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Edge-transitive core-free Nest graphs*

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Abstract

A finite simple graph Γ is called a Nest graph if it is regular of valency 6 and admits an automorphism ρ with two orbits of the same length such that at least one of the subgraphs induced by these orbits is a cycle. We say that Γ is core-free if no non-trivial subgroup of the group generated by ρ is normal in $\mathrm{Aut}(\Gamma)$. In this paper we show that, if Γ is edge-transitive and core-free, then it is isomorphic to one of the following graphs: the complement of the Petersen graph, the Hamming graph $H(2,4)$, the Shrikhande graph and a certain normal 2-cover of $K_{3,3}$ by $\mathbb{Z}_2^4$.

Keywords: Bicriculant, edge-transitive, primitive permutation group.

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1 Introduction

All groups in this paper will be finite and all graphs will be finite and simple. A graph admitting an automorphism with two orbits of the same length is called a *bicirculant*. Symmetry properties of bicirculants have attracted considerable attention (see, e.g., [1, 5, 7, 16, 22, 23, 25, 29]). Following [17], for an integer $d \geq 3$, we denote by $\mathcal{F}(d)$ the family of regular graphs having valency d and admitting an automorphism with two orbits of the same length such that at least one of the subgraphs induced by these orbits is a cycle. Jajcay et al. [12] initiated the investigation of the edge-transitive graphs in the classes $\mathcal{F}(d)$, $d \geq 6$. The families $\mathcal{F}(d)$ with $3 \leq d \leq 5$ were studied under different names. The graphs in $\mathcal{F}(3)$ were introduced by Watkins [27] under the name *generalised Petersen graphs*, the

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graphs in $\mathcal{F}(4)$ by Wilson [28] under the name *Rose Window graphs*, and the graphs in $\mathcal{F}(5)$ by Arroyo et al. [2] under the name *Tabačjn graphs*. The automorphism groups of these graphs form the subject of the papers [9, 10, 15, 18], and the question which of them are edge-transitive has been answered in [2, 10, 15].

Jajcay et al. [12] asked whether there exist edge-transitive graphs in $\mathcal{F}(d)$ for $d \geq 6$. Following [26], they call the graphs in $\mathcal{F}(6)$ *Nest graphs*. Several infinite families of edge-transitive Nest graphs were exhibited, which turn out to have interesting properties (e.g., half-arc-transitivity). However, no edge-transitive graph of valency larger than 6 was found. Recently, it was proved by the author and Ruff [17] that the complement of the Petersen graph is the only edge-transitive graph in $\mathcal{F}(d)$ with $d \geq 6$, which has twice an odd number of vertices. The main result of [12] is the classification of the edge-transitive Nest graphs of girth 3 (see [12, Theorem 3.6]), and the task to classify all edge-transitive Nest graphs was posed as [12, Problem 1.2]. In what follows, the Nest graphs will be described via their representation due to [12, Construction 3.1], which goes as follows. Let $n \geq 4$ and let $a, b, c, k \in \mathbb{Z}_n$ such that each of them is distinct from 0 (the zero element of $\mathbb{Z}_n$), the elements a, b and c are pairwise distinct, and in the case when n is even, $k \neq n/2$. Then the *Nest graph* $\mathcal{N}(n; a, b, c; k)$ is defined to have vertex set $\{u_i : i \in \mathbb{Z}_n\} \cup \{v_i : i \in \mathbb{Z}_n\}$, and three types of edges such as

- $\{u_i, u_{i+1}\}$ for $i \in \mathbb{Z}_n$ (*rim edges*),
- $\{v_i, v_{i+k}\}$ for $i \in \mathbb{Z}_n$ (*hub edges*),
- $\{u_i, v_i\}$, $\{u_i, v_{i+a}\}$, $\{u_i, v_{i+b}\}$ and $\{u_i, v_{i+c}\}$ for $i \in \mathbb{Z}_n$ (*spoke edges*),

where the sums in the subscripts are computed in $\mathbb{Z}_n$. It is easy to see that the permutation ρ of $V(\Gamma)$, defined as $\rho = (u_0, u_1, \ldots, u_{n-1})(v_0, v_1, \ldots, v_{n-1})$, is an automorphism of Γ with orbits $\{u_i : i \in \mathbb{Z}_n\}$ and $\{v_i : i \in \mathbb{Z}_n\}$, and the subgraph induced by the former orbit is a cycle. It is not hard to show that all the graphs $\mathcal{N}(n; a, b, c; k)$ comprise the whole family $\mathcal{F}(6)$.

