Polytopes, Configurations and Symmetries – Bled 2019

In memory of Branko Grünbaum (1929 – 2018).

Branko Grünbaum was born in 1929 in Osijek, Croatia (which was then part of the Kingdom of Yugoslavia). He studied at the University of Zagreb, where his professors included Stanko Bilinski (one of the people to the memory of whom he dedicated the last of his four books [9]; see also the paper [10]). In 1949 he emigrated to Israel, where he earned a Master's degree in 1954 at the Hebrew University of Jerusalem, and completed his PhD in 1957 under the supervision of Aryeh Dvoretzky. Soon afterwards he moved to the United States. He was appointed to a full professorship in 1966 at the University of Washington, Seattle, and he remained there until he retired in 2001. He supervised 19 PhD students, including Leah Wrenn Berman and Steve Wilson (two of the authors of the research papers in this issue).

Branko Grünbaum authored over 250 papers, mostly in combinatorial, convex and discrete geometry. To give a full account of the phenomenal influence he had in the mathematical world, through his papers, books, lectures and professional and personal interactions, would overwhelm the scope of an editorial like this. Instead, here we mention just a few key points, some of which motivated the choice of topics in this issue of *ADAM*. Many further details not touched upon here (including personal reminiscences) can be read in the memorial article [17] in *Ars Mathematica Contemporanea*. The contribution by Moshe Rosenfeld in this issue is also full of nice reminiscences of their friendship and work.

One of the earliest papers by Branko Grünbaum deals with common transversals [5], attesting how far back went his interest in combinatorial geometry. A paper by Ted Bisztriczky, Károly Böröczky and Károly J. Böröczky [P#3.12] represents this topic here.

Branko Grünbaum's classic monograph *Convex Polytopes* (published for the first time in 1967) is undoubtedly a milestone in the theory of convex polytopes, which has since undergone massive development. The significance of this book is illustrated well by the fact that Günter Ziegler published a paper on the occasion of the fiftieth anniversary of its publication [18]. In fact it was also Ziegler who together with Volker Kaibel and Victor Klee prepared a second edition [8], updating each chapter with numerous additions and comments, which served to bridge the three and a half decades between the two editions.

In the meantime, the theory of abstract polytopes also began to take shape. In writing two papers [6, 7], Branko Grünbaum became one of the pioneers of the theory of abstract regular polytopes, as acknowledged in a chapter on the historical background of this topic in the huge monograph by Peter McMullen and Egon Schulte [14]; see also the papers by Andreas Dress [1, 2]. More recently, new combinatorial objects such as hypertopes and maniplexes were spawned as further generalisations of abstract polytopes. In this issue, papers by Gabe Cunningham [#P3.06], Antonio Montero and Asia Ivić Weiss [#P3.07], and Daniel Pellicer and Steve Wilson [#P3.02], exemplify this development.

Branko Grünbaum's wide interests extended also to other forms of highly symmetrical geometric objects, and this led him to publish his monograph *Tilings and Patterns*, jointly with Geoffrey Shephard [13]. This comprehensive work has attracted the attention of not only mathematicians, but also crystallographers, chemists, physicists, architects and artists, and has proved to be an amazing source of inspiration for anyone working with geometric patterns.

Branko Grünbaum's legacy includes not only his numerous publications (and a huge number of citations), but also several interesting geometric objects and constructions that bear his name. We mention some of these explicitly, since they represent his interests as well as many of the topics covered in this issue.

Polyhedra and their various symmetry properties were a constant source of attraction and challenge for his ever-growing geometric imagination, even to an extent which led him to joke about 'otherhedra' in the title of one of his papers [10]. Especially interesting to him were the combinatorially regular polyhedra, which form polyhedral models for regular maps on surfaces of arbitrary topological genus, making them generalisations of the Platonic solids.

While studying various vertex-transitive polyhedra (partly in collaboration with Geoffrey Shephard), he discovered a regular polyhedron that proved to be a geometric realisation of the famous regular map of Fricke and Klein. Later this polyhedron became known as the *Grünbaum polyhedron* [3] – not to be confused with the polyhedra treated in [1, 2]. Relevant papers in this issue are those by Gábor Gévay and Egon Schulte [#P3.04], Jürgen Bokowski and Gábor Gévay [#P3.09], and Jürgen Bokowski [#P3.10]. Symmetry also plays a role in a related topic, namely the theory of hypermaps, the subject of the paper by Maria Elisa Fernandes and Claudio Alexandre Piedade [#P3.13].

Another interest of Branko Grünbaum has roots in the theory of regular maps, and in particular, in the works of Felix Klein. In a joint paper [11], Branko Grünbaum and John Rigby gave a geometric point-line realisation of an abstract configuration discovered by Klein in the study of his famous quartic surface. The paper [11] may be considered as the starting point of a renaissance in the investigation of geometric configurations, and the configuration in question is now known as the *Grünbaum–Rigby configuration*. Branko Grünbaum was not only the initiator but also a leading figure of the revival of this field of research. Its rapid development quickly resulted in the publication of two monographs: one by himself [9], and the other by Tomaž Pisanski and Brigitte Servatius [15].

In the latter, the authors introduced the term *Grünbaum calculus*, to describe a collection of ingenious constructions by which one can build new configurations from old ones. It is impressive to see (for example) in the paper by Leah Wrenn Berman, Gábor Gévay and Tomaž Pisanski [#P3.14], how these techniques are very useful in improving what is known about the existence of certain geometric configurations. In other directions, the contributions by Jürgen Bokowski and Hendrik Van Maldeghem [#P3.08], William Kocay [#P3.15] and Vito Napolitano [#P3.05] add further new aspects to the rapidly developing theory of configurations, and in their paper on more general incidence structures [#P3.01], Natalia Garcia-Colin and Dimitri Leemans declare how their research was inspired by the relevant writings of Branko Grünbaum.

Branko Grünbaum's unflagging interest in graphs (which arise in various contexts such as combinatorics, geometry and group theory) may be seen in many of his papers. Even in his third paper reviewed in *Mathematical Reviews* (now MathSciNet), namely [4], he used graph theory to prove a conjecture of the Hungarian-born American mathematician Endre Vázsonyi (who was a friend of Erdős from the days of their youth in Budapest). The *Levi graph* (or *incidence graph*) of an incidence structure such as a configuration is a useful tool for translating a problem from that context into the language of graphs [9, 15], and the term *Grünbaum graph* introduced by Tomaž Pisanski and Thomas Tucker [16] is a tribute to Branko Grünbaum's influence. In this issue, his influence on graph theory is exemplified by the paper by Gareth Jones [#P3.03], which begins with a reference to the influential paper [12] by Grünbaum and Shephard on edge-transitive planar graphs, while the paper by Edward Dobson and Joy Morris [#P3.011] deals with Cayley graphs, the study of which interweaves graph theory with group theory.

Finally, we make some comments about the inception of this issue of *ADAM*.

The *9th Slovenian International Conference on Graph Theory* (also known as Bled'19) was held in Bled, Slovenia, the week 23 – 29 June 2019, continuing a series of conferences held every four years in Slovenia, and by tradition, mostly at Lake Bled. There were more than 300 participants, including 11 plenary speakers, and a large number of others giving talks that were organised into 17 invited special sessions. Topics of three of these special sessions were closely related to Branko Grünbaum's mathematical works:

- Configurations,
- Polytopes, and
- Symmetries of Graphs and Maps.

These sessions could be regarded as honours to Branko Grünbaum and his life's work, as are the papers accepted for this issue. Also we believed that *The Art of Discrete and Applied Mathematics* (ADAM) is a highly appropriate venue for publishing this collection: Branko Grünbaum was an enthusiastic supporter of the founding of both *Ars Mathematica Contemporanea* (AMC) and *ADAM*, and was a member of the Advisory Board of both journals. We are delighted to have been involved in helping to pay this tribute to him.

Marston Conder, Gábor Gévay, Asia Ivić Weiss

Guest Editors

Front row, left to right: Leah Wrenn Berman, Tomaž Pisanski and Branko Grünbaum. Back row, left to right: Jürgen Bokowski and Marko Boben. Three of the people in this photo (LWB, JB and TP) are authors of papers in this issue. (By courtesy of Tomaž Pisanski.)

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An infinite family of incidence geometries whose incidence graphs are locally X*

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Abstract

We construct a new infinite family of incidence geometries of arbitrarily large rank. These geometries are thick and residually connected and their type-preserving automorphism groups are symmetric groups. We also compute their Buekenhout diagram. The incidence graphs of these geometries are locally X graphs, but more interestingly, the automorphism groups act transitively, not only on the vertices, but more strongly on the maximal cliques of these graphs.

DEDICATED TO THE MEMORY OF BRANKO GRÜNBAUM.

Keywords: Kneser graph, locally X graph, incidence geometry.

Math. Subj. Class.: 05C75, 51E24

The influence of Branko in the world combinatorics is unbounded. Although the authors of this work did not have the luck of meeting him in person, they have being inspired by him through his influential writtings on the theories of Convex Polytopes and Configurations of Points and Lines.

This paper contributes to the study of incidence geometries whose type-preserving automorphism groups are symmetric groups, this subject posseses a straight-forward connection to Branko's work on point-line configurations, the latter being incidence geometries too.

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1 Introduction

In [9, 7], Francis Buekenhout, Philippe Cara and Michel Dehon determined the so-called inductively minimal geometries, that are the thin residually connected regular incidence geometries of rank $n-1$ with type-preserving automorphism group S_n. Recently, Maria Elisa Fernandes and Dimitri Leemans, extended this study to rank $n-2$ thin geometries [19], and Fernandes and Leemans together with Mark Mixer classified the thin residually connected regular incidence geometries of rank $n-i$ ($i = 1, \dots, 4$) with a linear Buekenhout diagram and type preserving automorphism group S_n [18, 20], these geometries being abstract regular polytopes. Here we start from Kneser graphs and construct thick geometries whose underlying incidence graphs turn out to be locally X graphs. This provides yet another example of the natural connections between incidence geometries and other combinatorial structures.

Our construction is as follows (see Section 2 for terminology and notation).

Construction 1.1. *For any positive integer $r \geq 2$ and a given Kneser graph $KG(n,k)$, define a rank r incidence system $\Gamma(KG(n,k),r) := (Y,*,t,I)$ as follows. Let $\Omega := \{1,\dots,n+k(r-2)\}$. Let $I := \{1,\dots,r\}$. Take r copies $Y_1,\dots,Y_r$ of the set of all the subsets of size k of Ω, and let $Y = Y_1 \cup \ldots \cup Y_r$. For any $x \in Y_i$, define $t(x) = i$. For any elements $x_i \in Y_i$ and $x_j \in Y_j$, we say that $x_i * x_j$ if and only if $i \neq j$ and x_i and x_j are disjoint as subsets of Ω.*

These geometries have several nice properties summarized as follows.

Theorem 1.2. *$\Gamma(KG(n,k),r)$ is a thick, residually connected and flag-transitive incidence geometry of rank r. Its automorphism groups are $Aut_I(\Gamma) \cong S_{n+k(r-2)}$ and $Aut(\Gamma) \cong S_{n+k(r-2)} \times S_r$. The Buekenhout diagram of $\Gamma(KG(n,k),r)$ is a complete graph. The orders of the diagram are $\binom{n-k}{k-1}$, the number of elements of each type is $\binom{n+k(r-2)}{k}$, the edges are labelled as $d-g-d$ where $g = 3$ if $n = 2k+1$ and $g = 2$ otherwise, and $d = 2\lceil\frac{k}{n-2k}\rceil + 1$.*

We want to highlight one remarkable member of this infinite family. The geometry $\Gamma(KG(5,2),r)$ has arbitrary large rank r and all of its rank two residues are isomorphic to the Desargues configuration.

We found these new geometries while searching for large locally X graphs. For a given graph X, a graph $\mathcal{G}$ is *locally* X if the graphs induced on the neighbours of every vertex of $\mathcal{G}$ are isomorphic to X. In the literature, $\mathcal{G}$ is also referred to as an *extension of* X or a *locally homogeneous graph.*

The construction of the incidence graph corresponding to the incidence geometry in Theorem 1.2 immediately gives the following corollary, adding explicitly one large family to the list of known locally X graphs.

Corollary 1.3. *The incidence graph of $\Gamma(KG(n,k),r)$ is a locally $KG(n+k(r-3),k) \times K_{r-1}$ graph.*

These graphs are isomorphic to $KG(n+k(r-2),k) \times K_r$, where $\times$ stands for the tensor product. Thus their vertex-transitivity is easy to prove, implying that they are locally $KG(n+k(r-3),k) \times K_{r-1}$. However, Theorem 1.2 proves more than the above, namely:

Corollary 1.4. *The symmetric group $S_{n+k(r-2)}$ acts transitively on the set of maximal cliques of $KG(n+k(r-2),k) \times K_r$.*

We believe that the connection between incidence geometries and locally X graphs is worthy of further study. It is interesting to explore under which conditions incidence geometries can lead to locally X graphs.

The paper is organized as follows. In Section 2 we present some structural properties of Kneser graphs which will be used in the construction of the incidence geometries we study and we present the necessary background on incidence geometries. In Section 3 we compute the neighbourhood geometry of a Kneser graph, which is used in Section 4 to construct a new infinite family of locally X graphs.

2 Preliminaries

2.1 Graph theory

A *Kneser graph*[1] $KG(n, k)$ is a graph whose vertex set is the set of all k-subsets of $\{1, \ldots, n\}$ and any pair of disjoint k-subsets is joined by an edge. Some Kneser graphs are very familiar objects, for instance the complete graphs,K_n, are Kneser graphs $KG(n, 1)$ and the Petersen graph is a $KG(5, 2)$. Its very simple to see that Kneser graphs are locally Kneser graphs, furthermore Jonathan Hall proved in [21] that there are exactly three pairwise non-isomorphic locally Petersen graphs, only one of them being a Kneser graph, namely $KG(7, 2)$.

The following lemmas will cover some structural characteristics of Kneser graphs which, in turn, will be used in the later sections for determining structural characteristics of our constructions.

Lemma 2.1. *The smallest odd cycle of a Kneser graph, $KG(n, k)$, with $n > 2k + 1$ has length $2\lceil \frac{k}{n-2k} \rceil + 1$.*

Proof. We may assume that $n < 3k$, as $KG(n, k)$ with $n \geq 3k$ has triangles, and the statement holds. Let $n = 2k + r$ for some $r < k$ and $A_1, \ldots, A_{2l+1}$ be the vertices of the smallest odd cycle, as subsets of $[n]$.

By construction, we have $A_1 \cap A_2 = \emptyset$ and $A_2 \cap A_3 = \emptyset$ thus $A_1 \cup A_3 \subset A_2^c$ and they have non empty intersection as, $|A_2^c| = n - k = k + r < 2k$, thus $|A_1 \cap A_3| \geq k - r$ and A_1 and A_3 cannot be adjacent. Similarly, we can argue that $|A_3 \cap A_5| \geq k - r$ thus $|A_1 \cap A_5| \geq k - 2r$. We may continue this process to conclude that $|A_1 \cap A_{2i+1}| \geq k - ir$.

Hence $A_1 \cap A_{2l+1} = \emptyset$ if and only if $k - lr \leq 0$. This happens precisely when $\lceil \frac{k}{n-2k} \rceil \leq l$ and the result follows. □

Lemma 2.2. *Between any two vertices of a Kneser graph, A, B, such that $|A \cap B| = c$ there is an even path of length $2\lceil \frac{k-c}{n-2k} \rceil$ and an odd path of length $2\lceil \frac{c}{n-2k} \rceil + 1$.*

Proof. Let A and B be two vertices of $KG(n, k)$, $C = A \cap B$, $|C| = c$, $D = (A \cup B)^c$, and $|D| = n - 2k + c$. Let $X_1, \ldots, X_l$ and $Y_1, \ldots, Y_l$ with $l = \lceil \frac{k-s}{n-2k} \rceil$ be partitions of $A \setminus B$ and $B \setminus A$, respectively, in sets of size $n - 2k$, perhaps except the last one.

Let $A_{2i-1} = (\cup_{j=1}^{i} X_j) \cup (\cup_{j=i+1}^{l} Y_j) \cup D'$ for some $D' \subset D$ of cardinality c and $A_{2i} = (\cup_{j=1}^{i} Y_j) \cup (\cup_{j=i+1}^{l} X_j) \cup C$ for $0 \leq i \leq l$. Clearly $A_j \cap A_{j+1} = \emptyset$, $A = A_0$, $B = A_{2l}$, thus $A, A_1 \ldots, A_{2l-1}, B$ is a path of even length $2\lceil \frac{k-c}{n-2k} \rceil$.

For the odd path, take $D' \subset D$ of cardinality c and let $A' = B \setminus A \cup D'$. Then A and A' are adjacent, and we may construct an even path of size $2\lceil \frac{c}{n-2k} \rceil$ between A' and B as before, given that $|A' \cap B| = k - c$, and the result follows. □

[1] Lovász introduced the term Kneser graph in [30] after a problem posed by Kneser in [27].

2.2 Incidence geometry

An *incidence system* [8], $\Gamma := (Y, *, t, I)$ is a 4-tuple such that

- Y is a set whose elements are called the *elements* of Γ;
- I is a set whose elements are called the *types* of Γ;
- $t : Y \to I$ is a *type function*, associating to each element $x \in Y$ of Γ a type $t(x) \in I$;
- $*$ is a binary relation on Y called *incidence*, that is reflexive, symmetric and such that for all $x, y \in Y$, if $x * y$ and $t(x) = t(y)$ then $x = y$.

The *incidence graph* of Γ is the graph whose vertex set is Y and where two vertices are joined provided the corresponding elements of Γ are incident.

A *flag* is a set of pairwise incident elements of Γ, i.e. a clique of its incidence graph. The *type* of a flag F is $\{t(x) : x \in F\}$. A *chamber* is a flag of type I. An element x is *incident* to a flag F and we write $x * F$ for that, when x is incident to all elements of F. An incidence system Γ is a *geometry* or *incidence geometry* if every flag of Γ is contained in a chamber (or in other words, every maximal clique of the incidence graph is a chamber). The *rank* of Γ is the number of types of Γ, namely the cardinality of I.

Observe that the incidence graph of an incidence system of rank n is an n-partite graph. This will play a key role in the construction of an infinite family of locally X graphs.

Let $\Gamma := (Y, *, t, I)$ be an incidence system. Given $J \subseteq I$, the *J–truncation* of Γ is the incidence system $\Gamma^J := (t^{-1}(J), *_{|t^{-1}(J)\times t^{-1}(J)}, t_{|J}, J)$. In other words, it is the subgeometry constructed from Γ by taking only elements of type J and restricting the type function and incidence relation to these elements.

Let $\Gamma := (Y, *, t, I)$ be an incidence system. Given a flag F of Γ, the *residue* of F in Γ is the incidence system $\Gamma_F := (Y_F, *_F, t_F, I_F)$ where

- $Y_F := \{x \in Y : x * F, x \notin F\}$;
- $I_F := I \setminus t(F)$;
- t_F and $*_F$ are the restrictions of t and $*$ to Y_F and I_F.

An incidence system Γ is *residually connected* when each residue of rank at least two of Γ has a connected incidence graph. It is called *firm* (resp. *thick*) when every residue of rank one of Γ contains at least two (resp. three) elements.

Let $\Gamma := (Y, *, t, I)$ be an incidence system. An *automorphism* of Γ is a mapping $\alpha : (Y, I) \to (Y, I) : (x, t(x)) \mapsto (\alpha(x), t(\alpha(x))$ where

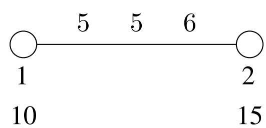
- α is a bijection on Y;
- for each $x, y \in Y$, $x * y$ if and only if $\alpha(x) * \alpha(y)$;
- α induces a bijection on I such that for each x, $y \in Y$, $t(x) = t(y)$ if and only if $t(\alpha(x)) = t(\alpha(y))$.

An automorphism α of Γ is called *type preserving* when for each $x \in Y$, $t(\alpha(x)) = t(x)$. The set of all automorphisms of Γ together with the composition forms a group that is called the *automorphism group* of Γ and denoted by $Aut(\Gamma)$. The set of all type-preserving automorphisms of Γ is a subgroup of $Aut(\Gamma)$ that we denote by $Aut_I(\Gamma)$. An incidence system Γ is *flag-transitive* if $Aut_I(\Gamma)$ is transitive on all flags of a given type J for each type $J \subseteq I$.

A rank two geometry with points and lines is called a *generalised digon* if every point is incident to every line and, it is called a *partial linear space* if there is at most one line through any pair of points. An incidence geometry is said to satisfy the *intersection property of rank 2*, denoted by $(IP)_2$, when all its rank two residues are either partial linear spaces or generalised digons.

Let Γ be a firm, residually connected and flag-transitive geometry. The *Buekenhout diagram* of Γ is a graph whose vertices are the elements of I and with an edge $\{i, j\}$ with label $d_{ij} - g_{ij} - d_{ji}$ whenever every residue of type $\{i, j\}$ is not a generalised digon. The number g_{ij} is called the *gonality* and is equal to half the girth of the incidence graph of a residue of type $\{i, j\}$. The number d_{ij} is called the i-diameter of a residue of type $\{i, j\}$ and is the longest distance from an element of type i to any element in the incidence graph of a residue of type $\{i, j\}$. Moreover, to every vertex i is associated a number s_i, called the i-order, which is equal to the size of a residue of type i minus one, and a number n_i which is the number of elements of type i of the geometry.

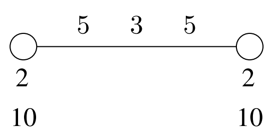
The Petersen graph, for instance, can be seen as a geometry of rank two whose elements are the vertices and edges of the graph. Its Buekenhout diagram is the following.



Let G be a graph. Denote its set of vertices (resp. edges) by G_0 (resp. G_1). For distinct p, $q \in G_0$, we say that p and q are *adjacent* – and we write $p \sim q$ – whenever $\{p, q\} \in G_1$. As in [29], to the graph G, we associate a new rank 2 geometry $\tilde{G}$, called the *neighborhood geometry* of G, whose elements are, roughly speaking, the vertices and the neighborhoods of vertices of G. More precisely, we define $\tilde{G}$ to be the geometry $(G_0 \times \{0\} \cup G_0 \times \{1\}, \tilde{*}, \tilde{t}, \{0, 1\})$ with

- $\tilde{t}(G_0 \times \{i\}) = i$, for $i = 0, 1$;
- $(p, 0)\tilde{*}(q, 1)$ iff $p \sim q$, for $p, q \in G_0$.

As pointed out in [29, Table 1], the neighborhood geometry of the Petersen graph is Desargues' configuration. Figure 1 gives the Petersen graph and Desargues' configuration. The Buekenhout diagram of Desargues' configuration is the following.



3 The neighborhood geometry of a Kneser graph

In this section, we compute the neighborhood geometry of a given Kneser graph. These geometries will then be used in the next section to construct locally X graphs as incidence graphs of some particular incidence geometries. The incidence graphs of these geometries are sometimes called the bipartite Kneser graphs.

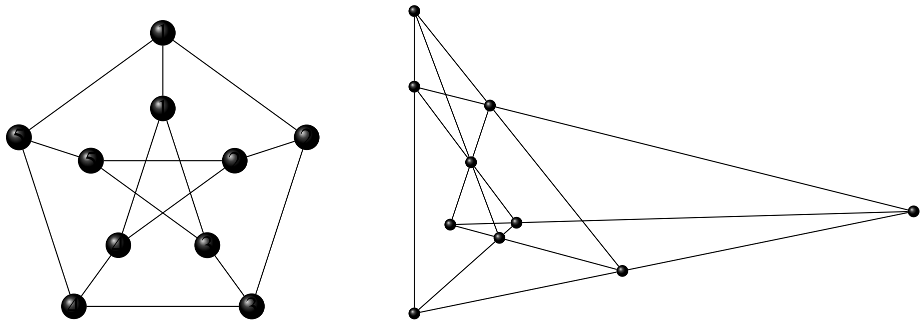


Figure 1: Petersen graph $KG(5,2)$ and Desargues' configuration $\tilde{KG}(5,2)$

Lemma 3.1. *The* 0*–diameter and* 1*–diameter of* $\tilde{KG}(n,k)$ *is* $2\lceil \frac{k}{n-2k} \rceil + 1$.

Proof. As the construction of $\tilde{KG}(n,k)$ is symmetric in the set of types, the 0–diameter and the 1–diameter are the same.

By Lemma 2.2 the distance between any two vertices of $\tilde{KG}(n,k)$ is at most $2\lceil \frac{k-c}{n-2k} \rceil$ or $2\lceil \frac{c}{n-2k} \rceil + 1$, for some $0 \leq c \leq k$. This achieves a maximum of $2\lceil \frac{k}{n-2k} \rceil + 1$ when $c = k$, that is, when we are tracing a path from the two copies of the same vertex in $\tilde{KG}(n,k)$.

We now argue that this bound can't be improved, as such improvement would contradict Lemma 2.1. □

Lemma 3.2. *The gonality of* $\tilde{KG}(n,k)$ *is* 3 *when* $n = 2k+1$ *and* 2 *when* $n \geq 2k+2$.

Proof. If $n = 2k+1$, the following is a circuit of length 6 in the incidence graph of $\tilde{KG}(n,k)$: $(\{1,\dots,k\},0)\tilde{*}(\{k+1,\dots,2k\},1)\tilde{*}(\{1,\dots,k-1,2k+1\},0)\tilde{*}(\{k,\dots,2k-1\},1)\tilde{*}(\{1,\dots,k-1,2k\},0)\tilde{*}(\{k+1,\dots,2k-1,2k+1\},1)\tilde{*}(\{1,\dots,k\},0)$.

If $n > 2k+1$, the following is a circuit of length 4 in the incidence graph of $\tilde{KG}(n,k)$: $(\{1,\dots,k\},0)\tilde{*}(\{k+1,\dots,2k\},1)\tilde{*}(\{1,\dots,k-1,2k+1\},0)\tilde{*}(\{k+1,\dots,2k-1,2k+2\},1)\tilde{*}(\{1,\dots,k\},0)$. □

Lemma 3.3. $\tilde{KG}(n,k)$ *is a connected graph.*

Proof. This follows from the fact that $KG(n,k)$ has cycles of odd length as proven in Lemma 2.1. □

4 Proof of Theorem 1.2

In this section we prove the main theorem of this paper. Observe that the case $r = 2$ in Construction 1.1 gives the neighbourhood geometry $\tilde{KG}(n,k)$ of the Kneser graph $KG(n,k)$ that was defined in the previous section.

Lemma 4.1. $\Gamma(KG(n,k),r)$ *is an incidence geometry.*

Proof. As $\mid \Omega \mid = n + k(r-2)$ and $n \geq 2k+1$, it is always possible to find r pairwise disjoint subsets of size k in Ω. Hence every maximal flag of Γ must be a chamber and Γ is an incidence geometry. □

Lemma 4.2. *The symmetric group $S_\Omega \cong S_{n+k(r-2)}$ acts transitively on the chambers of $\Gamma(KG(n,k),r)$ (or in other words, $\Gamma(KG(n,k),r)$ is flag-transitive).*

Proof. This is due to the fact that S_Ω is $(n+k(r-2))$–transitive on Ω. □

Lemma 4.3. $\Gamma(KG(n,k),r)$ *is residually connected.*

Proof. We prove this by induction on r. The case $r = 2$ is dealt with in Lemma 3.3.

Suppose that $\Gamma(KG(n,k),r)$ is residually connected. In order to prove that $\Gamma(KG(n,k),r+1)$ is residually connected, we only need to show that the incidence graph of $\Gamma(KG(n,k),r+1)$ is connected, as all residues of $\Gamma(KG(n,k),r+1)$ of rank $< r+1$ are connected by the induction hypothesis and the fact that $\Gamma(KG(n,k),r+1)$ is flag-transitive as shown in Lemma 4.2. Take x_i an element of type i and x_j an element of type $j \neq i$. The $\{i,j\}$–truncation of $\Gamma(KG(n,k),r)$ is the neighborhood geometry $\tilde{KG}(n*k(r-2),k)$ of a Kneser graph $KG(n*k(r-2),k)$. By Lemma 3.3, $\tilde{KG}(n*k(r-2),k)$ is connected. Hence, every rank two truncation of $\Gamma(KG(n,k),r)$ is connected and therefore $\Gamma(KG(n,k),r)$ is connected. □

Lemma 4.4. *For any $i = 1,\ldots,r$, the i–order of $\Gamma(KG(n,k),r)$ is equal to $\binom{n-k}{k-1}$.*

Proof. A flag F of rank $r-1$ consists of $r-1$ pairwise disjoint subsets of size k. Hence these subsets cover $k(r-1)$ points of Ω. So there are $n+k(r-2)-k(r-1) = n-k$ points not covered by F in Ω. There are thus $\binom{n-k}{k}$ subsets of size k that are disjoint with all subsets of F. □

The previous lemma immediately implies the following corollary.

Corollary 4.5. $\Gamma(KG(n,k),r)$ *is thick.*

Lemma 4.6. *Every rank two residue of $\Gamma(KG(n,k),r)$ is isomorphic to $\tilde{KG}(n,k)$.*

Proof. Every rank two residue is obtained from a flag F that has $r-2$ elements. These $r-2$ elements cover $k(r-2)$ elements of Ω and therefore there are n elements of Ω remaining. □

Lemma 4.7. *$Aut_I(\Gamma(KG(n,k),r)) \cong S_{n+k(r-2)}$ and $Aut(\Gamma(KG(n,k),r)) \cong S_{n+k(r-2)} \times S_r$*

Proof. By Lemma 4.2, we know that $Aut_I(\Gamma(KG(n,k),r)) \geq S_{n+k(r-2)}$. Obviously, it cannot be strictly bigger. Construction 1.1 is symmetric in the set of types. Hence $Aut(\Gamma(KG(n,k),r) \cong S_{n+k(r-2)} \times S_r$. □

Theorem 1.2 is a summary of all the properties we have proved on $\Gamma(KG(n,k),r)$ above. Note that the gonality and diameters of the rank two residues of the Buekenhout diagram were computed in Lemma 3.2 and Lemma 3.1.

We highlight that $\Gamma(KG(5,2),3)$ was already given in [10, page 86]. Furthermore, geometries satisfying the intersection property of rank two, $(IP)_2$, attracted much attention in the nineties and noughties[2] (see, for instance, [10, 28]). It turns out that some of the incidence geometries obtained by Construction 1.1 satisfy $(IP)_2$, as shown in the next corollary.

[2]The noughties mean the years 2000–2010.

Corollary 4.8. *$\Gamma(KG(n,k),r)$ is $(IP)_2$ if and only if $n = 2k+1$.*

Proof. For a rank two residue of $\Gamma(KG(n,k),r)$ to satisfy $(IP)_2$, we need it to be a generalised digon or its gonality to be at least 3. Generalised digons have gonality and diameters equal to two. Lemma 3.1 and Lemma 3.2 then finish the proof. □

Observe that when $n = 2k+1$, the diameters and gonality written on the edges of the Buekenhout diagram are respectively n and 3.

5 Connection with locally X graphs

A. Zykov [38, 39] posed the problem of characterizing the graphs, X, for which there are locally X graphs. Finding any general solution for this problem is difficult, if at all possible.

Apart from the inherent interest of this problem for graph theorists, another motivation for the study of locally homogeneous graphs is observed in [23]:

"The theorems may find application in the characterization of the Johnson scheme among the primitive association schemes and distance regular graphs. It can also be used to characterize alternating and symmetric groups (of sufficiently large degree) by centralizers of various of their elements (the initial motivation for the theorem)."

The progress thus far has followed three general lines of enquiry; the undecidability of the problem; the construction of locally X graphs for some selected graphs, X; and sufficient conditions for a graph X to have an extension. Our construction adds to the second line. We now prove Corollary 1.3.

Proof of Corollary 1.3. Since $\Gamma(KG(n,k),r)$ is a flag-transitive geometry and since $Aut(\Gamma(KG(n,k),r) \cong S_{n+k(r-2)} \times S_r$, all the incidence graphs of the residues of rank $r-1$ are isomorphic. □

Corollary 5.1. *There exists a locally Desargues graph whose automorphism group is isomorphic to S_7.*

Proof. Such a graph can be obtained as the incidence graph of $\Gamma(KG(5,2),3)$. □

It would be interesting to explore what are the characteristics of incidence geometries that lead to interesting locally X graphs.

We would like to highlight that in [22], a complete list of all graphs X of order up to six having an extension is given; in some cases all such extensions are characterized. Constructions of locally X graphs, for instance, cycles [16, 15], unions of paths [34], trees [2], polyhedra [3, 13, 14], the Petersen graph [21], other Kneser graphs [23, 32], dense graphs [12, 33] and others have been investigated in [4, 5, 6, 11, 17, 24, 26, 37]. A rich compilation of locally X graphs can also be found in [36]. Some structural characteristics for a graph X to have an extension have been given in [1, 5, 6, 25, 35].

Finally, as pointed out by a referee, there are also lots of interesting examples of locally X graphs that come from abstract regular polytopes. For example, as mentioned in [31, page 165], if $\mathcal{P}$ is a regular n-polytope with triangular faces, the 1-skeleton (or edge graph) of $\mathcal{P}$ is a locally X graph where X is the 1-skeleton of the vertex-figure of $\mathcal{P}$. Every polytope with a Schläfli symbol of the form $\{3, p_2, \ldots, , p_n\}$ is of this kind. For example, there are lots of interesting locally X graph where X is the graph of a toroidal map (and these are obtained from polytopes of type $\{3,4,4\}$, $\{3,3,6\}$ or $\{3,6,3\}$.

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Rotary one-facet maniplexes*

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Abstract

Maniplexes are combinatorial objects that generalize, simultaneously, maps on surfaces and abstract polytopes. We are interested on studying *rotary* maniplexes, that is, those having maximal 'rotational' symmetry.

This note classifies rotary 4-dimensional maniplexes with the property of having exactly one facet, and gives examples and related results.

Keywords: Graph, automorphism group, symmetry, polytope, maniplex, map, flag, transitivity, rotary, reflexible, chiral.

Math. Subj. Class.: 05E18, 05C25.

1 Introduction

Maps on surfaces are structures which are said to be of rank 3 since they have three kinds of objects: vertices, edges and faces. We have been particularly interested in those exhibiting a good deal of symmetry. Such maps have been studied from topological, algebraic, geometric and combinatorial points of view (see for example [9, 10, 15, 17]).

Abstract polytopes are partially ordered sets satisfying some of the main properties of the face-lattices of convex polytopes. Every abstract polytope of rank 3 can be regarded as a map. Conversely, many maps on surfaces can be regarded as abstract polytopes of rank 3, but many others violate some of the axioms for polytopality. In this sense, the concept of

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map on a surface is more general than that of an abstract 3-polytope. Again, we are most interested in those possessing substantial symmetry.

Maniplexes are combinatorial objects that generalize both, maps on surfaces and abstract polytopes. Recently, many results on maps and polytopes have been strenghtened being stated now for maniplexes.

In recent years there has been attention on finding the smallest polytopes with given properties (see for example [1, 2, 5]); partially motivated by the gigantic size of the chiral polytopes of ranks 6 and higher known so far ([3, 6, 13]).

As we begin to study any combinatorial structure, we always want to know what the smallest example is of such an object. In this work we study n-maniplexes having rotary symmetry and only one facet, for n up to 4. Intuitively, these will be 'small' maniplexes, and the constructions shown below may yield the smallest 4-maniplexes that satisfy certain sets of properties.

2 Maniplexes

2.1 Definitions

We consider the notion, introduced in [16], of a *maniplex*. An *(n+1)-dimensional maniplex* $\mathcal{M}$ is a pair $(\Omega, [r_0, r_1, \ldots, r_n])$, where Ω is a set of things called *flags* and each r_i is a partition of Ω into unordered pairs. We often use algebraic language, and refer to each r_i as an involutory permutation on Ω, where "$r_i x = y$" means that the pair $\{x, y\}$ appears in r_i. We require these to be such that (1) the *connection group* $C = \langle r_0, r_1, \ldots, r_n \rangle$ acts transitively on Ω, and (2) for all $0 \leq i < j - 1 \leq n - 1$, we have that $(r_i r_j)^2$ is trivial. One can easily verify that every map on a surface is a 3-maniplex with Ω being its set of (triangular) flags. Furthermore, every 3-maniplex can be realised as a map on a surface. When we desire to avoid degeneracies, such as semi-edges or maps on a surface with boundary, we also require that (3) each r_i and $r_i r_j$ are fixed-point-free, whenever $i \neq j$.

The *type* of a maniplex is the sequence $\{p_1, p_2, \ldots, p_n\}$, where each p_i is the order of $r_{i-1} r_i$ in C. The cube, then, is of type $\{4, 3\}$, the simplex is of type $\{3, 3, \ldots, 3\}$, and the 600-cell is of type $\{3, 3, 5\}$ (see [4, Chapter VII]).

Let C_i be the subgroup of C generated by all of the r_j's except r_i. Then an orbit of flags under C_i is called an *i-face*. A 0-face is a *vertex,* a 1-face is an *edge*, a 2-face is a *face*, an n-face is a *facet*. A facet of a facet is a *subfacet*; this is an orbit under $\langle r_0, r_1, \ldots, r_{n-2} \rangle$. The restriction to a subfacet of the permutation r_n acts as an isomorphism from that subfacet to some subfacet. We often view the $(n + 1)$-maniplex as being assembled from a collection of n-maniplexes, and think of r_n as 'glueing' them together along their subfacets.

We wish to assign colors, red and white, to flags so that for any given two i-adjacent flags, either one is colored red (and not white) and one is colored white (and not red), or both flags are colored both red and white. Choose a *root* flag (sometimes called also *base flag*) and call it I. Let $\mathcal{R}_0 = \{I\}$. Recursively let $\mathcal{W}_i$ be the set of all flags adjacent to flags in $\mathcal{R}_i$, and let $\mathcal{R}_{i+1}$ be the set of all flags adjacent to flags in $\mathcal{W}_i$. Finally, let $\mathcal{R}$ be the union of all $\mathcal{R}_i$'s and similarly let $\mathcal{W}$ be the union of all $\mathcal{W}_i$'s. We often say this differently: let C^+ be the subgroup of C generated by all products of the form $r_i r_j$. Then $\mathcal{R}$ is the orbit of I under C^+ and $\mathcal{W}$ is the orbit of $r_0 I$ under C^+. Consider these as assignments of the colors red and white, respectively to the flags. There are two possibilities for the result:

1. it could happen that $\mathcal{R}$ and $\mathcal{W}$ are disjoint; in this case we say that $\mathcal{M}$ is *orientable*;

2. otherwise it must happen that $\mathcal{R} = \mathcal{W} = \Omega$, and in this case we say that $\mathcal{M}$ is *non-orientable*.

See [11] for more information about bi-colorings of flags.

2.2 Symmetry

We define a *symmetry* of a maniplex $\mathcal{M}$ as a permutation of the flags which preserves the connections. We write symmetries on the right, so that the image of the flag f under the symmetry α is $f\alpha$. We denote the group of symmetries of $\mathcal{M}$ by $\mathrm{Aut}(\mathcal{M})$, and the notation gives the nice statement that for all $i \in \{0, 1, 2, \ldots, n\}$ and all $\alpha \in \mathrm{Aut}(\mathcal{M})$, we have that

$$(r_i f)\alpha = r_i(f\alpha).$$

There are two levels of symmetry that are particularly interesting in maps and maniplexes. First, we say that $\mathcal{M}$ is *rotary* provided that $\mathrm{Aut}(\mathcal{M})$ acts transitively on $\mathcal{R}$, the set of red flags. Also, $\mathcal{M}$ is *reflexible* provided that $\mathrm{Aut}(\mathcal{M})$ acts transitively on Ω. It follows trivially, then, that if $\mathcal{M}$ is rotary and non-orientable, then it is reflexible. If $\mathcal{M}$ is rotary but not reflexible, we say it is *chiral*. If $\mathcal{M}$ is orientable, it is often useful to consider $\mathrm{Aut}^+(\mathcal{M})$; this is the group of all symmetries which send $\mathcal{R}$ (the set of red flags) to itself (and so send $\mathcal{W}$ to itself). For orientable $\mathcal{M}$, we refer to symmetries in $\mathrm{Aut}^+(\mathcal{M})$ as 'preserving orientation' and those not in $\mathrm{Aut}^+(\mathcal{M})$ as 'reversing orientation'.

The group $\mathrm{Aut}^+(\mathcal{M})$ acts semi-regularly on Ω. That is, for any two flags, there is at most one symmetry sending one to the other. Thus, for any flag f and any symmetries β, γ, if $f\beta = f\gamma$, then $\beta = \gamma$.

2.3 One Facet

The purpose of this note is to discuss when a rotary n-maniplex might have just one facet for $n \leq 4$.

In the case $n = 1$, there is only one isomorphism class of maniplex and it has two facets.

For $n = 2$, a maniplex is a polygon, a facet is an edge and a polygon with one edge is the degenerate maniplex that consists of a single vertex joined to itself by a loop.

For $n = 3$, a maniplex is a map. In [18] it was noted that a rotary one-face map with k edges must be one of the two reflexible maps called M_k, δ_k in that paper.

The rest of this paper is devoted to classifying rotary 4-maniplexes with one facet. Whereas some reflexible one-facet n-maniplexes can be constructed with techniques similar to the ones presented here, a formal treatment of all rotary n-maniplexes for $n > 4$ appears to present difficulties not encountered in lower dimensions. It is not known yet whether there are any chiral n-maniplexes for $n > 4$ which have just one facet.

3 Constructions

In all of this section, suppose that $\mathcal{F} = (\Omega, [r_0, r_1, r_2])$ is a rotary 3-maniplex having root flag $I \in \mathcal{R}$. Let ρ be the symmetry of $\mathcal{F}$ sending I to $r_0 r_1 I$. Let G stand for $\mathrm{Aut}(\mathcal{F})$, G^+ stand for $\mathrm{Aut}^+(\mathcal{F})$, and let e be the identity in those groups. If $\mathcal{F}$ is reflexible, let α be the symmetry of $\mathcal{F}$ such that $I\alpha = r_0 I$. A symmetry $\tau \in G$ is *blue* provided that both τ and $\tau\rho$ are of order two.

For a blue τ, we define two kinds of elligibility for the pair $(\mathcal{F}, \tau)$ based on the flag $f = r_0 I\tau$:

1. We will say that the pair $(\mathcal{F}, \tau)$ is *white-elligible* provided that f is not red. This can happen only if (a) $\mathcal{F}$ is chiral or (b) $\mathcal{F}$ is orientable and reflexible, and τ preserves orientation.
2. We will say that the pair $(\mathcal{F}, \tau)$ is *red-elligible* provided first that f *is* red. This can happen only if (c) $\mathcal{F}$ is non-orientable (and hence reflexible) or (d) $\mathcal{F}$ is orientable and reflexible, and τ reverses orientation. In both case (c) and case (d), we further require that $\tau\alpha$ is of order 2.

Note that the blue condition shows that

$$r_0 r_1 f = r_0 r_1 r_0 I\tau = r_0 I \rho^{-1}\tau = r_0 I\tau\rho = f\rho.$$

Theorem 3.1 (One-Facet Maniplexes). *If the pair $(\mathcal{F}, \tau)$ is white-elligible, let $r_3 = \{I, f\}G^+$. Then $\mathcal{M} = (\Omega, [r_0, r_1, r_2, r_3])$ is an orientable, rotary maniplex, invariant under G^+. It is reflexible if $\mathcal{F}$ is reflexible and $\tau\alpha$ is of order 2.*

If the pair $(\mathcal{F}, \tau)$ is red-elligible, let $r_3 = \{I, f\}G$. Then $\mathcal{M} = (\Omega, [r_0, r_1, r_2, r_3])$ is a non-orientable, reflexible maniplex, invariant under G.

Proof. First suppose that $(\mathcal{F}, \tau)$ is white-elligible. Note that, for any pair $\{g, h\}$ in r_3, we may assume with no loss of generality that for some symmetry β in G^+, $g = I\beta$ is the red flag and $h = f\beta$ is the white.

As G^+ is transitive on each color, the union of pairs in r_3 is all of Ω. And as for each red flag, there is only one element of G^+ sending I there, r_3 is a partition of Ω. Now, r_3 contains the pair $\{I, f\}\tau = \{I\tau, f\tau\} = \{r_0 f, r_0 I\}$. Thus, for every pair $\{g, h\}$ in r_3, the pair $\{r_0 g, r_0 h\}$ is also in r_3, showing that $r_0 r_3 = r_3 r_0$. Further,

$$r_1 g = r_1 I\beta = r_0 r_0 r_1 I\beta = r_0 I\rho\beta = f\tau\rho\beta$$

and

$$r_1 h = r_1 f\beta = r_0 r_0 r_1 f\beta = r_0 f\rho\beta = r_0 r_0 I\tau\rho\beta = I\tau\rho\beta,$$

and so the pair $\{r_1 g, r_1 h\} = \{I, f\}\tau\rho\beta$ is in r_3. Thus r_1 and r_3 commute, and so $\mathcal{M}$ is a maniplex.

Every symmetry in G^+ is a symmetry of $\mathcal{M}$, so $\mathrm{Aut}(\mathcal{M})$ is transitive on the red flags of $\mathcal{F}$. Since $\mathcal{F}$ is orientable, and every pair in r_3 is of two colors, $\mathcal{M}$ is orientable. The red flags of $\mathcal{M}$ are exactly the red flags of $\mathcal{F}$, and so $\mathcal{M}$ is rotary.

If $\mathcal{F}$ is reflexible, assume that $\tau\alpha$ is of order 2, so that τ and α commute. Then

$$f\alpha = r_0 I\tau\alpha = r_0 I\alpha\tau = I\tau = r_0 f.$$

Consider the typical element $\{g, h\}$ of r_3, with $g = I\beta$ and $h = f\beta$. Since G^+ is normal in G, $\alpha\beta\alpha = \beta'$, for some β' in G^+. Then

$$g\alpha = I\beta\alpha = I\alpha\beta' = r_0 I\beta',$$

and, similarly,

$$h\alpha = f\beta\alpha = f\alpha\beta' = r_0 f\beta'.$$

Now, $\{I\beta', f\beta'\}$ is in r_3 and so $\{g\alpha, h\alpha\}$ is, as well. Thus α acts on $\mathcal{M}$ as a symmetry and so $\mathcal{M}$ is reflexible.

On the other hand, suppose that $(\mathcal{F}, \tau)$ is red-elligible. Then $r_3 = \{I, f\}G$, and the symmetries τ and α commute. Suppose that $\{g, h\}$ and $\{g, h'\}$ are distinct pairs in r_3. Then we can assume that for some symmetries β and γ, we have

$$g = I\beta = f\gamma, h = f\beta, h' = I\gamma.$$

Then

$$I\beta = f\gamma = r_0 I\tau\gamma = I\alpha\tau\gamma,$$

and so $\beta = \alpha\tau\gamma$. Thus, $\gamma = \tau\alpha\beta$. Then

$$h' = I\gamma = I\tau\alpha\beta = r_0 f\alpha\beta = f\beta = h.$$

Thus, r_3 is a partition of Ω.

As in the first case, r_0 and r_1 commute with r_3, and so $\mathcal{M}$ is a maniplex. Because r_3 connects flags which have, in $\mathcal{F}$, the same color, all flags of $\mathcal{M}$ have both colors, and so $\mathcal{M}$ is non-orientable. As each r_i is a partition invariant under G, which acts transitively on Ω, all of G acts on $\mathcal{M}$, and so $\mathcal{M}$ is reflexible. □

Theorem 3.2. *If $\mathcal{M}$ is any rotary 4-maniplex, let τ in $\mathrm{Aut}^+(\mathcal{M})$ be the symmetry such that $I\tau = r_3 r_0 I$. If $\mathcal{M}$ has exactly one facet $\mathcal{F}$, then the pair $(\mathcal{F}, \tau)$ is white- or red-elligible, and $\mathcal{M}$ is isomorphic to one of the maniplexes constructed from $\mathcal{F}$ and τ in Theorem 3.1.*

Proof. Suppose that $\mathcal{M} = (\Omega, [r_0, r_1, r_2, r_3])$ is a rotary 4-maniplex containing exactly one facet, $\mathcal{F} = (\Omega, [r_0, r_1, r_2])$, and that $I \in \mathcal{R}$ is the root flag of each. Let f be $r_3 I = r_0 I\tau$.

Then $I\tau = r_3 r_0 I$ and so I is fixed by τ^2, making τ of order 2.Further, $I\rho\tau = r_0 r_1 I\tau = r_0 r_1 r_0 r_3 I$. As r_3 commutes with both r_0 and r_1, the flag I is fixed by $(\rho\tau)^2$. Thus, $\tau\rho$ is also an involution and so τ is blue.

In the case in which $\mathcal{M}$ is orientable, $\mathrm{Aut}^+(\mathcal{M})$ acts regularly on the red flags, and on the white flags, and each of the r_i's is an orbit of the red-white pair $\{I, r_i I\}$. Thus, such an $\mathcal{M}$ is constructed in the first half of Theorem 3.1.

In the case in which $\mathcal{M}$ is non-orientable, and thus reflexible, there is $\alpha \in \mathrm{Aut}(\mathcal{M})$ such that $I\alpha = r_0 I$. Then

$$I\tau\alpha = r_3 r_0 I\alpha = r_3 I = f = r_0 I\tau = I\alpha\tau.$$

Thus, $\tau\alpha = \alpha\tau$ and so $(\mathcal{F}, \tau)$ is red-elligible. Further, $\mathrm{Aut}(\mathcal{M})$ acts regularly on the flags, and each r_i is an orbit of $\{I, r_i I\}$ under $\mathrm{Aut}(\mathcal{M})$. Thus, such an $\mathcal{M}$ is constructed in the second half of Theorem 3.1. □

4 Examples and results

We present here some examples of and results about one-facet maniplexes to display some of the variety possible:

4.1 Opposites

If $\mathcal{M} = (\Omega, [r_0, r_1, r_2, \dots, r_n])$ is an $n+1$-maniplex, let $s_2 = r_0 r_2$. Then $(\Omega, [r_0, r_1, s_2, \dots, r_n])$ is also a maniplex, called *opp*$(\mathcal{M})$. If $\mathcal{M}$ is reflexible, so is *opp*$(\mathcal{M})$, while if $\mathcal{M}$ is chiral, then *opp*$(\mathcal{M})$ is not rotary. See [16] for more on this and other operators on maniplexes.

From this it should be clear that:

Lemma 4.1. *If $\mathcal{M}$ is a reflexible one-facet maniplex whose facet is $\mathcal{F}$, then opp$(\mathcal{M})$ is a reflexible one-facet maniplex whose facet is opp$(\mathcal{F})$.*

Suppose that $\mathcal{F}$ is a map of type $\{p, q\}$ which has Petrie paths of length r, and suppose that $\mathcal{M}$ is a reflexible one-facet 4-maniplex with facet $\mathcal{F}$ and of type $\{p, q, t\}$. Then *opp*$(\mathcal{M})$ is a reflexible one-facet 4-maniplex with facet *opp*$(\mathcal{F})$ which has type $\{p, r, \frac{2t}{(2,t)}\}$.

4.2 Twists

If $\mathcal{M} = (\Omega, [r_0, r_1, r_2, r_3])$ is an orientable 4-maniplex, let $T_j(\mathcal{M})$ be $(\Omega, [s_0, s_1, s_2, s_3])$, where $s_0 = r_0, s_1 = r_1, s_2 = r_2$ and

$$s_3 f = \begin{cases} (r_0 r_1)^j r_3 f & \text{if } f \in \mathcal{R} \\ (r_1 r_0)^j r_3 f & \text{if } f \in \mathcal{W} \end{cases}$$

for all $f \in \Omega$. This construction first appeared in [7] and is described in more detail in [8]. In each of those papers, it is shown that each orientation-preserving symmetry of $\mathcal{M}$ is a symmetry of $\mathcal{T}_j(\mathcal{M})$, and so if $\mathcal{M}$ is rotary then $\mathcal{T}_j(\mathcal{M})$ is rotary, though if $\mathcal{M}$ is reflexible then $\mathcal{T}_j(\mathcal{M})$ might be chiral or reflexible.

An easy consequence of these results is:

Lemma 4.2. *If $\mathcal{M}$ is an orientable rotary one-facet maniplex with facet $\mathcal{F}$, then each $T_j(\mathcal{M})$ is an orientable rotary one-facet maniplex with facet $\mathcal{F}$.*

4.3 Examples: the Cube

Under this heading, we present four examples of rotary orientable one-facet maniplexes for each of which its one facet is isomorphic to the cube.

1. **Central Inversion**

   In this example, the flag f is diametrically opposite I. More precisely, if I belongs to face F, vertex v and edge e then f is the flag belonging to face F' farthest from F, the vertex v' farthest from v and the edge e' farthest from e. It follows that every flag is 3-adjacent to its antipodal flag. This gives a reflexible maniplex of type $\{4, 3, 2\}$. It has 3 faces, 4 vertices and 6 edges.

2. **Toroidal Identification**

   Imagine the cube as one cell of the tessellation of 3-space into cubes. Identify each flag with the flag translated by $(\pm 1, 0, 0), (0, \pm 1, 0)$ or $(0, 0, \pm 1)$. The resulting maniplex is the quotient space of the tessellation under the group generated by those vectors. The maniplex is reflexible and is of type $\{4, 3, 4\}$. It has 3 faces, one vertex and 3 edges.

The central inversion and toroidal identifications generalize to construct one-facet maniplex of all dimensions, of types $\{4, 3, 3, 3, \ldots, 3, 3, 2\}$ and $\{4, 3, 3, 3, \ldots, 3, 3, 4\}$, respectively.

3. **The Krughoff cube (and its reverse)**

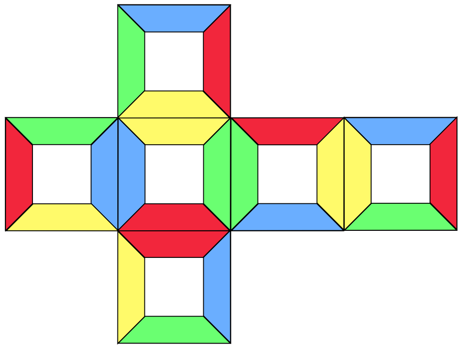

Figure 1: Krughoff's cube

Consider the cube shown in Figure 1 [12]. The edges have been colored in such a way that each of the six possible circular orderings of the four colors appears exactly once clockwise about some face. Notice that this coloring is chiral; i.e., every rotation of the cube permutes the colors, while any reflection sends edges of any one fixed color to edges of different colors.

Also note that opposite faces have the reverse circular order of colors. If we now form a 4-maniplex by glueing each face to the opposite face so that colors match, we get a chiral 4-maniplex. Each color, then, becomes a single edge. As each color appears three times, the maniplex is of type $\{4, 3, 3\}$. It has 3 faces, 2 vertices and 4 edges The mirror image of the coloring produces another maniplex, the mirror image of the first.

4. **Twists of each other** If we let $\mathcal{C}$ be the 4-maniplex made from the cube by using the central inversion, then $T_1(\mathcal{C})$ is one of the Krughoff maniplexes, $T_2(\mathcal{C})$ is the one using the toroidal identification, and $T_3(\mathcal{C})$ is the other Krughoff maniplex. In each case the corresponding involution τ is the rotation by $180°$ about the face containing $r_2 I$, followed by ρ^i for some i.

4.4 Octahedron examples

The octahedron has an edge-coloring dual to that of the cube shown above. Using that coloring, we build two mirror-image one-facet chiral maniplexes, each of them a twisted form of the maniplex made by using the central inversion.

4.5 The $f = IR^{\frac{p}{2}}$ construction for all even p

Suppose that $\mathcal{F}$ is a reflexible map whose faces have even length $2k$. Then making r_3 equal to the set of all pairs of the form $\{g, (r_0 r_1)^k g\}$ forms a 4-maniplex having just the one facet, isomorphic to $\mathcal{F}$, and is reflexible and non-orientable. Each face is 3-connected to itself after a $180°$ turn. In the case of the cube, the result is of type $\{4,3,4\}$, and has 6 faces, 3 edges and 2 vertices.

4.6 Non-orientable facet

When Lemma 4.1 is applied to a maniplex having an orientable facet, the result might have a non-orientble facet. Let the map $\mathcal{C}$ be the cube, and $\mathcal{M}$ be the 4-maniplex made from it by using the toroidal identification. Then $\mathcal{F} = opp(\mathcal{C})$ is a non-orientable map of type $\{4,6\}$, and $opp(\mathcal{M})$ is of type $\{4,6,4\}$.

4.7 Chiral facet

Assume that a chiral 3-maniplex $(\Omega, [r_0, r_1, r_2])$ is the facet of a one-facet 4-maniplex $\mathcal{M} = (\Omega, [r_0, r_1, r_2, r_3])$. By the blue condition there exists a symmetry τ such that $\tau\rho\tau = \rho^{-1}$. This is not a common phenomenon in 3-maniplexes with few flags; however, it certainly occurs if $\mathrm{Aut}(\mathcal{M})$ is a symmetric group.

It was proven in [14, Section 3] that for $n \geq 6$, the symmetric group S_n is the symmetry group of some chiral 3-maniplex. In particular, when $n = 6$ the permutations

$$(1,2,3,4,5,6),$$
$$(2,6,3)(4,5)$$

represent the symmetry ρ and a symmetry mapping I to $r_1 r_2 I$ of a chiral 3-maniplex whose symmetry group is isomorphic to the symmetric group on six points.

By choosing as τ the symmetry acting as $(2,6)(3,5)$ we can construct a one-facet chiral 4-maniplex whose facet is itself chiral.

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Edge-transitive embeddings of complete graphs

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Abstract

Building on earlier work of Biggs, James, Wilson and the author and on the Graver–Watkins description of the 14 classes of edge-transitive maps, we complete the classification of the edge-transitive embeddings of complete graphs by classifying their non-regular embeddings in non-orientable surfaces and their embeddings with non-empty boundary.

DEDICATED TO THE MEMORY OF BRANKO GRÜNBAUM

Keywords: Edge-transitive map, complete graph, Biggs map, James map.

Math. Subj. Class.: 05C10, 20B25.

1 Introduction

The purpose of this paper is to complete the classification of edge-transitive embeddings of complete graphs (see Theorem 1.8) by classifying the non-regular embeddings in non-orientable surfaces in Theorem 1.7 and the embeddings with non-empty boundary in Theorem 6.1. First we review the earlier results in this area.

In 1987 Grünbaum and Shephard [6] published an influential paper on edge-transitive planar graphs, extending earlier work on finite edge-transitive graphs to the infinite case. An important consequence of this was the monograph [4] by Graver and Watkins; this included a partition of all edge-transitive maps $\mathcal{M}$ into 14 classes, which can be interpreted as corresponding to the 14 possibilities for $\mathcal{M}/\mathrm{Aut}\,\mathcal{M}$. These classes were studied further by Širáň, Tucker and Watkins [20], by Orbanič, Pellicer, Pisanski and Tucker [17] and by the author [12], investigating the possible surfaces and automorphism groups for maps in

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the various classes, mainly in the finite case. The aim of this note is to approach edge-transitive maps from the point of view of their embedded graphs, specifically when these are complete graphs.

A map on a surface is *regular* (sometimes called *fully regular*) or *edge-transitive* if its automorphism group acts transitively on vertex-edge-face flags or edges respectively. A map on an orientable surface is *orientably regular* if its orientation-preserving automorphism group acts transitively on arcs, and *chiral* if it has no orientation-reversing automorphisms, so that it is not isomorphic to its mirror image. Thus an orientably regular map is either regular or chiral, but not both. The regular and orientably regular embeddings of complete graphs are all known, as are their edge-transitive orientable embeddings. This paper will deal with the remaining cases, the non-regular embeddings in non-orientable surfaces, together with those in surfaces with non-empty boundary. (Unless otherwise stated, as in Section 6, all maps considered from now on are assumed to be without boundary.)

In 1971 Biggs [1], building on earlier work of Heffter [7], proved:

Theorem 1.1. *The complete graph K_n has an orientably regular embedding if and only if n is a prime power.*

The maps Biggs constructed to prove that this condition is sufficient are Cayley maps (see [5, §1.2.4]) $\mathcal{M}_n(c)$ for the additive groups of finite fields $\mathbb{F}_n$; in each case the generating set is the multiplicative group $\mathbb{F}_n^* = \mathbb{F}_n \setminus \{0\}$, taken in the cyclic order $1, c, c^2, \dots, c^{n-2}$ where c is a primitive element of $\mathbb{F}_n$, that is, a generator for the cyclic group $\mathbb{F}_n^*$. They are chiral for all $n \geq 5$.

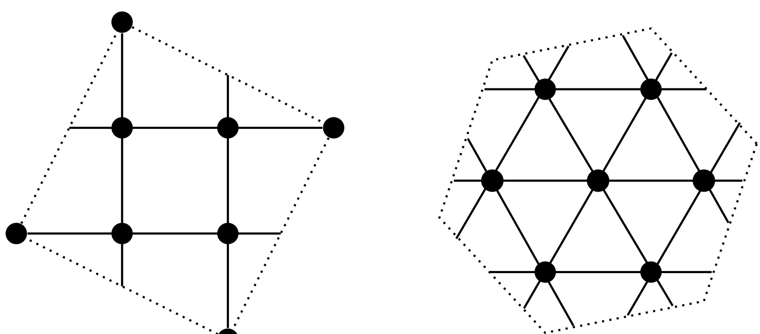

Figure 1: The Biggs maps $\mathcal{M}_5(2)$ and $\mathcal{M}_7(3)$

Example 1.2. For $n = 4$ we have $\mathbb{F}_4 = \{0, 1, c, c^2 = c^{-1} = c+1\}$, and the corresponding map $\mathcal{M}_4(c)$ is the tetrahedral map on the sphere. The Biggs maps $\mathcal{M}_5(2)$ and $\mathcal{M}_7(3)$ are shown in Figure 1, with opposite sides of the outer square and hexagon identified to form the torus maps $\{4,4\}_{1,2}$ and $\{3,6\}_{1,2}$ in the notation of [3, §8.3 and §8.4]. Their mirror images are the maps $\mathcal{M}_5(3) \cong \{4,4\}_{2,1}$ and $\mathcal{M}_7(5) \cong \{3,6\}_{2,1}$.

In 1985 James and the author [11] proved that the Biggs maps $\mathcal{M}_n(c)$ are the only orientably regular embeddings of complete graphs:

Theorem 1.3. *A map $\mathcal{M}$ is an orientably regular embedding of K_n if and only if $\mathcal{M} \cong \mathcal{M}_n(c)$ for some primitive element c of $\mathbb{F}_n$. Moreover, $\mathcal{M}_n(c)$ and $\mathcal{M}_n(c')$ are isomorphic (as oriented maps) if and only if c and c' are equivalent under a field automorphism of $\mathbb{F}_n$.*

It follows that $n = p^e$ for some prime p and there are, up to isomorphism, $\phi(n-1)/e$ orientably regular embeddings of K_n, one for each orbit of the Galois group of $\mathbb{F}_n$ (isomorphic to C_e, generated by the Frobenius automorphism $t \mapsto t^p$) on the $\phi(n-1)$ primitive elements of the field. The orientation-preserving automorphism group of $\mathcal{M}_n(c)$ is the affine group $AGL_1(\mathbb{F}_n)$; this map is regular if and only if $n = 2, 3$ or 4, in which case the full automorphism group is isomorphic to $V_4 \cong C_2 \times C_2$, $D_6 \cong S_3 \times C_2$ or S_4.

If $n \geq 3$ the Petrie dual $P(\mathcal{M}_n(c))$ of $\mathcal{M}_n(c)$ is a non-orientable edge-transitive embedding of K_n, with the same automorphism group as $\mathcal{M}_n(c)$.

In [9] James classified the non-orientable regular embeddings of complete graphs (see also [22] for an independent proof due to Wilson):

Theorem 1.4. *The non-orientable regular embeddings of complete graphs K_n are the maps $\{6,2\}_3$, $\{4,3\}_3$, $\{3,5\}_5$ and $\{5,5\}_3$ of characteristic $1, 1, 1$ and -3 for $n = 3, 4, 6$ and 6.*

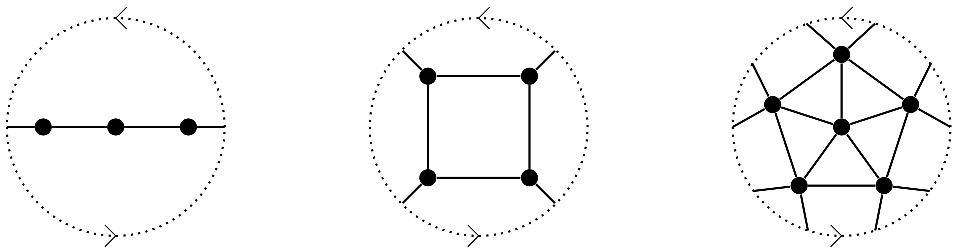

Figure 2: Regular embeddings of K_n on the projective plane, $n = 3, 4, 6$

Here the notation $\{p, q\}_r$, from [3, §8.6 and Table 8], denotes the largest map of type $\{p, q\}$ with Petrie length r. The first three of these maps, on the real projective plane, are the antipodal quotients of a hexagon, a cube and an icosahedron on the sphere (see Figure 2, where antipodal boundary points of the discs are identified). The first two are the Petrie duals of the unique regular embeddings $\mathcal{M}_n(c)$ of K_3 and K_4 on the sphere, with the same automorphism groups D_6 and S_4, while the last two are a Petrie dual pair with automorphism group $PSL_2(5) \cong A_5$.

In [10] James extended Theorem 1.3 to a classification of the orientable edge-transitive embeddings of K_n. If $3 < n = p^e \equiv 3 \bmod (4)$ where p is prime, and c is a primitive element of $\mathbb{F}_n$, let $\mathcal{M}_n(c, j)$ be the Cayley map for $\mathbb{F}_n$ with generating set $\mathbb{F}_n^*$, where now the cyclic ordering is

$$1, c^j, c^2, c^{j+2}, c^4, c^{j+4}, \ldots, c^{n-3}, c^{j+n-3}$$

for some odd element $j \in \mathbb{Z}_{n-1} \setminus \{1\}$. (Taking $j = 1$ gives the orientably regular Biggs map $\mathcal{M}_n(c)$.)

Theorem 1.5. *A map $\mathcal{M}$ is an orientable edge-transitive embedding of K_n, which is not orientably regular, if and only if $\mathcal{M} \cong \mathcal{M}_n(c, j)$ for some n, c and j as above. As oriented maps, $\mathcal{M}_n(c, j)$ and $\mathcal{M}_n(c', j')$ are isomorphic if and only if c and c' are equivalent under* $\mathrm{Gal}\,\mathbb{F}_n$ *and* $j' \equiv j$ *or* $2 - j \bmod (n-1)$.

These James maps $\mathcal{M}_n(c, j)$ have automorphism group $AHL_1(\mathbb{F}_n)$, the unique subgroup of index 2 in $AGL_1(\mathbb{F}_n)$; their Petrie duals are non-orientable edge-transitive embeddings of K_n. Each map $\mathcal{M}_n(c, j)$ is chiral, with mirror image $\mathcal{M}_n(c^{-1}, 2 - j) \cong \mathcal{M}_n(c^{-1}, j)$.

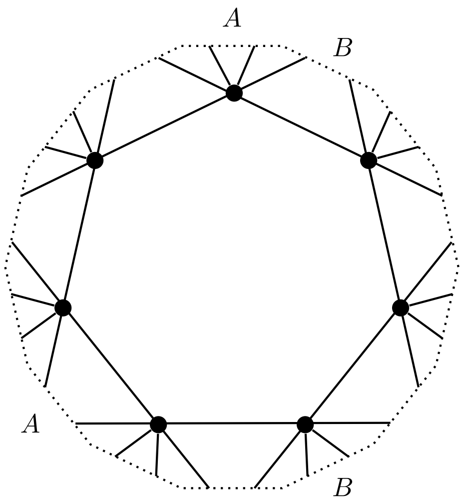

Figure 3: An edge-transitive embedding $\mathcal{M}_7(5, 5)$ of K_7

Example 1.6. The James map $\mathcal{M}_7(5, 5)$ is shown in Figure 3, taken from [10]. Two of the identifications of sides of the outer 14-gon are indicated by the letters A and B; the others can be found by C_7 rotational symmetry. The vertices are identified with the elements $0, 1, \ldots, 6$ of $\mathbb{F}_7$, in clockwise order. There are seven triangular faces and three heptagons. The mirror image $\mathcal{M}_7(3, 3)$ of this map is given by reflection in the horizontal axis. Each of these maps is a representation of the Fano plane $\mathbb{P}^2(\mathbb{F}_2)$, with the vertices as its points and the triples incident with its triangular faces forming its lines. (Similar representations of this and other finite projective planes are discussed by Singerman in [19].)

These two maps $\mathcal{M}$ can be drawn as follows on Klein's Riemann surface (or quartic curve) $\mathcal{K}$ of genus 3 with automorphism group $PSL_2(7)$ (see [15], for instance), so that $\mathrm{Aut}\,\mathcal{M}$ ($\cong C_7 \rtimes C_3$) is contained in $\mathrm{Aut}\,\mathcal{K}$. Each of the eight Sylow 7-subgroups $S \cong C_7$ of $\mathrm{Aut}\,\mathcal{K}$ preserves three of the 24 faces of the $\mathrm{Aut}\,\mathcal{K}$-invariant tessellation $\mathcal{T} = \{7, 3\}_8$ of $\mathcal{K}$. These three faces are incident with 21 of the 56 vertices of $\mathcal{T}$, and there are 21 more vertices adjacent in $\mathcal{T}$ to these. This leaves 14 vertices of $\mathcal{T}$, at graph-theoretic distance 2 from the S-invariant faces, forming two orbits under S. The vertices in each orbit, joined pairwise by geodesics, determine a chiral pair of maps $\mathcal{M} \cong \mathcal{M}_7(5, 5)$ and $\mathcal{M}_7(3, 3)$, each

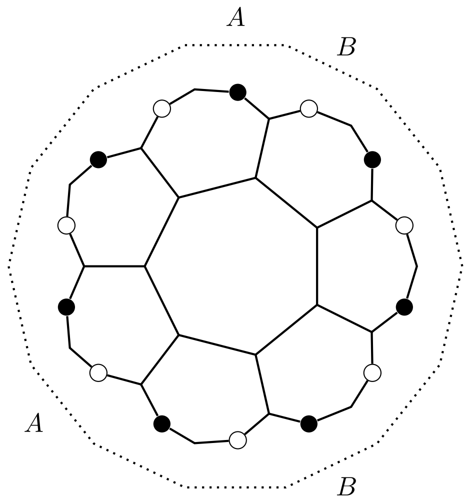

Figure 4: Vertices of $\mathcal{M}_7(5,5)$ and $\mathcal{M}_7(3,3)$ on Klein's surface

invariant under the normaliser $\mathrm{Aut}\,\mathcal{M} = S \rtimes C_3$ of S in $\mathrm{Aut}\,\mathcal{K}$. Figure 4 shows part of $\mathcal{T}$, with the central face invariant under the group S generated by rotation through $2\pi/7$, and the orbits of S on vertices of $\mathcal{T}$ yielding these two maps indicated in black and white. The vertices of each map are the centres of the triangular faces of the other. The three heptagonal faces of each map each contain one of the S-invariant faces of $\mathcal{T}$.

In order to complete the classification of edge-transitive embeddings of complete graphs, it remains for us to deal with the non-regular non-orientable cases. The main result of this paper, proved in Section 4 after some preparatory work in Sections 2 and 3, is as follows:

Theorem 1.7. *A map $\mathcal{M}$ is a non-orientable non-regular edge-transitive embedding of a complete graph K_n if and only if $\mathcal{M}$ is isomorphic to the Petrie dual of a Biggs map $\mathcal{M}_n(c)$ for $n \geq 5$ or of a James map $\mathcal{M}_n(c,j)$ for $n \geq 7$.*

As an immediate corollary we have the following classification of the edge-transitive embeddings of complete graphs, showing that they are simply the maps listed above:

Theorem 1.8. *A map $\mathcal{M}$ is an edge-transitive embedding of a complete graph K_n if and only if $\mathcal{M}$ is isomorphic to a Biggs map $\mathcal{M}_n(c)$ or its Petrie dual, a James map $\mathcal{M}_n(c,j)$ or its Petrie dual, or one of the Petrie dual pair $\{3,5\}_5$ and $\{5,5\}_3$ of non-orientable regular embeddings of K_6.*

A detailed description of these maps is given in Section 5. Edge-transitive embeddings of K_n in surfaces with boundary are classified in Section 6.

Finally, we note that by results of Korzhik and Voss [14], *vertex*-transitive embeddings of K_n are much more abundant and harder to classify than its edge-transitive embeddings: for example, any cyclic ordering of the non-identity elements of a group of order n will yield such a Cayley map for that group. Nevertheless, Li gives a good description of the possibilities in [16].

2 Algebraic theory of maps

Here we sketch the algebraic theory of maps developed in more detail elsewhere: see [13], for example, and [5] for further background in topological graph theory.

Each map $\mathcal{M}$ (possibly non-orientable or with non-empty boundary) determines a permutation representation of the group

$$\Gamma = \langle R_0, R_1, R_2 \mid R_i^2 = (R_0 R_2)^2 = 1 \rangle \cong V_4 * C_2$$

on the set Φ of flags $\phi = (v, e, f)$ of $\mathcal{M}$, where v, e and f are a mutually incident vertex, edge and face. For each $\phi \in \Phi$ and each $i = 0, 1, 2$, there is at most one flag $\phi' \neq \phi$ with the same j-dimensional components as ϕ for each $j \neq i$ (possibly none if ϕ is a boundary flag). Define r_i to be the permutation of Φ transposing each ϕ with ϕ' if the latter exists, and fixing ϕ otherwise, in which case the incident edge is called a *free edge* or *semi-edge*. (See Figures 5 and 6 for these two cases. In Figure 6 the broken line represents part of the boundary of the map.) Since $r_i^2 = (r_0 r_2)^2 = 1$ there is a permutation representation

$$\theta : \Gamma \to G := \langle r_0, r_1, r_2 \rangle \leq \mathrm{Sym}\, \Phi$$

of Γ on Φ, given by $R_i \mapsto r_i$.

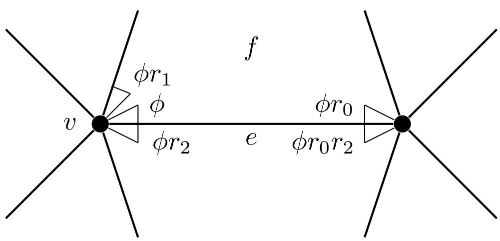

Figure 5: Generators r_i of G acting on a flag $\phi = (v, e, f)$.

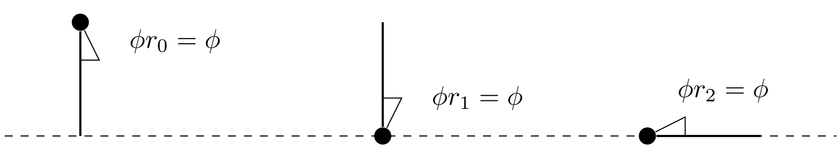

Figure 6: Flags fixed by r_0, r_1 and r_2.

Conversely, any permutation representation of Γ on a set Φ determines a map $\mathcal{M}$ in which the vertices, edges and faces are identified with the orbits on Φ of the subgroups $\langle R_1, R_2 \rangle \cong D_\infty$, $\langle R_0, R_2 \rangle \cong V_4$ and $\langle R_0, R_1 \rangle \cong D_\infty$, incident when they have non-empty intersection.

The map $\mathcal{M}$ is connected if and only if Γ acts transitively on Φ, as we will always assume. In this case the stabilisers in Γ of flags $\phi \in \Phi$ form a conjugacy class of subgroups $M \leq \Gamma$, called *map subgroups*.

The map $\mathcal{M}$ is finite (has finitely many flags) if and only if M has finite index in Γ, and it has non-empty boundary if and only if M contains a conjugate of some R_i, or equivalently some r_i has a fixed point in Φ. In particular, $\mathcal{M}$ is orientable and with empty boundary if and only if M is contained in the even subgroup Γ^+ of index 2 in Γ, consisting of the words of even length in the generators R_i.

The *automorphism group* $A = \mathrm{Aut}\,\mathcal{M}$ of $\mathcal{M}$ is the centraliser of G in $\mathrm{Sym}\,\Phi$. Then $A \cong N/M$ where $N := N_\Gamma(M)$ is the normaliser of M in Γ. The map $\mathcal{M}$ is called *regular* if A is transitive on Φ, or equivalently G is a regular permutation group, that is, M is normal in Γ; in this case $A \cong G \cong \Gamma/M$, and one can identify Φ with G, so that A and G are the left and right regular representations of G. The map $\mathcal{M}$ is *edge-transitive* if A acts transitively on its edges, or equivalently $\Gamma = NE$ where $E := \langle R_0, R_2\rangle \cong V_4$.

The (classical) dual $D(\mathcal{M})$ of $\mathcal{M}$ corresponds to the image of M under the automorphism δ of Γ fixing R_1 and transposing R_0 and R_2. The Petrie dual $P(\mathcal{M})$ embeds the same graph as $\mathcal{M}$, but the faces are transposed with Petrie polygons, closed zig-zag paths turning alternately first right and first left at the vertices of $\mathcal{M}$; this operation P corresponds to the automorphism π of Γ transposing R_0 with R_0R_2 and fixing R_1 and R_2. Both of these operations D and P preserve automorphism groups and (full) regularity, but D may change the embedded graph, and P may change the underlying surface; in particular, if $\mathcal{M}$ is orientable, then $P(\mathcal{M})$ is orientable if and only if the embedded graph is bipartite. The group $\Omega = \langle D, P\rangle$ of map operations generated by D and P, introduced by Wilson in [21], is isomorphic to S_3, permuting vertices, faces and Petrie polygons; it corresponds to the outer automorphism group $\mathrm{Out}\,\Gamma \cong \mathrm{Aut}\,E \cong S_3$ of Γ acting on maps by permuting conjugacy classes of map subgroups [13].

3 Edge-transitive maps

In 1997 Graver and Watkins [4] partitioned the edge-transitive maps $\mathcal{M}$ into 14 classes, distinguished by the isomorphism class of the quotient map $\mathcal{M}/\mathrm{Aut}\,\mathcal{M}$; in that year, Wilson [23] gave a similar classification. These classes T correspond to the 14 isomorphism classes of maps $\mathcal{N}(T)$ with one edge, shown in Figure 7, or equivalently to the 14 conjugacy classes of *parent groups*, subgroups $N = N(T)$ of Γ satisfying $\Gamma = NE$ (see [12, §4]).

The maps in Figure 7 are all on the closed disc, apart from $\mathcal{N}(2^P\mathrm{ex})$, $\mathcal{N}(5)$ and $\mathcal{N}(5^*)$ on the sphere, $\mathcal{N}(4^P)$ on the Möbius band and $\mathcal{N}(5^P)$ on the real projective plane. The 14 edge-transitive classes include class 1, consisting of the regular maps, and class $2^P\mathrm{ex}$, consisting of the chiral (non-regular) orientably regular maps. Each map $\mathcal{M}$ in class T, with automorphism group A, is a regular covering by A of the basic map $\mathcal{N}(T)$ for that class; it corresponds to a map subgroup $M \le \Gamma$ with $N_\Gamma(M) = N(T)$ and $N(T)/M \cong A$. The six rows of Figure 7 correspond to the orbits of Ω on the 14 classes: the duals of the maps in class 2 form class 2^*, while the Petrie duals of the latter form class 2^P; similar remarks apply to the classes $2\,\mathrm{ex}$, 4 and 5, whereas the classes 1 and 3 are invariant under Ω. Thus, being chiral and vertex- but not face-transitive, $\mathcal{M}_n(c, j)$ is in class 5^*, so $P(\mathcal{M}_n(c, j))$ is in class 5^P.

The Reidemeister-Schreier process, applied to the inclusions $N(T) \le \Gamma$, gives the

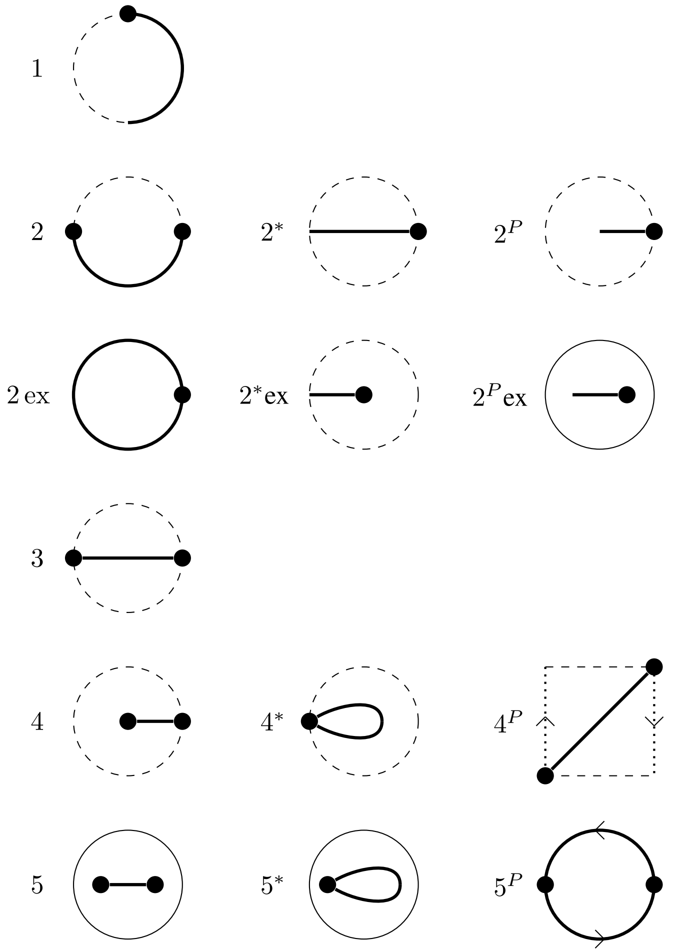

Figure 7: The basic maps $\mathcal{N}(T)$ for the 14 edge-transitive classes T

following presentations and free product decompositions:

$$N(1) = \Gamma = \langle R_0, R_1, R_2 \mid R_i^2 = (R_0R_2)^2 = 1\rangle \cong V_4 * C_2,$$
$$N(2) = \langle S_1 = R_1, S_2 = R_1^{R_0}, S_3 = R_2 \mid S_1^2 = S_2^2 = S_3^2 = 1\rangle \cong C_2 * C_2 * C_2,$$
$$N(2\,\mathrm{ex}) = \langle S_1 = R_2, S = R_0R_1 \mid S_1^2 = 1\rangle \cong C_2 * C_\infty,$$
$$N(3) = \langle S_0 = R_1,\ S_1 = R_1^{R_0},\ S_2 = R_1^{R_2},\ S_3 = R_1^{R_0R_2} \mid S_i^2 = 1\rangle \cong C_2 * C_2 * C_2 * C_2,$$
$$N(4) = \langle S_1 = R_1, S_2 = R_1^{R_2}, S = (R_1R_2)^{R_0} \mid S_1^2 = S_2^2 = 1\rangle \cong C_2 * C_2 * C_\infty,$$
$$N(5) = \langle S = R_1R_2, S' = S^{R_0} \mid -\rangle \cong C_\infty * C_\infty \cong F_2.$$

(Here F_2 denotes a free group of rank 2.) Applying elements of $\mathrm{Aut}\,E \cong \Omega$, permuting R_0, R_2 and R_0R_2, gives presentations for the other (isomorphic) parent groups in each orbit of Ω.

4 Proof of Theorem 1.7

It is easy to see that $P(\mathcal{M}_n(c))$ and $P(\mathcal{M}_n(c,j))$ have the stated properties.

Conversely, if $\mathcal{M}$ is a non-regular non-orientable edge-transitive embedding of K_n, then it is in one of the 10 edge-transitive classes $T \neq 1$, 2^Pex, 5 or 5^*, since the maps in class 1 are regular, while those in the other three classes are orientable. The cases $n \leq 3$ are easily dealt with, so assume that $n \geq 4$. If $T = 2^*$ex or 5^P then $\mathcal{M} = P(\mathcal{M}')$ for some edge-transitive embedding $\mathcal{M}'$ of K_n in class 2^Pex or 5^*; as such, $\mathcal{M}'$ is orientable, so by Theorems 1.3 and 1.5 it is isomorphic to a map $\mathcal{M}_n(c)$ or $\mathcal{M}_n(c,j)$ respectively as it is or is not orientably regular. This leaves only $T = 2$, 2^*, 2^P, 2 ex, 3, 4, 4^* and 4^P, so it is sufficient to eliminate these cases.

The edges of K_n may be identified with the distinct pairs of vertices, so edge-transitivity implies that the group $A = \mathrm{Aut}\,\mathcal{M}$ permutes these pairs transitively, that is, it acts 2-homogenously on the vertices. It permutes the vertices faithfully since $n \geq 4$, and transitively since K_n is not bipartite; the latter property rules out classes $T = 2, 3$ and 4, since the corresponding maps $\mathcal{N}(T)$ have two vertices, so only the classes $T = 2^*$, 2^P, 2 ex, 4^* and 4^P remain. If A has odd order then, as a quotient of the group $N(T)$ with a free product decomposition given above, it must be cyclic; however, a cyclic group cannot be 2-homogenous of degree $n \geq 4$, so A has even order. It thus has an element transposing two vertices, so it is 2-transitive on the vertices.

Each edge of $\mathcal{M}$ is therefore reversed by some automorphism (a reflection or half turn), so the single edge of $\mathcal{N}(T)$ must be free, ruling out the classes $T = 2$ ex, 4^* and 4^P. Only the classes 2^* and 2^P remain, with A sharply 2-transitive on the vertices (since $\mathcal{M}$ is not regular), and the stabiliser of an edge generated by a reflection or a half-turn respectively, transposing its two incident vertices. Zassenhaus [24] showed that any sharply 2-transitive finite group can be identified with $AGL_1(F)$ acting naturally on a near-field F. Since $\mathcal{M}$ is a map, the stabiliser A_0 of the vertex 0 must be a cyclic or dihedral group of order $n-1$, acting regularly on the set $F \setminus \{0\}$ of neighbours of 0. Now A acts on the vertices as a Frobenius group $F \rtimes A_0$, so A_0 is a Frobenius complement; these contain at most one involution (see [8, Satz V.8.18(a)] or [18, Theorem 18.1(iii)]), so they cannot be dihedral, and hence A_0 is cyclic. Since $T = 2^*$ or 2^P, $N(T)$ is generated by involutions, and hence so are its epimorphic images A and A_0, giving $n - 1 \leq 2$, a contradiction. $\square$

5 Summary of properties of the maps

The following basic properties of the maps classified in Theorem 1.8 are taken from [1, 9, 10, 11] or have been demonstrated earlier in this paper.

