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Calculation of electroproduction amplitudes in the K-matrix formalism

Bojan Golli

Faculty of Education, University of Ljubljana, and J. Stefan Institute, Ljubljana, Slovenia

Abstract. We present the K-matrix approach to calculate pion electroproduction amplitudes in the framework of chiral quark models. We derive the relation between the K-matrix and the experimentally measured T-matrix and show how to separate the resonant contribution to the amplitudes from the background.

The work is being done in collaboration with Manuel Fiolhais (Coimbra), Pedro Alberto (Coimbra), Ze Amoreira (Covilha) and Simon Širca (Ljubljana).

1 Motivation

In our previous work [1] on electroweak excitations we have shown that different versions of chiral quark models may successfully describe the properties of the low lying nucleon resonances. In these calculations the excited states have been treated as bound states which is justified if we are interested only in the resonant part of the production amplitudes. The total amplitudes as measured in the experiment however include also non-resonant (background) processes related to the outgoing pion. Incorporating the decaying channel in the model calculation may therefore represent a stringent test for the model as well as yield interesting information on the production mechanism and in particular on the role of non-quark degrees in freedom in baryons.

In several models such as the linear σ-model or the chromodielectric model, which include nonlinear effects and in which the interplay between quark and non-quark degrees of freedom is treated in a self-consistent way, the calculation is only feasible in a variational approach. While such an approach can be easily implemented in the bound state calculations its application to the description of scattering processes is much more complicated. In [2] the Kohn variational approach has been adopted to calculate the πN phase shifts and the structure of the resonant state in the Cloudy-Bag-type models. In this work we extend the approach to be able to calculate also the electroproduction amplitudes for pions.

We shall limit ourself to the description of the Δ resonance.

2 The variational approach to the K-matrix

We shall be interest in the class of models in which the pion is coupled linearly to the quark source:

$$H=\int dk \sum_{mt}\left[\omega_k\, a_{mt}^\dagger(k)a_{mt}(k)+V_{mt}(k)a_{mt}(k)+V_{mt}^\dagger(k)\, a_{mt}^\dagger(k)\right]+\sum_B E_B^\circ c_B^\dagger c_B \tag{1}$$

where

$$V_{mt}(k)=-V(k)\sum_{i=1}^{3}\tau_t{}^i\sigma_m{}^i\ , \qquad V_{mt}^\dagger(k)=(-1)^{t+m}V_{-m-t}(k)\ . \tag{2}$$

Here $a_{mt}(k)$ and $a_{mt}^\dagger(k)$ are annihilation and creation operators for the $l=1$ pions with the third component of the spin m and isospin t; c_B and $c_B^\dagger$ are annihilation and creation operators for the "bare" baryons made up of three quarks, E_B° are the corresponding bare energies, $V(k)$ is the source function determined from the quark profiles. We shall limit here only to two state, the nucleon N, and the Δ. In the present stage we do not include meson self-interactions.

In the variational approach to the K-matrix when only a single channel is opened (such as the resonant scattering in the P33 channel below the 2 pion threshold) the resonant state is assumed in the form:

$$|\Psi\rangle=c_\Delta|\Phi_\Delta\rangle+\int dk\,\eta(k,k_0)\left[a_{mt}^\dagger(k)|\Phi_N^E\rangle\right]^{\frac{3}{2}\frac{3}{2}}\ . \tag{3}$$

Here $|\Phi_N^E\rangle$ and $|\Phi_\Delta\rangle$ represent the nucleon and the Δ bound states normalized as $\langle\Phi_N^E|\Phi_N^E\rangle=1$ and $\langle\Phi_\Delta|\Phi_\Delta\rangle=1$, respectively; $\eta(k,k_0)$ describes the scattering pion and $[]^{ST}$ denotes the coupling of spin and isospin of the pion and the bare quark core to the quantum numbers of the Δ. Asymptotically the pion wave function behaves as

$$\eta(r,k_0)=k_0\, j_1(k_0r)-\tan\delta\, k_0\, y_1(k_0r)\ ,\quad r\to\infty\ . \tag{4}$$

Note that in the K-matrix approach the standing waves rather than outgoing (and incoming) waves are used. In k-space this leads to

$$\eta(k,k_0)=\sqrt{\frac{\pi}{2}}\,\delta(k-k_0)+\frac{\chi(k,k_0)}{\omega_k-\omega_0}\ ,\quad K\equiv\tan\delta=\sqrt{2\pi}\,\frac{\omega_0}{k_0}\,\chi(k_0,k_0)\ . \tag{5}$$

The nucleon state $|\Phi_N^E\rangle$ in (3), modified in presence of the scattering pion, i.e. it depends on k and k_0 of the pion, should asymptotically go over to the ground states, $|\Phi_N^E\rangle\to|\Phi_N\rangle$ for $k\to k_0$.

