. A functional $p : S \to [0, \infty)$ is called *monotone* if $0 \leq a \leq b$ implies $p(a) \leq p(b)$. We say that the product $*$ is *monotone* on V if $0 \leq a \leq b$ and $0 \leq c \leq d$ imply $0 \leq a * c \leq b * d$. Observe that in the case when the functional p is monotone and the product $*$ is monotone, the inequality $0 \leq a \leq b$ implies $r_p(a) \leq r_p(b)$.

An element $a \in S$ is called p-power bounded if there exists $M > 0$ such that for each $n \in \mathbb{N}$ we have $p(a_*^n) \leq M$. Obviously, $r_p(a) \leq 1$ whenever a is p-power bounded.

Theorem 2.1. *Let V be a partially ordered vector space with a positive cone S, such that S is also a semigroup with respect to the monotone product $*$. Let $p : S \to [0, \infty)$ be a monotone $*$-submultiplicative functional. Let a, b, c be elements of $\mathcal{S}$ such that $a * b \leq c * b * a$, $a * c \leq c * a$ and $b * c \leq c * b$. If c is p-power bounded, then*

$$r_p(a * b) \leq r_p(c * b * a) \leq r_p(a)r_p(b) \tag{2.5}$$

and

$$r_p(c * b * a - a * b) \leq r_p(a)r_p(b). \tag{2.6}$$

If $p(c) < 1$, then $r_p(a * b) = r_p(c * b * a) = r_p(c * b * a - a * b) = 0$.

Proof. At first we will prove by induction that for each integer $n \geq 2$ the following inequality holds

$$(b * a)^n_* \leq c_*^{\frac{n(n-1)}{2}} * b^n_* * a^n_*. \tag{2.7}$$

For $n = 2$, the claim is true by the assumption. Suppose the formula (2.7) holds for some positive integer $n \geq 2$. Then we have

$$(b*a)^{n+1}_* \leq c_*^{\frac{n(n-1)}{2}} * b^n_* * a^n_* * b * a \leq c_*^{\frac{n(n-1)}{2}+n} * b^{n+1}_* * a^{n+1}_* = c_*^{\frac{n(n+1)}{2}} * b^{n+1}_* * a^{n+1}_*,$$

which completes the induction step.

Inequality (2.7) implies

$$(c * b * a)^n_* \leq c^n_* * (b * a)^n_* \leq c_*^{\frac{n(n+1)}{2}} * b^n_* * a^n_* \tag{2.8}$$

for every positive integer n. Since the functional p is $*$-submultiplicative and monotone, inequality 2.8 implies

$$p((c * b * a)^n_*) \leq p\left(c_*^{\frac{n(n+1)}{2}}\right) p(b^n_*)p(a^n_*)$$

for all integers $n \geq 1$. Since the element c is p-power bounded, there exists a positive number M such that for all $n \in \mathbb{N}$ we have $p(c^n_*) \leq M$. Therefore, for each $n \in \mathbb{N}$ we have

$$p((c * b * a)^n_*) \leq M\, p(a^n_*)p(b^n_*).$$

Applying n-th roots and passing to limits we obtain

$$r_p(c * b * a) \leq r_p(a)r_p(b). \tag{2.9}$$

The left hand side inequality in 2.5 follows from the monotonicity of the functional p.

The inequalities

$$0 \leq c * b * a - a * b \leq c * b * a, \tag{2.10}$$

the monotonicity of r_p and (2.5) imply

$$r_p(c * b * a - a * b) \leq r_p(c * b * a) \leq r_p(a)r_p(b),$$

which proves (2.6).

Now suppose that $p(c) < 1$. From the inequality (2.8) we obtain

$$p((c * b * a)^n_*)^{\frac{1}{n}} \leq p(c)^{\frac{n+1}{2}} p(b^n_*)^{\frac{1}{n}} p(a^n_*)^{\frac{1}{n}}. \tag{2.11}$$

for every positive integer n. Since $p(c) < 1$, the right hand side of (2.11) tends to zero and thus $r_p(c * b * a) = 0$, which completes the proof. □

The *commutator* of elements $a, b \in S$ is defined by $[a, b]_* = a * b - b * a$. The following proposition can be shown similarly as Theorem 2.1.