For each prime power $n = p^e$ there are $\phi(n-1)/e$ Biggs maps $\mathcal{M}_n(c)$, all orientable. The mirror image of $\mathcal{M}_n(c)$ is $\mathcal{M}_n(c^{-1})$. For each $n = 2, 3, 4$ the unique map $\mathcal{M}_n(c)$ is regular (in class 1), of type $\{2,1\}_2$, $\{3,2\}_6$ or $\{3,3\}_4$ and genus 0, with automorphism group V_4, D_6 or S_4. For $n \geq 5$ these maps are orientably regular (in class 2^Pex) with automorphism group $AGL_1(\mathbb{F}_n)$; they have type $\{m, n-1\}_{2p}$ where $m = (n-1)/2$ or $n-1$ as $n \equiv 3 \bmod (4)$ or not, and have genus $(n^2 - 7n + 4)/4$ or $(n-1)(n-4)/4$ respectively.

The maps $P(\mathcal{M}_n(c))$ for $n \geq 3$ are non-orientable, with the same automorphism group as $\mathcal{M}_n(c)$; they are in class 1 or 2^*ex as $n \leq 4$ or $n \geq 5$. They have type $\{2p, n-1\}_m$

where m is as above, and characteristic

$$\chi = n\left(1 - \frac{n-1}{2} + \frac{n-1}{2p}\right).$$

If $3 < n = p^e \equiv 3 \bmod (4)$, there are $(n-3)\phi(n-1)/4e$ James maps $\mathcal{M}_n(c,j)$, all orientable, with mirror image $\mathcal{M}_n(c^{-1}, 2-j)$. They are in class 5^*, with automorphism group $AHL_1(\mathbb{F}_n)$ consisting of the transformations $v \mapsto av + b$ in $AGL_1(\mathbb{F}_n)$ such that a is a non-zero square in $\mathbb{F}_n$. This group has two orbits on the faces: if $j, 2-j \not\equiv (n-1)/2 \bmod (n-1)$ they have cardinality $n(n-1,j)$ and $n(n-1,2-j)$ and the genus is $\frac{n}{4}\left((n-3) - 2(n-1,j) - 2(n-1,2-j)\right) + 1$, whereas if j or $2-j \equiv (n-1)/2 \bmod (n-1)$ they have cardinality n and $n(n-1)/2p$ and the genus is $(n-1)(n(p-1)-4p)/4p$. There is a single orbit of $n(n-1)/l$ Petrie polygons, each polygon having length $l = 2(n-1)/(n-1, 2(j-1))$.

The maps $P(\mathcal{M}_n(c,j))$ are non-orientable, with automorphism group $AHL_1(\mathbb{F}_n)$; they are in class 5^P, and have type $\{l, n-1\}$ and characteristic

$$\chi = n\left(1 - \frac{n-1}{2} + \frac{n-1}{l}\right).$$

For $n = 6$ the Petrie dual pair $\{3,5\}_5$ and $\{5,5\}_3$ are non-orientable, with $\chi = 1$ and -3; they are regular, with automorphism group $PSL_2(5) \cong A_5$.

These maps $\mathcal{M}$ are all vertex-transitive. Indeed, in all cases $\mathrm{Aut}\,\mathcal{M}$ has a subgroup acting regularly on the vertices, so that $\mathcal{M}$ is a Cayley map: when n is a prime power there is a unique such subgroup, namely the additive group of $\mathbb{F}_n$ acting by translations, whereas when $n = 6$ there is a conjugacy class of ten such subgroups, isomorphic to S_3. Apart from the James maps $\mathcal{M}_n(c,j)$, these maps are all face-transitive. They are all arc-transitive with the exception of the James maps and their Petrie duals.

6 Edge-transitive embeddings with boundary

Here we will classify the edge-transitive maps $\mathcal{M}$ which embed complete graphs K_n in surfaces with non-empty boundary $\partial\mathcal{M}$. (A general theory of maps with boundary was developed by Bryant and Singerman in [2].)

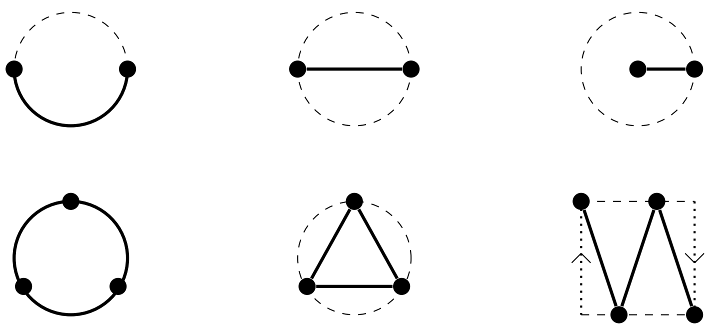

Figure 8: Edge-transitive embeddings of K_n with boundary, $n = 2, 3$

There are three maps each for $n = 2$ and 3, shown in Figure 8, and none for $n \geq 4$. The first five are on the closed disc, while the last is on the Möbius band. The first two embeddings of K_2 are in class 1, while the third is in class 2; their automorphism groups are isomorphic to C_2, V_4 and C_2. The three embeddings of K_3, all with automorphism group S_3, are in classes 1, 2^* and 2^P respectively. The last two form a Petrie dual pair.

Theorem 6.1. *The only edge-transitive maps which embed complete graphs K_n in surfaces with non-empty boundary are the six shown in Figure 8.*

Proof. Let $\mathcal{M}$ be such a map. If $n = 2$ then $\mathcal{M}$ has one edge, so it is one of the 14 basic maps $\mathcal{N}(T)$ with one edge shown in Figure 7. By inspection, $\mathcal{N}(T)$ embeds K_2 if and only if $T = 2, 3, 4$ or 5. Only the first three of these have non-empty boundary, giving the maps in the first row of Figure 8. We may therefore assume that $n \geq 3$. It follows that by edge-transitivity $\mathrm{Aut}\,\mathcal{M}$ acts transitively (in fact, 2-homogeneously) on the vertices.

Since $\partial\mathcal{M} \neq \emptyset$, some flag ϕ of $\mathcal{M}$ must be fixed by some r_i for $i = 0, 1, 2$. If $i = 0$ then $\mathcal{M}$ has a free edge (see Figure 6), contradicting the fact that it embeds a complete graph. If $i = 2$ then the corresponding edge, and hence every edge, is contained in $\partial\mathcal{M}$; thus each vertex has valency at most 2, so $n = 3$ and we have the first graph in the second row of Figure 8.

We may therefore assume that only r_1 fixes a flag. By vertex-transitivity, every vertex is in $\partial\mathcal{M}$, with two of its $2(n-1)$ incident flags fixed by r_1. Thus the proportion of flags fixed by r_1 is $1/(n-1)$. By edge-transitivity, each edge in incident with the same number of flags fixed by r_1, which must be $1, 2, 3$ or 4, so $1/(n-1) = 1/4$, $1/2$, $3/4$ or 1 and hence $n = 5, 3, 7/3$ or 2. The last two possibilities are absurd or excluded, so $n = 3$ or 5.

If $n = 3$, each edge is incident with two flags fixed by r_1, and $\mathcal{M}$ is the second or third map in the second row of Figure 8 as these are transposed by r_0 or by r_0r_2. If $n = 5$, each edge is incident with one flag fixed by r_1, so $\mathrm{Aut}\,\mathcal{M}$ acts regularly on the edges and thus sharply 2-homogeneously on the vertices. However, there is no sharply 2-homogeneous group of degree 5, so this case cannot arise. □

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Realizations of lattice quotients of Petrie-Coxeter polyhedra*

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Abstract

The Petrie-Coxeter polyhedra naturally give rise to several infinite families of finite regular maps on closed surfaces embedded into the 3-torus. For the dual pair of Petrie-Coxeter polyhedra $\{4, 6 \,|\, 4\}$ and $\{6, 4 \,|\, 4\}$, we describe highly-symmetric embeddings of these maps as geometric, combinatorially regular polyhedra (polyhedral 2-manifolds) with convex faces in Euclidean spaces of dimensions 5 and 6. In each case the geometric symmetry group is a subgroup of index 1 or 2 in the combinatorial automorphism group.

IN MEMORY OF BRANKO GRÜNBAUM.

Keywords: Regular polyhedron, regular map, Petrie-Coxeter polyhedron, polyhedral embedding, polyhedral 2-manifold, automorphism group.

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1 Introduction

In the 1930's, Coxeter and Petrie discovered three remarkable infinite regular polyhedra (apeirohedra) with convex faces and skew vertex-figures in Euclidean 3-space $\mathbb{E}^3$ (see [7]). These polyhedra are often called the *Petrie-Coxeter polyhedra* and are known to be the only infinite regular polyhedra in $\mathbb{E}^3$ that have convex faces (see [10, 11, 16, 23]).

The Petrie-Coxeter polyhedra naturally give rise to several infinite families of finite regular maps (abstract regular polyhedra) on closed surfaces embedded into the 3-torus. In this paper we focus on the dual pair of Petrie-Coxeter polyhedra $\{4, 6 \,|\, 4\}$ and $\{6, 4 \,|\, 4\}$ (see Section 3 for notation) and investigate the maps in certain infinite families in more detail. These maps occur in dual pairs of types $\{4, 6\}$ and $\{6, 4\}$, and have genus $1 + 2t^3$ and automorphism groups $D_t^3 \rtimes D_6$, $t \geq 1$. In particular, we derive presentations for their automorphism groups and prove isomorphism with Coxeter's regular maps $\{4, 6 \,|\, 4, 2t\}$ and their duals [7, p. 57], which were rediscovered in [23, pp. 259]. For $t \geq 3$, we describe highly-symmetric polyhedral realizations as geometric polyhedra with convex faces in dimensions 5 and 6. These geometric polyhedra are free of self-intersections and are polyhedral 2-manifolds (polyhedral embeddings) in the sense of [5, 25, 26]. The 6-dimensional polyhedra are particularly remarkable, in that these are combinatorially regular polyhedra whose geometric symmetry group is a subgroup of index 1 or 2 in the full automorphism group. For each map of type $\{4, 6\}$ or $\{6, 4\}$, exactly one of the 6-dimensional realizations is a geometrically regular polyhedron (in this case the index is 1). In all other cases, the realizations are "combinatorially regular polyhedra of index 2" (that is, combinatorially regular polyhedra with the property that exactly one half of all combinatorial symmetries is realized by geometric isometries of the ambient space [9, 19, 37]).

The study of polyhedral 2-manifolds has attracted a lot of attention. Remarkable and beautiful 3-dimensional polyhedral embeddings of regular maps of relatively small genus have been discovered (see [33] for a survey of the maps up to genus 6, as well as [32] and [3]), including realizations for some well-known regular maps such as Klein's maps $\{3, 7\}_8$ [30] and $\{7, 3\}_8$ [22] of genus 3 (the latter realized with non-convex planar faces), Dyck's maps $\{3, 8\}_6$ of genus 3 [4, 1], Coxeter's maps $\{4, 6|3\}$ and $\{6, 4|3\}$ of genus 6 [31], and quite recently, Hurwitz's regular map of type $\{3, 7\}$ and genus 7 (sometimes called the Macbeath map) [2], which is the second smallest among the Hurwitz maps (regular maps of type $\{3, 7\}$), the smallest being Klein's $\{3, 7\}_8$. These realizations naturally have small symmetry groups compared with the much larger automorphism group. Appealing topological pictures of regular maps have been created in [35, 36] and [34].

There are also interesting infinite families of regular maps that have polyhedral embeddings in $\mathbb{E}^3$. In [25, 26], two remarkable infinite series of polyhedra of types $\{4, r\}$ and $\{r, 4\}$, $r \geq 3$, and genus $1 + (r - 4)2^{r-3}$ were described, and these are polyhedral embeddings of Coxeter's regular maps $\{4, r \,|\, 4^{\lfloor r/2 \rfloor - 1}\}$ and their duals [24]. The maps in each series also admit polyhedral embeddings in $\mathbb{E}^4$ as subcomplexes of the boundary complex of certain convex 4-polytopes [20, 21, 29].

2 Maps, polyhedra, and polyhedral realizations

A *map* $\mathcal{M}$ on a closed surface S is a decomposition (tessellation) of S into non-overlapping simply-connected regions, called the *faces* of $\mathcal{M}$, by arcs, called the *edges* of $\mathcal{M}$, joining pairs of points, called the *vertices* of $\mathcal{M}$, such that two conditions are satisfied: first, each edge belongs to exactly two faces; and second, if two distinct edges intersect, they meet in one vertex or in two vertices (see [15, 8]).

For most maps on surfaces, the underlying set of vertices, edges, and faces, ordered by inclusion, forms an abstract polyhedron (abstract polytope of rank 3). As we will explain in a moment, conversely, every (locally finite) abstract polyhedron gives rise to a map on a surface. Following [23] (with minor modifications), an *abstract polyhedron* $\mathcal{P}$ is a partially ordered set with a *rank* function with range $\{0, 1, 2\}$. For $j = 0, 1, 2$, an element of $\mathcal{P}$ of rank j is called a j*-face* of $\mathcal{P}$, or *vertex*, *edge*, and *face*, respectively. A *flag* of $\mathcal{P}$ is an incident triple consisting of a vertex, an edge, and a face. Two flags are *adjacent* if they differ by one element. Further, $\mathcal{P}$ is *strongly flag-connected*, meaning that any two flags Φ and Ψ of $\mathcal{P}$ can be joined by a sequence of flags $\Phi = \Phi_0, \Phi_1, \dots, \Phi_k = \Psi$, where the flags Φ_{i-1} and Φ_i are *adjacent* for each $i \geq 1$, and $\Phi \cap \Psi \subseteq \Phi_i$ for each $i \geq 0$. Finally, every edge contains exactly two vertices and lies in exactly two faces, and every vertex of every face lies in exactly two edges of that face.

From now on, we are making the implicit assumption that all maps considered in this paper are abstract polyhedra. For a discussion about this polytopality assumption see also [13].

Every (locally finite) abstract polyhedron lives naturally as a map on a closed surface. This surface is the underlying topological space of the order complex ("combinatorial barycentric subdivision") of $\mathcal{P}$. Recall that the *order complex* is the 2-dimensional abstract simplicial complex, whose vertices are the vertices, edges and faces of $\mathcal{P}$, and whose simplices are the chains (subsets of flags) of $\mathcal{P}$ (see [23, Ch. 2C]). The maximal simplices are in one-to-one correspondence with the flags of $\mathcal{P}$, and are 2-dimensional (triangles). Adjacency of flags in $\mathcal{P}$ corresponds to adjacency of triangles on the surface.

We require further terminology and notation that applies to both abstract polyhedra and maps. For $0 \leq j \leq 2$, every flag Φ is adjacent to just one flag, denoted Φ^j, differing by the j-face; the flags Φ and Φ^j are said to be j*-adjacent* to each other. For integers $j_1, \dots, j_l$ (with $l \geq 2$ and $0 \leq j_1, \dots, j_l \leq 2$) we inductively define the new flag

$$\Phi^{j_1 \cdots j_l} := (\Phi^{j_1 \cdots j_{l-1}})^{j_l}.$$

Note that $\Phi, \Phi^{j_1}, \Phi^{j_1 j_2}, \dots, \Phi^{j_1 \cdots j_l}$ is a sequence of successively adjacent flags.

If each face of an abstract polyhedron $\mathcal{P}$ (or a map $\mathcal{M}$) has p vertices and each vertex lies in q faces, then $\mathcal{P}$ (or $\mathcal{M}$) is said to be of (*Schläfli*) *type* $\{p, q\}$.

An abstract polyhedron $\mathcal{P}$ (or a map $\mathcal{M}$) is called *regular* if its automorphism group $\Gamma(\mathcal{P})$ (or $\Gamma(\mathcal{M})$) is transitive on the flags. Suppose $\mathcal{P}$ is an abstract regular polyhedron and $\Phi := \{F_0, F_1, F_2\}$ is a (fixed) *base* flag of $\mathcal{P}$. Then $\Gamma(\mathcal{P})$ is generated by *distinguished generators* ρ_0, ρ_1, ρ_2 (*with respect to* Φ), where ρ_j is the unique automorphism which fixes all elements of Φ but the j-face. Thus $\rho_j(\Phi) = \Phi^j$ for each $j = 0, 1, 2$. These generators satisfy the standard Coxeter-type relations

$$\rho_0^2 = \rho_1^2 = \rho_2^2 = (\rho_0\rho_1)^p = (\rho_1\rho_2)^q = (\rho_0\rho_2)^2 = 1 \tag{2.1}$$

determined by the type $\{p, q\}$; in general there are also other independent relations.

A k*-hole* of an abstract polyhedron $\mathcal{P}$ (or a map $\mathcal{M}$) is an edge-path which leaves a vertex by the k-th edge from which it entered, in the same sense (that is, always keeping to the left, say, in some local orientation). If a regular polyhedron $\mathcal{P}$ has k-holes of length h_k, then the distinguished generators of $\Gamma(\mathcal{P})$ satisfy the additional relation

$$(\rho_0\rho_1(\rho_2\rho_1)^{k-1})^{h_k} = 1. \tag{2.2}$$

When $k = 2$ we refer to a k-hole simply as a *hole*. Thus a hole of a regular polyhedron of length h contributes the relation $(\rho_0\rho_1\rho_2\rho_1)^h = 1$. A *Petrie polygon* of $\mathcal{P}$ (or $\mathcal{M}$) is an edge-path with the property that any two successive edges, but not three, are edges of a face of $\mathcal{P}$. A Petrie polygon of length t contributes the relations $(\rho_0\rho_1\rho_2)^t = 1$.

For the purpose of this paper, a *geometric polyhedron* P is a closed surface embedded in a Euclidean space made up of finitely many *convex* polygons, the *faces* of P, such that any two distinct polygons intersect, if at all, in a common vertex or a common edge (see [15, 5]). Thus P has convex faces and is free of self-intersections. We require additionally that no two faces with a common edge lie in the same 2-dimensional plane. In [5], these polyhedra were called *polyhedral 2-manifolds*. We usually identify P with the map on the underlying surface, which, in our applications, will be orientable, or with the abstract polyhedron consisting of the vertices and edges (of the polygons) and the faces, partially ordered by inclusion. We also call P a *polyhedral realization* of the map or abstract polyhedron. By $G(P)$ we denote the geometric symmetry group consisting of the isometries of the ambient space that map P to itself. A geometric polyhedron P is *geometrically regular* if $G(P)$ acts transitively on the flags.

The edge-graph of a (geometric or an abstract) polyhedron is also called the *1-skeleton* of the polyhedron; its vertices and edges are formed by the vertices and edges of the polyhedron. Similarly, for a convex d-polytope K and $0 \leq k \leq d$, the *k-skeleton* of K is the subcomplex of the face-lattice of K consisting of the faces of dimension less than or equal to k.

A convex hexagon is called *semi-regular* if its symmetry group acts transitively on the vertices. In a semi-regular hexagon which is not regular, the edges are of two kinds and alternate in length.

3 The Petrie-Coxeter polyhedra revisited

Among the geometrically regular polyhedra in Euclidean 3-space $\mathbb{E}^3$, the three Petrie-Coxeter polyhedra $\{4, 6\,|\,4\}$, $\{6, 4\,|\,4\}$ and $\{6, 6\,|\,3\}$ are characterized as the infinite polyhedra (apeirohedra) with convex faces (see [7, 23]). Each polyhedron $\{p, q\,|\,h\}$ forms a periodic polyhedral surface which bounds a pair of congruent non-compact "polyhedral handlebodies" (the inside and outside) which tile $\mathbb{E}^3$. Each polyhedron has p-gonal convex faces and q-gonal skew vertex-figures, and has 2-holes of length h. The polyhedra were discovered by Petrie and Coxeter in 1930's (see [7]).

Like any geometrically regular polyhedron in $\mathbb{E}^3$, a Petrie-Coxeter polyhedron P has a flag-transitive symmetry group $G = G(P)$ isomorphic to the combinatorial automorphism group $\Gamma(P)$. The translation subgroup $T(P)$ of $G(P)$ is generated by three translations in independent directions and can be identified with the 3-dimensional *translation lattice* $\Sigma = \Sigma(P)$ in $\mathbb{E}^3$ spanned by the three corresponding translation vectors.

In this paper we focus on the dual pair of polyhedra $\{4, 6\,|\,4\}$ and $\{6, 4\,|\,4\}$. The duality is very explicit in this case, in that, up to similarity, either polyhedron could be constructed from the other by choosing the vertices at the face centers of the other; if related in this fashion, the polyhedra share the same symmetry group and translation subgroup. The translation lattice Σ for either polyhedron is the body-centered cubic lattice and thus contains a translation sublattice $\Lambda = \Lambda(P)$ of index 4 generated by three orthogonal vectors of equal lengths. When convenient, we may take Λ to be the standard integral lattice $\mathbb{Z}^3$, and Σ to be the lattice generated by the vectors $(1, 0, 0)$, $(0, 1, 0)$, $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ (denoted $\Lambda_{(1,1,1)}$ in [23, Section 6D] and [28]). Note that Λ is also a normal subgroup of $G(P)$.

The polyhedron $\{6,4\,|\,4\}$ is built from copies of an Archimedean truncated octahedron (or rather, parts of its surface). An Archimedean truncated octahedron Q has eight regular hexagons and six squares as faces [12]. Its eight regular hexagons (and their vertices and edges) form a subcomplex of the boundary complex of Q which we will call a *hexagonal octet complex*, or simply, an *HO complex* (see Figure 1). Clearly, both Q and its HO complex are naturally inscribed in a cube C bounded by the planes containing the six square faces of Q. Then the polyhedron $\{6,4\,|\,4\}$ can be constructed from the standard cubical tessellation $\mathcal{T}$ in $\mathbb{E}^3$ by translates of C by inscribing copies of the HO complex into the cubical tiles of $\mathcal{T}$ (see Figure 2). The copies of the HO complex placed into adjacent cubical tiles attach along the "missing square faces" of the corresponding truncated octahedra, and thus a polyhedral surface is formed. Note that the symmetry group of an HO complex coincides with that of its underlying truncated octahedron.

Figure 1: Hexagonal octet complex (HO complex).

Note that the polyhedron $\{6,4\,|\,4\}$ is also related to the Voronoi tiling of the body-centered cubic lattice [7], whose tiles are truncated octahedra tessellating the "outside" and the "inside" of the polyhedron. This tiling is among the 28 uniform tilings of $\mathbb{E}^3$ enumerated by Grünbaum [17]. The polyhedron then consists of all the hexagonal 2-faces of this tiling.

The dual polyhedron $\{4,6\,|\,4\}$ can also be constructed from a standard cubical tessellation $\mathcal{T}$ of $\mathbb{E}^3$. In this case, a copy of the 1-skeleton of a smaller cube of half the size is placed concentrically into each cubical tile of $\mathcal{T}$, and then the copies in adjacent cubical tiles of $\mathcal{T}$ are connected by cylindrical square-faced tunnels so that, at either end, a tunnel meets the corresponding copy of the 1-skeleton in a "missing square face" (see Figure 3). Alternatively, and less formally, each cubical tile of $\mathcal{T}$ is shrunk concentrically to half its size and then adjacent shrunk cubes are connected by cylindrical square-faced tunnels. We refer to the part of the polyhedron lying inside a single cube of $\mathcal{T}$ as a *6-elbow*. Thus a 6-elbow consists of six *half-tunnels* attached to a shrunk cube in a cross-like fashion, where each half-tunnel is formed by four rectangles with sides of lengths 1 and 1/2 emanating from a square face of the shrunk cube. The construction shows that the polyhedron has a natural *6-elbow decomposition*.

The polyhedron $\{4,6\,|\,4\}$ also admits a 3-elbow decomposition relative to the standard

Figure 2: A $3 \times 3 \times 3$ block of the Petrie-Coxeter polyhedron $\{6, 4 \,|\, 4\}$.

cubical tessellation $\mathcal{T}$ of $\mathbb{E}^3$. Let e_1, e_2, e_3 denote the standard basis vectors of $\mathbb{E}^3$, let $a_i := \frac{1}{2}e_i$ for $i = 1, 2, 3$, and let

$$C := \{(x_1, x_2, x_3) \mid 0 \leq x_1, x_2, x_3 \leq 1\}.$$

Then $\{4, 6 \,|\, 4\}$ can be constructed as follows. The vertex set is again $\frac{1}{2}\,\mathbb{Z}^3$. The face set of $\{4, 6 \,|\, 4\}$ consists of the translates, under $\mathbb{Z}^3$, of a set of twelve particular faces contained in C. More precisely, the intersection of the entire polyhedron with C, denoted E, consists of the twelve square faces of the smaller cubes $a_i + \frac{1}{2}C$ ($i = 1, 2, 3$) which do not lie in the planes $x_i = \frac{1}{2}$ or $x_i = 1$. Six of these square faces of E meet at the vertex $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ of the polyhedron. The entire polyhedron then consists of all translates $z + E$ with $z \in \mathbb{Z}^3$. Each translate $z + E$ coincides with the intersection of the polyhedron with the cubical tile $z + C$ of $\mathcal{T}$. We refer to these intersections as *3-elbows*. Then it is clear that the polyhedron admits a natural *3-elbow decomposition* such that every cubical tile of $\mathcal{T}$ contributes exactly one 3-elbow.

4 Petrie-Coxeter-type polyhedra in the 3-torus

The three Petrie-Coxeter polyhedra P naturally give rise to regular maps embedded in the 3-torus which usually are abstract regular polyhedra. This construction was also independently described by Montero [28]. Suppose as above that the translation subgroup $T(P)$ of the symmetry group $G(P)$ of P is identified with the translation lattice $\Sigma = \Sigma(P)$. Then the maps arise as quotients of P by subgroups of $T(P)$ which are normal in $G(P)$, or equivalently, as quotients by sublattices Ω of Σ, denoted P/Ω, which are invariant under $G(P)$.

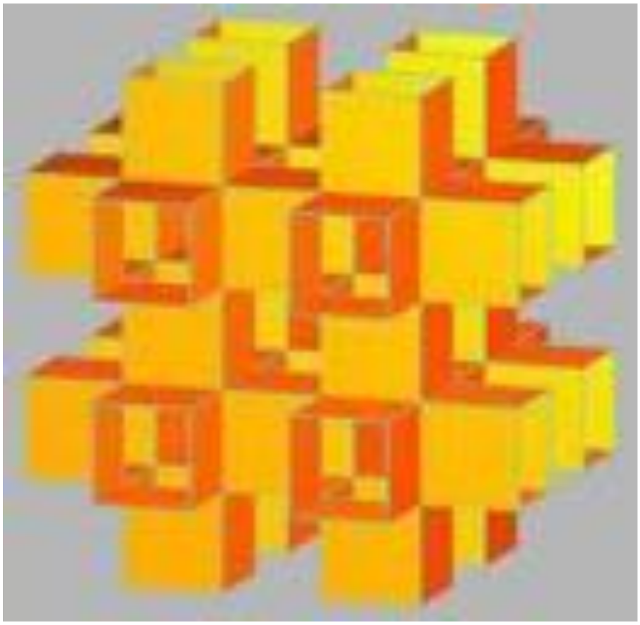

Figure 3: The Petrie-Coxeter polyhedron $\{4, 6\,|\,4\}$.

In this paper, we only consider the maps derived from the dual pair of polyhedra $\{6, 4\,|\,4\}$ and $\{4, 6\,|\,4\}$, and focus on those that later are realized polyhedrally. We choose, as sublattices Ω, certain scaled copies of the sublattice Λ of Σ (of index 4) described earlier, where here we identify Λ with $\mathbb{Z}^3$. The resulting maps are embedded in the 3-torus $\mathbb{E}^3/\Omega$ determined by Ω. We have not investigated quotients of P by other types of sublattices Ω of Σ.

4.1 The polyhedron $\mathcal{P}_t := \{6, 4\,|\,4\}/t\mathbb{Z}^3$

We begin with the polyhedron $P := \{6, 4\,|\,4\}$. Suppose P has been constructed from a standard cubical tessellation $\mathcal{T}$ by inscribing HO complexes into the cubical tiles of $\mathcal{T}$ so that the HO complexes in adjacent cubes are attached along "missing squares". Further, suppose that the origin o lies at the center of one of these HO complexes, at $\mathrm{HO_o}$ (say), and that Λ is generated by the translations by e_1, e_2, e_3, the standard basis vectors of $\mathbb{E}^3$. Thus $\Lambda = \mathbb{Z}^3$.

Now let $t \geq 1$ be an integer, and let $\Lambda_t := t\Lambda = t\mathbb{Z}^3$. Then Λ_t is invariant under $G(P)$ and hence

$$\mathcal{P}_t := P/\Lambda_t = \{6, 4\,|\,4\}/\Lambda_t$$

is a regular map of type $\{6, 4\}$ embedded in the 3-torus $\mathbb{E}^3/\Lambda_t$. The numbers of vertices, edges and faces of $\mathcal{P}_t$ are given by $12t^3$, $24t^3$ and $8t^3$, respectively. Thus the Euler characteristic is

$$\chi(\mathcal{P}_t) := 12t^3 - 24t^3 + 8t^3 = -4t^3,$$

and since $\mathcal{P}_t$ is orientable (as we will see), it must have genus $1 + 2t^3$. Clearly, since $\mathcal{P}_t$ is a regular map with 4-valent vertices, the order of the automorphism group $\Gamma(\mathcal{P}_t)$ of $\mathcal{P}_t$ is given by $96t^3 = 8 \cdot 12t^3$. The map is an abstract polyhedron for each $t \geq 1$. When $t = 1$ the map has the property that any two hexagon faces that meet in an edge also meet in the opposite edge, so in particular the map cannot admit a polyhedral realization with convex faces in any Euclidean space.

The structure of $\Gamma(\mathcal{P}_t)$ can be determined more explicitly. First note that

$$\Gamma(\mathcal{P}_t) = \Gamma(P/\Lambda_t) \cong \Gamma(P)/\Lambda_t \cong G(P)/\Lambda_t; \tag{4.1}$$

the second equality follows, for example, from [23, 2E18]. Clearly, by construction, $\Gamma(\mathcal{P}_t)$ contains a normal abelian subgroup $\mathbb{Z}_t^3$, where $\mathbb{Z}_t := \mathbb{Z}/t\mathbb{Z}$. This subgroup is generated by the three "translations" corresponding to three generators of Λ/Λ_t. Its normality is inherited from the normality of Λ_t in $G(P)$. It follows directly from (4.1) that a presentation for $\Gamma(\mathcal{P}_t)$ can be obtained from a presentation of $G(P) = \Gamma(P)$ by adding a single extra relation determining the quotient of P by Λ_t. More explicitly, since Λ_t is generated by the conjugates of the translation by the single vector te_1 (or by te_2, or by te_3), denoted T (say), it suffices to express this translation in terms of the standard generators of $G(P)$ and add the corresponding relation to the standard relations for $G(P)$. The details are as follows.

We begin by choosing a base flag $\Phi := \{F_0, F_1, F_2\}$ of $P = \{6, 4\,|\,4\}$. Then $G(P)$ is generated by three involutory isometries R_0, R_1, R_2 of $\mathbb{E}^3$ defined by the conditions $R_j(F_i) = F_i$, $R_i(F_i) \neq F_i$, for $i, j = 0, 1, 2$, $j \neq i$. The generators R_0 and R_2 are plane reflections and R_1 is a half-turn. In terms of these generators, $G(P)$ has the presentation

$$R_0^2 = R_1^2 = R_2^2 = (R_0R_1)^6 = (R_1R_2)^4 = (R_0R_2)^2 = (R_0R_1R_2R_1)^4 = 1. \tag{4.2}$$

Note that the relator $R_0R_1R_2R_1$ of the last relation shifts a 2-hole of P one step along itself and hence has period 4.

To explain the additional relation for the automorphism group $\Gamma(\mathcal{P}_t)$ of the quotient $\mathcal{P}_t$ we use the notation for flags introduced in Section 2. Consider the flags Φ and $\Phi^{101012101012}$ of the original polyhedron P, as well as the corresponding implied sequence of successively adjacent flags of P joining them,

$$\Phi, \Phi^{1}, \Phi^{10}, \Phi^{101}, \Phi^{1010}, \ldots, \Phi^{10101210101}, \Phi^{101012101012}.$$

On the underlying surface of P, flags correspond to triangles of the barycentric subdivision, and adjacency of flags corresponds to adjacency of triangles. Thus the sequence of flags translates into a sequence of triangles that starts with the triangle for the base flag, and inspection shows that the last triangle in the sequence is just the translate of the first triangle under the translation T. The expression of T in terms of the generators R_0, R_1, R_2 then can be derived from the flag sequence. In fact, taking the sequence of indices in reverse order we see that

$$T = R_2R_1R_0R_1R_0R_1R_2R_1R_0R_1R_0R_1 = (R_2R_1(R_0R_1)^2)^2$$

and hence $T^t = (R_2R_1(R_0R_1)^2)^{2t}$. Thus, a presentation for $\Gamma(\mathcal{P}_t)$ consists of the relations in (4.2) and the single extra relation

$$(R_2R_1(R_0R_1)^2)^{2t} = 1. \tag{4.3}$$

For a justification of the method employed see [23, Sect. 2B] or [27]. (The proper setting for the above argument is the monodromy group (connection group) of P, but since P

regular, this group is isomorphic to the automorphism group of P.) Note that each defining relation of $\Gamma(\mathcal{P}_t)$ involves an even number of generators. In particular this shows that $\mathcal{P}_t$ is orientable.

In Section 4.2 we will show that $\mathcal{P}_t$ is isomorphic to the orientable regular map $\{4,6\,|\,4,2t\}^*$, the dual of Coxeter's regular map $\{4,6\,|\,4,2t\}$, with automorphism group $D_t^3 \rtimes D_6$.

The length of the Petrie polygons of $\mathcal{P}_t$ is given by the order of $R_0R_1R_2$ in $\Gamma(\mathcal{P}_t)$, which is $6t$. In fact, the element $(R_0R_1R_2)^k$ of $G(P)$ lies in Λ if and only if $6\,|\,k$, and hence $(R_0R_1R_2)^k$ lies in Λ_t if and only if $6t\,|\,k$. This can either be seen directly geometrically, or by using a coordinate representation for $G(P)$ and its generators, for example the presentation of [23, p. 231]. Thus the element $R_0R_1R_2$ of $\Gamma(\mathcal{P}_t)$ has order $6t$.

Recall from [23, Sect. 6D] that the 3-torus $\mathbb{E}^3/\Lambda_t$ admits a regular tessellation by t^3 cubes obtained as the quotient of the standard cubical tessellation $\{4,3,4\}$ of $\mathbb{E}^3$ by Λ_t; this quotient, $\{4,3,4\}/\Lambda_t$, is called a *cubic toroid* and is denoted $\{4,3,4\}_{(t,0,0)}$. More informally, $\{4,3,4\}_{(t,0,0)}$ is obtained from a $t\times t\times t$ block of cubes by identifying opposite sides of the block. Note that we can think of $\mathcal{P}_t$ as being constructed from $\{4,3,4\}_{(t,0,0)}$ by inscribing copies of the HO complex into the cubical tiles, in the same way in which copies of the HO complex were inscribed into the cubical tiles of the cubical tessellation of $\mathbb{E}^3$ to construct P.

4.2 The polyhedron $\mathcal{Q}_t := \{4,6\,|\,4\}/t\mathbb{Z}^3$

For the dual polyhedron $Q := \{4,6\,|\,4\}$ of $P := \{6,4\,|\,4\}$, we may assume that its vertices lie at the face centers of P. Then $G(Q) = G(P)$ and $\Lambda = \Lambda(Q) = \Lambda(P)$. The distinguished generators of $G(Q)$ are just the distinguished generators of $G(P)$ taken in reverse order. If we label these new generators of $G(Q)$ by R_0, R_1, R_2 so that $R_j(F_i) = F_i$, $R_i(F_i) \neq F_i$, for $i,j = 0,1,2$, $j \neq i$, and some base flag $\{F_0,F_1,F_2\}$ of Q, then $G(Q)$ admits the presentation

$$R_0^2 = R_1^2 = R_2^2 = (R_0R_1)^4 = (R_1R_2)^6 = (R_0R_2)^2 = (R_0R_1R_2R_1)^4 = 1, \qquad (4.4)$$

where the order of the relator $R_0R_1R_2R_1$ gives the length of the 2-hole of Q, which again is 4.

If we set again $\Lambda_t := t\Lambda = t\mathbb{Z}^3$ for $t \geq 1$, then

$$Q_t := Q/\Lambda_t = \{4,6\,|\,4\}/\Lambda_t$$

is a regular map of type $\{4,6\}$ and genus $1 + 2t^3$ embedded in the 3-torus $\mathbb{E}^3/\Lambda_t$, and is the dual of $\mathcal{P}_t := P/\Lambda_t$. The numbers of vertices, edges, and faces of $\mathcal{Q}_t$ are given by $8t^3$, $24t^3$, and $12t^3$, respectively. The automorphism groups of $\mathcal{Q}_t$ and $\mathcal{P}_t$ are isomorphic. In particular, if we reverse the order of the generators in (4.2) and (4.3), we find that a presentation of $\Gamma(\mathcal{Q}_t)$ is given by the relations in (4.4) and the extra relation

$$(R_0R_1(R_2R_1)^2)^{2t} = 1. \qquad (4.5)$$

Now recall from Coxeter [7, p. 57] that the set of relations in (4.4) and (4.5) abstractly define the automorphism group Γ (say) of the regular map $\{4,6\,|\,4,2t\}$ determined by the lengths 4 and $2t$ of the 2-holes and 3-holes, respectively. Thus $\{4,6\,|\,4,2t\}$ is the "universal" regular map with 2-holes and 3-holes of lengths 4 and $2t$, respectively, and by

[7, p. 57] and [23, p. 259] is known to be finite and have an automorphism group isomorphic to $D_t^3 \rtimes D_6$, of order $96t^3$. As $\Gamma(\mathcal{Q}_t)$ satisfies all the defining relations of Γ, it must be a quotient of Γ, and since $\Gamma(\mathcal{Q}_t)$ has the same order as Γ, it must coincide with Γ. Thus $\Gamma(\mathcal{Q}_t) = \Gamma$ and

$$\mathcal{Q}_t = \{4, 6 \,|\, 4, 2t\},$$

and by duality,

$$\mathcal{P}_t = \{4, 6 \,|\, 4, 2t\}^*.$$

For $t = 2$ we obtain $Q_2 = \{4, 6 \,|\, 4, 4\}$, which also occurs (with $r = 6$) as the first non-toroidal map in Coxeter's infinite series of regular maps $\{4, r \,|\, 4^{\lfloor r/2 \rfloor - 1}\}$ mentioned in the Introduction. The Petrie polygons of the regular map $\mathcal{Q}_t$ are of course all of length $6t$, just like those of $\mathcal{P}_t$.

Note that we can construct $\mathcal{Q}_t$ from $\{4, 3, 4\}_{(t,0,0)}$ by inscribing copies of a 6-elbow into the cubical tiles, in the same way in which Q can be obtained by inscribing copies of a 6-elbow into the cubical tiles of the cubical tessellation of $\mathbb{E}^3$.

5 Embeddings in $\mathbb{E}^6$

The polyhedral embeddings of the Petrie-Coxeter type polyhedra $\mathcal{P}_t$ and $\mathcal{Q}_t$, $t \geq 3$, in Euclidean 6-space $\mathbb{E}^6$ will be constructed from an embedding of the 3-torus as a 3-dimensional cubical subcomplex in the boundary complex of a convex 6-polytope K_t. Then, via Schlegel diagrams [18], polyhedral embeddings can also be constructed in Euclidean 5-space $\mathbb{E}^5$. We begin by describing the polytope K_t in $\mathbb{E}^6$. Our construction requires that $t \geq 3$.

5.1 A polyhedral embedding of the cubic toroid

It is well-known that the 3-torus admits a natural embedding into the 5-sphere $\sqrt{3}\ \mathbb{S}^5$ (of radius $\sqrt{3}$) as the submanifold M consisting of all points $x := (x_1, \ldots, x_6)$ of $\mathbb{E}^6$ satisfying the equations

$$x_1^2 + x_2^2 = x_3^2 + x_4^2 = x_5^2 + x_6^2 = 1.$$

This expresses M as the product $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$ lying in $\mathbb{E}^6$.

Let $t \geq 3$. We now construct a convex 6-polytope K_t in $\mathbb{E}^6$ with vertices in M and exploit its structure to find polyhedral realizations of the regular maps $\mathcal{P}_t$ and $\mathcal{Q}_t$ constructed in the previous section. For $j = 0, \ldots, t-1$, set $a_j := 2\pi j/t$ and define the points

$$\begin{aligned} u_j &:= (\cos a_j, \sin a_j, 0, 0, 0, 0), \\ v_j &:= (0, 0, \cos a_j, \sin a_j, 0, 0), \\ w_j &:= (0, 0, 0, 0, \cos a_j, \sin a_j), \end{aligned}$$

all of which are contained in M. Then the convex 6-polytope

$$K_t := \operatorname{conv}\{u_j + v_k + w_l \,|\, j, k, l = 0, \ldots, t-1\} \tag{5.1}$$

is the cartesian product of the three regular convex t-gons K_t^1, K_t^2 and K_t^3 with vertex-sets $\{u_0, \ldots, u_{t-1}\}$, $\{v_0, \ldots, v_{t-1}\}$ or $\{w_0, \ldots, w_{t-1}\}$, respectively, lying in the 2-dimensional coordinate subspaces containing the first two, the third and the fourth, or the last two standard coordinate axes of $\mathbb{E}^6$. Thus

$$K_t = K_t^1 \times K_t^2 \times K_t^3.$$

Note that K_t is a 6-cube when $t = 4$. Now recall that the nonempty faces of a cartesian product of convex polytopes are just the cartesian products of the nonempty faces of its component polytopes [18]. In particular, the cartesian products of three edges from different component polygons K_t^j give t^3 3-dimensional faces of K_t that are 3-cubes. When $t \neq 4$ there are no other 3-faces of K_t which are 3-cubes; however, when $t = 4$ the polytope K_t is the 6-cube and thus each of its 3-faces is a 3-cube. In any case, the union of these t^3 3-faces obtained as products of three edges from different components forms a subcomplex $\mathcal{C}_t$ of the boundary complex of K_t which is homeomorphic to the 3-torus. This follows from the fact that $\mathcal{C}_t$ is topologically the product of the three unit circles determined by the boundary complexes of K_t^1, K_t^2 and K_t^3. Thus $\mathcal{C}_t$ is a 3-dimensional cubical complex embedded into the boundary complex of K_t and is isomorphic to the regular cubic toroid $\{4,3,4\}_{(t,0,0)}$.

When $t \neq 4$ the symmetry group of K_t is isomorphic to the wreath product $D_t \wr S_3$ ($\cong D_t^3 \rtimes S_3$), of order $48t^3$. However, when $t = 4$ the symmetry group of K_t is isomorphic to $C_2 \wr S_6$, of order $2^6\,6!$; this group still contains a subgroup isomorphic to $D_4 \wr S_3$, which is of index 15. In any case, for each $t \geq 3$, the subgroup $D_t \wr S_3$ of the symmetry group of K_t leaves the cubical complex $\mathcal{C}_t$ invariant. In fact, $D_t \wr S_3$ is the symmetry group of $\mathcal{C}_t$, and being of the right order, is also the full automorphism group of $\mathcal{C}_t$. It acts flag-transitively on $\mathcal{C}_t$.

5.2 A polyhedral embedding of $\mathcal{P}_t = \{6, 4\,|\,4\}/t\mathbb{Z}^3$

We begin by rescaling the 6-polytope so that the 3-cubes in the cubical complex become unit 3-cubes. Each 3-cube in $\mathcal{C}_t$ has edge length $2\sin(\pi/t)$, which is the edge length of a regular convex t-gon inscribed in a unit circle. Thus the new convex polytope $K_t' := rK_t$, with $r := 1/(2\sin(\pi/t))$, as well as the corresponding cubical complex $\mathcal{C}_t' := r\,\mathcal{C}_t$, have the property that all faces which are 3-cubes are unit 3-cubes. In particular, $\mathcal{C}_t'$ is a faithful realization of the regular cubic toroid $\{4,3,4\}_{(t,0,0)}$. In other words, $\mathcal{C}_t'$ is isomorphic to $\{4,3,4\}_{(t,0,0)}$, and the automorphism group of $\mathcal{C}_t'$ (of order $48t^3$) is realized by a group of Euclidean isometries, namely $D_t \wr S_3$.

Now the construction of the embedding for $\mathcal{P}_t$ is straightforward: in each 3-cube of $\mathcal{C}_t'$ (or rather its support) we inscribe a basic building block for $\mathcal{P}_t$, that is, a copy of an HO complex. Note that the insertion in each 3-cube is unique. Thus we arrive at a polyhedral 2-manifold in $\mathbb{E}^6$ with regular convex hexagons as faces, which is isomorphic to $\mathcal{P}_t$ and invariant under $D_t \wr S_3$. We will show in the next section that the latter group provides only one half of all the geometric symmetries of the polyhedral 2-manifold. In fact, the polyhedral embedding for $\mathcal{P}_t$ is a geometrically regular polyhedron in $\mathbb{E}^6$ (with regular convex hexagons as faces), that is, its symmetry group is isomorphic to the full automorphism group $D_t^3 \rtimes D_6$.

There is also a host of 6-dimensional polyhedral realizations of $\mathcal{P}_t$ in $\mathbb{E}^6$ with convex hexagonal faces which are not regular. To construct these we can proceed in the same way as above, but now from a modified HO complex. The modified HO complex can be obtained from the HO complex of Figure 1 by shrinking or expanding the "missing squares" in a uniform fashion, so that the resulting octet consists of eight congruent semi-regular convex hexagons (with two kinds of edges). The resulting polyhedral 2-manifold has $D_t \wr S_3$ as its symmetry group, that is, it is a combinatorially regular polyhedron of index 2 in $\mathbb{E}^6$.

Via a Schlegel diagram of the 6-polytope K_t' we can also produce polyhedral realizations of $\mathcal{P}_t$ in Euclidean 5-space $\mathbb{E}^5$. The (possibly modified) HO complexes sitting inside the 3-cubes of $\mathcal{C}_t'$ project to distorted HO complexes sitting inside the 3-dimensional parallelepipeds which are the projections of the 3-cubes and make up the image of $\mathcal{C}_t'$ under the projection. There are no self-intersections, since $\mathcal{C}_t'$ is a subcomplex of the boundary complex of K_t', and Schlegel diagrams faithfully represent the boundary complex of a convex polytope.

5.3 A polyhedral embedding of $\mathcal{Q}_t = \{4, 6 \mid 4\}/t\mathbb{Z}^3$

We describe two possible ways of constructing polyhedral realizations of the regular maps $\mathcal{Q}_t = \{4, 6 \mid 4\}/t\mathbb{Z}^3$ derived from $Q := \{4, 6 \mid 4\}$. The first is based on the 3-elbow decomposition of Q, and the second employs the 6-elbow decomposition of Q and has a larger symmetry group.

The first construction of the polyhedral embedding for $\mathcal{Q}_t$ proceeds as follows: in each 3-cube of $\mathcal{C}_t'$ we inscribe a basic building block for $\mathcal{Q}_t$, that is, a copy of a 3-elbow. Note that the insertion in each 3-cube is uniquely determined once a 3-elbow has been placed in one 3-cube. Thus we arrive at a polyhedral 2-manifold with square faces in $\mathbb{E}^6$ which is isomorphic to $\mathcal{Q}_t$ and lies in the support of the 3-skeleton of K_t'. The symmetry group of this realization is $C_t \wr S_3$, of order $6t^3$, which is a subgroup of index 16 in the full automorphism group of $\mathcal{Q}_t$. Note that the special position of the 3-elbow at a corner inside the unit cube prevents additional symmetries from occuring.

The second construction of a polyhedral realization of $\mathcal{Q}_t$ must proceed in a different way, since adjacent 6-elbows in the 6-elbow decomposition attach along the boundary of half-tunnels, not along 4-cycles of edges as in the previous case. In this case we will construct a realization that lies in the support of the 4-skeleton of the polytope K_t' but not in the support of the 3-skeleton. However, the 3-dimensional cubical complex $\mathcal{C}_t'$ still provides the blueprint for the construction.

The basic idea is simple but the details are tedious. Let K_t' and $\mathcal{C}_t'$ be as above. We shrink all 3-cubes of $\mathcal{C}_t'$ concentrically (with respect to their centers, by a fixed scaling factor $\lambda < 1$), and then connect the shrunk copies of any two adjacent 3-cubes of $\mathcal{C}_t'$ by small cylindrical tunnels whose “bases” and “tops” are given by the shrunk copies of the square face (with sides of length λ) shared by the adjacent 3-cubes, and whose side faces are rectangles. The set of all side faces of cylindrical tunnels obtained in this way, as well as their vertices and edges, forms a polyhedral realization of the regular map $\mathcal{Q}_t$ with rectangular faces (with one side of length λ). Moreover, as we explain below, if the 3-cubes are shrunk to the right size, then the rectangular faces become squares.

The realization of $\mathcal{Q}_t$ lies inside the support of the 4-skeleton of K_t', and we show that it is free of self-intersections. Recall that each 3-cube in $\mathcal{C}_t'$ is the cartesian product of three edges from different component polygons of K_t'. Two 3-cubes C_1 and C_2 of $\mathcal{C}_t'$ are adjacent if their sets of three edges have two edges in common, I_1 and I_2 (say), and the remaining two edges, I_3 and I_4 respectively (say), are adjacent edges of a component polygon, F (say), of K_t' meeting at a vertex z (say) of F. In the shrinking process, the common square face $I_1 \times I_2 \times \{z\}$ of C_1 and C_2 yields a pair of parallel squares of smaller size contained in C_1 and C_2 (the base and top of the cylindrical tunnel). The cylindrical tunnel connecting these smaller squares lies entirely in the cartesian product $F' := I_1 \times I_2 \times F$, which is a 4-dimensional face of K_t' that has C_1 and C_2 as a pair of adjacent facets. Note that F' is a double prism over the regular convex t-gon F. More explicitly, since $C_1 = I_1 \times I_2 \times I_3$ and $C_2 = I_1 \times I_2 \times I_4$, their shrunk copies are given by $C_1' = I_1' \times I_2' \times I_3'$ and $C_2' = I_1' \times I_2' \times I_4'$, respectively, where $I'_j =: [a_j, b_j] \subset I_j$ for $j = 1, 2, 3, 4$ and the labeling is such that the points a_3 and a_4 are closer to z than b_3 and b_4; here $[a_j, b_j]$ denotes the line segment with endpoints a_j and b_j. It now follows that the four rectangular side faces of the cylindrical tunnel connecting C'_1 and C'_2 are given by

$$\begin{aligned} &\{a_1\} \times I'_2 \times [a_3, a_4], \quad I'_1 \times \{a_2\} \times [a_3, a_4], \\ &\{b_1\} \times I'_2 \times [a_3, a_4], \quad I'_1 \times \{b_2\} \times [a_3, a_4]. \end{aligned}$$

Then it is clear that two cylindrical tunnels inserted during the process could only intersect nontrivially (that is, other than in vertices, or in edges lying in their bases or tops) if they lied in the same 4-face F' of K'_t. Thus the question whether or not there are unwanted intersections reduces to a 4-dimensional problem.

Each 4-face $F' = I_1 \times I_2 \times F$ of K'_t contains exactly t cyclindrical tunnels, one for each pair of adjacent edges of F. As the shrinking process is uniform, the subgroup $D_t \wr S_3$ of the symmetry group of K'_t preserves the realization of $\mathcal{Q}_t$. In particular, the symmetry group of F, which is isomorphic to D_t, appears as a subgroup of $D_t \wr S_3$ and permutes the t cylindrical tunnels in F' according to a standard dihedral action. Thus the collection of t cylindrical tunnels inside F' is invariant under the dihedral group D_t determined by F.

To see that nontrivial self-intersections cannot occur, choose vertices u and v of I_1 and I_2, respectively, and project F' orthogonally along the direction of $I_1 \times I_2$ onto its 2-face $\{u\} \times \{v\} \times F$. A square face shared by a pair of adjacent 3-cubes from $\mathcal{C}'_t$ that lie in F' is necessarily of the form $I_1 \times I_2 \times \{w\}$, where w is a vertex of F, and is parallel to its two shrunk copies (the base and top of the cylindrical tunnel) in the 3-cubes. Hence each cylindrical tunnel in F' projects orthogonally to a line segment strictly inside $\{u\}\times\{v\}\times F$. For the cylindrical tunnel associated with C_1 and C_2 as described above, the line segment is $[a_3, a_4]$. There are no nontrivial intersections among the t resulting line segments. Since these line segments were obtained by orthogonal projections of cylindrical tunnels, there also cannot be any nontrivial intersections among the cylindrical tunnels. Hence there are no intersections of cylindrical tunnels inside a 4-face F'. Thus the realizations of $\mathcal{Q}_t$ are free of self-intersections, for all scaling factors λ.

As will become clearer in a moment, the above analysis also shows that each realizations for $\mathcal{Q}_t$ is a subcomplex of the 3-skeleton of a convex 6-polytope (depending on λ) defined as the cartesian product of three semi-regular convex $2t$-gon (with two kinds of edges); moreover, the realization of $\mathcal{Q}_t$ has the same vertex-set as the 6-polytope. The tunnels then are formed by four rectangular faces of 3-dimensional faces (rectangular boxes) given by the cartesian product of three edges of these $2t$-gons, with one edge from each $2t$-gon and exactly two edges of the same length. This observation provides an alternative proof of the fact that the realization for $\mathcal{Q}_t$ is free of self-intersections.

As in the previous section, in all but one case only one half of the combinatorial symmetries appear as symmetries of the realization. The symmetry group of the realization for $\mathcal{Q}_t$ is $D_t \wr S_3$ (even if $t = 4$), except when the rectangular faces become squares (in which case it is $D_t^3 \rtimes S_3$), as can be seen as follows.

If the unit 3-cubes are shrunk to smaller 3-cubes of the right size, then all rectangular faces of the realization become squares. In fact, the correct scaling factor λ is the inverse ratio between the edge lengths of a convex regular t-gon and of a convex regular $2t$-gon obtained from the t-gon by vertex-truncation. It is straightforward to check that

$$\lambda = \frac{\cos\frac{\pi}{t}}{1 + \cos\frac{\pi}{t}}.$$

Thus $\mathcal{Q}_t$ admits a polyhedral embedding with square faces in $\mathbb{E}^6$ and then, via a Schlegel diagram, a polyhedral embedding with quadrangular faces in $\mathbb{E}^5$. The vertex-set of the 6-dimensional square-faced realization coincides with the vertex-set of the cartesian product of three regular $2t$-gons, namely the regular $2t$-gons obtained by vertex-truncation from the three regular t-gons appearing as factors in the cartesian product decomposition of K_t'. This also shows that this square-faced realization is identical with the realization of $\mathcal{Q}_t$ outlined in [7, p. 57] and [23, p. 259] and further described below. The square-faced realization of $\mathcal{Q}_t$ has the desirable property that all combinatorial symmetries are realized by geometric symmetries, or more exactly, that the geometric symmetry group is isomorphic to $\Gamma(\mathcal{Q}_t) = D_t^3 \rtimes D_6$.

The regular maps $\mathcal{Q}_t = \{4, 6\,|\,4, 2t\}$ are isomorphic to certain abstract polyhedra $2^{\{6\},\mathcal{G}(s)}$ (see [23, p. 259]). These polyhedra admit 6-dimensional realizations (in the sense of [23, Ch. 5]) as subcomplexes of the 2-skeleton of the cartesian product of three regular $2t$-gons (see [23, p. 264] and [7, p. 57]), and thus are free of self-intersections and form a polyhedral 2-manifold. Our approach to realizations of the maps $\mathcal{Q}_t$ as polyhedral 2-manifolds builds on a 6-dimensional realization of the cubic 4-toroid $\{4, 3, 4\}_{(t,0,0)}$ and exploits the quotient relationships of $\mathcal{Q}_t$ and $\mathcal{P}_t$ with the Coxeter-Petrie polyhedra $\{4, 6\,|\,4\}$ and $\{6, 4\,|\,4\}$ in a very explicit manner. In particular, we based our construction of rectangle-faced polyhedral embeddings for $\mathcal{Q}_t$ on the cartesian product of three regular t-gons, rather than of three regular $2t$-gons as in [23]. However, as hinted at before, the rectangular-faced embeddings can also be viewed as subcomplexes of the 3-skeleton of a cartesian product of three semi-regular convex $2t$-gons. On the other hand, the square-faced realization of [7, p. 57] and [23, p. 264] has the advantage that all combinatorial symmetries are realized by geometric symmetries.

Note that the 6-dimensional polyhedral embeddings for $\mathcal{Q}_t$ could also be obtained directly from those of the previous section for their combinatorial duals $\mathcal{P}_t$, by choosing as vertices of an embedding for $\mathcal{Q}_t$ the centers of the hexagonal faces of the embeddings for $\mathcal{P}_t$ and then proceeding according to duality. (This process results in rectangular faces for the realization of $\mathcal{Q}_t$. In fact, the four vertices of every face of the realization of $\mathcal{Q}_t$ must lie on four of the edges of a tetrahedron which has two of its vertices located at the centers of a pair of adjacent 3-cubes, and the other two vertices located at adjacent vertices of the common square face of these 3-cubes. These four edges of the tetrahedron form a skew equilateral 4-gon, and the vertices of the realization for $\mathcal{Q}_t$ all have the same distance from the center of the 3-cube in which they lie. Hence the vertices are those of a rectangle.) Under this dual correspondence, the polyhedral embedding with regular hexagons as faces for $\mathcal{P}_t$ corresponds to the square-faced polyhedral embedding for $\mathcal{Q}_t$, and in particular, both polyhedral embeddings are geometrically regular polyhedra in $\mathbb{E}^6$ with symmetry group $D_t^3 \rtimes D_6$. In all other cases, polyhedral embeddings with semi-regular hexagon faces (with two kinds of edges) for $\mathcal{P}_t$ correspond to rectangle-faced polyhedral embeddings for $\mathcal{Q}_t$, and both kinds give combinatorially regular polyhedra of index 2, with symmetry group $D_t \wr S_3$.

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Small stopping sets in finite projective planes of order q

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Abstract

A *configuration* $\mathcal{C}$ in a (finite) incidence structure is a subset $\mathcal{C}$ of blocks. If every point on a block of $\mathcal{C}$ belongs to at least one other block of $\mathcal{C}$, then $\mathcal{C}$ is called *stopping set* (or equivalently *full configuration*). If $s_{min}(q)$ is the minimal size of a stopping set in a finite projective plane of odd order q, then either $s_{min}(q) \geq q+5$ if $3 \not| q$ or $s_{min}(q) \geq q+3$ if $3|q$. In this note, we prove that $s_{min}(q) \geq q+5$ for any odd $q \neq 3$. If $q = 3$, then $s_{min}(3) = 6$ and a stopping set of minimal size 6 in $\mathrm{PG}(2,3)$ is the dual set of the symmetric difference of two lines. Also, we study stopping sets of size $q+4$ in a finite projective plane of order q.

Keywords: Low density parity check codes, projective planes, KM*–arcs, stopping sets, linear spaces.*

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1 Introduction

Low density parity check (LDPC) codes based on finite geometries and combinatorial designs are codes with good minimum distance properties. Since the performance of a LDPC code over the binary erasure channel is determined by certain combinatorial objects called *stopping sets*, in [6] the authors define and analyze the notion of stopping set in the underlying design.

Let $\mathbb{G}$ be a finite incidence structure, with set of points $\mathcal{P}$ and with set of blocks $\mathcal{B} \subseteq 2^{\mathcal{P}}$. Let $1 \leq s \leq |\mathcal{B}|$ be an integer, a set $\Sigma = \{B_1, \ldots, B_s\}$ of blocks of $\mathbb{G}$ is a *stopping set* (or, equivalently, a *full configuration* [3]) if every point on a block of Σ belongs to at least one other block of Σ.

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In [6], the authors give a lower bound and an upper bound for the size of a stopping set and for the smallest size s_{min} of a stopping set in a $2-(v,k,\lambda)$–design, respectively. They provide examples of designs having stopping set whose sizes achieve the lower bound. Also, they give some necessary conditions for the existence of a stopping set in finite projective planes, and one of their results reads as follows.

Result 1.1. *Let $s_{min}(q)$ denote the minimal size for a stopping set in a finite projective plane π_q of order q. If q is odd, then*

$$s_{min}(q) \geq \begin{cases} q+3 \text{ if } 3|q \\ q+5 \text{ if } 3\nmid q \end{cases}.$$

In projective planes of even order q, we have $s_{min}(q) = q+2$, and the dual of a hyperoval (in planes containing such sets) is an example of stopping set of size $q+2$.

Since the integer s_{min} is related to the size of the smallest stopping set in a low-density parity check (LDPC) code, which determines, to some extent, the performance of iterative decoding methods over the binary erasure channel, it seems natural to study stopping sets of minimal size in designs.

In [3] Colbourn and Fujiwara studied the existence of small stopping sets in Steiner and partial Steiner triple systems. Recently, in [5] stopping sets in finite projective spaces and affine spaces of any finite dimension $m \geq 2$ have been considered.

In this paper, first we describe stopping sets having the minimum number of points in a finite incidence structure with no assumption on the size of its blocks. Then we prove the following two results.

Result 1.2. *If a projective plane of order q has a stopping set Σ of size $q+3$, then $q=3$ and Σ is the stopping set described in Example 1.2.*

Thus, $s_{min}(q) \geq q+5$ in any finite projective plane of odd order q, $q \neq 3$.

Result 1.3. *If a finite projective plane π_q of order q contains a stopping set Σ of size $q+4$, then either Σ consists of six of the seven lines of* $\mathrm{PG}(2,2)$*, or $4|q$ and either $q=4$ and Σ is the stopping set described in Example 1.2 or $q>4$ and every point covered by the lines of Σ belongs to exactly 2 or 4 lines of Σ.*

If $\pi_q = \mathrm{PG}(2,q)$, there are examples of stopping sets of size $q+4$, $q>4$ (cf. e.g. [7, 8, 9]). Moreover, all the $q/4+1$ points through which there pass exactly four lines of Σ are collinear on a line not in Σ (cf. Section 3).

Below, there are three examples of stopping sets of small size[1] in finite projective and affine geometries which give upper bounds for the minimal size of a stopping set in these geometries.

Example 1.1. The set of all the lines of a regulus $\mathcal{R}$ union those of the opposite regulus to $\mathcal{R}$ is a stopping set in $\mathrm{PG}(3,q)$ of size $2q+2$.

Example 1.2. Let π_q be a finite projective plane of order q, consider two distinct points p and p' in π_q. The set Σ consisting of all the lines of π_q through p and p', respectively, and different from the line pp' is a stopping set of size $2q$. Thus the minimum size of a stopping set in a projective plane of order q is at most $2q$.

[1]Examples 1.2 and 1.3 are given by a construction in [6].

Example 1.3. Let α_q be an affine plane of order q, consider a line ℓ, and all the lines of α_q parallel to ℓ. Let L be one of the lines of the parallel class of ℓ, and let p be a point of L. The set of lines Σ consisting of ℓ, all the lines parallel to ℓ and different from L and all the lines containing p and different from L has size $q-1+q$ and is a stopping set, so in affine plane of order q the minimum size of a stopping set is at most $2q-1$. Note that in an affine plane the set of all the lines of at least two parallel classes give rise to a stopping set of size $2q$.

2 Stopping sets with the smallest size

The next result describes the structure of a stopping set with the smallest possible size.

Proposition 2.1. *Let Σ be stopping set in an incidence structure $\mathbb{G}$ and let m be the minimum size of the blocks of Σ. Then $|\Sigma| \geq m+1$. If $|\Sigma| = m+1$ then the blocks of Σ have constant size m, are pairwise intersecting each other, no three of them are confluent at a same point and they cover $\binom{m+1}{2}$ points of $\mathbb{G}$.*

Proof. If B is a block of a stopping set Σ, since on each point of B there is at least one block of Σ different from B, it follows that $|\Sigma| \geq |B|+1$. Thus, if B is a block of minimum size m, we have that $|\Sigma| \geq m+1$.

Let Σ be a stopping set of size $m+1$ and with minimum block size m. Then, on every point on a block of Σ there are exactly two blocks of Σ and any block of Σ has size m. If there are two disjoint blocks, say B_1 and B_2, since B_1 is intersected by at least m blocks of Σ, then $|\Sigma| \geq m+2$, a contradiction. So any two blocks of Σ intersect each other.

Thus the blocks of Σ and the points they cover form a finite linear space on $|\Sigma| = m+1$ points with constant point degree m and constant line size 2, that is the complete graph on $m+1$ vertices and so with $\binom{m+1}{2}$ edges. It follows that the lines of Σ cover $\binom{m+1}{2}$ points of $\mathbb{G}$. ∎

If one assumes that any two distinct points of $\mathbb{G}$ are incident with at most one block (*semilinearity condition*), then the converse of the last part of the statement of Proposition 2.1 holds, (cf. [6]).

The Desarguesian projective plane $\mathrm{PG}(2,q)$, q even, is an example of a 2-$(v,k,1)$-design with stopping sets of minimum size $k+1$. Also, Example 1.3 for $q=2$ yields a set of size $q+1$ in an affine plane of order q, whose lines are pairwise intersecting.

Let us end this section with a remark on stopping sets whose lines are pairwise intersecting, and so in particular for those contained in projective planes. The lines of such a stopping set are the points of a linear space[2] $\mathbb{S} = (\Sigma, \mathcal{P}_\Sigma)$ with constant point degree $q+1$. Since the list of all possible finite linear spaces with at most 18 points is known (cf. [1] and its bibliography), in the enumeration question (cf. [6]), for stopping sets with at most 18 lines this classification may be of help.

3 Small stopping sets in finite projective planes

By the results in the previous section, for every stopping set Σ in a projective plane we may assume that its size is at least $q+3$. So, let $q+t$, $t \geq 3$ denote the size of a stopping set.

[2] Throughout the paper, $\mathcal{P}_\Sigma$ denotes the set of points covered by the lines of a stopping set Σ.

Since we are interested in minimal stopping sets, in view of Example 1.2, we may assume $t \leq q$. Thus, from now on, Σ denotes a stopping set of size $|\Sigma| = q+t$, $3 \leq t \leq q$, in a projective plane of order q.

Proposition 3.1. *Every line of Σ contains at least three points on exactly two lines of Σ.*

Proof. Let $\mathbb{S} = (\Sigma, \mathcal{P}_\Sigma)$ be the linear space dual of Σ, it has $q+t$ points and all its points have degree $q+1$.

Let p be a point of $\mathbb{S}$, the lines through p contain at least $q+2$ points of $\mathbb{S}$. If there are at least $q-1$ lines of size at least 3 on p then:

$$2q \geq q+t \geq 2(q-1)+2+1,$$

a contradiction. □

Now, we recall some definitions which will be useful in the following.

If X is a subset of points of a finite projective plane π_q of order q, and $0 \leq i \leq q+1$ is an integer, a line ℓ of π_q is an i*–line* if it intersects X in exactly i points. The line ℓ is an *external* line if $i=0$, and a *tangent* line if $i=1$.

Definition 3.2. A $\mathrm{KM}_{q,t}$–arc in $\mathrm{PG}(2,q)$ (or a $(q+t)$–arc of type $(0,2,t)$) is a set S of $q+t$ points in $\mathrm{PG}(2,q)$ intersected by every line in either 0, 2 or t points.

If $t=1$, the set is an arc (degenerate case), if $t=q$, there is only one example: the symmetric difference of two lines. So $\mathrm{KM}_{q,t}$–arcs, are studied for $1 < t < q$.

Examples of these sets first appeared in [8] and [7]. Moreover, in [7] these sets were studied and the following result[3] was proved

- *$\mathrm{KM}_{q,t}$–arcs in $\mathrm{PG}(2,q)$ of type $(0,2,t)$, $1 < t < q$ can only exist if q is even. Moreover, t needs to be a divisor of q.*

Also, the following structural result [4] is known.

- *All t–secant lines of a $\mathrm{KM}_{q,t}$–arc in $\mathrm{PG}(2,q)$ with $t > 2$ are concurrent in a point outside the set, which is called the* **nucleus***.*

If Σ is a stopping set in a projective plane π_q and $\mathbb{S}$ is the linear space it forms in the dual plane π^*, let b_i denote the number of lines of size i in $\mathbb{S}$. Since $\mathbb{S}$ is embedded in the projective plane π^* of order q, Σ is a subset of points of π^*, and the lines of $\mathbb{S}$ are the set of points of π^* obtained by intersecting the lines of π^* with Σ and having at least two points in Σ. So, b_i also indicates the number of i–lines of Σ in π^*. Finally, since every point of $\mathbb{S}$ has degree $q+1$, the set Σ of points of π^* has no tangent lines.

[3] Since we use the recent terminology of KM–arc, we state this result in a Desarguesian plane, but it holds for any projective plane.

3.1 The case $|\Sigma| = q + 3$

We are going to prove the following result:

Proposition 3.3. *If π_q contains a stopping set Σ of size $q + 3$ then $q = 3$ (and so π_q is Desarguesian) and Σ is the stopping set described in Example 1.2.*

Proof. If $t = 3$, arguing as in Proposition 3.1 one has that for every point of $\mathbb{S} = (\Sigma, \mathcal{L}_\Sigma)$ the numbers of incident 3-lines and 2-lines are 1 and q, respectively. Double counting gives

$$q + 3 = 3b_3 \qquad \text{and} \qquad q(q + 3) = 2b_2.$$

So, $3|q$.

Now, assume $q > 3$. Recall that $\mathbb{S}$ is embedded in a projective plane of order q, namely, the dual of π_q. So Σ is a subset of points of π^*, with b_3 3–lines and b_2 2–lines. For any point p of π^* not in Σ, let $x_i(p)$ denote the number of i–lines through p.

Thus, for any point $p \in \pi^* \setminus \Sigma$ we have

$$2x_2(p) + 3x_3(p) = q + 3. \tag{3.1}$$

Let ℓ be a 3–line. At least one of the points of ℓ outside Σ is on exactly one 3–line, otherwise $q/3 \geq q - 2$, and so $q = 3$ contradicting our assumption.

Let p be such a point of ℓ on exactly one 3–line, then

$$q + 3 = 3 + 2x_2(p),$$

and so q is even.

Then, by (3.1), $x_3(p) > 0$ for every point p outside Σ. So,

$$q^2 + q + 1 - q - 3 \leq \sum_{p \notin \mathbb{S}} x_3(p) = b_3(q - 2) = (\frac{q}{3} + 1)(q - 2),$$

$$3(q - 2)(q + 2) + 6 = 3(q^2 - 2) \leq (q + 3)(q - 2),$$

$$3q + 6 + \frac{6}{q - 2} \leq q + 3,$$

a contradiction.

Thus, $q = 3$, π_q is Desarguesian and $\mathbb{S}$ is the linear space all whose points lie on two disjoint lines of size 3 and so Σ is the stopping set described in Example 1.2. □

3.2 The case $|\Sigma| = q + 4$

In this case, $\mathbb{S}$ has $q+4$ points and for each point the numbers of incident 3-lines and 2-lines are either 1 and q, respectively, or 2 and $q-1$. Let u denote the number of points contained in a 3–line. Hence $3b_3 = 2u \leq 2(q + 4)$.

Proposition 3.4. *q is even, and if $q \neq 2$ then $\mathbb{S}$ contains no 3–lines.*

Proof. For any point p outside Σ let $x_i(p)$ denote the number of i–lines on p. So,

$$2x_2(p) + 3x_3(p) + 4x_4(p) = q + 4. \tag{3.2}$$

If q is odd, then $x_3(p) > 0$. It follows that:

$$q^2 - 3 = q^2 + q + 1 - q - 4 \leq \sum_{p \notin \mathbb{S}} x_3(p) = b_3(q-2) \leq \frac{2(q+4)}{3}(q-2),$$

$$3q^2 - 9 \leq 2q^2 + 4q - 16,$$

$$q^2 - 4q + 7 \leq 0,$$

which cannot occur.

Thus, q is even.

Assume that $\mathbb{S}$ contains a 3–line. Then, in π^* there is a line ℓ intersecting Σ in three points. By Equation (3.2) and since q is even it follows that on any point of ℓ outside Σ there are at least two 3–lines. Thus,

$$3 + 1 + q - 2 \leq b_3 \leq \frac{2}{3}(q+4)$$

and so $q \leq 2$. Thus, either $q = 2$ or $\mathbb{S}$ contains no 3–line. □

If $q = 2$, then $|\Sigma| = 6$ and so Σ is given by six of the seven lines of $\mathrm{PG}(2,2)$ (i.e. the $STS(7)$–*Line* full configuration in [3]).

If $q \neq 2$, then on each point of $\mathbb{S}$ there is exactly one 4–line and so $4b_4 = q + 4$, that is $4|q$ and a projective stopping set of size $q + 4$ gives rise to a set of points of π^* intersected by any line in 0, 2 or 4 points and each point of a line of Σ belongs to exactly one 4–line. If $q = 4$ then Σ is the set of lines decribed in Example 1.2. Hence, also Result 1.3 is proved.

Let $q > 4$ and assume that π_q is Desarguesian, hence the dual of Σ is a $KM_{q,4}$–arc [9] and examples of $KM_{q,4}$–arc exist (cf. e.g. [7, 8, 9]).

For the next cases, that is for stopping sets of size $q + t$, $t \geq 5$, we recall that the dual of a stopping set is a set of points of a projective plane with no tangent lines. So, in the Desarguesian case, when t is small with respect to q, the results on these sets of points (cf. [2, 10]) show that, if q is odd then it is upper bounded by a quadratic function of t. For q even, if the dual of the stopping set has an i–line with i odd, again q is upper bounded by a quadratic function of t. If q even, and i is even for any i–line, then not much is known about the size of such sets.

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Flat extensions of abstract polytopes

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Abstract

We consider the problem of constructing an abstract $(n+1)$-polytope $\mathcal{Q}$ with k facets isomorphic to a given n-polytope $\mathcal{P}$, where $k \geq 3$. In particular, we consider the case where we want $\mathcal{Q}$ to be $(n-2, n)$-flat, meaning that every $(n-2)$-face is incident to every n-face (facet). We show that if $\mathcal{P}$ admits such a *flat extension* for a given k, then the facet graph of $\mathcal{P}$ is $(k-1)$-colorable. Conversely, we show that if the facet graph is $(k-1)$-colorable and $k-1$ is prime, then $\mathcal{P}$ admits a flat extension for that k. We also show that if $\mathcal{P}$ is facet-bipartite, then for every even k, there is a flat extension $\mathcal{P}|k$ such that every automorphism of $\mathcal{P}$ extends to an automorphism of $\mathcal{P}|k$. Finally, if $\mathcal{P}$ is a facet-bipartite n-polytope and $\mathcal{Q}$ is a vertex-bipartite m-polytope, we describe a *flat amalgamation* of $\mathcal{P}$ and $\mathcal{Q}$, an $(m+n-1)$-polytope that is $(n-2, n)$-flat, with n-faces isomorphic to $\mathcal{P}$ and co-$(n-2)$-faces isomorphic to $\mathcal{Q}$.

Keywords: Polytope, extension, amalgamation, perfect 1-factorization.

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1 Introduction

Fix an abstract n-polytope $\mathcal{P}$ and a positive integer k, and suppose that you want to glue together copies of $\mathcal{P}$ to build an $(n+1)$-polytope $\mathcal{Q}$ such that each $(n-2)$-face of $\mathcal{Q}$ is surrounded by k copies of $\mathcal{P}$. What is the smallest possible $\mathcal{Q}$?

Clearly, the best we could hope for is to use only k copies of $\mathcal{P}$, building $\mathcal{Q}$ so that every $(n-2)$-face is surrounded by every copy of $\mathcal{P}$. For which polytopes $\mathcal{P}$ and which integers k is this possible? When $k = 2$, this is always possible; this is called the *trivial extension* of $\mathcal{P}$. More generally, we will show that if k is even, then this is always possible if $\mathcal{P}$ is *facet-bipartite* (in other words, if we can color the facets with two colors such that adjacent facets have different colors). On the other hand, we will show that if $\mathcal{P}$ is not $(k-1)$-facet-colorable, then it is impossible to glue together k copies of $\mathcal{P}$ in this manner.

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The polytopes that we are working with are *abstract polytopes*, which are usually defined in terms of a poset that is similar to the face-lattice of a polytope [7]. For the constructions discussed here, it is more natural to consider polytopes as a subclass of *maniplexes*, which can be viewed as a kind of edge-colored graphs [10]. The paper [5] provides a characterization of which maniplexes are the flag graphs of polytopes, which is a key ingredient to our approach.

We start by giving some background on maniplexes and polytopes in Section 2. Then we consider the problem of building a flat extension of $\mathcal{P}$ that uses k copies in Section 3. Corollary 3.2 shows that the facet graph of $\mathcal{P}$ must be $(k-1)$-colorable. In Subsection 3.1, we will show that if $\mathcal{P}$ is facet-bipartite, then any even $k \geq 2$ will work (see Theorem 3.4) and we determine some further properties related to the automorphism group of the extension (see Proposition 3.9). Then, in Subsection 3.2, we describe a more general construction that works for any $\mathcal{P}$ whose facet graph is $(k$–$1)$-colorable, subject to some restrictions on k (see Theorem 3.14). In Section 4, we generalize the first construction in another way, building a flat amalgamation of a facet-bipartite polytope $\mathcal{P}$ and a vertex-bipartite polytope $\mathcal{Q}$. Finally, we briefly discuss some open questions that remain in Section 5.

2 Maniplexes and polytopes

Abstract polytopes are posets that, broadly speaking, look something like the incidence relation of a convex polytope or a tiling of a surface or space. Their basic theory is outlined in [7]. Another way to view a polytope is in terms of its flag graph, and in [5], Garza-Vargas and Hubard characterize which properly-edge-colored regular simple graphs are the flag graphs of abstract polytopes. Since the constructions in this paper operate on the flag graphs of polytopes, it will be natural for us to define polytopes in terms of graphs instead of posets.

Let us start with a (non-standard) definition. Let $\mathcal{G}$ be a graph whose nodes we will call *flags*. Then $\mathcal{G}$ is an n*-pre-maniplex* if it is an n-regular simple graph where the edges are colored $\{0, 1, \ldots, n-1\}$ and each flag is incident to exactly one edge of each color. For each color i and each flag Φ, we define Φ^i to be the other endpoint of the edge of color i that touches Φ, and we say that Φ^i is i*-adjacent* to Φ. We further define $\Phi^{i,j}$ to be $(\Phi^i)^j$.

If $\mathcal{G}$ is an n-pre-maniplex, then let $\mathcal{G}[i_1, \ldots, i_m]$ denote the subgraph of $\mathcal{G}$ with all of the same flags as $\mathcal{G}$ and with only the edges of colors $i_1, \ldots, i_m$. The $(i_1, \ldots, i_m)$*-color-components* of $\mathcal{G}$ are the connected components of $\mathcal{G}[i_1, \ldots, i_m]$.

In an n-pre-maniplex $\mathcal{G}$, we say that colors i and j *commute* if, for each flag Φ, $\Phi^{i,j} = \Phi^{j,i}$. Equivalently, i and j commute if $\mathcal{G}[i, j]$ is a union of 4-cycles. Note that if A and B are sets of colors such that every color in A commutes with every color in B, then whenever there is a path from Φ to Ψ using edges of colors in $A \cup B$, there must be a flag Λ such that there is a path from Φ to Λ using color set A and then a path from Λ to Ψ using color set B.

We define an n*-maniplex* to be an n-pre-maniplex such that, for every pair of colors i and j such that $|i - j| > 1$, those colors commute. For each $i \in \{0, \ldots, n-1\}$, the i*-faces* of an n-maniplex are the connected components of $\mathcal{G}[0, \ldots, i-1, i+1, \ldots, n-1]$. We say that two faces are *incident* if they have nonempty intersection. The $(n-1)$-faces of an n-maniplex are called its *facets*.

Finally, an n-maniplex is an n*-polytope* if it satisfies the following *Path Intersection Property*: for every pair of flags Φ and Ψ and every $i < j$, if there is a path between Φ and

Ψ that uses colors $i, \dots, n-1$ and another path between them that uses colors $0, \dots, j$, then there must be a path between them that uses only the colors $i, \dots, j$ (see [5, Theorem 5.3]).

In the context of graphs, an *automorphism* of an n-polytope is a graph automorphism that preserves the edge colors, and we denote the automorphism group of $\mathcal{P}$ by $\Gamma(\mathcal{P})$. In other words, φ is an automorphism of $\mathcal{P}$ if it is a bijection on the flags such that, for every flag Φ and every edge color i, we have $\Phi^i \varphi = (\Phi\varphi)^i$. If $\mathcal{P}$ and $\mathcal{Q}$ are n-polytopes, then $\mathcal{P}$ *covers* $\mathcal{Q}$ if there is a surjective graph homomorphism from $\mathcal{P}$ to $\mathcal{Q}$ that preserves the edge colors. A polytope is *regular* if the automorphism group acts transitively on the flags. The *symmetry type graph* of a polytope $\mathcal{P}$ is the quotient of $\mathcal{P}$ by the orbits of the nodes under $\Gamma(\mathcal{P})$; see [3].

The *facet graph* of a polytope $\mathcal{P}$ is a simple graph whose nodes correspond to the facets of $\mathcal{P}$, and where two nodes are connected if the corresponding facets are connected by an edge labeled $n-1$ in $\mathcal{P}$. A polytope is *facet-bipartite* if its facet graph is bipartite. Equivalently, a polytope is facet-bipartite if and only if there are no cycles in $\mathcal{P}$ with an odd number of edges labeled $n-1$.

The *dual* of a polytope $\mathcal{P}$ is the polytope $\mathcal{P}^*$ obtained by changing every edge label from i to $n-1-i$. The 1-*skeleton* of $\mathcal{P}$ is the facet graph of $\mathcal{P}^*$. That is, the nodes of the 1-skeleton correspond to the 0-faces of $\mathcal{P}$, and two nodes are connected if there is an edge labeled 1 between the corresponding faces in $\mathcal{P}$. The polytope $\mathcal{P}$ is *vertex-bipartite* if there are no cycles in $\mathcal{P}$ with an odd number of edges labeled 0.

A polytope $\mathcal{P}$ is (i, j)-*flat* if every i-face is incident to every j-face. In other words, $\mathcal{P}$ is (i, j)-flat if, for every flag Φ and every j-face, there is a path from Φ to some flag in that j-face that does not use any edges of color i.

Proposition 2.1. *Suppose $i < j$. Then the n-polytope $\mathcal{P}$ is (i, j)-flat if and only if, for every flag Φ and every j-face, there is a path from Φ to some flag in that j-face that only uses edges of colors $\{i+1, \dots, n-1\}$.*

Proof. Suppose that $\mathcal{P}$ is (i, j)-flat and consider an arbitrary flag Φ and a j-face. Suppose that Ψ is a flag in the j-face such that there is a path from Φ to Ψ that never uses color i. So the path from Φ to Ψ uses colors $\{0, \dots, i-1\}$ and $\{i+1, \dots, n-1\}$. Since these two color sets commute, there must be a flag Λ such that there is a path from Φ to Λ using colors $\{i+1, \dots, n-1\}$ and then a path from Λ to Ψ using colors $\{0, \dots, i-1\}$. Since $i < j$, the latter color set does not include j, and so Λ is in the same j-face as Ψ. Then there is a path from Φ to the j-face that only uses edges of colors $\{i+1, \dots, n-1\}$. That proves one direction, and the other direction is clear. □

3 Flat extensions

Our goal is to take k copies of an n-polytope $\mathcal{P}$ and glue them together into an $(n+1)$-polytope $\mathcal{Q}$. Furthermore, we would like for every $(n-2)$-face of $\mathcal{Q}$ to be surrounded by all k copies of $\mathcal{P}$ — in other words, we would like $\mathcal{Q}$ to be $(n-2, n)$-flat. How do we get started?

If such a polytope $\mathcal{Q}$ exists, then removing all edges labeled n yields k copies of $\mathcal{P}$. So in order to build $\mathcal{Q}$, let us take k copies of $\mathcal{P}$ (which we will call the *layers* of $\mathcal{Q}$), labeled $\mathcal{P}_1, \dots, \mathcal{P}_k$. For each flag Φ of $\mathcal{P}$, we will write Φ_i for the image of Φ in $\mathcal{P}_i$. For convenience, we will always interpet the subscripts of $\mathcal{P}_i$ and Φ_i modulo k. Now, we create $\mathcal{Q}$ from these k copies of $\mathcal{P}$ by adding a perfect matching using new edges labeled n. How do we do so in a way that ensures that $\mathcal{Q}$ is a polytope?

First we need to make sure that color n commutes with each color c in $\{0,\dots,n-2\}$. To do so, once we decide to match some flag Φ_i to Ψ_j, we must also match $(\Phi_i)^c$ to $(\Psi_j)^c$ for every $c \in \{0,\dots,n-2\}$. Applying this restriction recursively shows that the matching of flags must induce a matching of the $\{0,\dots,n-2\}$-color components, which correspond to the facets of $\mathcal{P}$. (See Figure 1.)

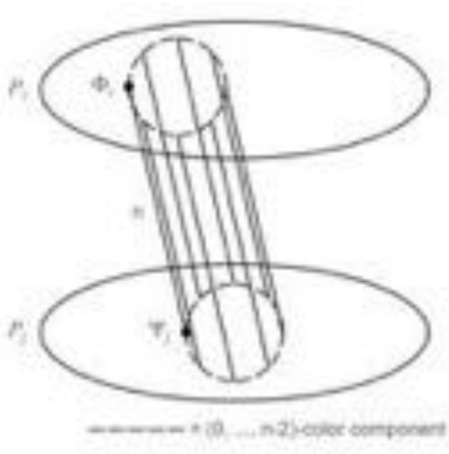

Figure 1: Matching Φ_i to Ψ_j induces a matching of the $(0,\dots,n-2)$-color components.

Next, we want $\mathcal{Q}$ to be $(n-2,n)$-flat. By Proposition 2.1, this is equivalent to making every $(n-1,n)$-color-component intersect every $\mathcal{P}_j$.

We have already observed that once we match a flag Φ, that induces a matching of Φ^c for each $c \in \{0,\dots,n-2\}$. Now we will see that requiring that $\mathcal{Q}$ be flat restricts our choice of how we match Φ^{n-1}.