Before introducing the variational principle which determines the parameters of the trial function of the type (3), i.e. the pion wave function $\eta(k,k_0)$, the parameter c_Δ as well as structure of the states Φ_Δ and Φ_N^E, we first prove an important relation which hold in this type of models:

$$\chi(k_0,k_0)=\langle\Psi|(H-E)\left[a_{mt}^\dagger(k_0)|\Phi_N\rangle\right]^{\frac{3}{2}\frac{3}{2}}\ . \tag{6}$$

We assume that both $|\Psi\rangle$ and $|\Phi\rangle$ are exact states, i.e. $H|\Psi\rangle = E|\Psi\rangle$ and $H|\Phi\rangle = E_N|\Phi\rangle$. We should keep in mind that H *is not Hermitian* since Ψ is not a square integrable function, hence $\langle\Psi|H \neq \langle\Psi|E$, and the above expression does not vanish. We write $H = H^\dagger + (H - H^\dagger)$:

$$\langle\Psi|(H-E)\left[a^\dagger_{mt}(k_0)|\Phi_N\rangle\right]^{\frac{3}{2}\frac{3}{2}} = \langle\Psi|(H-H^\dagger)\left[a^\dagger_{mt}(k_0)|\Phi_N\rangle\right]^{\frac{3}{2}\frac{3}{2}} . \qquad (7)$$

The non-Hermitian part of H is the pion kinetic energy term:

$$H_{kin} = \int d\mathbf{r} \sum_t (-1)^t \pi_{-t}(\mathbf{r}) \left[-\overrightarrow{\Delta}_r\right] \pi_t(\mathbf{r}) . \qquad (8)$$

Only those terms in Ψ that asymptotically ($r \to \infty$) behave as r^{-1} contribute to (7), i.e. the terms involving $\eta(k, k_0)$. Expression (7) yields

$$\int dk\, \eta(k,k_0) \int d\mathbf{r} \sum_t \left[\langle\Phi^E_N|a_{m't'}(k)\right]^{\frac{3}{2}\frac{3}{2}} \pi^\dagger_t(\mathbf{r}) \left[\overleftarrow{\Delta}_r - \overrightarrow{\Delta}_r\right] \pi_t(\mathbf{r}) \left[a^\dagger_{mt}(k_0)|\Phi_N\rangle\right]^{\frac{3}{2}\frac{3}{2}} . \qquad (9)$$

We now commute a and $a^\dagger$ through the pion field; only the commutators produce a non vanishing contribution. We next perform the k-integration yielding the pion wave function in r-space (4). After performing the angular integration we end up with the following integral

$$\frac{k_0^2}{2\omega_0}\sqrt{\frac{2}{\pi}} \int dr \left(j_1(k_0 r) - \sqrt{2\pi}\,\frac{\omega_0}{k_0}\,\chi(k_0,k_0)\, y_1(k_0 r)\right) \left[\overleftarrow{\frac{d}{dr} r^2 \frac{d}{dr}} - \overrightarrow{\frac{d}{dr} r^2 \frac{d}{dr}}\right] j_1(k_0 r) . \qquad (10)$$

The first term in () vanishes. We perform an integration *per partes* and we are left with the Wronskian of j_1 and y_1, $W(j(k_0 r), y_1(k_0 r)) = 1/(k_0 r)^2$, multiplied by $k_0^2 r^2 \chi(k_0, k_0)$. The resulting integral is finite for $r \to \infty$ and equal to $\chi(k_0, k_0)$ which proves (6). Relation (6) holds quite generally since the only assumption in the derivation was on the asymptotic form of the wave function Ψ.

For the class of models (2) we can derive another useful relation by commuting $a^\dagger$ in (6) through (2): $(H-E)a^\dagger_{mt}(k_0) = \omega_0 a^\dagger_{mt}(k_0) + V_{mt}(k_0) + a^\dagger_{mt}(k_0)(H-E)$. Since $(H-E)|\Phi\rangle = -\omega_0|\Phi\rangle$ the last term cancels the first one and we are left only with the matrix element of (2):

$$K_{\pi\pi} \equiv K(k_0,k_0) = \sqrt{2\pi}\,\frac{\omega_0}{k_0}\,\chi(k_0,k_0) = -\sqrt{2\pi}\,\frac{\omega_0}{k_0}\,\langle\Psi||\sum \sigma\tau||\Phi\rangle\, V(k_0) . \qquad (11)$$

This is an exact relation since in the derivation we have not made any assumption about the structure of the resonant state or the ground state; we have only referred to the form of the pion field far away from the source. The result is similar to the expression derived by Chew and Low for the T-matrix [3].

We now sketch a more general derivation of the Kohn variational principle for the K-matrix in the case of pion scattering compared to the derivation given in [2]. It is valid for a more general class of Hamiltonians than (2); we only require that the exact solution (as well as the trial state) has the form of (3) where the pion field asymptotically behaves as (4).