Proposition 2.2. *Let V be a partially ordered vector space with a positive cone S, such that S is also a semigroup with respect to the monotone product $*$. Let $p : S \to [0, \infty)$ be a monotone $*$-submultiplicative functional. If $a, b \in S$ are such that $[a, b]_* \in S$, then*

$$r_p([a, b]_*) \leq r_p(a * b) \leq r_p(a)r_p(b) \tag{2.12}$$

Remark 2.3. It often occurs that the associative product $*$ and the functional p are defined on the whole space V (for instance if V is Banach algebra of bounded operators on a certain ordered Banach space and S is a cone of positive operators). In this case (2.1) defines the radius $r_p(a)$ for arbitrary $a \in V$. The above proof shows that under these modifications the inequalities (2.5) and (2.6) remains valid if $a, b, c \in V$ and $0 \le a * b \le c * b * a$, $a * c \le c * a$, $b * c \le c * b$. Note also that we need here (2.2) to hold for arbitrary $a, b \in V$.

Definition 2.4. Let V be a partially ordered vector space with a positive cone S, such that S is also a semigroup with respect to monotone product $*$. The product $*$ is said to be subdistributive, if for every $a, b, c \in S$ the following inequalities hold:

$$(a+b) * c \le (a * c) + (b * c) \qquad \text{and} \qquad c * (a+b) \le (c * a) + (c * b).$$

Remark 2.5. Obviously every distributive product $*$ on a partially ordered vector space is subdistributive.

Theorem 2.6. *Let V be a partially ordered vector space with a positive cone S, such that S is also a semigroup with respect to a monotone subdistributive product $*$. Let $p : S \to [0, \infty)$ be a monotone $*$-submultiplicative seminorm. Let $a, b, c \in S$ be elements of $\mathcal{S}$ such that $a * b \le c * b * a$, $a * c \le c * a$ and $b * c \le c * b$. If c is p-power bounded, then*

$$r_p(a+b) \le r_p(a) + r_p(b). \tag{2.13}$$

Proof. At first, we will prove by induction that for each $n \in \mathbb{N}$ we have

$$(a+b)_*^n \le \sum_{k=0}^{n} q_k^n(c) * b_*^{n-k} * a_*^k, \tag{2.14}$$

where $q_k^n(c)$ denotes the sum of $\binom{n}{k}$ (possibly distinct) powers of the element c. For $n = 1$, the claim is trivially true as $q_0^1(c) = q_1^1(c) = e$. Suppose now that the formula is true for some $n \in \mathbb{N}$. Then we have

$$(a+b)_*^{n+1} \le \left(\sum_{k=0}^{n} q_k^n(c) * b_*^{n-k} * a_*^k \right) * (a+b) \le$$

$$\le \sum_{k=0}^{n} q_k^n(c) * b_*^{n-k} * a_*^{k+1} + \sum_{k=0}^{n} q_k^n(c) * c_*^k * b_*^{n-k+1} * a_*^k.$$

From here it follows that

$$q_k^{n+1}(c) = q_{k-1}^n(c) + q_k^n(c) * c_*^k.$$

We complete the induction step noticing that the number of the summands where the power of the element c occurs in $q_k^{n+1}(c)$ is equal $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$.

Since c is p-power bounded, there exists $M > 0$ such that $p(c_*^n) \le M$ for each $n \in \mathbb{N}$. Let $N = \max\{M, 1\}$. From here it easily follows

$$p((a+b)_*^n) \le N \sum_{k=0}^{n} \binom{n}{k} p(a_*^k) p(b_*^{n-k}).$$

For any $\epsilon > 0$ there exists a positive scalar $L > 0$ such that for any $n \in \mathbb{N}$ we have

$$p(a_*^n) \le L(r_p(a) + \epsilon)^n \qquad \text{and} \qquad p(b_*^n) \le L(r_p(b) + \epsilon)^n.$$

This implies

$$p((a+b)_*^n) \le N \sum_{k=0}^{n} \binom{n}{k} p(a_*^k) p(b_*^{n-k}) \le$$

$$\le NL^2 \sum_{k=0}^{n} \binom{n}{k} (r_p(a)+\epsilon)^k (r_p(b)+\epsilon)^{n-k} = NL^2 (r_p(a)+r_p(b)+2\epsilon)^n.$$

This implies

$$(p(a+b)_*^n)^{\frac{1}{n}} \le N^{\frac{1}{n}} L^{\frac{2}{n}} (r_p(a)+r_p(b)+2\epsilon).$$

Passing to limits we obtain $r_p(a+b) \le r_p(a)+r_p(b)+2\epsilon$. Since $\epsilon > 0$ was arbitrary, we obtain the inequality (2.13).