Proposition 3.1. *Suppose $\mathcal{Q}$ is an $(n+1)$-polytope that is $(n-2,n)$-flat, with k facets isomorphic to $\mathcal{P}$, where $k \geq 3$. Then for every Φ_i, the flags $(\Phi_i)^n$ and $(\Phi_i)^{n-1,n}$ are in different layers $\mathcal{P}_j$.*

Proof. Suppose $(\Phi_i)^n$ and $(\Phi_i)^{n-1,n}$ are in the same layer. Then there is a path from $(\Phi_i)^n$ to $(\Phi_i)^{n-1,n}$ using edges labeled $\{0,\dots,n-1\}$. There is also a path from $(\Phi_i)^n$ to $(\Phi_i)^{n-1,n}$ using edges labeled only $n-1$ and n. Then the Path Intersection Property implies that there is a path using only edges labeled $n-1$, which means that $(\Phi_i)^{n,n-1} = (\Phi_i)^{n-1,n}$. Thus the $(n-1,n)$-color component that contains Φ_i consists of only four flags in two layers, and since $k \geq 3$ this implies that $\mathcal{Q}$ is not $(n-2,n)$-flat. $\square$

Let us reinterpret this result in terms of the facet graph of $\mathcal{P}$. For each facet of $\mathcal{P}$ (corresponding to a $(0,\dots,n-2)$-color component of $\mathcal{Q}$), consider the flags in the last layer $\mathcal{P}_k$ that are contained in that facet. By the discussion earlier, all of these flags are matched to flags in some single layer $\mathcal{P}_i$ with $i \in \{1,\dots,k-1\}$. Then we may color each facet of $\mathcal{P}$ by that number i, and Proposition 3.1 implies that this is a *proper* coloring! Therefore,

Corollary 3.2. *Let $k \geq 3$. If $\mathcal{P}$ is an n-polytope such that its facet graph is not $(k-1)$-colorable, then there are no $(n+1)$-polytopes $\mathcal{Q}$ with k facets isomorphic to $\mathcal{P}$ such that $\mathcal{Q}$ is $(n-2, n)$-flat.*

Example 3.3. Since the facet graph of the n-simplex is the complete graph K_{n+1}, there are no $(n-2, n)$-flat $(n+1)$-polytopes $\mathcal{Q}$ with $n+1$ simplicial facets.

3.1 Flat extensions of facet-bipartite polytopes

When trying to define a matching in order to build $\mathcal{Q}$, the most straightforward way would be for each Φ_i to be matched to some Φ_j. That is, each flag is matched to the 'same' flag in a different layer. The easiest such matching would have each flag Φ_i matched to either Φ_{i-1} or Φ_{i+1}. (Recall that the subscripts are interpreted modulo k, so that Φ_1 could be matched to Φ_k.) Then the argument for Corollary 3.2 works in essentially the same way to show that, since each layer is matched to only two other layers, $\mathcal{P}$ must be facet-bipartite in order for this to work. We will show that this necessary condition is also sufficient.

So, suppose that $\mathcal{P}$ is a facet-bipartite n-polytope, and let k be an even positive integer. Given a proper coloring of the facet graph of $\mathcal{P}$ with two colors (say red and blue), we can color each flag of $\mathcal{P}$ according to the color of its facet. Then, for each red flag Φ, we will match Φ_1 to Φ_2, Φ_3 to Φ_4, and so on. For each blue flag Ψ, we will match Ψ_2 to Ψ_3, Ψ_4 to Ψ_5, and so on. We refer to the graph that we obtain by $\mathcal{P}|k$. (See Figure 2.)

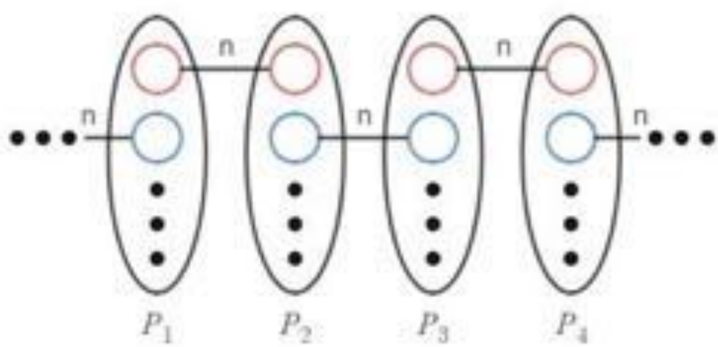

Figure 2: Flags are matched according to the coloring of the facet graph of $\mathcal{P}$.

First, let us show that this construction really yields a polytope with the desired properties.

Theorem 3.4. *The graph $\mathcal{P}|k$ is an $(n-2, n)$-flat $(n+1)$-polytope with k facets isomorphic to $\mathcal{P}$.*

Proof. By construction, it is clear that $\mathcal{P}|k$ has k facets isomorphic to $\mathcal{P}$. If Φ is a red flag and $\Psi = \Phi^{n-1}$, then Ψ is blue and the $(n-1, n)$-color component that contains Φ_1 is the cycle

$$(\Phi_1, \Phi_2, \Psi_2, \Psi_3, \Phi_3, \Phi_4, \ldots, \Psi_k, \Psi_1),$$

which intersects every layer. It is clear then that each $(n-1, n)$-color component intersects every $\mathcal{P}_i$, and so $\mathcal{P}|k$ is $(n-2, n)$-flat. It is also clear that $\mathcal{P}|k$ is a maniplex, since we forced the new edges labeled n to commute with the edges labeled $0, 1, \ldots, n-2$.

It remains to show that $\mathcal{P}|k$ is a polytope by showing that it satisfies the Path Intersection Property. Consider colors i and j satisfying $0 \leq i < j \leq n$. Suppose there are two flags such that there is a path between them using colors $0, \ldots, j-1$ and $i+1, \ldots, n$. Since $j - 1 < n$, it follows that the two flags are in the same layer, and without loss of generality we will assume they are in layer 1. So there are two flags Φ_1 and Ψ_1 such that there is a path between them that uses colors $i+1, \ldots, n$. Since edges of color n always connect two flags with the same underlying flag in $\mathcal{P}$, such a path induces a path between Φ and Ψ in $\mathcal{P}$ that only uses colors $i+1, \ldots, n-1$. Similarly, there is an induced path between Φ and Ψ in $\mathcal{P}$ that uses colors $0, \ldots, j-1$. Then, since $\mathcal{P}$ is a polytope, it follows that there is a path from Φ to Ψ that uses colors $i+1, \ldots, j-1$, and then this path also lifts to an isomorphic path from Φ_1 to Ψ_1 using only those colors, as desired. $\square$

Example 3.5. If $\mathcal{P}$ is the unique 1-polytope, then $\mathcal{P}|k$ is a k-gon.

Example 3.6. If $\mathcal{P}$ is a square, then $\mathcal{P}|4$ is the map $\{4, 4\}_{(2,0)}$ on the torus (see [7, Section 1D]).

Example 3.7. If $k = 2$, then we don't even need for $\mathcal{P}$ to be facet-bipartite — we can just match each Φ_1 to Φ_2. Indeed, $\mathcal{P}|2$ is the *trivial extension* of $\mathcal{P}$, also denoted $\{\mathcal{P}, 2\}$.

Example 3.8. Nothing goes wrong if we try $k = \infty$ and index the layers $\mathcal{P}_i$ by letting i be any integer. We still get an $(n-2, n)$-flat polytope with infinitely many facets isomorphic to $\mathcal{P}$.

Now let us determine the automorphism group of $\mathcal{P}|k$. Fix a base flag Φ of $\mathcal{P}$, and consider an automorphism φ of $\mathcal{P}$ that sends Φ to Ψ. Can we extend φ to an automorphism $\tilde{\varphi}$ of $\mathcal{P}|k$?

Without loss of generality, let us assume that Φ is red. Then the other red flags are those that can be reached from Φ using an even number of edges labeled $n-1$, and the blue flags are those that can be reached from Φ using an odd number of edges labeled $n-1$. Furthermore, φ respects these color classes since, for each flag Λ, we have $\Lambda^{n-1}\varphi = (\Lambda\varphi)^{n-1}$.

Now, if Ψ is also red, then φ preserves the color of every flag. Then we define $\tilde{\varphi}$ so that, for each flag Λ of $\mathcal{P}$,

$$(\Lambda_i)\tilde{\varphi} = (\Lambda\varphi)_i.$$

In other words, $\tilde{\varphi}$ fixes each layer setwise, and acts on each layer in the same way that φ acts on $\mathcal{P}$. To see that this defines an automorphism, it suffices to show that $\tilde{\varphi}$ preserves the edges of color n, and this is true since

$$(\Lambda_i)^n\tilde{\varphi} = \Lambda_{i\pm1}\tilde{\varphi} = (\Lambda\varphi)_{i\pm1} = ((\Lambda\varphi)_i)^n = (\Lambda_i\tilde{\varphi})^n.$$

If Ψ is blue instead, then the action of φ on $\mathcal{P}$ reverses the color of every flag. Then we define $\tilde{\varphi}$ so that, for each flag Λ of $\mathcal{P}$,

$$(\Lambda_i)\tilde{\varphi} = (\Lambda\varphi)_{k+2-i}.$$

Again, this will define an automorphism if and only if $\tilde{\varphi}$ preserves the edges of color n, and this is true since

$$(\Lambda_i)^n\tilde{\varphi} = \Lambda_{i\pm 1}\tilde{\varphi} = (\Lambda\varphi)_{k+2-i\mp 1} = ((\Lambda\varphi)_{k+2-i})^n = (\Lambda_i\tilde{\varphi})^n,$$

where the third equality follows because $\Lambda\varphi$ is the opposite color of Λ, and so the matching of $\Lambda\varphi$ is in the opposite direction of the matching of Λ (that is, $\mp$ instead of $\pm$). So in either case, we see that each automorphism of $\mathcal{P}$ lifts to an automorphism of $\mathcal{P}|k$; in other words, $\mathcal{P}|k$ is *hereditary* (see [8]).

In addition to these automorphisms $\tilde{\varphi}$, which all fix the first layer setwise, there are automorphisms of $\mathcal{P}|k$ that simply permute the layers. Indeed, it is clear from the symmetry of the graph (see Figure 2) that there is an automorphism α that sends each Λ_i to Λ_{k+3-i} and an automorphism β that sends each Λ_i to Λ_{k+5-i} (with the subscripts of Λ reduced modulo k). The subgroup $\langle\alpha,\beta\rangle$ acts transitively on the layers, and the orbit of the flag Λ_1 is all flags Λ_j.

We can now characterize the automorphism group of $\mathcal{P}|k$.

Proposition 3.9. *Let $\mathcal{P}$ be a facet-bipartite n-polytope and let k be a positive even integer. Let $\tilde{\varphi}$, α and β be defined as above.*

(a) *$\mathcal{P}|k$ is hereditary.*

(b) $\Gamma(\mathcal{P}|k) \cong \Gamma(\mathcal{P}) \ltimes \langle\alpha,\beta\rangle$.

(c) *The symmetry type graph of $\mathcal{P}|k$ is obtained from the symmetry type graph of $\mathcal{P}$ by adding semi-edges labeled n to each node. In particular, $\mathcal{P}|k$ is regular if and only if $\mathcal{P}$ is regular.*

Proof. The first part was already proved in the previous discussion. For the second part, let us first show that every automorphism in $\Gamma(\mathcal{P}|k)$ may be written as $\tilde{\varphi}\gamma$, with $\varphi \in \Gamma(\mathcal{P})$ and $\gamma \in \langle\alpha,\beta\rangle$. Fix a base flag Φ of $\mathcal{P}$, and suppose that an automorphism ψ of $\mathcal{P}|k$ sends Φ_1 to Ψ_j. Then there must be an automorphism φ of $\mathcal{P}$ that sends Φ to Ψ, and the induced automorphism $\tilde{\varphi}$ sends Φ_1 to Ψ_1. Then there is some $\gamma \in \langle\alpha,\beta\rangle$ that sends Ψ_1 to Ψ_j, and so $\tilde{\varphi}\gamma$ sends Φ_1 to Ψ_j. Since polytope automorphisms are determined by their action on any one flag, this shows that $\psi = \tilde{\varphi}\gamma$.

Next, we note that α and β both only change the subscript of a flag independently of the underlying flag of $\mathcal{P}$. Similarly, $\tilde{\varphi}$ only changes the underlying flag, independent of the subscript. So if $\gamma \in \langle\alpha,\beta\rangle$, then $\tilde{\varphi}^{-1}\gamma\tilde{\varphi}$ also only changes the subscript of each flag independently of the underlying flag, and so $\tilde{\varphi}^{-1}\gamma\tilde{\varphi} \in \langle\alpha,\beta\rangle$. So $\langle\alpha,\beta\rangle$ is normal in $\Gamma(\mathcal{P}|k)$. Finally, since each $\tilde{\varphi}$ fixes the first layer setwise, whereas no nontrivial element of $\langle\alpha,\beta\rangle$ fixes the first layer, we find that $\langle\alpha,\beta\rangle \cap \Gamma(\mathcal{P}) = \langle 1\rangle$, and so $\Gamma(\mathcal{P}|k) \cong \Gamma(\mathcal{P}) \ltimes \langle\alpha,\beta\rangle$.

For the last part, note that the orbit of each Λ_i under $\langle\alpha,\beta\rangle$ consists of all k flags of the form Λ_j, and so these flags are all identified under the quotient by $\Gamma(\mathcal{P}|k)$. In particular, each flag is in the same orbit as its n-adjacent flag. Furthermore, any pair of flags Φ_i and Ψ_j that lie in the same orbit must have underlying flags Φ and Ψ that lie in the same orbit of $\Gamma(\mathcal{P})$, and so the symmetry type graph of $\mathcal{P}|k$ is just the symmetry type graph of $\mathcal{P}$ with extra semi-edges labeled n at each node. $\square$

Let us now show some nice properties of $\mathcal{P}|k$ related to covers.

Proposition 3.10. *If $\mathcal{P}$ and $\mathcal{Q}$ are facet-bipartite polytopes such that $\mathcal{Q}$ covers $\mathcal{P}$, then $\mathcal{Q}|k$ covers $\mathcal{P}|k$ for every even positive integer k.*

Proof. To say that $\mathcal{Q}$ covers $\mathcal{P}$ is to say that there is a color-preserving graph epimorphism φ from $\mathcal{Q}$ to $\mathcal{P}$. Fix a flag Ψ of $\mathcal{Q}$ and let $\Phi = (\Psi)\varphi$. Without loss of generality, we may color both Φ and Ψ red, so that Φ_1 is matched to Φ_2 and Ψ_1 is matched to Ψ_2. Then the obvious extension of φ that acts separately on each layer of $\mathcal{Q}|k$ will also respect the edges of color n, and thus $\mathcal{Q}|k$ covers $\mathcal{P}|k$. □

Proposition 3.11. *If $\mathcal{P}$ is a facet-bipartite polytope and k_1 and k_2 are positive even integers with k_2 a multiple of k_1, then $\mathcal{P}|k_2$ covers $\mathcal{P}|k_1$. In particular, for every even positive integer k, the polytope $\mathcal{P}|k$ covers the trivial extension $\mathcal{P}|2$.*

Proof. The function taking each Φ_i to $\Phi_{i(\text{mod } k_1)}$ is a color-preserving graph epimorphism. (But note that we denote Φ_0 by Φ_k instead.) □

Next, we note that it is possible to repeatedly apply this construction:

Proposition 3.12. *If $\mathcal{P}$ is a facet-bipartite n-polytope, then for every finite sequence $k_1, \dots, k_m$ with each k_i a positive even integer, there is a facet-bipartite polytope $\mathcal{Q} = \mathcal{P}|k_1|k_2|\cdots|k_m$ that is $(i, i+2)$-flat for each i in $\{n-2, \dots, n+m-3\}$. Furthermore, $\mathcal{Q}$ is regular if $\mathcal{P}$ is regular.*

Proof. The first part follows immediately from the fact that the facet graph of $\mathcal{P}|k$ is an even cycle (consisting of the k layers $\mathcal{P}_i$), and so $\mathcal{P}|k$ is facet-bipartite. The second part follows from Proposition 3.9(c). □

Example 3.13. For any sequence of positive even integers $k_1, \dots, k_m$, we can take $\mathcal{P}$ to be a k_1-gon and then extend it by $k_2, \dots, k_m$. This yields a regular $(m+1)$-polytope that is $(i, i+2)$-flat for each i in $\{0, \dots, m-1\}$. In fact, this is a *tight polytope of type* $\{k_1, \dots, k_m\}$; see [2].

3.2 Flat extensions of other polytopes

We have seen that if $\mathcal{P}$ is facet-bipartite, then there is a straightforward matching on k copies of $\mathcal{P}$ that yields a polytope $\mathcal{P}|k$. What can we do with other polytopes $\mathcal{P}$? Let us fix an even $k \geq 4$ and try to build an $(n–2, n)$-flat $(n+1)$-polytope $\mathcal{Q}$ with k-facets isomorphic to $\mathcal{P}$. As before, we will focus on the case where each flag Φ_i is matched to some Φ_j.

First, recall that Corollary 3.2 says that in order for $\mathcal{Q}$ to exist, $\mathcal{P}$ must be $(k-1)$-facet-colorable. Naturally, we wonder whether this necessary condition is also sufficient. Suppose μ is a proper coloring of the facet graph of $\mathcal{P}$, with colors $1, 2, \dots, k-1$, (though some colors may not be used). As before, we can extend this to a (non-proper) coloring of $\mathcal{P}$ itself by coloring each flag according to the color of its facet. Take k copies of $\mathcal{P}$ as before: $\mathcal{P}_1, \dots, \mathcal{P}_k$, with each Φ_i colored the same as Φ. For each color c, we designate a perfect matching σ_c of the layers, and if Φ is color c, then we match Φ_i to $\Phi_{\sigma_c(i)}$. Since μ is a proper coloring of the facet graph, this ensures that flags in $\mathcal{P}_k$ that are $(n-1)$-adjacent are matched to distinct layers, as required (see Proposition 3.1).

To determine whether the matchings σ_c satisfy the desired properties, it is helpful to represent them using a new graph called the *layer graph*. This is a graph on k nodes, corresponding to the k layers $\mathcal{P}_1, \dots, \mathcal{P}_k$, where there is an edge of color c between two nodes if σ_c matches the corresponding layers. See Figure 3 for an example with $k = 6$.

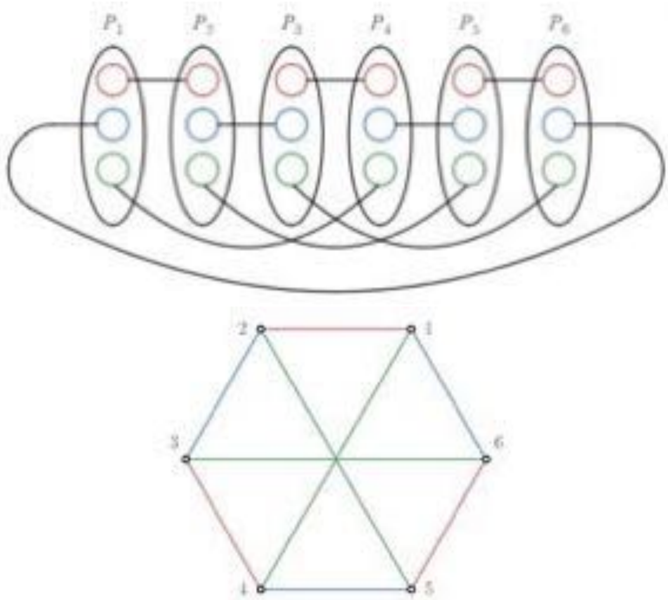

Figure 3: A matching of layers by color (above) and the corresponding layer graph (below).

Our goal is to pick matchings so that we obtain an $(n-2,n)$-flat $(n+1)$-polytope. Recall that to be $(n-2,n)$-flat means that, for every flag Φ_i, the cycle that starts from Φ_i and follows edges labeled $n-1$ and n should intersect every layer. Note that such a cycle consists only of flags of the form Φ_j and Ψ_j, where $\Psi = \Phi^{n-1}$. Therefore, the cycle is completely determined by the matchings corresponding to the colors of Φ and Ψ. Thus, if every pair of matchings of the layers yields a single cycle, then the result will be $(n-2,n)$-flat. In terms of the layer graph, this means that it suffices for every pair of colors to yield a single cycle. Such a collection of matchings is called a *perfect* 1*-factorization* of the graph. Kotzig conjectured in 1964 that every complete graph on an even number of vertices has a perfect 1-factorization [6]. This conjecture remains open; see [9] for a recent survey on this and related problems.

In any case, let us suppose that the complete graph K_k admits a perfect 1-factorization, and match flags Φ_i accordingly. As discussed, this will give us something that is $(n-2,n)$-flat. We still need to demonstrate that it is a polytope.

Theorem 3.14. *Let k be a positive even integer, $k \geq 4$, and let $\mathcal{P}$ be $(k-1)$-facet-colorable. Suppose that the complete graph K_k has a perfect* 1*-factorization. Then the preceding construction defines a polytope.*

Proof. Let $\mathcal{G}$ be the graph defined above. First, let us show that it is connected. The facet graph of $\mathcal{P}$ must use at least two colors, and by construction, the matchings corresponding to those two colors must induce a cycle that intersects each layer. Since each layer is connected, this shows that $\mathcal{G}$ is itself connected.

The remainder of the proof is analogous to the proof of Theorem 3.4. The key element is that each Φ_i is matched to some Φ_j – that is, each flag is matched to "itself" in another layer. □

Example 3.15. If $k-1$ is prime or $k/2$ is prime, then there is a perfect 1-factorization of K_k; see [6] and [1], respectively. Thus, every finite polytope $\mathcal{P}$ has infinitely many flat extensions — simply take $k-1$ to be a prime that is greater than or equal to the number of facets of $\mathcal{P}$.

4 Flat amalgamations

There is another way of thinking about $\mathcal{P}|k$ that readily admits one last generalization. It starts with seeing $\mathcal{P}|k$ as a *mix* of $\mathcal{P}$ with the flag graph of a k-gon. A similar construction for regular polytopes was described in [7, Section 4F], using their automorphism groups instead of their flag graphs. For non-regular polytopes, the construction may provide different results depending on the choice of a base flag, and so we define the construction using *rooted polytopes* $(\mathcal{P}, \Phi)$ (see [4]).

Definition 4.1. Suppose that $\mathcal{P}$ is an n-polytope with base flag Φ and that $\mathcal{Q}$ is an m-polytope with base flag Ψ. Let $0 \leq r \leq n-1$ with also $r \geq n-m$. Then the *r-mix of* $(\mathcal{P}, \Phi)$ *with* $(\mathcal{Q}, \Psi)$, denoted $(\mathcal{P}, \Phi) \diamond_r (\mathcal{Q}, \Psi)$, is the connected, properly edge-colored, $(m+r)$-regular graph $\mathcal{M}$ defined as follows.

(a) The base flag of $\mathcal{M}$ is the pair (Φ, Ψ).

(b) For each $i \in \{0, \dots, m+r-1\}$ and for each flag (Λ, Δ) of $\mathcal{M}$ (with Λ a flag of $\mathcal{P}$ and Δ a flag of $\mathcal{Q}$), we define $(\Lambda, \Delta)^i$ to be $(\Lambda^i, \Delta^{i-r})$, with the understanding that if a superscript is "out of bounds" then we treat it as empty. In other words:

$$(\Lambda, \Delta)^i = \begin{cases} (\Lambda^i, \Delta) & \text{if } 0 \leq i < r, \\ (\Lambda^i, \Delta^{i-r}) & \text{if } r \leq i \leq n-1, \\ (\Lambda, \Delta^{i-r}) & \text{if } n \leq i \leq m+r-1 \end{cases}.$$

(c) The flags of $\mathcal{M}$ are all pairs (Λ, Δ) that are in the same connected component as (Φ, Ψ).

Definition 4.2. Suppose that $\mathcal{P}$ is an n-polytope and that $\mathcal{Q}$ is an m-polytope. Then the *flat amalgamation of* $(\mathcal{P}, \Phi)$ *with* $(\mathcal{Q}, \Psi)$ is $(\mathcal{P}, \Phi) \diamond_{n-1} (\mathcal{Q}, \Psi)$, denoted $(\mathcal{P}, \Phi)|(\mathcal{Q}, \Psi)$. If the base flags are understood in context, then we simply write $\mathcal{P}|\mathcal{Q}$. Note that, for each $i \in \{0, \dots, m+n-2\}$ and for each flag (Λ, Δ),

$$(\Lambda, \Delta)^i = \begin{cases} (\Lambda^i, \Delta) & \text{if } 0 \leq i < n-1, \\ (\Lambda^{n-1}, \Delta^0) & \text{if } i = n-1, \\ (\Lambda, \Delta^{i-n+1}) & \text{if } n \leq i \leq m+n-2 \end{cases}.$$

Recall that $\mathcal{P}$ is facet-bipartite if and only if there are no cycles in $\mathcal{P}$ with an odd number of edges labeled $n-1$, and that $\mathcal{Q}$ is vertex-bipartite if and only if there are no cycles in $\mathcal{Q}$ with an odd number of edges labeled 0.

Proposition 4.3. *Let $\mathcal{P}$ be an n-polytope with base flag Φ and let $\mathcal{Q}$ be an m-polytope with base flag Ψ. Let $\mathcal{M} = \mathcal{P}|\mathcal{Q}$.*

(a) *Each connected component of* $\mathcal{M}[0,\dots,n-1]$ *is isomorphic to* $\mathcal{P}$ *if and only if* $\mathcal{P}$ *is facet-bipartite.*

(b) *Each connected component of* $\mathcal{M}[n-1,\dots,m+n-2]$ *is isomorphic to* $\mathcal{Q}$ *(with edge labels increased by* $n-1$*) if and only if* $\mathcal{Q}$ *is vertex-bipartite.*

Proof. Without loss of generality, consider the connected component of $\mathcal{M}[0,\dots,n-1]$ that contains (Φ,Ψ). Recall that for $i<n-1$ we have that $(\Lambda,\Delta)^i=(\Lambda^i,\Delta)$, and so each flag in this connected component has either the form (Λ,Ψ) or (Λ,Ψ^0). Now let $\pi:\mathcal{M}\to\mathcal{P}$ be the projection in the first coordinate, sending each (Λ,Δ) to Λ. Since $\mathcal{P}$ is an n-polytope and we have edges of labels 0 through $n-1$, π is surjective. Furthermore, π will be injective (and thus bijective) if and only if there is no flag Λ such that both (Λ,Ψ) and (Λ,Ψ^0) are in the connected component. A path from (Λ,Ψ) to (Λ,Ψ^0) exists if and only if there is a cycle in $\mathcal{P}$ that includes Λ and has an odd number of edges labeled $n-1$. Thus, π is bijective if and only if no such cycle exists, which is to say if and only if $\mathcal{P}$ is facet-bipartite.

The proof of the second part is analogous. □

In the usual language of polytopes, we say that if $\mathcal{P}$ is facet-bipartite and $\mathcal{Q}$ is vertex-bipartite, then the n-faces of $\mathcal{P}|\mathcal{Q}$ are isomorphic to $\mathcal{P}$ and the co-$(n-2)$-faces are isomorphic to $\mathcal{Q}$.

We now collect a few properties of $\mathcal{P}|\mathcal{Q}$. Let $\mathcal{F}(\mathcal{M})$ denote the set of flags of the maniplex $\mathcal{M}$. As in Subsection 3.1, we can properly color the facet graph of $\mathcal{P}$ with two colors, and then extend this coloring to the flag graph. Similarly, we can properly color the 1-skeleton of $\mathcal{Q}$ with two colors and extend this coloring to the flag graph.

Proposition 4.4. *Let* $\mathcal{P}$ *be a facet-bipartite* n*-polytope with base flag* Φ *and let* $\mathcal{Q}$ *be a vertex-bipartite* m*-polytope with base flag* Ψ*. Color the flags of* $\mathcal{P}$ *red and blue according to a bipartition of its facet graph, and color the flags of* $\mathcal{Q}$ *red and blue according to a bipartition of its* 1*-skeleton, and let us assume that* Φ *and* Ψ *are both red.*

(a) $\mathcal{F}(\mathcal{P}|\mathcal{Q})=\{(\Lambda,\Delta)\in\mathcal{F}(\mathcal{P})\times\mathcal{F}(\mathcal{Q}):\ \Lambda$ *and* Δ *are the same color*$\}$.

(b) $|\mathcal{F}(\mathcal{P}|\mathcal{Q})|=\frac{1}{2}|\mathcal{F}(\mathcal{P})|\cdot|\mathcal{F}(\mathcal{Q})|$.

(c) $\mathcal{P}|\mathcal{Q}$ *is* $(n-2,n)$*-flat.*

Proof. Suppose that (Λ,Δ) is a flag of $\mathcal{P}|\mathcal{Q}$. By the definition of $(\Lambda,\Delta)^j$, either both components change color (when $j=n-1$) or neither component changes color. Since $\mathcal{P}|\mathcal{Q}$ consists of only those flags that are reachable from (Φ,Ψ), which are both red, it follows that all flags of $\mathcal{P}|\mathcal{Q}$ have the same color in both components.

Now, suppose that Λ and Δ are arbitrary flags of $\mathcal{P}$ and $\mathcal{Q}$ (respectively) that are the same color. There is a path in $\mathcal{P}$ from Φ to Λ, and this induces a path in $\mathcal{P}|\mathcal{Q}$ that uses only edges of colors in $\{0,\dots,n-1\}$. Such a path will either take (Φ,Ψ) to (Λ,Ψ) or to (Λ,Ψ^0). In the latter case, we may follow an additional edge labeled $n-1$ to arrive at (Λ^{n-1},Ψ). Now, there is a path in $\mathcal{Q}$ from Ψ to Δ, and this induces a path in $\mathcal{P}|\mathcal{Q}$ that uses only edges of colors in $\{n-1,\dots,m+n-2\}$. Such a path will take us from (Λ,Ψ) or (Λ^{n-1},Ψ) to (Λ,Δ) or (Λ^{n-1},Δ). By the previous paragraph, since Λ^{n-1} has a different color to Δ, the flag (Λ^{n-1},Δ) cannot be in $\mathcal{P}|\mathcal{Q}$, and so we have found a path from (Φ,Ψ) to (Λ,Δ), proving that the latter is a flag of $\mathcal{P}|\mathcal{Q}$. The second part follows immediately from the first.

For the third part, we need to show that, given flags flags (Φ, Ψ) and (Λ, Δ) of $\mathcal{P}|\mathcal{Q}$, there is a path from (Φ, Ψ) to (Λ, Δ) that can be written as the concatenation of a path that never uses color n with a path that never uses color n–2. The path described in the previous paragraph already satisfies this condition. $\square$

Theorem 4.5. *Let $\mathcal{P}$ be a facet-bipartite n-polytope and let $\mathcal{Q}$ be a vertex-bipartite m-polytope. Let $\mathcal{M} = \mathcal{P}|\mathcal{Q}$. Then $\mathcal{M}$ is an $(m+n-1)$-polytope that is $(n-2, n)$-flat.*

Proof. It is straightforward to check that if i and j are in $\{0, \ldots, m+n-2\}$ with $|i-j| > 1$, then $\mathcal{M}[i, j]$ consists of 4-cycles; this shows that $\mathcal{M}$ is a maniplex. Flatness was proved in Proposition 4.4. To show that $\mathcal{M}$ is a polytope, it suffices to show that it satisfies the Path Intersection Property. Consider two arbitrary flags of $\mathcal{M}$, say (Φ, Ψ) and (Λ, Δ). Suppose that there is a path from (Φ, Ψ) to (Λ, Δ) that uses only colors in $\{0, \ldots, j\}$ and another path that uses only colors in $\{i, \ldots, m+n-2\}$. We want to show that there must be a path that uses only the colors $\{i, \ldots, j\}$.

Since colors greater than n–1 do not affect the first component, the path that uses colors in $\{i, \ldots, m+n–2\}$ induces a path in $\mathcal{P}$ from Φ to Λ that uses colors in $\{i, \ldots, n–1\}$. Since colors less than $n-1$ do not affect the second component, following the same sequence of colors in $\mathcal{M}$ gives us a path from (Φ, Ψ) to either (Λ, Ψ) or (Λ, Ψ^0). In the latter case, we can follow one more edge of color $n-1$ to arrive at $(\Lambda^{n–1}, \Psi)$. Now, the path from (Φ, Ψ) to (Λ, Δ) that uses colors in $\{0, \ldots, j\}$ induces a path from Ψ to Δ that uses colors in $\{n-1, \ldots, j\}$, and following this sequence of colors in $\mathcal{M}$ gives us a path from wherever we stopped (either (Λ, Ψ) or $(\Lambda^{n–1}, \Psi)$) to either (Λ, Δ) or $(\Lambda^{n–1}, \Delta)$. Since we supposed that (Λ, Δ) was a flag of $\mathcal{M}$, Proposition 4.4 implies that $(\Lambda^{n–1}, \Delta)$ is not a flag of $\mathcal{M}$, and so we must have arrived at (Λ, Δ). Thus, we have a path from (Φ, Ψ) to (Λ, Δ) that only uses colors in $\{i, \ldots, n-1\} \cup \{n-1, \ldots, j\} = \{i, \ldots, j\}$, as desired. $\square$

Example 4.6. If $\mathcal{Q}$ is a k-gon with k even, then $\mathcal{P}|\mathcal{Q} \cong \mathcal{P}|k$. Essentially, each flag of the k-gon corresponds to a choice of one of the k layers and one of the colors red or blue.

Proposition 4.7. *Let $\mathcal{P}$ be a facet-bipartite n-polytope and let $\mathcal{Q}$ be a vertex-bipartite m-polytope. If $\mathcal{Q}$ is facet-bipartite, then $\mathcal{P}|\mathcal{Q}$ is facet-bipartite.*

Proof. If there is a cycle in $\mathcal{P}|\mathcal{Q}$ with an odd number of edges labeled $m+n$–2, this induces a cycle in $\mathcal{Q}$ with an odd number of edges labeled $m-1$. $\square$

Proposition 4.7 implies that, if $\mathcal{Q}_1, \ldots, \mathcal{Q}_k$ are all vertex-bipartite and facet-bipartite, then we may construct a flat amalgamation $\mathcal{P}|\mathcal{Q}_1|\cdots|\mathcal{Q}_k$.

Finally, let us determine the automorphism group of $\mathcal{P}|\mathcal{Q}$. Given an automorphism φ of $\mathcal{P}$ that sends Φ to Λ, let us say that φ is $(n-1)$-*even* (respectively $(n-1)$-odd) if the number of edges labeled $n-1$ in any path from Φ to Λ is even (respectively odd). (As long as $\mathcal{P}$ is facet-bipartite, this is well-defined.) We will similarly define automorphisms of $\mathcal{Q}$ to be 0-even or 0-odd.

Theorem 4.8. *Let $\mathcal{P}$ be a facet-bipartite n-polytope with base flag Φ and let $\mathcal{Q}$ be a vertex-bipartite m-polytope with base flag Ψ. Then*

$$\Gamma(\mathcal{P}|\mathcal{Q}) = \{(\varphi, \psi) \in \Gamma(\mathcal{P}) \times \Gamma(\mathcal{Q}) : \varphi \text{ is } (n-1)\text{-even if and only if } \psi \text{ is } 0\text{-even}\}.$$

In particular, if all automorphisms of $\mathcal{P}$ are $(n-1)$-even and all automorphisms of $\mathcal{Q}$ are 0-even, then $\Gamma(\mathcal{P}|\mathcal{Q}) = \Gamma(\mathcal{P}) \times \Gamma(\mathcal{Q})$, and otherwise $\Gamma(\mathcal{P}|\mathcal{Q})$ is an index-2 subgroup of $\Gamma(\mathcal{P}) \times \Gamma(\mathcal{Q})$.

Proof. Clearly, each automorphism of $\Gamma(\mathcal{P}|\mathcal{Q})$ induces an automorphism φ of $\mathcal{P}$ and an automorphism ψ of $\mathcal{Q}$, and so $\Gamma(\mathcal{P}|\mathcal{Q}) \leq \Gamma(\mathcal{P}) \times \Gamma(\mathcal{Q})$. Conversely, given automorphisms φ and ψ, we may try to build an automorphism (φ, ψ) of $\mathcal{P}|\mathcal{Q}$ that acts component-wise. Clearly, this will only work if $(\Phi\varphi, \Psi\psi)$ is in $\mathcal{P}|\mathcal{Q}$, and this is true if and only if the parity of the number of edges labeled $n-1$ from Φ to $\Phi\varphi$ is the same as the parity of the number of edges labeled 0 from Ψ to $\Psi\psi$. If that is the case, then note that for each flag (Λ, Δ),

$$\begin{aligned}(\Lambda, \Delta)^i(\varphi, \psi) &= (\Lambda^i, \Delta^{i-n+1})(\varphi, \psi) \\ &= (\Lambda^i\varphi, \Delta^{i-n+1}\psi) \\ &= ((\Lambda\varphi)^i, (\Delta\psi)^{i-n+1}) \\ &= (\Lambda\varphi, \Delta\psi)^i,\end{aligned}$$

proving that (φ, ψ) is an automorphism. That proves the first part and the second follows immediately. □

Example 4.9. Suppose $\mathcal{P}$ is the cuboctahedron and $\mathcal{Q}$ is its dual, the rhombic dodecahedron. Then $\mathcal{P}$ is facet-bipartite: we can color all of the square faces with one color and the triangles with another. Every automorphism of $\mathcal{P}$ is 2-even. Similarly, $\mathcal{Q}$ is vertex-bipartite, and its automorphisms are all 0-even. Thus $\Gamma(\mathcal{P}|\mathcal{Q}) = \Gamma(\mathcal{P}) \times \Gamma(\mathcal{Q})$, a group of order 48^2.

5 Conclusions

We have shown that every finite polytope $\mathcal{P}$ has a flat extension, where we glue together an even number of copies of $\mathcal{P}$ in a flat way. The strategy used does not work if we want to use an odd number of copies of $\mathcal{P}$. In particular, if we use an odd number of copies, then we cannot match each flag Φ_i to some Φ_j — some flags Φ_i must get matched to Ψ_j with $\Phi \neq \Psi$. When is this possible and how can we do this in a consistent way?

Problem 5.1. Describe a construction that takes an n-polytope $\mathcal{P}$ and produces an $(n-2, n)$-flat $(n+1)$-polytope with 3 facets all isomorphic to $\mathcal{P}$. What restrictions on $\mathcal{P}$ are there?

Another interesting problem would be to further investigate the properties of the flat extensions that were described in Subsection 3.2.

Problem 5.2. Determine the automorphism groups of the flat extensions described in Subsection 3.2.

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Locally spherical hypertopes from generalised cubes*

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Abstract

We show that every non-degenerate regular polytope can be used to construct a thin, residually-connected, chamber-transitive incidence geometry, i.e. a regular hypertope. These hypertopes are related to the semi-regular polyotopes with a tail-triangle Coxeter diagram constructed by Monson and Schulte. We discuss several interesting examples derived when this construction is applied to generalised cubes. In particular, we produce an example of a rank 5 finite locally spherical proper hypertope of hyperbolic type. No such examples were previously known.

Keywords: Regularity, thin geometries, hypermaps, hypertopes, abstract polytopes.

Math. Subj. Class.: 52B15, 51E24, 51G05

1 Introduction

Hypertopes are a special type of incidence geometries that generalise the notions of abstract polytopes and of hypermaps. The concept was introduced in [9] with particular emphasis on regular hypertopes (that is, the ones with highest degree of symmetry). Although in [8, 10, 11] a number of interesting examples of regular hypertopes have been constructed, within the theory of abstract regular polytopes much more work has been done. Notably, [26] and [28] deal with universal constructions of polytopes, while in [5, 23, 24] some constructions with prescribed combinatorial conditions are explored. In another direction, in [3, 7, 14, 22] the questions of existence of polytopes with prescribed (interesting) groups

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are investigated. Much of the impetus to the development of the theory of abstract polytopes, as well as the inspiration with the choice of problems, was based on work of Branko Grünbaum [13] from the 1970s.

In this paper we generalise the halving operation on polyhedra (see 7B in [18]) to a certain class of regular abstract polytopes to construct regular hypertopes. More precisely, given a regular non-degenerate n-polytope $\mathcal{P}$, we construct a regular hypertope $\mathcal{H}(\mathcal{P})$ whose type-preserving automorphism group is a subgroup of $\mathrm{Aut}(\mathcal{P})$ of index at most 2. The hypertope $\mathcal{H}(\mathcal{P})$, as we shall see in Section 3, is closely related to the semi-regular polytopes with tail-triangle Coxeter diagram described by Monson and Schulte in [19, 20].

The paper is organised as follows. In Section 2 we review the basic theory of hypertopes (with particular focus on regular hypertopes) and revisit the notion of a regular polytope (first introduced in the early 1980s) within the theory of hypertopes. In Section 3 we explore the halving operation on an abstract polytope and show that the resulting incidence system is a regular hypertope. Finally, in Section 4 we give concrete examples arising from our construction. In particular, we focus on locally spherical hypertopes arising from Danzer's construction of generalised cubes. As a result we produce a number of new regular hypertopes including an example of a finite regular rank 5 proper hypertope which, because of the size of its automorphism group, could not previously be found (see [11, Section 6]).

2 Regular hypertopes

In this section we review the definition and basic properties of regular hypertopes. We introduce abstract polytopes as a special class of hypertopes. However, if the reader is interested in a classic and more detailed definition of abstract polytopes, we suggest [18, Section 2A].

The notion of a regular hypertope was introduced in [9] as a common generalisation of an abstract regular polytope and of a regular hypermap. In short, a *regular hypertope* is a *thin*, *residually-connected*, *chamber-transitive* geometry (the concepts are defined below). More details and an account of general theory can be found in [2].

An *incidence system* is a 4-tuple $\Gamma := (X, *, t, I)$ where X and I are sets, $t : X \to I$ is the *type function* and $*$ is a binary relation in X called *incidence*. The elements of X and I are called the *elements* and the *types* of Γ, respectively. The cardinality of I is the *rank* of Γ. An element x is said to be of type i, or an i*-element*, whenever $t(x) = i$, for $i \in I$. The relation $*$ is reflexive, symmetric and such that, for all $x, y \in X$, if $x * y$ and $t(x) = t(y)$, then $x = y$.

A *flag* F is a subset of X in which every two elements are incident. An element x is incident to a flag F, denoted by $x * F$, when x is incident to all elements of F. For a flag F the set $t(F) := \{t(x) \,|\, x \in F\}$ is called *the type of* F. When $t(F) = I$, F is called a *chamber*.

An incidence system Γ is a *geometry* (or an *incidence geometry*) if every flag of Γ is contained in a chamber, that is, if all maximal flags of Γ are chambers.

The *residue* of a flag F of an incidence geometry Γ is the incidence geometry $\Gamma_F := (X_F, *_F, t_F, I_F)$ where $X_F := \{x \in X : x * F, x \notin F\}$, $I_F := I \setminus t(F)$, and where t_F and $*_F$ are restrictions of t and $*$ to X_F and I_F respectively.

An incidence system Γ is *thin* when every residue of rank one of Γ contains exactly two elements. If an incidence geometry is thin, then given a chamber C there exists exactly one chamber differing from C in its i-element. An incidence system Γ is *connected* if its incidence graph is connected. Moreover, Γ is *residually connected* when Γ is connected and each residue of Γ of rank at least two is also connected. It is easy to see that this condition is equivalent to *strong connectivity* for polytopes (as defined in [18, pg. 23] and reviewed below) and the thinness is equivalent to the diamond condition for polytopes. A *hypertope* is a thin incidence geometry which is residually connected.

An *abstract polytope* of rank n is usually defined as a strongly-connected partially ordered set $(\mathcal{P}, \leqslant)$ that satisfies the *diamond condition* and in such a way that all maximal chains of $\mathcal{P}$ have the same length $(n+2)$. In the language of incidence geometries, an abstract polytope is an incidence system $(\mathcal{P}, *_\leqslant, \mathrm{rk}, \{-1, \dots, n\})$, where $*_\leqslant$ is the incidence relation defined by the order of $\mathcal{P}$ (i.e., $x *_\leqslant y$ if and only if $x \leqslant y$ or $y \leqslant x$) and rk is the *rank function*. We require that $\mathcal{P}$ has a unique (minimum) element of rank (type) -1 and a unique (maximum) element of rank n. Note that a flag (in the language of incidence systems) is what has been called a *chain* in the theory of abstract polytopes. Therefore, maximal chains of $\mathcal{P}$ are precisely the chambers of the corresponding incidence system. The fact that every maximal chain of $\mathcal{P}$ has $(n+2)$ elements implies that $\mathcal{P}$ defines a geometry. It is well-known that for any two incident elements $F_i \leqslant F_j$ of $\mathcal{P}$, with $\mathrm{rk}(F_i) = i$ and $\mathrm{rk}(F_j) = j$, the *section* $F_j/F_i = \{x \in \mathcal{P} : F_i \leqslant x \leqslant F_j\}$ is a $(j-i-1)$-polytope. We note that for polytopes, the residue of a chain F is a union of sections of $\mathcal{P}$ defined by the intervals of I_F.

Observe that the rank 2 hypertopes are precisely the abstract polygons and the rank 3 hypertopes are the non-degenerate hypermaps.

A *type-preserving automorphism* of an incidence system $\Gamma := (X, *, t, I)$ is a permutation α of X such that for every $x \in X$, $t(x) = t(x\alpha)$ and if $x, y \in X$, then $x * y$ if and only if $x\alpha * y\alpha$. The set of type-preserving automorphisms of Γ is denoted by $\mathrm{Aut}_I(\Gamma)$.

The group of type-preserving automorphisms of an incidence geometry Γ generalises the automorphism group of an abstract polytope. Some familiar symmetry properties of polytopes extend naturally to incidence geometries. For instance, $\mathrm{Aut}_I(\Gamma)$ acts faithfully on the set of chambers of Γ. Moreover, if Γ is a hypertope this action is semi-regular. In fact, if $\alpha \in Aut_I(\Gamma)$ fixes a chamber C, it also fixes its i-adjacent chamber C^i. Since Γ is residually connected, α must be the identity.

We say that Γ is *chamber-transitive* if the action of $Aut_I(\Gamma)$ on the chambers is transitive, and in that case the action of Γ on the set of chambers is regular. For that reason Γ is then called a *regular hypertope*. As expected, this generalises the concept of a regular polytope.

Observe that, when Γ is a geometry, chamber-transitivity is equivalent to *flag-transitivity* (meaning that for each $J \subseteq I$, there is a unique orbit on the flags of type J under the action of $Aut_I(\Gamma)$; see for example Proposition 2.2 in [9]).

Let $\Gamma := (X, *, t, I)$ be a regular hypertope and let C be a fixed (base) chamber of Γ. For each $i \in I$ there exists exactly one automorphism ρ_i mapping C to C^i. If $F \subseteq C$ is a flag, then the automorphism group of the residue Γ_F is precisely stabiliser of F under the action of $\mathrm{Aut}_I(\Gamma)$. We denote this group by $\mathrm{Stab}_\Gamma(F)$. It is easy to see that

$$\mathrm{Stab}_\Gamma(F) = \langle \rho_i : i \in I_F \rangle .$$

If $I_F = \{i\}$, that is Γ_F is of rank $|I|-1$, the thinness of Γ implies that

$$\rho_i^2 = 1. \tag{2.1}$$

If $I_F = \{i, j\}$, then there exists $p_{ij} \in \{2, \dots, \infty\}$ such that

$$(\rho_i \rho_j)^{p_{ij}} = 1; \tag{2.2}$$

in this situation the residue of F is an abstract p_{ij}-gon. Moreover, if J and K are arbitrary subsets of I and $F, G \subseteq C$ are flags such that $I_F = J$ and $I_G = K$, then

$$\mathrm{Stab}_\Gamma(F) \cap \mathrm{Stab}_\Gamma(G) = \mathrm{Stab}_\Gamma(F \cup G),$$

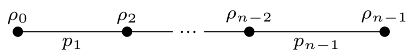
or equivalently

$$\langle \rho_j : j \in J \rangle \cap \langle \rho_k : k \in K \rangle = \langle \rho_i : i \in J \cap K \rangle. \tag{2.3}$$

We call the condition in (2.3) the *intersection property*. Following [9], a C-*group* is a group generated by involutions $\{\rho_i : i \in I\}$ that satisfies the intersection property. It follows that the type-preserving automorphism group of a regular hypertope is a C-group ([9, Theorem 4.1]).

Every Coxeter group U is a C-group and in particular, it is the type-preserving automorphism of a regular hypertope [32, Section 3] called the *universal regular hypertope* associated with the Coxeter group U. Moreover, every C-group G is a quotient of a Coxeter group U. If $\mathcal{H}$ is a regular hypertope whose type-preserving automorphism group is G, the *universal cover* of $\mathcal{H}$ is the regular hypertope associated with U.

The *Coxeter diagram* of a C-group G is a graph with $|I|$ vertices corresponding to the generators of G and with an edge $\{i, j\}$ whenever the order p_{ij} of $\rho_i \rho_j$ is greater than 2. The edge is endowed with the label p_{ij} when $p_{ij} > 3$. The automorphism group of an abstract polytope is a *string C-group*, that is, a C-group having a linear Coxeter diagram. If $\mathcal{P}$ is a regular n-polytope, then we say that $\mathcal{P}$ is of (Schläfli) type $\{p_1, \dots, p_{n-1}\}$ whenever the Coxeter diagram of $\mathrm{Aut}(\mathcal{P})$ is



One of the most remarkable results in the theory of abstract regular polytopes is that the string C-groups are precisely the automorphism groups of the regular polytopes. In other words, given a string C-group G, there exists a regular polytope $\mathcal{P} = \mathcal{P}(G)$ such that $G = \mathrm{Aut}(\mathcal{P})$ (see [18, Section 2E]). This result was proved in [25, 27] for so-called *regular incidence complexes*, (combinatorial objects slightly more general than abstract polytopes). However, the results were essentially already known to Tits who constructed coset geometries from string Coxeter groups in [30], which preceded the introduction of the intersection property in a working paper from 1961 (see [32]). Nevertheless, Schulte was not aware of this. In [29] he gives a nice historical note on the development of the theory.

Analogously, it is also possible to construct, under certain conditions, a regular hypertope from a group, and particularly from a C-group, using the following proposition.

Proposition 2.1 (Tits Algorithm [32]). *Let n be a positive integer and $I := \{0, \dots, n-1\}$. Let G be a group together with a family of subgroups $(G_i)_{i \in I}$, X the set consisting of all cosets $G_i g$ with $g \in G$ and $i \in I$, and $t : X \to I$ defined by $t(G_i g) = i$. Define an incidence relation $*$ on $X \times X$ by:*

$$G_i g_1 * G_j g_2 \quad \textit{if and only if} \quad G_i g_1 \cap G_j g_2 \neq \emptyset.$$

Then the 4*-tuple* $\Gamma := (X, *, t, I)$ *is an incidence system having* $\{G_i : i \in I\}$ *as a chamber. Moreover, the group* G *acts by right multiplication as an automorphism group on* Γ*. Finally, the group* G *is transitive on the flags of rank less than* 3*.*

The incidence system constructed using the proposition above will be denoted by $\Gamma(G; (G_i)_{i\in I})$ and called a *coset incidence system*.

Theorem 2.2 ([9, Theorem 4.6])**.** *Let* $I = \{0, \dots, n-1\}$*, let* $G = \langle \rho_i \,|\, i \in I\rangle$ *be a C-group, and let* $\Gamma := \Gamma(G; (G_i)_{i\in I})$ *where* $G_i := \langle \rho_j \,|\, j \neq i\rangle$ *for all* $i \in I$*. If* G *is flag-transitive on* Γ*, then* Γ *is a regular hypertope.*

In other words, the coset incidence system $\Gamma = \Gamma(G, (G_i)_{i\in I})$ is a regular hypertope if and only if the group G is a C-group and Γ is flag-transitive. In order to prove that a given group G is a C-group, we can use the following result.

Proposition 2.3 ([7, Proposition 6.1])**.** *Let* G *be a group generated by* n *involutions* $\rho_0, \dots, \rho_{n-1}$*. Suppose that* G_i *is a C-group for every* $i \in \{0, \dots, n-1\}$*. Then* G *is a C-group if and only if* $G_i \cap G_j = G_{i,j}$ *for all* $0 \leqslant i, j \leqslant n-1$*.*

At the end of this section we introduce Lemma 2.4 whose proof is straightforward and will be used in Section 3 to prove our main results.

Lemma 2.4. *Let* $G = \langle \rho_0, \dots, \rho_{r-1}\rangle$ *and* $H = \langle \rho_r, \dots, \rho_{r+s-1}\rangle$ *be two C-groups. Then the group*

$$G \times H = \langle \rho_0, \dots, \rho_{r-1}, \rho_r, \dots, \rho_{r+s-1}\rangle$$

is a C-group.

3 Halving operation

In this section the halving operation is applied to the automorphism group of a non-degenerate regular polytope $\mathcal{P}$ producing $\mathrm{H}(\mathcal{P})$, which is a subgroup of $\mathrm{Aut}(\mathcal{P})$ of index at most 2. We prove that the group $\mathrm{H}(\mathcal{P})$ is a C-group and that the corresponding incidence system is flag-transitive. Therefore the group $\mathrm{H}(\mathcal{P})$ is the type-preserving automorphism group of a regular hypertope.

Let $n \geqslant 3$ and $\mathcal{P}$ be a regular, non-degenerate n-polytope of type $\{p_1, \dots, p_{n-2}, p_{n-1}\}$ and automorphism group $\mathrm{Aut}(\mathcal{P}) = \langle \varrho_0, \dots, \varrho_{n-1}\rangle$. The *halving operation* is the map

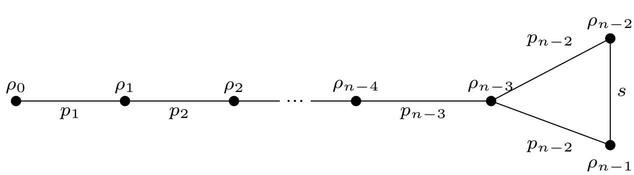
$$\eta : \langle \varrho_0, \dots, \varrho_{n-1}\rangle \to \langle \rho_0, \dots, \rho_{n-1}\rangle\,,$$

where

$$\rho_i = \begin{cases} \varrho_i, & \text{if } 0 \leqslant i \leqslant n-2, \\ \varrho_{n-1}\varrho_{n-2}\varrho_{n-1}, & \text{if } i = n-1, \end{cases} \tag{3.1}$$

The *halving group* of $\mathcal{P}$, denoted by $\mathrm{H}(\mathcal{P})$, is the image of $\mathrm{Aut}(\mathcal{P})$ under η.

Observe that the group $\mathrm{H}(\mathcal{P}) = \langle \rho_1, \dots, \rho_{n-1}\rangle$ has the following diagram



(3.2)

where $s = p_{n-1}$ if p_{n-1} is odd, otherwise $s = \frac{p_{n-1}}{2}$. We denote by $\mathcal{H}(\mathcal{P})$ the coset incidence system $\Gamma\left(\mathrm{H}(\mathcal{P}), (H_i)_{i\in\{0,\dots,n-1\}}\right)$, where H_i is the subgroup of $\mathrm{H}(\mathcal{P})$ generated by $\{\rho_j : j \neq i\}$. In Theorem 3.1 we show that the group $\mathrm{H}(\mathcal{P})$ satisfies the intersection property and in Proposition 3.2 we show that the corresponding incidence $\mathcal{H}(\mathcal{P})$ is flag-transitive. We conclude the section with Corollary 3.3 which states that $\mathcal{H}(\mathcal{P})$ is in fact a thin, chamber-transitive coset geometry, i.e. a regular hypertope.

The halving operation has been used before in the context of regular polyhedra of type $\{q, 4\}$ (see [17] and [18, Section 7B]) and the resulting incidence system is a regular polyhedron of type $\{q, q\}$.

The operation described above doubles the fundamental region of $\mathrm{Aut}(\mathcal{P})$ by gluing together the base flag Φ and the flag Φ^{n-1}.

As an example we explore the halving operation applied to the cubic tessellation $\{4, 3, 4\}$. The elements of type 0 and 1 of the resulting incidence system are the vertices and edges of the $\{4, 3, 4\}$, respectively. The elements of type 2 are half of the cubes and the elements of type 3 are the other half. This is the construction of the infinite hypertope described in [11, Example 2.5] and can also be seen as a semi-regular polytope (see [19, Section 3]).

It is easy to see that $\mathrm{H}(\mathcal{P})$ has index 2 in $\mathrm{Aut}(\mathcal{P})$ if and only if the set of facets of $\mathcal{P}$ is bipartite. This is only possible if p_{n-1} is even. If this is the case, then the elements of type i, for $i \in \{0, \dots, n-3\}$ are the faces of rank i of $\mathcal{P}$. The elements of type $n-2$ are half of the facets of $\mathcal{P}$ (those belonging to the same partition as the base facet) and the elements of type $n-1$ are the other half of the facets, namely, those in the same partition as the facet of Φ^{n-1}.

In the remainder of the section we let $\mathcal{P}$ be a fixed regular n-polytope with a base flag Φ, the automorphism group $\mathrm{Aut}(\mathcal{P}) = \langle \varrho_0, \dots \varrho_{n-1}\rangle$ and $H = \langle \rho_0, \dots, \rho_{n-1}\rangle$ the halving group of $\mathcal{P}$. For $i, j \in \{0, \dots, n-1\}$ we let H_i and $H_{i,j}$ be the groups $\langle \rho_k : k \neq i\rangle$ and $\langle \rho_k : k \notin \{i, j\}\rangle$, respectively. Finally, by $\mathcal{H}(\mathcal{P})$ we denote the incidence system $\Gamma\left(H, (H_i)_{i\in\{0,\dots,n-1\}}\right)$ and by Γ_i the residue of $\mathcal{H}(\mathcal{P})$ induced by H_i, that is $\Gamma_i = \Gamma\left(H_i, (H_{i,j})_{j\in\{0,\dots,n-1\}\setminus\{i\}}\right)$.

Theorem 3.1. *Let $n \geqslant 3$ and $\mathcal{P}$ be a regular, non-degenerate n-polytope of type $\{p_1, \dots, p_{n-1}\}$. Then the halving group $\mathrm{H}(\mathcal{P})$ is a C-group.*

Proof. The strategy of this proof is to use Proposition 2.3. To do so, we proceed by induction over n. Let $\Phi = \{F_{-1}, \dots, F_n\}$ be the base flag of $\mathcal{P}$. Let F'_{n-1} be the facet of Φ^{n-1}.

If $n = 3$, we need to prove that the group $H_0 = \langle \rho_1, \rho_2\rangle = \langle \varrho_1, \varrho_2\varrho_1\varrho_1\rangle$ is a C-group. However, this group is a subgroup of the automorphism group of the polygonal section F_3/F_1 isomorphic to the dihedral group $\mathbb{D}_s$. The groups $H_1 = \langle \rho_0, \rho_2\rangle$ and $H_2 = \langle \rho_0, \rho_1\rangle$ are the automorphism groups of the polygonal sections F'_2/F_{-1} and F_2/F_{-1}, respectively. It follows that they are C-groups.

To finish our base case we only need to show

$$\langle \rho_0, \rho_1\rangle \cap \langle \rho_0, \rho_2\rangle = \langle \rho_0\rangle, \tag{3.3}$$
$$\langle \rho_0, \rho_1\rangle \cap \langle \rho_1, \rho_2\rangle = \langle \rho_1\rangle, \tag{3.4}$$
$$\langle \rho_0, \rho_2\rangle \cap \langle \rho_1, \rho_2\rangle = \langle \rho_2\rangle. \tag{3.5}$$

To prove (3.3), just observe that $\langle\rho_0\rangle = \mathrm{Stab}_{\mathcal{P}}(\{F_1, F_2\})$. Let $\gamma \in \langle\rho_0, \rho_1\rangle \cap \langle\rho_0, \rho_2\rangle$. Since $\gamma \in \langle\rho_0, \rho_1\rangle$, γ fixes F_2. Similarly, since $\gamma \in \langle\rho_0, \rho_2\rangle$, γ must fix F_2'. This implies that γ fixes F_1, since this is the only 1-face of $\mathcal{P}$ incident to both F_2 and F_2'. Therefore, $\gamma \in \mathrm{Stab}_{\mathcal{P}}(\{F_1, F_2\}) = \langle\rho_0\rangle$. The other inclusion is obvious.

Similarly, we have that $\langle\rho_0, \rho_1\rangle \cap \langle\rho_1, \rho_2\rangle \subseteq \mathrm{Stab}_{\mathcal{P}}(\{F_0, F_1\})$. This follows from the fact that the group $\langle\rho_0, \rho_1\rangle$ fixes F_2 and the group $\langle\rho_1, \rho_2\rangle$ fixes F_0. Then, $\langle\rho_0, \rho_1\rangle \cap \langle\rho_1, \rho_2\rangle \subseteq \langle\rho_1\rangle$. Again, the other inclusion is obvious. The proof of (3.4) follows from the same argument but now with respect to the flag $\Phi^2 = \{F_0, F_1, F_2'\}$ of $\mathcal{P}$. This completes the base case.

Assume that the halving group $\mathrm{H}(\mathcal{F})$ of every non-degenerate regular polytope $\mathcal{F}$ of rank r with $3 \leqslant r < n$ is a C-group.

Observe that the groups $H_{n-1} = \langle\rho_0, \dots, \rho_{n-2}\rangle$ and $H_{n-2} = \langle\rho_0, \dots, \rho_{n-3}, \rho_{n-1}\rangle$ are the automorphism groups of the sections F_{n-1}/F_{-1} and F'_{n-1}/F_{-1}, respectively. Hence, these groups are C-groups (see [18, Proposition 2B9]).

Observe that if $i \in \{0, 1, \dots, n-3\}$ then $H_i = H_i^- \times H_i^+$ where $H_i^- = \langle\rho_j : j < i\rangle$ and $H_i^+ = \langle\rho_j : i < j\rangle$. Note that H_i^- is just the automorphism of the section F_i/F_{-1}, hence a C-group. If $i < n-3$ then H_i^+ is the halving group of the section F_n/F_i, which is a C-group by the inductive hypothesis. If $i = n-3$, then H_i^+ is isomorphic to a dihedral group $\mathbb{D}_s$. In any case, it follows from Lemma 2.4 that H_i is a C-group.

In order to use Proposition 2.3, we need to prove that for every pair $i, j \in \{0, \dots, n-1\}$, with $i < j$, the equality

$$H_i \cap H_j = H_{i,j} \tag{3.6}$$

holds. We proceed in a similar way as in rank 3. If $\{i, j\} = \{n-1, n-2\}$, then observe that $H_i = H_{n-2}$ fixes F_{n-1} and $H_j = H_{n-1}$ fixes F'_{n-1}. This implies that an element $\gamma \in H_{n-2} \cap H_{n-1}$ must fix F_{n-2}. Thus $\gamma \in \mathrm{Stab}_{\mathcal{P}}(\{F_{n-2}, F_{n-1}\}) = \langle\rho_0, \dots, \rho_{n-3}\rangle = H_{i,j}$. The other inclusion is obvious.

If $j \in \{n-1, n-2\}$ and $i \leqslant n-3$ then (3.6) follows from the fact that H_j is a string C-group.

Assume that $0 \leqslant i < j \leqslant n-3$. Let $\mathcal{F}$ be the section F_j/F_{-1} of $\mathcal{P}$. Let $\gamma \in H_i \cap H_j$. Observe that $H_j = H_j^- \times H_j^+$ and that $\mathrm{Aut}(\mathcal{F}) = H_j^-$. Let $\alpha \in H_j^-$ and $\beta \in H_j^+$ be such that $\gamma = \alpha\beta$. Note that β fixes the face F_i of $\mathcal{P}$ and since $\gamma \in H_i$ it follows that α must fix F_i. Since H_j^- is a string C-group, it follows that $\alpha \in \langle\rho_0, \dots, \rho_{i-1}, \rho_{i+1}, \dots, \rho_{j-1}\rangle$. Then

$$\gamma \in \langle\rho_0, \dots, \rho_{i-1}, \rho_{i+1}, \dots, \rho_{j-1}\rangle \times \langle\rho_{j+1}, \dots, \rho_{n-1}\rangle = H_{i,j}.$$

The other inclusion is obvious. □

The halving group of a regular polytope $\mathcal{P}$ is a C-group with Coxeter diagram (3.2). The groups generated by involutions with this diagram are called *tail-triangle groups* (even when $s = 2$). In [19, 20] Monson and Schulte show that when a tail-triangle group is a C-group, it is the automorphism group of an alternating semi-regular polytope. We denote by $\mathcal{S}(\mathcal{P})$ the semi-regular polytope obtained by the halving operation on $\mathcal{P}$. The polytope $\mathcal{S}(\mathcal{P})$ has two orbits of isomorphic regular facets, namely the right cosets of $\langle\rho_0, \dots, \rho_{n-3}, \rho_{n-2}\rangle$ and $\langle\rho_0, \dots, \rho_{n-3}, \rho_{n-1}\rangle$. These *base facets* are incident with a regular polytope $\mathcal{R}$ of rank $n-2$. In fact, every flag of $\mathcal{R}$ can be extended to a flag of $\mathcal{S}(\mathcal{P})$ in two different ways. Moreover, any flag of $\mathcal{S}(\mathcal{P})$ belongs to the orbit of one of these two flags. The automorphisms of $\mathcal{S}(\mathcal{P})$ can now be used to show flag transitivity of $\mathcal{H}(\mathcal{P})$.

Proposition 3.2. *The incidence system $\mathcal{H}(\mathcal{P})$ associated with a non-degenerate regular polytope $\mathcal{P}$ is flag-transitive.*

Proof. Let $J \subseteq \{0, \ldots, n-1\}$ and let $F = \{H_i h : i \in J\}$ for some $h \in \mathrm{H}(\mathcal{P})$ be a flag of $\mathcal{H}(\mathcal{P})$ of type J. If $|J \cap \{n-2, n-1\}| \leqslant 1$, then F is a chain of $\mathcal{S}(\mathcal{P})$ of type J. By [19, Lemma 4.5b], the group $\mathrm{H}(\mathcal{P})$ is transitive on the chains of this type.

If $\{n-2, n-1\} \subseteq J$, then $F = \Upsilon \cup \Upsilon^{n-1}$, where Υ is the chain of $\mathcal{S}(\mathcal{P})$ of type $J' = J \setminus \{n-1\}$ whose faces are contained in F. Again, [19, Lemma 4.5b] implies that $\mathrm{H}(\mathcal{P})$ is transitive in chains of type J'. Finally, observe that if $\beta \in \mathrm{H}(\mathcal{P})$, then $(\Upsilon\beta)^{n-1} = (\Upsilon^{n-1})\beta$. It follows that $\mathrm{H}(\mathcal{P})$ is also transitive on flags of type J. $\square$

Corollary 3.3. *Let $\mathcal{P}$ be a non-degenerate, regular n-polytope and $I = \{0, \ldots, n-1\}$. Let $\mathrm{H}(\mathcal{P})$ be the halving group of $\mathcal{P}$. Then the incidence system $\mathcal{H}(\mathcal{P}) = \Gamma\left(\mathrm{H}(\mathcal{P}), (H_i)_{i \in I}\right)$ is a regular hypertope such that $\mathrm{Aut}_I\left(\mathcal{H}(\mathcal{P})\right) = \mathrm{H}(\mathcal{P})$.*

The assumption that $\mathcal{P}$ is non-degenerate is very important. When the halving operation is applied on the dual of the 4-hemicube the resulting incidence system is not a hypertope (see [11, Example 3.3]).

Theorem 2.5 in [20] implies that the semi-regular polytope $\mathcal{S}(\mathcal{P})$ is regular because the associated C-group admits a group automorphism interchanging the generators ρ_{n-1} and ρ_{n-2} (this automorphism is given by conjugation by ϱ_{n-1}). The polytope $\mathcal{S}(\mathcal{P})$ is in fact isomorphic to $\mathcal{P}$.

4 Locally spherical hypertopes from generalised cubes.

A *spherical hypertope* is a universal regular hypertope whose Coxeter diagram is a union of diagrams of finite irreducible Coxeter groups. This definition is slightly different from that in [11]. A *locally spherical hypertope* is a regular hypertope whose all proper residues are spherical hypertopes. An irreducible regular hypertope is of *euclidean* (resp. *spherical*) *type* if the type-preserving automorphism group of its universal regular cover is an affine (resp. finite irreducible) Coxeter group. Similarly, an irreducible locally spherical hypertope is of *hyperbolic type* if the type-preserving automorphism group of its universal cover is a compact hyperbolic Coxeter group. It is well known that compact hyperbolic Coxeter groups exist only in ranks 3, 4 and 5.

In [11] the authors show that a locally spherical hypertope has to be of spherical, euclidean or hyperbolic type. The complete list of the Coxeter diagrams of these groups can be found in [11, Tables 1 and 2]. Whereas the first two classes are well understood, not much is known about the hyperbolic type. In particular, the authors were not successful in producing a finite example of rank 5 locally spherical proper hypertope of this type. In what follows we will use the halving operation on a certain class of polytopes first described by Danzer in [5] (see also [18, Theorem 8D2]) and in particular we produce an example of finite rank 5 proper regular hypertope of hyperbolic type.

We briefly review Danzer's construction of generalised cubes.

Let $\mathcal{K}$ be a regular finite non-degenerate n-polytope with vertex set $V = \{v_1, \ldots, v_m\}$. Consider the set

$$2^V = \prod_{j=1}^{m} \{0, 1\} = \{\bar{\mathrm{x}} = (x_1, \ldots, x_m) : x_j \in \{0, 1\}\}.$$

Given an i-face F of $\mathcal{K}$ and $\bar{x} \in 2^V$ define the sets

$$F(\bar{x}) = \left\{\bar{y} = (y_1, \ldots, y_m) \in 2^V : y_j = x_j \text{ if } v_j \not\leqslant F\right\}.$$

Then the polytope $2^{\mathcal{K}}$ is the set

$$\{\emptyset\} \cup 2^V \cup \left\{F(\bar{x}) : F \in \mathcal{K}, \bar{x} \in 2^V\right\}$$

ordered by inclusion. The improper face of rank -1 of $2^{\mathcal{K}}$ is $\emptyset$ and if $i \geqslant 0$ the i-faces of $2^{\mathcal{K}}$ are the sets $F(\bar{x})$ for F a certain $(i-1)$-face of $\mathcal{K}$ and some $\bar{x} \in 2^V$. Note that $F_{-1}(\bar{x}) = \{\bar{x}\}$ for every $\bar{x} \in 2^V$, hence $2^{\mathcal{K}}$ has $2^{|V|}$ vertices.

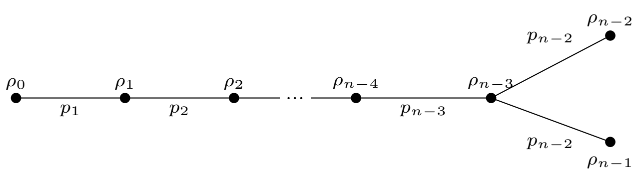
If $\mathcal{K}$ is a regular of type $\{p_1, \ldots, p_{n-2}\}$ then $2^{\mathcal{K}}$ is a regular polytope of type $\{4, p_1, \ldots, p_{n-2}\}$. In fact, all the vertex figures of $2^{\mathcal{K}}$ are isomorphic to $\mathcal{K}$. The polytope $2^{\mathcal{K}}$ is called a *generalised cube* since when $\mathcal{K}$ is the $(n-1)$-simplex, the polytope $2^{\mathcal{K}}$ is isomorphic to the n-cube.

For our purposes it is convenient to denote by $\hat{2}^{\mathcal{K}}$ the polytope $\left(2^{\mathcal{K}^*}\right)^*$, so that $\hat{2}^{\mathcal{K}}$ is a regular polytope of type $\{p_1, \ldots, p_{n-2}, 4\}$ whose facets are isomorphic to $\mathcal{K}$.

The automorphism group of $\hat{2}^{\mathcal{K}}$ is isomorphic to $\mathbb{Z}_2^m \rtimes \mathrm{Aut}(\mathcal{K})$, where m denotes the number of facets of $\mathcal{K}$ and the action of $\mathrm{Aut}(\mathcal{K})$ on $\mathbb{Z}_2^m$ is given by permuting coordinates in the natural way. In particular, the size of this group is $2^m \times |\mathrm{Aut}(\mathcal{K})|$ (see [18, Theorem 2C5] and [23]).

Remark 4.1. In [23] Pellicer generalises Danzer's construction of $2^{\mathcal{K}}$. Given a finite non-degenerate regular $(n-1)$-polytope $\mathcal{K}$ of type $\{p_1, \ldots, p_{n-2}\}$ and $s \in \mathbb{N}$, Pellicer's construction gives as a result an n-polytope $\mathcal{P}_s$ of type $\{p_1, \ldots p_{n-2}, 2s\}$. If $s = 2$, the polytope $\mathcal{P}_2$ is isomorphic to $\hat{2}^{\mathcal{K}}$. However, when our construction is applied to the polytopes $\mathcal{P}_s$ for $s \geqslant 3$, the resulting hypertopes are not locally spherical and therefore not included in this paper.

Now we discuss the locally spherical hypertopes resulting from applying the halving operation to the polytopes obtained from Danzer's construction. Since $\hat{2}^{\mathcal{K}}$ is of type $\{p_1, \ldots p_{n-2}, 4\}$, the hypertope $\mathcal{H}(\hat{2}^{\mathcal{K}})$ has the following Coxeter diagram:



We naturally extend the Schläfli symbol and say that $\mathcal{H}(\hat{2}^{\mathcal{K}})$ is of type $\{p_{n-1}, \ldots, p_{n-3}, {}^{p_{n-2}}_{p_{n-2}}\}$.

In rank 3 the polytope $\hat{2}^{\{p\}}$ is obtained by applying the construction on a regular polygon $\{p\}$ and the induced hypertope is in fact a self-dual polyhedron of type $\{p, p\}$. This polyhedron has 2^{p-1} vertices, $2^{p-2}p$ edges and 2^{p-1} faces and it is a map on a surface of genus $2^{p-3}(p-4)+1$. For $p = 3$ the resulting hypertope is a spherical polyhedron $\{3, 3\}$, i.e. the tetrahedron. When $p = 4$ the polytope $\hat{2}^{\{4\}}$ is the toroid $\{4, 4\}_{(4,0)}$ and the induced hypertope is also of euclidean type, more precisely, it is the toroid $\{4, 4\}_{(2,2)}$.

To obtain locally spherical hypertopes in rank 4, $\mathcal{K}$ must be of type $\{p, 3\}$ with $p = 3, 4, 5$. The resulting hypertopes are of spherical, euclidean, and hyperbolic type, respectively. If $p = 3$, the hypertope $\mathcal{H}(\hat{2}^{\mathcal{K}})$ is the universal hypertope of Coxeter diagram D_4

and type $\{3, {}^3_3\}$. When $p = 4$ the polytope $\hat{2}^{\mathcal{K}}$ is the toroid $\{4, 3, 4\}_{(4,0,0)}$ and $\mathcal{H}(\hat{2}^{\mathcal{K}})$ is a toroidal hypertope described by Ens in [6, Theorem 4.3], which we denote by $\{4, {}^3_3\}_{(4,0,0)}$. The automorphism group of this hypertope has Coxeter diagram $\tilde{B}_3$. If $p = 5$, the resulting hypertope is of type $\{5, {}^3_3\}$ with automorphism group of size $2^{11} \times 120 = 245, 760$. This example is different from any of the examples listed in [11].

In rank 5 the polytope $\mathcal{K}$ must be of type $\{p, 3, 3\}$ with $p = 3, 4, 5$, the resulting hypertopes are of spherical, euclidean and hyperbolic type, respectively. If $p = 3$ then the hypertopes is the universal spherical hypertope of type $\{3, 3, {}^3_3\}$, i.e. the universal hypertope of Coxeter diagram D_5. If $p = 4$ the polytope $\hat{2}^{\mathcal{K}}$ is the regular toroid $\{4, 3, 3, 4\}_{(4,0,0,0)}$. The induced hypertope is of euclidean type, hence a toroidal hypertope which we denote by $\{4, 3, {}^3_3\}_{(4,0,0,0)}$. The Coxeter diagram of its automorphism group is $\tilde{B}_4$. For $p = 5$ the regular polytope $\hat{2}^{\mathcal{K}}$ is constructed from the 120-cell. The hypertope $\mathcal{H}(\hat{2}^{\mathcal{K}})$ is of type $\{5, 3, {}^3_3\}$ and its automorphism group has size $2^{119} \times 14400$. It is not surprising that the authors of [11] could not find this example using a computational approach.

For rank $n \geqslant 6$ we can only obtain locally spherical hypertopes from our construction if $\mathcal{K}$ is the $(n-1)$-simplex $\{3^{n-2}\}$ or the $(n-1)$-cube $\{4, 3^{n-3}\}$. The polytope $\hat{2}^{\mathcal{K}}$ is the n-cross-polytope $\{3^{n-2}, 4\}$ or the toroid $\{4, 3^{n-3}, 4\}_{(4,0,\ldots 0)}$, respectively. In the former case the resulting hypertope is the universal spherical hypertope of type $\{3^{n-3}, {}^3_3\}$ associated with the Coxeter diagram D_n while in the latter it is a toroidal hypertope associated with the Coxeter diagram $\tilde{B}_{n-1}$ which we denote by $\{4, 3^{n-4}, {}^3_3\}_{(4,0,\ldots 0)}$.

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On a smallest topological triangle free (n_4) point-line configuration*

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Abstract

We study an abstract object, a finite generalised quadrangle $W(3)$, due to Jacques Tits, that can be seen as the Levi graph of a triangle free (40_4) point-line configuration. We provide for $W(3)$ representations as a topological (40_4) configuration, as a (40_4) circle representation, and a representation in the complex plane. These come close to a still questionable (real) geometric (40_4) point-line configuration realising this finite generalised quadrangle. This abstract (40_4) configuration has interesting triangle free realisable geometric subconfigurations, which we also describe. A topological (n_4) configuration for $n < 40$ must contain a triangle, so our triangle free example is minimal.

Keywords: Finite generalised quadrangles, computational synthetic geometry, point-line configurations, oriented matroids, pseudoline arrangements.

Math. Subj. Class.: 52C30

1 Introduction

In Computational Synthetic Geometry, see [5], we search for an unknown geometric object when its abstract mathematical structure is given. Oriented matroids, see [2] or [6], play a central part within this field. Our article can be seen in this context. It provides a connection from the theory of finite generalised quadrangles, see [12], to the study of point-line configurations in the sense of recent books about these topics, see [9] and [14].

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Definition 1.1. An (n_k) *configuration* is a set of n points and n *lines* such that every point lies on precisely k of these lines and every line contains precisely k of these points. We distinguish three concepts.

Definition 1.2. When the lines are *straight lines* in the projective plane, we have a *geometric* (n_k) *configuration*.

Definition 1.3. When the lines are *pseudolines* forming a rank 3 oriented matroid, we have a *topological* (n_k) *configuration*.

Definition 1.4. When the lines are *abstract lines*, we have an *abstract* (n_k) *configuration*.

We assume the reader to know basic facts about rank 3 oriented matroids or pseudoline arrangements in the real projective plane.

This article provides, among other results, a triangle free topological (40_4) configuration. We remark that triangle free configurations have been studied so far only for smaller (n_3) configurations, see e.g. [3], [10], or [15].

Definition 1.5. The *generalised quadrangle* $W(3)$ is the point-line geometry where the points are the points of the projective 3-space $\mathbb{P}_3(3)$ over the field of 3 elements, and the lines are the lines of $\mathbb{P}_3(3)$ fixed under a symplectic polarity. A *symplectic polarity* is a permutation of the set of points, lines and planes of $\mathbb{P}_3(3)$ mapping the points to planes, lines to lines and planes to points, such that incidence and non-incidence are both preserved (that is, containment of points in lines and planes, and of lines in planes is transferred into reversed containment), and the permutation has order 2, that is, if a point p is mapped to the plane α, then the plane α is mapped to the point p. Such a polarity can be described, after suitable coordinatization, as mapping the point (a, b, c, d) to the plane with equation $bX - aY + dZ - cU = 0$, from which all other images follow.

As such, the full automorphism group of $W(3)$ is isomorphic to $\mathrm{Aut}(\mathsf{PSp_4}(3))$, a group of order 51840, containing $\mathsf{PSp_4}(3)$ as normal simple subgroup of index 2.

The geometry $W(3)$ is a member of the family of so-called *symplectic generalised quadrangles* $W(q)$, where q is any prime power. Each line of $W(q)$ contains $q+1$ points and each point is contained in $q+1$ lines. Moreover, $W(q)$ contains q^3+q^2+q+1 points. Hence it is an abstract $((q^3+q^2+q+1)_{q+1})$ configuration. For $q = 3$, we obtain an abstract (40_4) configuration.

The name "generalised quadrangle" comes from the fact that the geometry does not contain any triangle, but every two elements are contained in a quadrangle. Hence every generalised quadrangle $W(q)$ defines a triangle free abstract configuration. We will not need the general definition of a generalised quadrangle, we content ourselves with mentioning that, conversely, when an abstract $((q^3+q^2+q+1)_{q+1})$ configuration is triangle free, then it is a generalised quadrangle, i.e., every pair of elements is contained in a quadrangle. We also say that the generalised quadrangle has order q. When $q > 4$ is a power of 2, there are many non-isomorphic generalised quadrangles with order q known. For q a power of an odd prime, we know exactly two generalised quadrangles of order q. One of those is $W(q)$. The other one is obtained from the first one by interchanging the names "point" and "line". We say that the latter is *the dual* of the former. The dual of $W(q)$ is usually denoted by $Q(4, q)$; it arises as a non-singular parabolic quadric in the projective 4-space $\mathbb{P}_4(q)$ over the field of q elements, that is, a quadric with equation $X_1X_2 + X_3X_4 = X_0^2$, after suitable coordinatization. That $W(q)$ is really not isomorphic to $Q(4, q)$, q odd, can be seen by noting that $Q(4,q)$ admits substructures isomorphic to a $(q+1)\times(q+1)$ grid, while this is not the case for $W(q)$. If q is even, then $W(q)$ is isomorphic to $Q(4,q)$, and the isomorphism can be realised by projecting $Q(4,q)$ from its nucleus, that is, the intersection point of all tangent hyperplanes of $Q(4,q)$ (a hyperpane is *tangent* if it intersects $Q(4,q)$ in a cone).

For $q = 3$, it follows that there are at least two triangle free (40_4) configurations. However, it is shown in 6.2.1 of [12] that these are the only examples.

The question whether a given generalised quadrangle of order q is a geometric $((q^3+q^2+q+1)_{q+1})$ configuration seems to be extremely difficult. The only such quadrangle for which we know the answer is the one with $q = 2$: $W(2)$ is a geometric (15_3) configuration. Already for the next cases $W(3)$ and $Q(4,3)$ nothing is known. In the present paper, we focus on $W(3)$. We motivate this in Section 6.

For now we have only a conjecture concerning the main question:

Conjecture 1.6. *There is no geometric* (40_4) *configuration that represents the given finite generalised quadrangle* $W(3)$.

Here are some aspects about the missing methods for solving this problem. One way to prove that there is no such geometric configuration would be to show that there is even no corresponding topological configuration. Our theorem shows that this cannot be done. Another method would have been to start with a projective base and to apply the construction sequence method, see [4], that was very useful for the investigation of smaller (n_4) configurations. However, because of the missing triangles property, the number of variables for an algebraic investigation exceeds very soon the problem size that can be handled with computer algebra support. With a symmetry assumption we reduce the number of variables, however, by using these assumptions we very soon realised that the best results are those that we present in this article. Without any symmetry assumption, we never found a triangle free projective incidence theorem that should occur towards the end of a construction sequence; a property occurring in so many non-symmetric (n_4) configurations. For instance, if a configuration contains two triangles in perspective from a point, then we know by Desargues' theorem that an extra incidence occurs in the real plane, even if the "axis" of the corresponding Desargues' configuration is not a line of the configuration. But $W(3)$ does not contain triangles, and we are not aware of any incidence theorem in the real plane (like Desargues' theorem), which can be applied to $W(3)$. In particular, such an incidence theorem should be triangle free. What remains after this observation is a question.

Problem 1.7. Does there exist a triangle free incidence theorem in the real plane?

2 Description of the given abstract object

Our abstract object, an abstract (40_4) configuration, is known in the literature as $W(3)$. The authors attribute the discovery of classical finite quadrangles (including $W(3)$) to J. Tits and they are first described in 1968 in the book by P. Dembowski [8]

The second author mentioned the problem of realising $W(3)$ long ago to the first author hoping for a solution with methods from computational synthetic geometry.

2.1 The Levi graph of a triangle free abstract (40_4) point-line configuration

The *Levi graph* of a (point-line) configuration is the graph with vertices the points and the lines of the configuration, adjacent when incident. The Levi graph of the triangle free abstract (40_4) configuration $W(3)$ is given by the following list of vertices with its following four neighbors. We have used the first 40 labels for the points.

(1, 41 42 43 44) (2, 45 46 47 48) (3, 49 50 51 52) (4, 53 54 55 56) (5, 41 45 49 53) (6, 41 57 58 59) (7, 41 60 61 62) (8, 45 63 64 65) (9, 49 66 67 68) (10, 53 69 70 71) (11, 45 72 73 74) (12, 53 75 76 77) (13, 49 78 79 80) (14, 42 46 50 54) (15, 42 63 66 69) (16, 42 72 75 78) (17, 46 60 70 79) (18, 50 61 64 76) (19, 54 62 67 73) (20, 46 57 68 77) (21, 54 58 65 80) (22, 50 59 71 74) (23, 43 47 51 55) (24, 43 65 68 71) (25, 43 73 76 79) (26, 47 61 69 80) (27, 51 62 63 77) (28, 55 60 66 74) (29, 47 59 67 75) (30, 55 57 64 78) (31, 51 58 70 72) (32, 44 48 52 56) (33, 44 64 67 70) (34, 44 74 77 80) (35, 48 62 71 78) (36, 52 60 65 75) (37, 56 61 68 72) (38, 48 58 66 76) (39, 56 59 63 79) (40, 52 57 69 73)

(41, 1 5 6 7) (42, 1 14 15 16) (43, 1 23 24 25) (44, 1 32 33 34) (45, 2 5 8 11) (46, 2 14 17 20) (47, 2 23 26 29) (48, 2 32 35 38) (49, 3 5 9 13) (50, 3 14 18 22) (51, 3 23 27 31) (52, 3 32 36 40) (53, 4 5 10 12) (54, 4 14 19 21) (55, 4 23 28 30) (56, 4 32 37 39) (57, 6 20 30 40) (58, 6 21 31 38) (59, 6 22 29 39) (60, 7 17 28 36) (61, 7 18 26 37) (62, 7 19 27 35) (63, 8 15 27 39) (64, 8 18 30 33) (65, 8 21 24 36) (66, 9 15 28 38) (67, 9 19 29 33) (68, 9 20 24 37) (69, 10 15 26 40) (70, 10 17 31 33) (71, 10 22 24 35) (72, 11 16 31 37) (73, 11 19 25 40) (74, 11 22 28 34) (75, 12 16 29 36) (76, 12 18 25 38) (77, 12 20 27 34) (78, 13 16 30 35) (79, 13 17 25 39) (80, 13 21 26 34)

2.2 A combinatorial construction

The generalised quadrangle $W(3)$ can be coordinatized and so a description using coordinates in the field of order 3 can be given, see [11]. However, we rather present a combinatorial description, which we will use later in Subsection 3.1 and in Section 6.