We start by observing that the matrix element between the exact wave function Ψ_e and a trial wave function Ψ_t, satisfying the boundary condition (4) with an approximate phase shift δ_t, can be written in the form

$$\langle\Psi_e|H - E|\Psi_t\rangle = -\frac{k_0}{2\omega_0}\left(\tan\delta_e - \tan\delta_t\right) . \tag{12}$$

The proof goes similarly as from (7) to (10), the only difference is that in (10) the expression $\sqrt{2/\pi}(j_1(k_0 r) - \tan\delta_t\, y_1(k_0 r)$ appears instead of the second $j_1(k_0 r)$. Introducing $\delta\Psi = \Psi_t - \Psi_e$ and taking into account $(H - E)|\Psi_e\rangle = 0$ we can write

$$\tan\delta_e = \tan\delta_t - \frac{2\omega_0}{k_0}\left(\langle\Psi_t|H - E|\Psi_t\rangle - \langle\delta\Psi|H - E|\delta\Psi\rangle\right) \tag{13}$$

which means that the functional

$$\mathcal{F}_K(\Psi_t) = \tan\delta_t - \frac{2\omega_0}{k_0}\langle\Psi_t|H - E|\Psi_t\rangle \tag{14}$$

is stationary with respect to variations of Ψ_t.

3 The K-matrix for the pion production

We may now extend the model by introducing the coupling to the photon, $H \to H + H_\gamma$, where H_γ has the usual form of the EM Hamiltonian with the minimal coupling to the hadronic EM current. Then the resonant state will include the emission and absorption of the M1 and E2 photons (denoted by the common index $\mathcal{M}$) which will modify our ansatz (3) in the following way:

$$\begin{aligned}|\Psi\rangle \;&= c_\Delta|\Phi_\Delta\rangle + \left[a^\dagger_{mt}(k_0)|\Phi_N\rangle\right]^{\frac{3}{2}\frac{3}{2}} + \int dk\,\frac{\chi_{\pi\pi}(k,k_0)}{\omega_k - \omega_0}\left[a^\dagger_{mt}(k)|\Phi^E_N\rangle\right]^{\frac{3}{2}\frac{3}{2}} \\ &+ \sum_{\mathcal{M}}\int dq\,\frac{\chi^{\mathcal{M}}_{\gamma\pi}(q,k_\gamma)}{\omega_q - \omega_\gamma}\left[a^\dagger_{\mathcal{M}}(q)|\Phi_N\rangle\right]^{\frac{3}{2}\frac{3}{2}} .\end{aligned} \tag{15}$$

We make the standard assumption that the EM coupling is much weaker compared to the strong coupling and does not modify the structure of the resonant state. The functions χ^{M1} and χ^{E2} are related to the corresponding K-matrices for the pion electroproduction. The proof of these relations is analogous to the derivation of the K-matrix for the elastic pion scattering. We find:

$$\frac{k_\gamma}{\sqrt{2\pi}\,\omega_\gamma}K^{\mathcal{M}}_{\gamma\pi} \equiv \chi^{\mathcal{M}}_{\gamma\pi}(k_\gamma,k_\gamma) = -\langle\Psi|(H - E)\left[a^\dagger_{\mathcal{M}}(k_\gamma)|\Phi_N\rangle\right]^{\frac{3}{2}\frac{3}{2}} \tag{16}$$

leading to

$$K^{\mathcal{M}}_{\gamma\pi} = -\sqrt{2\pi}\,\frac{\omega_\gamma}{k_\gamma}\langle\Psi||\left[H_\gamma, a^\dagger_{\mathcal{M}}(k_\gamma)\right]||\Phi\rangle . \tag{17}$$

4 Splitting the amplitudes in the resonant and the non resonant part

The K-matrix calculated in a model as discussed above exhibits a typical resonant behavior (provided of course the bare Δ-N splitting is sufficiently large). The energy at which the phase goes through 90° corresponds to the energy of the physical Δ, E_Δ. It can be parametrized in the form suggested by Davidson et al.[5]:

$$K_{\pi\pi} \equiv \tan\delta = \frac{C}{E_\Delta - E} + D \equiv \tan\delta^K_\Delta + \tan\delta_b \,. \tag{18}$$

Here δ is the full phase shift, δ^K_R is called the resonant phase shift and δ_b the background phase shift (BPS), which – as shown below – is identical to the BPS in the T-matrix approach. Note that both, δ and δ^K_R go through 90° at the same E, (e.g. $E = 1232$ MeV for Δ). The width of the resonance is simply related to the parameter C by $\Gamma^K_\Delta = 2C$.