□

Remark 2.7. In the proofs of Theorems 2.1 and 2.6 we actually do not need the cone S to be a positive cone of some partially ordered vector space V. We only need the operation "minus" to act on the elements of the cone S. It therefore suffices V to be an additive group. In this case we can again define the partial order on V by $a \le b$ for $a, b \in V$ if $b - a \in S$.

3. The spectral radius and the spectral radius in max algebra

In this section we apply our results to the spectral radius of non-negative matrices and its max-version. For the sake of simplicity we restrict our attention at the moment to the finite dimensional case, i.e., $S = \mathbb{R}_+^{n\times n}$.

Let $\|\cdot\|$ denote an operator norm with respect to a monotone vector norm on $\mathbb{R}^n$. The usual commutator of the matrices A and B is denoted by $[A,B] = AB - BA$. Applying Theorem 2.1, Theorem 2.6 and Proposition 2.2 we obtain the following results for the usual spectral radius.

Theorem 3.1. *Let A, B and C be non-negative $n\times n$ matrices satisfying $AC \le CA$, $BC \le CB$ and $AB \le CBA$. If C is power bounded, then we have*

$$r(AB) \le r(A)r(B), \quad r(A+B) \le r(A) + r(B) \tag{3.1}$$

and

$$r(CBA - AB) \le r(CBA) \le r(A)r(B). \tag{3.2}$$

If $\|C\| < 1$, then AB, CBA and $CBA - AB$ are nilpotent.

Corollary 3.2. *If A and B are non-negative $n \times n$ matrices such that $[A,B] \ge 0$, then we have*

$$r([A,B]) \le r(AB) \le r(A)r(B) \;\; \text{and} \;\; r(A+B) \le r(A)+r(B). \tag{3.3}$$

Note that the inequalities $r([A,B]) \le r(A)r(B)$, $r(AB) \le r(A)r(B)$ and $r(A+B) \le r(A)+r(B)$ from (3.3) are known (see e.g. [27], [10]). On the other hand, Theorem 3.1 seems to be new even in this finite dimensional setting.

The following example shows that in Corollary 3.2 the positivity of the commutator cannot be omitted.

Example 3.3. Let

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Then

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad [A,B] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \text{and} \qquad A+B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and so $r(AB) = r([A,B]) = r(A+B) = 1$ while $r(A) = r(B) = 0$.

The following example shows that the assumptions in Theorem 3.1 can be satisfied also when $C \neq I$.

Example 3.4. Let

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \qquad \text{and} \qquad C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Then C is a power bounded matrix that satisfies $AB \le CBA$, $AC \le CA$ and $BC \le CB$.

Next we apply our results from section 2 in the max algebra setting. Since the property of $\|\cdot\|$-power boundedness in this case is independent of a choice of a norm $\|\cdot\|$ on $V = \mathbb{R}^{n\times n}$, we use the term $\otimes$-power boundedness instead, i.e, a matrix $A \in \mathbb{R}_+^{n\times n}$ is called $\otimes$-*power bounded* if there exists $M > 0$ such that for each $n \in \mathbb{N}$ we have $\|A_\otimes^n\| \le M$.

Let us consider the vector norm $\|A\|_\infty = \max_{i,j=1,\dots n} |a_{ij}|$ on $\mathbb{R}^{n\times n}$. It is obvious that $\|\cdot\|_\infty$ is monotone and $\otimes$-submultiplicative on $\mathbb{R}_+^{n\times n}$. It is also easy to verify that the product $\otimes$ is monotone and subdistributive with respect to the usual sum $+$. Therefore the following result is a direct consequence of Theorems 2.1 and 2.6. It seems to be new even in this finite dimensional case.