In the case of both the Rose Window and the Tabačjn graphs, the classification of the edge-transitive graphs was obtained in two main steps. The so called core-free graphs were found first and the rest was retrieved from the core-free graphs using covering techniques (see [2, 15]). Here is the formal definition of a core-free Nest graph.

Definition 1.1. Let $\Gamma = \mathcal{N}(n; a, b, c; k)$ be a Nest graph and ρ be the permutation of $V(\Gamma)$ defined as $\rho = (u_0, u_1, \ldots, u_{n-1})(v_0, v_1, \ldots, v_{n-1})$. Then Γ is *core-free* if no non-trivial subgroup of $\langle \rho \rangle$ (the group generated by ρ) is normal in $\mathrm{Aut}(\Gamma)$.

Remark 1.2. The term "core-free" comes from group theory. For a subgroup $A \leq B$, the *core* of A in B is the largest normal subgroup of B contained in A. In the case when A has trivial core, it is also called *core-free*. In this context, Definition 1.1 can be rephrased by saying that Γ is core-free if and only if $\langle \rho \rangle$ is core-free in $\mathrm{Aut}(\Gamma)$.

Our goal in this paper is to determine the edge-transitive core-free Nest graphs. For an explanation why this task is more subtle than in the case of Rose Window and Tabačjn graphs, we refer to [12, page 9]. The edge-transitive non-core-free Nest graphs are handled in the paper [14].

The main result of this paper is the following theorem.

Theorem 1.3. *If $\mathcal{N}(n; a, b, c; k)$ is an edge-transitive core-free graph, then it is isomorphic to one of the following graphs:*

$$\mathcal{N}(5; 1, 2, 3; 2),\ \mathcal{N}(8; 1, 3, 4; 3),\ \mathcal{N}(8; 1, 2, 5; 3) \text{ and } \mathcal{N}(12; 2, 4, 8; 5).$$

Remark 1.4. The fact that each of the Nest graphs in Theorem 1.3 is core-free was mentioned by Jajcay et al., see [12, page 9]. The first three of them are well-known strongly regular graphs. The Nest graph $\mathcal{N}(5; 1, 2, 3; 2)$ is the complement of the *Petersen graph*, $\mathcal{N}(8; 1, 3, 4; 3)$ is the *Hamming graph* $H(2, 4)$, and $\mathcal{N}(8; 1, 2, 5; 3)$ is the *Shrikhande graph*. The fourth Nest graph $\mathcal{N}(12; 2, 4, 8; 5)$ is not strongly-regular, it can be described as a normal 2-cover of the complete bipartite graph $K_{3,3}$ by $\mathbb{Z}_2^4$ (for the definition of a normal 2-cover, see the 2nd paragraph of Subsection 2.1).

The paper is organised as follows. Section 2 contains the needed results from graph and group theory. In Section 3 we review some results about Nest graphs obtained in [12, 17]. Section 4 is devoted to the Nest graphs in the form $\mathcal{N}(2m; 2, m, 2 + m; 1)$, m is odd. The main result (Proposition 4.1) is a characterisation, which was mentioned in [12] without a proof, and which is needed for us in the proof Theorem 1.3. The latter proof is presented in Section 5.