The T-matrix which is directly related to experimentally measured amplitudes is related to the K-matrix as

$$T = \frac{K}{1 - iK} = \frac{C + (E_\Delta - E)D}{(E_\Delta - E) - i(C + (E_\Delta - E)D)} \,. \tag{19}$$

In contradistinction to the K-matrix which is a real quantity the T-matrix is complex. The position of the pole in the complex plain can be easily determined if we assume that the coefficients C and D do not depend on the energy. Then the above expression can be brought in the familiar form:

$$T_{\pi\pi}(E) = e^{2i\delta_b}\,\frac{\Gamma^T_\Delta/2}{M_\Delta - E - i\Gamma^T_\Delta/2} + \sin\delta_b e^{i\delta_b} \tag{20}$$

with

$$M_\Delta = E_\Delta + \frac{CD}{1+D^2} = E_\Delta + \tfrac{1}{2}\Gamma^T_\Delta \tan\delta_b \,, \qquad \Gamma^T_\Delta = \frac{2C}{1+D^2} = \Gamma^K_\Delta \cos^2\delta_b \,. \tag{21}$$

From the experimental phase shift in the P33 channel the following values are extracted $M_\Delta = 1210$ MeV, $\Gamma^T_\Delta = 100$ MeV and $\delta_b \approx -23.5°$.

Turning to the T-matrix matrix for the electroproduction, $T_{\gamma\pi}$, we note that the effect of the EM coupling to the photon has a negligible effect on the structure of the state (15). The position and the width of the pole is not changed with respect to the pure πN channel; the phase shift of the amplitudes is that of the πN scattering. This is the so called *Watson theorem* which in our case can be expressed in the form [5]

$$T_{\gamma\pi} = K_{\gamma\pi}(1 + iT_{\pi\pi}) = \frac{K_{\gamma\pi}}{1 - iK_{\pi\pi}} \,. \tag{22}$$

Using the popular parametrization

$$K_{\gamma\pi} = \frac{A}{E_\Delta - E} + B \tag{23}$$

we can express (22) in the form:

$$T_{\gamma\pi} = \frac{A + B(E_\Delta - E)}{(E_\Delta - E) - i[C + D(E_\Delta - E)]} \,. \tag{24}$$

Assuming again that the parameters A, B, C and D do not depend on the energy, we can relate our result to the parametrization of $T_{\gamma\pi}$ used in parameterizing the experimental data [4]:

$$T_{\gamma\pi} = re^{i\Phi} T^R_{\pi\pi} + \beta e^{i\delta_b} \,, \qquad T^R_{\pi\pi} = \frac{\Gamma^T_\Delta/2}{M_\Delta - E - i\Gamma^T_\Delta/2} \,. \tag{25}$$

We obtain:

$$re^{i\Phi} = \frac{A}{C} - 2\,\frac{B}{D}\,\frac{D^2}{1+D^2} + i\,D\left(\frac{A}{C} + \frac{B}{D}\,\frac{1-D^2}{1+D^2}\right) \quad \text{and} \quad \beta = \frac{B}{D}\sin\delta_b \,. \tag{26}$$

We have to comment on the above assumption of constant parameters. If we identify the parameters in (18) and (23) with what comes out from a model calculation it turns out that the parameters exhibit a strong dependence on the pion momenta k_0 and therefore also on the energy. This can be seen already from the lowest order expression for the scattering matrix which takes the form:

$$K_{\pi\pi} = \pi\,\frac{\omega_0}{k_0}\,V(k_0)^2\left[\frac{\langle\Delta\|\sum\sigma\tau\|N\rangle^2}{E_\Delta - E} + \frac{4}{9}\,\frac{\langle N\|\sum\sigma\tau\|N\rangle^2}{E_N + 2\omega_0 - E} + \frac{1}{9}\,\frac{\langle\Delta\|\sum\sigma\tau\|N\rangle^2}{E_\Delta + 2\omega_0 - E}\right] \tag{27}$$

where we immediately identify the parameter C in (18) with the numerator of the first term and D with the other two terms corresponding to the crossed processes with either the nucleon or the Δ in the intermediate state. Since the pions are p-waves, $V(k_0) \propto k_0^2$, and $K_{\pi\pi}$ behaves as k_0^3 close to the threshold – as it should. The resulting background phase shift δ_b is positive for all k_0.

In the analysis of experimental data it is more convenient to parametrize the resonant T matrix using the familiar form (25) with constant Γ and M. A consequence of such an assumption is that the background phase shift δ_b stays negative for the whole energy range. We should be aware that the negative δ_b does not correspond to any physical process; it is merely a convenient tool to analyze the experimental data. In order to extract the parameters M, Γ, r, and δ_b from the model calculation we should therefore numerically fit the calculated $K_{\pi\pi}$ and $K_{\gamma\pi}$ to the forms (18) and (23) and use (21) and (26) to relate them to the values of A, B, C and D obtained from the fit.

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