Theorem 3.5. *Let $A, B, C \in \mathbb{R}_+^{n\times n}$ such that $A\otimes B \le C \otimes B \otimes A$, $A \otimes C \le C \otimes A$ and $B \otimes C \le C \otimes B$. If C is $\otimes$-power bounded, then we have*

$$\mu(A\otimes B) \le \mu(A)\mu(B), \quad \mu(A+B) \le \mu(A)\mu(B) \ \textit{and} \tag{3.4}$$

$$\mu(C\otimes B\otimes A - A\otimes B) \le \mu(C\otimes B\otimes A) \le \mu(A)\mu(B). \tag{3.5}$$

If $\|C\|_\infty < 1$, then $\mu(A\otimes B) = \mu(C\otimes B\otimes A) = \mu(C\otimes B\otimes A - A\otimes B) = 0$.

Corollary 3.6. *If $A, B \in \mathbb{R}_+^{n\times n}$ such that $[A,B]_\otimes \in \mathbb{R}_+^{n\times n}$, then*

$$\mu([A,B]_\otimes) \le \mu(A\otimes B) \le \mu(A)\mu(B) \ \textit{and} \ \mu(A+B) \le \mu(A)\mu(B).$$

Remark 3.7. The matrices A, B from Example 3.3 show that in Corollary 3.6 the positivity of the commutator in max algebra cannot be omitted. Indeed, in this case we have $\mu(A\otimes B) = \mu(A+B) = 1$ while $\mu(A) = \mu(B) = 0$.

The following example shows that Theorem 3.5 is in fact stronger then Corollary 3.6.

Example 3.8. Let

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Then C is a $\otimes$-power bounded matrix that satisfies $A \otimes B \le C \otimes B \otimes A$, $A \otimes C \le C \otimes A$ and $B \otimes C \le C \otimes B$. Also $\mu(A) = \mu(B) = \mu(A \otimes B) = 1$ and $\mu(A+B) = \sqrt{2}$, which is consistent with Theorem 3.5. However,

$$[A, B]_\otimes = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}.$$

4. Bonsall‘s cone spectral radius and the essential cone spectral radius

In this section we apply our results to a large class of linear or non-linear order preserving maps. The Bonsall's cone spectral radius of such maps was introduced in [6] and it has received a lot of attention since (see e.g. [20],[2], [23] and the references cited there). It enables us to consider the spectral theory of a larger class of nonlinear infinite dimensional maps. In particular, this includes the eigenvalue problem of certain max-type operators $F : C[0,a] \to C[0,a]$ of the form

$$(F(x))(s) = \max_{t \in [\alpha(s), \beta(s)]} k(s,t)x(t),$$

where $x \in C[0,a]$ and $\alpha, \beta : [0,a] \to [0,a]$ are given continuous functions satisfying $\alpha \le \beta$. The kernel $k : \mathcal{S} \to [0,\infty)$ is a given non-negative continuous function, where $\mathcal{S}$ denotes the compact set

$$\mathcal{S} = \{(s,t) \in [0,a] \times [0,a] : t \in [\alpha(s), \beta(s)]\}.$$

This problem arises in the study of periodic solutions of a class of differential-delay equations

$$\varepsilon y'(t) = g(y(t), y(t-\tau)), \quad \tau = \tau(y(t)),$$

with state-dependent delay (see e.g. [20], [21]).

Let E be an ordered Banach space and let its positive cone K be closed. A cone K is called *normal* if there exists a constant N such that $\|x\| \le N\|y\|$ whenever $0 \le x \le y$. It is known (see e.g. [29]) that in this case there exists an equivalent norm $\|\!|\cdot|\!\|$ on E such that $\|\!|x|\!\| \le \|\!|y|\!\|$ whenever $0 \le x \le y$.

Let S be a cone of all maps $f : K \to K$, which are continuous, positively homogeneous and monotone. A cone S is a semigroup with respect to the composition $(f_1 \circ f_2)(x) = f_1(f_2(x))$, which is monotone on S. The composition of f with itself n times is denoted by f^n. Note that S is a generating cone of a vector space $S - S = \{x - y : x, y \in S\}$ and that the product $\circ$ is subdistributive on the subcone $S_A \subset S$ of all subadditive maps $f \in S$.