2 Preliminaries

2.1 Graph theory

Given a graph Γ, let $V(\Gamma)$, $E(\Gamma)$, $A(\Gamma)$ and $\mathrm{Aut}(\Gamma)$ denote its *vertex set*, *edge set*, *arc set* and *automorphism group*, respectively. The number $|V(\Gamma)|$ is called the *order* of Γ. The set of vertices adjacent with a given vertex v is denoted by $\Gamma(v)$. If $G \leq \mathrm{Aut}(\Gamma)$ and $v \in V(\Gamma)$, then the *stabiliser* of v in G is denoted by G_v, the *orbit* of v under G by v^G, and the set of all G-orbits by $\mathrm{Orb}(G, V(\Gamma))$. If $B \subseteq V(\Gamma)$, then the *setwise stabiliser* of B in G is denoted by $G_{\{B\}}$. If G is transitive on $V(\Gamma)$, then Γ is said to be *G-vertex-transitive*, and Γ is simply called *vertex-transitive* when it is $\mathrm{Aut}(\Gamma)$-vertex-transitive; (G-)*edge-* and (G-)*arc-transitive* graphs are defined correspondingly.

Let π be an arbitrary partition of $V(\Gamma)$ and for a vertex $v \in V(\Gamma)$, let $\pi(v)$ denote the class containing v. The *quotient graph* of Γ with respect to π, denoted by Γ/π, is defined to have vertex set π, and edges $\{\pi(u), \pi(v)\}$, where $\{u, v\} \in E(\Gamma)$ such that $\pi(u) \neq \pi(v)$. Now, if there exists a constant r such that

$$\forall \{u, v\} \in E(\Gamma) :\ \pi(u) \neq \pi(v) \text{ and } |\Gamma(u) \cap \pi(v)| = r,$$

then Γ is called an *r-cover* of Γ/π. The term *cover* will also be used instead of 1-cover. In the special case when $\pi = \mathrm{Orb}(N, V(\Gamma))$ for an intransitive normal subgroup $N \lhd \mathrm{Aut}(\Gamma)$, Γ/N will also be written for Γ/π and when Γ is also an r-cover (cover, respectively) of Γ/N, then the term *normal r-cover* (*normal cover*, respectively) will also be used. It is well-known that this is always the case when Γ is edge-transitive. More precisely, if Γ is a G-edge-transitive graph, Γ is regular with valency κ, and $N \lhd G$ is intransitive, then Γ is a normal r-cover of Γ/N for some r and r divides κ.

A graph admitting a regular cyclic group of automorphisms is called a *circulant*. A recursive classification of finite arc-transitive circulants was obtained independently by Kovács [13] and Li [19]. The paper [13] also provides an explicit characterisation (see

[13, Theorem 4]), which was rediscovered recently by Li et al. [20]. The characterisation presented below follows from the proof of [13, Theorem 4] or from [20, Theorem 1.1].

In what follows, given a cyclic group C and a divisor d of $|C|$, we denote by C_d the unique subgroup of C of order d.

Theorem 2.1 ([13])**.** *Let Γ be a connected arc-transitive graph of order n and of valency κ and suppose that $C \leq \mathrm{Aut}(\Gamma)$ is a regular cyclic subgroup. Then one of the following holds.*

(a) *Γ is the complete graph.*

(b) *C is normal in $\mathrm{Aut}(\Gamma)$.*

(c) *$\mathcal{B} = \mathrm{Orb}(C_d, V(\Gamma))$ is a block system for $\mathrm{Aut}(\Gamma)$ for some divisor d of $\gcd(n, \kappa)$, $d > 1$. Γ is a normal d-cover of $\Gamma/\mathcal{B}$, and $\Gamma/\mathcal{B}$ is a connected arc-transitive circulant of valence κ/d.*

(d) *$\mathcal{B}_1 = \mathrm{Orb}(C_d, V(\Gamma))$ and $\mathcal{B}_2 = \mathrm{Orb}(C_{n/d}, V(\Gamma))$ are block systems for $\mathrm{Aut}(\Gamma)$ for some divisor d of n such that $d > 3$, $\gcd(d, n/d) = 1$ and $d-1$ divides κ. $\Gamma/\mathcal{B}_1$ is a connected arc-transitive circulant of valency $\kappa/(d-1)$, $\Gamma/\mathcal{B}_2 \cong K_d$, and*

$$\mathrm{Aut}(\Gamma) = G_1 \times G_2, \tag{2.1}$$

where $C_d \leq G_1$, $G_1 \cong S_d$, $C_{n/d} < G_2$, and $G_2 \cong \mathrm{Aut}(\Gamma/\mathcal{B}_1)$.