Let $N = \{1, 2, 3, 4\}$. Then the points of $W(3)$ are the elements $(i+)$, $(i-)$, $(ij+)$ and $(ij-)$, with $i, j \in N$. Sixteen of the forty lines can be described as the sets $L_{ij} := \{(i+), (j-), (ij+), (ij-)\}$, $i, j \in N$ (we emphasise that $i \geq j$ is allowed). Two of the remaining lines can be described as $L_\epsilon := \{(ii\epsilon) : i \in N\}$, $\epsilon \in \{+, -\}$. For each fixed point free involution σ of N we have the two lines $L_\sigma^{\epsilon_1} := \{(11^\sigma\epsilon_1), (22^\sigma\epsilon_2), (33^\sigma\epsilon_3), (44^\sigma\epsilon_4) : \epsilon_i = \epsilon_j \Leftrightarrow i^\sigma = j, \forall i, j \in \{1, 2, 3, 4\}\}$, which accounts for six more lines. Finally, let θ_0 be a fixed permutation of N with exactly one fixed point, say $i_0 \in N$. For each permutation θ of N with exactly one fixed point we define the two lines $L_\sigma^\epsilon := \{(ii^\theta\epsilon) : i \in N\}$, $\epsilon \in \{+, -\}$, if $\theta_0\theta$ has exactly one fixed point, and $L_\sigma^{\epsilon_j} := \{(ii^\theta\epsilon_i) : i \in N, \{\epsilon_i, \epsilon_j\} = \{+, -\}$ for $i \neq j = j^\theta\}$, otherwise (i.e., if $\theta_0\theta$ has no or four fixed points). Since there are exactly eight permutations with exactly one fixed point, this accounts for the remaining sixteen lines.

It is elementary to check that the abstract configuration defined in the previous paragraph is a triangle free (40_4) configuration. The fact that it defines $W(3)$ can be deduced from the observation that it contains a dual 4×4 grid, namely, all points of the 4-set $\{(i+) : i \in N\}$ are collinear to all points of the 4-set $\{(i-) : i \in N\}$.

There are essentially two different choices for θ_0. We will choose θ_0 to be the permutation $(2\ 3\ 4)$, fixing 1.

Concretely, we see the correspondence with the construction in the previous section as follows (it is only one of the 51840 possible identifications).

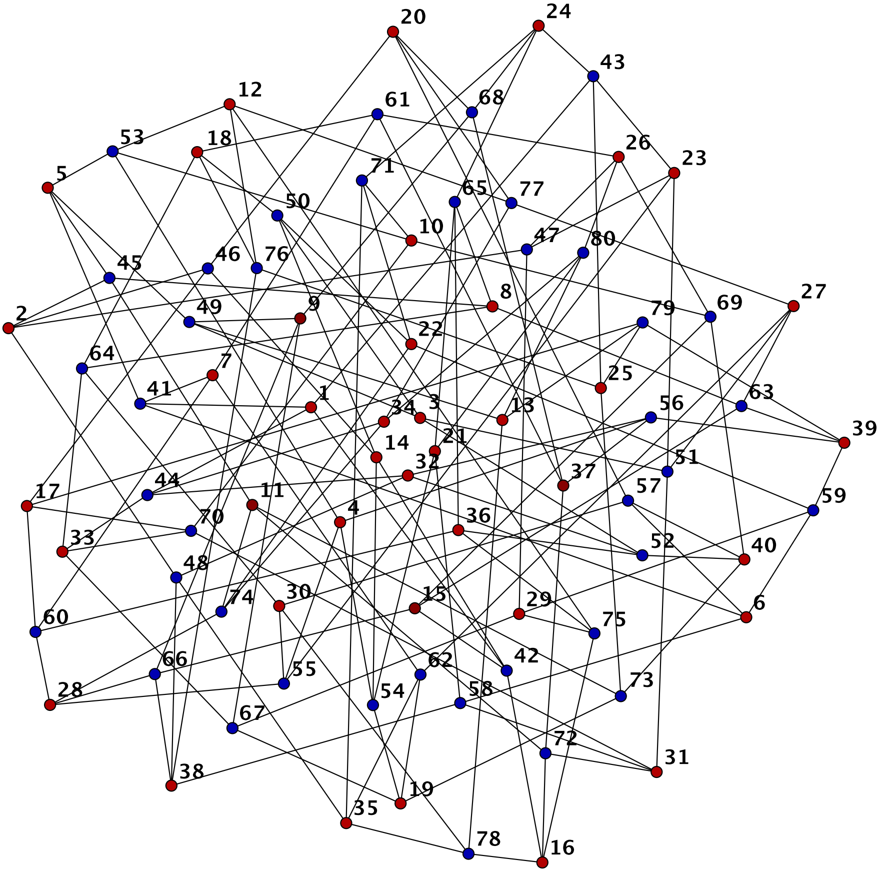

Figure 1: The Levi graph with a five-fold rotational symmetry of the triangle free (40_4) point-line configuration, we use red labels 1, 2, . . . , 40 as points and blue labels 41, 42, . . . 80 as (abstract) lines.

$$
\begin{array}{llll}
(1+) \mapsto 1 & (2+) \mapsto 35 & (3+) \mapsto 36 & (4+) \mapsto 37 \\
(1-) \mapsto 7 & (2-) \mapsto 24 & (3-) \mapsto 16 & (4-) \mapsto 32 \\
(11+) \mapsto 6 & (22+) \mapsto 22 & (33+) \mapsto 29 & (44+) \mapsto 39 \\
(11-) \mapsto 5 & (22-) \mapsto 10 & (33-) \mapsto 12 & (44-) \mapsto 4 \\
(12+) \mapsto 23 & (21+) \mapsto 27 & (12-) \mapsto 25 & (21-) \mapsto 19 \\
(13+) \mapsto 15 & (31+) \mapsto 28 & (13-) \mapsto 14 & (31-) \mapsto 17 \\
(14+) \mapsto 34 & (41+) \mapsto 26 & (14-) \mapsto 33 & (41-) \mapsto 18 \\
(23+) \mapsto 30 & (32+) \mapsto 8 & (23-) \mapsto 13 & (32-) \mapsto 21 \\
(24+) \mapsto 2 & (42+) \mapsto 20 & (24-) \mapsto 38 & (42-) \mapsto 9 \\
(34+) \mapsto 40 & (43+) \mapsto 11 & (34-) \mapsto 3 & (43-) \mapsto 31
\end{array}
$$

This provides the following identification of the lines.

$$
\begin{array}{rrrr}
L_{11} \mapsto 41 & L_{22} \mapsto 71 & L_{33} \mapsto 75 & L_{44} \mapsto 56 \\
L_{12} \mapsto 43 & L_{23} \mapsto 78 & L_{34} \mapsto 52 & L_{41} \mapsto 61 \\
L_{13} \mapsto 42 & L_{24} \mapsto 48 & L_{31} \mapsto 60 & L_{42} \mapsto 68 \\
L_{14} \mapsto 44 & L_{21} \mapsto 62 & L_{32} \mapsto 65 & L_{43} \mapsto 72 \\
L_{+} \mapsto 59 & L_{-} \mapsto 53 & L^{+}_{(12)(34)} \mapsto 51 & L^{-}_{(12)(34)} \mapsto 73 \\
L^{+}_{(13)(24)} \mapsto 66 & L^{-}_{(13)(24)} \mapsto 46 & L^{+}_{(14)(23)} \mapsto 80 & L^{-}_{(14)(23)} \mapsto 64 \\
L^{+}_{(123)} \mapsto 79 & L^{-}_{(123)} \mapsto 55 & L^{+}_{(321)} \mapsto 63 & L^{-}_{(321)} \mapsto 54 \\
L^{+}_{(124)} \mapsto 47 & L^{-}_{(124)} \mapsto 76 & L^{+}_{(421)} \mapsto 67 & L^{-}_{(421)} \mapsto 77 \\
L^{+}_{(134)} \mapsto 50 & L^{-}_{(134)} \mapsto 69 & L^{+}_{(431)} \mapsto 74 & L^{-}_{(431)} \mapsto 70 \\
L^{+}_{(234)} \mapsto 57 & L^{-}_{(234)} \mapsto 49 & L^{+}_{(432)} \mapsto 58 & L_{(432)} \mapsto 45
\end{array}
$$

We now study some interesting subconfigurations.

3 Geometric subconfigurations

3.1 The geometric unique triangle free (20_3) configuration

There exist a lot of triangle free (v_3) configurations, for $v \geq 15$. The one with $v = 15$ is often called the *Cremona–Richmond configuration* and it is the unique generalised quadrangle $W(2)$ with three points per line and three lines per point. Its Levi graph is Tutte's 8-cage.

There is another remarkable triangle free (v_3) configuration with v relatively small, and that is the unique flag-transitive (20_3) configuration, denote it by $\mathcal{T}$. Note that there are 162 triangle-free (20_3) configurations altogether [1].

The Levi graph of $\mathcal{T}$ is the Kronecker cover (also sometimes called the bipartite double) of the dodecahedron graph. It can be described as follows. The point set $P_{\mathcal{T}}$ of $\mathcal{T}$ is the set of ordered non-identical pairs (a, b), with $a, b \in \{1, 2, 3, 4, 5\}$. The lines of $\mathcal{T}$ are the triples $\{(a, b), (b, c), (c, a)\}$, with a, b, c three distinct members of $\{1, 2, 3, 4, 5\}$. We denote the line set by $\mathcal{L}_{\mathcal{T}}$. The full collineation group $\mathsf{Sym}(5) \times \mathbb{Z}_2$ is now easy to see (the involution in the center corresponds to the "opposition" mapping $(a, b) \mapsto (b, a)$; we denote (b, a) by $\overline{(a, b)}$ and call these two points *opposite*).

The configuration $\mathcal{T}$ is realisable, see [3]. But it is also a subconfiguration of $W(3)$. This can be easily seen using the construction of Subsection 2.2. We present an embedding of $\mathcal{T}$ in $W(3)$, given explicitly as follows:

$$\begin{aligned}
(5,i) &\mapsto (i+), && i \in \{1,2,3,4\},\\
(i,5) &\mapsto (i-), && i \in \{1,2,3,4\},\\
(i,j) &\mapsto (ij+), && i,j \in \{1,2,3,4\}, i \neq j.
\end{aligned}$$

Using these explicit descriptions, the following properties can easily be checked. (A *hyperbolic line* in $W(3)$ is the set of points of an ordinary line of $\mathbb{P}_3(3)$ which is not a line of $W(3)$ in Definition 1.5 of $W(3)$.)

- *For every point p of $\mathcal{T}$, the point $\overline{p}$ is the unique point of $\mathcal{T}$ at distance 6 from p in the Levi graph.*
- *For every line $L = \{p_1, p_2, p_3\}$ of $\mathcal{T}$, the line $\overline{L} := \{\overline{p}_1, \overline{p}_2, \overline{p}_3\}$ is the unique line at distance 6 from L in the Levi graph.*
- *Two distinct points p, q of $\mathcal{T}$ are collinear in $W(3)$ if and only if either they are collinear in $\mathcal{T}$, or $p = \overline{q}$. In the latter case, no other points of $\mathcal{T}$ are contained in the line of $W(3)$ determined by p and q. In the former case, only the points of the line in $\mathcal{T}$ determined by p and q are contained in the line of $W(3)$ determined by p and q.*
- *The lines of $W(3)$ corresponding to two distinct lines of $\mathcal{T}$ intersect in $W(3)$ if and only if they intersect in $\mathcal{T}$.*
- *For each $i \in \{1,2,3,4,5\}$, the point set $\{(i,j) : j \in \{1,2,3,4,5\} \setminus \{i\}\}$ forms a hyperbolic line in $W(3)$; the same thing holds for $\{(j,i) : j \in \{1,2,3,4,5\} \setminus \{i\}\}$.*
- *There are 20 lines of $W(3)$ containing exactly three (necessarily collinear) points of $\mathcal{T}$; there are 10 lines of $W(3)$ containing exactly two (necessarily opposite) points of $\mathcal{T}$; there are 10 lines of $W(3)$ disjoint from $\mathcal{T}$.*
- *The ten lines of $W(3)$ containing opposite points of $\mathcal{T}$ form a spread of $W(3)$, that is, a partition of the point set of $W(3)$ into lines.*

The geometry $\mathcal{T}$ is self-dual, even self-polar, see [3]. A polarity using our description is for instance given by the mapping

$$(i_1, i_2) \mapsto \{(i_3, i_4), (i_4, i_5), (i_5, i_3)\},$$

where $\{i_1, i_2, i_3, i_4, i_5\} = \{1,2,3,4,5\}$ and the permutation $(i_1\ i_2\ i_3\ i_4\ i_5)$ belongs to a preassigned conjugacy class of elements of order 5 in $\mathsf{Alt}(5)$. There are two such conjugacy classes of elements of order 5, and this gives rise to two distinct polarities, which differ by the opposition map.

This polarity cannot be induced by a duality of $W(3)$ as the latter is not self-dual. The question can be asked whether every collineation of $\mathcal{T}$ is induced by a collineation of $W(3)$. We now show that the answer is positive. To that aim we prove that $W(3)$ can be canonically recovered from $\mathcal{T}$. Given the abstract configuration $\mathcal{T}$, we define the following geometry $\Gamma = (P, \mathcal{L})$. The point set P consists of the union of the point set of $\mathcal{T}$ and the set

$$\{(\{p, \overline{p}\}, L) : p \in P_{\mathcal{T}}, L \in \mathcal{L}_{\mathcal{T}}, L \cap (p^{\perp} \cup \overline{p}^{\perp}) = \emptyset\},$$

where $p^{\perp}$ denotes the set of points of $\mathcal{T}$ collinear to p. It is easy to see that, for each $p \in P_{\mathcal{T}}$, there are exactly two lines of $\mathcal{T}$ not containing any point collinear to p or $\overline{p}$ (and those two lines are mutually opposite). Also, given a line $L \in \mathcal{L}_\mathcal{T}$, there is a unique pair of opposite points $p, \overline{p}$ with the property that neither of them is collinear to a point of L. Hence each line L defines a unique new point $(\{p, \overline{p}\}, L)$, which we denote by p_L. So in total, we have 40 points. We now define three types of lines of Γ.

Type 1 If L is a line of $\mathcal{T}$, then $L \cup \{p_L\}$ is a new line of Type 1.

Type 2 If $p \in P_\mathcal{T}$, then $\{p, \overline{p}\} \cup \{(\{p, \overline{p}\}, L) : L \in \mathcal{L}_\mathcal{T}, L \cap (p^\perp \cup \overline{p}^\perp) = \emptyset\}$ is a line of Type 2.

Type 3 Let L_1, L_2, L_3, L_4 be four pairwise disjoint lines of $\mathcal{T}$. Let $(\{p_i, \overline{p}_i\}, L_i)$, $i = 1, 2, 3, 4$, be the corresponding new points. If $P_\mathcal{T} = \{p_i, \overline{p}_i : i \in \{1, 2, 3, 4\}\} \cup L_1 \cup L_2 \cup L_3 \cup L_4$, then $\{p_{L_1}, p_{L_2}, p_{L_3}, p_{L_4}\}$ is a line of Type 3.

Type 3 lines require some explanation. First of all, it is clear that, if $p_1 = (a, b)$ and $p_2 = (c, d)$, then $\{a, b\} \cap \{c, d\} \neq \emptyset$, because otherwise L_2 contains one of (a, b) or (b, a), and we cannot obtain $P_\mathcal{T}$. Hence without loss, we can assume that $p_i = (5, i)$, for all $i \in \{1, 2, 3, 4\}$. Then there are only two possibilities for L_1, L_2, L_3, L_4 anymore. Indeed, if $L_1 = \{(2, 3), (3, 4), (4, 2)\}$, then L_j, $j = 2, 3, 4$ must be equal to $\{(4, 3), (3, 1), (1, 4)\}$, $\{(2, 4), (4, 1), (1, 2)\}$ and $\{(3, 2), (2, 1), (1, 3)\}$, respectively. The other possibility is by applying opposition all these points. Hence, for each member $j \in \{1, 2, 3, 4, 5\}$, we have exactly two lines of Type 3. So in total we have $20 + 10 + 10 = 40$ lines. Now it can be checked easily that Γ is a generalised quadrangle isomorphic to $W(3)$.

Hence every collineation (not duality) of $\mathcal{T}$ extends to a (unique) collineation of $W(3)$.

It can also be shown that the inclusion $\mathcal{T} \subseteq W(3)$ is unique, but we shall not insist on that.

3.2 Subconfigurations from (dual) geometric hyperplanes

A *geometric hyperplane* of a configuration is a subset H of the point set with the property that every line either has all its points in H, or intersects H in a unique point. A *dual geometric hyperplane* is the dual of that, hence a subset G of the line set with the property that for every point p either all the lines through p are in G, or a unique line through p is.

The interest in (dual) geometric hyperplanes for us lies in the fact that, removing a (dual) geometric hyperplane of size s, together with all lines (points) completely contained in it, from a configuration (v_k), always gives a $((v - s)_k, v'_{k-1})$-configuration, or (in the dual case) a $(v'_{k-1}, (v - s)_k)$-configuration, where $v' = k\frac{v-s}{k-1}$. To find geometric realisations of a given configuration, it can help to first find those of such subconfigurations. We give two examples, one with a dual geometric hyperpane and one with a geometric hyperplane. First a dual geometric hyperplane.

In the description of $W(3)$ given in Subsection 2.2, the lines $\{(i+), (j-), (ij+), (ij-)\}$, $i, j \in \{1, 2, 3, 4\}$, $i \neq j$, form a dual geometric hyperplane G. Removing all lines of G and all points $(i+)$ and $(i-)$, $i \in \{1, 2, 3, 4\}$, from $W(3)$ gives rise to a geometric $(32_3, 24_4)$ configuration. A realisation is provided in Figure 2. It has a rotational symmetry of order 4.

An example of a geometric hyperplane is given by the set H_p of all points collinear to a given point p. Removing such a set of points, together with all lines through p, gives rise to a $(27_4, 36_3)$-configuration, which is a subconfiguration of the unique triangle free $(27_5, 45_3)$-configuration, which is the unique generalised quadrangle with 3 points per line and 5 lines per point. It is realisable by Theorem 1.4 of [16].

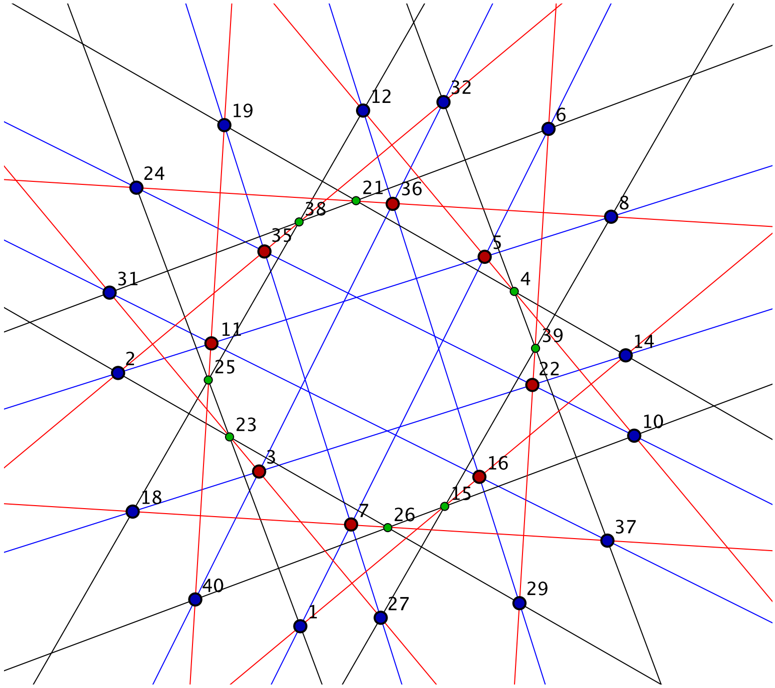

Figure 2: A realised part of the questionable triangle free (40_4) configuration, an incidence structure with 32 3-valent points and 24 4-valent lines.

Remark 3.1. Another example of a dual geometric hyperplane G_L is given by a line L and the set of lines intersecting L nontrivially. The intersection of the configurations arising as complements of H_p and G_L, for a point p incident with L is the dual of the so-called *Gray configuration*, that is, dual to the triple Cartesian product $K \times K \times K$ of a line K of size 3 with itself. More information about the Gray configuration can be found in [13].

3.3 An incidence structure with 40 points and 35 lines

The group $\mathrm{Aut}(\mathsf{PSp}_4(3))$ contains a single conjugacy class of elements of order 5. Each such element acts fixed-point freely on $W(3)$, and hence semi-regularly (this can immediately be deduced from the character table in [7]). Therefore, $W(3)$ is a *polycyclic* configuration. Remarkably, if we remove one line orbit, then we can realise the rest of $W(3)$. This is shown in Figure 3.

Starting with this geometric point-line incidence structure of 40 points and 35 lines, we will be able to construct a circle configuration in Section 5.

Figure 3: A realised part of the questionable triangle free (40_4) configuration, an incidence structure with 40 points and 35 lines

4 Topological solution

In this section we provide our first main result of this article.

Theorem 4.1. *We have a topological* (40_4) *configuration that represents the given finite generalised quadrangle* $W(3)$

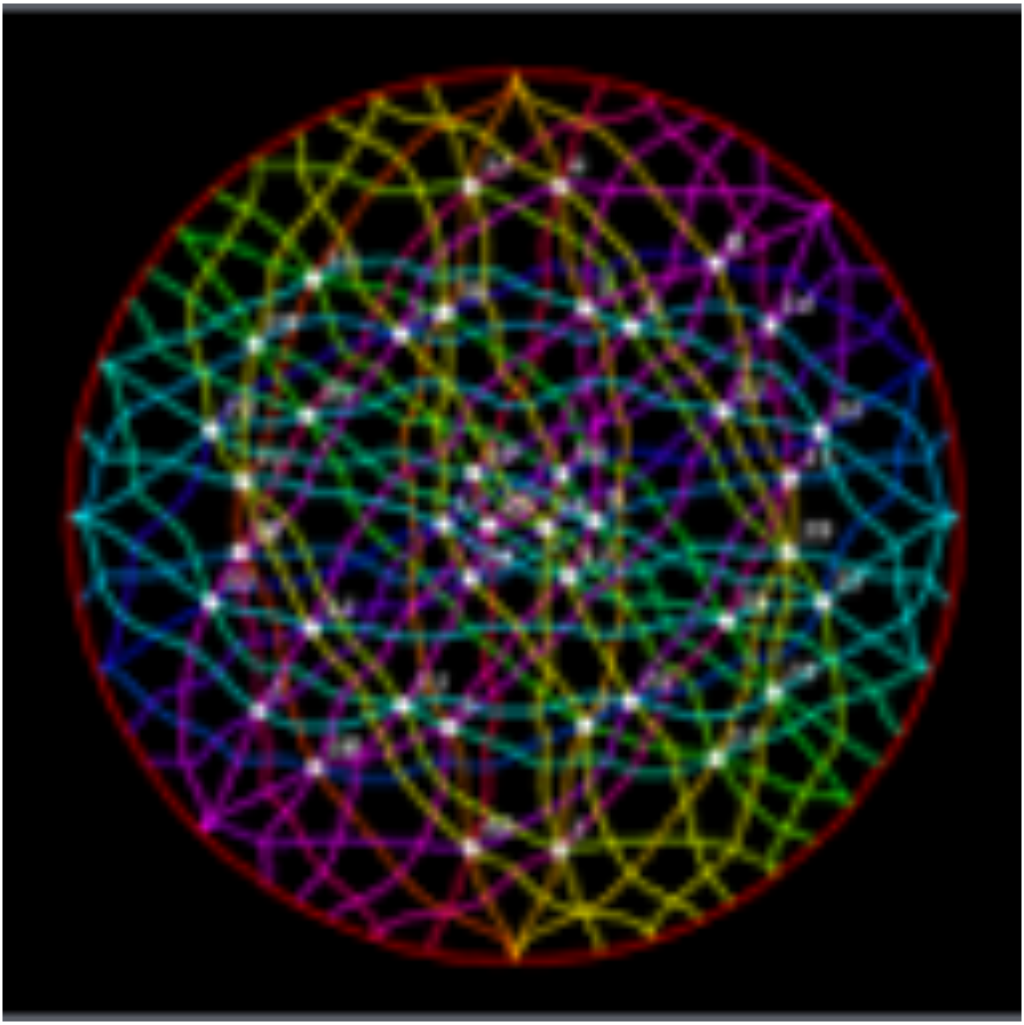

Figure 4: A topological triangle free (40_4) point-line configuration, i.e., a pseudoline arrangement, with a symmetry of order 2.

Proof. We describe our result according to the picture in Figure 4. It shows the pseudoline arrangement with a two-fold symmetry about a vertical axis (which is not drawn). The circular disc provides a model of the projective plane, the outer circle is not an element of the configuration. It is easy to confirm the properties of this configuration. □

The following proposition implies that our configuration is minimal.

Proposition 4.2. *A triangle free (n_4)-configuration must have at least 40 lines.*

Proof. This can be seen as follows: Consider a first point P of a triangle free (n_4)-configuration with its four lines L_1, L_2, L_3, and L_4 that are incident with P. On each of these four lines L_i, $i \in \{1, 2, 3, 4\}$ we have three additional points $P_{(i,j)}$, $j \in \{1, 2, 3\}$. There are three additional lines incident with each of these twelve points $P_{(i,j)}$, $j \in \{1, 2, 3\}$. These 36 lines have to be all different. Otherwise such a line forms a triangle together with P. This was our claim, the four lines we started with, together with these 36 lines have to be part of any triangle free (n_4)-configuration. □

5 A circle configuration representing $W(3)$

We also have a "realisation" of W(3) as a "circle configuration" in which 35 "circles" are degenerated, they are lines, see Figure 5. It has a rotational symmetriy of order 5, the same symmetry as the Levi graph of Figure 1. A realisation with 40 proper circles can be obtained by applying inversion.

6 Realisation in higher dimensions and over other fields

In this section, we further motivate the study of the (40_4) configuration $W(3)$.

In [16], the geometric realisations of all so-called *classical* generalised quadrangles in finite projective spaces of dimension at least 3 are studied, except for the class of symplectic generalised quadrangles. The reason is that all methods break down for these examples. Now, for the other classes, the generic result is that, up to a very few exceptions, if a quadrangle defined over a finite field $\mathbb{F}_q$ of order q admits a representation spanning a projective space of dimension at least 3 defined over the field $\mathbb{F}_{q'}$ of q' elements, then $\mathbb{F}_q$ is a subfield of $\mathbb{F}_{q'}$ and the representation in obtained by a field extension and a (possibly trivial) projection of the standard representation. Although the results in [16] are stated and proved for finite projective spaces, most results also hold for the infinite case, in particular over the reals and the complex numbers. We summarise the results for $Q(4, q)$ below in Theorem 6.1, but first we'd like to point out that, as a consequence of the results in [16], in the generic case, the characteristic of the field over which the quadrangle is defined coincides with the characteristic of the field over which the projective space is defined. If this is not the case, the representation has been called *grumbling* in [17]. Hence, any representation of a finite (classical) quadrangle in a real or complex projective space is necessarily grumbling.

The following theorem can be proved similar to the results in [16].

Theorem 6.1. *Let $Q(4, q)$ be the dual of $W(q)$ and let $\mathbb{P}_n(k)$ be the n-dimensional projective space over the field k, with $n \geq 3$. Then $Q(4, q)$ admits a grumbling representation spanning $\mathbb{P}_n(k)$, for some $n \geq 3$, if and only if either $q = 2$ (and k is any field), or $q = 3$ and k admits a nontrivial cubic root of -1, say ζ. Let k' be the prime field of k. Then, if $q = 2$, the embedding is a (possibly trivial) projection of a projectively unique embedding in $\mathbb{P}_4(k')$. If $q = 3$, then the embedding is a (possibly trivial) projection of a projectively unique embedding in $\mathbb{P}_4(k')$ (if $\zeta \in k'$) or $\mathbb{P}_4(k'(\zeta))$ (if $\zeta \notin k'$).*

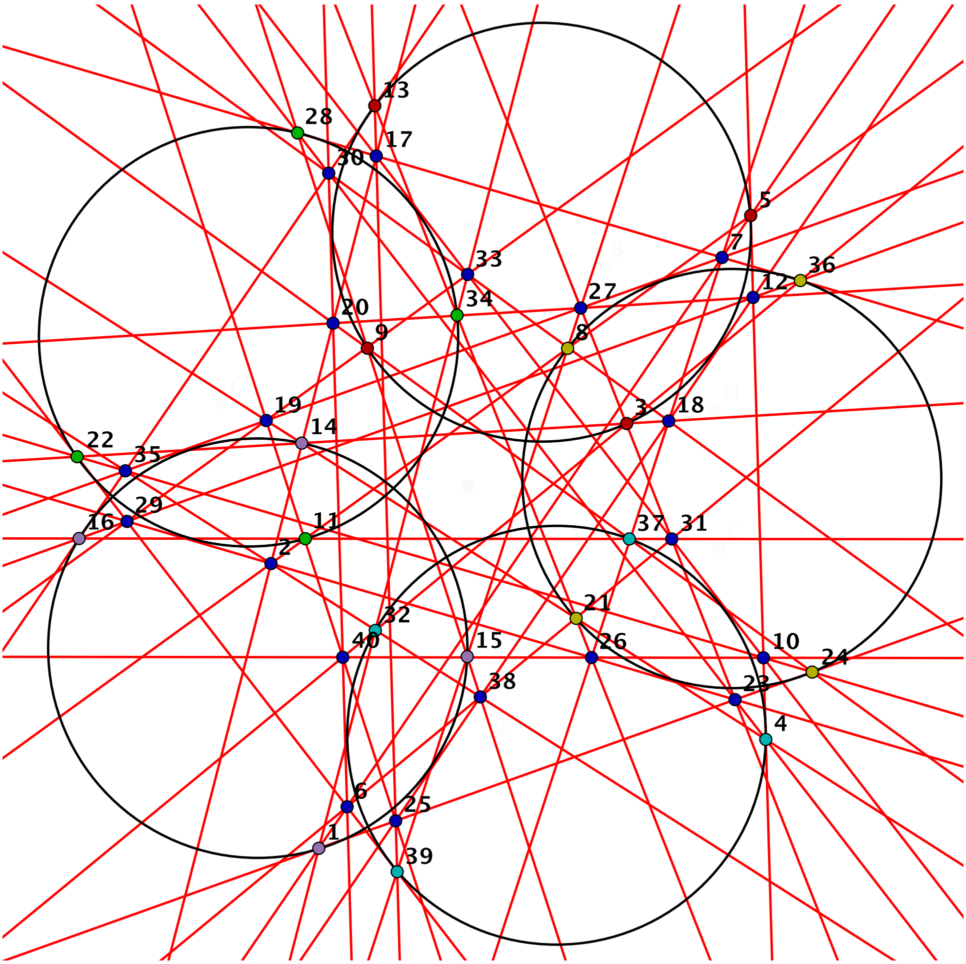

Figure 5: (40_4) circle configuration.

A similar result for $W(q)$, q odd, is now known, and probably out of reach for the moment (although we do know that $W(q)$ does not admit a representation spanning $\mathbb{P}_n(k)$, for $n \geq 4$ and any skew field k). That is why the geometry $W(3)$ is interesting to us. It is the smallest case for which we do not know a result like the previous theorem, and it is small enough to possibly behave exceptionally. In general, it is the belief that $W(q)$ does not admit a grumbling embedding, but the case $q = 3$ could be exceptional. In fact, we will now show that it does admit a grumbling embedding, but unfortunately, not over the reals, though it does over the complex numbers. To that aim, we classify its embeddings spanning a projective 3-space and such that, with the notation of Section 2.2, the points $(i+)$, $i \in N$, are contained in a single line, and the same thing holds for the points $(i-)$, $i \in N$.

Theorem 6.2. *The abstract* (40_4) *configuration* $W(3)$ *admits no representation spanning* $\mathbb{P}_n(k)$ *for* $n \geq 4$*. It admits a unique grumbling representation spanning* $\mathbb{P}_3(k)$*, for* k *a field, with the property that, with the notation of Section 2.2, the points* $(i+)$*,* $i \in N$*, are contained in a single line, and the same thing holds for the points* $(i-)$*,* $i \in N$*, if and only the characteristic of* k *is not equal to* 2 *and* k *admits a nontrivial cubic root of* -1*.*

Proof. Suppose $W(3)$ admits a representation spanning $\mathbb{P}_n(k)$, $n \geq 3$. We show that $n = 3$. The lines $\{(11+), (22+), (33+), (44+)\}$ and $\{(11-), (22-), (33-), (44-)\}$ span a subspace of dimension at most 3. But now all points must be contained in that subspace, since $(i\epsilon)$ is contained in the line defined by $(ii+)$ and $(ii-)$, for all $i \in N$, and the arbitrary point $(ij\epsilon)$, with $i, j \in N$ and $\epsilon \in \{+, -\}$ is contained in the line defined by $(i+)$ and $(j-)$. Hence $W(3)$ spans a subspace of dimension at most 3 and so $n \leq 3$.

Now suppose $n = 3$ and the points $(i+)$, $i \in N$, are contained in a single line, and the same thing holds for the points $(i-)$, $i \in N$. We can introduce coordinates in $\mathbb{P}_3(k)$ in the following way (where "$\longrightarrow$" means "gets the coordinates").

$$\begin{array}{rcl}
(1+) & \longrightarrow & (1,0,0,0), \\
(1-) & \longrightarrow & (0,1,0,0), \\
(2+) & \longrightarrow & (0,0,1,0), \\
(2-) & \longrightarrow & (0,0,0,1), \\
(3+) & \longrightarrow & (1,0,1,0), \\
(11+) & \longrightarrow & (1,1,0,0), \\
(22+) & \longrightarrow & (0,0,1,1).
\end{array}$$

We denote the line of $\mathbb{P}_3(k)$ joining the points P and Q by $\langle P, Q\rangle$. Expressing that $\langle (3+), (3-)\rangle$ and $\langle (11+), (22+)\rangle$ meet in $(33+)$, and that $(3-)$ belongs to $\langle (1-), (2-)\rangle$ by assumption, we obtain

$$\begin{array}{rcl}
(3-) & \longrightarrow & (0,1,0,1), \\
(33+) & \longrightarrow & (1,1,1,1).
\end{array}$$

Since the point $(4+)$ belongs to $\langle (1+), (2+)\rangle$, there exists $x \in k$ so that $(4+)$ has coordinates $(x, 0, 1, 0)$. Expressing that $(4+), (4-)$ and $(44+)$ are collinear, that $(4-) \in \langle (1-), (2-)\rangle$ and $(44+) \in \langle (11+), (22+)\rangle$, we easily see that $(4-)$ has coordinates $(0, x, 0, 1)$ and $(44+)$ has coordinates $(x, x, 1, 1)$.

Now we consider the line defined by θ_0 and $(11+)$. Since $\theta_0\theta_0$ has exactly one fixed point in N, the points $(23+), (34+)$ and $(41+)$ are on a line with $(11+)$ and belong to $\langle (2+), (3-)\rangle$, $\langle (3+), (4-)\rangle$ and $\langle (4+), (1-)\rangle$, respectively. Hence, we can give $(23+)$ the coordinates $(0,1,a,1)$, for some $a \in k$, so that $(34+)$ gets assigned $(b, b+1, a, 1)$, for some $b \in k$. Since $(34+) \in \langle (3+), (4-) \rangle = \langle (1,0,1,0), (0,x,0,1) \rangle$, we see that $a = b = x-1$. Finally, the point $(41+)$ is the intersection of $\langle (4+), (1-) \rangle = \langle (x,0,1,0), (0,0,0,1) \rangle$ and $\langle (11+), (23+) \rangle = \langle (1,1,0,0), (0,1,x-1,1) \rangle$, which easily implies $(41+) = (x^2 - x, 0, x-1, 1) = (-1, 0, x-1, 1)$. This is only possible if $x^2 - x + 1 = 0$, hence if x is a nontrivial third root of -1, since our assumption "grumbling" implies that the characteristic of k is unequal to 3. So we can put $x = \zeta$, with ζ one of the two nontrivial cubic roots of -1. We now calculate:

$$\begin{array}{rcl}
(4+) & \longrightarrow & (\zeta, 0, 1, 0), \\
(4-) & \longrightarrow & (0, \zeta, 0, 1), \\
(44+) & \longrightarrow & (\zeta, \zeta, 1, 1), \\
(23+) & \longrightarrow & (0, -\zeta, 1, -\zeta), \\
(34+) & \longrightarrow & (1, -\zeta^2, 1, -\zeta), \\
(42+) & \longrightarrow & (\zeta, 0, 1, -\zeta).
\end{array}$$

In a similar way, we calculate the points on the line defined by θ_0^{-1} and $(11+)$.

$$\begin{array}{rcl}
(24-) & \longrightarrow & (0, \zeta, -\zeta, 1), \\
(32-) & \longrightarrow & (-\zeta, 0, -\zeta, 1), \\
(43-) & \longrightarrow & (-\zeta^2, 1, -\zeta, 1).
\end{array}$$

We continue similarly with calculating the coordinates of the points of the lines $\{(22+), (13-), (34-), (41-)\}$ and $\{(22+), (14+), (31+), (43+)\}$:

$$\begin{array}{rcl}
(13-) & \longrightarrow & (-\zeta, 1, 0, 1), \\
(34-) & \longrightarrow & (1, \zeta^2, 1, \zeta), \\
(41-) & \longrightarrow & (\zeta, -1, 1, 0), \\
(14+) & \longrightarrow & (1, -\zeta, 0, -1), \\
(31+) & \longrightarrow & (1, -\zeta, 1, 0), \\
(43+) & \longrightarrow & (\zeta^2, 1, \zeta, 1).
\end{array}$$

Comparing $(43+)$ and $(43-)$, or equivalently, $(34+)$ and $(34-)$, we see that $\zeta \neq -\zeta$, implying that the characteristic of k cannot be equal to 2.

Continuing like this, we obtain the coordinates of all remaining points.

$$\begin{array}{rcl}
(11-) & \longrightarrow & (1, -1, 0, 0), \\
(12+) & \longrightarrow & (1, 0, 0, \zeta), \\
(12-) & \longrightarrow & (1, 0, 0, -\zeta), \\
(13+) & \longrightarrow & (\zeta, 1, 0, 1), \\
(14-) & \longrightarrow & (1, \zeta, 0, 1), \\
(21+) & \longrightarrow & (0, 1, -\zeta, 0), \\
(21-) & \longrightarrow & (0, 1, \zeta, 0), \\
(22-) & \longrightarrow & (0, 0, 1, -1), \\
(23-) & \longrightarrow & (0, \zeta, 1, \zeta), \\
(24+) & \longrightarrow & (0, \zeta, \zeta, 1), \\
(31-) & \longrightarrow & (1, \zeta, 1, 0), \\
(32+) & \longrightarrow & (\zeta, 0, \zeta, 1), \\
(33-) & \longrightarrow & (1, -1, 1, -1), \\
(41+) & \longrightarrow & (\zeta, 1, 1, 0), \\
(42-) & \longrightarrow & (\zeta, 0, 1, \zeta), \\
(44-) & \longrightarrow & (\zeta, -\zeta, 1, -1).
\end{array}$$

It is not difficult to check now that the four points on any line of $W(3)$ are collinear in $\mathbb{P}_3(k)$. This concludes the proof of Theorem 6.2. $\square$

One also checks that the group induced on $W(3)$ by the linear transformation of $\mathbb{P}_3(k)$ is isomorphic to $2 \times \mathsf{Alt}(4) \times \mathsf{Alt}(4)$. The first 2 is realised by the involution sending (x_0, x_1, x_2, x_3) to $(x_0, -x_1, x_2, -x_3)$ and fixes all points of shape $(i\epsilon)$, $i \in N$ and $\epsilon \in \{+, -\}$. The $\mathsf{Alt}(4)$ part can be derived from the mapping

$$(x_0, x_1, x_2, x_3) \mapsto (x_0, x_1 - \zeta x_3, x_2, -\zeta x_3)$$

and the uniqueness of the representation.

Discussion. Since $\mathbb{R}$ does not admit nontrivial cubic roots of -1, and $\mathbb{C}$ does, we deduce that $W(3)$ is not embeddable in $\mathbb{P}_3(\mathbb{R})$ with the restrictions of Theorem 3, but it is embeddable in the complex plane, and also as a spanning set of points in complex 3-space (this was not known before). Hence it feeds our conjecture stated before.

It is perhaps remarkable that the condition of k having a nontrivial cubic root of -1 turns up in both theorems of this section. The explanation could be that every planar grumbling representation of $W(3)$ and of $Q(4, q)$ over a field k arises from a projection of a 3-dimensional spacial grumbling representation over the field k. In that case, a planar grumbling embedding of $Q(4, q)$ in $\mathbb{P}_2(k)$ exists if and only if k admits a nontrivial cubic root of -1. In the dual plane, this gives rise to a grumbling embedding of $W(3)$. Hence the existence conditions for grumbling embeddings of $W(3)$ and $Q(4, 3)$ are exactly the same! This, however, leaves us wondering about the additional condition of Theorem 6.2, namely that the characteristic of k is not 2. This could be explained by the fact that the condition, for each $\epsilon \in \{+, -\}$, of the four points $(i\epsilon)$, $i \in N$, being collinear, is too strong in the characteristic 2 case.

If our claim that every planar grumbling embedding of $W(3)$ is obtained from a 3-dimensional one is right, then there certainly exist embeddings spanning $\mathbb{P}_3(k)$, with k not of characteristic 2 or 3, and k admitting nontrivial roots of unity, such that the points of no dual grid are contained in two lines of $\mathbb{P}_3(k)$. Indeed, there are projections of $Q(4, 3)$ that do not satisfy the dual of this condition (as the dual of that condition is never satisfied in any 4-dimensional representation of $Q(4, 3)$ (meaning to span the 4-space), and we can choose the projection line appropriately).

However, as already mentioned, it is not clear whether proving the claims in this discussion is feasible. For the moment we either have to make assumptions that make the calculations feasible, or use ad hoc methods and trial and error to find a representation.

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On polyhedral realizations of Hurwitz's regular map $\{3, 7\}_{18}$ of genus 7 with geometric symmetries*

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Abstract

In 2017 a first selfintersection-free polyhedral realization of Hurwitz's regular map $\{3, 7\}_{18}$ of genus 7 was found by Michael Cuntz and the first author. For any regular map which had previously been realized as a polyhedron without self-intersections in 3-space, it was also possible to find such a polyhedron with nontrivial geometric symmetries. So it is natural to ask of whether we can find for the above-mentioned regular map a corresponding version with some non-trivial geometric symmetry. The orientation-preserving combinatorial automorphism group of this Hurwitz map is the projective special linear group $\mathbf{PSL}(2, 8)$ of order $504 = 2^3 \cdot 3^2 \cdot 7$. All non-trivial subgroups of $\mathbf{PSL}(2, 8)$ are candidates for such a geometric symmetry. Using the GAP software for exploring the subgroup structure, we found that it is sufficient to consider only four cyclic subgroups whose order is 9, 7, 3, and 2, respectively. We prove that there are obstructions for selfintersection-free polyhedral realizations of the Hurwitz map $\{3, 7\}_{18}$ of genus 7 with geometric rotational symmetries of order 9 or 3. We provide new small integer coordinates within the realization space known from 2017, which are also suitable for making a 3D-printed model. We present Kepler–Poinsot type realizations, both with 7-fold and with 3-fold rotational symmetry, the latter with integer coordinates.

IN MEMORY OF BRANKO GRÜNBAUM.

Keywords: Hurwitz surface, regular map, Kepler–Poinsot type polyhedron.

Math. Subj. Class.: 51M20, 52B70

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1 Introduction

A *Hurwitz surface*, named after Adolf Hurwitz, is a compact Riemann surface with the maximum number $84(g-1)$ of conformal automorphisms, where g is the genus of the surface [17]. The smallest Hurwitz surface has genus 3, and in fact it was already known to Felix Klein [19] (called after him the *Klein quartic*). The next Hurwitz surface, of genus 7, was discovered by Fricke [21], and it was Macbeath who proved its uniqueness. Macbeath also proved the existence of an infinite family of Hurwitz surfaces [22, 23]; therefore, it is also called the *Fricke–Macbeath surface* [18].

For our purposes in this paper we use the classical notion of a *regular map* as given by Coxeter and Moser [12]. Thus, a *map* $\mathcal{M}$ is defined as a decomposition of a 2-manifold into simply connected, non-overlapping regions called *faces* by means of arcs called *edges*. The intersections of the edges are called the *vertices* of $\mathcal{M}$. If all the faces are p-sided polygons, and q faces meet in each vertex, then we use the symbol $\{p, q\}$ for giving the *Schäfli type* of $\mathcal{M}$. If, moreover, we require that the automorphism group of $\mathcal{M}$ is transitive on the flags, i.e., on the mutually incident triples of a vertex, an edge, and a face, then we say that $\mathcal{M}$ is a *regular map*. A *Petrie polygon* of a regular map is an edge-path such that any two but no three successive edges are the edges of a common face; in a finite regular map each Petrie polygon is of the same length. A regular map of type $\{p, q\}$, determined by the length r of its Petrie polygon, is denoted by $\{p, q\}_r$. (For more details on regular maps, see [7, 11, 24, 33]).

A *Hurwitz group* is defined as a finite abstract group which can be generated by an element t of order 2 and an element u of order 3 whose product tu has order 7 [9, 23]. The automorphism group of a Hurwitz surface is a Hurwitz group, and it can be obtained in the following way. Consider the regular map $\{3, 7\}$ on the hyperbolic plane, and take the group of its orientation-preserving automorphisms (following Coxeter, we denote this group by $[3, 7]^+$). Now the automorphism groups of Hurwitz surfaces are precisely the non-trivial finite quotient groups of $[3, 7]^+$, by some suitable normal subgroup. The quotients of the regular map $\{3, 7\}$ by the same normal subgroups are the *Hurwitz maps* $\{3, 7\}_r$ on the corresponding Hurwitz surfaces. The smallest Hurwitz map, obtained in this way on the Klein quartic, is $\{3, 7\}_8$. The next one, the Fricke–Macbeath map, is $\{3, 7\}_{18}$–the subject of this paper.

Macbeath has also shown that there is a subclass of the projective special linear groups $\mathbf{PSL}(2, q)$ consisting of Hurwitz groups. In particular, $\mathbf{PSL}(2, 7)$ and $\mathbf{PSL}(2, 8)$ is isomorphic to the automorphism group of the Hurwitz surface of genus 3 and of genus 7, respectively.

By a *polyhedron* P we mean a compact 2-manifold embedded in Euclidean 3-space, made up of finitely many convex polygons, called the *faces* of P, such that any two distinct polygons are disjoint, or intersect in a common vertex (the *vertex* of P), or in a common edge (the *edge* of P) [7, 8, 26, 27, 29]. We require that no two adjacent faces lie in the same plane.

In analogy with the regular maps, we say that a polyhedron P is *combinatorially regular* if its combinatorial automorphism group is transitive on the flags of P. Thus, a combinatorially regular polyhedron is a polyhedral realization in $\mathbb{E}^3$ of a regular map on an orientable surface of some genus g. A summary of regular maps whose geometric realizations are known is given in Table 1 (as an extension of Table 1 in [34]).

Note that in our definition of a polyhedron, the condition that it is *embedded*, excludes self-intersections.

Genus g	Type	f-Vector	Author	Polyhedral realization	Group order	Conder's notation [10]
			Sporadic examples			
3	$\{3,7\}_8$	$(24,84,56)$	Klein [19, 20]	[30]	336	$R3.1$
3	$\{3,8\}_6$	$(12,48,32)$	Dyck [13, 14]	[6, 1]	192	$R3.2$
5	$\{3,8\}$	$(24,96,64)$	Klein & Fricke [21]	[16]	384	$R5.1$
5	$\{4,5\,\vert\,4\}$	$(32,80,40)$	Coxeter [11]	[26, 27, 25]	320	$R5.3$
5	$\{5,4\,\vert\,4\}$	$(40,80,32)$	Coxeter [11]	[26, 27, 25]	320	$R5.3^*$
6	$\{4,6\,\vert\,3\}$	$(20,60,30)$	Boole Stott [5], Coxeter [11]	[26, 31]	240	$R6.2$
6	$\{6,4\,\vert\,3\}$	$(30,60,20)$	Boole Stott [5], Coxeter [11]	[26, 31]	240	$R6.2^*$
7	$\{3,7\}_{18}$	$(72,252,168)$	Hurwitz [17]	[3]	1008	$R7.1$
73	$\{4,8\,\vert\,3\}$	$(144,576,288)$	Boole Stott [5], Coxeter [11]	[26, 31]	2304	$R73.13$
73	$\{8,4\,\vert\,3\}$	$(288,576,144)$	Boole Stott [5], Coxeter [11]	[26, 31]	2304	$R73.13^*$
			Infinite series with $r \geq 3$			
$1+(r-4)2^{r-3}$	$\{4,\mathrm{r}\}$		Coxeter [11]	[26, 27, 25]		
$1+(r-4)2^{r-3}$	$\{\mathrm{r},4\}$		Coxeter [11]	[26, 27, 25]		

Table 1: Regular maps of genus $g \geq 2$ having polyhedral realization.

Sometimes, for a regular polyhedron we drop this condition while preserving as many combinatorial symmetries of the map as possible as geometric symmetries for the polyhedron; in this case we speak of a *Kepler–Poinsot polyhedron*, or, in short, a *KP-type polyhedron* [8, 24, 29, 32, 33].

In this paper, we investigate the polyhedral realizability of the Hurwitz map $\{3,7\}_{18}$ of genus 7 from a symmetry point of view. In spite of its high combinatorial symmetry of order 1008, the polyhedral realization of this regular map in [3] had no geometric symmetry at all. Although all former polyhedral realizations of regular maps must have so-called *hidden symmetries*, see the corresponding references in [3], they all do have some non-trivial geometric symmetries as well. So, a natural problem has remained to find a polyhedral realization of this regular map $\{3,7\}_{18}$ with a non-trivial geometric symmetry. Such a non-trivial geometric symmetry must form a subgroup of the orientation-preserving automorphism group of $\{3,7\}_{18}$. As mentioned above, this is isomorphic to the projective special linear group $\mathbf{PSL}(2,8)$ of order $504 = 2^3 \cdot 3^2 \cdot 7$. This is a simple group, i.e., it has no normal subgroup. All non-trivial subgroups of $\mathbf{PSL}(2,8)$ are candidates for forming a geometric symmetry for our polyhedral realization in question. Each of these subgroups has among its generators elements of order 9, 7, 3, or 2. Hence, our aim was to find obstructions for each of the 4 cyclic subgroups generated by these elements, which would tell us that no non-trivial geometric symmetry can exist. We have used the GAP software [15] to confirm the subgroup structure. We study the various subgroup classes in Section 4. For these subgroups we show the following theorem.

Theorem 1.1. *A polyhedral realization of the Hurwitz map $\{3,7\}_{18}$ of genus 7 cannot have a geometric symmetry group generated by a single element of order 9 or 3.*

This theorem is proved below in two different parts, by the proofs given for Propositions 5.1 and 5.2, respectively, in Section 5.

We neither have a corresponding proof for a geometric symmetry of order 7, nor for a geometric symmetry by reflection in a line. However, we strongly believe after a series of attemps to find such polyhedral realizations, that there does not exist a symmetric polyhedral realization of Hurwitz's regular map of genus 7 at all. We provide some heuristic arguments in the next section to support our belief.

Even when some selfintersection-free polyhedral realization of a regular map exists (with or without non-trivial geometric symmetry), representing the same 2-manifold by a polyhedral object in which self-intersections do occur can also be interesting. Such an object, called a Kepler–Poinsot type polyhedron, can sometimes be constructed with higher geometric symmetry than under the former, more restricted condition; this may provide more insight into the structure of the map in question. In our case we found Kepler–Poinsot type realizations with 7-fold and 3-fold rotational symmetry; they are described in the last section.

Some further details of our investigation omitted from here can be found in the book [2].

2 The polyhedral realization with small integers

Based on the previous paper [3] on the polyhedral realization of the Hurwitz map $\{3,7\}_{18}$ of genus 7, here we provide a new version with small integer coordinates (see Table 2). A 3D printed model of this version is shown in Figure 1(a). We also present an exploded view of this model, made with the aid of the Blender 2.80 software, see Figure 1(b).

Table 2: Small integer coordinates for the polyhedral realization of the Hurwitz map $\{3,7\}_{18}$ of genus 7.

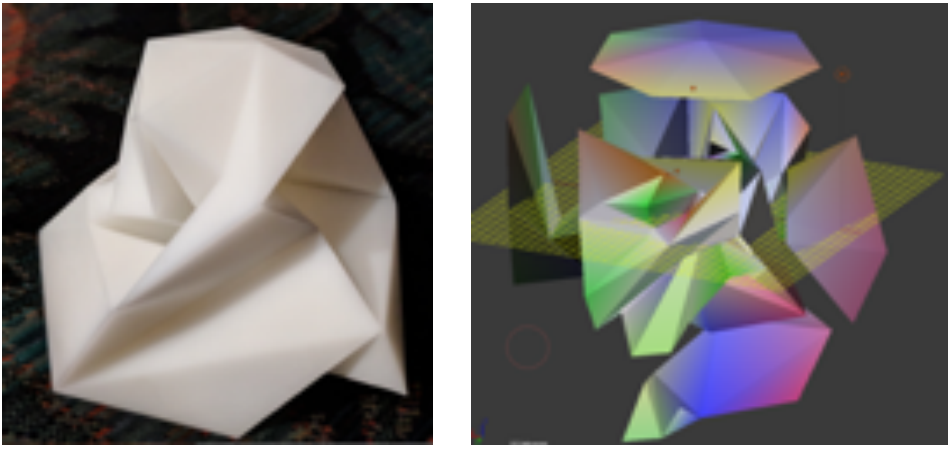
No.	x	y	z	No.	x	y	z	No.	x	y	z
1	0	0	27	2	13	-24	22	3	28	-6	22
4	22	18	22	5	0	29	22	6	-23	21	22
7	-28	-7	22	8	-12	-26	22	9	4	-20	18
10	12	-2	9	11	16	7	-12	12	5	6	9
13	-10	14	9	14	-24	24	10	15	-17	-14	6
16	-2	5	-15	17	14	5	-7	18	16	18	4
19	-14	12	18	20	-7	28	0	21	-14	-26	-14
22	-11	-10	10	23	4	0	-11	24	7	25	-21
25	18	22	1	26	-3	-2	12	27	-6	16	3
28	-30	-20	-12	29	-8	-9	0	30	12	23	-13
31	-14	5	-11	32	-17	10	19	33	-3	3	1
34	3	-26	-16	35	-7	-10	4	36	-10	23	-21
37	0	14	-13	38	-8	22	-5	39	-1	24	2
40	2	-28	5	41	-3	3	5	42	-7	5	-30
43	6	-16	-16	44	-24	24	4	45	-8	-6	-2
46	6	-22	8	47	10	15	-14	48	11	-6	-25
49	0	-22	-13	50	-17	15	-11	51	-22	8	15
52	0	2	-7	53	13	-27	-4	54	6	13	2
55	-7	27	-28	56	14	25	-26	57	-18	6	-8
58	-30	-1	-24	59	3	-8	-14	60	-1	14	-25
61	4	26	-31	62	17	25	-18	63	-10	-23	-9
64	-20	9	-7	65	-25	2	-8	66	-11	-2	-22
67	-2	4	-19	68	9	13	-22	69	10	1	-25
70	8	-6	-22	71	-18	-4	-15	72	-9	-4	-24

3 The $(7,2)$ torus knot within an embedding with a symmetry of order 7

In this section we provide some arguments why we believe that a symmetry of order 7 cannot lead to a polyhedral realization of our Hurwitz map with a symmetry of order 7. (We note that earlier considerations in this respect occur in paper [4].)

A combinatorial symmetry of order 7 and even an additional combinatorial symmetry of order 2 can be seen from Figure 2. We use the labels from [3]. Our Figures 3 and 4 show how one can split the topological Hurwitz surface of genus 7 into a sphere and a torus when we add 7 hexagons as *windows* both, for the sphere and for the torus. There is a closed polygon of length 14 in Figure 3 that does not touch any of these seven windows (connections) between the sphere and the torus of the surface. This closed polygon has to form a knot within a symmetric embedding that topologists call a $(7,2)$ *torus knot*.

We see this knot also when we look at the symmetric topological model in Figure 5 in which some labels have been marked. The symmetry of order 7 of this model with fixed points 1 and 72 and the same symmetry as in Figures 3 and 4 helps to confirm this $(7,2)$



(a) (b)

Figure 1: (a) 3D print of a polyhedral realization of Hurwitz's regular map of genus 7, based on the article [3] with small integer coordinates. (b) Exploded view of the same model; the upper part has been lifted; the bottom part has been dropped; two side parts have been moved to the outside. The *3D-Blender* software allows to move these parts on the screen.

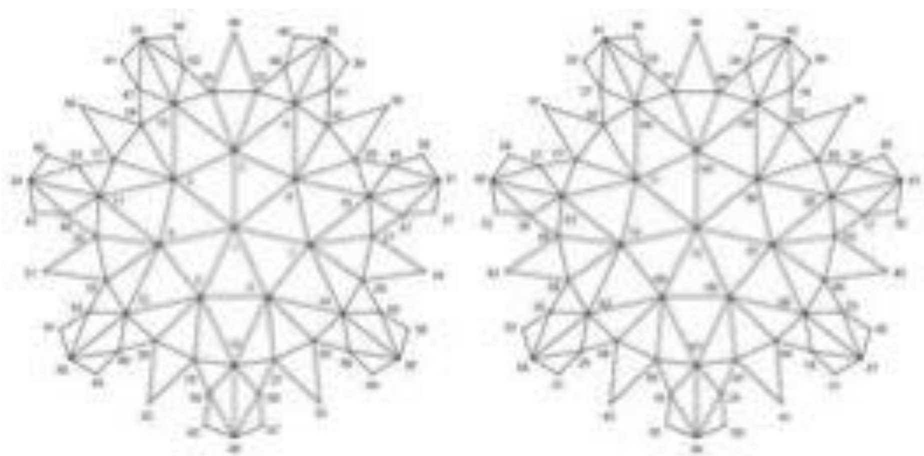


Figure 2: Diagram providing the combinatorial description of the Hurwitz map $\{3, 7\}_{18}$ of genus 7. The map is cut into two equal parts which are to be glued together at vertices with identical label.

torus knot. When we imagine adjacent triangles along this knot, we are tempted to believe that we find an obstruction at least for a symmetric polyhedral realization with a symmetry of order 7. However, for such a symmetry of order 7, we have found a polyhedral $(7, 2)$

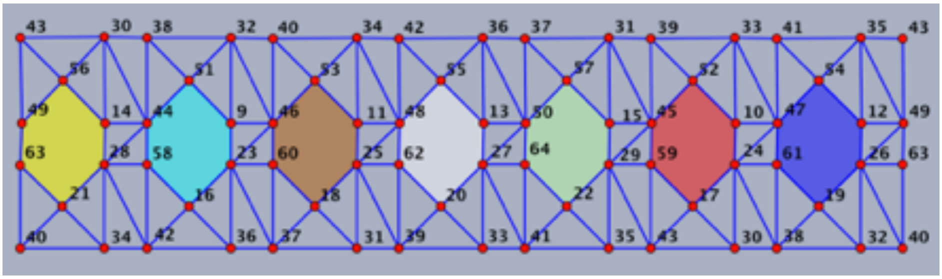


Figure 3: Triangles of the torus with seven holes each bounded by a polygon of length 6.

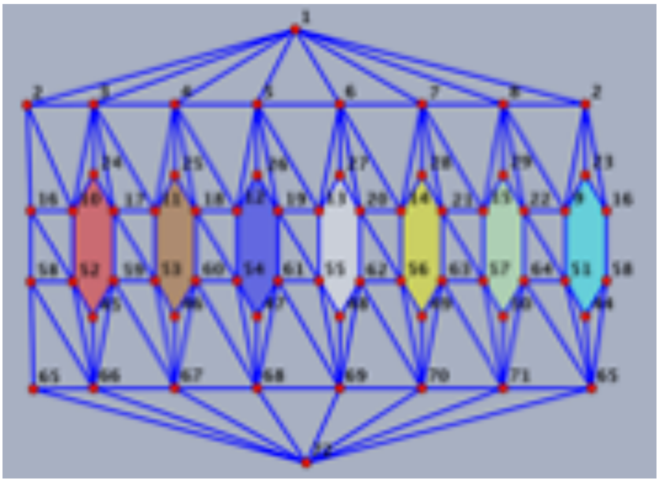


Figure 4: Triangles of the sphere with seven holes each bounded by a polygon of length 6.

torus knot in Figure 5. The situation changes when we continue to keep the symmetry of order 7 and when we add an upper topological disc with a topological circular boundary with vertices

$$16, 10, 24, 17, 11, 25, 18, 12, 26, 19, 13, 27, 20, 14, 28, 21, 15, 29, 22, 9, 23, 16,$$

and a lower topological disc with a topological circular boundary with vertices

$$58, 52, 45, 59, 53, 46, 60, 54, 47, 61, 55, 48, 62, 56, 49, 63, 57, 50, 64, 51, 44, 58.$$

The seven holes that we can see from above in the model should appear as well in the polyhedral realization to create the handles for a genus 7 manifold. All attempts to see at least these holes to come into being, have supported our belief. Observe that there are also many edges that connect the boundaries of both topological discs, and the $(7, 2)$ torus knot is not very flexible.

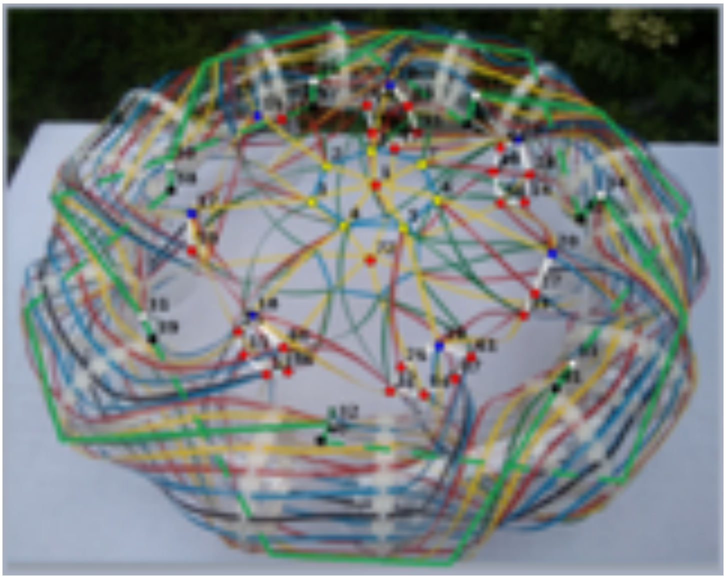


Figure 5: The green knotted polygon with vertices $43, 30, 38, 32, 40, 34, 42, 36, 37, 31, 39, 33, 41, 35, 43$ forms a $(7, 2)$ torus knot of the Hurwitz's regular map genus 7. We have called the white hexagons *windows* between the sphere and the torus.

A knot structure of an edge polygon of a 2-manifold is not determined by the face structure of the 2-manifold. You can have a torus and a knotted torus with the same boundary structure. As a consequence of a symmetry of order 7, we must have a $(7, 2)$ torus knot, however, our known realization has a different knot structure. It can be seen in Figure 6 in connection with Figure 7.

This knot structure cannot change within the realization space. This means that the realization space of the intersection-free polyhedral realization of our Hurwitz surface does not contain a symmetric version with a symmetry of order 7. The projection of a closed polygon that forms a $(7, 2)$ torus knot must have much more self-intersections.

By the way, observe that the sequence of coloured windows of the sphere is not the same as the sequence of coloured windows of the torus. The $(7, 2)$ torus knot structure provides the solution why this is not a contradiction. However, the $(7, 2)$ torus knot structure seems to be the key argument why a polyhedral realization was not found earlier by assuming a certain geometric symmetry. This torus knot structure is not very flexible when all triangles have to be flat (cf. Figure 8).

We add another heuristic argument that might even lead to a proof that there is an obstruction in case of a cyclic geometric symmetry of order 7. Figure 9 shows another embedding of a partial torus part with its $(7, 2)$ torus knot. In Figure 10 on the right one can see that the winding number of the boundary polygon of a circular topological disc around the axis of symmetry is different from 1. This leads clearly to an obstruction in this case. However, the realized torus structure with its $(7, 2)$ torus knot is not very flexible, and we can perhaps show that this leads probably in all cases to an obstruction.

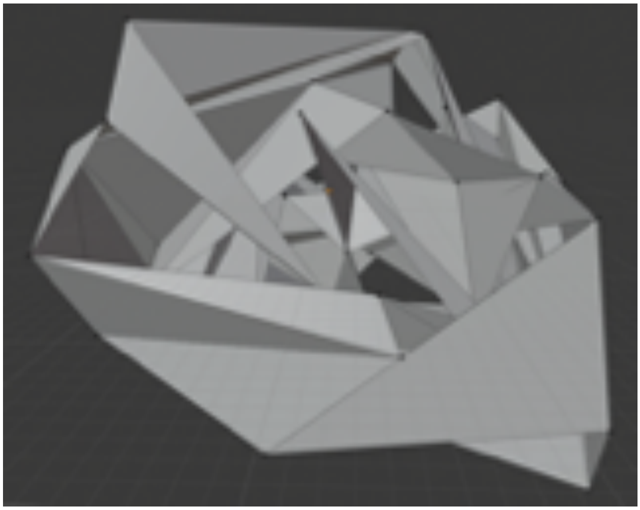


Figure 6: Knot structure within the embedded solution

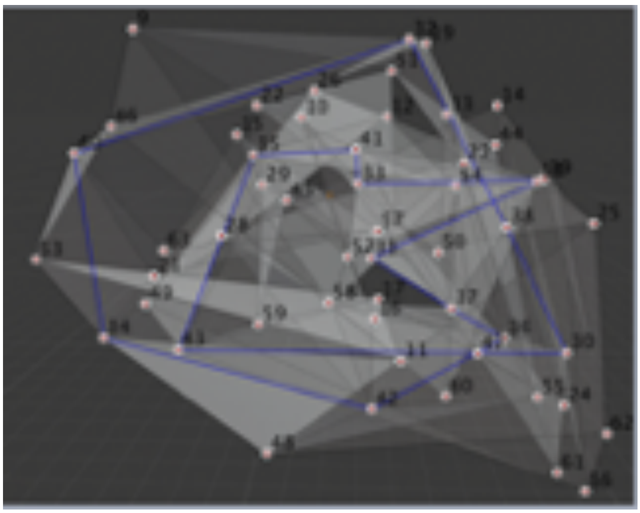


Figure 7: Knot structure with labels and transparent faces.

4 Conjugacy classes of subgroups of the orientation-preserving automorphism group of the Hurwitz map $\{3, 7\}_8$ of genus 7

For our purposes in this paper, it is sufficient to consider the group of orientation-preserving automorphisms of the Hurwitz map of genus 7, which we denote by $\mathbf{A}^0\{3,7\}_8$. This group is isomorphic to the group $\mathbf{PSL}(2,8)$. Its order is 504, and it is a subgroup of index 2 of the full automorphism group $\mathbf{A}\{3,7\}_8$ of the map.

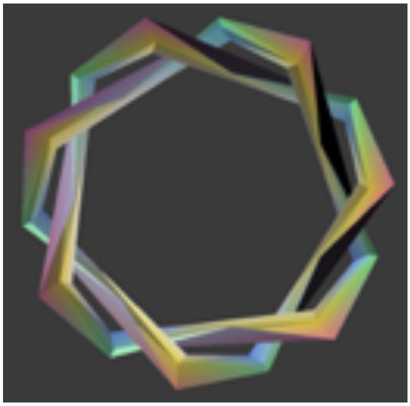


Figure 8: This figure shows a selfintersection-free $(7, 2)$ torus knot (formed by a polygon) with some but not all adjacent triangles along the former mentioned polygon with a rotational symmetry of order 7 (it might be that even all adjacent triangles of the knot cannot be realized).

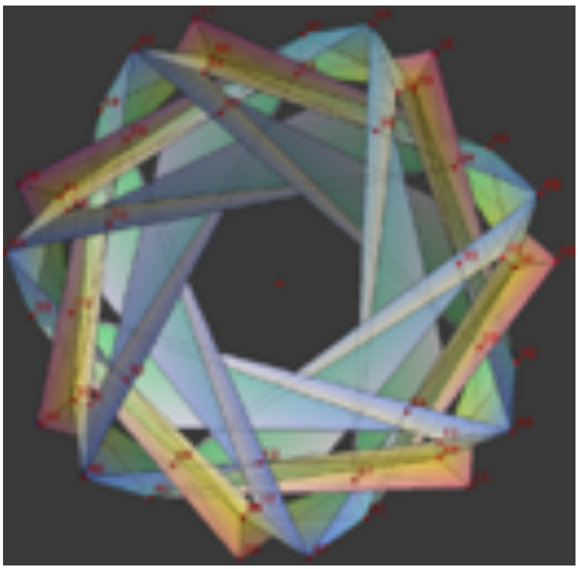


Figure 9: This is another partial embedding of a $(7, 2)$ torus knot with vertex labels.

Here we provide conjugacy classes of all subgroups of $\mathbf{PSL}(2, 8)$. In this investigation we have used the GAP software, see [15], installed on a Mac book Pro (OS X El Capitan) Version 10.11.6. We obtain the group as a permutation group on the set of vertices of our map $\{3, 7\}_{18}$, which is generated by the following two generators:

- a "rotation" r around a triangle that keeps the triangle fixed, and

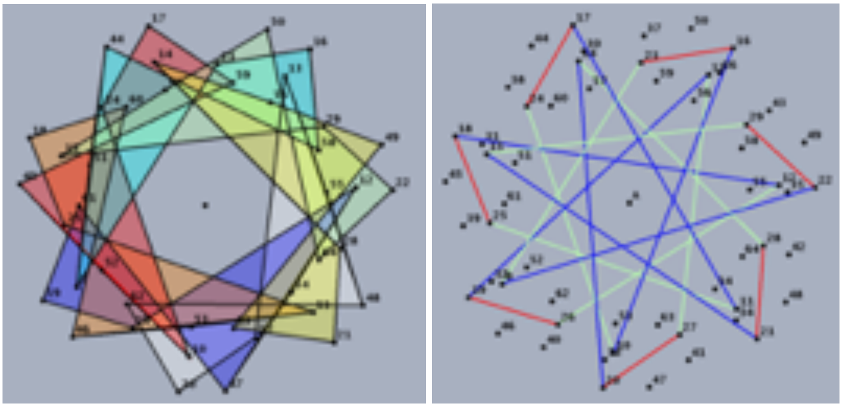


Figure 10: The same projection of Figure 9 with polygon structures.

- a "rotation" s around a fixed vertex.

These generators have been defined for GAP by using the labels as in the article [3]. The second generator s can easily be checked via Figure 2 that shows a symmetry of order 7 of the map. The first generator r can be checked likewise via Figure 18, which is given in Section 6 below.

Here one can see the GAP prompt and some GAP output.

```
gap> r;
(1,2,3)(4,8,10)(5,9,24)(6,23,17)(7,16,11)(12,22,47)(13,46,30)(14,36,53)(15,52,25)(18,29,33)(19,32,38)(20,37,59)
(21,58,48)(26,51,61)(27,60,43)(28,42,34)(31,45,39)(35,41,54)(40,44,55)(49,64,68)(50,67,56)(57,66,62)(63,65,69)(70,71,72)
gap> s;
(2,3,4,5,6,7,8)(9,10,11,12,13,14,15)(16,17,18,19,20,21,22)(23,24,25,26,27,28,29)(30,31,32,33,34,35,36)
(37,38,39,40,41,42,43)(44,45,46,47,48,49,50)(51,52,53,54,55,56,57)(58,59,60,61,62,63,64)(65,66,67,68,69,70,71)
```

The conjugacy classes of all subgroups of $\mathbf{PSL}(2,8)$ provided by the GAP program are listed below, in terms of permutations. We are in particular interested in subgroups of order 2, 3, 7, and 9; we make use of them in the next section.

```
gap> hurwitz := Group( r,s );
< permutation group with 2 generators >
gap> ConjugacyClassesSubgroups( hurwitz );
[ Group( () )^G,

  Group( [ (1,3)(2,4)(5,10)(6,24)(7,17)(8,11)(9,25)(12,16)(13,47)(14,30)(15,53)(18,23)(19,33)(20,38)(21,59)
           (22,48)(26,52)(27,61)(28,43)(29,34)(31,46)(32,39)(35,42)(36,54)(37,60)(40,45)(41,55)(44,56)(49,58)(50,68)
           (51,62)(57,67)(63,66)(64,69)(65,70)(71,72) ] )^G,

  Group( [ (1,36,32)(2,23,9)(3,37,51)(4,50,38)(5,13,19)(6,55,26)(7,42,40)(8,16,46)(10,60,22)(11,57,44)
           (12,27,61)(14,48,63)(15,58,53)(17,31,65)(18,64,24)(20,69,49)(21,28,34)(25,71,30)(29,52,67)(33,68,35)
           (39,72,43)(41,47,54)(45,66,59)(56,62,70) ] )^G,

  Group( [ (1,56)(2,70)(3,49)(4,43)(5,30)(6,14)(7,20)(8,62)(9,69)(10,63)(11,35)(12,17)(13,44)(15,39)(16,71)
           (18,59)(19,38)(21,33)(22,48)(23,72)(24,26)(25,29)(27,28)(31,45)(32,61)(34,41)(36,65)(37,66)(40,47)(42,64)
           (46,68)(50,58)(51,55)(52,57)(53,54)(60,67)(1,55)(2,48)(3,42)(4,36)(5,13)(6,19)(7,61)(8,69)(9,62)
           (10,34)(11,16)(12,50)(14,38)(15,68)(17,58)(18,37)(20,32)(21,47)(22,70)(23,25)(24,28)(26,27)(29,72)(30,44)
           (31,60)(33,40)(35,71)(39,46)(41,63)(43,65)(45,67)(49,64)(51,56)(52,53)(54,57)(59,66) ] )^G,

  Group( [ (1,2,3)(4,8,10)(5,9,24)(6,23,17)(7,16,11)(12,22,47)(13,46,30)(14,36,53)(15,52,25)(18,29,33)
           (19,32,38)(20,37,59)(21,58,48)(26,51,61)(27,60,43)(28,42,34)(31,45,39)(35,41,54)(40,44,55)(49,64,68)
           (50,67,56)(57,66,62)(63,65,69)(70,71,72)(1,35)(2,54)(3,41)(4,22)(5,29)(6,43)(7,49)(8,12)(9,18)
           (10,47)(11,64)(13,59)(14,56)(15,26)(16,68)(17,27)(19,45)(20,30)(21,63)(23,60)(24,33)(25,51)(28,70)
           (31,32)(34,71)(36,67)(37,46)(38,39)(40,57)(42,72)(44,62)(48,65)(50,53)(52,61)(55,66)(58,69) ] )^G,

  Group( [ (1,52,51,55,56,54,57)(2,58,38,69,49,18,15)(3,66,32,48,43,60,21)(4,45,9,42,30,68,63)
           (5,39,22,36,14,47,71)(6,33,64,13,20,41,50)(7,10,65,19,62,35,37)(8,16,44,61,70,12,31)
           (11,59,46,34,17,67,40)(23,28,24,72,26,25,29) ] )^G,

  Group( [ (1,56)(2,70)(3,49)(4,43)(5,30)(6,14)(7,20)(8,62)(9,69)(10,63)(11,35)(12,17)(13,44)(15,39)(16,71)
           (18,59)(19,38)(21,33)(22,48)(23,72)(24,26)(25,29)(27,28)(31,45)(32,61)(34,41)(36,65)(37,66)(40,47)(42,64)
```

```
         (46,68)(50,58)(51,55)(52,57)(53,54)(60,67)(1,52)(2,10)(3,16)(4,58)(5,66)(6,45)(7,39)(8,33)(9,47)
         (11,42)(12,65)(13,59)(14,31)(15,20)(17,36)(18,44)(19,67)(21,62)(22,41)(23,24)(25,28)(26,72)(27,29)(30,37)
         (32,68)(34,48)(35,64)(38,60)(40,69)(43,50)(46,61)(49,71)(51,54)(53,55)(56,57)(63,70)(1,55)(2,48)(3,42)
         (4,36)(5,13)(6,19)(7,61)(8,69)(9,62)(10,34)(11,16)(12,50)(14,38)(15,68)(17,58)(18,37)(20,32)(21,47)
         (22,70)(23,25)(24,28)(26,27)(29,72)(30,44)(31,60)(33,40)(35,71)(39,46)(41,63)(43,65)(45,67)(49,64)(51,56)
         (52,53)(54,57)(59,66) ] )^G,

Group( [ (1,8,22,64,71,70,62,25,4)(2,9,51,65,72,69,48,11,3)(5,7,29,41,50,63,56,39,18)
         (6,15,35,27,57,49,20,31,12)(10,23,32,44,66,68,55,34,17)(13,21,43,33,37,26,14,45,54)
         (16,46,38,58,67,61,42,53,24)(19,28,59,47,36,40,30,52,60) ] )^G,

Group( [ ((1,64)(2,50)(3,71)(4,65)(5,51)(6,22)(7,41)(8,27)(9,13)(10,57)(11,72)(12,44)(14,35)(15,33)(16,37)
         (17,70)(18,58)(19,32)(20,29)(21,47)(23,36)(24,63)(25,66)(26,38)(28,54)(30,49)(31,52)(34,68)(39,45)(40,61)
         (42,60)(43,56)(46,55)(48,67)(53,69)(59,62)(1,55)(2,48)(3,42)(4,36)(5,13)(6,19)(7,61)(8,69)(9,62)
         (10,34)(11,16)(12,50)(14,38)(15,68)(17,58)(18,37)(20,32)(21,47)(22,70)(23,25)(24,28)(26,27)(29,72)(30,44)
         (31,60)(33,40)(35,71)(39,46)(41,63)(43,65)(45,67)(49,64)(51,56)(52,53)(54,57)(59,66) ] )^G,

Group( [ (1,35)(2,54)(3,41)(4,22)(5,29)(6,43)(7,49)(8,12)(9,18)(10,47)(11,64)(13,59)(14,56)(15,26)(16,68)
         (17,27)(19,45)(20,30)(21,63)(23,60)(24,33)(25,51)(28,70)(31,32)(34,71)(36,67)(37,46)(38,39)(40,57)(42,72)
         (44,62)(48,65)(50,53)(52,61)(55,66)(58,69)(1,72)(2,71)(3,70)(4,69)(5,68)(6,67)(7,66)(8,65)(9,64)
         (10,63)(11,62)(12,61)(13,60)(14,59)(15,58)(16,57)(17,56)(18,55)(19,54)(20,53)(21,52)(22,51)(23,50)(24,49)
         (25,48)(26,47)(27,46)(28,45)(29,44)(30,43)(31,42)(32,41)(33,40)(34,39)(35,38)(36,37) ] )^G,

Group( [ (1,52)(2,10)(3,16)(4,58)(5,66)(6,45)(7,39)(8,33)(9,47)(11,42)(12,65)(13,59)(14,31)(15,20)(17,36)
         (18,44)(19,67)(21,62)(22,41)(23,24)(25,28)(26,72)(27,29)(30,37)(32,68)(34,48)(35,64)(38,60)(40,69)(43,50)
         (46,61)(49,71)(51,54)(53,55)(56,57)(63,70)(1,51,56,57,52,55,54)(2,38,49,15,58,69,18)(3,32,43,21,66,48,60)
         (4,9,30,63,45,42,68)(5,22,14,71,39,36,47)(6,64,20,50,33,13,41)(7,65,62,37,10,19,35)(8,44,70,31,16,61,12)
         (11,46,17,40,59,34,67)(23,24,26,29,28,72,25) ] )^G,

Group( [ (1,3)(2,4)(5,10)(6,24)(7,17)(8,11)(9,25)(12,16)(13,47)(14,30)(15,53)(18,23)(19,33)(20,38)(21,59)
         (22,48)(26,52)(27,61)(28,43)(29,34)(31,46)(32,39)(35,42)(36,54)(37,60)(40,45)(41,55)(44,56)(49,58)(50,68)
         (51,62)(57,67)(63,66)(64,69)(65,70)(71,72)(1,36,32)(2,23,9)(3,37,51)(4,50,38)(5,13,19)(6,55,26)
         (7,42,40)(8,16,46)(10,60,22)(11,57,44)(12,27,61)(14,48,63)(15,58,53)(17,31,65)(18,64,24)(20,69,49)
         (21,28,34)(25,71,30)(29,52,67)(33,68,35)(39,72,43)(41,47,54)(45,66,59)(56,62,70) ] )^G ]
```

5 Symmetry obstructions

5.1 Obstruction for a cyclic symmetry of order 9

Here we use the following element of order 9 taken from Section 4:

$$
\begin{gathered}
(01,08,22,64,71,70,62,25,04)(02,09,51,65,72,69,48,11,03)\\
(05,07,29,41,50,63,56,39,18)(06,15,35,27,57,49,20,31,12)\\
(10,23,32,44,66,68,55,34,17)(13,21,43,33,37,26,14,45,54)\\
(16,46,38,58,67,61,42,53,24)(19,28,59,47,36,40,30,52,60).
\end{gathered}
$$

Note that an isometry realizing this element can only be a rotation around an axis.

A planar representation of the underlying graph of the Hurwitz map displaying this symmetry is shown in Figure 11. (We note that so far, this is the only known topological realization of this graph with rotational symmetry of order 9.) We use this figure to present a suitable detail of the map that yields an easy obstruction argument, see Figure 12. With this, we have seen the following Proposition.

Proposition 5.1. *A polyhedral realization of the Hurwitz map $(3,7)_{18}$ of genus 7 cannot have rotational symmetry of order 9.* □

5.2 Obstruction for a cyclic symmetry of order 3

We use the following element of order 3 (cf. Section 4).

$$
\begin{gathered}
(01,36,32)(02,23,09)(03,37,51)(04,50,38)(05,13,19)(06,55,26)\\
(07,42,40)(08,16,46)(10,60,22)(11,57,44)(12,27,61)(14,48,63)\\
(15,58,53)(17,31,65)(18,64,24)(20,69,49)(21,28,34)(25,71,30)\\
(29,52,67)(33,68,35)(39,72,43)(41,47,54)(45,66,59)(56,62,70).
\end{gathered}
$$

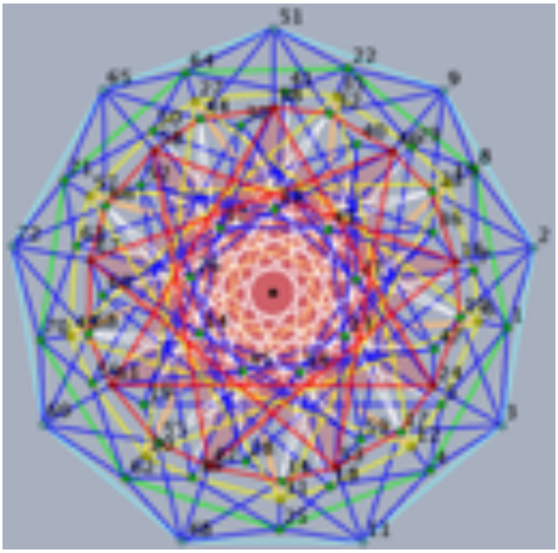


Figure 11: The underlying graph of the Hurwitz map $\{3,7\}_{18}$ of genus 7 with cyclic symmetry of order 9.

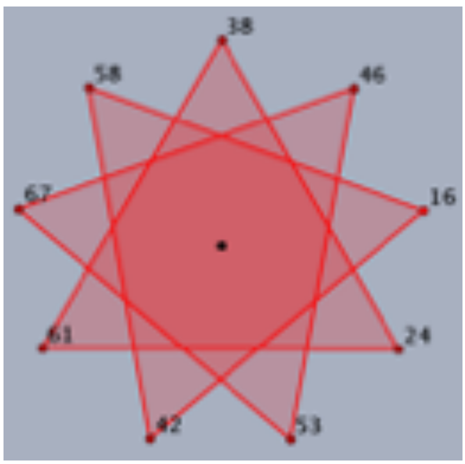


Figure 12: The red part of the graph in Figure 11 shows that in this case there are three triangular faces of the map which have to lie in a common plane.

Accordingly, we provide the underlying graph of the Hurwitz map in a representation exhibiting rotational symmetry of order 3, see Figure 13.

Proposition 5.2. *A polyhedral realization of the Hurwitz map* $\{3,7\}_{18}$ *of genus 7 cannot have rotational symmetry of order 3.*

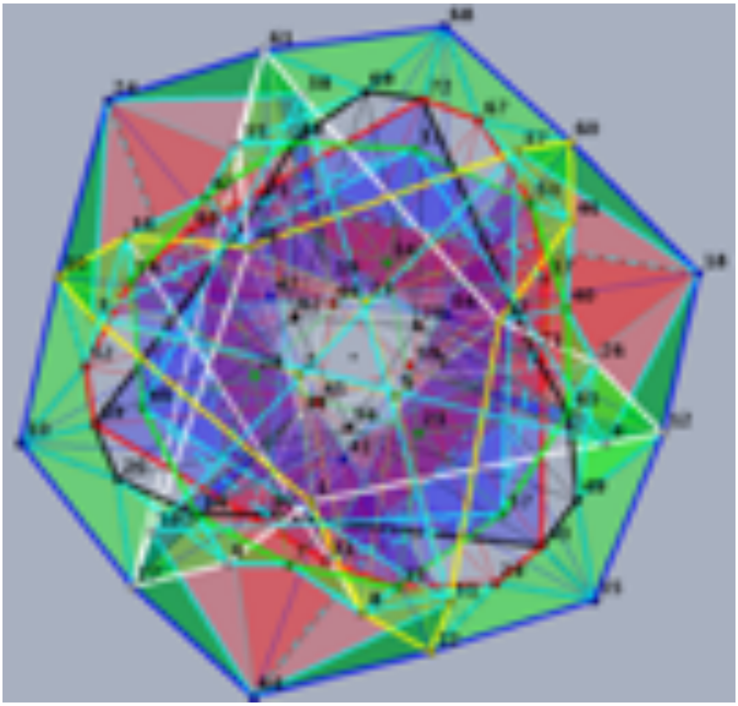


Figure 13: The underlying graph of the Hurwitz map $\{3, 7\}_{18}$ of genus 7 with cyclic symmetry of order 3.