A monotone submultiplicative seminorm $p : S \to [0,\infty)$ is defined by

$$p(f) = \sup\{\|\!|f(x)|\!\| : x \in K, \|\!|x|\!\| \le 1\}. \tag{4.1}$$

The *Bonsall's cone spectral radius* $r_B(f)$ of $f \in S$ is defined by

$$r_B(f) = \lim_{n\to\infty} p(f^n)^{1/n}.$$

If E is a Banach lattice and $f : E \to E$ a positive (linear) operator (see e.g. [1] for definitions), then $p(f)$ equals the operator norm of f and thus $r_B(f)$ equals the spectral radius of f. However, if E is not a Banach lattice, then this needs not be the case (see e.g. [6]).

By Theorems 2.1 and 2.6 we obtain the following result.

Theorem 4.1. *Let E be an ordered Banach space and let its positive cone K be closed and normal. Let S be a cone of all maps $f : K \to K$, which are continuous, positively homogeneous and monotone and let $p : S \to [0, \infty)$ be as in (4.1). Let f, g, h be elements of S such that $f \circ g \le h \circ g \circ f$, $f \circ h \le h \circ f$ and $g \circ h \le h \circ g$. If h is p-power bounded, then*

$$r_B(f \circ g) \le r_B(h \circ g \circ f) \le r_B(f) r_B(g) \tag{4.2}$$

and

$$r_B(h \circ g \circ f - f \circ g) \le r_B(f) r_B(g). \tag{4.3}$$

If f, g, h are subadditive, then we have

$$r_B(f + g) \le r_B(f) + r_B(g).$$

If $p(h) < 1$, then $r_B(f \circ g) = 0$ and $r_B(h \circ g \circ f - f \circ g) = 0$.

The solution of the eigenproblem in [20] is obtained under certain compactness conditions. In particular, the condition $\rho(f) < r_B(f)$ plays a crucial role. Here $\rho(f)$ denotes the *essential cone spectral radius* with respect to some homogeneous generalized measure of noncompactness ν (see definitions bellow).

If ν is a map which assigns to each bounded subset A of E a non-negative, finite number $\nu(A)$, then ν is called a *homogeneous generalized measure of noncompactness* if it satisfies the following conditions:

(i) $\nu(A) = 0$ if and only if $\overline{A}$ is compact;

(ii) $\nu(A + B) \le \nu(A) + \nu(B)$;

(iii) $\nu(\overline{co(A)}) = \nu(A)$;

(iv) $\mu(A \cup B) = \max\{\nu(A), \nu(B)\}$;

(v) $\nu(\lambda A) = \lambda \nu(A)$ for $\lambda \ge 0$.

Here we denote $A + B = \{a + b : a \in A, b \in B\}$ and $\overline{co(A)}$ denotes the smallest closed convex set containing A. We refer to [20] for interesting examples.

Let the cone K be closed and normal and let $f : K \to K$ be a continuous, positively homogeneous and monotone map. We may define the quantity

$$\nu(f) = \inf\{\lambda > 0 : \nu(f(A)) \le \lambda \nu(A) \ \text{ for every bounded set } \ A \subset K\},$$

where we set $\inf \emptyset = \infty$. It is not hard to verify that $\nu : S \to [0, \infty]$ defines a $\circ$-submultiplicative seminorm on the subcone $\{f \in S : \nu(f) < \infty\}$. This seminorm may not be monotone in general. However, there exist conditions on the space E under which the seminorm ν may be monotone on certain subcones of S. We refer the interested reader to the books [1] and [22] for details.

The *cone essential spectral radius* of f with respect to $\nu(f)$ is defined by

$$\rho(f) = \limsup_{n \to \infty} \nu(f^n)^{1/n}.$$

If $\rho(f) < \infty$, then

$$\rho(f) = \lim_{n \to \infty} \nu(f^n)^{1/n} = \inf_{n \in \mathbb{N}} \nu(f^n)^{1/n}.$$

Applying Theorems 2.1 and 2.6 we obtain the following result for the cone essential spectral radius.