Remark 2.2. Although not used later, it is worth mentioning that the graph Γ in part (c) is isomorphic to the lexicographical product $\Gamma/\mathcal{B}[\overline{K}_d]$, where $\overline{K}_d$ is the edgeless graph on d vertices, and the graph Γ in part (d) is isomorphic to the tensor (direct) product $K_d \times \Gamma/\mathcal{B}_1$ (see [13, 20]).

In the rest of the section we restrict ourselves to arc-transitive circulants of small valency.

Lemma 2.3 ([3, part (ii) of Corollary 1.3])**.** *Let Γ be a connected arc-transitive graph of order n and of valency κ, where $\kappa = 3$ or 4, and suppose that $C \leq \mathrm{Aut}(\Gamma)$ is a regular cyclic subgroup. Then one of the following holds.*

(1) *Γ is isomorphic to one of the graphs: $K_4, K_5, K_{3,3}$ and $K_{5,5} - 5K_2$.*

(2) *$\kappa = 4$ and C is normal in $\mathrm{Aut}(\Gamma)$.*

(3) *$\kappa = 4$, n is even, $\mathcal{B} = \mathrm{Orb}(C_2, V(\Gamma))$ is a block system for $\mathrm{Aut}(\Gamma)$, and Γ is a normal 2-cover of $\Gamma/\mathcal{B}$, which is a cycle.*

Lemma 2.4. *Let Γ be a connected arc-transitive graph of order $n > 14$ and of valency 6, and suppose that $C \leq \mathrm{Aut}(\Gamma)$ is a regular cyclic subgroup. Then $\mathrm{Aut}(\Gamma)$ contains a normal subgroup N such that one of the following holds.*

(1) *$N = C$, or*

(2) *$n \equiv 4 \pmod 8$ and $N = C_{n/4}$, or*

(3) *$N \cong \mathbb{Z}_3^\ell$ for $\ell \geq 2$ and $C_3 < N$.*

Proof. Γ belongs to one of the families (a) – (d) in Theorem 2.1.

Family (a): This case cannot occur as $n > 14$.

Family (b): Part (1) follows.

Family (c): In this case $\mathrm{Orb}(C_d, V(\Gamma))$ is a block system for $\mathrm{Aut}(\Gamma)$, where $d = 2$ or $d = 3$ or $d = 6$. Let $\mathcal{B} = \mathrm{Orb}(C_d, V(\Gamma))$.

If $d = 2$, then $\Gamma/\mathcal{B}$ has valency 3. It follows from Lemma 2.3 that $n \le 12$, but this is excluded.

If $d = 3$, then choose N to be the Sylow 3-subgroup of the kernel of the action of $\mathrm{Aut}(\Gamma)$ on $\mathcal{B}$. It is clear that $C_3 \le N$. The quotient graph $\Gamma/\mathcal{B}$ is a cycle. Using this, one can see that a Sylow 3-subgroup of a vertex stabilizer in $\mathrm{Aut}(\Gamma)$ is contained in N, in particular, $N \ne C_3$. It follows that $N \cong \mathbb{Z}_3^\ell$ for some $\ell \ge 2$. Also, N is characteristic in the latter kernel, which implies that $N \lhd \mathrm{Aut}(\Gamma)$. Finally, $\mathrm{Orb}(N, V(\Gamma)) = \mathcal{B} = \mathrm{Orb}(C_3, V(\Gamma))$, and so $C_3 < N$, i.e., part (3) holds.

If $d = 6$, then $\Gamma \cong K_{6,6}$. This contradicts the assumption that $n > 14$.