Proof. Consider the two topological discs of Figure 14 with its fixed triangles $(41, 47, 54)$ and $(5, 13, 19)$ and consider a substructure of the Hurwitz map consisting of the following three parts, as it is shown in Figure 15:

(i) a topological annulus, bounded by the following 12-gon and 33-gon, respectively:

$$(60, 68, 61, \ldots, 18, 60), \quad (46, 67, 72, \ldots, 23, 46)$$

(in Figure 15 it is shown by light grey);

(ii) three strips, denoted in Figure 15 by red, blue and green, formed by the triples of triangles

$$((19, 13, 55), (55, 13, 36), \ (36, 13, 50)), ((13, 5, 6), (6, 5, 1), (1, 5, 4)),$$
$$((5, 19, 26), (26, 19, 32), (32, 19, 38)),$$

respectively;

(iii) the fixed triangle $(19, 13, 5)$ (denoted in Figure 15 by dotted yellow edges).

Note that the three strips are cyclically permuted by the rotation around the axis that perpendicularly intersects triangle $Y = (5, 13, 19)$ in its centre.

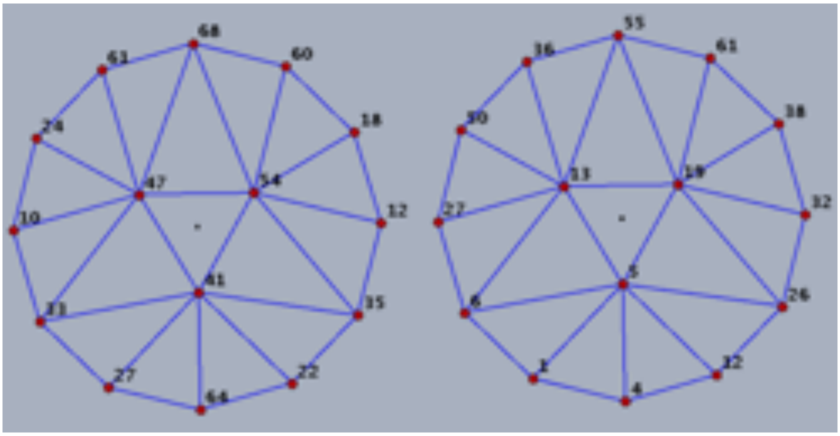


Figure 14: Illustration 1 for the proof of Proposition 5.2.

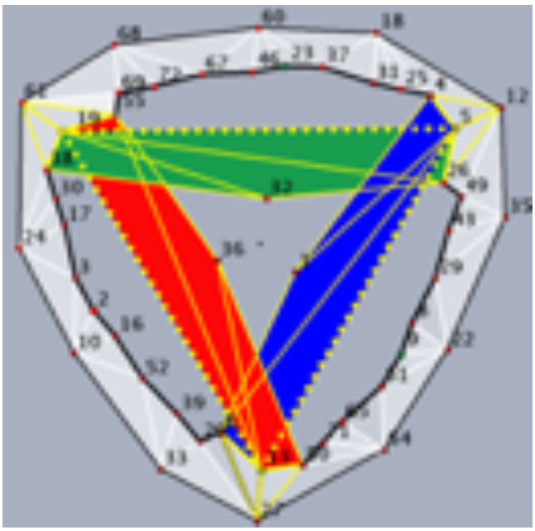


Figure 15: Illustration 2 for the proof of Proposition 5.2.

Start from the (red) triangle $(13, 19, 38)$, and place it in a (say) horizontal plane. Then assume that the mutual position of the red strip and of the blue strip is such that in the vicinity of the (common) vertex 19 the latter is below the former. By symmetry, the mutual position of the blue/green and of the green/red strips in the vicinity of vertex 5 and in the vicinity of vertex 19, respectively, is similar.

Choose now any vertex, say 19, of triangle Y, and assume that in the vicinity of this vertex the corner of this triangle is located below the red strip; then it is below the green strip, too. Since it shares the edge $(19, 13)$ with the red strip, it gets over the blue strip in the vicinity of vertex 13. We can proceed in the same way in a cyclic order, until we return to vertex 19, in the vicinity of which the corner of Y gets over the green strip–a contradiction. □

5.3 Symmetry obstruction for symmetry of order 2

Consider the following subgroup of order 2 of the group $\mathbf{PSL}(2,8)$ given in Section 4:

$$\begin{aligned}
&(01,03)(02,04)(05,10)(06,24)(07,17)(08,11)(09,25)(12,16)(13,47)\\
&(14,30)(15,53)(18,23)(19,33)(20,38)(21,59)(22,48)(26,52)(27,61)\\
&(28,43)(29,34)(31,46)(32,39)(35,42)(36,54)(37,60)(40,45)(41,55)\\
&(44,56)(49,58)(50,68)(51,62)(57,67)(63,66)(64,69)(65,70)(71,72).
\end{aligned} \tag{5.1}$$

Note that this is, up to conjugacy, the only subgroup of order 2 of $\mathbf{PSL}(2,8)$, as it can directly be seen from the GAP output in Section 4.

Proposition 5.3. *If a polyhedral realization of the Hurwitz map $\{3,7\}_{18}$ of genus 7 is symmetrical by a half-turn around any axis, then this axis passes through the midpoints of precisely 4 edges.*

Proof. Comparing (5.1) with the diagram of the Hurwitz map $\{3,7\}_8$ in Figure 2, one observes that there are precisely 4 edges in the map which are invariant under this subgroup, namely $(1,3)$, $(14,30)$, $(37,60)$ and $(71,72)$. The statement follows from this by conjugation. □

We conclude this section with our conjecture that there is an obstruction for symmetry of order 2, too, but we postpone our heuristic arguments to Subsection 6.3 below.

6 Kepler–Poinsot type realizations

6.1 KP type realization with 7-fold rotational symmetry

We start from the following topological construction of an (orientable) surface of genus 7: take a sphere, cut in it 7 pairs of holes, and connect each pair of holes by a tube. When building our polyhedral model of the Hurwitz surface, we realize geometrically just this construction. Closer investigation of the Hurwitz map shows that it contains (altogether 56) 9-sided polygons, each forming a *3-hole* in the sense of Coxeter; this latter means that it is a closed edge-path which leaves a vertex by the 3rd edge from which it entered, in the same sense (that is, always keeping to the left, say, in some local orientation) [11, 24]. We use in our construction 7 pairs of such 9-gonal holes. Accordingly, the tubes, by which these pairs are connected, are combinatorially equivalent with the mantle of an antiprism with 9-gonal bases. All these are arranged by a 7-fold rotational symmetry; in fact, it turns out that a higher degree of symmetry can be realized: the symmetry group of our model will be the same as that of a regular heptagonal antiprism, which is $\mathbf{D}_{7d}$ (in Schoenflies notation) of order 28.

Figure 16 serves as a "design" of our KP model. The shaded region highlights a triangulated topological annulus bounded by two 9-gonal holes. Note that this triangulated annulus can be considered as a Schlegel diagram of an antiprismatic tube; thus it represents one of the 7 connecting tubes mentioned above. A separated 9-gon is also highlighted, the vertex labels of which show that it is a copy of one of the bounding 9-gons of the shaded region; thus one can follow how the topological gluing is to be performed (the rest of the 9-gonal holes that are to be glued can easily be identified as well).

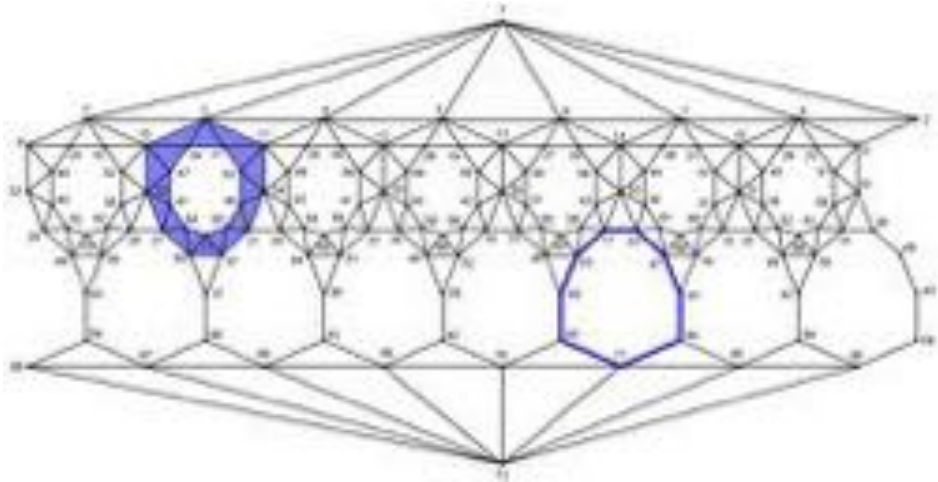


Figure 16: Representation of the Hurwitz map $\{3,7\}_{18}$ of genus 7 as a design for a KP type realization with 7-fold symmetry.

Note that the side edges of each of the connecting antiprismatic tubes form a Petrie polygon (recall that the length of the Petrie polygons in this map is 18). Thus, one can observe that these tubes clearly reveal 7 of the 28 Petrie polygons of the map.

The complete model, together with its convex hull, is depicted in Figure 17. One can see that it is highly self-intersecting: in fact, the antiprismatic tubes pairwise penetrate each other (the $7 \times 18 = 126$ faces of these tubes are shown by green). For a better understanding the overall shape of the model, the convex hull of the polyhedron is depicted in the same figure. It is a convex polytope bounded by 14 9-gonal and 42 triangular faces.

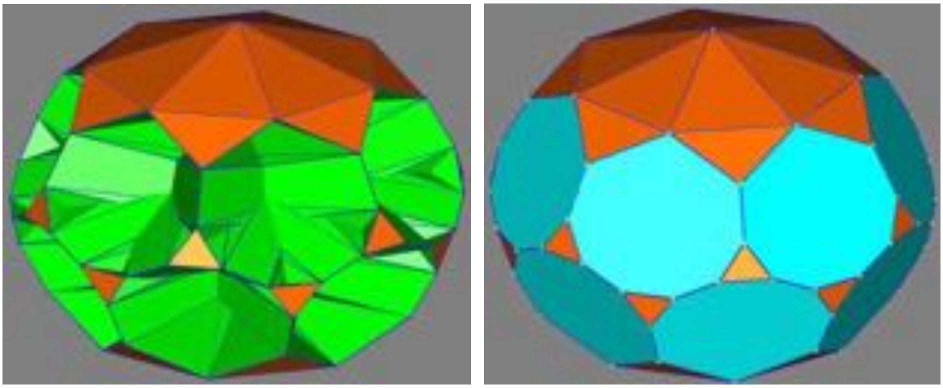
As we already mentioned above, the symmetry group of this model is isomorphic to the group $\mathbf{D}_{7d}$. This means that there is an axis of rotation of order 7, and there are 7 mirror planes the common intersection of which is this axis. In addition, there are 7 twofold axes intersecting perpendicularly the 7-fold axis in a common point and in angle bisector position with respect to the mirror planes.

The rotational subgroup of index 2 of this group is isomorphic to the dihedral group $\mathbf{D}_7$, and also to a subgroup given in our list in Section 4.

Note that by Proposition 5.3, each of the twofold axes passes through the midpoints of precisely 4 edges. A quadruple of such edges is as follows: $(18,54)$, $(28,44)$, $(32,40)$, $(33,39)$; the additional 6 such quadruples can easily be identified in Figure 16 by cyclic shift.

6.2 KP type realization with 3-fold rotational symmetry

In this case we start from a diagram of the Hurwitz map exhibiting 3-fold rotational symmetry, see Figure 18. It turns out that there are six regions in the diagram (arranged by the 3-fold symmetry) which can be considered as Schlegel diagrams of triangulated tube-like



(a) (b)

Figure 17: A KP type realization of the Hurwitz map $\{3, 7\}_{18}$ of genus 7 with 7-fold symmetry (a), and its convex hull (b).

substructures. In the figure these regions are shaded by different colours (note that each is shown in two pieces belonging to different parts of the diagram, and are to be glued along the vertices with identical labels). The corresponding triangulated tube can be considered as the mantle of a generalized antiprism: one base of such an antiprism is a hexagon, while the other is an octagon (accordingly, the tube is composed of 14 triangles).

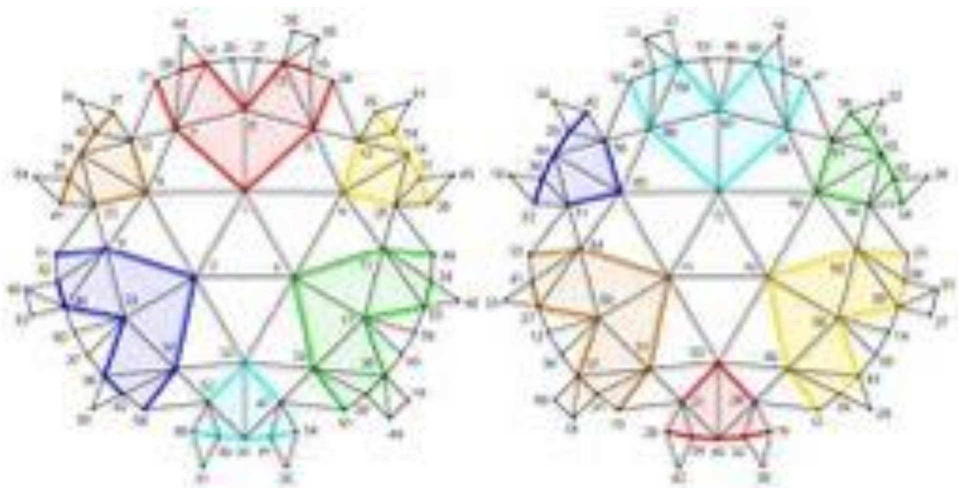


Figure 18: A representation of the Hurwitz map $\{3, 7\}_{18}$ of genus 7 as a design for a KP type realization with 3-fold symmetry.

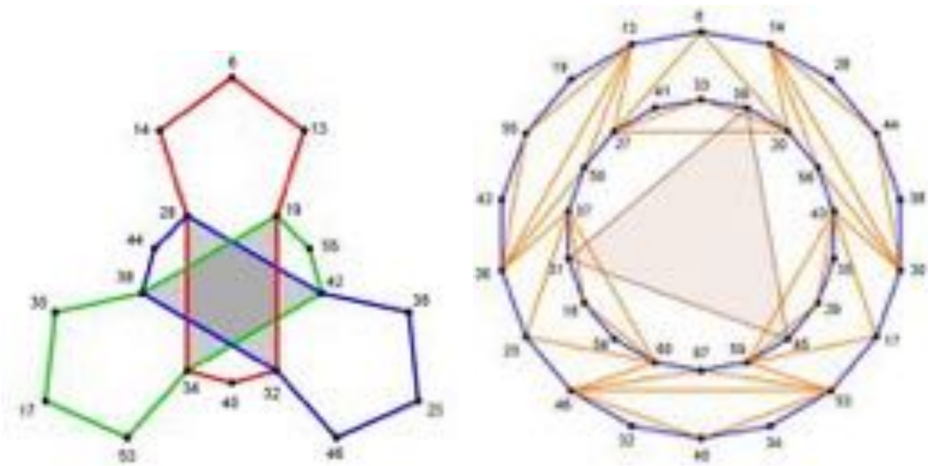
With their hexagonal boundaries, these tubes are connected to six corresponding holes of a sphere-like unit which form an outer shell of our model (similarly to our preceding model with heptagonal symmetry). With their other end, bounded by octagons, three of the tubes (mutually penetrating each other) are connected together via two equilateral triangles, while leaving free a 18-gon which will serve as the boundary of a "hole" of this complex. This way of connection is shown in Figure 19(a): the octagons are

$$(6, 14, 28, 34, 40, 32, 19, 13), (17, 53, 34, 42, 55, 19, 38, 30), (23, 36, 42, 28, 44, 38, 32, 46),$$

the connecting triangles are shaded, and the "free" 18-gon is

$$(6, 14, 28, 44, 38, 30, 17, 53, 34, 40, 32, 46, 23, 36, 42, 55, 19, 13).$$

The other three antiprismatic tubes are connected together in an analogous way (hence providing a second free 18-gonal hole). Finally, the two 18-gonal holes are connected by a 7th tube, which is topologically a triangulated annulus (it is composed of 36 triangles). A diagram of this latter tube is shown in Figure 19(b).



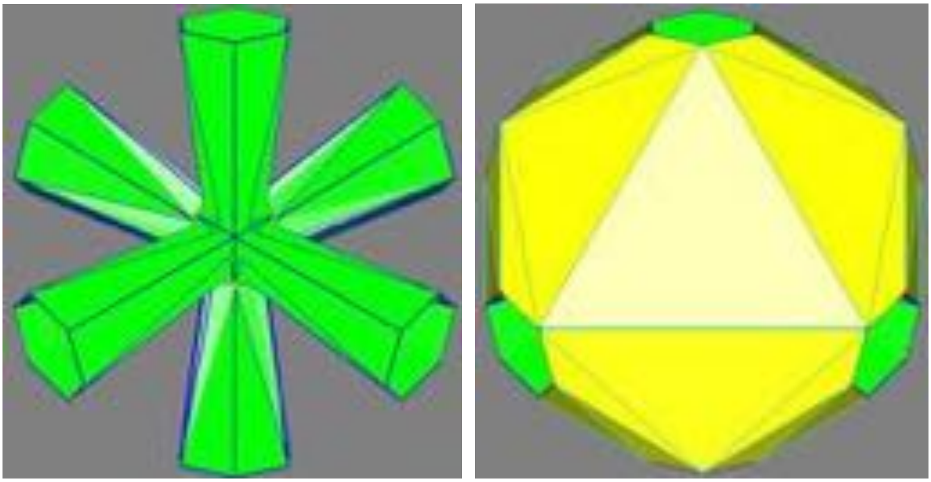
(a) (b)

Figure 19: Explanation of the central block of our trigonally symmetric KP type model: (a) a triple of octagons occurring in the Hurwitz map; (b) diagram of the 7th, central tube of the model (with one of the 6 invariant triangles).

We note that there are precisely six triangles in the Hurwitz map $\{3, 7\}_{18}$ which are invariant under an automorphism of order three. One such 6-tuple can be identified in Figure 18 by direct inspection:

$$(1, 2, 3), (31, 39, 45), (35, 41, 54), (19, 32, 38), (28, 34, 42), (70, 71, 72).$$

When such an automorphism is geometrically realized, it becomes a rotation around an axis, and the corresponding invariant (equilateral) triangles are located in parallel planes which are perpendicular to this axis. In our geometric model, 4 of these invariant triangles belong to the central block of the model (two of them is shown in Figure 19(a), and one in Figure 19(b)). The remaining two, $(1, 2, 3)$ and $(70, 71, 72)$, form in turn the uppermost and lowermost face, respectively, of the outer shell of the model (assuming that the model



(a) (b)

Figure 20: The trigonally symmetric KP type realization of the Hurwitz map $\{3,7\}_{18}$ of genus 7, seen from the direction of its threefold axis: (a) arrangement of the 6 equal tubes (each consisting of 14 triangles); (b) the complete model (its outer shell consists of 44 triangles).

is located with its threefold axis in vertical position). The outer appearance of the model can be seen in Figure 20.

The symmetry group of the model is isomorphic to $\mathbf{D}_{3d}$ (in Schoenflies notation) of order 12. It is closely related to $\mathbf{D}_{7d}$ considered above; a regular antiprism with triangular base and isosceles (non-equilateral) triangles on the side has $\mathbf{D}_{3d}$ as its symmetry group.

Finally, we mention the following observation in connection with mutual position of the invariant triangles.

Observation 6.1. *The generalized Petersen graph* $\mathrm{GP}(9,3)$ *occurs as a subgraph of the underlying graph of the Hurwitz map* $\{3,7\}_{18}$ *of genus 7. The inner triangles of this graph form one triple of 6 triangles which are invariant under the same automorphism of order 3 of the map.*

(For generalized Petersen graphs, see e.g. [28], 2.2.9). It is easy to see that there are altogether 56 isomorphic copies of $\mathrm{GP}(9,3)$ in the Hurwitz map $\{3,7\}_{18}$. Besides that given in Figure 21, a different example can be found in Figure 11. This latter shows that Proposition 6.1 provides an alternative (but similarly easy) proof of the obstruction for 9-fold symmetry.

6.3 What has to be done in case of a line reflection?

So far we have not mentioned heuristic arguments for our belief that a line reflection seems to have no chance to lead to a selfintersection-free realization, too. To understand the problem in this case, we provide Figure 22. It shows the orthogonal projection of all the 72 vertices along an axis of symmetry.

Table 3: Integer coordinates of our trigonally symmetric KP realization of the Hurwitz map $\{3, 7\}_{18}$ of genus 7.

No.	x	y	z	No.	x	y	z	No.	x	y	z
1	48	-38	48	2	48	48	-38	3	-38	48	48
4	-30	-30	76	5	30	-48	58	6	5	26	5
7	58	-48	30	8	76	-30	-30	9	58	30	-48
10	-30	76	-30	11	-48	30	58	12	-20	-48	66
13	- 2	22	9	14	9	22	- 2	15	66	-48	-20
16	30	58	-48	17	26	5	5	18	- 8	- 8	12
19	- 8	12	3	20	- 9	2	-22	21	48	-66	20
22	66	-20	-48	23	5	5	26	24	-48	58	30
25	-48	-20	66	26	20	-66	48	27	-22	2	- 9
28	0	11	- 9	29	12	-8	- 8	30	22	9	- 2
31	-11	0	9	32	3	- 8	12	33	- 8	12	- 8
34	11	- 9	0	35	8	- 3	-12	36	- 2	9	22
37	-22	- 9	2	38	12	3	-8	39	0	9	-11
40	8	-12	8	41	-12	8	-3	42	- 9	0	11
43	2	- 9	-22	44	8	8	-12	45	9	-11	0
46	9	- 2	22	47	-48	66	-20	48	-66	20	48
49	-30	-58	48	50	-26	- 5	- 5	51	48	20	-66
52	-20	66	-48	53	22	- 2	9	54	-3	-12	8
55	-12	8	8	56	- 5	- 5	-26	57	48	-58	-30
58	20	48	-66	59	2	-22	- 9	60	- 9	-22	2
61	-66	48	20	62	-58	-30	48	63	30	-76	30
64	48	-30	-58	65	30	30	-76	66	-30	48	-58
67	- 5	-26	- 5	68	-58	48	-30	69	-76	30	30
70	-48	-48	38	71	38	-48	-48	72	-48	38	-48

There are precisely 4 edges that have to pierce this axis of symmetry orthogonally; in our example these are: $(1, 3)$, $(14, 30)$, $(37, 60)$ and $(71, 72)$ (cf. the proof of Proposition 5.3). Such edges play a special role, namely they determine 4 topological discs as their neighbourhoods, such that none of them can have another point on the axis of symmetry. Otherwise we would have a self-intersection. In other words, we have the necessary condition that no inner point of a triangle should be on the axis of symmetry.

We tried very hard to find an example for which this necessary condition is fulfilled for all the 168 triangles, but we did not succeed. Our Figure 22 shows a particular test example. One can see in this example four topological discs in orange, in blue, in red, and in yellow. The boundaries of the last three have been marked.

It is clear from this example that for a proper test example, the orders in which these discs intersect, and as all "bridges" between the disc boundaries appear along the axis of symmetry, have still to be determined, and thus finding a solution with a symmetry of order 2 is still not clear at all.

Our experimental work on such projections was completed by a careful study of the corresponding 3D-coordinates using the powerful 3D-software *Blender*, but without any

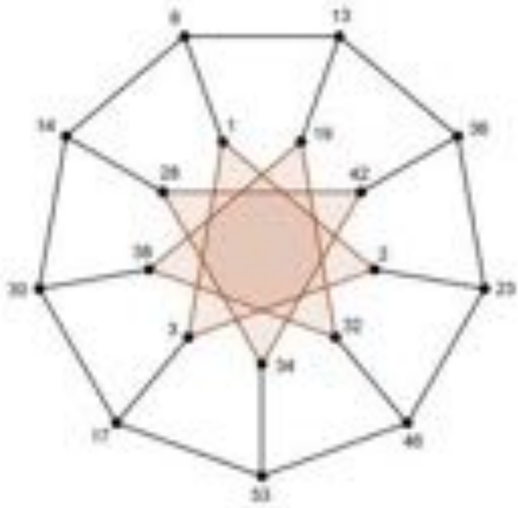


Figure 21: The generalized Petersen graph $\mathrm{GP}(9,3)$ occurs as a substructure of the Hurwitz map $\{3,7\}_{18}$ of genus 7.

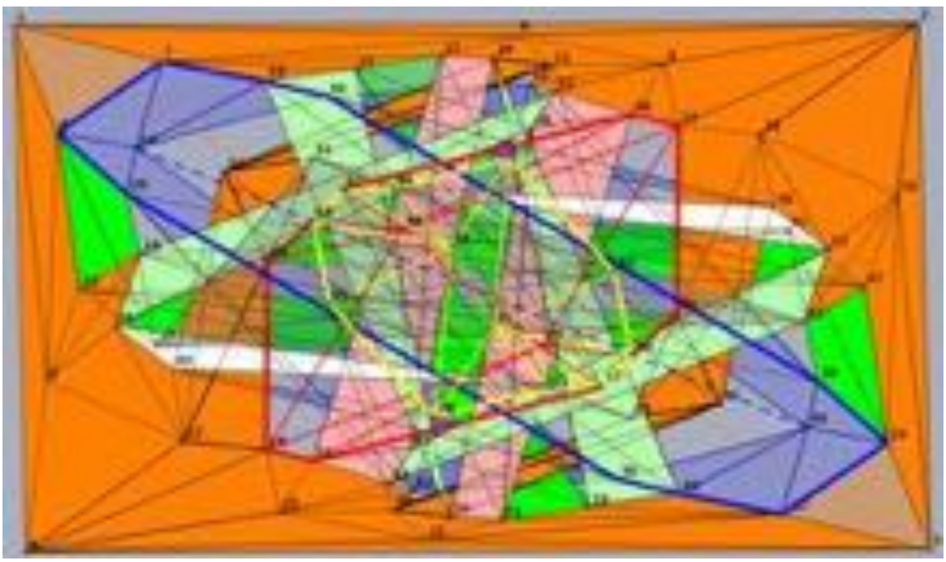


Figure 22: Orthogonal projection along an axis of symmetry.

success. There was not even an optimistic starting position that we can offer for an additional research towards an affirmative solution. Whereas we believe that an obstruction can be found in case of a symmetry of order 7, we consider the problem for finding an obstruction in case of a line reflection to be much more involved.

Plenty of KP models in case of a line reflection symmetry have appeared, however the interior was always far from being without intersections. We have depicted one example in Figure 23.

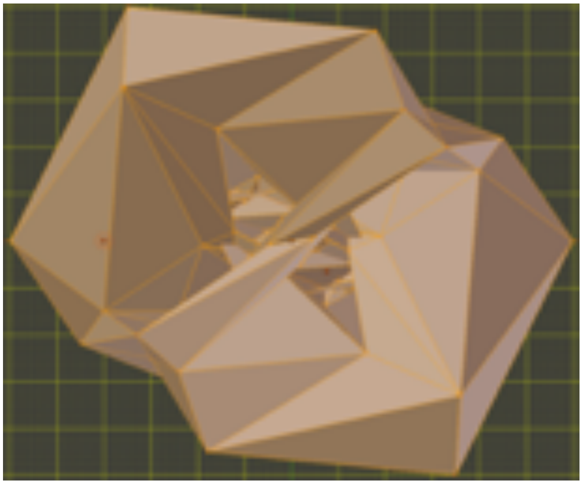


Figure 23: A KP model with a two-fold rotational axis.

We conclude with a remark on mirror symmetry, i.e. symmetry with respect to reflection in a plane. It is beyond the scope of this paper, since here we restricted ourselves to orientation-preserving automorphisms. Yet, we have also made experiments to study the possibility of this kind of symmetry. Based on these experiments, we do not believe that a reflection in a plane does allow a symmetrical embedding. One such reflection would interchange, for example, vertex 1 and vertex 72, together with their stars (cf. Figure 2). A topological KP model exhibiting precisely the corresponding mirror symmetry can be constructed; it is shown in Figure 24.

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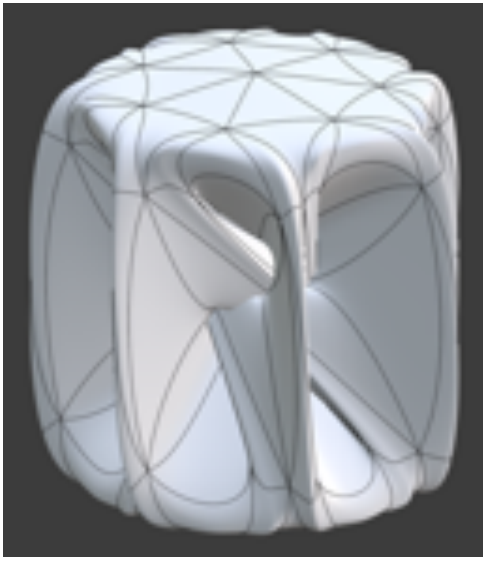

Figure 24: A topological KP model with mirror symmetry.

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A Kepler–Poinsot-type polyhedron of the genus 7 Hurwitz surface

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Abstract

In 2017 a first polyhedral embedding of the genus 7 Hurwitz surface of type $\{3,7\}_{18}$ was found by M. Cuntz and the author. For all previously determined polyhedral embeddings of regular maps, there exist those with non-trivial geometric symmetries as well. The orientation-preserving combinatorial automorphism group of this regular map of Hurwitz is the projective special linear group $\mathbf{PSL}(2,8)$. For its subgroups, their possible corresponding geometric polyhedral embeddings have been investigated by G. Gévay and the author in this volume. There is an additional symmetry of order 2 that reverses the orientation. For this symmetry with eight fixed points, this paper provides a Kepler–Poinsot-type polyhedron which realizes this symmetry together with two additional symmetries of order 2. This polyhedron might serve as a starting point for proving that a geometric symmetry of order 2 for an embedding cannot exist.

Keywords: Hurwitz surface, regular map, Kepler–Poinsot-polyhedron.

Math. Subj. Class.: 51M20, 52B70.

1 Introduction

We refer the reader to the article [2] in this volume for a more detailed introduction and additional references for symmetric polyhedral embeddings of the *Hurwitz surface of genus* 7 also named as the *Fricke–Macbeath surface*. In this article, we use its additional orientation reversing combinatorial symmetry that was not investigated in that paper. The main result is a Kepler–Poinsot-type polyhedron that has not only the mentioned orientation reversing symmetry of order 2 but it has also two additional symmetries of order 2. Since a

*The author is grateful to Marston Conder and Michael Cuntz for providing some MAGMA Code for obtaining the orientation reversing symmetry.

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proof of the conjecture is still missing that all combinatorial symmetries of order 2 do not allow a geometric polyhedral embedding, this KP-polyhedron might be a starting object for additional investigations in this direction.

2 The orientation reversing symmetry of the genus 7 surface of Hurwitz

When the reader is interested to get the automorphisms of regular maps expressed in terms of permutations, the software MAGMA is a very good tool, however it is not freely available. I have received from Marston Conder the following decisive input expression for MAGMA and Michael Cuntz printed for me the element T.

```
> G<R,S,T>:=Group<R,S,T | T^2, R^-3, (R * S)^2, (R * T)^2,
(S * T)^2, S^-7,
> S^-2 * R * S^-3 * R * S^-2 * R^-1 * S^2 * R^-1 * S^2
* R^-1 * S^-2 * R * S^-1 >;
```

However, the labeling of the vertices of the genus 7 Hurwitz surface that we obtain when we use this input in MAGMA is different from the one used in of [2] and in [1]. When we carry over the MAGMA labeling to that used in these former articles, we obtain the orientation reversing symmetry of order 2 via the permutation $(2,3)(4,8)(5,7)(9,11)(12,15)$ $(13,14)$ $(16,24)(17,23)(18,29)(19,28)(20,27)$ $(21,26)(22,25)(30,36)(31,35)(32,34)$ $(37,43)(38,42)(39,41)(44,55)(45,54)(46,53)(47,52)$ $(48,51)(49,57)(50,56)(58,61)$ $(59,60)(62,64)(65,69)(66,68)(70,71)$

The vertices $1,6,10,33,40,63,67,72$ are fixed under this symmetry.

In the next section we provide a KP-polyhedron of the genus 7 Hurwitz surface in which the above permutation descibes the reflection of all vertices at a plane.

3 A KP-polyhedron of the genus 7 Hurwitz surface with three plane reflections

For investigations concerning geometric embeddings of abstract objects with symmetries, as well as for the finding of this KP-polyhedron, both, the software Cinderella and the software Blender are useful freely available packages.

In Blender we have the x-axis in red, the y-axis in green and the z-axis in blue. In Figure 1 we see the orthogonal pojection of the KP-polyhedron with transparent triangles onto the xy-plane of the Blender software that has been inserted as a background in the Cinderella software in order to add all the labels of the vertices. We see that the xy-plane can be used as a mirror plane for the orientation reversing symmetry. Which vertices should lie above this plane, or below this plane, respectively, has still to be determined. The eight points $(22,25)(30,36)(37,43)(48,51)$ have been moved to the center in order to obtain the xz-plane and the yz-plane as additional plane reflection symmetries. When we are interested in having 72 different vertices, we would lose at least one of the additional symmetries.

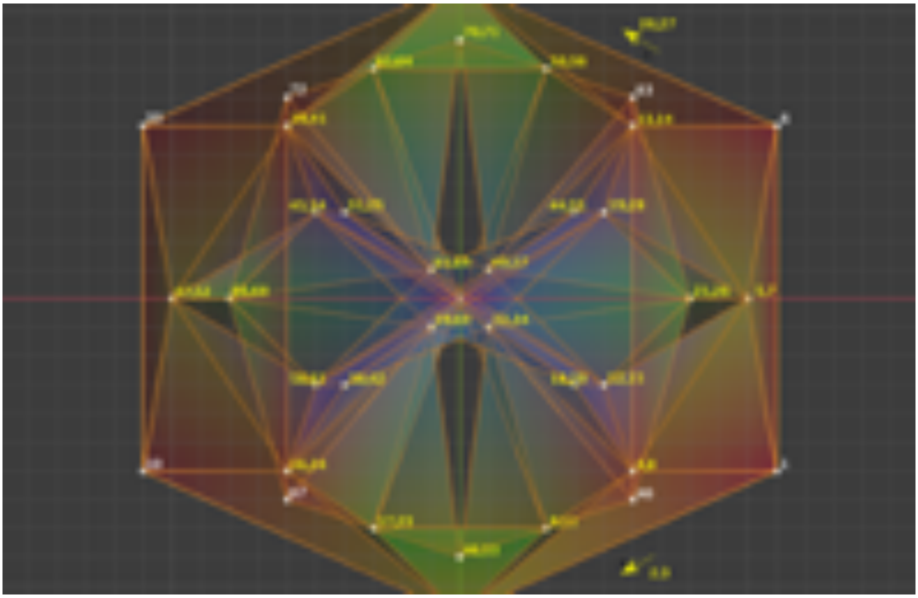

Figure 1: A transparent KP-polyhedron projected orthogonally onto the xy-plane.

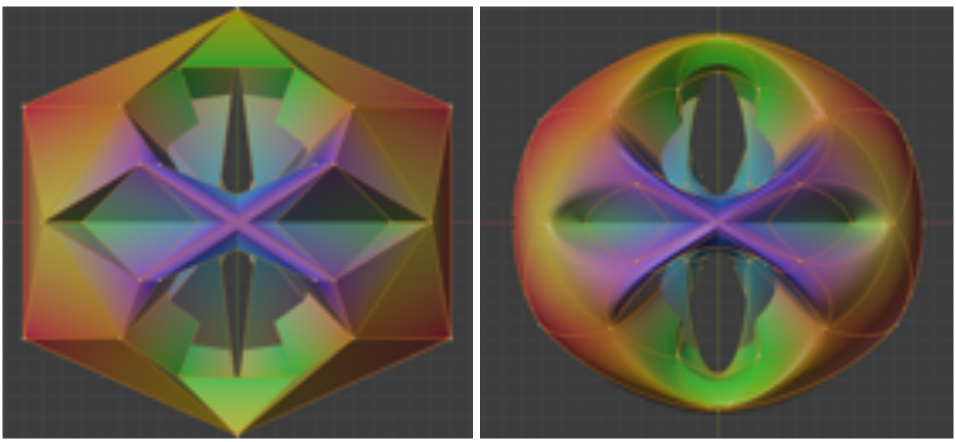

Figure 2: The non-transparent KP-polyhedron and its subdivision to improve its topological aspects.

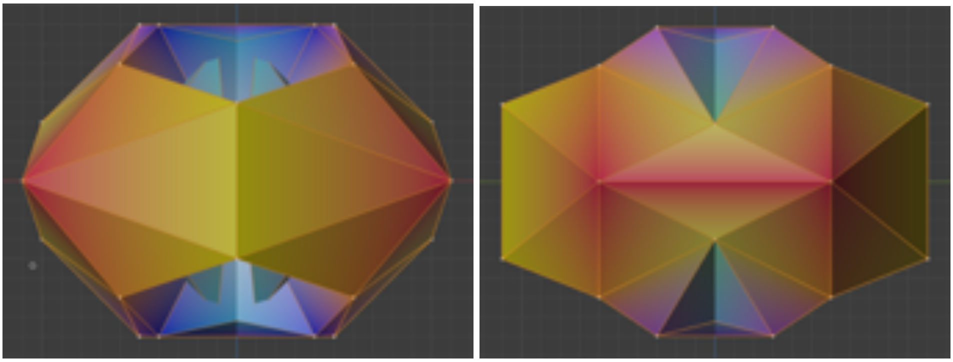

Figure 3: The KP-polyhedron in two additional orthogonal projections along coordinate axis.

Table 1: Coordinates of the KP-polyhedron

No.	x	y	z	No.	x	y	z	No.	x	y	z
1	11	-6	0	2	0	-11	-4	3	0	-11	4
4	6	-6	6	5	10	0	3	6	11	6	0
7	10	0	-3	8	6	-6	-6	9	3	-8	-3
10	-11	-6	0	11	16	7	-12	12	5	-3	8
13	6	6	6	14	6	6	-6	15	5	-3	-8
16	-6	-6	- 6	17	-3	-8	3	18	4	-3	8
19	5	3	8	20	0	11	-4	21	8	0	-1
22	0	0	0	23	-3	-8	-3	24	-6	-6	6
25	0	0	0	26	8	0	1	27	0	11	4
28	5	3	-8	29	4	-3	-8	30	0	0	0
31	-4	3	-8	32	1	-1	7	33	-3	3	1
34	1	-1	-7	35	-4	3	8	36	0	0	0
37	0	0	0	38	-4	-3	8	39	-6	6	-6
40	6	-7	0	41	-6	6	6	42	-4	-3	-8
43	0	0	0	44	4	3	-8	45	-5	3	-8
46	0	-9	-3	47	-10	0	3	48	0	0	0
49	1	1	7	50	3	8	3	51	0	0	0
52	-10	0	-3	53	0	-9	3	54	-5	3	8
55	4	3	8	56	3	8	-3	57	1	1	-7
58	-5	-3	-8	59	-1	-1	-7	60	-1	-1	7
61	-5	-3	8	62	-3	8	-3	63	6	7	0
64	-3	8	3	65	-1	1	-7	66	-8	0	-1
67	-6	-7	0	68	-8	0	1	69	-1	1	7
70	0	9	-3	71	0	9	3	72	-6	7	0

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Cayley graphs of more than one abelian group*

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Abstract

We show that for certain integers n, the problem of whether or not a Cayley digraph Γ of $\mathbb{Z}_n$ is also isomorphic to a Cayley digraph of some other abelian group G of order n reduces to the question of whether or not a natural subgroup of the full automorphism group contains more than one regular abelian group up to isomorphism (as opposed to the full automorphism group). A necessary and sufficient condition is then given for such circulants to be isomorphic to Cayley digraphs of more than one abelian group, and an easy-to-check necessary condition is provided.

Keywords: Cayley graph, circulant graph, group.

Math. Subj. Class.: 05C25, 20B05

1 Introduction

It is well known that a Cayley digraph of a group G may also be isomorphic to a Cayley digraph of group H where G and H are not isomorphic. A natural question is then to determine exactly when a Cayley digraph is isomorphic to a Cayley digraph of a nonisomorphic group. Perhaps the first work on this problem was by Joseph in 1995 [16] where she determined necessary and sufficient conditions for a Cayley digraph of order p^2, p a

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prime, to be isomorphic to a Cayley digraph of both groups of order p^2 (see [12, Lemma 4] for a group theoretic version of this result). The second author [24] subsequently extended this result and determined necessary and sufficient conditions for a Cayley digraph of $\mathbb{Z}_{p^k}$, $k \geq 1$ and p an odd prime, to be isomorphic to a Cayley digraph of some other abelian group (see Theorem 1.5 for the statement of this result). Additionally, she found necessary and sufficient conditions for a Cayley digraph of $\mathbb{Z}_{p^k}$ to be isomorphic to a Cayley digraph of any group of order p^k. The equivalent problem for $p = 2$ (when both groups are abelian) was solved by Kovács and Servatius [17]. Digraphs of order pq that are Cayley graphs of both groups of order pq, where $q \mid (p-1)$ and p, q are distinct primes were determined by the first author in [6, Theorem 3.4]. Finally, Marušič and the second author studied the question of which normal circulant graphs of square-free order are also Cayley graphs of a nonabelian group [20].

We show in this paper that for some values of n, we can reduce the problem of which circulant digraphs of order n are also Cayley digraphs of some other abelian group of order n, to the prime-power case previously solved by the second author. Specifically, let $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$ and let $k = p_1 \cdots p_r$, where each p_i is prime. Then the reduction works if $\gcd(k, \varphi(k)) = 1$.

At first glance, this arithmetic condition may seem odd. However, this condition is also contained in a well-known result of Pálfy [27] where he characterized all finite groups which are CI-groups with respect to all classes of combinatorial objects.

Theorem 1.1 (Theorem A, [27])**.** *A finite group G is a CI-group with respect to every class of combinatorial objects if and only if $|G| = 4$ or G is cyclic of order n, with* $\gcd(n, \varphi(n)) = 1$.

An equivalent statement of this theorem is that a group G of order n has the property that any subgroup $H \leq S_n$ with a regular subgroup isomorphic to G has one conjugacy class of regular subgroups isomorphic to G if and only if $n = 4$ or $\gcd(n, \varphi(n)) = 1$. See [4, 9] for some generalizations of Pálfy's Theorem (Theorem 1.1). In addition to completely answering the question of which groups are CI-groups with respect to every class of combinatorial objects, Pálfy's Theorem has also been used to classify various classes of vertex-transitive graphs [3, 7, 10], and using these results, the first author with Pablo Spiga [13] showed that there are Cayley numbers with arbitrarily many prime divisors, settling an old problem of Praeger and McKay. Thus Pálfy's Theorem and its generalizations not only have the obvious applications to the isomorphism problem for Cayley objects, but also to classification problems.

Our approach is to consider the values of n with the following property: Any subgroup $H \leq S_n$ that contains a regular subgroup isomorphic to $\mathbb{Z}_n$ and some other regular abelian group G has a nilpotent subgroup which contains conjugates of every regular abelian subgroup of H. We show in Theorem 2.15 that n has this property if and only if $\gcd(k, \varphi(k)) = 1$ (with k as defined above). In a sense, our result shows that for these special values, the question of when a Cayley object of $\mathbb{Z}_n$ is also a Cayley object of some other abelian group reduces to the prime-power case as a finite nilpotent group is the direct product of its Sylow subgroups. As our result is a characterization of such values of n, the full automorphism groups of classes of combinatorial objects may have the above property for all values of n, but the structure of the combinatorial object will have to be used to prove this - permutation group theoretic techniques will not suffice.

Next, we apply this result when the combinatorial object are digraphs - the only combinatorial objects for which the prime-power case has been solved. We show in Theorem 3.10 that if all automorphism groups of circulant digraphs have the permutation group theoretic property in the previous paragraph (which is the case if $\gcd(k, \varphi(k)) = 1$), then the question of when a circulant digraph is also a Cayley digraph of some other abelian groups reduces to the prime-power case.

Theorem 2.15 also turns out to be a generalization of a restricted form of Pálfy's Theorem (Theorem 1.1). This ultimately follows as the only regular abelian subgroup of $\mathbb{Z}_n$ when n is square-free is $\mathbb{Z}_n$, and a transitive nilpotent group of square-free order is necessarily $\mathbb{Z}_n$. For a complete explanation, see Remark 2.16. This gives that Pálfy's Theorem and its generalizations also have applications to a third question, the question of when a combinatorial object is a Cayley object of more than one group.

To summarize, we determine an algebraic condition which reduces the problem of when a circulant digraph is also a Cayley digraph of some other abelian group to the prime-power case. We determine for which values of n this condition holds for *all* permutation groups of degree n. We then combine these results to determine necessary and sufficient conditions for a circulant digraph to also be a Cayley digraph of some other abelian group for those values of n.

In the remainder of this section, we state the second author's result (Theorem 1.5), first providing the necessary definitions for that statement. In Section 2, we provide the necessary group theoretic results to prove our main theorem. In Section 3, we provide the necessary graph theoretic results to prove our main result, which is Corollary 3.11.

Definition 1.2. Let G be a group and $S \subset G$. Define a **Cayley digraph of** G, denoted $\mathrm{Cay}(G, S)$, to be the digraph with $V(\mathrm{Cay}(G, S)) = G$ and $A(\mathrm{Cay}(G, S)) = \{(g, gs) : g \in G, s \in S\}$. We call S the **connection set** of $\mathrm{Cay}(G, S)$.

Definition 1.3. Let Γ_1 and Γ_2 be digraphs. The **wreath product of** Γ_1 **and** Γ_2, denoted $\Gamma_1 \wr \Gamma_2$, is the digraph with vertex set $V(\Gamma_1) \times V(\Gamma_2)$ and arcs $((u, v), (u, v'))$ for $u \in V(\Gamma_1)$ and $(v, v') \in A(\Gamma_2)$ or $((u, v), (u', v'))$ where $(u, u') \in A(\Gamma_1)$ and $v, v' \in V(\Gamma_2)$.

For any terms from permutation group theory that are not defined in this paper, see [2]. For a set X we denote the symmetric and alternating groups on X by $\mathrm{Sym}(X)$ and $\mathrm{Alt}(X)$. If $|X| = n$ and the set is unimportant, we write $\mathrm{Sym}(n)$ and $\mathrm{Alt}(n)$.

Definition 1.4. Let X and Y be sets, $G \leq \mathrm{Sym}(X)$, and $H \leq Sym(Y)$. Define the **wreath product of** G **and** H, denoted $G \wr H$, to be the set of all permutations of $X \times Y$ of the form $(x, y) \mapsto (g(x), h_x(y))$, where for each $x \in X$, h_x is an element of H that acts on Y, but for different $x \in X$, the choice of h_x is independent.

We caution the reader that these definitions of wreath products are not completely standard, in that some mathematicians use $H \wr G$ for what we have defined as $G \wr H$, and similarly use $\Gamma_2 \wr \Gamma_1$ for our $\Gamma_1 \wr \Gamma_2$. Both orderings appear frequently in the literature.

We will often times consider wreath products of multiple digraphs or groups, and sometimes the specific digraphs or groups are unimportant. In this circumstance, rather than write out and define the digraphs or groups, we will just say that a digraph Γ or group G is a **multiwreath product**. Formally, a digraph Γ is a multiwreath product if there exist nontrivial digraphs $\Gamma_1, \ldots, \Gamma_r$ such that $\Gamma = \Gamma_1 \wr \Gamma_2 \wr \cdots \wr \Gamma_r$, and a group G is a multiwreath product if there exist nontrivial groups $G_1, \ldots, G_r$ such that $G = G_1 \wr G_2 \wr \cdots \wr G_r$. By a trivial digraph we mean a digraph on a single vertex; along similar lines, we say that a

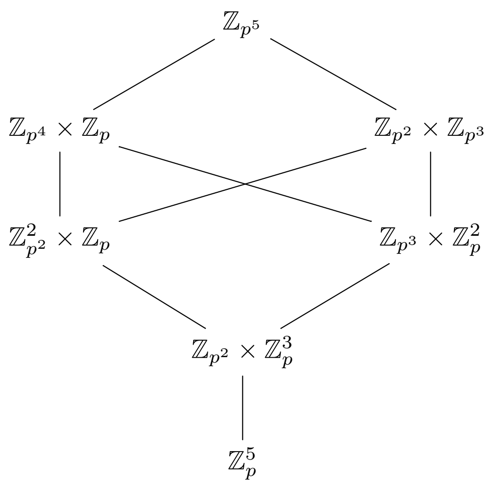

Figure 1: The partial order for groups of order p^5

group is a trivial multiwreath product if one of the factors is the same as the product, so the other factors are all trivial and the product is not a genuine decomposition.

Following [24], define a partial order on the set of abelian groups of order p^n as follows: We say $G \preceq_p H$ if there is a chain $H_1 < H_2 < \cdots < H_m = H$ of subgroups of H such that $H_1, H_2/H_1, \ldots, H_m/H_{m-1}$ are all cyclic, and

$$G \cong H_1 \times \frac{H_2}{H_1} \times \cdots \times \frac{H_m}{H_{m-1}}.$$

There is an equivalent definition for this partial order. Take the natural partial order on partitions of a fixed integer n, so for partitions $\mu : n = i_1 + \cdots + i_m$ and $\nu : n = j_1 + \cdots + j_{m_0}$ of n, we say $\mu \leq \nu$ if ν can be obtained from μ after possible rearrangement, by grouping some summands. Now, $G \preceq_p H$ precisely if $G \cong \mathbb{Z}_{p^{i_1}} \times \mathbb{Z}_{p^{i_2}} \times \cdots \times \mathbb{Z}_{p^{i_m}}$ and $H \cong \mathbb{Z}_{p^{j_1}} \times \mathbb{Z}_{p^{j_2}} \times \cdots \times \mathbb{Z}_{p^{j_{m_0}}}$ where $\mu \leq \nu$. In Figure 1 this partial order is depicted for abelian groups of order p^5.

The following result was proven in [24] (see also [23]) in the case where p is an odd prime and in [17] when $p = 2$.

Theorem 1.5. *Let $\Gamma = \mathrm{Cay}(G, S)$ be a Cayley digraph on an abelian group G of order p^k, where p is prime. Then the following are equivalent:*

1. *The digraph Γ is isomorphic to a Cayley digraph on both $\mathbb{Z}_{p^k}$ and H, where H is an abelian group with $|H| = p^k$, say $H = \mathbb{Z}_{p^{k_1}} \times \mathbb{Z}_{p^{k_2}} \times \cdots \times \mathbb{Z}_{p^{k_{m_0}}}$, where $k_1 + \cdots + k_{m_0} = k$.*
2. *There exists a chain of subgroups $G_1 \leq \cdots \leq G_{m-1}$ in G such that*
   - *(a) $G_1, G_2/G_1, \ldots, G/G_{m-1}$ are cyclic groups;*
   - *(b) $G_1 \times G_2/G_1 \times \cdots \times G/G_{m-1} \preceq_p H$;*
   - *(c) For all $s \in S \backslash G_i$, we have $sG_i \subseteq S$, for $i = 1, \ldots, m-1$. (That is, $S \backslash G_i$ is a union of cosets of G_i.)*

3. *There exist Cayley digraphs $U_1, \ldots, U_m$ on cyclic p-groups $H_1, \ldots, H_m$ such that $H_1 \times \cdots \times H_m \preceq_p H$ and $\Gamma \cong U_m \wr \cdots \wr U_1$.*

Furthermore, any of these implies:

4. *Γ is isomorphic to Cayley digraphs on every abelian group of order p^k that is greater than H in the partial order.*

2 Group theoretic results

In the section we collect all permutation group theoretic results that we will need for our main result.

Theorem 2.1 (Theorem 3 of [15], or Corollary 1.2 of [18])**.** *A primitive permutation group K acting on Ω of finite degree n has a cyclic regular subgroup if and only if one of the following holds:*

1. *$\mathbb{Z}_p \leq K \leq \mathrm{AGL}(1,p)$, where $n = p$ is prime;*
2. *$K = \mathrm{Sym}(n)$ for some n, or $K = \mathrm{Alt}(n)$ for some odd n;*
3. *$\mathrm{PGL}(d,q) \leq K \leq \mathrm{P\Gamma L}(d,q)$ where $n = (q^d - 1)/(q-1)$ for some $d \geq 2$;*
4. *$K = \mathrm{PSL}(2,11)$, M_{11} or M_{23} where $n = 11$, 11 or 23 respectively.*

When we refer to any of the groups listed in the above theorem, we will be considering it not as an abstract group, but as a permutation group endowed with its natural action.

If G has a block system $\mathcal{B}$ with blocks of minimal size, then the action of the set-wise stabilizer H in G of the block $B \in \mathcal{B}$ is primitive. We will only be concerned with the case when G contains a regular cyclic subgroup, and it is not hard to show that the action of H on B is one of the groups in Theorem 2.1. We now consider conjugation results concerning the groups in Theorem 2.1 that we will require later.

'The following result can be deduced from Theorem 1.1 and Corollary 1.2 of [18].

Lemma 2.2. *If K satisfies $\mathrm{PGL}(d,q) \leq K \leq \mathrm{P\Gamma L}(d,q)$, then every regular abelian subgroup of K is cyclic. Furthermore, any such subgroup is a Singer subgroup (and so any two are conjugate in $\mathrm{PGL}(d,q)$) unless $d = 2$ and $q = 8$, in which case $n = 9$.*

The conjugacy of regular cyclic subgroups is also noted in Corollary 2 of [15], but the fact that all regular abelian subgroups are cyclic is proved in the Li paper.

The following result is well-known.

Lemma 2.3. *If two elements of $\mathrm{Alt}(n)$ are conjugate in $\mathrm{Sym}(n)$ but not in $\mathrm{Alt}(n)$, then their cycle structures are the same, they have no cycle of even length, and the lengths of all of their odd cycles are distinct.*

Lemma 2.4. *Let $K, K' \leq G$ be conjugate in G. If $N \triangleleft G$ and $KN = G$, then there exists $n \in N$ with $n^{-1}Kn = K'$.*

In particular, any two regular cyclic subgroups of $\mathrm{PGL}(d,q)$ are conjugate by an element of $\mathrm{PSL}(d,q)$. Also, if n is even, then any two regular cyclic subgroups of $\mathrm{Sym}(n)$ are conjugate by an element of $\mathrm{Alt}(n)$.

Proof. Let $g \in G$ such that $g^{-1}Kg = K'$. As $KN = G$ there exists $k \in K$, $n \in N$ such that $kn = g$. Now $n^{-1}Kn = g^{-1}kKk^{-1}g = g^{-1}Kg = K'$.

By [15, Corollary 2] or [18, Corollary 1.2], there is one conjugacy class of regular cyclic subgroups of $\mathrm{PGL}(d, q)$, and clearly there is always one conjugacy class of regular cyclic subgroups in $\mathrm{Sym}(n)$. If K is a regular cyclic subgroup of $\mathrm{PGL}(d, q)$, then $K\mathrm{PSL}(d, q) = \mathrm{PGL}(d, q)$ by [18, Lemma 2.3]. Similarly, if K is a regular cyclic subgroup of $\mathrm{Sym}(n)$ where n is even, then $K\mathrm{Alt}(n) = \mathrm{Sym}(n)$. The result follows by the first paragraph of this proof, with $\mathrm{PSL}(d, q)$ or $\mathrm{Alt}(n)$ taking the role of N. □

The concept of Ω-step imprimitivity will be important in this paper. Intuitively, on a set of cardinality n, the action of a transitive group is Ω-step imprimitive if there is a sequence of nested block systems that is as long as possible (given n). The terms "nested" and "as long as possible" may not be clear, so we provide formal definitions below, including an explicit formula for $\Omega = \Omega(n)$.

Definition 2.5. Let G be a transitive permutation group. Let Y be the set of all block systems of G. Define a partial order on Y by $\mathcal{B} \leq \mathcal{C}$ if and only if every block of $\mathcal{C}$ is a union of blocks of $\mathcal{B}$. We say that a strictly increasing sequence of $m + 1$ block systems under this partial order is an ***m*-step imprimitivity sequence** admitted by G.

An m-step imprimitivity sequence is what we referred to in our intuitive description as a "nested" sequence.

Definition 2.6. Let G be a transitive group of degree n. Let $n = \Pi_{i=1}^{r} p_i^{a_i}$ be the prime factorization of n and let $\boldsymbol{\Omega = \Omega(n)} = \sum_{i=1}^{r} a_i$. (The number $\Omega(n)$ is known as the total number of prime divisors of n, see for example [28].) Then G is **Ω-step imprimitive** if it admits an Ω-step imprimitivity sequence.

A block system $\mathcal{B}$ will be said to be **normal** if the elements of $\mathcal{B}$ are the orbits of a normal subgroup. We will say that G is **normally Ω-step imprimitive** if G is Ω-step imprimitive with a sequence in which each block system is normal.

Let $\mathcal{B}_0 < \cdots < \mathcal{B}_\Omega$ be an Ω-step imprimitivity sequence of G, where G is acting on X. Then $\mathcal{B}_0$ consists of singleton sets, $\mathcal{B}_\Omega = \{X\}$, and if $B_i \in \mathcal{B}_i$ and $B_{i+1} \in \mathcal{B}_{i+1}$, then $|B_{i+1}|/|B_i|$ is a prime. Thus it is not possible to have a k-step imprimitivity sequence for any $k > \Omega$, satisfying our intuitive description of the system as being "as long as possible".

Recall that we would like to characterise digraphs Γ with $G, G' \leq \mathrm{Aut}(\Gamma)$, where G and G' are regular (and nonisomorphic), and G is cyclic. Our method will be to find a subgroup N of $\mathrm{Aut}(\Gamma)$ that is normally $\Omega(n)$-step imprimitive. If we can then find regular subgroups $H, H' \leq N$ with $H \cong G$ and $H' \cong G'$, then we will be able to use the fact that H and H' also admit the many block systems of N to determine a lot about the structure of Γ.

In fact, in Lemma 2.12, we will show that there is some conjugate $\delta^{-1}G'\delta \leq \mathrm{Aut}(\Gamma)$ of G', with $\delta \in \langle G, G' \rangle$, such that $K = \langle G, \delta^{-1}G'\delta \rangle$ is normally $\Omega(n)$-step imprimitive. Then in Theorem 2.15, we show that if we assume a numerical condition on n, there is a nilpotent group $N \leq K$ that contains subgroups isomorphic to both G and G'. Clearly N is still $\Omega(n)$-step imprimitive since it must admit all of the blocks that K admits.

The next two lemmas, the intervening corollary, and the definitions surrounding them are required for the proof of Lemma 2.12, which is in turn used in the proof of Theorem 2.15.

Definition 2.7. Let X be a set, let $G \leq Sym(X)$ be transitive and $\mathcal{B}$ a block system of G. For $g \in G$ we denote by $g/\mathcal{B}$ the permutation of $\mathcal{B}$ induced by g, and $G/\mathcal{B} = \{g/\mathcal{B} : g \in G\}$.

By $\mathbf{fix}_{\boldsymbol{G}}(\boldsymbol{\mathcal{B}})$ we mean the subgroup of G which fixes each block of $\mathcal{B}$ set-wise. That is,

$$\mathrm{fix}_G(\mathcal{B}) = \{g \in G : g(B) = B \text{ for all } B \in \mathcal{B}\}.$$

This can also be thought of as the kernel of the projection from G to $G/\mathcal{B}$.

For $B \in \mathcal{B}$, we denote the set-wise stabilizer of the block B by $\mathbf{Stab}_{\boldsymbol{G}}(\boldsymbol{B})$. That is,

$$\mathrm{Stab}_G(B) = \{g \in G : g(B) = B\}.$$

The **support of** $\boldsymbol{G}$, denoted $\mathrm{supp}(G)$, is the set of all $x \in X$ that are acted on nontrivially by some $g \in G$. That is,

$$\mathrm{supp}(G) = \{x \in X : \text{there exists } g \in G \text{ such that } g(x) \neq x\}.$$

Lemma 2.8 ([8, Lemma 2.2])**.** *Let $H \leq \mathrm{Sym}(n)$ be transitive such that H admits a block system $\mathcal{B}$. If $T = (\mathrm{soc}(\mathrm{fix}_H(\mathcal{B})))^B$ is a transitive nonabelian simple group where $B \in \mathcal{B}$, then $\mathcal{C} = \{\mathrm{supp}(L) : L$ is a minimal normal subgroup of $\mathrm{fix}_H(\mathcal{B})\}$ is a block system of H, $\mathcal{B} \leq \mathcal{C}$, and $\mathrm{soc}(\mathrm{fix}_H(\mathcal{B}))$ is a direct product of simple groups isomorphic to T.*

Some of the arguments in the following corollary are also used in the proof of the above lemma, but since they are not included in the final statement, we repeat them for clarity.

Corollary 2.9. *Let H, $\mathcal{B}$, T, and $\mathcal{C}$ be as in the statement of Lemma 2.8. If the blocks of $\mathcal{C}$ are $C_1, \ldots, C_k$, then we can write $\mathrm{soc}(\mathrm{fix}_H(\mathcal{B})) = T_1 \times \cdots \times T_k$ where the support of T_i is C_i for every $1 \leq i \leq k$.*

Let $C \in \mathcal{C}$ and let B, B' be two blocks of $\mathcal{B}$ that lie inside C. If T also has trivial centralizer in $\mathrm{fix}_H(\mathcal{B})^B$ then there is no element $\gamma \in \mathrm{fix}_H(\mathcal{B})$ such that $\gamma^B = 1$ but $\gamma^{B'} \neq 1$.

In particular, this applies if T is primitive on B.

Proof. First notice that as Lemma 2.8 states, $\mathrm{soc}(\mathrm{fix}_H(\mathcal{B}))$ is a direct product of isomorphic finite simple groups isomorphic to T, so the kernel of any homomorphism from $\mathrm{soc}(\mathrm{fix}_H(\mathcal{B}))$ onto T is the same as the kernel of some projection onto a single factor.

Thus, each block of $\mathcal{B}$ is in the support of a unique direct factor of $\mathrm{soc}(\mathrm{fix}_H(\mathcal{B}))$, so there is a well-defined map from $\mathcal{B}$ to the direct factors of $\mathrm{soc}(\mathrm{fix}_H(\mathcal{B}))$ determined by the direct factor that includes a given $B \in \mathcal{B}$ in its support.

Now, taking any $h \in \mathrm{fix}_H(\mathcal{B})$, since h fixes every block in the support of any direct factor T_i of $\mathrm{soc}(\mathrm{fix}_H(\mathcal{B}))$, the map we've just mentioned means that T_i is normalized by h. Thus, the direct factors in $\mathrm{soc}(\mathrm{fix}_H(\mathcal{B}))$, which are the minimal normal subgroups of $\mathrm{soc}(\mathrm{fix}_H(\mathcal{B}))$, are also normal subgroups (and therefore minimal normal subgroups) in $\mathrm{fix}_H(\mathcal{B})$. In other words, there is a one-to-one correspondence between the blocks of $\mathcal{C}$ and the direct factors of $\mathrm{soc}(\mathrm{fix}_H(\mathcal{B}))$, establishing the first claim of this corollary.

Towards a contradiction, suppose that there exists $\gamma \in \mathrm{fix}_H(\mathcal{B})$ such that $\gamma^B = 1$ but $\gamma^{B'} \neq 1$. Let $N = \langle\gamma\rangle^{\mathrm{fix}_H(\mathcal{B})}$ be the normal closure of $\langle\gamma\rangle$ in $\mathrm{fix}_H(\mathcal{B})$, and notice that since $\gamma^B = 1$ we also have $N^B = 1$.

Let $T_{B'} = (\mathrm{soc}(\mathrm{fix}_H(\mathcal{B})))^{B'}$. By hypothesis, $T_{B'}$ has trivial centralizer in $\mathrm{fix}_H(\mathcal{B})^{B'}$, so there is some element $t' \in T_{B'}$ such that $[t', \gamma^{B'}]$ is nontrivial on B'. Let M be the direct factor of $\mathrm{soc}(\mathrm{fix}_H(\mathcal{B}))$ with $\mathrm{supp}(M) = C$. As $C = \mathrm{supp}(M)$ and $B \subseteq C$, $M^B \neq 1$. Then there exists $t \in M$ such that $t^B = t'$. Now, $[t, \gamma] \in N$, since $\gamma \in N$, $N \trianglelefteq \mathrm{fix}_H(\mathcal{B})$, and $t \in \mathrm{fix}_H(\mathcal{B})$. Also, $[t, \gamma] \in M$, since $t \in M$, $M \trianglelefteq \mathrm{fix}_H(\mathcal{B})$, and $\gamma \in \mathrm{fix}_H(\mathcal{B})$. So $N \cap M$ is a nontrivial normal subgroup of M. Since M is simple, $N \geq M$. But then $N^B = 1$ and so $M^B = 1$, a contradiction.

We complete this proof by showing that if T is primitive then its centralizer in $\mathrm{fix}_H(\mathcal{B})^B$ is trivial, so this corollary applies. This result follows from [2, Theorem 4.2A (vi)]. □

Lemma 2.10. *Let G, G' be regular abelian subgroups of a primitive group K of degree n, with G cyclic. Let p be any prime divisor of n, let G_p be the unique subgroup of G of order p, and let G'_p be any subgroup of G' of order p. Then there exists $\delta \in \mathrm{soc}(K)$ such that $G_p = \delta^{-1}G'_p\delta$.*

Proof. By our hypotheses, K must be given in Theorem 2.1.

Suppose first that $n = p$ is prime (this deals with parts (1) and (4) of Theorem 2.1, as well as with some cases of part (2)). In this case, G and G' are Sylow p-subgroups of K. Furthermore, either $K \leq \mathrm{AGL}(1, p)$ has a unique regular subgroup (this covers $p = 2$ in particular), K is simple and $\mathrm{soc}(K) = K$, or $K = \mathrm{Sym}(p)$ and $\mathrm{soc}(K) = \mathrm{Alt}(p)$ with $p > 2$. In the first case, $G = G' = \mathrm{soc}(K)$ and the result is immediate. In the third case p is odd and $G, G' \leq \mathrm{Alt}(p) = \mathrm{soc}(K)$, so in both the second and third cases $G, G' \leq \mathrm{soc}(K)$ are Sylow p-subgroups of $\mathrm{soc}(K)$ and we can use a Sylow theorem.

Next we suppose that $\mathrm{soc}(K) = \mathrm{Alt}(n)$. Let $G_p \leq G$, $G'_p \leq G'$ be subgroups of order p. Unless $p = 2$ and n/p is odd, the semiregular cyclic groups G_p and G'_p are generated by elements g and g' respectively, where $g, g' \in \mathrm{Alt}(n)$ each has a cycle structure consisting of $n/p > 1$ cycles of length p, since n is composite. By Lemma 2.3, there is some $\delta \in \mathrm{Alt}(n)$ such that $g = \delta^{-1}g'\delta$, and so $G_p = \delta^{-1}G'_p\delta$. If $p = 2$ and n/p is odd, then the elements g, g' are not in $\mathrm{Alt}(n)$. However, there is some element γ of $\mathrm{Sym}(n)$ such that $\gamma^{-1}g\gamma = g'$, so if one of the cycles of g is $(a\ b)$ then $(a\ b)\gamma \in \mathrm{Alt}(n)$ and $\gamma^{-1}(a\ b)g(a\ b)\gamma = g'$. This deals with part (2) of Theorem 2.1.

We may now assume that part (3) of Theorem 2.1 holds, so $\mathrm{PGL}(d, q) \leq K \leq \mathrm{P\Gamma L}(d, q)$. Unless $d = 2$, $q = 8$, and $n = 9$, we may apply Lemma 2.2 to deduce that both G and G' are cyclic Singer subgroups that are conjugate in $\mathrm{PGL}(d, q)$. Furthermore, by Lemma 2.4 there exists $\delta \in \mathrm{soc}(K) = \mathrm{PSL}(d, q)$ such that $\delta^{-1}G'\delta = G$, and we are done.

To complete our proof, we need to address the possibility that $\mathrm{PGL}(2, 8) \leq K \leq \mathrm{P\Gamma L}(2, 8)$ so $n = 9$ and both G and G' are cyclic groups of order 9. Since the index of $\mathrm{PGL}(2, 8)$ in $\mathrm{P\Gamma L}(2, 8)$ is 3, we have only these two possibilities for K. If $K = \mathrm{PGL}(2, 8)$ (so its Sylow 3-subgroups have order 9), then every regular subgroup of K is a Sylow subgroup and the result follows by a Sylow theorem.

As might be presumed from the exception in Lemma 2.2, $\mathrm{P\Gamma L}(2, 8)$ does contain more than one conjugacy class of regular cyclic subgroups. However, we will show that the semiregular subgroups of order 3 in these regular subgroups are conjugate, completing the proof of this result. In order to do this, first observe that for any Sylow 3-subgroup P of $\mathrm{P\Gamma L}(2, 8)$, P is nonabelian of order 27, and the center of P has order 3. Now any cyclic subgroup of order 9 in P is normal and hence contains $Z(P)$, so $Z(P)$ must be the unique semiregular subgroup of order 3 in any such subgroup. Since all of the Sylow subgroups are conjugate in $\mathrm{P\Gamma L}(2, 8)$, so are their centers.

Now let G and G' be as in our hypothesis. Let P be the Sylow 3-subgroup of K that contains G, and let Q be the Sylow 3-subgroup of K that contains G'. Let $S \leq P$ and

$T \leq Q$ be the unique Singer cycles in P and Q. We claim that G_3 is the unique subgroup of S of order 3. Since nothing in our hypotheses distinguishes S from T or G from G', this will imply that G_3' is the unique subgroup of T of order 3, and therefore since there exists $\delta \in \mathrm{PGL}(2,8) = \mathrm{PSL}(2,8)$ such that $\delta^{-1}T\delta = S$, we have $\delta^{-1}G_3'\delta = G_3$. To prove our claim, it certainly suffices to show that P itself has a unique semiregular subgroup of order 3 (which is G_p); this is what we will do.

We may assume that $P = \{x \mapsto ax + b : a \in \{1,4,7\}, b \in \mathbb{Z}_9\}$ as the multiplicative orders of 4 and 7 are 3 in $\mathbb{Z}_9$, and the subgroup $\{x \mapsto ax : a = 1,4,7\}$ normalizes $x \mapsto x+1$ and P has order 27. Let $H \leq P$ be semiregular of order 3, and note that such an H exists as $H' = \{x \mapsto x + b : b \in \langle 3\rangle\}$ is semiregular of order 3. Let $f \in H$ with $\langle f\rangle = H$, where $f(x) = ax + b$, $a \in \{1,4,7\}$ and $b \in \mathbb{Z}_9$. Then $f^3(x) = x + (a^2 + a + 1)b = x + 3b = x$, from which we conclude that $b \equiv 0 \pmod 3$. Thus $b = 3i$, where $i \in \{0,1,2\}$. Set $a = 1 + 3j$, where $j \in \{0,1,2\}$. If $j = 0$, then $H = H'$. Otherwise, let $\ell \in \mathbb{Z}_9$ such that $j\ell \equiv -i \pmod 3$. Then

$$f(\ell) = a\ell + b = (1+3j)\ell + 3i = \ell + 3j\ell + 3i \equiv \ell - 3i + 3i \pmod 9 = \ell.$$

We conclude that f has a fixed point and H is not semiregular. Hence there is a unique semiregular subgroup of P as required. This completes the proof. $\square$

Let $G \leq \mathrm{Sym}(n)$ admit a block system $\mathcal{B}$. It is straightforward to observe that there is a block system $\mathcal{D}$ in $G/\mathcal{B}$ if and only if there is a block system $\mathcal{C} \geq \mathcal{B}$ of G where a block of $\mathcal{C}$ consists of the union of all blocks of $\mathcal{B}$ contained within a block of $\mathcal{D}$.

Definition 2.11. If G admits block systems $\mathcal{B}$ and $\mathcal{C}$ with $\mathcal{B} \leq \mathcal{C}$, then we denote the corresponding block system $\mathcal{D}$ in $G/\mathcal{B}$ by $\boldsymbol{\mathcal{C}/\mathcal{B}}$.

We are now ready to show that we can find a group that contains our regular cyclic subgroup and a conjugate of any other given regular abelian subgroup, and is $\Omega(n)$-step imprimitive. Although it is not immediately clear from the statements of the results as written, [25, Theorem 4.9 (i)] is a consequence of this lemma.

Lemma 2.12. *Let G, G' be regular abelian subgroups of a permutation group of odd degree n, with G cyclic. Let $n = p_1^{a_1} \cdots p_r^{a_r}$ be the prime-power decomposition of n, and $\Omega = \Omega(n) = \sum_{i=1}^{r} a_i$. Then there exists $\delta \in \langle G, G'\rangle$ such that $\langle G, \delta^{-1}G'\delta\rangle$ is normally Ω-step imprimitive.*

Proof. We proceed by induction on $\Omega = \Omega(n)$. If $\Omega(n) = 1$, then n is prime and the result follows from a Sylow theorem. Let G, G', and n satisfy the hypotheses, with $\Omega(n) \geq 2$. Assume the result holds for all permutation groups of degree n' with $\Omega(n')$ at most $\Omega(n) - 1$. If $H = \langle G, G'\rangle$ is primitive, then by Lemma 2.10, there exists $\delta \in H$ such that $\langle G, \delta^{-1}G'\delta\rangle$ has a nontrivial center and hence is imprimitive, so we may assume without loss of generality that H is imprimitive.

Suppose that $\mathcal{B}$ is a block system of H with m blocks of size k. Observe that since G, G' are regular and abelian, the orbits of $\mathrm{fix}_H(\mathcal{B}) \triangleleft H$ are the blocks of $\mathcal{B}$, so $\mathcal{B}$ is a normal block system. We first show that if $k = p$ is prime, we can complete the proof; then we will devote the remainder of the proof to demonstrating that if k is composite, then for any prime $p \mid k$ there exists $\delta \in H$ such that $\langle G, \delta^{-1}G'\delta\rangle$ admits a normal block system with blocks of prime size p. Replacing G' by this conjugate and $\mathcal{B}$ by this system then completes the proof.

Suppose $k = p$ is prime, and set $\mathcal{B}_1 = \mathcal{B}$. By the induction hypothesis, since $\Omega(n/p) = \Omega(n) - 1 = \Omega - 1$, there exists $\delta \in \langle G/\mathcal{B}_1, G'/\mathcal{B}_1\rangle$ such that $\langle G/\mathcal{B}_1, \delta^{-1}(G'/\mathcal{B}_1)\delta\rangle$ is normally $(\Omega - 1)$-step imprimitive with $(\Omega - 1)$-step imprimitivity sequence $\mathcal{B}_1/\mathcal{B}_1 < \mathcal{B}_2/\mathcal{B}_1 < \cdots < \mathcal{B}_\Omega/\mathcal{B}_1$. Then taking $\delta_1 \in H$ such that $\delta_1/\mathcal{B}_1 = \delta$, we see that $\langle G, \delta_1^{-1}G'\delta_1\rangle$ is normally $\Omega(n)$-step imprimitive with $\mathcal{B}_0 < \mathcal{B}_1 < \cdots < \mathcal{B}_m$. This completes the proof when $k = p$.

Suppose now that k is composite. We assume that k is chosen to be minimal, and so by [2, Exercise 1.5.10] $\mathrm{Stab}_H(B)^B$ is primitive. Since k is composite, for any block $B \in \mathcal{B}$, $\mathrm{Stab}_H(B)^B$ is doubly-transitive as $\mathbb{Z}_k$ is a Burnside group [2, Theorem 3.5A]. Since the groups in Theorem 2.1(1) are not of composite degree, $\mathrm{Stab}_H(B)^B$ has nonabelian simple socle, T_B. In fact, $T_B \cong \mathrm{PSL}(d,q)$ for some d, q, or $T_B \cong \mathrm{Alt}(k)$, so T_B is doubly-transitive. Since $\mathrm{fix}_H(\mathcal{B})^B \triangleleft \mathrm{Stab}_H(B)^B$, we see that $(\mathrm{fix}_H(\mathcal{B})^B) \cap T_B$ is nontrivial and normal in T_B; since T_B is a simple group, we conclude that $\mathrm{soc}(\mathrm{fix}_H(\mathcal{B})^B) = T_B$. Thus $\mathrm{fix}_H(\mathcal{B})^B$ has a doubly-transitive socle, so must itself be doubly-transitive. Let $K = \mathrm{soc}(\mathrm{fix}_H(\mathcal{B}))$.

We now show that $T_B = \mathrm{soc}(\mathrm{fix}_H(\mathcal{B})^B) = (\mathrm{soc}(\mathrm{fix}_H(\mathcal{B})))^B = K^B$; that is, we get the same group whether we restrict to B before or after taking the socle. Clearly, $K = \mathrm{soc}(\mathrm{fix}_H(\mathcal{B})) \triangleleft \mathrm{fix}_H(\mathcal{B})$, so $K^B \triangleleft (\mathrm{fix}_H(\mathcal{B}))^B$. Therefore, some minimal normal subgroup of $(\mathrm{fix}_H(\mathcal{B}))^B$ lies in K^B. We have just shown that $\mathrm{soc}(\mathrm{fix}_H(\mathcal{B})^B) = T_B$ is a nonabelian simple group, so $(\mathrm{fix}_H(\mathcal{B}))^B$ has only one minimal normal subgroup, meaning $T_B = \mathrm{soc}(\mathrm{fix}_H(\mathcal{B})^B) \leq K^B$. As n is odd, A_n contains a regular cyclic subgroup. Then K^B (since it contains T_B) must also contain a composite regular cyclic subgroup, so just as above it too has a nonabelian simple socle. Since any socle is a direct product of simple groups and T_B is already (doubly) transitive on B, it is not possible that $K^B > T_B$. Thus $K^B = T_B$.

Since $K^B = T_B$, it is a transitive nonabelian simple group, so the hypotheses of Lemma 2.8 are satisfied. Using Lemma 2.8 together with the first conclusion of Corollary 2.9 (which requires no additional assumptions, we may write $K = T_1 \times \ldots \times T_r$, where for each $i \in \{1, \ldots, r\}$ we have $T_i \cong T_B$ and $\mathrm{supp}(T_i)$ is a block of H. Furthermore if we take $C_i = \mathrm{supp}(T_i)$ and $\mathcal{C} = \{C_1, \ldots, C_r\}$ then $\mathcal{B} \leq \mathcal{C}$ (so $r \mid m$).

Let $\mathcal{B} = \{B_{i,j} : 1 \leq i \leq r \text{ and } 1 \leq j \leq m/r\}$ be labelled so that $\cup_{j=1}^{m/r} B_{i,j} = C_i$. Let p be a prime divisor of k. Let $G_p \leq \mathrm{fix}_G(\mathcal{B})$ be the unique subgroup of G of order p, and $G'_p \leq \mathrm{fix}_{G'}(\mathcal{B})$ a subgroup of order p. Certainly, $G_p, G'_p \leq \mathrm{fix}_H(\mathcal{B})$, so $G_p^B, (G'_p)^B \leq \mathrm{fix}_H(\mathcal{B})^B$ for any $B \in \mathcal{B}$.

For any $1 \leq i \leq r$, let T_i take the role of K in Lemma 2.10. We know that $K \leq \mathrm{fix}_H(\mathcal{B})$ and that T_i is acting primitively on each block $B_{i,j}$ in C_i, so the hypotheses of the lemma are satisfied. The lemma tells us that there exists $\delta_i \in T_i$ such that $(\delta_i^{-1}G'_p\delta_i)^{B_{i,1}} = G_p^{B_{i,1}}$. Let $\delta_i^{-1}G'_p\delta_i = \langle h\rangle$, and $G_p = \langle g\rangle$. Then there exists $b_i \in \mathbb{Z}$ such that $(gh^{b_i})^{B_{i,1}} = 1$. Since $T_B = K^B$ is doubly transitive and therefore primitive, the hypotheses of Corollary 2.9 are also satisfied, so since $gh^{b_i} \in \mathrm{fix}_H(\mathcal{B})$, it must be the case that for every $1 \leq j \leq m/r$, $(gh^{b_i})^{B_{i,j}} = 1$, and so $(\delta_i^{-1}G'_p\delta_i)^{C_i} = (G_p)^{C_i}$. Also, $(\delta_i^{-1}G'_p\delta_i)^{C_j} = (G'_p)^{C_j}$ for any $j \neq i$, since $\delta_i^{C_j} = 1$. So if $\delta = \Pi_{i=1}^r \delta_i$ then $\delta \in K$ and $\delta^{-1}G'_p\delta = G_p$ is a central subgroup of $\langle G, \delta^{-1}G'\delta\rangle$, whose orbits are blocks of size p. Thus, replacing G' with $\delta^{-1}G'\delta$, we assume without loss of generality that $k = p$. $\square$

Definition 2.13. Let $K \leq \mathrm{Sym}(n)$ contain a regular abelian subgroup G. We say that a subgroup N with $G \leq N \leq K$ **mimics every regular abelian subgroup of** K, if the following two statements are equivalent:

- a regular abelian group $M \leq \mathrm{Sym}(n)$ is contained in K; and
- N contains a regular abelian subgroup isomorphic to M.

If in addition, the subgroup of N isomorphic to M is conjugate in K to M, we say that N **mimics by conjugation every regular abelian subgroup of** K.

We first give a sufficient condition for a group K to contain a nilpotent subgroup that mimics by conjugation every regular abelian subgroup of K.

Lemma 2.14. *Let G be a regular cyclic subgroup of K. Suppose that whenever $R \leq K$ is a regular abelian subgroup, then there exists $\delta \in K$ such that $\langle G, \delta^{-1}R\delta\rangle$ is nilpotent. If N is a maximal nilpotent subgroup of K that contains G, then N mimics by conjugation every regular abelian subgroup of K.*

Proof. Let N be a maximal nilpotent subgroup of K that contains G, and $R \leq K$ a regular abelian subgroup of K. We will show that there exists $\delta \in K$ with $\delta^{-1}R\delta \leq N$, which will establish the result.

Let $n = p_1^{a_1} \cdots p_r^{a_r}$ be the prime-power decomposition of n. Note that since G is a regular cyclic group, for any $1 \leq i \leq r$ it admits block systems $\mathcal{B}_i$ with blocks of size $p_i^{a_i}$ and $\mathcal{C}_i$ with blocks of size $n/p_i^{a_i}$, and if a group containing G as a subgroup admits a block system whose blocks have one of these sizes, it must be one of these block systems. As N is nilpotent and $G \leq N$, by [4, Lemma 10] for every $1 \leq i \leq r$, N admits $\mathcal{B}_i$ and $\mathcal{C}_i$ as normal block systems. Also, by hypothesis, there is some $\delta_1 \in K$ such that $\langle G, \delta_1^{-1}R\delta_1\rangle$ is nilpotent and contains G, so admits each $\mathcal{B}_i$ and $\mathcal{C}_i$. Since we are only aiming for a conjugate in K, we can replace R by $\delta_1^{-1}R\delta_1$, if necessary, to assume that $\langle G, R\rangle$ is nilpotent and admits each $\mathcal{B}_i$ and $\mathcal{C}_i$. Then $\langle R, N\rangle = \langle G, R, N\rangle$ admits $\mathcal{B}_i$ and $\mathcal{C}_i$ as block systems, and so $\langle R, N\rangle \leq \Pi_{i=1}^r \mathrm{Sym}(p_i^{a_i})$ in its natural action on, say $B_1 \times \cdots \times B_r$ where $B_i \in \mathcal{B}_i$, by [5, Lemma 10].

Now, $R/\mathcal{C}_1$ and $N/\mathcal{C}_1$ are p_1-subgroups of $\mathrm{Sym}(p_1^{a_1})$, and so there exists $\omega_1 \in \langle R, N\rangle$ such that $\omega_1^{-1}R\omega/\mathcal{C}_1$ and $N/\mathcal{C}_1$ are contained in the same Sylow p_1-subgroup. Then $\langle \omega_1^{-1}R\omega_1, N\rangle/\mathcal{C}_1$ is a p_1-group. Inductively suppose that $\langle \omega_j^{-1} \cdots \omega_1^{-1}R\omega_1 \cdots \omega_j, N\rangle/\mathcal{C}_i$ is a p_i-group, $1 \leq i \leq j < r$. Then there is some $\omega_{j+1} \in \langle \omega_j^{-1} \cdots \omega_1^{-1}R\omega_1 \cdots \omega_j, N\rangle$ such that (by conjugation of Sylow p_{j+1}-subgroups) $\langle \omega_{j+1}^{-1} \cdots \omega_1^{-1}R\omega_1 \cdots \omega_{j+1}, N\rangle/\mathcal{C}_{j+1}$ is a p_{j+1}-group. As $\omega_{j+1} \in \langle \omega_j^{-1} \cdots \omega_1^{-1}R\omega_1 \cdots \omega_j, N\rangle$ we have

$$\langle \omega_{j+1}^{-1} \cdots \omega_1^{-1}R\omega_1 \cdots \omega_{j+1}, N\rangle \leq \langle \omega_j^{-1} \cdots \omega_1^{-1}R\omega_1 \cdots \omega_j, N\rangle,$$

so our inductive hypothesis implies that $\langle \omega_{j+1}^{-1} \cdots \omega_1^{-1}R\omega_1 \cdots \omega_{j+1}, N\rangle/\mathcal{C}_i$ is still a p_i-group for each $1 \leq i \leq j < r$, completing the induction. Let $\delta = \omega_1 \cdots \omega_r \in K$, so that $\langle \delta^{-1}R\delta, N\rangle/\mathcal{C}_i$ is a p_i-group for every $1 \leq i \leq r$. As $\langle R, N\rangle \leq \Pi_{i=1}^r \mathrm{Sym}(p_i^{a_i})$, we have $\langle \delta^{-1}R\delta, N\rangle \leq \Pi_{i=1}^r \mathrm{Sym}(p_i^{a_i})$. Let $G_i \leq \mathrm{Sym}(p_i^{a_i})$ be minimal such that $\langle \delta^{-1}R\delta, N\rangle \leq \Pi_{i=1}^r G_i$. Then $\langle \delta^{-1}R\delta, N\rangle/\mathcal{C}_i \leq G_i$. Hence each G_i is a p_i-group. This shows that $\langle \delta^{-1}R\delta, N\rangle$ is a direct product of its Sylow p-subgroups, so is nilpotent. By the maximality of N, we have $\langle \delta^{-1}R\delta, N\rangle = N$, meaning $\delta^{-1}R\delta \leq N$, as required. □

We now characterize those values of n for which transitive groups of degree n that contain a regular cyclic subgroup always have this extremely useful property.)