Theorem 4.2. *Let E be an ordered Banach space such that its positive cone K is closed and normal and let ν be a homogeneous generalized measure of noncompactness on E. Let S be a cone of all maps $f : K \to K$, which are continuous, positively homogeneous and monotone and let S_M denote a subcone and a semigroup ideal of S such that ν is monotone on S_M. Suppose that there exists $f, g \in S_M, h \in S$ such that $f \circ g \leq h \circ g \circ f$, $f \circ h \leq h \circ f$ and $g \circ h \leq h \circ g$.*

If $\rho(f) < \infty$, $\rho(g) < \infty$ and h is ν-power bounded, then

$$\rho(f \circ g) \leq \rho(h \circ g \circ f) \leq \rho(f)\rho(g) \tag{4.4}$$

and

$$\rho(h \circ g \circ f - f \circ g) \leq \rho(f)\rho(g). \tag{4.5}$$

If, in addition, f, g, h are subadditive, then we have

$$\rho(f + g) \leq \rho(f) + \rho(g).$$

If, in addition, $\nu(h) < 1$, then $\rho(f \circ g) = 0$ and $\rho(h \circ g \circ f - f \circ g) = 0$.

5. Related results in general Banach algebras

In this final section we make some related remarks in the setting of Banach algebras. Let X be a Banach space, $\mathcal{L}(X)$ the Banach algebra of all bounded linear operators on X with respect to the operator norm and by $\mathcal{K}(X)$ we denote the closed subalgebra of all compact operators of the algebra $\mathcal{L}(X)$. Operators $A, B \in \mathcal{L}(X)$ on a Banach space are said to be proportional or quasi-commuting if there exists a scalar t such that $AB = tBA$. R. L. Moore proved in [25] that either $|t| = 1$ or AB and BA are quasinilpotent operators. Proportional operators are also important in the theory of invariant subspaces. For example, it was proved in [25] that if for a given operator A the algebra $\mathcal{A}$ generated by

$$\{B \in \mathcal{L}(X) : \; AB = tBA \text{ for some } |t| \leq 1\}$$

contains a nonzero compact operator, then there exists a nontrivial closed subspace invariant under $\mathcal{A}$. R. Drnovšek and T. Košir in [11] considered an existence of invariant subspaces under a semigroup of pairwise proportional compact operators. Among other results, they proved that if there exists an irreducible semigroup of pairwise proportional compact operators on a complex Banach space, then the underlying space has to be finite dimensional.

Let $\mathcal{A}$ be a unital Banach algebra and let $a, b \in \mathcal{A}$ be commuting elements. Then it is well known that

$$r(ab) \leq r(a)r(b) \text{ and } r(a+b) \leq r(a) + r(b). \tag{5.1}$$

The following theorem extends inequalities (5.1) for commuting elements and it also extends [25, Theorem 1.6 (i)]. Since its proof is very similar to the proofs of Theorem 2.1 and Theorem 2.6, we omit it.

Theorem 5.1. *Let $\mathcal{A}$ be a unital Banach algebra and let the elements a, b, c satisfy $ab = cba$, $ac = ca$ and $bc = cb$. If c is power bounded, then*

$$r(ab) \leq r(a)r(b) \text{ and } r(a+b) \leq r(a) + r(b).$$

If $\|c\| < 1$, *then* $r(ab) = 0$.

Applying the previous result to the Calkin algebra $\mathcal{L}(X)/\mathcal{K}(X)$, we obtain the following result for the essential spectral radius r_{ess}.

Corollary 5.2. *Let* X *be a Banach space and let bounded operators* A, B, C *satisfy* $AB = CBA$, $AC = CA$ *and* $BC = CB$. *If* C *is power bounded, then the following inequalities hold*

$$r_{\mathrm{ess}}(AB) \le r_{\mathrm{ess}}(A) r_{\mathrm{ess}}(B) \;\; \text{and} \;\; r_{\mathrm{ess}}(A+B) \le r_{\mathrm{ess}}(A) + r_{\mathrm{ess}}(B).$$