Family (d): In this case it follows from the assumption that $n > 14$ that $\mathrm{Orb}(C_4, V(\Gamma))$ and $\mathrm{Orb}(C_{n/4}, V(\Gamma))$ are block systems for] $\mathrm{Aut}(\Gamma)$ and $n \equiv 4 \pmod 8$. Furthermore,

$$\mathrm{Aut}(\Gamma) = G_1 \times G_2,$$

where $C_4 < G_1$, $G_1 \cong S_4$, $C_{n/4} < G_2$ and $G_2 \cong \mathrm{Aut}(\Gamma/\mathcal{B}_1)$, where $\mathcal{B}_1 = \mathrm{Orb}(C_4, V(\Gamma))$. The graph $\Gamma/\mathcal{B}_1$ is connected of valency 2, hence it is a cycle of length $n/4$. It follows that $C_{n/4}$ is characteristic in G_2, and as $G_2 \lhd \mathrm{Aut}(\Gamma)$, part (2) follows. $\square$

2.2 Group theory

Our terminology and notation are standard and we follow the books [8, 11]. The *socle* of a group G, denoted by $\mathrm{soc}(G)$, is the subgroup generated by the set of all minimal normal subgroups (see [8, page 111]). The group G is called *almost simple* if $\mathrm{soc}(G) = T$, where T is a non-abelian simple group. In this case G is embedded in $\mathrm{Aut}(T)$ so that its socle is embedded via the inner automorphisms of T, and we also write $T \le G \le \mathrm{Aut}(T)$ (see [8, page 126]).

Our proof of Theorem 1.3 relies on the classification of primitive groups containing a cyclic subgroup with two orbits due to Müller [24]. Here we need only the special case when the cyclic subgroup is semiregular.

Theorem 2.5 ([24, Theorem 3.3])**.** *Let G be a primitive permutation group of degree $2n$ containing an element with two orbits of the same length. Then one of the following holds, where G_0 denotes the stabiliser of a point in G.*

(1) *(Affine action) $\mathbb{Z}_2^m \lhd G \le \mathrm{AGL}(m, 2)$, where $n = 2^{m-1}$. Furthermore, one of the following holds.*

   (a) *$n = 2$, and $G_0 = \mathrm{GL}(2, 2)$.*

   (b) *$n = 2$, and $G_0 = \mathrm{GL}(1, 4)$.*

   (c) *$n = 4$, and $G_0 = \mathrm{GL}(3, 2)$.*

   (d) *$n = 8$, and G_0 is one of the following groups: $\mathbb{Z}_5 : \mathbb{Z}_4$, $\Gamma\mathrm{L}(1, 16)$, $(\mathbb{Z}_3 \times \mathbb{Z}_3) : \mathbb{Z}_4$, $\Sigma\mathrm{L}(2, 4)$, $\Gamma\mathrm{L}(2, 4)$, A_6, $\mathrm{GL}(4, 2)$, $(S_3 \times S_3) : \mathbb{Z}_2$, S_5, S_6 and A_7.*

(2) *(Almost simple action) G is an almost simple group and one of the following holds.*

(a) *$n \geq 3$, $\mathrm{soc}(G) = A_{2n}$, and $A_{2n} \leq G \leq S_{2n}$ in its natural action.*

(b) *$n = 5$, $\mathrm{soc}(G) = A_5$, and $A_5 \leq G \leq S_5$ in its action on the set of 2-subsets of $\{1, 2, 3, 4, 5\}$.*

(c) *$n = (q^d - 1)/2(q - 1)$, $\mathrm{soc}(G) = \mathrm{PSL}(d, q)$, and $\mathrm{PSL}(d, q) \leq G \leq \mathrm{P\Gamma L}(d, q)$ for some odd prime power q and even number $d \geq 2$ such that $(d, q) \neq (2, 3)$.*

(d) *$n = 6$ and $\mathrm{soc}(G) = G = M_{12}$.*

(e) *$n = 11$, $\mathrm{soc}(G) = M_{22}$, and $M_{22} \leq G \leq \mathrm{Aut}(M_{22})$.*

(f) *$n = 12$ and $\mathrm{soc}(G) = G = M_{24}$.*

If G is a group in one of the families (a) – (f) in part (2) above, then it follows from [8, Theorem 4.3B] that $\mathrm{soc}(G)$ is the unique minimal normal subgroup of G. Therefore, we have the following corollary.