Theorem 2.15. *Let $k = p_1 \cdots p_r$ where the p_i are prime distinct primes, and $n = p_1^{a_1} \cdots p_r^{a_r}$. Let G be a regular cyclic group of degree n. The condition $\gcd(k, \varphi(k)) = 1$ is both necessary and sufficient to guarantee that whenever $G \le K \le S_n$ there exists a nilpotent subgroup N, with $G \le N \le K$, that mimics by conjugation every regular abelian subgroup of K.*

Proof. Suppose $\gcd(k, \varphi(k)) = 1$. Let $R \le K$ be a regular abelian group. By Lemma 2.12 there exists $\delta_1 \in K$ such that $\langle G, \delta_1^{-1} R \delta_1 \rangle$ is normally $\Omega(n)$-step imprimitive. By [4, Theorem 12], (applying this requires our hypothesis that $\gcd(k, \varphi(k)) = 1$), there exists $\delta_2 \in \langle G, \delta_1^{-1} R \delta_1 \rangle \le K$ such that $\langle G, \delta_2^{-1} \delta_1^{-1} R \delta_1 \delta_2 \rangle$ is nilpotent. By Lemma 2.14, any maximal nilpotent subgroup N that contains G mimics by conjugation every regular abelian subgroup of K.

Conversely, we will show that if $\gcd(k, \varphi(k)) > 1$, then for every $a_1, \ldots, a_r$ with $n = p_1^{a_1} \cdots p_r^{a_r}$, there exists a regular cyclic subgroup G of degree n, a group $K \ge G$, and a regular abelian group $R \le K$ such that for every conjugate T of R in K, $\langle G, T \rangle$ is not nilpotent. First, if $n = k$, then n is square-free and the only abelian group of order n is the cyclic group of order n. So every regular abelian subgroup is cyclic. By Pálfy's Theorem (Theorem 1.1) there exists $G \le K \le S_n$ and a regular cyclic subgroup $T \le K$ such that G and T are not conjugate in K. Also, the only transitive nilpotent subgroups of S_n are the regular cyclic subgroups, so for any $\delta \in K$, $\langle G, \delta^{-1} T \delta \rangle$ is nilpotent if and only if $\delta^{-1} T \delta = G$. So the result follows in this case.

Suppose $n \ne k$. We have just shown that if $G_k \le S_k$ is a regular cyclic subgroup of degree k, there exists $K_k \le S_k$ and $T_k \le K_k$ a regular cyclic subgroup such that no conjugate of T_k is contained in a nilpotent subgroup of K_k that contains G_k. We now use this to build the groups of degree n that we require.

Choose K_k and T_k as in the previous paragraph. Let $K = K_k \wr \mathrm{Sym}(n/k)$ acting on the same set that G is acting on. Note K admits a block system $\mathcal{B}$ consisting of blocks of size n/k formed by the orbits of the normal subgroup $1 \wr \mathrm{Sym}(n/k)$. By the Embedding Theorem [21, Theorem 1.2.6] we may ensure that the orbits of the copies of $\mathrm{Sym}(n/k)$ are the orbits of the semiregular subgroup of G of order n/k, so that G is a subgroup of K and therefore $G/\mathcal{B} \le K_k$. Clearly, K is a subgroup of S_n. Suppose that T is a regular abelian subgroup of K. Since K admits $\mathcal{B}$, these must also be blocks of the subgroup T. Then $T/\mathcal{B}$ is square-free and so cyclic, and since $T_k \le K_k$ again using the Embedding Theorem, we may choose such a T so that $T/\mathcal{B} = T_k$.

Suppose there exists $\delta \in K$ with $\delta^{-1} T \delta \le N$ where $G \le N$ and N is nilpotent. Since our goal is to show that $\langle G, \delta^{-1} T \delta \rangle$ is not nilpotent, and we have $G, T \le K$ and $\delta \in K$, we may further assume that $N \le K$. Therefore N also admits the block system $\mathcal{B}$.

As $\delta \in K = K_k \wr \mathrm{Sym}(n/k)$, we see $\delta/\mathcal{B} \in K_k$. We must have $(\delta/\mathcal{B})^{-1} T_k (\delta/\mathcal{B}) = (\delta^{-1} T \delta)/\mathcal{B}$ and $G_k = G/\mathcal{B}$, but then

$$\langle G_k, (\delta/\mathcal{B})^{-1} T_k (\delta/\mathcal{B}) \rangle = \langle G, \delta^{-1} T \delta \rangle / \mathcal{B} \le N/\mathcal{B} \le K/\mathcal{B} = K_k.$$

Since $N/\mathcal{B}$ is nilpotent, this contradicts our choice of K_k and T_k and consequently establishes the result. □

Remark 2.16. If, in Pálfy's Theorem (Theorem 1.1), we restrict the hypothesis to only cyclic groups of square-free order, then Theorem 2.15 generalizes this restricted form. This follows as the only nilpotent group of square-free degree is a cyclic group, and the only abelian group is also a cyclic group. So a nilpotent group of square-free degree mimics by conjugation every abelian group of square-free degree if and only if every two regular cyclic subgroups of G are conjugate in G.

3 Graph theoretic results

Lemma 3.1. *Let p be prime and $k_1, \dots, k_j$ be positive integers. There exist circulant digraphs $\Gamma_1, \dots, \Gamma_j$ such that* $\mathrm{Aut}(\Gamma_1 \wr \cdots \wr \Gamma_j) = \mathbb{Z}_{p^{k_1}} \wr \cdots \wr \mathbb{Z}_{p^{k_j}}$.

Proof. Let $\vec{D_k}$ denote the directed cycle of length k, which is a circulant digraph. When $k > 2$, $\vec{D_k}$ has automorphism group $\mathbb{Z}_k \ncong \mathrm{Sym}(k)$. When $k = 2$, $\mathrm{Aut}(K_2) = \mathrm{Aut}(\bar{K_2}) = \mathbb{Z}_2 = \mathrm{Sym}(2)$ and K_2 and its complement are also circulant (di)graphs.

For each $1 \le i \le j$, let

$$\Gamma_i = \begin{cases} \vec{D_{p^{k_i}}} \text{ if } p^{k_i} > 2; \\ K_2 \text{ if } p^{k_i} = 2, \text{ and either } i = 1 \text{ or } \Gamma_{i-1} \neq K_2; \text{ and} \\ \bar{K_2} \text{ otherwise.} \end{cases}$$

Let $\Gamma = \Gamma_1 \wr \cdots \wr \Gamma_j$. By [11, Theorem 5.7], $\mathrm{Aut}(\Gamma) = \mathbb{Z}_{p^{k_1}} \wr \cdots \wr \mathbb{Z}_{p^{k_j}}$. □

Definition 3.2. Let $G \le \mathrm{Sym}(X)$ be transitive, and $\mathcal{O}_1, \dots, \mathcal{O}_r$ the orbits of G in its natural action on $X \times X$. The **orbital digraphs** of G are the digraphs Γ_i whose vertices are the elements of X and arcs are $\mathcal{O}_r$, $1 \le i \le r$.

We now require Wielandt's notion of the 2-closure of a group.

Definition 3.3. Let $G \le \mathrm{Sym}(X)$ be transitive, and $\mathcal{O}_1, \dots, \mathcal{O}_r$ the orbits of G in its natural action on $X \times X$. The **2-closure of G**, denoted $G^{(2)}$, is the largest group whose orbits on $X \times X$ are $\mathcal{O}_1, \dots, \mathcal{O}_r$. We say that G is **2-closed** if $G = G^{(2)}$.

It is easy to verify that $G^{(2)}$ is a subgroup of $\mathrm{Sym}(X)$ containing G and, in fact, $G^{(2)}$ is the largest (with respect to inclusion) subgroup of $\mathrm{Sym}(X)$ that preserves every orbital digraph of G. Equivalently, $G^{(2)}$ is the automorphism group of the Cayley colour digraph obtained by assigning a unique colour to the edges of each orbital digraph of G. It follows that the automorphism group of a graph is 2-closed.

Corollary 3.4. *Let p be prime and G, H regular abelian groups of degree p^k with G cyclic. Then there exists $H' \le \langle G, H \rangle$ such that $H' \cong H$ is regular and $(\langle G, H' \rangle)^{(2)} \cong \mathbb{Z}_{p^{k_1}} \wr \cdots \wr \mathbb{Z}_{p^{k_m}}$ for some $k_1, \dots, k_m$ with $k_1 + \cdots + k_m = k$.*

Proof. Let Γ be the Cayley colour digraph formed by assigning a unique colour to the edges of each orbital digraph of $\langle G, H \rangle$. Since $\mathrm{Aut}(\Gamma)$ has regular subgroups isomorphic to G and to H, we see that Γ is a Cayley colour digraph on both of these groups, similar to condition (1) of Theorem 1.5.

Although the theorem is not stated for colour digraphs, most of the proof involves only permutation groups, so it is not hard to see that the same result is true for colour digraphs. By Theorem 1.5 (3), we can find $k'_1, \dots, k'_{m'}$ such that $\Gamma \cong U_1 \wr \cdots \wr U_{m'}$ where each U_i

is a circulant colour digraph on $\mathbb{Z}_{p^{k'_i}}$, and $\mathbb{Z}_{p^{k'_1}} \times \cdots \times \mathbb{Z}_{p^{k'_{m'}}} \preceq_p H$. It is a standard and straightforward observation that $(\langle G, H\rangle)^{(2)} = \mathrm{Aut}(\Gamma) \geq \mathrm{Aut}(U_1) \wr \cdots \wr \mathrm{Aut}(U_{m'}) \geq \mathbb{Z}_{p^{k'_1}} \wr \cdots \wr \mathbb{Z}_{p^{k'_{m'}}}$. The group $\mathbb{Z}_{p^{k'_1}} \wr \cdots \wr \mathbb{Z}_{p^{k'_{m'}}}$ certainly contains a regular cyclic subgroup which is conjugate to G. So there is a conjugate K of $\mathbb{Z}_{p^{k'_1}} \wr \cdots \wr \mathbb{Z}_{p^{k'_{m'}}}$ that contains G. By Lemma 3.1, K is 2-closed.

Since $\mathbb{Z}_{p^{k'_1}} \times \cdots \times \mathbb{Z}_{p^{k'_{m'}}} \preceq_p H$, the group K contains a regular subgroup $H' \cong H$. Since $G, H' \leq K$ and K is 2-closed, we have $(\langle G, H'\rangle)^{(2)} \leq K$. Repeating the argument to this point, with H' taking the role of H, we can find a Sylow p-subgroup K' of $(\langle G, H'\rangle)^{(2)}$ that contains both G and H' (since G and H' are already in a p-group together, we do not need to pass to a regular subgroup isomorphic to H' this time). Since $(\langle G, H'\rangle)^{(2)}$ is a p-group, we must in fact have $K' = (\langle G, H'\rangle)^{(2)}$. And as before there must exist $k_1, \ldots, k_m$ with $K' = \mathbb{Z}_{p^{k_1}} \wr \cdots \wr \mathbb{Z}_{p^{k_m}}$. □

Given regular groups G_1 and H_1 of degree a and G_2 and H_2 of degree b, there is an obvious method for constructing digraphs that are simultaneously Cayley digraphs of $G_1 \times G_2$ and of $H_1 \times H_2$. Namely, construct a Cayley digraph Γ_1 of order a that is a Cayley digraph of G_1 and H_1 and a digraph Γ_2 of order b that is a Cayley digraph of G_2 and H_2, and then consider some sort of "product construction" of Γ_1 and Γ_2 to produce a digraph Γ of order ab with $\mathrm{Aut}(\Gamma_1) \times \mathrm{Aut}(\Gamma_2) \leq \mathrm{Aut}(\Gamma)$. We write "product construction" as there are two obvious products of Γ_1 and Γ_2 that ensure that $\mathrm{Aut}(\Gamma_1) \times \mathrm{Aut}(\Gamma_2) \leq \mathrm{Aut}(\Gamma)$: the wreath product, and the Cartesian product.

Definition 3.5. Let $\Gamma_1, \ldots, \Gamma_r$ be digraphs. We say that Γ is of **product type** $\Gamma_1, \ldots, \Gamma_r$ if $\mathrm{Aut}(\Gamma_1) \times \cdots \times \mathrm{Aut}(\Gamma_r) \leq \mathrm{Aut}(\Gamma)$.

Theorem 3.6. *Let $n = p_1^{a_1} \cdots p_r^{a_r}$. Let G, H be regular abelian groups of degree n with G cyclic and let G_{p_i}, H_{p_i} be Sylow p_i-subgroups of G and H respectively, and Γ a Cayley digraph on G. If*

> *Γ is of product type $\Gamma_1, \ldots, \Gamma_r$, where each Γ_i is a Cayley digraph on both G_{p_i} and a group isomorphic to H_{p_i}, $1 \leq i \leq r$,*

then

> *Γ is isomorphic to a Cayley digraph of H.*

Furthermore, the converse holds whenever $G, H \leq N \leq \mathrm{Aut}(\Gamma)$ for some nilpotent group N, and we can also conclude that each $\mathrm{Aut}(\Gamma_i)$ is a (possibly trivial) multiwreath product of cyclic groups.

Proof. First suppose that Γ is of product type $\Gamma_1, \ldots, \Gamma_r$. Then $\mathrm{Aut}(\Gamma_1) \times \cdots \times \mathrm{Aut}(\Gamma_r) \leq \mathrm{Aut}(\Gamma)$. For each i, there is some $H'_{p_i} \cong H_{p_i}$ such that $H'_{p_i} \leq \mathrm{Aut}(\Gamma_i)$. Thus $H \cong H'_{p_1} \times \cdots \times H'_{p_r} \leq \mathrm{Aut}(\Gamma)$, so Γ is isomorphic to a Cayley digraph of H.

Conversely, suppose that Γ is a Cayley digraph on G that is also isomorphic to a Cayley digraph of the abelian group H. Then G and some $H' \cong H$ are regular subgroups of $\mathrm{Aut}(\Gamma)$. By assumption, $G, H' \leq N \leq \mathrm{Aut}(\Gamma)$ for some nilpotent group N. Let $N = P_1 \times \cdots \times P_r$ where P_i is a Sylow p_i-subgroup of N. Notice that for each i, $\langle G_{p_i}, H'_{p_i}\rangle \leq P_i$, so we may choose N so that $P_i = \langle G_{p_i}, H_{p_i}\rangle$ for each i. Furthermore, since the 2-closure of a p-group is a p-group [29, Exercise 5.28], and $N^{(2)} = (P_1)^{(2)} \times \cdots \times (P_r)^{(2)}$

[1, Theorem 5.1] we see that $N^{(2)}$ is a nilpotent group that contains G and H', and since $\mathrm{Aut}(\Gamma)$ is 2-closed we have $N^{(2)} \le \mathrm{Aut}(\Gamma)$. We can therefore choose $N = \langle G, H'\rangle^{(2)} = (P_1)^{(2)} \times \cdots \times (P_r)^{(2)}$.

Let B_i be one of the orbits of $P_i^{(2)}$. As an orbit of a normal subgroup, B_i is a block of N. By Corollary 3.4, there exists $(H''_{p_i})^{B_i} \le \langle G_{p_i}^{B_i}, (H'_{p_i})^{B_i}\rangle$ such that $(H''_{p_i})^{B_i} \cong (H'_{p_i})^{B_i}$ acts regularly, and the group $(\langle (G_{p_i})^{B_i}, (H''_{p_i})^{B_i}\rangle)^{(2)}$ is a multiwreath product of cyclic p_i-groups. By Lemma 3.1 there exists a vertex-transitive digraph Γ_i with $\mathrm{Aut}(\Gamma_i) = (\langle G_{p_i}^{B_i}, (H''_{p_i})^{B_i}\rangle)^{(2)}$. Now, $(\langle G_{p_i}^{B_i}, (H''_{p_i})^{B_i}\rangle)^{(2)} \le (P_i^{B_i})^{(2)} = P_i^{B_i}$ The first containment follows from the fact that $\langle G_{p_i}, (H''_{p_i})\rangle \le P_i^{B_i}$, while the second follows from [1, Theorem 5.1 (a)]. So $\mathrm{Aut}(\Gamma_i) \le P_i^{B_i}$. We claim that $\mathrm{Aut}(\Gamma_1)\times\cdots\times\mathrm{Aut}(\Gamma_r) \le N \le \mathrm{Aut}(\Gamma)$, so that Γ is of product type $\Gamma_1, \ldots, \Gamma_r$. This is because for any i, $N = P_i \times N'_i$, where $p_i \nmid |N'_i$, so we can define $\mathrm{Aut}(\Gamma_i)$ to act semiregularly by commuting with every element of N'_i, and in this form, $\mathrm{Aut}(\Gamma_i) \le P_i$ for each i.

To complete the proof, notice for each i that since $\mathrm{Aut}(\Gamma_i)$ contains regular subgroups isomorphic to $G_{p_i}^{B_i}$ (which is cyclic) and to $(H''_{p_i})^{B_i} \cong (H'_{p_i})^{B_i}$, the digraph Γ_i is a circulant digraph of order $p_i^{a_i}$ that is also a isomorphic to a Cayley digraph of H_{p_i}. □

Although the condition requiring a nilpotent group to achieve the converse in the above theorem may seem quite limiting, we observe that in Muzychuk's solution to the isomorphism problem for circulant digraphs [26], he shows that when $G \cong H$ are cyclic groups, the isomorphism problem for general n can be reduced to the prime power cases; this implies that G and a conjugate of H always lie in some nilpotent group together.

We point out in the coming corollary that for some values of n, the nilpotent group required to achieve the converse of the above theorem will always exist, even if G and H are not both cyclic.

Corollary 3.7. *Let $k = p_1 \cdots p_r$ be such that $\gcd(k, \varphi(k)) = 1$ where each p_i is prime, and $n = p_1^{a_1} \cdots p_r^{a_r}$. Let G, H be regular abelian groups of degree n with G cyclic and let G_{p_i}, H_{p_i} be Sylow p_i-subgroups of G and H respectively, and Γ a Cayley digraph on G. Then Γ is isomorphic to a Cayley digraph of H if and only if Γ is of product type $\Gamma_1, \ldots, \Gamma_r$, where each Γ_i is a Cayley digraph on both G_{p_i} and a group isomorphic to H_{p_i}, $1 \le i \le r$.*

Proof. The first implication is already proven in Theorem 3.6.

Conversely, suppose that Γ is a Cayley digraph on G that is also a Cayley digraph of the abelian group H. Then G and some $H' \cong H$ are regular subgroups of $\mathrm{Aut}(\Gamma)$. By Theorem 2.15, there exists a nilpotent subgroup $N \le \mathrm{Aut}(\Gamma)$ that contains G and a regular subgroup H' isomorphic to H. Replacing H by H' and applying Theorem 3.6 yields the result. □

Definition 3.8. Let G and H be abelian groups of order n, and for each prime $p_i|n$, denote a Sylow p_i-subgroup of G or H by G_{p_i} or H_{p_i}, respectively. Define a partial order $\preceq$ on the set of all abelian groups of order n by $G \preceq H$ if and only if $G_{p_i} \preceq_{p_i} H_{p_i}$ for every prime divisor $p_i|n$.

Remark 3.9. If, in Theorem 3.6, H is chosen to be minimal with respect to $\preceq$ subject to being contained in $\mathrm{Aut}(\Gamma)$, then the rank of each Sylow p_i-subgroup of H (i.e. the number of elements in any irredundant generating set) will be equal to the number of factors in the multiwreath product $\mathrm{Aut}(\Gamma_i)$. Moreover, $\mathrm{Aut}(\Gamma_1) \times \cdots \times \mathrm{Aut}(\Gamma_r)$ will contain a regular subgroup isomorphic to the abelian group R if and only if $\mathrm{Aut}(\Gamma)$ contains a regular subgroup isomorphic to the abelian group R.

Theorem 3.10. *Let $\Gamma = \mathrm{Cay}(G, S)$ for some abelian group G of order n. Let R be a regular cyclic subgroup of $\mathrm{Aut}(\Gamma)$. Suppose that there is a nilpotent group N with $G \leq N \leq \mathrm{Aut}(\Gamma)$, such that N mimics every regular abelian subgroup of $\mathrm{Aut}(\Gamma)$. Then the following are equivalent:*

1. *The digraph Γ is isomorphic to a Cayley digraph on both R and H, where H is a regular abelian group; furthermore, if H is chosen to be minimal with respect to the partial order $\preceq$ from amongst all regular abelian subgroups of $\mathrm{Aut}(\Gamma)$, then for each i, the Sylow p_i-subgroup of H has rank m_i.*
2. *Let P_i be a Sylow p_i-subgroup of G. There exist a chain of subgroups $P_{i,1} \leq \cdots \leq P_{i,m_i-1}$ in P_i such that*

   (a) *$P_{i,1}, P_{i,2}/P_{i,1}, \ldots, P_{i,m-1}/P_{i,m-2}, P_i/P_{i,m_i-1}$ are cyclic p_i-groups;*

   (b) *$P_{i,1} \times P_{i,2}/P_{i,1} \times \cdots \times, P_{i,m-1}/P_{i,m-2}, P_i/P_{i,m_i-1} \preceq_p H_i$, where H_i is a Sylow p_i-subgroup of H;*

   (c) *For all $s \in S \backslash (P_{i,j} \times G_i')$, we have $sP_{i,j} \subseteq S$, for $j = 1, \ldots, m_i - 1$, where G_i' is a Hall p_i'-subgroup of G (of order $n/p_i^{a_i}$). That is, $S \backslash (P_{i,j} \times G_i')$ is a union of cosets of $P_{i,j}$.*
3. *The digraph Γ is of product type $\Gamma_1, \ldots, \Gamma_r$, where each $\Gamma_i \cong U_{i,m_i} \wr \cdots \wr U_{i,1}$ for some Cayley digraphs $U_{i,1}, \ldots, U_{i,m_i}$ on cyclic p_i-groups $K_{i,1}, \ldots, K_{i,m_i}$ such that $K_{i,1} \times \cdots \times K_{i,m_i} \preceq_p H_i$, where H_i is a Sylow p_i-subgroup of H.*

Furthermore, any of these implies:

4. *Γ is isomorphic to Cayley digraphs on every abelian group of order n that is greater than H in the partial order $\preceq$.*

Proof. Throughout this proof, let $n = p_1^{a_1} \cdots p_r^{a_r}$, where the p_i are distinct primes.

(1)⇒(2): By hypothesis there is a transitive nilpotent subgroup N of $\mathrm{Aut}(\Gamma)$ that contains regular subgroups isomorphic to G, R, and H. By Theorem 3.6, there exist $\Gamma_1, \ldots, \Gamma_r$ where each Γ_i is a circulant digraph of order $p_i^{a_i}$, such that Γ is of product type $\Gamma_1, \ldots, \Gamma_r$, and each $\mathrm{Aut}(\Gamma_i)$ is a (possibly trivial) multiwreath product of cyclic groups. Additionally, by Remark 3.9, we may assume that R is a regular abelian subgroup of $\mathrm{Aut}(\Gamma)$ if and only if $\mathrm{Aut}(\Gamma_1) \times \cdots \times \mathrm{Aut}(\Gamma_r)$ contains a regular abelian subgroup isomorphic to R.

Let $\mathrm{Aut}(\Gamma_i) = \mathbb{Z}_{p_i^{b_{m_i,i}}} \wr \cdots \wr \mathbb{Z}_{p_i^{b_{1,i}}}$ (by Remark 3.9, this multiwreath product does have m_i factors). Then $\mathrm{Aut}(\Gamma_i)$ admits block systems $\mathcal{D}_{i,j}$, $1 \leq j \leq m_i$ consisting of blocks of size $x_{i,j} = \Pi_{\ell=1}^{j} p_i^{b_{\ell,i}}$. Observe that $\mathrm{Aut}(\Gamma_i)/\mathcal{D}_{i,j} = \mathbb{Z}_{p_i^{b_{m_i,i}}} \wr \cdots \wr \mathbb{Z}_{p_i^{b_{j+1,i}}}$ and $\mathrm{fix}_{\mathrm{Aut}(\Gamma_i)}(\mathcal{D}_{i,j})^D = \mathbb{Z}_{p_i^{b_{j,i}}} \wr \cdots \wr \mathbb{Z}_{p_i^{b_{1,i}}}$, $D \in \mathcal{D}_{i,j}$. Additionally, $\mathrm{fix}_{\mathrm{Aut}(\Gamma_i)}(\mathcal{D}_{i,j+1})/\mathcal{D}_{i,j}$ in its action on $D/\mathcal{D}_{i,j} \in \mathcal{D}_{i,j+1}/\mathcal{D}_{i,j}$ is cyclic of order $p_i^{b_{j+1,i}}$. Let P_i be a Sylow p_i-subgroup of G. Now, as G is a transitive abelian group, $P_{i,j} = \mathrm{fix}_G(\mathcal{D}_{i,j})$ is semiregular and transitive on $D \in \mathcal{D}_{i,j}$. As $\mathrm{fix}_{\mathrm{Aut}(\Gamma_i)}(\mathcal{D}_{i,j+1})/\mathcal{D}_{i,j}$ in its action on $D/\mathcal{D}_{i,j} \in$

$\mathcal{D}_{i,j+1}/\mathcal{D}_{i,j}$ is cyclic of order $p_i^{b_{j+1,i}}$, we see that $P_{i,j+1}/P_{i,j}$ is cyclic of prime-power order, and for the same reason,

$$P_{i,1} \times P_{i,2}/P_{i,1} \times \cdots \times P_{i,m-1}/P_{i,m-2} \times P_i/P_{i,m_i-1} \preceq_p H_i,$$

where H_i is a Sylow p_i-subgroup of H. For $1 \leq j \leq m_i - 1$, N admits a block system $\mathcal{C}_{i,j}$ consisting of blocks of size $x_{i,j} \cdot n/p_i^{a_i}$, as well as a block system $\mathcal{B}_{i,j}$ consisting of blocks of size $x_{i,j}$, and of course $\mathcal{B}_{i,j} \leq \mathcal{C}_{i,j}$. As Γ is of product type $\Gamma_1, \ldots, \Gamma_r$, and each Γ_i is a circulant graph, we see that $\mathrm{Aut}(\Gamma_i) \times \mathbb{Z}_{n/p_i^{a_i}} \leq \mathrm{Aut}(\Gamma)$; furthermore, $\mathrm{Aut}(\Gamma_i) \geq \mathbb{Z}_{p_i^{a_i}/x_{i,j}} \wr \mathbb{Z}_{x_{i,j}}$. Thus, $P_{i,j}|_C \leq \mathrm{Aut}(\Gamma)$ for every $C \in \mathcal{C}_{i,j}$. Thus we see that between blocks $C, C' \in \mathcal{C}_i$, we have either every directed edge from a block of $\mathcal{B}_{i,j}$ contained in C to a block of $\mathcal{B}_{i,j}$ contained in C' or no directed edges. As $\mathcal{C}_i$ is formed by the orbits of $P_{i,j} \times G_i'$ and $\mathcal{B}_{i,j}$ is formed by the orbits of $P_{i,j}$, (2) follows.

(2) $\Rightarrow$ (3) Let $\mathcal{B}_{i,j}$ be the block system of G formed by the orbits of $P_{i,j}$, $0 \leq j \leq m_i$, and $\mathcal{C}_{i,j}$ the block system of G formed by the orbits of $P_{i,j} \times G_i'$. As for all $s \in S \backslash (P_{i,j} \times G_i')$, we have $sP_{i,j} \subseteq S$, for $j = 1, \ldots, m_i - 1$, we have $\mathrm{fix}_G(\mathcal{B}_{i,j})|_{C_{i,j}} \leq \mathrm{Aut}(\Gamma)$ for every $C_{i,j} \in \mathcal{C}_{i,j}$. Note that $\mathrm{fix}_G(\mathcal{B}_{i,m_i}) = P_i$ and that $\langle \mathbb{Z}_{p_i^{a_i}}, \mathrm{fix}_G(\mathcal{B}_{i,j})|_{C_{i,j}} : C_{i,j} \in \mathcal{C}_{i,j} \rangle$ in its action on $B_{i,m_i} \in \mathcal{B}_{i,m_i}$ is $\mathbb{Z}_{p_i^{a_i - \ell_{i,j}}} \wr \mathbb{Z}_{p_i^{\ell_{i,j}}}$ where the orbits of $P_{i,j}$ have order $p_i^{\ell_{i,j}}$. Now let $n_{i,j+1} = \ell_{i,j+1} - \ell_{i,j}$, $0 \leq j \leq m_i - 1$. Then $Q_i = \langle \mathrm{fix}_G(\mathcal{B}_{i,j})|_{C_{i,j}} : C_{i,j} \in \mathcal{C}_{i,j}, 1 \leq j \leq m_i \rangle$ in its action on $B_{i,m_i} \in \mathcal{B}_{i,m_i}$ is $\mathbb{Z}_{p_i^{n_{i,m_i}}} \wr \mathbb{Z}_{p_i^{n_{i,m_i-1}}} \wr \cdots \wr \mathbb{Z}_{p_i^{n_{i,1}}}$, and $Q_i = (\mathbb{Z}_{p_i^{n_{i,m_i}}} \wr \mathbb{Z}_{p_i^{n_{i,m_i-1}}} \wr \cdots \wr \mathbb{Z}_{p_i^{n_{i,1}}}) \times 1_{\mathrm{Sym}(n/p_i^{a_i})}$. Let $K_{i,j} = \mathbb{Z}_{p_i^{n_{i,j}}}$ for each $1 \leq j \leq m_i$. Clearly, these are cyclic p_i-groups. Also, $P_{i,j+1}/P_{i,j} \cong K_{i,j+1}$ as abstract groups for $0 \leq j \leq m_i - 1$, so assumption (b) tells us that $K_{i,1} \times \cdots \times K_{i,m_i} \preceq_p H_i$. Notice that $Q_1 \times Q_2 \times \cdots \times Q_r \leq \mathrm{Aut}(\Gamma)$, and by Lemma 3.1 there exists a digraph Γ_i with $\mathrm{Aut}(\Gamma_i) = Q_i$. The graphs $U_{i,j}$ are given in the proof of Lemma 3.1. Thus Γ is of product type $\Gamma_1, \ldots, \Gamma_r$, completing this part of the proof.

(3) $\Rightarrow$ (1), (4) For each i, $\mathrm{Aut}(\Gamma_i)$ is a multiwreath product of m_i Cayley digraphs on cyclic groups. It is straightforward to verify (or is an immediate consequence of the Universal Embedding Theorem) that $\mathbb{Z}_a \wr \mathbb{Z}_b$ contains regular subgroups isomorphic to both $\mathbb{Z}_{ab}$ and $\mathbb{Z}_a \times \mathbb{Z}_b$.

By assumption, $\mathrm{Aut}(\Gamma_i) \geq \mathbb{Z}_{p_i^{n_{i,m_i}}} \wr \cdots \wr \mathbb{Z}_{p_i^{n_{i,1}}}$. Hence the above facts tell us that $H_i = \mathbb{Z}_{p_i^{n_{i,1}}} \times \cdots \times \mathbb{Z}_{p_i^{n m_i}} \leq \mathrm{Aut}(\Gamma_i)$, and also that $H_i' \leq \mathrm{Aut}(\Gamma_i)$ for every $H_i' \succ_p H_i$. Therefore $H = H_1 \times \cdots \times H_r \leq \mathrm{Aut}(\Gamma)$, and also $H' \leq \mathrm{Aut}(\Gamma)$ for every $H' \succ H$. $\square$

The following result is obtained from the previous result by applying Theorem 2.15.

Corollary 3.11. *Let $k = p_1 \cdots p_r$ be such that $\gcd(k, \varphi(k)) = 1$ where each p_i is prime, and $n = p_1^{a_1} \cdots p_r^{a_r}$. Let $\Gamma = \mathrm{Cay}(G, S)$ for some abelian group G of order n. Then the following are equivalent:*

1. *The digraph Γ is isomorphic to a Cayley digraph on both $\mathbb{Z}_n$ and H, where H is a regular abelian group; furthermore, if H is chosen to be minimal with respect to the partial order $\preceq$ from amongst all regular abelian subgroups of $\mathrm{Aut}(\Gamma)$, then for each i, the Sylow p_i-subgroup of H has rank m_i.*
2. *Let P_i be a Sylow p_i-subgroup of G. There exist a chain of subgroups $P_{i,1} \leq \cdots \leq P_{i,m_i-1}$ in P_i such that*

(a) $P_{i,1}, P_{i,2}/P_{i,1}, \ldots, P_{i,m-1}/P_{i,m-2}, P_i/P_{i,m_i-1}$ *are cyclic* p_i*-groups;*

(b) $P_{i,1} \times P_{i,2}/P_{i,1} \times \cdots \times P_{i,m-1}/P_{i,m-2} \times P_i/P_{i,m_i-1} \preceq_p H_i$*, where* H_i *is a Sylow* p_i*-subgroup of* H*;*

(c) *For all* $s \in S \backslash (P_{i,j} \times G_i')$*, we have* $sP_{i,j} \subseteq S$*, for* $j = 1, \ldots, m_i - 1$*, where* G_i' *is a Hall* p_i'*-subgroup of* G *(of order* $n/p_i^{a_i}$*). That is,* $S \backslash (P_{i,j} \times G_i')$ *is a union of cosets of* $P_{i,j}$*.*

3. *The digraph* Γ *is of product type* $\Gamma_1, \ldots, \Gamma_r$*, where each* $\Gamma_i \cong U_{i,m_i} \wr \cdots \wr U_{i,1}$ *for some Cayley digraphs* $U_{i,1}, \ldots, U_{i,m_i}$ *on cyclic* p_i*-groups* $K_{i,1}, \ldots, K_{i,m_i}$ *such that* $K_{i,1} \times \cdots \times K_{i,m_i} \preceq_p H_i$*, where* H_i *is a Sylow* p_i*-subgroup of* H*.*

Furthermore, any of these implies:

4. Γ *is isomorphic to Cayley digraphs on every abelian group of order* n *that is greater than* H *in the partial order* $\preceq$*.*

A point about the previous results should be emphasized. That is, it is necessary to introduce the digraphs $\Gamma_1, \ldots, \Gamma_r$ from Lemma 3.1 - one cannot simply define $\Gamma_i = \Gamma[B_i]$ (the induced subgraph on the points of B_i), where $B_i \in \mathcal{B}_i$ and $\mathcal{B}_i$ is a block system whose blocks are the orbits of some Sylow p-subgroup of the regular cyclic subgroup. Rephrased, it is possible for $\Gamma[B_i]$ to be a Cayley digraph of more than one group even when Γ is only isomorphic to a Cayley digraph of a cyclic group, and even when the condition $\gcd(k, \varphi(k)) = 1$ is met. We give an example of such a digraph in the following result.

Example 3.12. Let p and q be distinct primes such that $\gcd(pq, \varphi(pq)) = 1$, and $\Gamma = \mathrm{Cay}(\mathbb{Z}_{p^2} \times \mathbb{Z}_q, S)$, where $S = \{(kp, 0), (1, 1) : k \in \mathbb{Z}_p\}$. The only abelian group H of order p^2q for which Γ is isomorphic to a Cayley digraph of H is the cyclic group of order p^2q. Nonetheless, let $\mathcal{B}$ be the block system of the left regular representation of $\mathbb{Z}_{p^2} \times \mathbb{Z}_q$, that has blocks of size p^2. Then for every $B \in \mathcal{B}$, the induced subdigraph $\Gamma[B]$ is a Cayley digraph on $\mathbb{Z}_{p^2}$ and on $\mathbb{Z}_p^2$, and is isomorphic to the wreath product of two circulant digraphs of order p.

Proof. Towards a contradiction, suppose that Γ is isomorphic to a Cayley digraph of the abelian group H', where H' is not cyclic. Let G be the left regular representation of $\mathbb{Z}_{p^2} \times \mathbb{Z}_q$. Then $H' = \mathbb{Z}_p^2 \times \mathbb{Z}_q$, and by Theorem 2.15 there exists $H \cong H'$ such that $N = \langle G, H \rangle$ is nilpotent. Then N admits $\mathcal{B}$ as a block system as well as block systems $\mathcal{B}_p$ and $\mathcal{B}_q$ consisting of blocks of size p and blocks of size q, respectively. As $H \cong \mathbb{Z}_p^2 \times \mathbb{Z}_q$, $N/\mathcal{B}_q$ is a p-group that contains regular subgroups isomorphic to $\mathbb{Z}_p^2$ and $\mathbb{Z}_{p^2}$. By [12, Lemma 4] we have that $N/\mathcal{B}_q \cong \mathbb{Z}_p \wr \mathbb{Z}_p$.

For $0 \leq i \leq p^2 - 1$, let $B_{i,q} \in \mathcal{B}_q$ denote the block that consists of $\{(i, j) : j \in \mathbb{Z}_q\}$. Each vertex of $B_{0,q}$ is at the start of a unique directed path in Γ (that does not include digons) of length q (travelling by arcs that come from $(1, 1) \in S$), and each of these paths ends at a vertex of $B_{q,q}$. Thus any automorphism of Γ that fixes $B_{0,q}$ must also fix $B_{q,q}$, contradicting $K/\mathcal{B}_q \cong \mathbb{Z}_p \wr \mathbb{Z}_p$.

Finally, $\Gamma[B]$ is isomorphic to $\mathrm{Cay}(\mathbb{Z}_{p^2}, \{kp : k \in \mathbb{Z}_p^*\}) \cong \bar{K}_p \wr K_p$, so by Theorem 1.5, $\Gamma[B]$ is also a Cayley graph on $\mathbb{Z}_p^2$. □

The converse though is true. That is, if the condition $\gcd(k, \varphi(k)) = 1$ is met, and Γ is a Cayley digraph of two abelian groups G and H with nonisomorphic Sylow p-subgroups

G_p and H_p, respectively, and G is cyclic, then it must be the case that $\Gamma[B]$ is a wreath product, where $\mathcal{B}$ is formed by the orbits of G_p and $B \in \mathcal{B}$.

Corollary 3.13. *Let $k = p_1 \cdots p_r$ be such that $\gcd(k, \varphi(k)) = 1$ where each p_i is prime, and $n = p_1^{a_1} \cdots p_r^{a_r}$. Let Γ be a circulant graph of order n, and let $\mathcal{B}_i$ be the block system of the left regular representation of $\mathbb{Z}_n$ consisting of blocks of size $p_i^{a_i}$. If $\Gamma[B_i]$, $B_i \in \mathcal{B}_i$, is not a nontrivial wreath product and H is an abelian group of order n such that Γ is isomorphic to a Cayley digraph of H, then a Sylow p_i-subgroup of H is cyclic. Consequently, if $\Gamma[B_i]$, $B_i \in \mathcal{B}_i$, is not isomorphic to a nontrivial wreath product for any $1 \le i \le r$ then Γ is not isomorphic to a Cayley digraph of any noncyclic abelian group.*

Proof. By Theorem 2.15, there is a transitive nilpotent subgroup N of $\mathrm{Aut}(\Gamma)$ that contains the left regular representation of $\mathbb{Z}_n$ as well as a regular subgroup isomorphic to H. The system $\mathcal{B}_i$ is a block system of N also, since N is nilpotent. Thus $\mathcal{B}_i$ is a block system of H. Let H_i denote a Sylow p_i-subgroup of H, and G_i a Sylow p_i-subgroup of $\mathbb{Z}_n$. If H_i is not cyclic, then the restrictions of H_i and G_i to any $B_i \in \mathcal{B}_i$ are nonisomorphic regular p-groups, so by Theorem 1.5, $\Gamma[B_i]$ is a nontrivial wreath product. □

4 Future work

The work in this paper provides a "template" that one can use to approach the problem of when a digraph is a Cayley digraph of two nonisomorphic nilpotent groups, as follows.

Let R, R' be two regular nilpotent groups of order n. If there exists $\delta \in \langle R, R' \rangle$ such that $\langle R, \delta^{-1}R'\delta \rangle$ is nilpotent, then $\langle R, \delta^{-1}R'\delta \rangle^{(2)}$ is also nilpotent, and writing the group $\langle R, \delta^{-1}R'\delta \rangle^{(2)}$ as $\Pi_{i=1}^r P_i$, where $P_1, \ldots, P_r$ are all Sylow subgroups of $\langle R, \delta^{-1}R'\delta \rangle^{(2)}$, then each P_i is 2-closed. Furthermore, if R_{p_i} is the Sylow p_i-subgroup of R, R'_{p_i} is the Sylow p_i-subgroup of R', and P_i is the Sylow p_i-subgroup of $\langle R, \delta^{-1}R'\delta \rangle^{(2)}$, then $P_i = \langle R_{p_i}, \delta^{-1}R'_{p_i}\delta \rangle^{(2)}$. Thus, the subgraph induced on each orbit of P_i is a Cayley graph on the Sylow p_i-subgroups of both R and R'. So from a group theoretic point of view, this "reduces" the group theoretic characterization to the corresponding prime-power cases.

Of course, we would ideally like conditions on the connection set of a Cayley digraph of one group to be a Cayley digraph of another group, but at this time such conditions are only known in the prime-power case when one of the groups is cyclic. We also suspect that the conditions given in Theorem 3.10 and Corollary 3.11 only hold when one of the groups is a cyclic group, and different conditions will be needed for different choices of nilpotent or even abelian groups. So we have the following problem:

Problem 4.1. Given p-groups P and P', determine necessary and sufficient conditions on $S \subseteq P$ so that $\mathrm{Cay}(P, S)$ is also isomorphic to a Cayley digraph of P'.

Some attempt to study this has been made in [22]. The authors do not study the connection sets, but show that the situation is vastly different when neither regular subgroup in the automorphism group is cyclic, in the following sense. Theorem 1.5 above shows that when one of the regular subgroups is cyclic (say of order p^k), the automorphism group of the graph is a multiwreath product, so the regular subgroup has index at least $p^{(k-1)(p-1)}$ in its automorphism group. However, Theorem 1.1 of [22] shows that if $k \ge 3$ then given any non-cyclic abelian group of order p^k where p is odd, there is a Cayley digraph on that group whose automorphism group has order just p^{k+1} (so the regular subgroup has index p in this group) that contains a regular nonabelian subgroup also.

So how can one determine if a δ that conjugates R' to lie in a nilpotent group with R exists? We have seen in this proof that sometimes in order to prove that they lie together in a nilpotent group, it is sufficient to show that they lie together in a group that is normally $\Omega(n)$-step imprimitive. Naively following the structure of the proof in this paper, we have the following problem:

Problem 4.2. Determine for which regular nilpotent groups N and N' of order n there exists $\delta \in \langle N, N' \rangle$ such that $\langle N, \delta^{-1}N'\delta \rangle$ is (normally) $\Omega(n)$-step imprimitive.

This condition is certainly a necessary condition for a nilpotent subgroup of $\langle N, N' \rangle$ to contain N and a conjugate of N', but this condition is also sufficient under the arithmetic condition in Corollary 3.11 by [9, Corollary 15]. The solution of Problem 4.2 will likely depend, like the proof of Theorem 2.1 and consequently Theorem 2.15, on the Classification of the Finite Simple Groups. In particular, it seems likely we will need at least a list of primitive groups which contain a regular nilpotent subgroup, as well as a perhaps a list of such nilpotent subgroups. It is worthwhile to point out that Liebeck, Praeger, and Saxl have determined all primitive almost simple groups which contain a regular subgroup [19, Theorem 1.1].

Finally, we conjecture that the condition $\gcd(k, \varphi(k)) = 1$ in Corollary 3.11 is unnecessary:

Conjecture 4.3. *Theorem 3.11 holds for all* $n \in \mathbb{N}$.

Settling this conjecture may be quite difficult, as, at least with our approach, a positive solution would require generalizing Muzychuk's solution to the isomorphism problem for circulant color digraphs in [26], which is a difficult and significant result.

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The $T(5)$ property of packings of squares*

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Abstract

According to a classical theorem of Gruenbaum, if any five of a family of pairwise disjoint translates of a square has a transversal line (the family satisfies T(5)), then the whole family has a transversal line (satisfies T). First we show that this result is optimal in the sense that the "T(5) implies T" property does not necessarily hold anymore if only the slightly shrinked versions of the squares are pairwise disjoint. Next we prove the "T(5) implies T" property for a family of translates of squares if the interiors are pairwise disjoint and there exist two translates meeting at a common vertex.

Keywords: Transversals, parallelograms, Minkowski plane.

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1 Introduction

A family $\mathcal{F}$ of *ovals* (compact convex sets with non-empty interior) in the Euclidean plane has the property T if there is a line (*transversal*) that intersects every member of $\mathcal{F}$. If each k-element subfamily has a transversal then $\mathcal{F}$ has the property $T(k)$.

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The history of the study of the conditions under which “$T(k)$ implies T” is extensive, and we refer to Holmsen [9], Jeronimo-Castro, Roldan-Pensado [10] and Holmsen, Wenger [8] for reviews.

Our interest here is the case $k = 5$. The main result, due to Tverberg in [15], is that it is sufficient that the ovals are pairwise disjoint translates. For specific ovals, earlier verifications of this are due to Danzer [3] in the case of disks, and Grünbaum [5] in the case of parallelograms.

We note that in [5], Grünbaum conjectured Tverberg’s result and presented an example (see Example 1.1) that showed that disjointedness is possibly a necessary condition for translates of a square. With that example in mind, we show that it is not. We note that the problem is affine invariant; therefore, considering translates of a parallelogram or translates of a square are equivalent.

For any oval C and $k \geq 3$, Grünbaum [5] indicated the problem of determining the infimum $\mu(C, k)$ of $\mu > 0$ such that if the finite family $\{c_i + C\}$ satisfies $T(k)$ and the translates $\{c_i + \mu\, C\}$ do not overlap, then the family $\{c_i + C\}$ has a common transversal.

Here $c_i + \mu\, C$ and $c_j + \mu\, C$ do not overlap means that their interiors are disjoint. This property can be written in the form $\|x_i - x_j\|_{DC} \geq 2\mu$ in terms of the norm $\|\cdot\|_{DC}$ with respect to the difference body $DC = \frac{1}{2}(C - C)$ where for $p \in \mathbb{R}^2$, we have

$$\|p\|_{DC} = \min\{\lambda \geq 0 : \, p \in \lambda\, DC\}.$$

In particular, $C = DC$ if C is origin symmetric, $\|\cdot\|_C$ is the Euclidean norm if C is a unit disk, and $\|(x, y)\|_C = \max\{|x|, |y|\}$ if $C = [-1, 1] \times [-1, 1]$.

Concerning $\mu(C, 5)$, the main result of Tverberg [15] cited above proves that

$$\mu(C, 5) \leq 1 \text{ for any oval } C. \tag{1.1}$$

The main focus of this paper is families of translates of parallelograms. First we recall Grünbaum’s example at the end of [5] on page 469. We consider a family $\mathcal{F}$ of translated squares S_i of edge length 20 with center c_i and edges parallel to the coordinates axes for $i = 1, \ldots, 6$ satisfying $T(5)$ but not T. We assume that (x, y) is the Cartesian coordinate system in $\mathbb{R}^2$.

Example 1.1 (Grünbaum)**.** Let $c_1 = (-22, 4)$, $c_2 = (0, 15)$, $c_3 = (12, 11)$, $c_4 = (22, 4)$, $c_5 = (12, -11)$ and $c_6 = (0, -15)$.

It is easy to see that there exists a line transversal t_i of $\mathcal{F}\backslash\{S_i\}$ for $i = 1, \ldots, 6$. We note the unique choice for t_3 is the line with equation $y = -\frac{x}{2}$, which is the only transversal of $\{S_2, S_4, S_6\}$ with negative slope, and unique choice for t_5 is the line with equation $y = \frac{x}{2}$, which is the only transversal of $\{S_1, S_2, S_6\}$ with positive slope. Thus, $\mathcal{F}$ has no transversal.

In particular, Grünbaum’s Example 1.1 shows that $\mu(C, 5) \geq \frac{1}{2}$ if C is a parallelogram. Our first result improves on this bound.

Theorem 1.2. *If C is a parallelogram, then $\mu(C, 5) = 1$.*

It is a natural question whether in Grünbaum’s result in [5], the disjointedness of the compact parallelograms is necessary, or it is enough to assume that the interiors of the translates are pairwise disjoint; namely, the translated parallelograms do not overlap.

Conjecture 1.3. *If a family $\mathcal{F}$ of non-overlapping translates of a parallelogram satisfies $T(5)$, then $\mathcal{F}$ has a common transversal.*

Actually, we even conjecture the following stronger statement about translates of a square by imposing a lower bound on the distance between distinct centers in terms of the Euclidean distance.

Conjecture 1.4. *Let $\mathcal{F}$ be a family of $n \geq 6$ translates, of a square of side length s, with the property that the Euclidean distance between distinct centres is at least s. Then $T(5)$ implies that $\mathcal{F}$ has a transversal.*

We prove a weaker version of Conjecture 1.3.

Theorem 1.5. *If a family $\mathcal{F}$ of non-overlapping translates of a parallelogram satisfies $T(5)$, and there exist two parallelograms in $\mathcal{F}$ that intersect in a common vertex, then $\mathcal{F}$ has a common transversal.*

Returning to $\mu(C, 5)$ for any oval, we verify the following bounds.

Theorem 1.6. *For any oval C, we have $\frac{2}{3} \leq \mu(C, 5) \leq 1$.*

We note that the paper Bisztriczky, Böröczky, Heppes [2] verifies that $\mu(C, 5) = 2/3$ if C is an ellipsoid, and Theorem 1.2 proves that $\mu(C, 5) = 1$ if C is a parallelogram. Therefore the bounds in Theorem 1.6 are optimal.

We recall that according to Santaló [12], if a family of parallelograms with parallel sides satisfies $T(6)$, then the family has a common transversal. Therefore $\mu(C, 6) = 0$ if C is a parallelogram.

Concerning notation for Theorem 1.2 and Theorem 1.5, we write h and v to denote the horizontal and the vertical axis, respectively, for the coordinate system (x, y) in $\mathbb{R}^2$, and write $c_i = (x_i, y_i)$ to denote the centers of the translated squares in the family $\mathcal{F}$. For different points $p, q \in \mathbb{R}^2$, their line is denoted by $\operatorname{aff}\{p, q\}$. For a line $\ell = \{(x, y) : y = Ax + B\}$, we set $A = \operatorname{slope} \ell$ and write $\ell^+ = \{(x, y) : y > Ax + B\}$ and $\ell^- = \{(x, y) : y < Ax + B\}$ to denote the open halfplane of points above and below, respectively, ℓ.

2 Proof of Theorem 1.2

We may assume that C is the square $[-1, 1] \times [-1, 1]$. It follows from (1.1) that $\mu(C, 5) \leq 1$, therefore it is sufficient to prove the following statement:

For any $\varepsilon \in (0, \frac{1}{3})$, there exist $c_1, \dots, c_6 \in \mathbb{R}^2$ such that $c_1 + (1 - 2\varepsilon)C, \dots, c_6 + (1 - 2\varepsilon)C$ do not overlap, the family $\mathcal{F} = \{S_1, \dots, S_6\}$ satisfies $T(5)$ for $S_i = c_i + C$, $i = 1, \dots, 6$, and $\mathcal{F}$ has no common transversal.

We define (see Figure 1)

$$\begin{aligned}
c_1 &= (-2, 1 - \varepsilon)\\
c_2 &= (0, 1 + \varepsilon)\\
c_3 &= (2 - \varepsilon, 1)\\
c_4 &= (4 - 4\varepsilon, 1 - 3\varepsilon)\\
c_5 &= (2 - \varepsilon, -1)\\
c_6 &= (0, -1 - \varepsilon).
\end{aligned}$$

We also consider some vertices of the S_is:

$$\begin{array}{rcl} a & = & (-1,\varepsilon) = c_2 + (-1,-1) \in S_1 \cap S_2 \\ b & = & (-1,-\varepsilon) = c_1 + (1,-1) = c_6 + (-1,1) \in S_1 \cap S_6 \\ u & = & (1-\varepsilon,0) = c_3 + (-1,-1) = c_5 + (-1,1) \in S_3 \cap S_5 \\ z & = & (1,\varepsilon) = c_2 + (1,-1) \in S_2 \cap S_3 \\ w & = & (1,-\varepsilon) = c_6 + (1,1) \in S_5 \cap S_6. \end{array}$$

For $i = 1, \dots, 6$, we write

$$\begin{array}{rcl} t_1 & = & \operatorname{aff}\{u,z\} = \{(x,y) : y = x - 1 + \varepsilon\}, \\ t_2 & = & \operatorname{aff}\{u,b\} = \left\{(x,y) : y = \dfrac{\varepsilon}{2-\varepsilon}\,(x-1+\varepsilon)\right\}, \\ t_3 & = & \operatorname{aff}\{a,w\} = \{(x,y) : y = -\varepsilon\, x\}, \\ t_4 & = & \operatorname{aff}\{u,w\} = \{(x,y) : y = -x + 1 - \varepsilon\}, \\ t_5 & = & \operatorname{aff}\{b,z\} = \{(x,y) : y = \varepsilon\, x\}, \\ t_6 & = & \operatorname{aff}\{u,a\} = \left\{(x,y) : y = \dfrac{-\varepsilon}{2-\varepsilon}\,(x-1+\varepsilon)\right\}. \end{array}$$

We claim that t_i, $i = 1, \dots, 6$, is a line transversal to $\mathcal{F}\backslash\{S_i\}$. We note that

$$c_4 + (-1,1) \in t_1 \cap S_4 \text{ and } c_4 + (-1,-1) \in t_3^- \cap S_4$$

for any $\varepsilon > 0$, and $\varepsilon < \frac{1}{2}$ yields that

$$c_4 + (-1,-1) \in t_6^- \cap S_4,$$

and hence $t_i \cap S_j \neq \emptyset$ for $i \neq j$ easily follows.

Finally, we observe that $S_2 \cap S_6 = \emptyset$ and both t_3 and t_5 are separating and supporting lines of S_2 and S_6. Thus if ℓ is a transversal of $\{S_2, S_6\}$, then

- either ℓ is parallel to v,
- or $\operatorname{slope} \ell \leq \operatorname{slope} t_3 = -\varepsilon$,
- or $\operatorname{slope} \ell \geq \operatorname{slope} t_5 = \varepsilon$.

In particular, if ℓ is parallel to v, then ℓ is disjoint from either S_1 or S_4.

Let $\operatorname{slope} \ell \leq \operatorname{slope} t_3 = -\varepsilon$, and we distinguish two cases. If $\operatorname{slope} \ell > \operatorname{slope} t_4 = -1$ and ℓ intersects S_6, then ℓ is disjoint from S_3. If $\operatorname{slope} \ell \leq \operatorname{slope} t_4 = -1$ and ℓ intersects S_6, then ℓ is disjoint from S_4.

Finally, let $\operatorname{slope} \ell \geq \operatorname{slope} t_5 = \varepsilon$, and we distinguish three cases. If $\operatorname{slope} \ell < \operatorname{slope} t_1 = 1$ and ℓ intersects S_2, then ℓ is disjoint from S_5. If $\operatorname{slope} \ell > \operatorname{slope} t_1 = 1$ and ℓ intersects S_2, then ℓ is disjoint from S_6. If $\operatorname{slope} \ell = \operatorname{slope} t_1 = 1$ and ℓ intersects S_2 and S_5, then $\ell = t_1$, and hence ℓ is disjoint from S_1.

Therefore, $\mathcal{F}$ has no transversal, proving $\mu(C,5) = 1$. $\square$

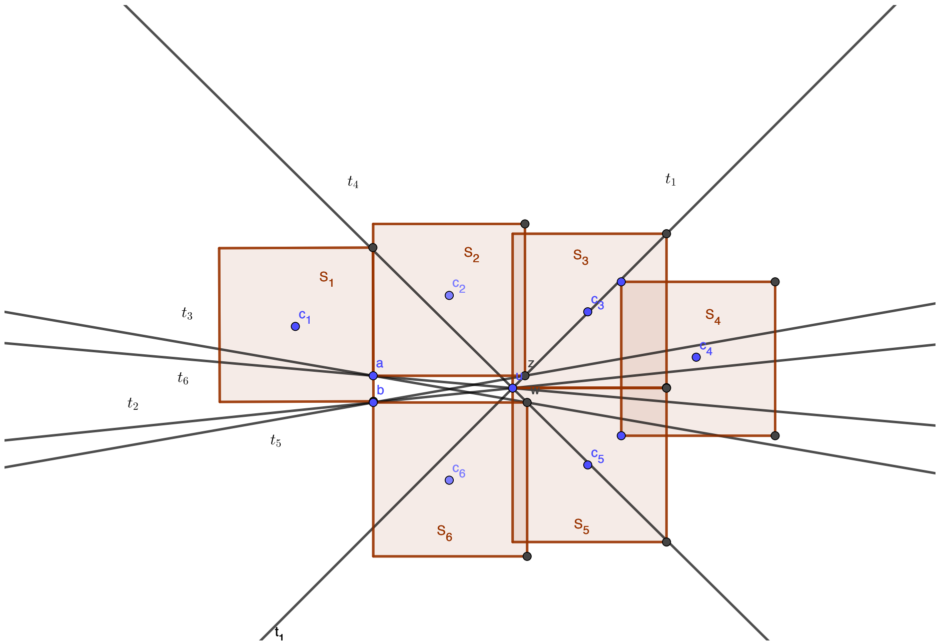

Figure 1: The transversals t_i of $\{S_1, \dots, S_6\}\backslash\{S_i\}$

3 Proof of Theorem 1.5

Let $\mathcal{F} = \{S_1, \dots, S_n\}$, $n \geq 6$, be a packing of n translates of the square $[-1,1] \times [-1,1]$ such that $\mathcal{F}$ satisfies $T(5)$ and two translates intersect in a common vertex. In addition, let $c_i = (x_i, y_i)$ be the center of S_i, $i = 1, \dots, n$. We may assume that $c_1 = (-1, 1)$ and $c_2 = (1, -1)$.

We say that S_j and S_k are *split* if $|x_j - x_k| \geq 2$, $|y_j - y_k| \geq 2$ and $|x_j - x_k| + |y_j - y_k| > 4$. It is well known (see Grünbaum [5]) that if $\mathcal{F}$ satisfies $T(5)$ and contains a split pair of squares, then $\mathcal{F}$ has a transversal. Accordingly, we assume that

$$\mathcal{F} \text{ contains no split pair.} \tag{3.1}$$

Case 1 $S_k \cap (h \cup v) = \emptyset$ for some $k \in \{3, \dots, n\}$

We may assume that $S_3 \cap (h \cup v) = \emptyset$ and $x_3, y_3 > 0$, and hence $x_3, y_3 > 1$. Since S_3 is disjoint and is not split from S_i, $i = 1, 2$, we deduce that

$$1 < x_3 < 3 \;\text{ and }\; 1 < y_3 < 3. \tag{3.2}$$

We claim that

$$S_m \cap (h \cup v) \neq \emptyset \;\text{ for } m \in \{4, \dots, n\}. \tag{3.3}$$

We suppose that $S_m \cap (h \cup v) = \emptyset$ for an $m \in \{4, \dots, n\}$, and hence $|x_m|, |y_m| > 1$, and seek a contradiction. If $x_m > 0$ and $y_m > 0$, then as S_3 and S_m do not overlap, (3.2) yields that either $x_m \geq x_3 + 2 > 3$ or $y_m \geq y_3 + 2 > 3$, thus S_m is split for either S_2 or S_1, respectively. If $x_m < 0$ and $y_m < 0$, then S_m is split from S_3, and if $x_m > 0$ and

$y_m < 0$, then S_m is split from S_1; futhermore, if $x_m < 0$ and $y_m > 0$, then S_m is split from S_2. In turn, we conclude (3.3). It follows from (3.3) that possibly after interchanging h and v, and a reflection to keep S_3 in the first quadrant, we may assume that $S_4 \cap v \neq \emptyset$ and $S_5 \cap v \neq \emptyset$.

If v is a transversal of $\mathcal{F}$, then Theorem 1.5 has been proved. Therefore we may assume that $S_6 \cap v = \emptyset$, and hence $S_6 \cap h \neq \emptyset$ by (3.3).

As $S_6 \cap v = \emptyset$ and $S_6 \cap h \neq \emptyset$, we have $|x_6| > 1$ and $|y_6| \leq 1$. Since S_6 does not overlap S_1, and is not split from either one of S_2 and S_3, we deduce that

$$\text{if } x_6 < 0, \text{ then } x_6 \leq -3 \text{ and } -1 < y_6 < 1. \tag{3.4}$$

On the other hand, since S_6 does not overlap S_2 and S_3, and is not split from S_1, we deduce that

$$\text{if } x_6 > 0, \text{ then } x_6 \geq 3 \text{ and } -1 < y_6 \leq 1. \tag{3.5}$$

Turning to S_4 and S_5, we may assume that $y_4 \leq y_5$, and if $y_4 = y_5$, then $x_4 = -1$ and $x_5 = 1$.

Case 1.1 $x_4 < 0$

Since S_4 does not overlap S_1 and S_2, and is not split from S_3, we have $y_4 \leq -3$. Thus $-1 \leq x_4 \leq 1$, (3.4) and (3.5) yield that S_4 and S_6 are split, contradicting (3.1).

Case 1.2 $x_4 > 0$

Since S_4 does not overlap S_1, S_2 and S_3, we have

$$y_4 \geq 3 \text{ and } -1 \leq x_4 \leq 1, \quad \text{and if } y_4 = 3, \text{ then even } x_4 \leq x_3 - 2 < 1. \tag{3.6}$$

Comparing (3.6) to (3.4) and (3.5) shows that S_4 and S_6 are split, contradicting again (3.1).

Case 2 $S_k \cap (h \cup v) \neq \emptyset$ for any $k \in \{3, \ldots, n\}$

We may assume that neither h nor v is a transversal of $\mathcal{F}$, thus we may assume that $|y_3| > 1$ and $|x_4| > 1$. In addition, we may assume that S_3 is farthest from h, S_4 is farthest from v, and S_3 is closer to h than S_4 to v; or in other words, $1 < y_3 \leq |x_4|$, $y_3 \geq |y_i|$ for $i \geq 3$ and $|x_4| \geq |x_i|$ for $i \geq 3$. It follows from $S_3 \cap v \neq \emptyset$ and $S_4 \cap h \neq \emptyset$ that

$$-1 \leq x_3 \leq 1 \text{ and } -1 \leq y_4 \leq 1.$$

Since S_3 and S_4 are not split, we have $y_3 \leq 3$ and if $y_3 = 3$, then either $c_3 = (1, 3)$ and $c_4 = (3, 1)$, or $c_3 = (-1, 3)$ and $c_4 = (-3, 1)$. However, if $c_3 = (-1, 3)$, then S_2 and S_3 are split, thus if $y_3 = 3$, then $c_3 = (1, 3)$ and $c_4 = (3, 1)$.

Case 2.1 $y_3 = 3$, and hence $c_3 = (1, 3)$ and $c_4 = (3, 1)$

In this case, the only common transversals of $\{S_1, S_2, S_3, S_4\}$ are $\ell_1 = \{(x, y) : y = x\}$ and $\ell_2 = \{(x, y) : y = 1 - x\}$. Let us assume that ℓ_2 is not a transversal of $\mathcal{F}$, thus we may assume that $S_5 \cap \ell_2 = \emptyset$ and ℓ_1 is a common transversal of $\{S_1, S_2, S_3, S_4, S_5\}$. In addition, we may assume that $|x_5| \leq |y_5|$.

As $S_5 \cap (h \cup v) \neq \emptyset$, $S_5 \cap \ell_1 \neq \emptyset$ and $S_5 \cap \ell_2 = \emptyset$, we deduce that $S_5 \subset l_2^-$. Therefore combining the conditions $|x_5| \leq |y_5|$, $S_5 \cap (h \cup v) \neq \emptyset$, $S_5 \cap \ell_1 \neq \emptyset$ and S_5 does not overlap S_1 and S_2 implies that $x_5 = -1$ and $-3 \leq y_5 \leq -1$. In particular, S_3 and S_5 are split, contradicting (3.1).

Case 2.2 $y_3 < 3$
Since S_3 does not overlap S_1 and S_2, we have $x_3 = 1$ and

$$x_3 = 1 \text{ and } 1 < y_3 < 3. \tag{3.7}$$

We claim that

$$y_3 - 2 \le y_i \le 1 \text{ and } |x_i| \ge 3 \text{ for } i = 4, \dots, n. \tag{3.8}$$

We suppose that there exists $j \in \{4, \dots, n\}$ with $y_j < y_3 - 2$, and seek a contradiction. As $1 > y_j \ge -|y_3| > -3$ and S_j does not overlap with S_1, S_2, S_3, we deduce that $|x_j| \ge 3$. Therefore $y_j < y_3 - 2$ and (3.7) imply that S_j and S_3 are split, contradicting (3.1), and verifying that $y_i \ge y_3 - 2$ for $i = 4, \dots, n$. For any $i = 4, \dots, n$, we have $-1 < y_3 - 2 \le y_i \le y_3 < 3$, S_i does not overlap S_1, S_2, S_3 and $S_i \cap (h \cup v) \ne \emptyset$, therefore $|x_i| \ge 3$ and $y_i \le 1$, as in (3.8).

We set $\ell_1 = \mathrm{aff}\{(0, y_3 - 1), (2, 0)\}$ and $\ell_2 = \mathrm{aff}\{(2, y_3 - 1), (0, 0)\}$, and note that they are separating and supporting lines of S_2 and S_3 with slope $\ell_1 < 0$ and slope $\ell_2 > 0$. We may assume that ℓ_1 is not a transversal of $\mathcal{F}$, and hence there exists $m \in \{4, \dots, n\}$ such that $\ell_1 \cap S_m = \emptyset$. In particular, either $x_m \ge 3$ and $S_m \subset \ell_1^+$, or $x_m \le -3$ and $S_m \subset \ell_1^-$.

We observe that if ℓ is a transversal of S_2 and S_3 with slope $\ell < 0$, then

$$\{(x, y) \in \ell_1^+ : x \ge 2\} \subset \ell^+ \text{ and } \{(x, y) \in \ell_1^- : x \le -2\} \subset \ell^-. \tag{3.9}$$

We claim that

$$S_i \cap \ell_2 \ne \emptyset \text{ for } i = 1, \dots, n. \tag{3.10}$$

Let ℓ be a transversal of S_1, S_2, S_3, S_m, S_i, and hence (3.9) yields that slope $\ell > 0$. Since ℓ is a transversal of S_1 and S_2, it contains the origin $(0, 0)$. As $S_i \cap h \ne \emptyset$ and ℓ_2 has minimal slope among transversals of S_2 and S_3, we deduce that $S_i \cap \ell_2 \ne \emptyset$. In turn, we conclude from (3.10) that ℓ_2 is a transversal of $\mathcal{F}$. □

4 Proof of Theorem 1.6

For references about Minkowski Geometry and properties of ovals in this section, see Schneider [13] and Thompson [14]. For an oval C, we say that a polygon P is circumscribed around C (inscribed into C) if each side of P touches P (each vertex of P lies on the boundary ∂P of P), respectively. We say that a polygon P is an affine regular hexagon if it is the image of a regular hexagon by a linear transformation. The proof of Theorem 1.6 rests on the following statement.

Proposition 4.1. *If C is an origin symmetric oval that is not a parallelogram, then there exists an affine regular hexagon H circumscribed around C such that no vertex of H lies in C.*

Since the proof of Proposition 4.1 is rather technical and uses ideas very different from the ones used in the rest of the paper, we present the argument in the Appendix (Section A).

The following observation due to Tverberg in [15] shows that it is sufficient to consider origin symmetric ovals in our study.

Lemma 4.2 (Tverberg)**.** *For any oval C and $x_1, \dots, x_k \in \mathbb{R}^2$, $x_1 + C, \dots, x_k + C$ has a transversal if, and only if, $x_1 + \frac{1}{2}(C - C), \dots, x_k + \frac{1}{2}(C - C)$ has a parallel transversal.*

Proof. We fix a line ℓ passing through the origin, and search for transversals parallel to ℓ. Let u be a unit vector orthogonal to ℓ, and let $b > a$ be defined by the property that $\ell + tu$ intersects C if, and only if, $a \leq t \leq b$, and hence $\ell + tu$ intersects $x + \frac{1}{2}(C - C)$ if, and only if, $\frac{a-b}{2} \leq t \leq \frac{b-a}{2}$. We write $u \cdot v$ to denote the scalar product of the vectors u and v. For an $x \in \mathbb{R}^2$ and $t, s \in \mathbb{R}$, it follows that $\ell + tu$ intersects $x + C$ if, and only if, $a + x \cdot u \leq t \leq b + x \cdot u$; moreover, $\ell + su$ intersects $x + \frac{1}{2}(C - C)$ if, and only if, $\frac{a-b}{2} + x \cdot u \leq s \leq \frac{b-a}{2} + x \cdot u$, which is in turn equivalent to saying that $\ell + (s + \frac{a+b}{2})u$ intersects $x + C$. We conclude that a line $\ell + su$ parallel to ℓ is a transversal of $x_1 + \frac{1}{2}(C - C), \dots, x_k + \frac{1}{2}(C - C)$ if, and only if, $\ell + (s + \frac{a+b}{2})u$ is a transversal of $x_1 + C, \dots, x_k + C$. □

Proof of Theorem 1.6: It follows from Tverberg [15] (see also (1.1)) that $\mu(C, 5) \leq 1$ for any oval C.

Let us turn to the proof of $\mu(C, 5) \geq \frac{2}{3}$ for any oval C. Since $\frac{1}{2}(C - C)$ is a parallelogram if, and only if, C is a parallelogram, we may assume that C is origin symmetric according to Lemma 4.2.

If the origin symmetric oval C is a parallelogram, then Theorem 1.2 verifies $\mu(C, 5) = 1$. Therefore we assume that C is an origin symmetric oval that is not a parallelogram, and hence Proposition 4.1 yields a circumscribed (origin symmetric) affine regular hexagon H such that no vertex of H is contained in ∂C.

Let $H_0 = \frac{2}{3}H$, and let $H_1, \dots, H_6$ be the six non-overlapping translates of H_0 in a way such that $H_0 \cap H_i$ is a common side for $i = 1, \dots, 6$, and $H_1, \dots, H_6$ are situated around H_0 in counterclockwise order. We write c_i to denote the center of H_i, and hence $c_1 + \frac{2}{3}\,C, \dots, c_6 + \frac{2}{3}\,C$ do not overlap.

Let us consider the family $\mathcal{F} = \{c_1 + C, \dots, c_6 + C\}$. We observe that $c_1 + H, c_3 + H, c_5 + H$ enclose a triangle T_{135}. For $i = 1, 3, 5$, T_{135} has a common side with $c_i + H$ which touches $c_i + C$, and let ℓ_i be the line containing this side. We observe that ℓ_i, $i = 1, 3, 5$, touches $c_1 + C, c_3 + C, c_5 + C$, it is a common transversal to $\mathcal{F} \backslash \{c_j + C\}$ where $j \in \{1, \dots, 6\}$ and $|j - i| = 3$, and $\ell_i \cap (c_j + C) = \emptyset$.

Similarly, $c_2 + H, c_4 + H, c_6 + H$ enclose a triangle T_{246}. For $i = 2, 4, 6$, T_{246} has a common side with $c_i + H$ which touches $c_i + C$, and let ℓ_i be the line containing this side. We observe that ℓ_i, $i = 2, 4, 6$, touches $c_2 + C, c_4 + C, c_6 + C$, it is a common transversal to $\mathcal{F} \backslash \{c_j + C\}$ where $j \in \{1, \dots, 6\}$ and $|j - i| = 3$, and $\ell_i \cap (c_j + C) = \emptyset$.

So far we have verified that $c_1 + \frac{2}{3}\,C, \dots, c_6 + \frac{2}{3}\,C$ do not overlap, $\mathcal{F}$ satisfies $T(5)$, and the fact that $\mathcal{F}$ has no transversal provided for any transversal ℓ of $c_1 + C, c_3 + C, c_5 + C$, we have

$$\ell \in \{\ell_1, \ell_3, \ell_5\}. \tag{4.1}$$

Since each ℓ_i, $i = 1, 3, 5$, separates two of $c_1 + C, c_3 + C, c_5 + C$, we may assume that ℓ is not parallel to ℓ_1, ℓ_3, ℓ_5. In this case, there exists a vertex v of T_{135} and a line ℓ' parallel to ℓ such that ℓ' passes through v and intersects $\operatorname{int} T_{135}$. We may assume that $\{v\} = (c_1 + H) \cap (c_3 + H)$. As ℓ' strictly separates $(c_1 + H) \backslash \{v\}$ and $(c_3 + H) \backslash \{v\}$ and $v \notin (c_i + C)$ for $i = 1, 3$, we deduce that ℓ' strictly separates $c_1 + C$ and $c_3 + C$. This contradicts that ℓ intersects both $c_1 + C$ and $c_3 + C$, and proves (4.1). In turn, we conclude Theorem 1.6. □

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A Appendix - proof of Proposition 4.1

We prove in fact Proposition A.1 (the equivalent form of Proposition 4.1 via polarity) through a series of simple statements Lemma A.2, Lemma A.3 and Lemma A.4.

If C is an oval with $o \in \operatorname{int} C$, then its polar is the oval

$$C^* = \{p \in \mathbb{R}^2 : \langle p, q \rangle \leq 1 \; \forall q \in C\}.$$

We note that $(C^*)^* = C^*$, and assuming that $C \subset K$ for an oval K, we have $K^* \subset C^*$. If C is a polygon, then so is C^*, and there exists a bijective correspondence between the vertices of P and the sides of P^*; namely, if v is a vertex of P, then $\{p \in C^* : \langle p, v\rangle = 1\}$ is the corresponding side of C^*. Since if A is a linear transformation and C is any oval, then $(AC)^* = A^{-t}C^*$ where A^{-t} is the inverse of the transpose of A, we have that P^* is an affine regular hexagon for any affine regular hexagon P centered at the origin, and P^* is a parallelogram for any parallelogram P centred at the origin.

Polarity shows that Proposition 4.1 is equivalent to Proposition A.1.

Proposition A.1. *If C is an origin symmetric oval that is not a parallelogram, then there exists an affine regular hexagon H inscribed into C such that no side of H lies in ∂C.*

Any origin symmetric oval C induces a Minkowski geometry where the length of a segment $[p, q]$ with endpoints p and q is $\|p - q\|_C$. For a polygon P, its correponding Minkowski perimeter $M_C(P)$ is the sum of the lengths of its sides with respect to $\|\cdot\|_C$. This notion of Minkowski perimeter can be extended to any oval K by approximation where $M_C(K_1) \leq M_C(K_2)$ holds for ovals K_1 and K_2 satisfying $K_1 \subset K_2$. The following statement is well known, see Lemma 4.1.1 in Thompson [14] or Martini, Swanepoel, Weiss [11], or Asplund and Grünbaum [1] for related results.

Lemma A.2. *If C is an origin symmetric oval, then for any $p \in \partial C$, there exists an origin symmetric affine regular hexagon H inscribed into C such that p is a vertex of H.*

Actually Lemma 4.1.1 in Thompson [14] states that there exists a $q \in \partial C$ in Lemma A.2 such that $q - p \in \partial C$, and therefore $\pm p, \pm q, \pm(q - p)$ are vertices of an inscribed affine regular hexagon. We observe that if H is an origin symmetric affine regular hexagon inscribed into an origin symmetric oval C, then each side of H is of length 1 with respect to both $\|\cdot\|_H$ and $\|\cdot\|_C$. The self-perimeter of any origin symmetric oval is between 6 and 8 according to Golab [4]. For the sake of the reader, we present the simple argument.

Lemma A.3 (Golab)**.** *If C is an origin symmetric oval, then $6 \leq M_C(C) \leq 8$.*

Remark We have $M_C(C) = 6$ if C is an affine regular hexagon, and $M_C(C) = 8$ if C is a parallelogram.

Proof. Let H be an affine regular hexagon inscribed into C, and let P be a parallelogram of minimal area containing C. Since the midpoints of P lie in ∂C, we have

$$6 = M_C(H) \leq M_C(C) \leq M_C(P) = 8.$$

□

We note that Golab [4] defined a notion of self perimeter for any (not necessarily centrally symmetric) oval. For this generalized notion of self perimeter, Grünbaum [6] verified that it is at least 6 (with equality for affine regular hexagons) and at most 9 (with equality for triangles) for any oval.

Lemma A.4. *If C is an origin symmetric oval that is not a parallellogram, then then there exists a $p \in \partial C$ not lying on any segment contained in ∂C of length longer than 1 with respect to $\|\cdot\|_C$.*

Proof. We suppose that ∂C is the union of segments of length longer than 1 with respect to $\|\cdot\|_C$, and seek a contradiction. Since C is origin symmetric, we deduce from Lemma A.3 that C is a hexagon. Let p_1, p_2, p_3 be vertices of C such that p_2 and p_3 are neighbors of p_1. Let P be the parallelogram such that $\pm p_1$ are opposite vertices and p_2 and p_3 lie on sides of P emanating from p_1. We may assume that P coincides with $[-1,1]\times[-1,1]$ in a way such that $p_1 = (1,1)$, $p_2 = (1-t,1)$ and $p_3 = (1, 1-s)$ where $0 < s,t < 2$. We may also assume that $s \leq t$.

We claim that

$$s > 1. \tag{A.1}$$

We suppose that $s \leq 1$, and seek a contradiction. Since $\|p_3 - p_1\|_C > 1$, it follows that the point $q = (0, -s)$ lies outside of C; therefore, there exists a line ℓ disjoint from C passing through q. Since $(1, 1-s) \in C$ and $(-1,-1) \in C$, we deduce that $0 < \operatorname{slope} \ell < 1$, and hence there exits $w = (-1+r, -1) \in \ell$ with $0 < r \leq s$. However $-p_2 = (-1+t, -1)$ lies on ∂C with $t \geq s$, thus $w \in [-p_1, -p_2] \subset \partial C$. This fact contradicts $\ell \cap C = \emptyset$, and in turn proves (A.1).

We deduce from $t \geq s > 1$ that $(1,0), (0,1) \in \partial C$, and in turn $p_3 - (-p_2) = (2-t, 2-s) \in \operatorname{int} C$, and hence the length of the side $[-p_2, p_3]$ of C is $\|p_3 - (-p_2)\|_C < 1$. This contradicts the conditions on C, and completes the proof of Lemma A.4. $\square$

Proof of Proposition 4.1 In fact, we prove the equivalent Proposition A.1. Let C be an origin symmetric oval that is not a parallelogram. It follows from Lemma A.3, that ∂C contains at most 8 maximal segments of length at least 1 with respect to $\|\cdot\|_C$.

According to Lemma A.4, there exists a $p \in \partial C$ not lying on any segment contained in ∂C of length longer than 1 with respect to $\|\cdot\|_C$. Possibly varying p, we may also assume that

(i) if p is contained in a segment s with $s \subset \partial C$ (thus the length of s is at most one), then p lies in the relative interior of s,

(ii) the line op is not parallel to any segment contained in ∂C of length at least 1 with respect to $\|\cdot\|_C$.

Let H be an affine regular hexagon inscribed into C such that p is a vertex of H. It follows that each side of H is of length 1 with respect to $\|\cdot\|_C$. The two sides of H parallel to p are not contained in ∂C by (ii). If a side s_0 of H containing p or $-p$ is part of ∂C then s_0 is a proper subset of the segment s of length at most 1 by (i); and that is a *reductio ad absurdum*. This completes the proof that no side of H is a subset of ∂C. $\square$

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The degrees of toroidal regular proper hypermaps*

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Abstract

Recently the classification of all possible faithful transitive permutation representations of the group of symmetries of a regular toroidal map was accomplished. In this paper we complete this investigation on a surface of genus 1 considering the group of a regular toroidal hypermap of type $(3, 3, 3)$.

Keywords: Regular polytopes, regular toroidal maps, regular toroidal hypermaps, permutation groups.

Math. Subj. Class.: 52B11, 05E18, 20B25.

1 Introduction

By Cayley's theorem, every group is isomorphic to some permutation group. A finite group G has a *faithful permutation representation of degree* n if there exists a monomorphism from G into the symmetric group S_n, or equivalently, if G acts faithfully on a set of n points. In this paper, only transitive actions will be considered. Faithful transitive permutation representations of a group G are in correspondence with core-free subgroups of G, that is, subgroups containing no nontrivial normal subgroups. The stabilizer of a point of a faithful transitive permutation representation is core-free and conversely, the action on the cosets of a core-free subgroup is faithful and transitive.

The minimal degree of a faithful permutation representation of G has been a subject of extensive study. In [6] it was shown that a faithful permutation representation of a simple group with minimal degree is primitive. The minimal degree of a faithful (transitive) permutation representation is known for all simple groups [8, Theorem 5.2.2].

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We have particular interest on the study of the faithful transitive permutation representations of the automorphism groups of abstract regular polytopes, which are quotients of Coxeter groups with linear diagram [9], or more generally, of the groups of regular hypertopes [4]. The minimal faithful permutation representations of finite irreducible Coxeter groups, which include the automorphism groups of spherical polytopes, was recently determined in [10].

This paper is a sequel to [3] in which faithful transitive permutation representations of the groups of symmetries of toroidal regular maps were determined and rectified in [2]. In the present paper we complete the classification of toroidal regular hypermaps, answering a question made by Gareth Jones in the Bled Conference in Graph Theory 2019, where the results accomplished in [3] were presented.

The results can be summarized as follows. Let $s \geq 2$.

- for the regular hypermap $(3,3,3)_{(s,0)}$, the possible degrees are s^2, $2ds$, $3ds$ and $6ds$ where d is a divisor of s. Moreover, the degree $2ds$ exists if and only if all prime divisors of s/d are congruent to 1 modulo 6;
- for the regular hypermap $(3,3,3)_{(s,s)}$, the possible degrees are those of the regular hypermap $(3,3,3)_{(s,0)}$ multiplied by 3.

Despite that $(3,3,3)_{\mathbf{s}}$ is an index two subgroup of $\{6,3\}_{\mathbf{s}}$ for $\mathbf{s} \in \{(s,0),(s,s)\}$, it is not true in general that if n is a degree of $\{6,3\}_{\mathbf{s}}$ then $n/2$ is the degree of a toroidal hypermap $(3,3,3)_{\mathbf{s}}$.

2 Toroidal hypermaps

Consider a regular tessellation of the plane by identical regular hexagons, whose full symmetry group is the Coxeter group $[6,3]$, generated by three reflections τ_0, τ_1 and τ_2, as shown in Figure 1.

By identifying opposite sides of a parallelogram with vertices $(0,0)$, (s,t), $(-t,s+t)$ and $(s-t,s+2t)$ of the tessellation, we obtain the toroidal map $\{6,3\}_{(s,t)}$, with $F = s^2+st+t^2$ faces, $3F$ edges and $2F$ vertices. This map is said to be regular when the group of symmetries acts regularly on the set of flags of the map (triples of mutually incident vertex, edge and face) [9], which is the case if and only if $st(s-t)=0$. Therefore, two families of toroidal regular maps of type $\{6,3\}$ arise: $\{6,3\}_{(s,0)}$ and $\{6,3\}_{(s,s)}$, which are factorizations of the infinite Coxeter group $[6,3]$ by $(\tau_0\tau_1\tau_2)^{2s}$ and $(\tau_0\tau_1\tau_0\tau_1\tau_2)^{2s}$, respectively. The number of flags of $\{6,3\}_{(s,0)}$ is $12s^2$ while the number of flags of $\{6,3\}_{(s,s)}$ is $36s^2$.

A hypermap can be defined as an embedding of a bipartite graph into a compact surface. The bipartition of vertices determines two types of vertices. We call hypervertices to the vertices of one type and hyperedges to the vertices of the other type (see [7] for more detail). A toroidal hypermap is obtained from a map of type $\{6,3\}$ by considering a bipartition on the set of its vertices (see Figure 2) and the translation subgroup of the map $\{6,3\}$ respects this bipartition. The toroidal hypermap constructed from $\{6,3\}_{(s,t)}$ is denoted by $(3,3,3)_{(s,t)}$, which is regular if and only if $st(s-t)=0$. All the proper toroidal regular hypermaps arise in this way [1].

Analogously a bipartition on the set of vertices of a toroidal regular map of type $\{4,4\}$ results in another map of type $\{4,4\}$ (where a face-rotation, preserving the bipartition, has order 2).

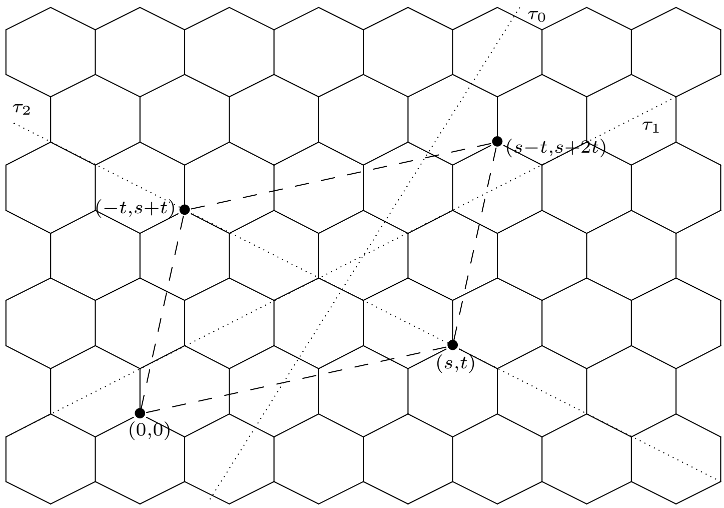


Figure 1: Toroidal map of type $\{6,3\}$ with $(s,t)=(4,1)$

The group G of symmetries of the hypermap $(3,3,3)_{(s,t)}$ is a subgroup of index 2 of the group of the map $\{6,3\}_{(s,t)}$,

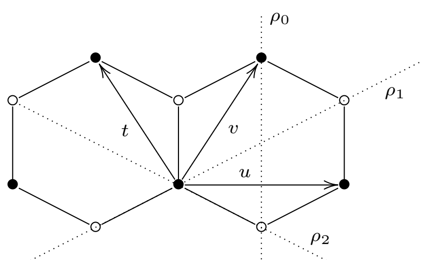
$$G := \langle \rho_0, \rho_1, \rho_2 \rangle, \text{ where } \rho_0 := \tau_0\tau_1\tau_0,\ \rho_1 := \tau_1 \text{ and } \rho_2 := \tau_2.$$

If the toroidal hypermap is regular, then G is the infinite Coxeter group $[3,3,3]$ factorized by either $(\rho_0\rho_1\rho_2\rho_1)^s$ or $(\rho_0\rho_1\rho_2)^{2s}$, depending on whether it is $(3,3,3)_{(s,0)}$ or $(3,3,3)_{(s,s)}$, respectively.