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References

1. Y. A. Abramovich, C. D. Aliprantis, *An invitation to Operator Theory*, American Mathematical Society (2002).
2. M. Akian, S. Gaubert, R.D. Nussbaum, The Collatz-Wielandt theorem for order-preserving homogeneous maps on cones, *preprint*, (2007).
3. F.L. Baccelli, G. Cohen, G.-J. Olsder, J.-P.Quadrat, *Synchronization and Linearity*, John Wiley, Chichester, New York, 1992.
4. R.B. Bapat, A max version of the Perron-Frobenius theorem, *Linear Algebra Appl.* 275-276, (1998), 3–18.
5. R.B. Bapat and T.E.S. Raghavan, *Nonnegative Matrices and Applications*, Encyclopedia of Mathematics and its Applications 64, Cambridge University Press, 1997.
6. F.F. Bonsall, Linear operators in complete positive cones, *Proc. London Math. Soc. (3)* 8 (1958), 53–75.
7. P. Butkovič, Max-algebra: the linear algebra of combinatorics?, *Linear Algebra Appl.* 367 (2003), 313–335.
8. R.A. Cuninghame-Green, *Minimax Algebra*, Lecture Notes in Economics and Math. Systems, vol. 166, Springer, Berlin, 1979.
9. R.A. Cuninghame-Green, *Minimax algebra and applications*, Advances in Imaging and Electron Physics, vol. 90, pp. 1–121, Academic Press, New York, 1995.
10. R. Drnovšek, M. Kandić, *More on Positive Commutators*, J. Math. Anal. Appl. **373** (2011), 580–584.
11. R. Drnovšek, T. Košir, *Irreducible operator semigroups such that AB and BA are proportional*, *Linear Algebra Appl.* 383 (2004), 113118
12. L. Elsner and P. van den Driessche, On the power method in max algebra, *Linear Algebra Appl.* 302-303 (1999), 17–32.
13. L. Elsner, D. Hershkowitz and A. Pinkus, Functional inequalities for spectral radii of nonnegative matrices, *Linear Algebra Appl.* 129 (1990), 103–130.
14. L. Elsner, C.R. Johnson and J.A. Dias Da Silva, The Perron root of a weighted geometric mean of nonnegative matrices, *Linear and Multilinear Algebra* 24 (1989), 1–13.
15. G.M. Engel, H. Schneider, Diagonal similarity and equivalence for matrices over groups with 0, *Czechoslovak. Math. J.* 25 (1975), 389–403.
16. S. Friedland, Limit eigenvalues of nonnegative matrices, *Linear Algebra Appl.* 74 (1986), 173–178.
17. S. Gaubert, *Théorie des systemes linéaires dans les dioïdes*, These, Ecole des Mines de Paris, 1992.
18. R.A. Horn, C.R. Johnson, *Topics in matrix analysis*, Cambridge University Press, 1999.
19. Y.Y. Lur, On the asymptotic stability of nonnegative matrices in max algebra, *Linear Algebra Appl.* 407 (2005), 149–161.
20. J. Mallet-Paret, R.D. Nussbaum, Eigenvalues for a class of homogeneous cone maps arising from max-plus operators, *Discrete and Continuous Dynamical Systems*, vol 8, num 3, (2002), 519–562.
21. J. Mallet-Paret, R.D. Nussbaum, Boundary layer phenomena for differential-delay equations with state-dependent time lags: III, *Journal of Differential Equations* 189, (2003), 640–692.
22. P. Meyer-Nieberg, *Banach lattices*, Springer-Verlag, Berlin, 1991.

23. R.D. Nussbaum, Convexity and log convexity for the spectral radius, *Linear Algebra Appl.* 73 (1986), 59–122.
24. R. D. Nussbaum, *The radius of the essential spectrum*, Duke Math. J., **37** (1970), 473–478.
25. R. L. Moore, *Contractive Commutants and Invariant Subspaces*, Proc. Amer. Math. Soc. **8**3 (1981), 747–750.
26. A. Peperko, On the max version of the generalized spectral radius theorem, *Linear Algebra Appl.* 428 (2008), 2312–2318.
27. A. Peperko, Inequalities for the spectral radius of non-negative functions, *Positivity* 13 (2009), 255-272.
28. A. Peperko, On the continuity of the generalized spectral radius in max algebra, *submitted*, 2010.
29. H.H. Schaefer, *Topological Vector Spaces*, Graduate Texts in Mathematics **3**, Springer Verlag, New York, 1971.
30. U. Zimmermann, *Linear and Combinatorial Optimization in Ordered Algebraic Structures*, Ann. Discrete Math., vol 10, North Holland, Amsterdam, 1981.

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