Corollary 2.6. *Let G be a primitive permutation group in one of the families (a) – (f) in part (2) of* Theorem 2.5*, and let $N \lhd G$, $N \neq 1$. Then N is also primitive.*

For a transitive permutation group $G \leq \mathrm{Sym}(\Omega)$, the *subdegrees* of G are the lengths of the orbits of a point stabiliser G_ω, $\omega \in \Omega$. Since G is transitive, it follows that the subdegrees do not depend on the choice of ω (see [8, page 72]). The number of orbits of G_ω is called the *rank* of G. The actions of a group G on sets Ω and Ω' are said to be *equivalent* if there is a bijection $\varphi\colon \Omega \to \Omega'$ such that

$$\forall \omega \in \Omega,\ \forall g \in G:\ \varphi(\omega^g) = (\varphi(\omega))^g.$$

Now, suppose that G is a group in one of the families (a) – (f) in part (2) of Theorem 2.5. If G is in family (a), then G is clearly 2-transitive. If $2n \neq 6$, then the action is unique up to equivalency. If $2n = 6$, then G admits two inequivalent actions. Suppose that $G \cong S_6$. Consider the action of $\mathrm{PGL}(2, 5)$ on the projective line $\mathrm{PG}(1, 5)$, denote this line by Ω. The group $\mathrm{PGL}(2, 5) \cong S_5$ and it has index 6 in the symmetric group $\mathrm{Sym}(\Omega)$. The action of $\mathrm{Sym}(\Omega)$ on the set of right cosets of $\mathrm{PG}(2, 5)$ is inequivalent with its natural action on Ω. If $G \cong A_6$, the above construction can be repeated with considering $\mathrm{PSL}(2, 5)$ instead of $\mathrm{PGL}(2, 5)$. Note that, if H is a stabilizer of a point in G with respect to either of the two inequivalent actions, then with respect to the other action, H acts transitively.

If G is in family (b), then the action is unique up to equivalence and the subdegrees are $1, 3$ and 6.

Let G be in family (c). The semiregular cyclic subgroup of G with two orbits is contained in a regular cyclic group, called the *Singer subgroup* of $\mathrm{PGL}(d, q)$ (see [11, Chapter 2, Theorem 7.3]). In this case the action is unique up to equivalence if and only if $d = 2$. If $d \geq 4$, then the action of G is equivalent to either its natural action on the set of points of the projective geometry $\mathrm{PG}(d - 1, q)$, or to its natural action on the set of hyperplanes of $\mathrm{PG}(d - 1, q)$. In both actions G is 2-transitive.

Let $G = M_{12}$ in the family (d). Then G has two inequivalent actions. These actions can be described using the action of the Mathieu group M_{24} on the Steiner System $\mathrm{S}(5, 8, 24)$ and the fact the setwise stabilizer of a dodecad in M_{24} is isomorphic to M_{12} (see [8, pages 207–208]). Note that, if H is a stabilizer of a point in G with respect to either of the two inequivalent actions, then with respect to the other action, H acts transitively.

Finally, if G is in the families (e) – (f), then the action is unique up to equivalence and G is 2-transitive (this can also be read off from [6]). All this information is summarised in the lemma below.

Lemma 2.7. *Let G be a primitive permutation group in one of the families (a) – (f) in part (2) of* Theorem 2.5.

(1) *G is 2-transitive, unless G belongs to family (b). In the latter case the subdegrees are $1, 3$ and 6.*

(2) *The action of G is unique up to equivalence, unless one of the following holds.*

 (i) *$G \cong A_6$ or S_6 or M_{12}, G admits two inequivalent faithful actions, and if H is a stabilizer of a point in G with respect to either of the two inequivalent actions, then with respect to the other action, H acts transitively.*

 (ii) *G is in family (c) and $d \geq 4$. In the latter case G admits two inequivalent faithful actions, namely, the natural actions on the set of points and the set of hyperplanes, respectively, of the projective geometry* $\mathrm{PG}(d-1, q)$.