The automorphism group of the map $\{6,3\}_{(s,s)}$ (resp. $\{6,3\}_{(3s,0)}$) has a subgroup of index 3 isomorphic to the automorphism group of the map $\{6,3\}_{(s,0)}$ (resp. $\{6,3\}_{(s,s)}$). The same relations hold for the corresponding toroidal hypermaps. Particularly, the group of the $(3,3,3)_{(s,0)}$ is a quotient of the group of $(3,3,3)_{(s,s)}$ by $\langle(\rho_0\rho_1\rho_2\rho_1)^s\rangle$, and the latter is a quotient of the group of $(3,3,3)_{(3s,0)}$ by $\langle(\rho_0\rho_1\rho_2)^{2s}\rangle$.

Let u and v be two translations of order s forming an oblique basis for the group of translations of the hypermap $(3,3,3)_{(s,0)}$ (or $\{6,3\}_{(s,0)}$).

$$u := \rho_0\rho_1\rho_2\rho_1,\ v := u^{\rho_1} = \rho_1\rho_0\rho_1\rho_2 \text{ and } t := u^{-1}v.$$



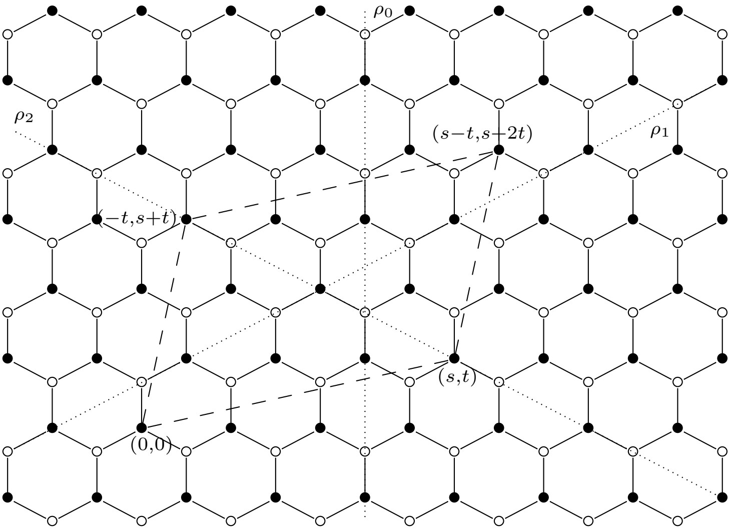
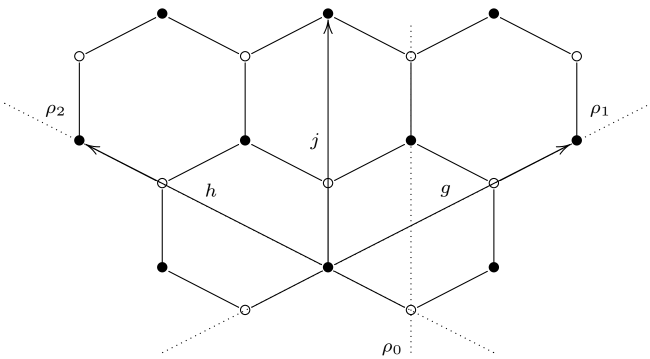


Figure 2: Toroidal map of type $(3, 3, 3)$

We have the equalities

$$u^{\rho_0} = u^{-1},\ u^{\rho_2} = t^{-1},\ v^{\rho_2} = v^{-1},\ v^{\rho_0} = t \ \text{ and } \ t^{\rho_1} = t^{-1}. \tag{2.1}$$

For the hypermap $(3, 3, 3)_{(s,s)}$, consider the translations $g := uv = (\rho_0\rho_1\rho_2)^2$, $h := g^{\rho_0}$ and $j := gh$.



In this case we have the following equalities

$$g^{\rho_1} = g,\ g^{\rho_2} = j^{-1} \ \text{ and } \ h^{\rho_1} = j^{-1}. \tag{2.2}$$

3 Degrees of maps of type $\{6, 3\}$ vs. degrees of toroidal hypermaps

The degrees of a faithful transitive permutation representation of the group of a regular map of type $\{3, 6\}$ (or equivalently $\{6, 3\}$) are given in [2] by the following two theorems.

Theorem 3.1 ([2, Theorem 5.1]). *Let $s \geq 2$. The degrees of a faithful transitive permutation representation of a toroidal regular map of type $\{3,6\}_{(s,0)}$ are*

(a) s^2,

(b) $3ds$, $6ds$ *or* $12ds$ *for any divisor d of s,*

(c) $2ds$ *and* $4ds$ *if and only if d is a divisor of s and all prime divisors of s/d are equal to* $1 \bmod 6$.

Theorem 3.2 ([2, Theorem 5.1]). *Let $s \geq 2$. The degrees of a faithful transitive permutation representation of a toroidal regular map of type $\{3,6\}_{(s,s)}$ are*

(a) $3s^2$,

(b) $9ds$, $18ds$ *or* $36ds$ *for any divisor d of s,*

(c) $6ds$ *and* $12ds$ *if and only if d is a divisor of s and all prime divisors of s/d are equal to* $1 \bmod 6$.

The basis for the proof of the above theorem is the following result that is a combination of both Lemma 3.4 of [3] and Lemma 2.1 of [2].

Lemma 3.3. *Let G be the group of a toroidal regular map. If $n \neq s^2$ then G is embedded into $S_k \wr S_m$ with $n = km$ $(m,\ k > 1)$ and*

(a) *$k = ds$ where d is a divisor of s and,*

(b) *m is a divisor of $\frac{|G|}{s^2}$.*

The above lemma assumes T is intransitive. Indeed we also have the following result.

Lemma 3.4 ([3, Lemma 3.2]). *If T is transitive, then $n = s^2$.*

The number m on Lemma 3.3 is the number of T-orbits, where T is the translation group with the translations as defined in Section 2 of this paper and of [3]. For the map $\{3,6\}_{(s,0)}$ and $T = \langle u, v\rangle$ we have proved the following.

Proposition 3.5 ([2, Proposition 3.3]). *If $m = 4$, then $k = sd$ where d is a divisor of s and all prime divisors p of s/d are such that $p \equiv 1 \bmod 6$.*

As seen in [3], there is a correspondence between core-free subgroups and faithful transitive actions. Moreover, if G has a faithful transitive permutation representation of degree n and is a subgroup of index α of U, then U has a faithful transitive permutation representation of degree αn. Similarly to Corollary 3.5 of [3], we have the following.

Corollary 3.6. *If n is a degree of $(3,3,3)_{(s,0)}$ (resp. $(3,3,3)_{(s,s)}$) then $3n$ is a degree of $(3,3,3)_{(s,s)}$ (resp. $(3,3,3)_{(3s,0)}$).*

Additionally, the group of symmetries of a toroidal hypermap $(3,3,3)_{(s,t)}$ is a subgroup of index 2 of the group of the toroidal map $\{6,3\}_{(s,t)}$ and, hence, we have the following.

Corollary 3.7. *If n is a degree of $(3,3,3)_{(s,0)}$ (resp. $(3,3,3)_{(s,s)}$) then $2n$ is a degree of $\{6,3\}_{(s,0)}$ (resp. $\{6,3\}_{(s,s)}$).*

It must be pointed out that this property works only in one direction, meaning that a degree n of the group of a map $\{6,3\}_{(s,t)}$ does not determine the degrees of $(3,3,3)_{(s,t)}$. However, by knowing the degrees of a map $\{6,3\}_{(s,t)}$ we can restrict the set of possible degrees for $(3,3,3)_{(s,t)}$.

4 The degrees of $(3,3,3)_{(s,0)}$

In what follows let $U := \langle \tau_0, \tau_1, \tau_2\rangle$ be the group of $\{6,3\}_{(s,0)}$, $G := \langle \rho_0, \rho_1, \rho_2\rangle$ be the group of $(3,3,3)_{(s,0)}$ and $T = \langle u, v\rangle$ be the translation group of order s^2 as defined in Section 2. We recall that the translation subgroups of $\{6,3\}_{(s,0)}$ and $(3,3,3)_{(s,0)}$ are the same.

Lemma 4.1. *If n is a degree of $(3,3,3)_{(s,0)}$, then $n \in \{s^2, 2ds, 3ds, 6ds\}$ for some divisor d of s.*

Proof. By Corollary 3.7 the set of possible degrees of $(3,3,3)_{(s,0)}$ is a subset of

$$\left\{\frac{s^2}{2},\ \delta s,\ \frac{3\delta s}{2},\ 2\delta s,\ 3\delta s,\ 6\delta s\right\},$$

where δ is a divisor of s. Moreover, the degrees δs and $2\delta s$ of this list must be considered only if all prime factors of s/δ are equal to 1 modulo 6. To prove this lemma we need to prove that each of the degrees $n = \frac{s^2}{2}$, $n = \frac{3\delta s}{2}$ and $n = \delta s$, either belongs to the list given in this lemma, or cannot be a degree of $(3,3,3)_{(s,0)}$.

These degrees are attained when a faithful transitive permutation representation of G on n cosets corresponds to a faithful transitive permutation representation of U on $2n$ cosets; while G acts on a set X of cosets of a core-free subgroup, U acts on $X \cup X\tau_0$.

Note that, for $x \in X$, x and $x\tau_0$ must be in different T-orbits. Thus, the number of T-orbits for the action of U on $X \cup X\tau_0$ is $2m$ where m is the number of T-orbits on X. By Lemma 3.3, $2m \in \{2, 4, 6, 12\}$. The size of a T-orbit, denoted by k, is the same in both actions.

Let first $n = \frac{s^2}{2}$. By Lemma 3.4, T is not transitive on X, which imply that $m \neq 1$. Hence $2m \in \{4, 6, 12\}$. If $2m = 4$ then by Proposition 3.5, $2n = 4ds$ with d a divisor of s, where all prime factors of s/d are equal 1 modulo 6. But then one get $\frac{s^2}{2} = 4ds$, hence $\frac{s}{d} = 4$, which is not 1 modulo 6, a contradiction. If $2m \in \{6, 12\}$ then $2n \in \{6ds, 12ds\}$ for some divisor d of s. In any case n is one of the degrees given in the statement of this lemma.

Now let $n = \frac{3\delta s}{2}$. First if δ is even then $n = 3ds$ with d being a divisor of s which is one degrees given in the statement of this lemma. Suppose that δ is odd. As $2m \in \{4, 6, 12\}$, hence $2n = 3\delta s \in \{2ds, 4ds, 6ds, 12ds\}$ for some divisor d of s, which implies that δ is even, a contradiction.

Now suppose that $n = \delta s$ with all prime factors of s/δ equal to 1 mod 6. Particularly s/δ must be odd. If $2m \in \{4, 6, 12\}$ then the degrees are among the ones listed in this lemma. We may assume that $2m = 2$. Then $m = 1$, which implies that $n = s^2$. $\square$

The dihedral groups $\langle \rho_i, \rho_j\rangle$ of order 6 are core-free subgroups of G (for $i, j \in \{0,1,2\}$ and $i \neq j$), hence there are faithful transitive permutation representations of G of degree s^2. Similarly to Proposition 5.1 (1) of [3], $\langle u^a, v^b\rangle$ is a core-free subgroup of G. Hence G has a faithful transitive permutation representation of degree $n = 6ab$ for any integers a and b such that $s = lcm(a,b)$, or equivalently, of degree $n = 6ds$ for any divisor d of s. In what follows we give other core-free subgroups of G.

Proposition 4.2. *Let G be the group of $(3,3,3)_{(s,0)}$ with $s \geq 2$ and d be a divisor of s.*

(a) *The group $\langle u^d\rangle \rtimes \langle \rho_0\rangle$ is a core-free subgroup of G of index $3ds$.*

(b) *Suppose that there exists* α, *coprime with* s/d, *such that* $\alpha^2 - \alpha + 1 \equiv 0 \bmod (s/d)$. *Then* $\langle (v^{-\alpha}u)^d,\ \rho_1\rho_2\rangle$ *is a core-free subgroup of* G *with index* $2sd$.

Proof. (a) Let $H := \langle u^d\rangle \rtimes \langle \rho_0\rangle$. Suppose that $x \in H \cap H^{\rho_1} = \langle u^d\rangle \rtimes \langle \rho_0\rangle \cap \langle v^d\rangle \rtimes \langle \rho_0^{\rho_1}\rangle$. If $x \notin T$ then $\rho_0\rho_0^{\rho_1} \in T$, a contradiction. Thus $x \in T$ and therefore $x \in \langle u^d\rangle \cap \langle v^d\rangle$, which implies that x is trivial. The order of H is $\frac{2s}{d}$ thus $|G : H| = 3ds$.

(b) Let now $H := \langle (v^{-\alpha}u)^d,\ \rho_1\rho_2\rangle$. First note that $\langle (v^{-\alpha}u)^d\rangle$ is a normal subgroup of H. Indeed we have $(v^{-\alpha}u)^{\rho_1\rho_2} = t^{\alpha}v^{-1} = u^{\alpha}v^{\alpha-1} = (v^{-\alpha}u)^{\alpha}$. Suppose that $x \in H \cap H^{\rho_1}$. Then $x = (v^{-\alpha}u)^{id}(\rho_1\rho_2)^j = (u^{-\alpha}v)^{i'd}(\rho_1\rho_2)^{j'}$. This implies that $j = j'$ and $i = i' = 0 \bmod (s/d)$. Thus $H \cap H^{\rho_1} = \langle \rho_1\rho_2\rangle$. Now the intersection of $\langle \rho_1\rho_2\rangle$ and H^{ρ_0} is trivial, otherwise we get that either $\rho_1\rho_2 \in T$, $(\rho_1\rho_2)^{\rho_0} \in T$, $\rho_0\rho_2 \in T$, $tu^{-1} \in \langle t^{-\alpha}u^{-1}\rangle$ or $uv \in \langle t^{-\alpha}u^{-1}\rangle$, which is never possible. □

Let us also recall the following proposition.

Proposition 4.3 ([2, Proposition 3.1]). *Let* q *be an odd number. The modular equation*

$$x^2 - x + 1 \equiv 0 \bmod q$$

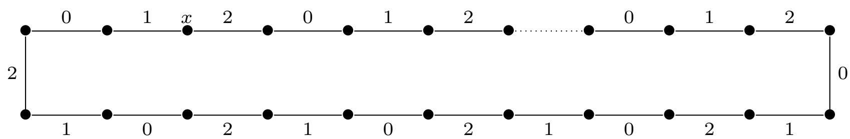
has a solution if and only if all prime divisors p *of* q *are such that* $p \equiv 1 \bmod 6$.

Theorem 4.4. *Let* $s \geq 2$. *A faithful transitive permutation representation of the group of symmetries of* $(3,3,3)_{(s,0)}$ *has degree* n *if and only if* $n \in \{s^2,\ 3ds,\ 6ds\}$ *where* d *is a divisor of* s *or;* $n = 2ds$ *where* d *is a divisor of* s *and all prime factors of* s/d *are equal* $1 \bmod 6$.

Proof. This is a consequence of Lemma 4.1 and the core-free subgroups indexes found in this section. □

A *Schreier coset graph* of a group G is a graph $\mathcal{G}$ associated with a subgroup $H \leq G$ and a set of generators $\langle \rho_i \mid i \in \{0, \ldots, r-1\}\rangle$ of G, where the vertices are the cosets G/H and there is an edge $\{Hx, Hy\}$ labeled i whenever $Hx\rho_i = Hy$ (for some $x, y \in G$). When H is core-free, a Schreier coset graph gives a faithful transitive permutation representation of the group G, of degree $n = |G : H|$.

Proposition 4.5. *Let* $s \geq 2$. *The following graph is a faithful transitive permutation representation graph of the automorphism group of* $(3,3,3)_{(s,0)}$ *with degree* $3s$.



Moreover, the stabilizer of a point is, up to conjugacy, $\langle u\rangle \rtimes \langle \rho_0\rangle$.

Proof. Let $G = \langle \rho_0, \rho_1, \rho_2\rangle$ be the group with the given permutation representation graph. It is clear from the graph that $\rho_0^2 = \rho_1^2 = \rho_2^2 = (\rho_0\rho_1)^3 = (\rho_0\rho_2)^3 = (\rho_1\rho_2)^3 = (\rho_0\rho_1\rho_2\rho_1)^s = 1$. Hence G must be a subgroup of the automorphism group of the regular hypermap $(3,3,3)_{(s,0)}$ and $|G| \leq 6s^2$.

Consider the vertex x of the permutation representation. Its stabilizer G_x contains the subgroup $\langle \rho_0, u\rangle$ of order $2s$. Then, $|G_x| \geq 2s$ and, by the Orbit-Stabilizer theorem, $|G| \geq 6s^2$. Consequently, the graph is a faithful transitive permutation representation of the automorfism group of $(3,3,3)_{(s,0)}$. □

Remark 4.6. The faithful transitive permutation representation given on Proposition 4.5 is of minimal degree whenever s is not a prime number congruent with 1 mod 6.

Similarly to what was done in [3], it is possible to obtain permutation representation graphs for other degrees. As some of the graphs are very complicated to draw we decide to include only the simplest one.

5 The degrees of $(3,3,3)_{(s,s)}$

In this section we determine the degrees of $(3,3,3)_{(s,s)}$ using the degrees of $(3,3,3)_{(s,0)}$ and $(3,3,3)_{(3s,0)}$, given by Theorem 4.4. Let n be the degree of a faithful transitive permutation representation of $(3,3,3)_{(s,s)}$ and $T = \langle u, v\rangle$ the translation group of $(3,3,3)_{(3s,0)}$ of order $(3s)^2$.

Theorem 5.1. *Let $s \geq 2$. A faithful transitive permutation representation of the group of symmetries of $(3,3,3)_{(s,s)}$ has degree n if and only if $n \in \{3s^2,\, 9ds,\, 18ds\}$ where d is a divisor of s or; $n = 6ds$ where d is a divisor of s and all prime factors of s/d are equal* 1 mod 6.

Proof. Let G be the group of $(3,3,3)_{(s,s)}$. From Theorem 4.4 and Corollary 3.6 there are faithful transitive permutation representations with the degrees given in the statement of this theorem. By Theorem 4.4, a degree of $(3,3,3)_{(3s,0)}$ is either equal to $(3s)^2$, $3\delta(3s)$ and $6\delta(3s)$, with δ being a divisor of $3s$, or to $2\delta(3s)$, with δ being a divisor of $3s$ and all prime factors of $3s/\delta$ equal 1 mod 6.

Dividing the possible degrees of $(3,3,3)_{(3s,0)}$ by 3, we get that

$$n \in \{3s^2,\, 2\delta s,\, 3\delta s,\, 6\delta s\}$$

with δ dividing $3s$.

The degree $n = 3s^2$ is in set given in the statement of the theorem. If $n = 2\delta s$ then, as in this case δ is a divisor of $3s$ and all prime divisors of $3s/\delta$ must be equal 1 mod 6, $\delta = 3d$ for some divisor d of s. Hence this degree is already included in the set given in the statement of this theorem. Let us prove that also on the remaining cases $\delta = 3d$ for some divisor d of s.

The hypermap $(3,3,3)_{(3s,0)}$ contains three copies of the hypermap $(3,3,3)_{(s,s)}$. To be more precise the group of $(3,3,3)_{(s,s)}$ is the group of $(3,3,3)_{(3s,0)}$ factorized by the translation $(uv)^s$ of order 3. Hence, the points x, $x(uv)^s$ and $x(uv)^{2s}$ of any faithful transitive permutation representation of $(3,3,3)_{(3s,0)}$ are identified under this factorization. Any faithful transitive permutation representation of an action of $(3,3,3)_{(3s,0)}$ on a set X gives a permutation representation, of degree $|X|/3$, of $(3,3,3)_{(s,s)}$ on triples of points of X of the form

$$\left\{x, x(uv)^s, x(uv)^{2s}\right\}.$$

with $x \in X$. Note that these points are in the same T-orbit. Hence the number m of T-orbits is unchanged under this factorization.

To prove that the action on the triple of points is faithful only if $\delta = 3d$, for some divisor d, we can follow an identical proof as the one presented for Theorem 5.3 of [2]. We note that Lemma 2.1 of [2], that establishes the size of T-orbit, can be used here. $\square$

6 Open Problems

The study of faithful transitive permutation representations can be extended to other regular polytopes, particularly to finite locally spherical regular polytopes, including the cubic tessellations and to the finite locally toroidal regular polytopes.

Problem 6.1. Determine the degrees of faithful transitive permutation representations of the groups of spherical and euclidean type.

Problem 6.2. Determine the degrees of faithful transitive permutation representations of the groups of the finite toroidal regular polytopes.

The problem of the classification locally toroidal regular polytopes dominated the theory of abstract polytopes for a while and it was originally posed by Grünbaum [5]. The meritoriously known as Grünbaum's Problem, is not yet totally solved [9].

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Connected geometric (n_k) configurations exist for almost all n

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Abstract

In a series of papers and in his 2009 book on configurations Branko Grünbaum described a sequence of operations to produce new (n_4) configurations from various input configurations. These operations were later called the "Grünbaum Incidence Calculus". We generalize two of these operations to produce operations on arbitrary (n_k) configurations. Using them, we show that for any k there exists an integer N_k such that for any $n \geq N_k$ there exists a geometric (n_k) configuration. We use empirical results for $k = 2, 3, 4$, and some more detailed analysis to improve the upper bound for larger values of k.

IN MEMORY OF BRANKO GRÜNBAUM

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1 Introduction

In a series of papers and in his 2009 book on configurations [11], Branko Grünbaum described a sequence of operations to produce new (n_4) configurations from various input configurations. These operations were later called the "Grünbaum Incidence Calculus" [13, Section 6.5]. Some of the operations described by Grünbaum are specific to producing 3- or 4-configurations. Other operations can be generalized in a straightforward way to produce (n_k) configurations from either smaller (m_k) configurations with certain properties, or from (m_{k-1}) configurations. Let N_k be the smallest number such that for any $n, n \geq N_k$ there exists a geometric (n_k) configuration. For $k = 2$ and $k = 3$, the exact value of N_k is known, and for $k = 4$ it is known that $N_4 = 20$ or 24. We generalize two of the Grünbaum Calculus operations in order to prove that for any integer k there exists an integer N_k and we give bounds on N_k for $k \geq 5$.

The existence of geometric 2-configurations is easily established. The only (connected) combinatorial configuration (n_2) is an n-lateral. For each $n, n \geq 3$, an n-lateral can be realized as a geometric multilateral (for the definition of a *multilateral*, see [11]). As a specific example, an (n_2) configuration can be realized as a regular n-gon with sides that are extended to lines. (For larger values of n it can also be realized as an n-gonal star-polygon, but the underlying combinatorial structure is the same.) Hence:

Proposition 1.1. *A geometric (n_2) configuration exists if and only if $n \geq 3$. In other words, $N_2 = 3$.*

For 3-configurations, N_3 is known to be 9 (see [11, Section 2.1]); for example, Branko Grünbaum provides a proof (following that of Schröter from 1888, see the discussion in [11, p. 65]) that the cyclic combinatorial configuration $\mathcal{C}_3(n)$, which has starting block $[0, 1, 3]$, can always be realized with straight lines for any $n \geq 9$. That is:

Proposition 1.2. *A geometric (n_3) configuration exists if and only if $n \geq 9$. In other words, $N_3 = 9$.*

Note that there exist two combinatorial 3-configurations, namely (7_3) and (8_3), that do not admit a geometric realization.

For $k = 4$, the problem of parameters for the existence of 4-configurations is much more complex, and the best bound N_4 is still not known. For a number of years, the smallest known 4-configuration was the (21_4) configuration which had been studied combinatorially by Klein and others, and whose geometric realization, first shown in 1990 [12], initiated the modern study of configurations. In that paper, the authors conjectured that this was the smallest (n_4) configuration. In a series of papers [6, 7, 8, 9] (summarized in [11, Sections 3.1-3.4]), Grünbaum showed that N_4 was finite and less than 43. In 2008, Grünbaum found a geometrically realizable (20_4) configuration [10]. In 2013, Jürgen Bokowski and Lars Schewe [3] showed that geometric (n_4) configurations exist for all $n \geq 18$ except possibly $n = 19, 22, 23, 26, 37, 43$. Subsequently, Bokowski and Pilaud [1] showed that there is no geometrically realizable (19_4) configuration, and they found examples of realizable (37_4) and (43_4) configurations [2]. In 2018, Michael Cuntz [5] found realizations of (22_4) and (26_4) configurations. However, the question of whether a geometric (23_4) geometric configuration exists is currently still open.

In this paper, $\bar{N}_k$ will denote any known upper bound for N_k and N_k^R will denote currently best upper bound for N_k.

Summarizing the above results, we conclude:

Proposition 1.3. *A geometric* (n_4) *configuration exists for* $n = 18, 20, 21, 22$ *and* $n \geq 24$. *Moreover, either* $N_4 = 20$ *or* $N_4 = 24$ *(depending on whether or not a* (23_4) *configuration exists). In other words,* $N_4^R = 24$.

The main result of the paper is the following result.

Theorem 1.4. *For each integer* $k \geq 2$ *the numbers* N_k *exist.*

To simplify subsequent discussions, we introduce the notion of *configuration-realizability*, abbreviated as *realizability*, of numbers. A number n is k*-realizable* if and only if there exists a geometric (n_k) configuration. We may rephrase Proposition 1.3 by stating that the numbers $n = 18, 20, 21, 22$ and $n \geq 24$ are 4-realizable. Also note that the number 9 is 2- and 3-realizable but not k-realizable for any $k \geq 4$.

2 Generalizing two constructions from the Grünbaum incidence calculus

In this section, we generalize two constructions of the Grünbaum Incidence Calculus which we will use to prove the existence of N_k for any k. As input to examples of these constructions, we often will use the standard geometric realization of the (9_3) Pappus configuration $\mathcal{P}$, shown in Figure 1.

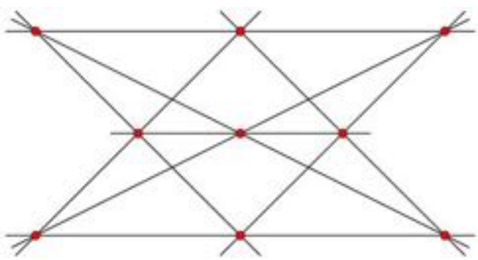


Figure 1: The standard geometric realization of the (9_3) Pappus configuration $\mathcal{P}$.

The first, which we call *affine replication* and denote $\mathrm{AR}(m, k)$, generalizes Grünbaum's $(\mathbf{5m})$ construction; it takes as input an (m_{k-1}) configuration and produces a $((k+1)m_k)$ configuration with a pencil of m parallel lines.

The second, which we call *affine switch*, is analogous to Grünbaum's $(\mathbf{3m}+)$ construction. It takes as input a single (m_k) configuration with a set of p parallel lines in one direction and a set of q parallel lines in a second direction which are disjoint (in terms of configuration points) from the pencil of p lines, and it produces a configuration $(((k-1)m+r)_k)$ for any r with $1 \leq r \leq p+q$. Applying a series of affine switches to a single starting (m_k) configuration with a pencil of q parallel lines produces a consecutive sequence (or "band") of configurations

$$(((k-1)m+1)_k), \ldots, (((k-1)m+q)_k)$$

which we will refer to as $\mathrm{AS}{+}(m, k, q)$.

2.1 Affine replication

Starting from an (m_{k-1}) configuration $\mathcal{C}$ we construct a new configuration $\mathcal{D}$ which is a $((k+1)m_k)$ configuration. A sketch of the construction is that $k-1$ affine images of $\mathcal{C}$ are carefully constructed so that each point P of $\mathcal{C}$ is collinear with the $k-1$ images of P, and each line of $\mathcal{C}$ and its images are concurrent at a single point. Then $\mathcal{D}$ consists of the points and lines of $\mathcal{C}$ and its images, the new lines corresponding to the collinearities from each point P, and the new points of concurrence corresponding to the lines of $\mathcal{C}$ and their images.

The details of the construction are as follows:

(1) Let A be a line that (i) does not pass through the intersection of two lines of $\mathcal{C}$, whether or not that intersection point is a point of the configuration; (ii) is perpendicular to no line connecting any two points of $\mathcal{C}$, whether or not that line is a line of the configuration; (iii) intersects all lines of $\mathcal{C}$.

(2) Let $\alpha_1, \alpha_2, \ldots, \alpha_{k-1}$ be pairwise different orthogonal axial affinities with axis A. Construct copies $\mathcal{C}_1 = \alpha_1(\mathcal{C})$, $\mathcal{C}_2 = \alpha_2(\mathcal{C})$,..., $\mathcal{C}_{k-1} = \alpha_{k-1}(\mathcal{C})$ of $\mathcal{C} = \mathcal{C}_0$.

(3) Let ℓ be any line of $\mathcal{C}$. Since A is the common axis of each α_i, the point $A \cap \ell$ is fixed by all these affinities. This means that the k-tuple of lines $\ell, \alpha_1(\ell), \ldots, \alpha_{k-1}(\ell)$ has a common point of intersection lying on A. We denote this point by F_ℓ. By condition (i) in (1), for different lines $\ell, \ell' \in \mathcal{C}$ the points $F_\ell, F_{\ell'}$ differ from each other; they also differ from each point of the configurations $\mathcal{C}_i$ $(i = 1, 2, \ldots, k-1)$. We denote the set $\{F_\ell : \ell \in \mathcal{C}\}$ of points lying on A by $\mathcal{F}$.

(4) Let P be any point of $\mathcal{C}$. Since the affinities α_i are all orthogonal affinities (with the common axis A), the k-tuple of points $P, \alpha_1(P), \ldots, \alpha_{k-1}(P)$ lies on a line perpendicular to A (and avoids A, by condition (i)). We denote this line by ℓ_P. Clearly, we have altogether m such lines, one for each point of $\mathcal{C}$, with no two of them coinciding, by condition (ii). We denote this set $\{\ell_P : P \in \mathcal{C}\}$ of lines by $\mathcal{L}$.

(5) Put $\mathcal{D} = \mathcal{C}_0 \cup \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_{k-1} \cup \mathcal{F} \cup \mathcal{L}$.

The conditions of the construction imply that $\mathcal{D}$ is a $((k+1)m_k)$ configuration. Moreover, by construction, $\mathcal{D}$ has a pencil of m parallel lines. Figures 2 and 3 show two examples of affine replication, first starting with a (4_2) configuration to produce a (16_3) configuration, and then starting with the (9_3) Pappus configuration to produce a (45_4) configuration.

Remark 2.1. The orthogonal affinities used in the construction are just a particular case of the axial affinities called *strains* [4]; they can be replaced by other types of axial affinities, namely, by oblique affinities (each with the same (oblique) direction), and even, by *shears* (where the direction of affinity is parallel with the axis) [4], while suitably adjusting conditions (i–iii) in (1).

We may summarize the above discussion as follows:

Lemma 2.2. *If affine replication* $\mathrm{AR}(m, k)$ *is applied to any* (m_{k-1}) *configuration, the result is a* $(((k+1)m)_k)$ *configuration with a pencil of* m *parallel lines.*

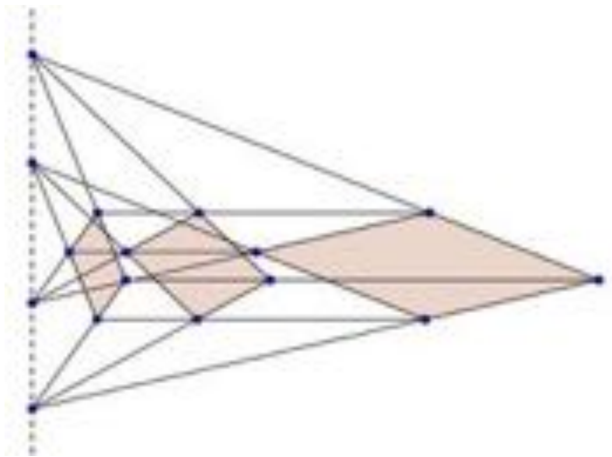


Figure 2: Affine replication $\mathrm{AR}(4,3)$ applied to a quadrilateral, i.e. a (4_2) configuration; it results in a (16_3) configuration. The corresponding ordinary quadrangles are shaded (the starting, hence each of the three quadrangles are parallelograms). The axis A is shown by a dashed line.

2.2 Affine switch

In our description of this construction, we are inspired by Grünbaum [11, §3.3, pp. 177–180] but we have chosen a slightly different approach (in particular, we avoid using 3-space). At the same time, we generalize it from (m_4) to (m_k).

A sketch of the construction is as follows: Suppose that $\mathcal{C}$ is an (m_k) configuration that contains a pencil $\mathcal{P}$ of p parallel lines in one direction, and a pencil $\mathcal{Q}$ of q parallel lines in a second direction, where the two pencils share no common configuration points; we say that the pencils are *independent*. For each subpencil $\mathcal{S}$ of $\mathcal{P}$ and $\mathcal{T}$ of $\mathcal{Q}$ containing s parallel lines and t parallel lines respectively, with $1 \le s \le p$ and $0 \le t \le q$, we form the subfiguration $\hat{\mathcal{C}}$ by deleting $\mathcal{S}$ and $\mathcal{T}$ from $\mathcal{C}$ (here we use the term *subfiguration* in the sense of Grünbaum [11]). We then carefully construct $k-2$ affine images of $\hat{\mathcal{C}}$ in such a way that for each (deleted) line ℓ in $\mathcal{S}$ and for each point $P_1, P_2, \dots, P_k$ on ℓ, the collection of lines through each P_i and its images all intersect in a single point Y_ℓ, and simultaneously, for each line ℓ' in $\mathcal{T}$ and for each point $Q_1, Q_2, \dots, Q_k$ on ℓ', the collection of lines through Q_i and its images all intersect in a single point $X_{\ell'}$. Let $\mathcal{D}$ be the collection of all the undeleted points and lines of $\hat{\mathcal{C}}$ and its affine images, and for each of the deleted ℓ and ℓ', the new lines through each point P_i Q_i and their images, the points Y_ℓ, and the points $X_{\ell'}$; then $\mathcal{D}$ is a $(((k-1)m+s+t)_k)$ configuration.

As a preparation, we need the following two propositions.

Proposition 2.3. *Let α be a (non-homothetic) affine transformation that is given by a diagonal matrix with respect to the standard basis. Note that in this case α can be written as a (commuting) product of two orthogonal affinities whose axes coincide with the x- and y-axis, respectively:*

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}.$$

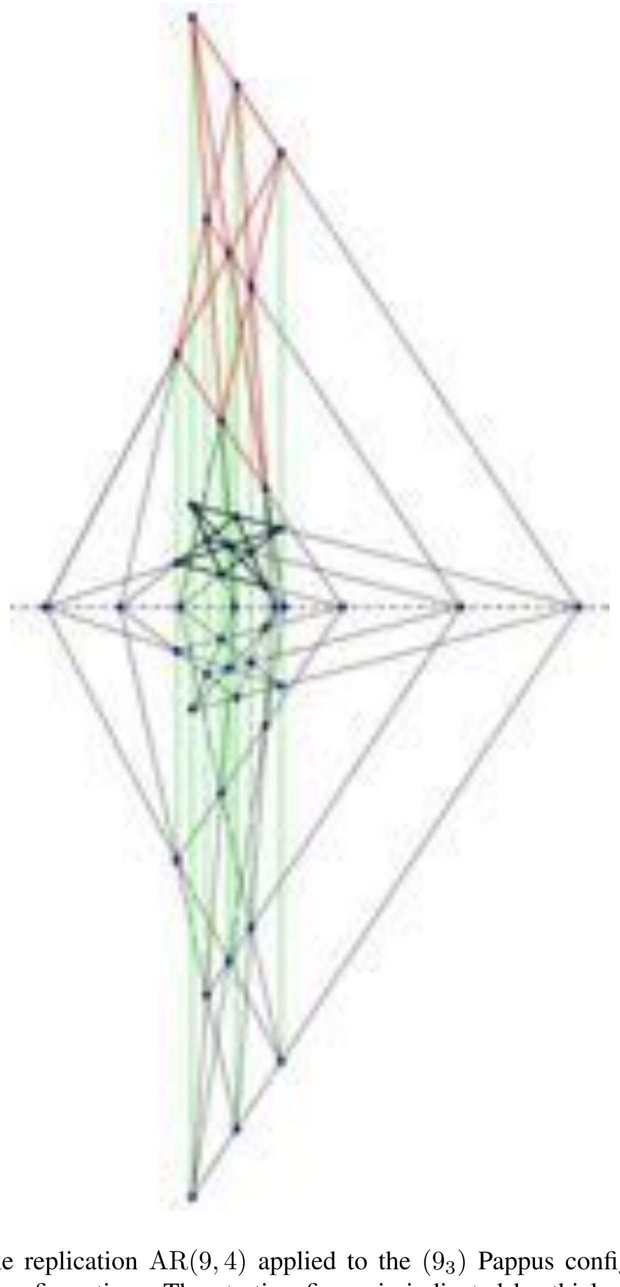


Figure 3: Affine replication $\mathrm{AR}(9, 4)$ applied to the (9_3) Pappus configuration, which yields a (45_4) configuration. The starting figure is indicated by thick segments, while the first image is highlighted by red segments. The axis A is shown by a dashed line. The construction is chosen so as to exemplify that ordinary mirror reflection can also be used. Note that the resulting configuration contains a pencil of 9 parallel lines arising from the construction, shown in green.

Let $P_0(x_0, 0), P_1(x_0, y_1), \dots, P_k(x_0, y_k)$ *be a range of* $k + 1$ *different points on a line which is perpendicular to the* x*-axis and intersects it in* P_0*. Then the* k *lines connecting the pairs of points* $(P_1, \alpha(P_1)), \dots, (P_k, \alpha(P_k))$ *form a pencil with centre* C_x *such that* C_x *lies on the* x*-axis, and its position depends only on* α *and* x_0*.*

Likewise, let $Q_0(x_0, y_0), Q_1(x_1, y_0), \dots, Q_k(x_k, y_0)$ *be a range of* $k+1$ *different points on a line which is perpendicular to the* y*-axis and intersects it in* Q_0*. Then the* k *lines connecting the pairs of points* $(Q_1, \alpha(Q_1)), \dots, (Q_k, \alpha(Q_k))$ *form a pencil with centre* C_y *such that* C_y *lies on the* y*-axis, and its position depends only on* α *and* y_0*.*

Proof. An elementary calculation shows that

$$C_x = C_x\left(0, \frac{a-b}{b-1}\,x_0\right), \text{ resp. } C_y = C_y\left(0, \frac{b-a}{a-1}\,y_0\right)$$

is the common point of intersection of any two, hence of all the lines in question. □

Proposition 2.4. *Let* $h \geq 3$ *be a positive integer, and for each* j *with* $j = 1, \dots, h-1$*, let the affine transformation* α_j *be given by the matrix*

$$M_j = \begin{pmatrix} \dfrac{h-j}{h} & 0 \\ 0 & \dfrac{h+j}{h} \end{pmatrix}. \tag{2.1}$$

Then for any point P*, the points* $P, \alpha_1(P), \dots, \alpha_{h-1}(P)$ *are collinear.*

Proof. Choose any j' and j'', and form the difference matrices $M_{j'} - U$ and $M_{j''} - U$ with the unit matrix U. Observe that these matrices are such that one is a scalar multiple of the other. Hence the vectors $\overrightarrow{PP'}$ and $\overrightarrow{PP''}$ are parallel, where $P' = \alpha_{j'}(P)$ and $P'' = \alpha_{j''}(P)$. This means that the points P, P' and P'' lie on the same line. □

Now we apply the following construction. Let $\mathcal{C}$ be an (m_k) configuration such that it contains a pencil $\mathcal{P}$ of $p \geq 1$ parallel lines and a pencil $\mathcal{Q}$ of $q \geq 1$ parallel lines, too, such that these pencils are perpendicular to each other and are independent. Note that any configuration containing independent pencils in two different directions can be converted by a suitable affine transformation to a configuration in which these pencils will be perpendicular to each other.

Choose a position of $\mathcal{C}$ (applying an affine transformation if necessary) such that these pencils are parallel to the x-axis and y-axis, respectively.

(1) Remove lines $\ell_1, \dots, \ell_s$ $(s \leq p)$ from the pencil $\mathcal{P}$ parallel to the x-axis and $\ell_{s+1}, \dots, \ell_{s+t}$ $(0 \leq t \leq q)$ from the pencil $\mathcal{Q}$ parallel to the y-axis. Let $\widehat{\mathcal{C}}$ denote the substructure of $\mathcal{C}$ obtained in this way.

(2) Let h be a positive integer (say, some suitable multiple of k), and for each j, $j = 1, \dots, k-2$, let α_j be an affine transformation defined in Proposition 2.4. Form the images $\alpha_j(\widehat{\mathcal{C}})$ for all j given here.

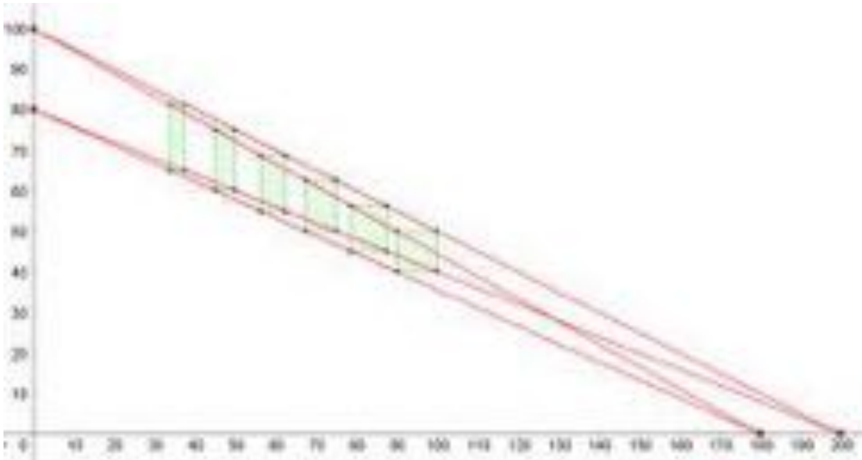


Figure 4: Illustration for Propositions 2.3 and 2.4. Affine transformations with parameters $h = 8$ and $j = 1, \dots 5$ are applied on a square.

(3) Let P be a point of $\widehat{\mathcal{C}}$ that was incident to one of the lines ℓ_i removed from $\mathcal{C}$. Take the images $\alpha_j(P)$ for all j given in (2). By Proposition 2.4, all the $k-1$ points $P, \alpha_j(P)$ are collinear. Let $c_i(P)$ denote this line.

(4) Take all the configuration points on ℓ_i and repeat (3) for each of them. By Proposition 2.3, the k-set of lines $\{c_i(P) : P \in \ell_i\}$ form a pencil whose centre lies on the x-axis or the y-axis according to which axis ℓ_i is perpendicular to.

(5) Let $r = (s+t) \in \{1, 2, \dots, p+q\}$ be the number of lines removed from the pencils of $\mathcal{C}$ in the initial step of our construction. Repeat (4) for all these lines. Eventually, we obtain rk new lines and r new points such that the set of the new lines is partitioned into r pencils, and the new points are precisely the centres of these pencils (hence they lie on the coordinate axes). Observe that there are precisely k lines passing through each of the new points, and likewise there are precisely k points lying on each of the new lines.

(6) Putting everything together, we form a $(((k-1)m+r)_k)$ configuration, whose

- points come from the $(k-1)m$ points of the copies of $\widehat{\mathcal{C}}$, completed with the r new points considered in (5).
- lines come from the $(k-1)(m-r)$ lines of the copies of $\widehat{\mathcal{C}}$, completed with the rk new lines considered in (5).

We use the notation $AS(m,k,r)$ to represent the $(((k-1)m+r)_k)$ configuration described above.

Summarizing the discussion above, we conclude:

Lemma 2.5. *Beginning with any (m_k) configuration with independent pencils of $p \geq 0$ and $q \geq 1$ parallel lines, for each integer r with $1 \leq r \leq p+q$, the affine switch construction produces an (n_k) configuration, where $n = (k-1)m+r$.*

Note that $p+q$ independent lines in an (m_k) configuration covers $k(p+q) \leq m$ points. This gives an upper bound $p+q \leq m/k$, where the equality is attained only if m divides k.

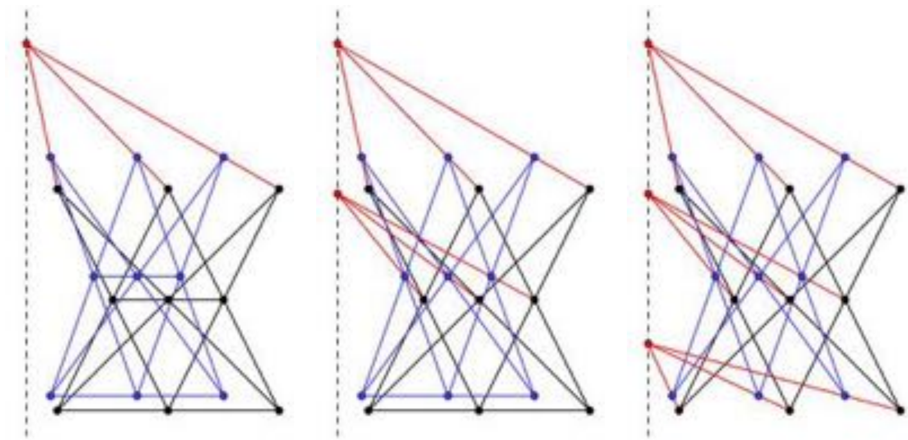
In this paper we use the above Lemma 2.5 in connection with Lemma 2.2 only for the case of a single pencil of parallel lines, such that $p = 0$.

Corollary 2.6. *From any starting (m_k) configuration that has a pencil of q parallel lines, we can apply a sequence of affine switches by removing $1, 2, \ldots, q$ lines in sequence, so as to construct a sequence of consecutive configurations*

$$[(((k-1)m + r)_k)]_{r=1}^{q} = [AS(m, k, r)]_{r=1}^{q}.$$

This collection of consecutive configurations is represented by the notation $\mathrm{AS+}(m, k, q)$. That is, $\mathrm{AS+}(m, k, q) = [AS(m, k, r)]_{r=1}^{q}$.

Example 2.7. Figure 5 illustrates an application of this construction to the Pappus configuration $\mathcal{P}$ (cf. Figure 1). Removing only one line from the horizontal pencil results in a (19_3) configuration, shown in Figure 5(a). Removing two or three lines results in a (20_3) or (21_3) configuration, respectively, shown in Figures 5(b) and 5(c). (Observe that since the Pappus configuration has 9 points, the maximal total number of lines in independent pencils is 3, since any three disjoint lines in the configuration contain all the points of the configuration.) Taken together the three configurations, we have: $[(19_3), (20_3), (21_3)] = \mathrm{AS+}(9, 3, 3)$.



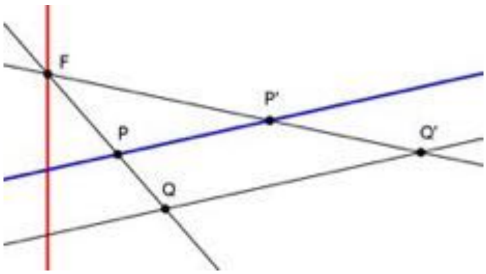
(a) A (19_3) configuration (b) A (20_3) configuration (c) A (21_3) configuration

Figure 5: Configurations (19_3), (20_3), and (21_3), constructed by applying the affine switch construction to the realization of the Pappus configuration with a pencil of 3 parallel lines, shown in Figure 1, by deleting one, two, or three lines respectively. (The vertical axis of affinity, denoted by dashed line, does not belong to the configuration.)

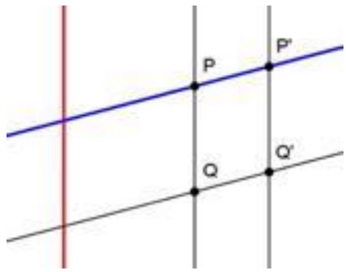
Since axial affinities play a crucial role in the constructions described above, we recall a basic property. The proof of the following proposition is constructive, hence it provides a simple tool for a basically synthetic approach to these constructions, which is especially useful when using dynamic geometry software to construct these configurations.

Proposition 2.8. *An axial affinity α is determined by its axis and the pair of points (P, P'), where P is any point not lying on the axis, and P' denotes the image of P, i.e. $P' = \alpha(P)$.*

Proof. In what follows, for any point X, we denote its image $\alpha(X)$ by X'. Let Q be an arbitrary point not lying on the axis and different from P. Take the line PQ, and assume that it intersects the axis in a point F (see Figure 6a). Thus $PQ = FP$. Take now the line $F'P'$, i.e., the image of FP. Since F is a fixed point, i.e. $F' = F$, we have $F'P' = FP'$. This means that Q' lies on FP', i.e. $P'Q' = FP'$. To find Q' on FP', we use the basic property of axial affinities that for all points X not lying on the axis, the lines XX' are parallel with each other (we recall that the direction of these lines is called the *direction* of the affinity). Accordingly, a line passing through Q which is parallel with PP' will intersect FP' precisely in the desired point Q'.



(a)



(b)

Figure 6: Construction of the image of a pont Q under axial affinity; the axis is the vertical red line, the direction of affinity is given by the blue line. Here we use oblique affinity, but the construction given in the proof is the same in any other types of axial affinities.

On the other hand, if PQ is parallel with the axis, then clearly so is $P'Q'$. In this case Q' is obtaned as the fourth vertex of the parallelogram determined by P', P and Q (see Figure 6b). $\square$

Remark 2.9. In using integer parameters h and j above, we followed Grünbaum's original concept [11] (as mentioned explicitly at the beginning of this subsection). However, the theory underlying Propositions 2.3 and 2.4 makes possible using continuous parameters as well, so that the procedure becomes in this way much more flexible. In what follows we outline such a more general version, restricted to using only one pencil of lines to be deleted.

Start again with a configuration $\mathcal{C}$, and assume that the pencil $\mathcal{P}$ is in horizontal position; accordingly, the axis that we use is in vertical position (see e.g. Figure 5). Choose a line ℓ in $\mathcal{P}$, and a configuration point P_0 on ℓ; then, remove ℓ. P_0 will be the initial point of our construction (e.g., in Figure 5 the "north-west" (black) point of the starting configuration). Choose a point C_ℓ on the axis such that the line $C_\ell P_0$ is not perpendicular to the axis (in our example, this is the red point in Figure 5a).

Now let $t \in \mathbb{R}$ be our *continuous parameter*. Take the point

$$P = tC_\ell + (1-t)P_0; \tag{2.2}$$

thus P is a point on the line $C_\ell P_0$, and as t changes, P slides along this line. Moreover, by Proposition 2.8 we see that the pair of points (P_0, P) determines two orthogonal affinities whose axes are perpendicular to each other. In particular, the axes are precisely the coordinate axes. These affinities act simultaneously, i.e. P_0 is sent to P by their (commuting) product. Using coordinates, such as $P_0(x_0, y_0)$ and $P(x, y)$, we also see that the ratio of these affinities is y/y_0 (for that with horizontal axis), respectively x/x_0 (for that with vertical axis). (Note that these ratios, using the relation (2.2), can also be expressed by the parameter t and by the prescribed coordinates of P_0 and C_ℓ. Furthermore, similarly, the matrix (2.1) above can also be parametrized by t; we omit the details.)

It is easily checked that both Proposition 2.3 and Proposition 2.4 remains valid with this continuous parameter t. Hence, for any P, we can construct the corresponding affine image of $\mathcal{C}$ (or its substructures $\hat{\mathcal{C}}$ with lines of any number r removed), together with the new lines (which are denoted by red in our example of Figure 5). In particular, in case of k-configurations, we need to choose altogether $k-2$ points on the line $C_\ell P_0$ (note the for $t=0$, the starting copy $\mathcal{C}$ returns; for $t=1$ the image of $\mathcal{C}$ collapses to a segment within the y-axis, and for a third value depending on the slope of $C_\ell P_0$, it collapses to a segment within the x-axis; these cases thus are to be avoided).

3 Proof of the main theorem

In this section we prove the main theorem of our paper. For notational convenienence, given integers $a < b$, let $[a : b]$ denote the range $\{a, a+1, \dots, b\}$. Similarly, for integer function $f(s)$ the range $\{f(a), f(a+1), \dots, f(b)\}$ will be denoted by $[f(s)]_{s=a}^{b}$. The crucial step in the proof will be provided by the following Lemma.

Lemma 3.1. *Assume that for some $k \geq 3$, N_{k-1} exists and that $\bar{N}_{k-1}$ is any known upper bound for it. Then N_k exists and: $\bar{N}_k = (k^2-1)\max(\bar{N}_{k-1}, k^2-2)$ is an upper bound for it. Moreover, if we have two upper bounds, say $\bar{N}_{k-1} < \tilde{N}_{k-1}$ for N_{k-1}, the better one will produce a better upper bound for N_k.*

This Lemma will be proven with the tools from previous section by applying affine replication and affine switch. More precisely, Lemma 2.2 and Corollary 2.6 will be used.

Proof of Lemma 3.1. Let $\bar{N}_{k-1}$ denote any known upper bound for N_{k-1}. By definition, the sequence of consecutive numbers

$$a = \bar{N}_{k-1}, a+1, \dots, a+s, \dots \tag{3.1}$$

are all $(k-1)$-realizable; in other words, for each s, $s = 0, 1, \dots,$ there exists a geometric $((a+s)_{k-1})$ configuration (recall the definition of realizability, given in the Introduction). Apply affine replication to these configurations; by Lemma 2.2, the sequence of numbers

$$(k+1)a, (k+1)(a+1), \dots, (k+1)(a+s), \dots \tag{3.2}$$

are all k-realizable. Note that this is an arithmetic sequence with difference $(k+1)$. Furthermore, observe that for each $X \geq a$, the geometric k-configuration realizing the number $(k+1)X$ that was produced by affine replication has X new parallel lines. Hence, we can apply a sequence of affine switch constructions to each of these configurations $((k+1)X_k)$.

By Corollary 2.6, the sequences AS+$((k+1)X, k, X)$ of configurations is produced. It follows that the sequences of numbers

$$\begin{aligned}&[(k-1)(k+1)a+1 : (k-1)(k+1)a+a],\\ &\qquad [(k-1)(k+1)(a+1)+1 : (k-1)(k+1)(a+1)+(a+1)],\\ &\qquad\quad [(k-1)(k+1)(a+2)+1 : (k-1)(k+1)(a+2)+(a+2)], \dots\end{aligned} \tag{3.3}$$

are all k-realizable.

Observe that from the initial outputs of affine replication, $n = X(k+1)$ is realizable as long as $X \geq \bar{N}_{k-1}$. Thus, every "band" of consecutive configurations produced by affine switches can be extended back one step, so there exists a band of consecutive k-configurations

$$[(k-1)(k+1)X : (k-1)(k+1)X + X)]$$

for each initial configuration (X_{k-1}). Another way to say this is that we can fill a hole of size 1 between the bands of configurations listed in equation (3.3) using the output of the initial affine replications, listed in equation (3.2).

To determine when we have either adjacent or overlapping bands, then, it suffices to determine when the last element of one band is adjacent to the first element of the next band; that is, when

$$(k-1)(k+1)X + X + 1 \geq (k-1)(k+1)(X+1).$$

It follows easily that $X \geq k^2 - 2$.

Hence, as long as we are guaranteed that a sequence of consecutive configurations $(q_{k-1}), ((q+1)_{k-1}), \dots$ exists, it follows that we are guaranteed the existence of consecutive k-configurations $Q_k, (Q+1)_k, \dots,$ where $Q = (k^2-1)(k^2-2)$. However, since we do not know whether that consecutive sequence exists, in the (extremely common) case where $\bar{N}_{k-1} > (k^2-1)(k^2-2)$, the best that we can do is to conclude that

$$N_k \leq (k^2-1)\max\{\bar{N}_{k-1}, k^2-2\}.$$

□

This result gives rise to an elementary proof by induction for the main theorem.

Proof of Theorem 1.4. Let $s = 2$. The number $N_s = N_2 = 3$ exists. This is the basis of induction. Now, let $s = k-1$. By assumption, N_{k-1} exists and some upper bound $\bar{N}_{k-1}$ is known. By Lemma 3.1, $\bar{N}_k = (k^2-1)\max(\hat{N}_{k-1}, k^2-2)$ is an upper bound for N_k. Therefore N_k exists and the induction step is proven. □

Recall that we let N_k^R denote the best known upper bound for N_k. The same type of result follows if we start with the best known upper bound N_s^R for some $s \geq 2$. However, the specific numbers for upper bounds depend on our starting condition. Table 1 shows the difference if we start with $s = 2, 3, 4$. The reason we are using only these three values for s follows from the fact that only $N_s^R, 2 \leq s \leq 4$ have been known so far.

The rightmost column of Table 1 summarises the information given in other columns by computing the minimum in each row and thereby gives the best bounds that are available using previous knowledge and direct applications of Lemma 3.1.

Table 1: Bounds on N_k from iterative applications of Lemma 3.1. Different bounds are produced if the iteration is started with $N_2^R = N_2 = 3$, $N_3^R = N_3 = 9$ or with $N_4^R = 24$. Boldface numbers give best bounds using this method and current knowledge.

k	$\bar{N}_k$ with $N_2^R = 3$	$\bar{N}_k$ with $N_3^R = 9$	$\bar{N}_k$ with $N_4^R = 24$	N_k^R
2	**3**	-	-	**3**
3	56	**9**	-	**9**
4	840	210	**24**	**24**
5	20 160	5 040	**576**	**576**
6	705 600	176 400	**20 160**	**20 160**
7	33 868 800	8 467 200	**967 680**	**967 680**
8	2 133 734 400	533 433 600	**60 963 840**	**60 963 840**
9	170 698 752 000	42 674 688 000	**4 877 107 200**	**4 877 107 200**
10	16 899 176 448 000	4 224 794 112 000	**482 833 612 800**	**482 833 612 800**

If new knowledge about best current values of N_k^R for small values of k becomes available, we may use similar applications of Lemma 3.1 to improve the bounds of the last column. Since, the values for $k = 2$ and $k = 3$ are optimal, the first candidate for improvement is $k = 4$. A natural question is what happens if someone finds a geometric (23_4) configuration. In this case Lemma 3.1 would give us for $k = 5$ the bound $(k^2 - 1)\max(N_{k-1}^R, k^2 - 2) = (5^2 - 1)\max(20, 5^2 - 2) = 24 \times 23 = 552$, an improvement over 576. An alternative feasible attempt to improve the bounds would be to use other methods in the spirit of Grünbaum calculus to improve the current bound 576 for $k = 5$. However, there is another approach that can improve the numbers even without introducing new methods. It is presented in the next section.

4 Improving the bounds

Recall that $N_3^R = N_3 = 9$, and $N_4^R = N_4 = 21$ or 24, according to whether or not a (23_4) configuration exists. If we apply the procedure in Lemma 3.1 using as input information $N_3 = N_3^R = 9$ (that is, beginning with a sequence of 3-configurations $(9_3), (10_3), (11_3)\ldots$), Lemma 3.1 says that

$$N_k \leq (k^2 - 1)\max\{N_{k-1}^R, k^2 - 2\} \implies N_4 \leq (15)\max\{9, 14\} = 210.$$

However, we know observationally that $N_4 = 21$ or 24. Thus, we expect that Theorem 1.4 is likely to give us significant overestimates on a bound for N_k for larger k.

For $k = 5$, the best we can do at this step with these constructions is the bound given by Lemma 3.1, beginning with the consecutive sequence of 4-configurations $((24_4), (25_4), (26_4), \ldots)$. In this case, Lemma 3.1 predicts that $N_5 \leq (24)\max(24, 23) = 576$. In a subsequent paper, we will show that this bound can be significantly decreased by incorporating other Grünbaum-calculus-type constructions and several ad hoc geometric constructions for 5-configurations.

However, we significantly decrease the bound on N_k for $k \geq 6$ by refining the construction sequence given in Lemma 3.1: instead of beginning with N_{k-1}^R determined by iterative applications of the sequence in Lemma 3.1, we consider all possible sequences determined by applying a series of affine replications, followed by a final affine switch.

First we introduce a function $N(k, t, a, d)$ with positive integer parameters k, t, a, d and $t < k$. Define for $t < k - 1$:

$$N(k, t, a, d) := (k^2 - 1) \left(\frac{k!}{(t + 1)!} \right) \max \left\{ a, (k^2 - 1)d \right\},$$

and for $t = k - 1$:

$$N(k, k - 1, a, d) := (k^2 - 1) \max \left\{ a, (k^2 - 1)d - 1 \right\}.$$

This value $N(k, t, a, d)$ is precisely the smallest n after which we are guaranteed there exists a sequence of consecutive k-configurations produced by starting with an initial sequence of t-configurations $a, a + d, \ldots$, and sequentially applying affine replications followed by a final affine switch as described above.

The following Lemma gives us a quite general and powerful tool for bound improvements without making any changes in constructions.

Lemma 4.1. *Let $t \geq 2$ be an integer and let $a, a + d, a + 2d, \ldots$ be an arithmetic sequence with integer initial term a and integer difference d such that for each $s = 0, 1, \ldots$ geometric configurations $((a + sd)_t)$ exist. Then for any $k > t$ the value $N(k, t, a, d)$ defined above is an upper bound for N_k; i.e., $N(k, t, a, d) \geq N_k$.*

Proof of Lemma 4.1. Beginning with an arithmetic sequence of t-configurations, we construct a consecutive sequence of k-configurations by iteratively applying a sequence of affine replications to go from t-configurations to $(k - 1)$-configurations; a final affine replication to go from $(k - 1)$-configurations to k-configurations with a known number of lines in a parallel pencil; and finish by applying affine switch on that final sequence of k-configurations to produce bands of consecutive configurations. We then analyze at what point we are guaranteed that the bands either are adjacent or overlap.

Specifically, starting with a sequence of t-realizable numbers $a, a + d, a + 2d, \ldots$ we successively apply $k - t$ affine replications to the corresponding sequence of configurations to form sequences of s-realizable numbers for $t \leq s \leq k$:

$$\begin{aligned}
a, a + d, a + 2d, \ldots &\xrightarrow[(t+1)\text{-cfgs}]{\mathrm{AR}(\cdot, t+1)} (t + 2)a, (t + 2)(a + d), (t + 2)(a + 2d), \ldots \\
&\xrightarrow[(t+2)\text{-cfgs}]{\mathrm{AR}(\cdot, t+2)} (t + 3)(t + 2)a, (t + 3)(t + 2)(a + d), \\
&\qquad (t + 3)(t + 2)(a + 2d), \ldots \\
&\vdots \\
&\xrightarrow[k\text{-cfgs}]{\mathrm{AR}(\cdot, k)} \frac{(k + 1)!}{(t + 1)!} a, \frac{(k + 1)!}{(t + 1)!} (a + d), \frac{(k + 1)!}{(t + 1)!} (a + 2d), \ldots
\end{aligned} \tag{4.1}$$

By Lemma 2.2, each of the k-configurations corresponding to the realizable numbers in equation (4.1) produced from a starting configuration X has a pencil of $\frac{k!}{(t+1)!} X$ parallel lines. To those configurations we apply the affine switch operation:

$$\frac{(k+1)!}{(t+1)!}a, \frac{(k+1)!}{(t+1)!}(a+d), \frac{(k+1)!}{(t+1)!}(a+2d), \ldots$$
$$\xrightarrow[k\text{-cfgs}]{\text{AS+}(\cdot,k,\cdot)} \left[(k-1)\frac{(k+1)!}{(t+1)!}a+1 : (k-1)\frac{(k+1)!}{(t+1)!}a + \frac{k!}{(t+1)!}q\right],$$
$$\left[(k-1)\frac{(k+1)!}{(t+1)!}(a+d)+1 : (k-1)\frac{(k+1)!}{(t+1)!}(a+d) + \frac{k!}{(t+1)!}(a+d)\right], \ldots \quad (4.2)$$

As in the proof of Theorem 1.4, observe that the (n_k) configurations described in (4.1) all have n as a multiple of $\frac{(k+1)!}{(t+1)!}$. That is, any n divisible by $\frac{(k+1)!}{(t+1)!}$ is k-realizable as long as when $n = \frac{(k+1)!}{(t+1)!}X$, X is larger than N_t^R. We thus can extend our band of consecutive realizable configurations back one step, to be of the form

$$\left[(k-1)\frac{(k+1)!}{(t+1)!}X : (k-1)\frac{(k+1)!}{(t+1)!}X + \frac{k!}{(t+1)!}X\right]$$

for a starting t-realizable number X.

Successive bands of this form are guaranteed to either exactly meet or to overlap when the end of one band, plus one, equals or is greater to the beginning of the next, that is, when

$$(k-1)\frac{(k+1)!}{(t+1)!}X + \frac{k!}{(t+1)!}X + 1 \geq (k-1)\frac{(k+1)!}{(t+1)!}(X+d) \implies$$
$$X \geq (k^2-1)d - \frac{(t+1)!}{k!}. \quad (4.3)$$

When $t = k-1$, $\frac{(t+1)!}{k!} = 1$, while when $t < k-1$, $\frac{(t+1)!}{k!} < 1$, and moreover, inequality (4.3) holds as long as X is greater than the bound on t-realizable configurations. $\square$

We refine and improve the upper bounds of Table 1 with Theorem 4.2. This proof proceeds by showing, given a starting arithmetic sequence of consecutive t-configurations, a construction method for producing a sequence of consecutive k-configurations.

Theorem 4.2. *Recursively define*

$$\hat{N}_k = (k^2-1)\min_{3\leq t<k}\{N(k,t,\hat{N}_t,1)\}$$

with $\hat{N}_3 = N_3 = 9$ *and* $\hat{N}_4 = N_4^R = 24$. *Then* $\hat{N}_k$ *is an upper bound for* N_k.

Proof. Observe that by unwinding definitions,

$$\hat{N}_k = (k^2-1)\min_{3\leq t\leq k-1}\left\{\frac{k!}{(t+1)!}\max\left\{\hat{N}_t, k^2-1\right\}\right\}.$$

By construction, since for each $\hat{N}_k$ we have shown there exists consecutive k-configurations for each $n \geq \hat{N}_k$, it follows that $N_k \leq \hat{N}_k$, and the result follows. $\square$

Table 2: Bounds on N_k produced from Theorem 4.2. The values for N_k^R given in this table agree with the record values listed in Table 1 for all $k \leq 5$ (boldface), and are strictly better for $k \geq 6$.

k	$\hat{N}_k = N_k^R$	formula	initial sequence
4	**24**	-	-
5	**576**	$(5^2-1)^2$	$t=4$
6	7 350	$6(6^2-1)^2$	$t=4$
7	96 768	$7 \cdot 6 \cdot (7^2-1)^2$	$t=4$
8	1 333 584	$\frac{8!}{5!}(8^2-1)^2$	$t=4$
9	19 353 600	$\frac{9!}{5!}(9^2-1)^2$	$t=4$
10	287 400 960	$\frac{10!}{6!} \cdot \mathbf{576} \cdot (10^2-1)$	$\mathbf{t=5}$
11	3 832 012 800	$\frac{11!}{6!} \cdot 576 \cdot (11^2-1)$	$t=5$
$\vdots$			
24	$\approx 2.85 \times 10^{26}$	$\frac{24!}{6!} \cdot 576 \cdot (24^2-1)$	$t=5$
25	$\approx 8.39 \times 10^{27}$	$\frac{25!}{6!} \cdot \mathbf{(25^2-1)^2}$	$t=5$
26	$\approx 8.02 \times 10^{30}$	$\frac{26!}{6!} \cdot (26^2-1)^2$	$t=5$
$\vdots$			
32	$\approx 3.82 \times 10^{38}$	$\frac{32!}{6!} \cdot (32^2-1)^2$	$t=5$
33	$\approx 1.38 \times 10^{40}$	$\frac{33!}{7!} \cdot \mathbf{7350} \cdot (33^2-1)$	$\mathbf{t=6}$
$\vdots$			
85	$\approx 2.97 \times 10^{132}$	$\frac{85!}{7!} \cdot \mathbf{7350} \cdot (85^2-1)$	$t=6$
86	$\approx 2.63 \times 10^{134}$	$\frac{86!}{7!} \cdot \mathbf{(86^2-1)^2}$	$t=6$
$\vdots$			
109	$\approx 4.04 \times 10^{180}$	$\frac{109!}{7!}(109^2-1)^2$	$t=6$
110	$\approx 4.61 \times 10^{182}$	$\frac{110!}{8!} \cdot \frac{7!}{5!} \cdot (7^2-1)^2 \cdot (110^2-1)$	$\mathbf{t=7}$

Applying Theorem 4.2 results in the bounds for N_k are shown in Table 2.

There are some interesting things to notice about the bounds from Theorem 4.2 shown in Table 2. First, note that $t=3$ is never used in determining $\hat{N}_k$. Second, for example, the bound $\hat{N}_{10}$ uses an initial sequence of 5-configurations, rather than starting with 4-configurations. To understand why, observe that

$$\begin{aligned}
\hat{N}_{10} &= (k^2-1) \min_{3 \leq t \leq 9} \{N(k,t,\hat{N}_t,1)\} \\
&= 99 \min\left\{ \frac{10!}{4!} \max\{\hat{N}_3 = 9, 99\}, \frac{10!}{5!} \max\{\hat{N}_4 = 24, 99\}, \frac{10!}{6!} \max\{\hat{N}_5 = 576, 99\}, \right. \\
&\qquad \left. \frac{10!}{7!} \max\{\hat{N}_6 = 7350, 99\}, \ldots, \frac{10!}{10!} \max\{\hat{N}_9, 99\} \right\} \\
&= 99 \min\left\{ \frac{10!}{4!} 99, \frac{10!}{5!} 99, \frac{10!}{6!} 576, \frac{10!}{7!} \hat{N}_6, \ldots, \hat{N}_9 \right\}
\end{aligned}$$

Since $6 \cdot 99 > 576$ (and the values $\hat{N}_t$ for $6 \leq t \leq 9$ much larger than either), the minimum of that list is actually $\frac{10!}{6!}576$, and the computation for $\hat{N}_{10}$ starts with the sequence of consecutive 5-configurations $(576_5), (577_5), \ldots$ rather than with $(24_4), (25_4), \ldots$. Sequences with $t = 5$ begin to dominate when $6(k^2 - 1) > 576 = (5^2 - 1)^2$; that is, when $k \geq \lceil\sqrt{97}\rceil = 10$. Sequences with $t = 6$ begin to dominate when $7(k^2 - 1) > 6(6^2 - 1)^2 = 7350$, or $k \geq \lceil\sqrt{1051}\rceil = 33$. Sequences with $t = 7$ will dominate when $8(k^2 - 1) > 7 \cdot 6 \cdot (7^2 - 1)^2$, that is $k \geq \lceil\sqrt{12097}\rceil = 110$. However, note that these bounds are absurdly large; $\hat{N}_{110} \approx 4.6 \times 10^{182}$.

In addition, observe that since $k = 25$ is the smallest positive integer satisfying $k^2 - 1 > 576$, the bounds for $\hat{N}_{25}$ use the $25^2 - 1$ choice rather than $\hat{N}_5$ in taking the maximum, even though both $\hat{N}_{24}$ and $\hat{N}_{25}$ are starting with the same initial sequence of 5-configurations, and there is a similar transition again at $k = 86$, when the function is using 6-configurations to produce the maximum. At this position, since $85^2 - 1 = 7224$ and $86^2 - 1 = 7395$, $\hat{N}_{85}$ uses $\hat{N}_6 = 7350$, but $\hat{N}_{86}$ transitions to using $86^2 - 1$ to compute the maximum.

5 Future work

With better bounds N_t^R developed experimentally for small values of t, in the same way that $N_4^R = 24$ has been determined experimentally, we anticipate significantly better bounds N_k^R, for $k > t$, without changing the methods for obtaining the bounds.

One obvious approach is to improve the bookkeeping even further. For instance, in Theorem 4.2 we only used arithmetic sequences with $d = 1$ in $N(k, t, a, d)$ and ignoring any existing configuration (m_t) for $m < N_t$. In particular, for $t = 4$, we could have used $N(k, 4, 18, 2)$ since $18, 20, 22, 24, \ldots$ form an arithmetic sequence of 4-realizable numbers. Our experiments indicate that this particular sequence has no impact in improving the bounds. However, by carefully keeping track of the existing t-configurations below N_t^R, other more productive arithmetic sequences may appear.

Another approach is to sharpen the bounds for N_k, for general k. This can be achieved, for instance, by generalizing some other "Grünbaum Calculus" operations, which we plan for a subsequent paper. We also plan to apply several ad hoc constructions for 5- and 6-configurations to further sharpen the bound for N_5 and N_6, which will, in turn, lead to significantly better bounds for N_k for higher values of k. However, based on the work involved in bounding N_4 and the fact that N_4 is not currently known (and on how hard it was to show the nonexistence of an (19_4) configuration), we anticipate that even determining N_5 exactly is an extremely challenging problem.

Finally, very little is known about existence results on *unbalanced* configurations, that is, configurations (p_q, n_k) where $q \neq k$. While some examples and families are known, it would be interesting to know any bounds or general results on the existence of such configurations.

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The Configurations (13_3)*

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Abstract

There are 2036 configurations (13_3). Here we establish that all of them are geometric; moreover, all have rational coordinatizations in the plane. This supports Grünbaum's conjecture that a geometric (n_3) configuration always has a rational coordinatization.

THIS PAPER IS IN HONOUR OF BRANKO GRÜNBAUM.

Keywords: $(n, 3)$*-configuration, geometric configuration, anti-Pappian, rational coordinatization*

Math. Subj. Class.: *51E20, 51E30*

1 Introduction

An (n_3) *configuration* is an incidence structure consisting of n *points* and n *lines* such that each point is contained in three lines, and each line contains three points. Any two lines are allowed to intersect in at most one point. Two recent reference books on configurations are Grünbaum [4] and Pisanski-Servatius [10]. The current paper builds upon the author's previous papers [7, 8], where one-point extensions and a coordinatization algorithm are presented.

Given an (n_3) configuration, a *one-point extension* is a construction that alters it slightly so as to produce an $((n + 1)_3)$ configuration. This construction is described in [7], where the configurations that can be obtained by it are characterized. The unique Fano (7_3) configuration cannot be obtained from it, nor can the Pappus (9_3) and Desargues (10_3) configurations, but the other two (9_3) configurations and the remaining nine (10_3) configurations can be, as well as all 31 (11_3) and all 229 (12_3) configurations. There is a family of

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Fano-type configurations, defined below, that are not generated by a one-point extension (see [7]). All other (n_3) configurations can be generated by it. The smallest Fano-type configuration is the Fano (7_3) configuration, and the next one is a (13_3) configuration — it will be discussed in Section 4. There is a single Fano-type (14_3) configuration — it will be discussed in Section 5.

The Fano configuration is the unique (7_3) configuration. Denote it by F. Let its points be $\{P_1, \ldots, P_7\}$ and its lines be $\{\ell_1, \ldots, \ell_7\}$. A point P_i can be deleted from F by removing P_i from all lines containing it. This leaves three lines with only two points. Denote the result by F_p, as all P_i give isomorphic results. Similarly, a line ℓ can be removed from all points containing it, leaving three points with only two lines. Denote the result by F_ℓ. Another substructure can be obtained as follows. Choose any P_i and any ℓ_j containing P_i. Remove the incidence between P_i and ℓ_j, leaving one point with only two incident lines, and one line with only two incident points. Denote the result by F', as every P_i and every ℓ_j containing P_i gives an isomorphic result.

Note that an F_p has three lines that "need another point", and F_ℓ has three points that "need another line". We can create a (13_3) configuration by combining an F_p and an F_ℓ which are disjoint by "tying them together", i.e., we add the three points of F_p that need another line to the thee lines of F_ℓ that need another point. The result is a Fano-type configuration (13_3). It is shown in Figure 5. F' can also be used as a sub-configuration to build larger configurations. Note that F' has one point that "needs another line" and one line that "needs another point", so two copies of F' can be tied together to obtain a Fano-type (14_3) configuration, shown in Figure 7. This construction is described in more detail in [7].

Definition 1.1. A *Fano-type* configuration is any (n_3) configuration that can be constructed from a collection of disjoint sub-configurations isomorphic to F_p, F_ℓ or F', by "tying them together".

There are 2036 configurations (13_3) (see [4], P.69). Of these, one is a Fano-type configuration. The remaining 2035 can all be generated by one-point extensions starting from the (12_3) configurations.

An (n_3) configuration is said to be *geometric* if it has a representation as n distinct points and n distinct straight lines in the real plane, such that the incidences in the configuration agree with the incidences in the planar drawing, and there are no additional incidences. The Fano (7_3) and Möbius-Kantor (8_3) configurations are not geometric, whereas all (9_3) configurations are geometric, and all but one (10_3) configurations are geometric. All (11_3) and all (12_3) configurations are also geometric, as shown by Sturmfels and White [11, 12]. Grünbaum has conjectured that every geometric configuration has a planar representation for which all point and line coordinates are rational. Bokowski and Sturmfels [2] and Sturmfels and White [11, 12] have shown that this is the case when $n \leq 12$. Their method is to construct polynomials representing the planar coordinates for each configuration, and then use ad-hoc methods to find rational roots. In this paper we extend this result to $n = 13$, using a coordinatization algorithm in conjunction with one-point extensions, and show that all (13_3) configurations are geometric and rational.

When a geometric (n_3) configuration is drawn in the plane, homogeneous real coordinates are assigned to its points and lines, such that a point with coordinates $P = (x, y, z)$ is incident on a line with coordinates $L = (a, b, c)$ iff $P \cdot L = ax + by + cz = 0$. A one-point

ℓ_1	ℓ_2	ℓ_3	ℓ_4	ℓ_5	ℓ_6	ℓ_7	ℓ_8	ℓ_9	(9_3)
1	4	1	1	2	2	3	3	7	
2	5	5	6	4	6	4	5	8	Pappus
3	6	9	7	9	8	7	8	9	
1	1	2	3	1	2	4	5	4	
2	3	3	7	7	8	6	6	5	$(9_3)\#2$
4	5	6	9	8	9	8	9	7	
1	1	2	3	2	1	5	4	6	
2	3	3	4	5	6	7	7	8	$(9_3)\#3$
4	5	6	9	8	7	9	8	9	

Table 1: The three distinct configurations (9_3)

point	Pappus	$(9_3)\#2$	$(9_3)\#3$
1	(1,0,1)	(2,4,-3)	(3,6,1)
2	(0,0,1)	(-1,1,0)	(3,1,1)
3	(1,0,0)	(1,2,-3)	(1,2,-8)
4	(-1,1,1)	(1,1,-1)	(0,1,0)
5	(1,-2,-1)	(0,0,1)	(0,0,1)
6	(0,1,0)	(1,0,-1)	(2,1,-1)
7	(1,1,1)	(2,2,-3)	(-3,3,4)
8	(0,2,1)	(0,1,0)	(3,1,-4)
9	(1,-1,0)	(1,0,0)	(1,1,-8)

Table 2: Integer coordinatizations of the configurations (9_3)

extension only extends the incidence structure of an (n_3) configuration, not the coordinatization. Using the coordinatization algorithm of [8], when a geometric configuration is derived by a one-point extension from a smaller geometric configuration, the planar drawing of the smaller configuration can usually be extended to a drawing of the configuration in question.

2 The rational coordinatizations

The smallest geometric configurations are the three (9_3) configurations. One of these is the Pappus configuration. Rational coordinatizations of them can be used as starting points for the one-point extension algorithm of [8]. Incidence tables of these three configurations are given in Table 1. Here the points are $1, 2, \ldots, 9$ and the lines are $\ell_1, \ell_2, \ldots, \ell_9$. The three points on line ℓ_i are given in the column labelled ℓ_i. Several rational coordinatizations of the configurations are given in Table 2, as homogeneous integer coordinates.

Starting with these rational coordinatizations, the one-point extension and coordinatization algorithm can be applied to obtain the geometric (10_3) configurations, except for the Desargues configuration. Rational coordinatizations of it can be found, either using polynomials, or by using the tables of [2]. When the one-point extension and coordinatization algorithm is applied to the geometric (10_3) configurations, thousands of rational coordinatizations of the (11_3) configurations result. Applying the algorithm again results in thousands of rational coordinatizations of the (12_3) configurations. Applying it again produces

ℓ_1	ℓ_2	ℓ_3	ℓ_4	ℓ_5	ℓ_6	ℓ_7	ℓ_8	ℓ_9	ℓ_{10}	ℓ_{11}	ℓ_{12}	ℓ_{13}
1	2	6	3	1	8	6	5	4	2	3	5	1
2	3	12	4	7	9	8	6	5	7	9	11	8
4	10	13	7	10	12	11	9	10	13	11	12	13

Table 3: The incidences of configuration $(13_3)\#2035$

ℓ_1	ℓ_2	ℓ_3	ℓ_4	ℓ_5	ℓ_6	ℓ_7	ℓ_8	ℓ_9	ℓ_{10}	ℓ_{11}	ℓ_{12}
1	1	2	3	1	8	6	5	4	2	3	5
4	3	6	4	7	9	8	6	5	7	9	11
12	10	12	7	8	12	11	9	10	10	11	12

Table 4: The incidences of configuration $(12_3)\#15$

thousands of rational coordinatizations of 2034 of the 2036 (13_3) configurations. Two (13_3) configurations are missing. One of them is the Fano-type configuration, $(13_3)\#2036$, treated in the section 4. We denote the other missing configuration by $(13_3)\#2035$. Investigation shows that it derives as a one-point extension from only *one* configuration (12_3). Further investigation shows that the reason it was not generated by the algorithm is that the number of digits in the coordinates was too large. In section 3, the algorithm is carried out manually, using the software Maple [13] with multi-precision integers, to obtain a rational coordinatization.

It is not feasible to list a table of coordinatizations of all (13_3) configurations here, as it would run to a hundred pages. Instead, we refer to a location on the internet [5] where the coordinatizations will be posted.

As n increases, the integers in the coordinates of an (n_3) configuration begin to grow quite large, as can be seen from the tables in the following sections. Additional programming using multi-precision integer arithmetic would be required to apply the algorithm to find rational coordinatizations of the 21,399 (14_3) configurations.

3 The configuration $(13_3)\#2035$

The (13_3) configurations are numbered $(13_3)\#1, (13_3)\#2, \ldots, (13_3)\#2035$, in the order in which they were constructed by the software. There is also a Fano-type (13_3) configuration that cannot be derived as a one-point extension. It is configuration $(13_3)\#2036$.

Given a list of (n_3) configurations with rational coordinatizations, the software that generates the configurations $((n+1)_3)$ by one-point extensions also finds rational coordinatizations. The algorithm is described in [8]. This software was used to find rational coordinatizations of most of the (10_3) configurations, and all of the (11_3) and (12_3) configurations. Tables of configurations and coordinatizations can be downloaded from [5]. When the (12_3) configurations are used as input for constructing one-point extensions, configuration $(13_3)\#2035$ turns out to be quite interesting. Its incidences are shown in Table 3, where the points are $\{1, 2, \ldots, 13\}$, and the lines are $\{\ell_1, \ell_2, \ldots, \ell_{13}\}$. It has a collineation group of order eight. $(13_3)\#2035$ derives as a one-point extension from only one (12_3) configuration, namely $(12_3)\#15$, whose incidences are shown in Table 4. It has a collineation group of order 32.

i	Point i	Line ℓ_i
1	$(-26, 12, 39)$	$(-132, 13, -92)$
2	$(-69, -368, 47)$	$(48, 13, 28)$
3	$(-2, -12, 9)$	$(-2, -12, 9)$
4	$(-38, 60, 63)$	$(18, 3, 8)$
5	$(127, 0, -138)$	$(3, 0, 2)$
6	$(1, 0, -1)$	$(0, 0, 1)$
7	$(-2, 4, 3)$	$(1, 0, 1)$
8	$(0, 1, 0)$	$(0, 1, 0)$
9	$(1, 0, 0)$	$(-2760, 919, -2540)$
10	$(816, -620, -1111)$	$(1292, -113, 1012)$
11	$(3, 4, -3)$	$(0, 3, 4)$
12	$(11, 184, 0)$	$(552, -33, 508)$

Table 5: Integer point and line coordinates of $(12_3)\#15$

We will also need point and line coordinates for $(12_3)\#15$. They are given as homogeneous integer coordinates in Table 5.

We first describe the structure of $(13_3)\#2035$, in terms of some sub-structures.

Definition 3.1. A *complete quadrilateral* in the plane is a set of four distinct lines, no three concurrent, and all six pairwise intersection points. A *complete quadrangle* in the plane is a set of four distinct points, no three collinear, and all six pairwise joining lines.

The Fano configuration (7_3) is shown in Figure 1. As is usual, one line is drawn as a circle, because the Fano configuration is non-geometric. Notice that it consists of a complete quadrangle $\{1, 2, 3, 4\}$, plus the three intersection points $\{5, 6, 7\}$ of the pairs of its six lines: $12 \cap 34$, $13 \cap 24$, $14 \cap 23$, plus a line containing the three points $\{5, 6, 7\}$.

Definition 3.2. Given a complete quadrangle $\{A, B, C, D\}$, the *diagonal points* are the three additional points determined by the intersections of the pairs of lines: $AB \cap CD$, $AC \cap BD$, $AD \cap BC$. Dually, given a complete quadrilateral $\{\ell_1, \ell_2, \ell_3, \ell_4\}$, the *diagonal lines* are the three additional lines determined by the pairs of its six intersection points.

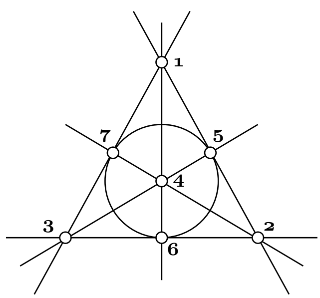

Figure 1: The Fano (7_3) configuration.

The (7_3) configuration and the non-geometric (10_3) configuration, known as the *anti-Pappian*, are both constructed from quadrangles and/or quadrilaterals. See [9, 3] for proofs that the anti-Pappian is non-geometric. It can be constructed as follows, as illustrated in Figure 2.

Construct a complete quadrangle determined by points $\{1, 2, 3, 4\}$, and a complete quadrilateral determined by lines $\{\ell_1, \ell_2, \ell_3, \ell_4\}$. We will associate point i with line ℓ_i, where $i = 1, 2, 3, 4$. The quadrangle has six lines, which can be labelled $\{\ell_5, \ell_6, \ell_7, \ell_8\, \ell_9\, \ell_{10}\}$, in some order. Each of these lines currently has just two points. The quadrilateral has six points of intersection, which are labelled $\{5, 6, 7, 8, 9, 10\}$, such that if line ℓ_i, where $5 \leq i \leq 10$, corresponds to points j and k, where $j, k \leq 4$, then point i corresponds to lines ℓ_j and ℓ_k. Each of $\{5, 6, 7, 8, 9, 10\}$ is currently on just two lines. In order to create a (10_3) configuration out of this substructure, we now place point m on line ℓ_m, where $m = 5, 6, 7, 8, 9, 10$. The result is the anti-Pappian. Thus, it consists of a quadrangle and a quadrilateral, tied together.