The following result about G-arc-transitive bicirculants can be found in Devillers et al. [7]. The proof works also for the edge-transitive bicirculants, in fact, it is an easy consequence of Theorem 2.5.

Proposition 2.8 ([7, part (1) of Proposition 4.2])**.** *Let Γ be a G-edge-transitive bicirculant such that G is a primitive group. Then Γ is one of the following graphs:*

(1) *The complete graph, and G is one of the 2-transitive groups described in part (2) of* Theorem 2.5.

(2) *The Petersen graph or its complement, and $A_5 \leq G \leq S_5$.*

(3) *The Hamming graph $H(2, 4)$ or its complement, and G is a rank 3 subgroup of* $\mathrm{AGL}(4, 2)$.

(4) *The Clebsch graph or its complement, and G is a rank 3 subgroup of* $\mathrm{AGL}(4, 2)$.

Using the computational result that there exists no edge-transitive graph in $\mathcal{F}(d)$ with $7 \leq d \leq 10$ and of order at most 100, one can easily deduce which of the graphs in the families (1) – (4) above belongs also to the family $\mathcal{F}(d)$ for some $d \geq 3$.

Corollary 2.9. *Let $\Gamma \in \mathcal{F}(d)$ be a G-edge-transitive graph for some $d \geq 3$. If G is primitive on $V(\Gamma)$, then Γ is isomorphic to one of the graphs: K_6, the Petersen graph and its complement, and the Hamming graph $H(2, 4)$.*

Finally, we also need a result of Lucchini [21] about core-free cyclic subgroups (this serves as a key tool in [2, 15] as well). For the definition of a core-free subgroup, see the remark following Definition 1.1.

Theorem 2.10 ([21])**.** *If C is a core-free cyclic proper subgroup of a group G, then $|C|^2 < |G|$.*

3 Nest graphs

In this section we review some previous results about Nest graphs, which were obtained in [12, 17].

Lemma 3.1 ([12, Lemma 3.2]). *Let $\Gamma = \mathcal{N}(n; a, b, c; k)$ and suppose that $c = a + b$ (in $\mathbb{Z}_n$). Then Γ is edge-transitive if and only if it is also arc-transitive.*

The next result establishes some obvious isomorphisms.

Lemma 3.2 ([12, Lemma 3.3]). *The graph $\mathcal{N}(n; a, b, c; k)$ is isomorphic to $\mathcal{N}(n; a', b', c'; k)$, where $\{a, b, c\} = \{a', b', c'\}$, as well as to any of the graphs:*

$$\mathcal{N}(n; a, b, c; -k),\ \mathcal{N}(n; -a, -b, -c; k) \text{ and } \mathcal{N}(n; -a, b - a, c - a; k).$$

The graphs in the next lemma will be further studied in the next section.

Lemma 3.3 ([12, Lemma 3.4]). *If $m \geq 3$ is an odd integer, then the graph $\mathcal{N}(2m; 2, m, 2 + m; 1)$ is arc-transitive having vertex stabilisers of order 12. Furthermore, the stabiliser of u_0 in $\mathrm{Aut}(\Gamma)$ is the dihedral group D_6 of order 12 generated by the involutions φ and η defined by*

$$u_i^{\varphi} = \begin{cases} u_{-i} & \text{if } i \text{ is even,} \\ v_{-i+1} & \text{if } i \text{ is odd} \end{cases} \quad \text{and} \quad v_i^{\varphi} = \begin{cases} u_{-i+1} & \text{if } i \text{ is even,} \\ v_{-i+2} & \text{if } i \text{ is odd,} \end{cases}$$

and $u_i^{\eta} = u_i$ and $v_i^{\eta} = v_{i+m}$ for every $i \in \mathbb{Z}_n$.