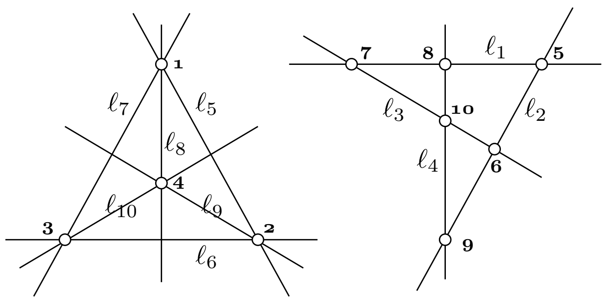

Figure 2: The anti-Pappian.

It is interesting to note that there are several ways of tying together a quadrilateral and a quadrangle. In Figure 2 of Boben, Gévay, Pisanski [1], is illustrated a Desargues configuration which can be viewed as a quadrilateral and quadrangle tied together. Six of the ten (10_3) configurations can be obtained in this way. In fact, the Desargues configuration and the anti-Pappian are very closely related. One can find distinct lines ℓ_1, ℓ_2 and distinct points $P_1 \in \ell_1$ and $P_2 \in \ell_2$ in the Desargues configuration such that, if the incidences $[P_1, \ell_1], [P_2, \ell_2]$ are changed to $[P_1, \ell_2], [P_2, \ell_1]$, the anti-Pappian results!

We now turn to configuration (13_3)#2035. Consideration of Table 3 will show that it contains a complete quadrilateral $\{\ell_1, \ell_2, \ell_4, \ell_5\}$ with its six points of intersection $\{1, 2, 3, 4, 7, 10\}$. In addition, there is a complete quadrilateral $\{\ell_6, \ell_7, \ell_8, \ell_{12}\}$ with its six points of intersection $\{5, 6, 8, 9, 11, 12\}$, as illustrated in Figure 3. In addition, the first quadrilateral has two diagonal lines, namely ℓ_9 formed by joining points 4 and 10, which is extended to contain point 5, and ℓ_{10} formed by joining points 2 and 7. The second quadrilateral also has two diagonal lines, ℓ_{11} formed by joining points 9 and 11, which is extended to contain point 3, and ℓ_3 formed by joining points 6 and 12. The line shaded gray is ℓ_{13}, which intersects ℓ_3 and ℓ_{10} in point 13.

It is also helpful to describe the structure of (12_3)#15. It consists of two quadrilaterals induced by lines $\{\ell_1, \ell_2, \ell_4, \ell_{10}\}$ and $\{\ell_6, \ell_7, \ell_8, \ell_{12}\}$ tied together through their diagonal lines. See Figure 4.

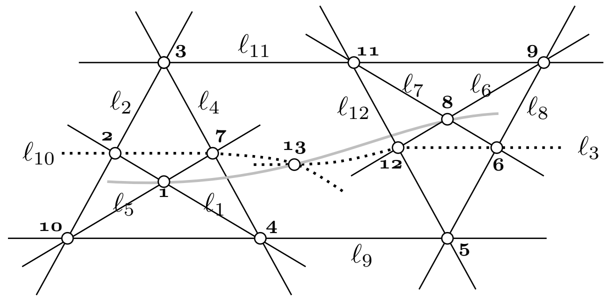

Figure 3: Configuration $(13_3)\#2035$.

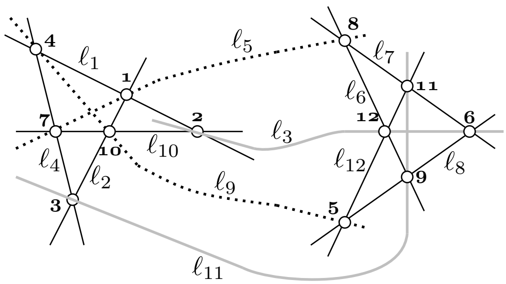

Figure 4: Configuration $(12_3)\#15$

The one-point extension in $(12_3)\#15$ adds a new point 13, and a new line ℓ_{13}, and alters lines $\ell_1, \ell_2, \ell_3, \ell_5.\ell_{10}$ slightly, so that the result is a (13_3) configuration. Refer to Tables 3 and 4. We now construct the incidence graph of the (13_3) configuration, and find three internally disjoint paths in it:

$[\ell_2, 2, \ell_{10}, 13]$
$[\ell_2, 10, \ell_{13}, 13]$
$[\ell_2, 3, \ell_{11}, 11, \ell_7, 6, \ell_3, 13]$

These paths will be used to find a coordinatization of $(13_3)\#2035$. Let the new coordinates of line ℓ_i be L_i, and the new coordinates of point i be P_i. Any point or line not lying on these paths is to have the same coordinates in $(13_3)\#2035$ as in $(12_3)\#15$, which are given in Table 5. We now set the new coordinates of ℓ_2 to be $L_2 = (x, y, z)$, being a homogeneous triple to be determined. Point 2 is incident on ℓ_1, which is not on one of the paths, so that $P_2 = L_2 \times L_1 = (-92y - 13z, 92x - 132z, 13x + 132y)$, which is a triple of linear homogeneous polynomials. Continuing like this, we find coordinates for all points and lines on the three paths as linear homogeneous triples. They are given in Table 6.

Lines $\ell_3, \ell_{10}, \ell_{13}$ must be concurrent in point 13. This gives an equation

$$p(x, y, z) = L_3 \cdot L_{10} \times L_{13} = 0$$

The cubic homogeneous polynomial $p(x, y, z)$ has enormous coefficients. We must choose

	Coordinates
L_2	(x, y, z)
P_2	$(-92y - 13z, 92x - 132z, 13x + 132y)$
P_3	$(8y - 3z, 18z - 8x, 3x - 18y)$
P_{10}	$(-2540y - 919z, -2760z + 2540y, 919x + 2760y)$
L_{11}	$(0, 3x - 18y, 8x - 18z)$
L_{10}	$(224x - 528y - 396z, -26x + 12y + 39z, 184x - 368y - 316z)$
L_{13}	$(88032x - 33120y - 107640z, -23894x + 27300y + 35841z,$ $66040x - 30480y - 82788z)$
P_{11}	$(1788x - 9144y - 594z, 4416x - 9936z, -1656x + 9936y)$
L_7	$(1656x - 9936y, 0, 1788x - 9144y - 594z)$
P_6	$(-1788x + 9144y + 594z, 0, 1656x - 9936y)$
L_3	$(-304704 + x1828224y, 18216x - 109296y,$ $-328992x + 1682496y + 109296z)$

Table 6: Homogeneous linear coordinates for the three paths

(x, y, z) so that $p(x, y, z) = 0$ and the resulting coordinates of Table 6 determine a coordinatization of $(13_3)\#2035$.

Observe that point 3 lies on lines ℓ_2, ℓ_4 and ℓ_{11}. If we choose $L_2 = (x, y, z) = L_4 = (18, 3, 8)$, then P_3 will be $(0, 0, 0)$, so that $p(18, 3, 8) = 0$. Now the equation $p(x, y, z) = 0$ is a curve in the projective plane. The tangent line at point $(x, y, z) = (18, 3, 8)$ has the equation

$$x\partial p/\partial x + y\partial p/\partial y + z\partial p/\partial z = 0$$

where the partial derivatives are evaluated at $(x, y, z) = (18, 3, 8)$. Removing a common factor from the coefficients of this equation results in

$$-2432x + 9048y + 2079z = 0$$

Solve for x in terms of y and z and substitute into $p(x, y, z)$ to obtain a cubic homogeneous polynomial $q(y, z) = 0$. Now the tangent has double contact with the curve at the point $(y, z) = (3, 8)$, so that $q(y, z)$ is divisible *twice* by $8y - 3z$. This division is easy to do. The result is

$$q(y, z) = (8y - 3z)^2(132451464y - 28581427z) = 0$$

We read off the solution (y, z), and use the previous substitution for x to obtain

$$(x, y, z) = (21956086, 28581427, 132451464)$$

These values are substituted into Table 6 to obtain point and line coordinates for $(13_3)\#2035$. The result is shown in Table 7. Here a common factor has been removed from some of the homogeneous coordinates whenever possible, in order to reduce the values of the coordinates.

These values of the coordinates ensure that all incidences of the configuration are satisfied. It is also necessary to check that these coordinates produce no unwanted incidences. This is straightforward by computer.

We summarize this calculation as a theorem.

Theorem 3.3. *Configuration* $(13_3)\#2035$ *is geometric, and has a rational coordinatization.*

i	Point i	Line ℓ_i
1	$(-26, 12, 39)$	$(-132, 13, -92)$
2	$(-2175680158, 1357990332, 3313517991)$	$(21956086, 28581427, 132451464)$
3	$(-2432, 9048, 2079)$	$(-11592, 693, -7652)$
4	$(-38, 60, 63)$	$(18, 3, 8)$
5	$(127, 0, -138)$	$(3, 0, 2)$
6	$(-1913, 0, 2898)$	$(0, 0, 1)$
7	$(-2, 4, 3)$	$(2898, 0, 1913)$
8	$(0, 1, 0)$	$(0, 1, 0)$
9	$(1, 0, 0)$	$(-2760, 919, -2540)$
10	$(-97159859998, 96058923900, 140330458527)$	$(-2295025242, -24998877, -1496684992)$
11	$(21043, -138736, -31878)$	$(0, 693, -3016)$
12	$(11, 184, 0)$	$(552, -33, 508)$
13	$(40949803542, -7065357484, -62674649163)$	$(506964732, 34572915, 327338668)$

Table 7: Integer point and line coordinates of $(13_3)\#2035$

4 The fano-type configuration (13_3)

The Fano-type configurations are constructed by tying together certain substructures of the Fano (7_3) configuration, as described in Definition 1. When an F_p and a disjoint F_ℓ are tied together, the result is a Fano-type (13_3) configuration. We denote it by $(13_3)\#2036$. It is the unique Fano-type configuration on 13 points, as it is the only (13_3) configuration not generated by a one-point extension from the (12_3) configurations. It is illustrated in Figure 5, and its incidence table is Table 8. Here the points are $\{1, 2, \ldots, 13\}$, and the lines are $\{\ell_1, \ell_2, \ldots, \ell_{13}\}$, where the three points in line ℓ_i are those in the column of ℓ_i.

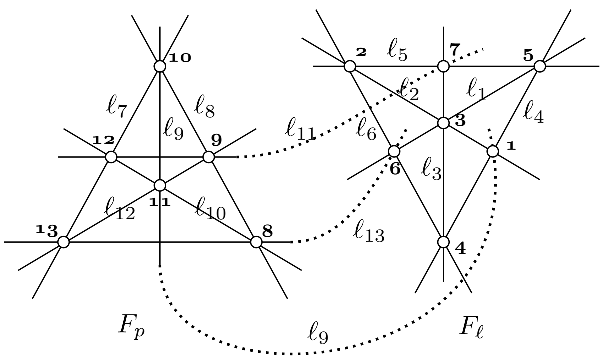

Figure 5: The Fano-type configuration $(13_3)\#2036$.

ℓ_1	ℓ_2	ℓ_3	ℓ_4	ℓ_5	ℓ_6	ℓ_7	ℓ_8	ℓ_9	ℓ_{10}	ℓ_{11}	ℓ_{12}	ℓ_{13}
3	1	3	1	2	2	10	8	1	8	7	9	6
5	2	4	4	5	4	12	9	10	11	9	11	8
6	3	7	5	7	6	13	10	11	12	12	13	13

Table 8: The incidences of the Fano-type configuration $(13_3)\#2036$

We carefully choose a subset of its points and lines, as independent as possible, so that the remaining points and lines are thereby determined. This is called a *determining set* in [6], where the term is defined precisely. Start by choosing points 2,3,4,5, and without loss of generality, assign them coordinates $P_2 = (1,0,0), P_5 = (0,1,0), P_3 = (0,0,1), P_4 = (1,1,1)$. This is possible because the real projective plane is 3-transitive on points if no three are collinear, and because the coordinates are homogeneous. These points completely determine lines $\ell_1, \ell_2, \ell_3, \ell_4, \ell_5, \ell_6$, namely $L_1 = (1,0,0), L_2 = (0,1,0), L_3 = (1,-1,0), L_4 = (1,0,-1), L_5 = (0,1,0), L_6 = (0,1,-1)$. These in turn determine points 1, 6, 7, namely $P_1 = (1,0,1), P_6 = (0,1,1), P_7 = (1,1,0)$. We construct a digraph, called a *construction sequence* for the configuration, whose vertices are the points and lines, and whose arcs indicate which points and lines determine others, e.g., points 2 and 3 uniquely determine ℓ_2. This is illustrated in Figure 6, where all edges are directed from left to right. The table of coordinates is given in Table 9.

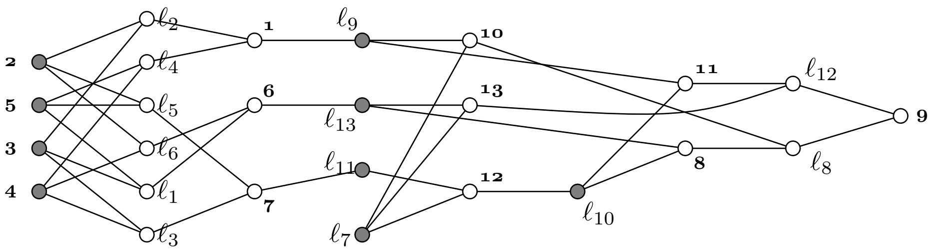

Figure 6: The construction sequence for $(13_3)\#2036$ used to find its coordinatization.

Line ℓ_9 is incident with point 1, so that $L_9 \cdot (1,0,1) = 0$. Consequently $L_9 = (u,p,-u)$ for some values p and u. But $p \neq 0$, for $p = 0$ would imply that L_9 and L_4 are equal. Without loss of generality, we can take $p = 1$. Similarly $L_{13} \cdot (0,1,1) = 0$, from which it follows that $L_{13} = (1,v,-v)$, for some value v. And $L_{11} \cdot (1,1,0) = 0$, from which it follows that $L_{13} = (w,-w,1)$, for some value w. $L_7 = (x,y,z)$ is chosen as part of the determining set. Points 10, 13, and 12 are now determined. L_{10} must be chosen so that $L_{10} \cdot P_{12} = 0$. But we already have $P_{12} \cdot L_{11} = P_{12} \cdot L_7 = 0$, where L_{11} and L_7 are linearly independent, from which it follows that $L_{10} = aL_{11} + bL_7$ for some values $a, b \neq 0$. Then $P_{11} = L_9 \times (aL_{11} + bL_7)$ and $P_8 = L_{13} \times (aL_{11} + bL_7)$ are determined.

We now calculate

$$A = L_9 \cdot P_{12} = uwy - uy + uwx - x + wz - wzu \neq 0$$

$$B = L_{11} \cdot P_{13} = wvz + wvy + wvx + wz + y - vx \neq 0$$

	$L_2 = (0,1,0)$		
$P_2 = (1,0,0)$	$L_4 = (1,0,-1)$	$P_1 = (1,0,1)$	$L_9 = (u,1,-u)$
$P_5 = (0,1,0)$	$L_5 = (0,0,1)$	$P_6 = (0,1,1)$	$L_{13} = (1,v,-v)$
$P_3 = (0,0,1)$	$L_6 = (0,1,-1)$	$P_7 = (1,1,0)$	$L_{11} = (w,-w,1)$
$P_4 = (1,1,1)$	$L_1 = (1,0,0)$		$L_7 = (x,y,z)$
	$L_3 = (1,-1,0)$		

$P_{10} = (z+uy, -ux-uz, uy-x)$
$P_{13} = (vz+vy, -vx-z, y-vx)$
$P_{12} = (-wz-y, x-wz, wy+wx)$
$L_{10} = aL_{11} + bL_7 = (aw+bx, -aw+by, a+bz)$

$P_{11} = (-a-bz+uaw-uby, uaw+ubx-ua-ubz, -uaw+uby+aw+bx)$
$P_8 = (va+vbz-vaw+vby, -vaw-vbx-a-bz, -aw+by-vaw-vbx)$

Table 9: Point and line coordinates for configuration $(13_3)\#2036$

$$C = L_{13} \cdot P_{10} = -z - uy + vux - vuz - vuy - vx \neq 0$$

$$D = L_{11} \cdot L_9 \times L_{13} = -2vuw + vu + vw - uw + 1 \neq 0$$

Note that $D \neq 0$, because the intersection of ℓ_9 and ℓ_{13} does not lie on ℓ_{11}, for this would imply that the Fano configuration is geometric. We then find that

$$L_9 \cdot P_{13} = C, \quad L_{11} \cdot P_{10} = -A, \quad L_{13} \cdot P_{12} = -B$$

This will result in factorization and cancellation in the coordinatizing polynomial, thereby making it possible to find rational roots.

We will also need the formulas

$$L_9 \times L_7 = P_{10} \quad \text{and} \quad L_{13} \times L_7 = P_{13}$$

Then

$$L_{12} = P_{11} \times P_{13} \quad \text{and} \quad L_8 = P_{10} \times P_8$$

These can both be expanded to large polynomial expressions. However, using the identity $(U \times V) \times W = (U \cdot W)V - (V \cdot W)U$, we can also write them as

$$L_{12} = P_{11} \times P_{13} = (L_9 \times (aL_{11} + bL_7)) \times P_{13} =$$

$$= (L_9 \cdot P_{13})(aL_{11} + bL_7) - ((aL_{11} + bL_7) \cdot P_{13})L_9 = C(aL_{11} + bL_7) - aBL_9$$

and

$$L_8 = P_{10} \times P_8 = P_{10} \times (L_{13} \times (aL_{11} + bL_7)) =$$

$$= -(L_{13} \cdot P_{10})(aL_{11} + bL_7) + ((aL_{11} + bL_7) \cdot P_{10})L_{13} = -C(aL_{11} + bL_7) - aAL_{13}$$

Then

$$P_9 = L_{12} \times L_8 = -(C(aL_{11} + bL_7) - aBL_9) \times (C(aL_{11} + bL_7) + aAL_{13})$$

$$= aBC(L_9 \times (aL_{11} + bL_7)) - aAC(aL_{11} + bL_7) \times L_{13} + a^2 ABL_9 \times L_{13}$$

$$= aBC(aL_9 \times L_{11} + bP_{10}) - aAC(aL_{11} \times L_{13} - bP_{13}) + a^2 ABL_9 \times L_{13}$$

point	coordinates	line	coordinates
P_1	$(1, 0, 1)$	L_1	$(1, 0, 0)$
P_2	$(1, 0, 0)$	L_2	$(0, 1, 0)$
P_3	$(0, 0, 1)$	L_3	$(1, -1, 0)$
P_4	$(1, 1, 1)$	L_4	$(1, 0, -1)$
P_5	$(0, 1, 0)$	L_5	$(0, 0, 1)$
P_6	$(0, 1, 1)$	L_6	$(0, 1, -1)$
P_7	$(1, 1, 0)$	L_7	$(2, 4, 3)$
P_8	$(-225, -263, -338)$	L_8	$(4958, 2368, -5143)$
P_9	$(-15809212, -16351336, -22769208)$	L_9	$(2, 1, -2)$
P_{10}	$(11, -10, 6)$	L_{10}	$(86, -80, 5)$
P_{11}	$(-155, -182, -246)$	L_{11}	$(42, -42, 1)$
P_{12}	$(-130, -124, 252)$	L_{12}	$(1850, 5476, -5217)$
P_{13}	$(21, -9, -2)$	L_{13}	$(1, 3, -3)$

Table 10: Rational point and line coordinates for configuration $(13_3)\#2036$

The missing incidence from the construction sequence is point 9 on ℓ_{11}. Thus, there will be a coordinatization of the configuration if $P_9 \cdot L_{11} = 0$. This reduces to

$$abBCP_{10} \cdot L_{11} - abACP_{13} \cdot L_{11} + a^2 ABL_{11} \cdot L_9 \times L_{13} =$$

$$= -abABC - abABC + a^2 ABD = 0$$

Cancelling aAB from the equation leaves

$$aD - 2bC = 0$$

We now look for values of a, b, x, y, z, u, v, w which satisfy this equation, and which make all point coordinates distinct, and all line coordinates distinct. In addition to $a, A, B, C, D \neq 0$, there are a number of constraints which can be written down to aid in this. For example, point 10 $\notin \ell_2$ gives the condition $P_{10} \cdot L_2 = -ux - uz \neq 0$, so that $x + z \neq 0$, etc. There are other conditions like this. Experimentation then leads to a solution with $a = 2, b = 1, x = 2, y = 4, z = 3, u = 2, v = 3, w = 42$. The resulting point and line coordinates are given in Table 10. The algebraic calculations were done using the software *Maple* [13]. The results of this calculation are stated as:

Theorem 4.1. *The Fano-type configuration* $(13_3)\#2036$ *is geometric, and has a rational coordinatization.*

5 The fano-type configuration (14_3)

There is a unique Fano-type configuration with 14 points. It can be constructed as follows. Take two copies of F' (see Definition 1). Each F' has 7 points and 7 lines. Tie them together as shown in Figure 7. The result is a (14_3) configuration.

It seems that most choices of determining set and construction sequence for this configuration lead to an enormous polynomial, for which it appears to be intractable to find roots. But with a judicious choice of determining set, magic happens, and a coordinatization

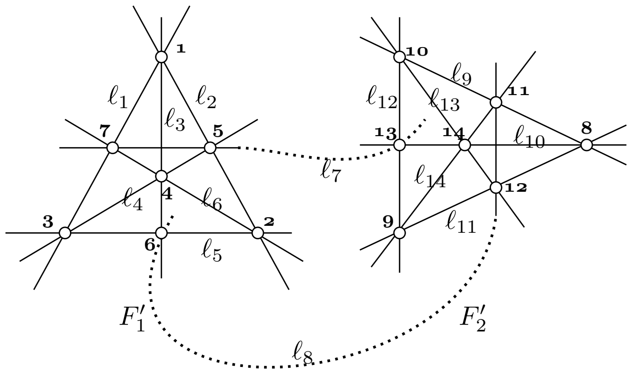

Figure 7: The Fano-type configuration on 14 points.

ℓ_1	ℓ_2	ℓ_3	ℓ_4	ℓ_5	ℓ_6	ℓ_7	ℓ_8	ℓ_9	ℓ_{10}	ℓ_{11}	ℓ_{12}	ℓ_{13}	ℓ_{14}
1	1	1	3	2	2	5	6	8	8	8	9	10	9
3	2	4	4	3	4	7	11	10	13	9	10	12	11
7	5	6	5	6	7	13	12	11	14	12	13	14	14

Table 11: The incidences of the Fano-type configuration on 14 points

can be found. Choose points 1,2,3,4 with coordinates $P_1 = (1, 0, 0), P_2 = (0, 1, 0), P_3 = (0, 0, 1), P_4 = (1, 1, 1)$. This determines lines $\ell_1, \ell_2, \ell_3, \ell_4, \ell_5, \ell_6$, which in turn determines points 5,6,7, and then line ℓ_7. At this point, choose lines $\ell_8, \ell_9, \ell_{11}$ as part of the determining set, with coordinates $L_8 = (u, v, -v), L_9 = (x, y, z), L_{11} = (a, b, c)$. And choose point 14 as part of the determining set, with coordinates $P_{14} = (p, q, r)$. All remaining coordinates are then determined. Refer to Figure 8 and Table 12.

The polynomials for P_9, P_{10} are quite large. Those for L_{12} and P_{13} are enormous. However, some factoring occurs. First note that

$$P_{11} \cdot L_{11} = -P_{12} \cdot L_9 = P_8 \cdot L_8 = (vza - ubz + vya - vcx - vbx + ucy) \neq 0$$

Calculating $P_9 \times P_{10}$, we find that it has a factor of $(vza - ubz + vya - vcx - vbx + ucy)$. As this is non-zero, we cancel it, and obtain a reduced (but still very long) expression for L_{12}. We then calculate $L_{12} \times L_{10}$ and find that it has two additional factors, namely

$$(xp + qy + rz) = P_{14} \cdot L_9 \neq 0$$

$$(ap + bq + cr) = P_{14} \cdot L_{11} \neq 0$$

which can be cancelled to obtain

$$P_{13} = [zpva + ypva - cpvx + 2cpuy - bpvx - 2bpuz + zrbv + yqvc - crvy - bqvz,$$

$$apuz + aqvy + 2aqvz - arvz - xpuc - bqvx + cquy - 2cqvx + crvx - zqub,$$

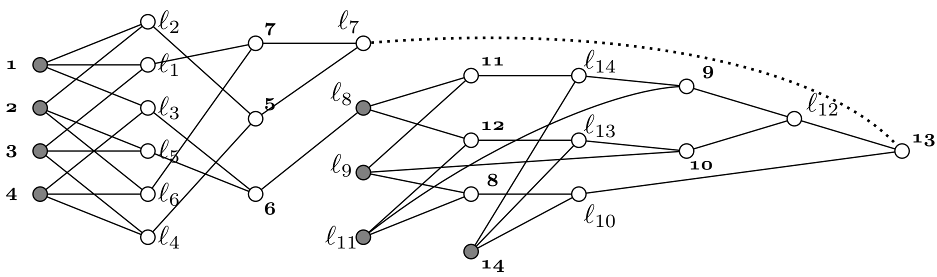

Figure 8: The construction sequence for the Fano-type configuration (14_3), used to find its coordinatization.

	$L_2 = (0, 0, 1)$		
$P_1 = (1, 0, 0)$	$L_1 = (0, 1, 0)$	$P_7 = (1, 0, 1)$	$L_7 = (1, -1, -1)$
$P_2 = (0, 1, 0)$	$L_3 = (0, 1, -1)$	$P_5 = (1, 1, 0)$	$L_8 = (u, v, -v)$
$P_3 = (0, 0, 1)$	$L_5 = (1, 0, 0)$	$P_6 = (0, 1, 1)$	$L_9 = (x, y, z)$
$P_4 = (1, 1, 1)$	$L_6 = (1, 0, -1)$		$L_{11} = (a, b, c)$
	$L_4 = (1, -1, 0)$		

$P_{11} = (vz + vy, -vx - uz, uy - vx)$
$P_{12} = (vc + vb, -va - uc, ub - va)$
$P_8 = (yc - zb, za - xc, xb - ya)$
$P_{14} = (p, q, r)$

$L_{14} = (-rvx - ruz - quy + qvx, puy - pvx - rvz - rvy, qvz + qvy + pvx + puz)$
$L_{13} = (-rva - ruc - qub + qva, pub - pva - rvc - rvb, qvc + qvb + pva + puc)$
$L_{10} = (rza - rxc - qxb + qya, pxb - pya - tyc + rzb, qyc - qzb - pza + pxc)$

Table 12: Point and line coordinates for the Fano-type configuration (14_3)

$$-apuy + arvz + 2arvy - aqvy + xpub + bqvx - crvx - bruz - 2brvx + yruc]$$

We then calculate $P_{13} \cdot L_7$, which must be zero if the configuration is geometric. The result is

$$P_{13} \cdot L_7 = (zpva + ypva + apuy - apuz - 2arvy - 2aqvz - cpvx - bpvx + 2cpuy + xpuc$$

$$-2bpuz - xpub - cquy + 2cqvx + yqvc - bqvz - crvy - yruc + zqub + bruz + 2brvx + zrbv)$$

There are a number of additional identities that must be satisfied, e.g.,

$$P_{14} \cdot L_8 = (up + vq - vr) \neq 0$$

$$P_7 \cdot L_9 = (x + z) \neq 0$$

$$P_6 \cdot L_9 = (y + z) \neq 0$$

$$P_7 \cdot L_8 = (u + v) \neq 0$$

etc.

point	coordinates	line	coordinates
P_1	$(1,0,0)$	L_1	$(0,1,0)$
P_2	$(0,1,0)$	L_2	$(0,0,1)$
P_3	$(0,0,1)$	L_3	$(0,-1,1)$
P_4	$(1,1,1)$	L_4	$(-1,1,0)$
P_5	$(1,1,0)$	L_5	$(1,0,0)$
P_6	$(0,1,1)$	L_6	$(1,0,-1)$
P_7	$(1,0,1)$	L_7	$(1,-1,-1)$
P_8	$(-9065,2345,-105)$	L_8	$(1,2,-2)$
P_9	$(412797,-141180,50641)$	L_9	$(3,117,2354)$
P_{10}	$(336847,-174745,8256)$	L_{10}	$(24710,90335,-115815)$
P_{11}	$(4942,-2360,111)$	L_{11}	$(1,4,3)$
P_{12}	$(14,-5,2)$	L_{12}	$(1844821,3277363,-5901117)$
P_{13}	$(310,137,173)$	L_{13}	$(-74,-134,183)$
P_{14}	$(3,12,10)$	L_{14}	$(-24932,-49087,66384)$

Table 13: Integer coordinates for the Fano-type configuration (14_3)

In order to find a rational solution, we try substituting various values into the variables. Substituting $x = a = p = u = 1$, $v = 2$, $c = 3$, $b = q = 4$, we obtain

$$-39z + 21y - 13ry + 33 + 12rz + 16r = 0$$

Then trying $y = 39$ and $r = 10/3$, we obtain

$$z = 2354/3$$

We then replace (x, y, z) with $(3x, 3y, 3z)$ to obtain integer coordinates for the configuration, as shown in Table 13.

These algebraic calculations were also done using the software *Maple* [13]. It can be verified that the inner products of non-incident points and lines are all non-zero. This gives

Theorem 5.1. *The Fano-type configuration* (14_3) *is geometric, and has a rational coordinatization.*

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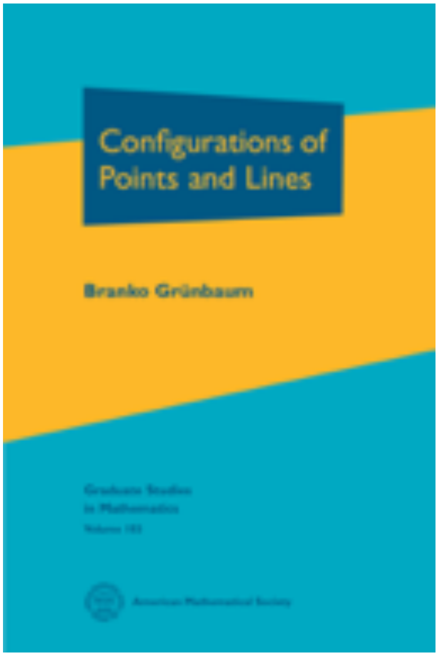
Configurations of Points and Lines
by Branko Grünbaum



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What else could be a simpler geometric figure than a point and a straight line? Yet, using some sets of points and lines, a kind of paradise can be created, and we may say, with Hilbert, that no one shall expel us from this paradise. With Branko Grünbaum's monograph, we have even got a guidebook to it.

The rules for organizing a set of points and lines into a structure, called a *configuration*, are very simple: let p, q, n, k be suitable positive integers, and take p points and n lines such that each point is incident with precisely q of the lines, and each line is incident with precisely k of the points. The type of a configuration with these parameters is denoted by (p_q, n_k). For a configuration with an equal number of points and lines, the more concise notation (n_k) is used, and it is called a *balanced configuration*; if the number k is emphasized, we speak of k-configurations.

The archetypal example is the (9_3) *Pappus configuration*. It is associated with the famous incidence theorem due to Pappus of Alexandria (3rd century A.D.); hence, it also exemplifies that certain configurations originate in incidence theorems.

The second half of the 19th century was an era when configurations caught the attention of many outstanding mathematicians including Burnside, Cayley, Cremona, Plücker, Reye, Schönflies, Steiner, Steinitz, Veronese. This led to the discovery of many (actually, infinitely many) examples of configurations. In the book under review, this period, together with the first decade of the 20th century, is called a "classical period".

In the next eight decades there were only few significant publications on this topic. But in 1990, a new era started that may rightly be called a "renaissance" of configurations. The initiator and the leading figure of this renaissance was Branko Grünbaum. Indeed, he published about twenty papers on configurations, starting with a paper in 1990 written jointly with John Rigby [25]; in addition, he gave graduate-level courses, lectures and talks. This work was crowned by the monograph *Configurations of Points and Lines*.

From the "Beginnings", which is also the title of Chapter 1, the monograph grabs the reader's attention; here in particular, by seven introductory sections with carefully chosen topics. One of them (Section 1.2) is entitled "An informal history of configurations"; some details from here we already mentioned above. It not only places the topic in a historical perspective, but also gives criticism of some earlier results and views.

Various notions regarding the symmetry properties of configurations are discussed in Section 1.5. Symmetry (both on a combinatorial and geometric level) appears in a large variety of forms in configurations, hence the notions introduced in this section are often referred to later in the book; among other things, they play important role in efficient construction methods.

The reader can also enjoy a particularly remarkable feature of the book even in the introductory chapter. Indeed, there are great many examples of configurations, throughout the book, which are presented by beautiful drawings (made by various dynamic geometry software). As the author himself remarks in the Preface, *"this is practically inevitable considering the topic"*.

The most well-known configurations are the (n_3) configurations. They are the subject of Chapter 2, the most extensive chapter of the book. The first part of this chapter is devoted to a careful, critical review of early results, going back more than a century. Not only are some deficiences of the original works pointed out and discussed, but relevant recent results are also presented. In particular, a detailed discussion of some parts of the 1894 doctoral thesis of Steinitz is included.

Two fundamental problems appear here: the enumeration of configurations, and the existence/nonexistence problem. Both occur on three different conceptual levels; namely, a sharp distinction is to be made between *combinatorial*, *topological*, and *geometric* configurations. Every geometric configuration has an underlying abstract incidence structure called a combinatorial (or abstract) configuration. Assume we are given a combinatorial configuration $\mathcal{C}$ and a geometric configuration $\bar{\mathcal{C}}$ such that they are isomorphic (informally speaking, this means that they have the same incidences). In this case we say that $\bar{\mathcal{C}}$ is a *geometric realization* of $\mathcal{C}$. Here we face the problem that combinatorial configurations are more abundant than geometric configurations; in other words, not every combinatorial configuration has a geometric realization. For example, the smallest combinatorial configuration is the (7_3) Fano configuration (also well known e.g. in finite geometry, where the term *Fano plane* is used). The second is the (8_3) Möbius-Kantor configuration. But neither of these can be realized geometrically, with straight lines.

Topological configurations represent an intermediate level: instead of lines, they have *pseudolines*, which are curves mimicking the lines of the real projective plane by their property that any two of them have a unique point of intersection. For a topological configuration which has a geometric realization, Grünbaum uses the term *stretchable*. Here we see again that not every topological configuration is stretchable.

The first enumeration results on 3-configurations occurred near the end of the 19th century, mainly due to Daublebsky von Sterneck, Kantor, Martinetti and Schröter. In more recent research, computer programs are used. For example, Sturmfels and White in 1990 [30], using methods from computer algebra, confirmed the result of Daublebsky who stated that there are 228 nonisomorphic types of combinatorial (12_3) configurations. Nearly a century after Daublebsky's result, Gropp pointed out that there is one additional type; Sturmfels and White also confirmed this, and now the list with 229 types is complete. Grünbaum went on and posed the question whether every geometric configuration has a planar representation for which all point and line coordinates are rational. This is answered in the affirmative for (12_3) configurations, also by Sturmfels and White. We note that this work continues also at present: by a most recent result, all the 2036 combinatorial (13_3) configurations found by Gropp [23] have rational geometric realization (cf. the paper of Kocay in this issue [26]). It seems that at present this is the most that is known about realizability of (n_3) configurations; besides, it also supports Conjecture 2.6.1 by Branko Grünbaum formulated in his book as follows:

Conjecture. *Every* 3*-connected combinatorial* 3*-configuration admits geometric realizations by points and straight lines with no incidences except the required ones.*

In addition, a list of the numbers of combinatorial (n_3) configurations is also known up to $n = 19$, and is included in Table 2.2.1, together with data obtained by various computational methods, among others by Betten, Brinkmann and Pisanski [8].

Four sections in the last part of Chapter 3 give an introduction to astral configurations, a remarkable class of highly symmetric objects of plane geometry. Symmetry is meant here as Euclidean symmetry; that is, the symmetry group of a configuration is the group of Euclidean isometries that map the configuration to itself (this does not conflict with the fact that configurations are sometimes considered as embedded into the projective plane, as it is explained in Section 1.5). Since a configuration consists of finitely many points and lines, this group can only be either the dihedral group $\mathbf{D}_m$ or the cyclic group $\mathbf{C}_m$ (here we use the old Schönflies group notation). An (n_k) configuration may have either of these groups as its symmetry group. If, in addition, both its set of points and set of lines decomposes into $h = \lfloor (k+1)/2 \rfloor$ orbits under the action of this group, then it is called a h-astral configuration. This definition has already been introduced also in Section 1.5, together with some refinements in several directions. Here astral 3-configurations with cyclic as well as with dihedral symmetry groups are studied, in two separate sections. Multiastral configurations (i.e. those where the number of orbits of points and of lines is not specified) are also presented (and this is the one of several places of the book where an important and closely related family called *polycyclic configurations* [10] is mentioned as well). Finally, some duality problems of astral 3-configurations are discussed.

The title of the next chapter is "4-Configurations", and ten sections are devoted to the subject. The gaps in our knowledge compared to 3-configurations is emphasized directly at the beginning. In Theorem 3.2.3 the result of Bokowski and Schewe [14] is cited by which there are no geometric (n_4) configurations for $n \leq 17$. In the same paper the authors also provided the first example of a geometric (18_4) configuration. (We note that here the contrast between combinatorial and geometric configurations is striking: there are precisely 971171 isomorphism classes of combinatorial (18_4) configurations!) However, the question of the next case, (19_4), still remained open. In fact, not long after publication of Branko Grünbaum's book it was proved by Bokowski and Pilaud that geometric (19_4) configuration does not exist [13]. Just in that period, the research for small geometric 4-configurations was particularly active. The current state of knowledge can briefly be summarized as follows: geometric (n_4) configurations exist for all $n \geq 18$ except possibly $n = 23$; the existence of (23_4) is still undecided. A brief summary of how this knowledge was acquired step by step, due to the works of Jürgen Bokowski, Michael Cuntz, Branko Grünbaum, Vincent Pilaud and Lars Schewe, can be found in the introductory part of the paper [6] in this issue; some other details of the story are also referred to (besides the original research articles) in the contribution [11], and in the newly published book by Bokowski [17].

Naturally, only the first part of this period is accounted on in the book under review. Even that account is a captivating reading with many interesting details, in particular with outlines of some proofs taken over from the original contributions.

Here an important characteristic of this research is to be stressed; it is the extensive use of computer-aided methods. On the one hand, this means the application of efficient techniques based on the theory of oriented matroids, mainly in the works of Bokowski and his co-authors. On the other hand, the computer is an indispensable tool in applying these techniques (for example, in the paper [12] the authors even mention that obtaining one of their results needed several months of CPU-time). In the present book Branko Grünbaum does not go into such details (it would fill another book); instead, he refers to relevant sources already in Chapter 1, regarding in particular the theoretical basis of Jürgen Bokowski's methods [16, 15].

In a next section of this chapter a collection of operations is described by which one can build new configurations from old ones. These are ingenious constructions with clever application of various isometric, affine and projective transformations. As in general in the book, they are also illustrated by spectacular examples. The set of these operations has been proved to be a valuable toolkit for constructing many new (n_k) configurations, even beyond the case of $n = 4$, so that later it is presented under the name *Grünbaum Incidence Calculus* in the monograph by Tomaž Pisanski and Brigitte Servatius [29]. It has already been applied in the very recent contribution [6] in proving results for the existence of (n_k) configurations; in particular, now we know that there is a bound such that for any $n \geq 576$ there exists a geometric (n_5) configuration. (Here we note that the smallest known (n_5) example is (48_5) [7]; in the time of writing Branko Grünbaum's book it was so new that it was cited there as a private communication, and only depicted in Figure 4.1.5. The question of existence of a smaller example is still open, and it is considered so important that e.g. Jürgen Bokowski puts it in his book as one of the "beautiful questions" in geometry, and devotes to it a small section [17, Section 2.8]; we also note that finding a smaller example would certainly reduce the bound 576 mentioned above.)

The second part of Chapter 3 deals again with astral configurations, more closely, with a particularly interesting subclass of them. This class is distinguished nowadays by the term *"celestial configurations"*, although the author in the book only mentions this name. In fact, it occurred for the first time in the work by Leah Berman [3] (former PhD student of Branko Grünbaum). Since this class has been studied very extensively, and is the most well-understood class of 4-configurations, it deserves citing its definition here. A h-astral

(n_4) configuration $\mathcal{C}$ is called *celestial* if the following conditions hold: (1) $h \geq 2$ and $n = h \cdot m$ for some $m \geq 7$; (2) the relative position of the h orbits of points of $\mathcal{C}$ is such that all angles subtended by these points from the centre of $\mathcal{C}$ are multiples of π/m (recall that since the symmetry group of $\mathcal{C}$ is finite, it has a unique common fixed point; thus the centre of $\mathcal{C}$ is naturally identified with this point); (3) each line of $\mathcal{C}$ contains two points from each of two point orbits, and likewise, each point is incident with two lines from each of two line orbits.

The particular action of the symmetry group determined by the third condition makes this class a really remarkable subject of study. Indeed, as it is emphasized in Section 3.5, we have an "easily implementable decision algorithm for checking the membership of either a given configuration to the class, or of a symbol for correspondence to a geometric configuration". The symbol mentioned here is called a "configuration symbol", and it is subject to certain axioms. All these properties are widely utilized in the subsequent sections, separately on 2-astral, 3-astral and k-astral $(k \geq 4)$ configurations. Moreover, the research in this direction continues later on, see e.g. [3, 1].

The title of Chapter 4 is "Other Configurations". Here the author overviews what is known on k-configurations for $k \geq 5$ and on unbalanced configurations. The simple reason that all these configurations can be reviewed in a single chapter (in contrast to the case of 3- and 4-configurations) is the "paucity of knowledge" in this case, as the author admits at the beginning of the chapter.

Some combinatorial aspects are also mentioned here (related to 5-configurations), but to a much lesser extent than in the previous chapters. For example, an interesting combinatorial property is being cyclic. A (p_q, n_k) configuration is *cyclic* if its points can be identified with the elements of the (abstract) cyclic group $\mathbb{Z}_p$ and its set of lines $\mathcal{L}$ is invariant under the action $x \mapsto (x+1) \bmod p$ (in case of a combinatorial configuration $\mathcal{L}$ is the set of abstract lines, or *blocks*). As it is noted in Section 2.1, study of configurations with this property goes back to Levi [27]. Here the contrast between the combinatorial and geometric side of configurations is even more striking: while cyclic combinatorial configurations are a subject of intensive research (see e.g. the recent paper [18] and the references therein), only a few papers deal with cyclic geometric configurations (as a more recent example, see [4]).

Similar differences can be seen in case of unbalanced configurations. For example, there are enumeration results from which we know that the number of nonisomorphic types of combinatorial $(15_6, 30_3)$ configurations is 10177328 (see e.g. Table 7.18 in [24] with data also for other types therein). Such results for geometric configurations appear rather rarely; here a paper by Leah Berman is cited, which reports, among other results, the determination of many $[6, 4]$ configurations (these are configurations in which each point is on six lines and each line passes through four points). Some infinite sequences of unbalanced configurations are also known, obtained mainly by combining the construction methods described earlier.

The situation is even worse regarding (n_k) configurations with $k \geq 6$; more precisely, it was so in the time of writing the book. In Section 4.2, dealing with these configurations, the author explicitly complains about the lack of relevant contributions. First he mentions an interesting observation (which occurs at several places in the book, first in Section 1.1) that geometric $((k^k)_k)$ configurations exist for all k (some graph-theoretical aspects of these configurations are discussed by Tomaž Pisanski [28], where they are called the "generalized

Gray configurations"). As a consequence, we know that for an arbitrary integer k there exists a k-configuration.

But it turns out that the same is known due to a much older observation by Cayley. Indeed, Cayley pointed out in 1846 that for all binomial coefficients of the form $B^{(k)} = \binom{2k-1}{k-1}$ with $k \geq 3$ there exists a $((B^{(k)})_k)$ configuration (we note that for $k = 3$ this is precisely the well-known Desargues configuration). (Much later, Cayley's idea was rediscovered, independently, by Ludwig Danzer, and has been elaborated in more detail by Boben, Gévay and Pisanski [9, 20].)

In the same section on (n_k) configurations with $k \geq 6$ results only from one additional contribution are mentioned [2]. One of them is the theorem stating that no 3-astral 6-configuration exists.

Some examples are also mentioned demonstrating how the "$(5m)$ construction" introduced in Section 3.3 can be generalized so that starting from any (m_k) configuration, it yields a $(((k+2)m)_{k+1})$ configuration. An interesting example is (880_7) obtained from a (110_6) configuration, presented here as the smallest known 7-configuration. We note that not much later this record was improved by reporting the existence of a (288_7) configuration [5]. Together with an example of type (96_6) reported in the same paper, it is still the known record-holder. Thus again, here one may put the following "beautiful question" (in the spirit of the book [17]).

Question. Does there exist an (n_6) configuration for $n < 96$, or an (n_7) configuration for $n < 288$?

As we noted previously, the $(5m)$ construction (and its generalization, utilized also in [6]) forms part of the "Grünbaum calculus". At the time of writing the book under review, this construction was the only known operation that can be applied to any configuration (n_k) for increasing the incidence number k. But not much later, a new binary operation was introduced, namely, the Cartesian product. This can be applied to any two (balanced or unbalanced) configurations, with the only restriction that the point/line incidence number must be same in both configurations; hence, to be precise, this is a *partial operation* (this term borrowed from universal algebra). It was defined independently for combinatorial configurations by Pisanski and Servatius [29], and for geometric configurations by Gévay [19].

By this operation, it is easy to obtain highly incident configurations, in fact those in which the incidence numbers exceed any bound; but in searching for minimal examples (such as in the question above) it can hardly be used, since the number of points grows very fast in comparison to the incidence number k. (A similar drawback occurs, to a lesser extent, for the generalization of the $(5m)$ operation as well.) Thus, novel clever and sophisticated methods are still badly needed, and hopefully, the wealth of ideas in the book by Branko Grünbaum will give inspiration in this direction as well.

The next two sections are concerned with the results of (at the time) quite recent research on "floral" configurations and topological configurations; both form promising subjects of further study.

The last section is on "unconventional configurations", necessarily a narrow selection from a large topic (that is essentially beyond the scope of this book). The general formula (p_q, n_k) giving the numbers of the elements and the mutual incidence numbers still applies here, precisely in an analogous way as in the case of configurations of points and lines; but instead of lines some other geometric constituents (e.g. circles, or planes, etc.) occur in the given structure. On the other hand, if lines are retained as second constituents, then one may consider configurations consisting of infinitely many points and lines.

In the first case, the examples are restricted to configurations of points and circles, and the beginnings are emphasized. Indeed, this topic goes back to the nice incidence theorem due to August Miquel (1844), which gives rise to a configuration of type $(8_3, 6_4)$. As a famous example, the infinite sequence found by Clifford in the second half of the 18th century is also mentioned; this consists of point-circle configurations, all related to incidence theorems. We note that more recent results on configurations of points and circles, together with further details on the Clifford configurations, appear in the paper by Gévay and Pisanski [22]. Moreover, in a quite recent paper other "unconventional configurations", namely, configurations of points and conics are also studied [21].)

The last chapter is entitled "Properties of Configurations", and covers eight distinct topics. Each one is so interesting that it would deserve a separate paper, but here we restrict ourselves to mention some details on one of them, the *dimension* of a configuration.

All configurations throughout the book are "planar" in the sense that they considered as embedded in either the Euclidean or in the projective plane. This may give the false illusion that they are confined in fact to one of these planes. However, closer scrutiny shows that some of these configurations, by an isomorphic but different arrangement of their points and lines, are able to span a space of dimension higher than two. The precise definition is the following. We say that a configuration $\mathcal{C}$ has dimension d if this is the largest integer for which $\mathcal{C}$ admits a geometric representation (by points and straight lines) in some Euclidean space, such that the affine hull of the embedding has dimension d.

Recall the well-known fact that the (10_3) Desargues configuration can be constructed in such a way that it spans a 3-dimensional space. In this section the author gives a short proof that the dimension of this configuration is 3. A theorem is also proved which states that there exist 3-configurations with arbitrary large dimensions. We note that the construction used in this proof is essentially a spatial version of one of those in the Grünbaum calculus and which is called "parallel switch" in [29]. This raises the question of the effect of various operations mentioned earlier on the dimension of the configurations involved. (The same question can be put related to the numerous spatial constructions given in [19], including the Cartesian product.)

The book is completed by a particularly extensive list of references, with back-references to pages of occurrence. We reproduced here some of them intentionally, and added some more recent ones. With the latter, and also with our remarks regarding the later developments, our aim was to indicate that the research in many directions presented in the book continues unbroken in the time elapsed since its publication. We are certain that it will be even more so in the future, and many research mathematicians will draw inspiration from it. Besides, we warmly recommend it to anybody who is delighted by the beauty of Geometry.

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Branko Grünbaum: the mathematician who beat the odds

Branko and Moshe, April 2018.

Branko Grünbaum was born in 1929 in Croatia. He survived the Great Depression that hit Yugoslavia when he was six, the Nazis' invasion and the holocaust when he was 12 (his father was Jewish), Marshal Tito's Communist regime when he was 16, and a new start from practically nothing in Israel when he was 19. These were the odds that Branko overcame.

He built a brilliant mathematical career. I knew him as a thesis advisor, teacher, mentor, colleague and close friend for 54 years.

The life story of Branko and his mathematics is fascinating. I was fortunate to know him and Zdenka, his wife, from 1964 until he passed away in 2018. Most of the following details I learned directly from them.

1 Yugoslavia 1929 – 1941

Branko was born in Osijek, Croatia in 1929. Vlado, Branko's father, was Jewish. His mother was Margareta Banderier. Her parents, Emma and Gustav Banderier, were French. She was Lutheran and he was Catholic. In the absence of civil marriages they were married in a Catholic church. This required them to register all their children as Catholics (*"not something to be excited about"*, Branko commented).

Gustav was employed by a French company that was supplying oak staves to French wineries. He managed a sawmill near Normanci, about 30 km from Osijek.

Vlado's father died young, leaving his widow Flora with two daughters and Vlado, the middle child. She supported the family as a hairdresser. These circumstances denied Vlado his study aspirations to become a doctor; he finished his studies in a commercial college. In 1925, a drunken dissatisfied worker assassinated Gustav. Vlado was promoted to manage the sawmill. During this period, a young German was staying with them. He was processing and sending lumber orders to Germany. In his spare time, he was playing with young Branko who learned German. This ended in 1935 when the Great Depression reached Europe and Yugoslavia and the French company who owned the sawmill went bankrupt.

Emma received some compensation from worker's insurance and a small pension from the French government. She bought a small house in Osijek, and supplemented her income by tutoring local high-school students in French and German. Branko continued his German "studies" and added French while playing under the table when Emma was tutoring.

The years 1935 – 1938 were tough for the family. In 1938 Vlado got a job managing a sawmill near Klenak, on the Sava river and the family moved to Sava, a town across the river.

2 Yugoslavia 1941 – 1945

Axis forces, led by Nazi Germany, invaded Yugoslavia on April 6, 1941. Only four days later, the Ustaše, an ultra-nationalist, racist organization declared the Nezavisna Država Hrvatska (NDH – The Independent State of Croatia), a puppet state of Nazi Germany. By the end of the month racial laws, targeting mostly Serbs, Jews and Roma were introduced. This sealed the tragic fate of many.

All Jews and former Jews, 14 years and older, were ordered to register with the authorities and wear a yellow armband with a black Star of David. Eventually, almost all were transported to concentration camps. Of the 40 000 Jews who lived in Croatia in 1941, fewer than 3 000 survived.

Branko and his family were among the few who survived. While the persecution of Jews in Osijek was proceeding, a decree by the Catholic Church in Zagreb exempted Jews in mixed marriages from registering as Jewish. They were permitted to maintain their daily routines and did not have to wear the Nazi armband. These "semi-jews" were ordered to provide basic services to the detainees in Jasenovac, one of the ten largest camps in Europe, and other concentration camps. When they walked into the camps, they were never sure whether they would be allowed to exit. Vlado, Branko's father, was one of the leaders of this group. In the early days they were able to pretend to take children to medical care, but in reality they managed to send them to safety in Israel via Italy. They risked their lives.

In 1978, in Jerusalem, Vlado was spotted and approached by a group of adult women from Jerusalem and Kibbutz Nirim in Israel whom he carried as children out of camp Đakovo and helped send to Israel via Italy. On June 12, 1978 Vlado and his wife Margareta received in Jerusalem a distinguished award for acting under great danger to themselves and their family during WW-II.

3 Yugoslavia 1945 – 1949

In May 1945, Marshal Tito's partisans liberated this part of Yugoslavia. To support the family, Branko's father and others started a successful lumberyard. Suddenly, Vlado was arrested. There were no charges or a trial. After about a month, he was released. The "happy" freedom was secured when Vlado donated his part of the lumberyard to the city. Young Branko started thinking about leaving Yugoslavia, but the borders had been closed.

Zdenka Bienenstock was born in Osijek, Croatia in 1930. Zdenka grew up in a Jewish family, her father owned a sporting goods store and Zdenka enjoyed with her father all the goods the store had to offer, swimming, hiking, ice-skating etc. *"She was quite a tomboy"* Branko commented. Like many Jewish families, Zdenka's family wanted to leave but no country was willing to accept them. They thought that converting to Catholicism might help. Dr Alfred Hoelender was a Jew from Germany who was ordained as a priest in 1940. He served in the Osijek and Đakovo Archdioceses where he oversaw the conversion, but the Nazis and Ustaše did not care. They were ordered to register with the authorities, wear the yellow armband and eventually, were transported to concentration camps.

As a Jew, Zdenka was humiliated in front of the whole public school population and ejected from the first grade of middle school. She lost the right to attend any school, which was a big blow and an ever-present incentive to complete her education. This was 11 years old Zdenka's cruel end of her childhood.

To increase Zdenka's chance of survival, the priest who converted the family brought her to the Catholic convent of the Sisters of the Holy Cross in Đakovo. Zdenka saw her parents for the last time during the winter of 1941 during a short recess from school. Harboring Jews was very risky. Zdenka could not officially attend a public school or the convent vocational school. To hide her identity, she was assigned a new last name, officially identified as an orphan form Bosnia. Only the mother superior knew she was Jewish.

Zdenka had spent four years in the convent under assumed new identity, last name and false documents obtained by Mother Superior. Zdenka was isolated from the world and her family. She spent four years in a large dormitory, with false documents and "new" identity. She shared a dormitory room with 23 girls, each with a bed, a night stand and separated from the others by a curtain. She took some classes in the convent school: math, geography, history, language and "a lot of sewing and embroidering" but with no documentation. The convent closed in 1945. Fifteen years old Zdenka was taken back to Osijek by Olga Mrljak, her mother's friend.

She started searching for her family, and she discovered that they all had been sent to Auschwitz. Although the three buildings her family owned were occupied, she was entitled to collect rent, which became her source of financial support. After a turbulent stay with a government-appointed guardian, Zdenka was permitted in 1947 to occupy her former bedroom but not the rest of the house. She was "adopted" by a friend's family and enrolled in a special government program that enabled her to do two years of study in one year.

Very few young Jews were left in Osijek after the war. About eight high school students connected with each other and began to meet frequently. This was where Branko and Zdenka met.

"Zdenka had difficulties with the mathematics she was supposed to study. I offered to help. She proved to be extraordinarily bright" Branko told us. The young couple fell in love and in early 1948 they considered marriage. *"At least wait until you have some means of supporting yourselves"* Vlado, Branko's father, suggested.

In October 1948, after finishing high school and successfully passing his matriculation exams, Branko went to study mathematics and physics at the University of Zagreb.

At the first day of school in an assembly of all students the leadership established resolutions about appropriate student behavior. By one of the proposed resolutions, an excellent student who was about to graduate was expelled from the university and all Yugoslav universities. His "crime": Insufficient ardor in the study of Marxism-Leninism.

Following this experience, Branko renewed his earlier thoughts about leaving Yugoslavia, but Yugoslavia's borders were closed. Despite the political constraints at the university, Branko was able to pursue his love for geometry under the instructions of Professor Stanko Bilinski. It came as a big surprise when in the summer of 1948 the government announced that Jews wishing to immigrate to Israel should register and would be provided with a ship to take them. Branko felt that this was a government attempt to identify people disloyal to the communist regime, so he did not register. However, in December, registered people were transported to Israel, which was in the middle of its War of Independence. When it was announced that another ship would be provided, Branko convinced his family to emigrate and take Zdenka with them.

4 Israel 1949 – 1957

Zdenka's recollections:

"We joined the second 'aliyah' in July 1949. The Yugoslav authorities required that we surrender our identification cards, renounce our citizenship, and abandon without compensation all property, such as our houses. They inspected our crates to prevent export of valuables from the country. The few items of my mother's jewelry that Ms Mrljak saved I sewed into the seams of my coat.

I was 18 years old and had finished seven out of eight grades of high school. I did not have a high school certificate and did not speak or read Hebrew. I could not take any money and did not know how I would survive. I was not strong enough for manual labor, and I had no marketable skills. But I had found Branko, the love of my life, and I was following him to Israel. We left Osijek by train to Rijeka, boarded the ship Radnik II and landed in Haifa on July 25th 1949."

When they landed, Israel was recovering from its War of Independence. The country was poor, food was rationed and was not prepared to receive the flood of immigrants. The extended Grünbaum family was taken to Atlit where they underwent the initial processing and instantly became Israeli citizens. Branko, his parents, grandmother, aunt, his cousin and Zdenka shared an eight-person tent. There was a communal kitchen where they received three meals a day, communal showers (cold water only) and communal toilets.

After three weeks they were transferred to another camp in Herzlia and after six weeks were transferred to their two one-room "panelaks" in the sand dunes of Rishon Lezion (near Tel Aviv). Regardless of the tough circumstances, Zdenka and Branko were determined to follow their dreams and get a good education. After learning five languages in Yugoslavia (Serbo-Croatian, German, Russian, French and English), Branko, and Zdenka, had to learn Hebrew. They also needed to secure a source of income. Vlado joined a cooperative of mechanics from Osijek who received a grant to open a car repair shop. He arranged for Zdenka a tuition-free admittance to the local high school in Rishon Lezion. Due to her language deficiency, she was required to repeat the 11^{th} grade and had a provisional acceptance to the 12^{th} grade.

Zdenka spent 18 hours each day studying. Her grades in the sciences, math and Hebrew grammar were very high, so she was permitted to take the matriculation exams. She passed them all except Hebrew composition. She was allowed to repeat the test again by the end of the summer. Zdenka sold her mother's jewelry to pay for a tutor in Hebrew composition. By the fall of 1951 she successfully completed her matriculation exams and was admitted to the chemistry school at the Hebrew University in Jerusalem.

Branko found employment as an "errand boy" in a shop in Tel Aviv that sold and serviced parts for textile machinery, an important part of the emerging economy of Israel. His job was delivering frames and combs to customers. Repairs and servicing the combs was done by a highly paid person who came to the shop intermittently. Branko watched him, and quickly learned how to do it. He offered to do the repairs himself and the store owner promoted him and increased his salary.

It was customary to take a three hours break in mid-day. Branko used that time to learn Hebrew. He found a tutor, a retired lawyer, who was passionate about the highly rational and algorithmic structure of the complicated Hebrew grammar. Branko found it very attractive. *"It was an excellent fit for me. I learned only a few words, but with them I could do all kind of verbs, nouns and sentences."*

In the fall of 1950, Branko was admitted provisionally to resume his mathematics studies at the Hebrew University in Jerusalem, with full admission status dependent on his performance. By then he determined that mathematics attracted him more than physics. It was an ideal solution for Branko. Through studying books, he became quickly familiar with the topics of most courses. He had no problem with the lectures in Hebrew as they were taught by newcomers who spoke Hebrew slowly. There was one exception, Professor Aryeh Dvoretzky, who was fluent in Hebrew. Branko skipped his probability class.

These are Branko's memories of his early days at the university:

"During the first week of classes I enrolled in a seminar by Professor Abraham Fraenkel, who assigned research articles or book chapters to students to present to the class. In the first week Professor Fraenkel asked for volunteers. There were none. After some hesitation, I tentatively agreed. When I explained that I was not familiar with the Hebrew terms corresponding to the English terms in the article, Professor Fraenkel invited me to come to his apartment the next day at 6:30 a.m. I showed up on time, learned the proper terms and presented the article to the class the following week. Volunteering was one of the most fortunate actions I ever took, as Professor Fraenkel 'took me under his wings' and helped me get small helpful jobs."

Branko received his MSc in 1954. At that time, Zdenka was ordered to report for her army duty so they decided to get married. They got married on June 30,1954, which exempted Zdenka from army service. After a "honey week" Branko started his PhD studies under the supervision of Professor Aryeh Dvoretzky. In the fall of 1955 Branko was called to do his mandatory army service. Professor Dvoretzky was the chief scientific advisor to the Defense Ministry. He recommended to the Air Force chief to start an Operation Research unit and recommended that they use Branko and another student, Eli Shamir (currently Professor Emeritus in Jerusalem).

Eli Shamir shared with me the following details of Branko's army service in the OR unit they started for the Air Force.

"We started the Operation Research unit of the air force. We were fortunate to have a supportive commander who let both of us spend one day a week in Jerusalem, officially, to consult with Professor Dvoretzky but practically it was to discuss Branko's PhD research and my MSc thesis."

With the aid of tables and mechanical calculators Branko developed simulations for the effect of enemy attacks on airfields, optimal location of additional air strips, storing airplanes, distribution of resources, the efficiency of the arms airplanes used against enemy tanks, but refused to deal with cluster bombs. Branko's wisdom and research were very influential and highly appreciated. His recommendation to use French rockets which his analysis showed to be more effective than the Israeli Air Industry made rockets was adopted. When French crews came to Israel in October 1956 Branko was appointed their liaison officer.

Zdenka completed her MSc degree in Chemistry in 1955 and started working for the Israeli Army in a unit preparing "spy tools" (such as vanishing inks).

In 1957 Branko submitted his thesis: *On Some Properties of Minkowski Spaces*. He was highly respected; although the air-force tried to lure him to stay with an attractive financial package, Branko opted to pursue an academic career.

In 1956 their first son Rami was born and Zdenka quit her job opting to take care of Rami. In the spring of 1958 Branko finished his army service and got a scholarship to the Institute of Advanced Study in Princeton. They traveled by boat to New York and by train to Princeton (those were the days ...). They were awed by the unbelievable luxurious housing in the institute. And thus a poor boy from Croatia began an inspiring mathematical career.

5 Stepping stones in a brilliant career

Branko became a prolific publisher. In his first paper, while still serving in the army, he paid homage to his sixth acquired language: it was published in Hebrew in *Riveon Lematematika* (*Mathematical Quarterly*). By the end of 1958 while still fulfilling his army duties, Branko had seven publications and two years later he had 21. During this period, a trend started to emerge. Branko included open problems, loose ends in many papers. Later he told me that he believed that it is nice to leave opportunities for further investigations. After two years in Princeton, one year at the University of Washington where their second son Danny was born in 1960, and summers at the Universities of Kansas and UCLA, Branko accepted a position as a Lecturer at the Hebrew University in Jerusalem. The Grünbaum family returned to Jerusalem in the Fall of 1961.

Zdenka enrolled in a PhD program at the Department of Chemistry at the Hebrew University. After only three years Branko was promoted to Associate Professor.

Branko's first PhD student was Micha Perles and I became his second student in 1964. After two years in Berkeley and one year in Northwestern University, Eli Shamir decided to return to Jerusalem primarily because Branko was there. In 1965 Branko told me that he will be going away for a one year sabbatical. He asked Professor Michael Rabin to temporarily supervise me.

In 1966 we were shocked to learn that Branko was not going to return. It was a difficult decision for Branko, Zdenka and their two young children. Zdenka wanted to return and complete her PhD in Chemistry, but under the new personal travel restrictions in Israel, she agreed to stay in the US.

Branko considered two universities, University of Washington in Seattle and University of Toronto. He chose Seattle where he could work with Victor Klee. They ran a seminar that continues meeting every Wednesday, at 4:00 p.m. in room 401 in the math department. Isabella Novik, Branko's mathematical great-grand-daughter, is currently running it.

It was not practical for Zdenka to resume her PhD in Chemistry, but she did not give up her pursuit of education. She studied Radiochemistry and Nuclear Medicine and was certified to practice them. She worked for a while in Swedish Hospital, and then she was recruited by the University of Washington School of Medicine. She finally retired in 1990 after publishing 30 scientific papers.

In 2007 Branko was one of the invited speakers in the Bled-2007 Graph Theory conference. He came with Zdenka and their two granddaughters. They traveled to Osijek, Đakovo, and visited the convent that saved Zdenka. One of the sisters recognized Zdenka and provided more details of the other women Mother Superior Amadeja Pavlović saved. Curiously, this is where and when Zdenka recalled Mother Superior's name. Zdenka submitted a report to "Yad Vashem" (the holocaust museum in Jerusalem) and Mother Superior Amadeja was designated in September 2008 as "Righteous among the Nations."

In 2011 Zdenka and Branko moved to a retirement home where 3 years later, they celebrated their 60$^{\text{th}}$ anniversary. This is how Zdenka ended her memoire: *"We welcomed Galya and Martha, our wonderful daughters in law, into our family and are delighted with grandchildren Mara, Sasha, Maks and Sam.*

The terrible journey that started in my early years ended with great joy later."

Figure 1: Branko, Zdenka, two sons, extended family and Branko's favorite hoby: geometric structures.

After battling illness, Zdenka passed away in her sleep on December 28, 2015. Branko wrote:

"While helping her with her Tora studies back in 1949 for some reason, the following from Jeremiah remained in my memory:

I remember the grace of your youth
The love of thy marriage
You following me to the desert
A land not sown."

We continued our weekly visits with Branko until he passed away three years later.

One of Branko's favorite activities was constructing three dimensional geometric objects. Hanging from the ceiling in his office was a dense "forest" of geometric objects. A few of them can be seen in Figure 1.

6 Branko's mathematics

One of Paul Erdős' favorite stories was the story of Wilhelm Röntgen. Röntgen observed a phenomenon seen by other people in the lab, including the lab manager, but only he was curious enough to further study it. He called the mysterious rays that exposed hidden films, X-rays (the unknown mathematical symbol). He went on to win the first Physics Nobel Prize in 1901.

Branko was endowed with a mathematical X-rays system. He saw in many articles or discussions "hidden" treasures. This was apparent almost from the beginning of his mathematical career. In 1958, *Helly's Theorem* (convex sets with a common point), made him wonder what about sets intersected by a line? He conjectured:

"A family of more than five disjoint translates of a compact convex set in the plane such that any five are met by a common line, is intersected by a single line."

In 1989, only 41 years later, Helge Tverberg proved that Branko was right.

We lived in Seattle about 200 meters away. Branko and I used to get together frequently. One day we conducted the usual math rambling when my wife interrupted us:

"I don't understand you mathematicians. You propose one problem, and within seconds you consider another problem."

"When we solve one problem, we do not consider a new one for months." She concluded: *"You mathematicians are men with problems."*

Indeed Branko proposed many problems that led to many threads of research.

For example, Tutte proved that every planar 4-connected graph is Hamiltonian. Branko, and independently Nash Williams asked: *"Is every 4-connected toroidal graph Hamiltonian?"*

This question created the following thread:

In 1972 Altshuler proved that 6-connected Toroidal graphs are Hamiltonian.

In 1996 Thomas and Yu proved that every edge in a 5-connected toroidal graph is contained in a Hamiltonian cycle;

and in 2005 Thomas, Yu and Zang proved that a 4-connected Toroidal graph is traceable.

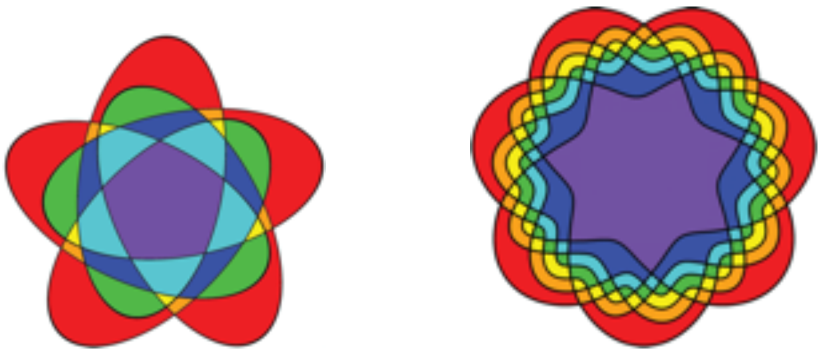
We are almost there, only one edge is missing . . .

This and many other attractive problems can be found in Branko's 1970 paper *Polytopes, Graphs, and Complexes*.

Among other popular aesthetically visual mathematical objects Branko helped popularize were Venn diagrams. Almost all books that include the basic theory of sets drew 3 circles that visualized all possible intersection patterns. *"Why not four or more?"* Branko wondered. This is Frank Ruskey's, Professor at the University of Victoria, Canada, testimonial about Branko's Venn diagrams contributions:

"I first became interested in the mathematical study of Venn diagrams when I invited one of Branko's visitors, Anthony Edwards, to make the trip from Seattle to Victoria and give a seminar. I then began corresponding with Branko and he was a tremendous help as Mark Weston and I were putting together the 'Survey of Venn Diagrams'. He really opened my eyes up to the wide range of interesting questions that could be asked about Venn diagrams and, of course, his early papers set the foundations and definitions that all later papers on Venn diagrams use."

In 1976 Branko received the Lester R. Ford Award for his expository article *Venn Diagrams and Independent Families of Sets.*



(a) Branko's 5 ellipses

(b) Branko's 7-set Venn diagram

Figure 2: Branko's five convex and seven non-monotone Venn diagrams.[1]

Branko published more than 200 papers, 4 books, and won multiple coveted prizes.

6.1 Branko's books

6.1.1 Convex Polytopes

His first book was *Convex Polytopes*. It was published in 1967. Prior to Euler's famous formula: $V - E + F = 2$, the first landmark in the theory of the combinatorial properties of convex polytopes (according to Victor Klee) there was hardly any activity dealing with the combinatorial properties of convex polytopes. Following Euler's formula, research activity on convex polytope flourished until the end of the 19$^{\text{th}}$ century. The end of the 19$^{\text{th}}$ century saw a steep decline in the interest in convex polytopes. Few new results were found. The

[1]The five ellipses were published in *Mathematics Magazine* in 1975. The seven non-monotone Venn diagram was published 17 years later in *Geombinatorics*. The colored diagram was done by Frank Ruskey. Branko did it in black and white.

feeling was that the interesting remaining problems are too hard. Interest reemerged in the 1950's with the emergence of operation research and in particular, linear programming.

Only sporadic research in polytopes existed in the first half of the 20$^{\text{th}}$ century, it was not considered a "main-line mathematics" topic. Caratheodory was aware of the cyclic polytopes. Neighborly polytopes were re-discovered in 1955 by David Gale. Motzkin in 1957 believed that all neighborly polytopes are cyclic and conjectured that they maximize the number of faces of all polytopes of the same size (*The Upper-Bound Conjecture*) and Victor Klee's 1962 paper on the Dehn-Sommerville equations inspired subsequent developments. Then Branko laid his X-rays eyes on this question and constructed neighborly polytopes that are not cyclic. His book grew out of lecture notes for a course on the combinatorial theory of convex polytopes Branko taught at the Hebrew University in 1964 – 65 and in Michigan State University in 1965 – 66.

Branko used to give us copies of the notes produced on a Ditto copier. One day Branko gave us notes in which he claimed to prove that every convex polytope can be realized with all vertices in rational coordinates. A mistake in the proof was quickly found. The Ditto copies were alcohol based, so after a while, when the alcohol evaporated, it became almost impossible to read them. *"Good planning"* remarked Branko soon after the discovery of the error, *"soon enough there will be no trace of my error."* Micha Perles, Branko's first PhD student, constructed an 8-dimensional polytope with 12 vertices that cannot be rationally realized.

Branko's book practically resuscitated a dying, important research area. *"The appearance of Grünbaum's book 'Convex Polytopes' in 1967 was a moment of grace to geometers and combinatorialists"* wrote Gil Kalai, Branko's mathematical grandson.

Peter McMullen wrote:

"The original edition of 'Convex Polytopes' inspired a whole generation of grateful workers in polytope theory. Without it, it is doubtful whether many of the subsequent advances in the subject would have been made. The many seeds it sowed have since grown into healthy trees, with vigorous branches and luxuriant foliage."

The book accumulated 4720 citations and still counting. In 2005, the AMS awarded Branko the Leroy P. Steele Prize for Mathematical Exposition.

The original 1967 edition is probably on its way to become a rare collector item. The second edition co-edited by Günter Ziegler was welcomed in 2003.

Many curious students will be intrigued by the existence of a trillion points in $\mathbb{R}^4$ that form the vertices of a convex polytope in which every pair of vertices form an edge. Many topologists will be curious to explore the counter intuitive existence of polytopes that cannot be realized in rational Euclidean spaces. Some will be baffled by Branko's 2010 paper *The Bilinski Dodecahedron, and Assorted Parallelohedra, Zonohedra, Monohedra, Isozonohedra and Otherhedra.*

6.1.2 Arrangements & Spreads

The classical theorems of Pappus and Desargues about points and lines in the plane led Branko to develop the "arrangements": theory of lines and points in special arrangements. In 1970 Branko spent a year visiting Michigan State University. He gave me a manuscript titled *Arrangements and Spreads*. It laid the foundations for his second book (published in 1980). In the introduction Branko wrote:

"In this present paper I hope to show that a lot of fun may be had with rather elementary mathematics, so elementary that its problems can be understood by under-graduates. While admitting that having fun is not considered one of the legitimate aims of mathematics, it is my firm belief that the problems discussed below are much more wholesome than many of the supercilious topics which are, in grim earnest, frequently presented as the pinnacle of contemporary mathematics."

An arrangement of lines is a collection of lines in the projective plane. It was Branko's preference as any two lines intersect in exactly one point. With an arrangement of lines, a cell complex is associated composed of the vertices, edges and cells. Two arrangements are isomorphic if their cell complexes are isomorphic.

The main subject of the manuscript was the counting of the number of non-isomorphic arrangements of a given number of lines.

A (p, t)-arrangement is an arrangement of t lines and p points in the plane (Euclidean or projective) such that each line contains exactly three points. The *Orchard Problem* is to find the largest t (the number of lines) for an arrangement with a given number of points.

In 1886 Sylvester, using points on the curve $y = x^3$, constructed $\sim \frac{n^2}{18}$ lines. In 1974 Branko, Stefan Burr and Neil Sloan published the *The Orchard Problem* paper. In this paper, using Weierstrass elliptic curves, they constructed $[p(p-3)/2]+1$ lines for p points, a new lower bound. In 2013 Ben Green and Terence Tao proved that this is also the upper bound. The book includes many attractive "elementary" problems sometime calling for non-elementary tools to solve them.

6.1.3 Tilings and Patterns

Branko's third book, *Tilings & Patterns* coauthored with Geoffrey Shephard, was published in 1986.

Branko expressed concern about the trend to squeeze geometry out of syllabus in schools and universities. He felt that this trend ignores engineers, architects, scientists, artists, mineralogists and others who wish to apply geometric ideas in their work. The downward trend can be traced to René Descartes' introduction of the Cartesian coordinates in 1637.

Geometry started to be "swallowed" by linear algebra. I remember a discussion over dinner between Branko and Jean Dieudonné who claimed that geometry is a small part of linear algebra. If my memory does not betray me, I think he contemplated writing a manuscript *"Geometry without Figures."* Traces of this trend still can be found more recently: T. Padmanabhan and V. Padmanabhan, famous Indian theoretical physicists, included in their book *The Dawn of Science*, published by Springer in 2019, a chapter *Geometry without Figures.*

Branko rejected the current fashion that geometry must be abstract, entirely without figures if it is to be regarded as "advanced mathematics". *"It seems to us as silly as to extol the virtues of silent music suggesting that the sign of true musical maturity is to appreciate it by merely looking at the printed score,"* he wrote. Fortunately, we still see *Proofs Without Words* being published.

Branko met Geoffrey Shephard in 1975. They decided to write a book on *Visual Geometry.* This was a huge undertaking. So they decided to start with tilings and patterns as a first step in their program. After 11 years of research, tracing ancient and current places where tilings were used, the book was published. Various prominent mathematicians considered tilings. They looked for patterns, shapes, symmetries and the unexpected. Most ancient tilings used triangles, quadrilaterals, pentagons or hexagons to tile the plane. The first chapter of Kepler's book, *Harmonices Mundi*, is devoted to regular polygons. He considered tiling of the plane with polygons, and noted that the regular pentagon cannot tile the plane without leaving gaps but the gaps can be filled with polygons. Existence of aperiodic tilings piqued interest with Wang's *Domino Problem* which attracted prominent mathematicians like Donald Knuth, Rafael Robinson, Robert Penrose, John Conway and many others. Various constructions of aperiodic tilings with few prototiles were constructed. The most famous one is Penrose' "kite-and-dart" two-prototile tiling.

6.1.4 Configurations of Points and Lines

This was Branko's fourth book, this time in color. It was published in 2009 though he became interested in this topic in 1946 in high school in Yugoslavia.

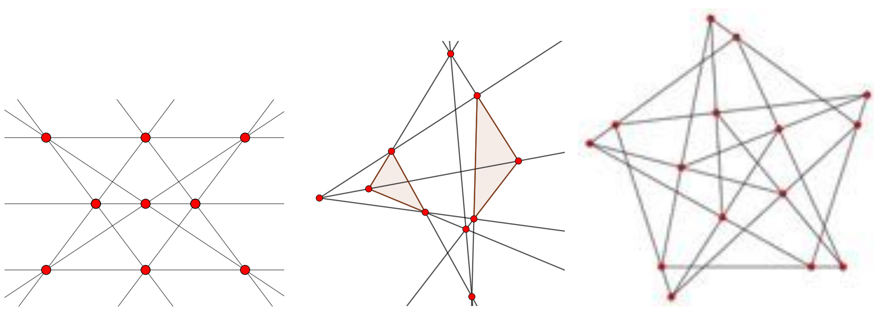


Figure 3: Pappus (9_3), Desargues (10_3), Cremona-Richmond (15_3).

The primary subject of the book was configurations of lines and points. The notation (n_k), for a configuration with n lines and k points, each line contains exactly k points and each point lies on k lines, was formulated by Theodor Reye in 1876.

Branko was motivated by the three "historic" arrangements, Pappus ((9_3), 4$^{\text{th}}$ century), Desargues ((10_3), 17$^{\text{th}}$ century) and Cremona's ((15_3), 19$^{\text{th}}$ century), which appear in the second page of his book (see Figure 3).

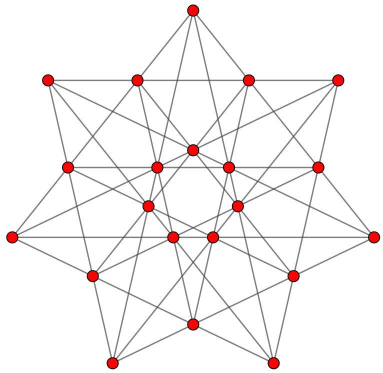


Figure 4: Grünbaum-Rigby (21_4) configuration.

The configuration (21_4) was studied by Felix Klein. He constructed 21 "lines" in the complex projective plane that realize the configuration (21_4).

A hundred years later, in 1990, Branko and John Rigby constructed the Grünbaum-Rigby (21_4) configuration (see Figure 4), which opened the flood door to the discovery of many configurations and the book which was published in the *AMS Graduate Studies in Mathematics* series.

Why graduate studies? Branko wrote in the introduction: *"Because on the road to this book I became aware of the interaction between configurations and advanced topics in combinatorics, algebraic geometry, computing, number theory and even analysis."*

Students completed their PhD thesis discovering new configurations, international cooperation such as the 2009 paper by L. Berman, J. Bokowski, Branko and T. Pisanski on floral configurations (see Figure 5) and other activities preceded and followed the publication of the book.

Branko was one of the founders of *Geombinatorics*, a journal dedicated to geometry. It became home to the famous Hadwiger-Nelson problem: *What is the chromatic number of the plane?* [26] Branko was very interested in this simple problem which is still open after more than 70 years. He was also interested in the odd-distance graph: coloring the points of the plane so that points whose distance is an odd integer receive distinct colors.

Following Branko's tradition I'd like to offer a few "simple" geometric problems:

1. Let $\mathbf{C_r}$ be a circle of radius r. What is the smallest number of colors needed to color the points of the circle so that two points at odd integral Euclidean distance will be colored differently?

2. Many tilings such as the square grid, the triangular grid, and the hexagonal grid are actually subgraphs of the odd-distance graph. As abstract graphs, they are 3-colorable. The square grid viewed as an abstract graph can be faithfully embedded

in the plane so that two points are connected by an edge iff their distance is an odd integer. Every finite subgraph of these graphs can be faithfully embedded in the odd distance graph. Question: Can the infinite graphs be faithfully embedded in the odd-distance plane?

Acknowledgement

I wish to acknowledge Tomaž (Tomo) Pisanski's helpful suggestions and great renditions of the figures in this article.

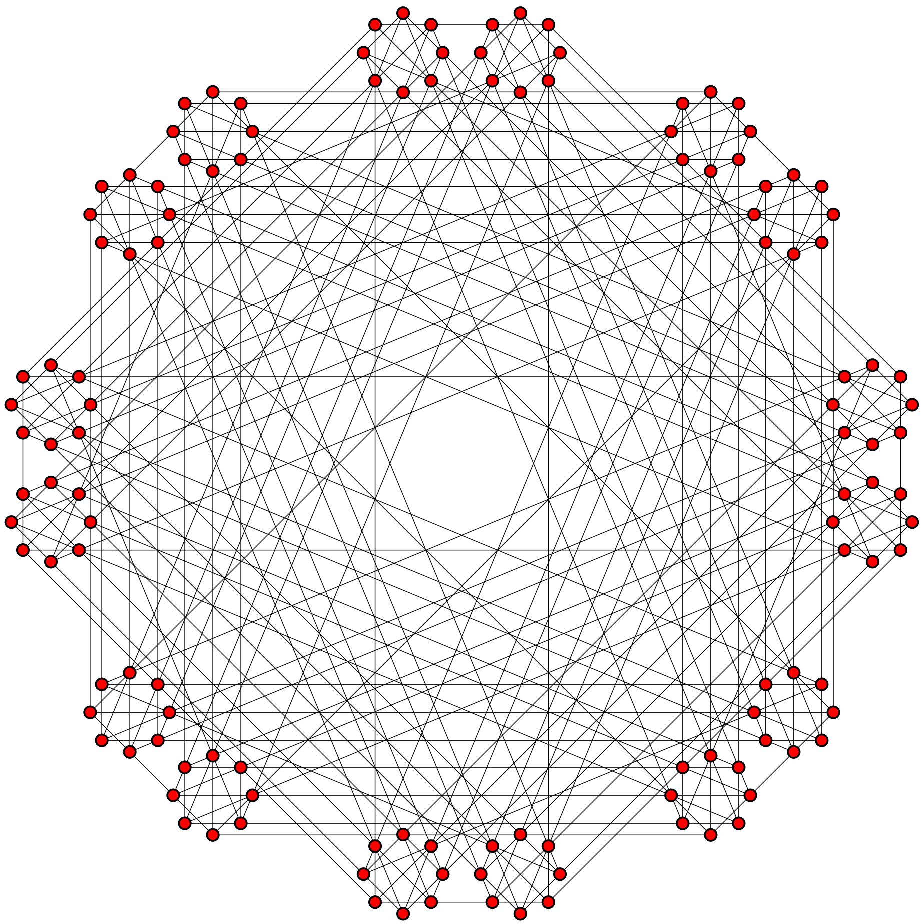


Figure 5: An aesthetic floral arrangement of the (128_4) configuration with dihedral symmetry from Branko's book is constructed by a novel combination of methods from the paper by L. Berman, J. Bokowski, Branko and T. Pisanski. (Drawing curtesy of Leah Berman.)

In May 2018 as we walked out of Magnuson Park in Seattle, none of us thought that it will be our last walk.

Branko passed away on September 14, 2018 but his legacy, inspiration and mathematics will live for many more years.

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SIGMAP 2022 Workshop – Announcement and Call for Papers

The Symmetries in Graphs, Maps, and Polytopes Workshop (SIGMAP) has been held every four years since Steve Wilson organized the first one in Flagstaff, Arizona in 1998. The workshop is devoted to the exploration of the symmetries of discrete objects, and has been an opportunity to share recent advances, discuss open problems, and start new collaborations. In addition to two daily sessions of talks, SIGMAP 2022 will set aside time each day for researchers to gather and meet to explore questions in the field. The plenary speakers for SIGMAP 2022 are:

- Gabriel Cunningham (University of Massachusetts Boston, USA)
- Maria Elisa Fernandes (University of Aveiro, Portugal)
- Gareth Jones (University of Southampton, United Kingdom)
- Klavdija Kutnar (University of Primorska, Slovenia)
- Primož Šparl (University of Ljubljana and University of Primorska, Slovenia)
- Pablo Spiga (Università degli Studi di Milano-Bicocca, Italy)
- Klara Stokes (Umeå University, Sweden)
- Gabriel Verret (University of Auckland, New Zealand)
- Jinxin Zhou (Beijing Jiaotong University, China)

The workshop will be held at the University of Alaska Fairbanks campus. UAF is Alaska's flagship research university, with an enrollment of over 7 000 students. The campus is situated on 2 250 acres amidst the boreal forest, and features 41.6 km of trails, the Museum of the North, botanical gardens, an experimental farm, and a viewing area for our herds of musk ox and reindeer. If you're lucky, you'll spot some of the campus's wild moose during a visit to the campus.

More information about the workshop is available at

https://www.alaska.edu/sigmap

This is also a **call for papers** for a special issue of the journal The Art of Discrete and Applied Mathematics (ADAM). Papers submitted for this special issue should be on topics presented or discussed at the workshop, or closely related to them. The Art of Discrete and Applied Mathematics (ADAM) is a modern, dynamic, platinum open access, electronic journal that publishes high-quality articles in contemporary discrete and applied mathematics (including pure and applied graph theory and combinatorics), with no costs to authors or readers. To be considered for inclusion in this special issue, papers should be submitted by December 31, 2022, via the ADAM website https://adam-journal.eu/. A template and style file for submissions can be downloaded from that website, or obtained from one of the guest editors on request. The ideal length of papers is 5 to 15 pages, but longer or shorter papers will certainly be considered. Papers that are accepted will appear on-line soon after acceptance, and papers that are not processed in time for the special issue may still be accepted and published in a subsequent regular issue of ADAM.

Leah Berman and Gordon Williams

Guest Editors