Suppose that Γ is a G-edge-transitive Nest graph. In the next two lemmas we consider block systems for G. A block system $\mathcal{B}$ is said to be *minimal* if it is non-trivial, and no non-trivial block for G is contained properly in a block of $\mathcal{B}$ (by *non-trivial* we mean that the block is neither a singleton subset nor the whole vertex set). We say that $\mathcal{B}$ is *normal* if $\mathcal{B} = \mathrm{Orb}(N, V(\Gamma))$ for some $N \lhd G$. Furthermore, we say that $\mathcal{B}$ is *cyclic* if any block in $\mathcal{B}$ is contained in either $\{u_i : i \in \mathbb{Z}_n\}$ or $\{v_i : i \in \mathbb{Z}_n\}$.

Lemma 3.4 ([17, Lemma 4.1]). *Let Γ be a G-edge-transitive Nest graph of order $2n$ such that*

$$C < G,\ C = \langle \rho \rangle,\ \text{and } \rho = (u_0, u_1, \ldots, u_{n-1})(v_0, v_1, \ldots, v_{n-1}),$$

and let $\mathcal{B}$ be a cyclic block system for G with blocks of size d, $d < n/2$. Then the following hold.

1. *The kernel of the action of G on $\mathcal{B}$ is equal to C_d (the subgroup of C of order d).*
2. *Γ is a normal cover of $\Gamma/\mathcal{B}$.*
3. *$\Gamma/\mathcal{B}$ is a $\bar{G}$-edge-transitive Nest graph of order $2n/d$, where $\bar{G}$ is the image of G induced by its action on $\mathcal{B}$.*

Remark 3.5. Suppose that the graph Γ in the lemma above is given as $\Gamma = \mathcal{N}(n; a, b, c; k)$ for $a, b, c, k \in \mathbb{Z}_n$. Note that, then $\Gamma/\mathcal{B} \cong \mathcal{N}(n/d; f(a), f(b), f(c); f(k))$, where f is the homomorphism from $\mathbb{Z}_n$ to $\mathbb{Z}_{n/d}$ such that $f(1) = 1$.

Lemma 3.6 ([17, Lemma 4.2]). *Let Γ be a G-edge-transitive Nest graph of order $2n$ such that*

$$C < G,\ C = \langle \rho \rangle,\ \text{and } \rho = (u_0, u_1, \ldots, u_{n-1})(v_0, v_1, \ldots, v_{n-1}),$$

and let $\mathcal{B}$ be a non-cyclic block system for G with blocks of size d. Then the following hold.

(1) *The number d is even and any block in $\mathcal{B}$ is a union of two $C_{d/2}$-orbits.*

(2) *The group C acts transitively on $\mathcal{B}$ and the kernel of the action of C on $\mathcal{B}$ is equal to $C_{d/2}$.*

(3) *If $d > 2$ and $\mathcal{B}$ is minimal, then $\mathcal{B}$ is normal.*

4 A property of the graphs $\mathcal{N}(2m; 2, m, 2+m; 1)$, m is odd

In this section we give the following characterisation of the Nest graphs in the title. As we said in the introduction, this was mentioned already in [12] without a proof.

Proposition 4.1. *Let $\Gamma = \mathcal{N}(n; a, b, c; k)$ be an edge-transitive graph such that $n > 8$ and suppose that there exists a non-identity automorphism of Γ, which fixes all vertices u_i, $i \in \mathbb{Z}_n$. Then $\Gamma \cong \mathcal{N}(2m; 2, m, 2+m; 1)$ for some odd number m.*

We prove first an auxiliary lemma.

Lemma 4.2. *Let $\Gamma = \mathcal{N}(2m; a, m, a+m; k)$, where $m > 2$, $a = 2$ or $m-2$, and $k = 1$ or $m-1$. Then Γ is edge-transitive if and only if m is odd and $\Gamma \cong \mathcal{N}(2m; 2, m, 2+m; 1)$.*

Proof. The "if" part follows from Lemma 3.3.

For the "only if" part, assume that Γ is edge-transitive. If $a = m-2$, then $-a = 2+m$, $-m = m$ and $-(a+m) = 2$. By Lemma 3.2, we find that

$$\Gamma \cong \Gamma' := \mathcal{N}(2m; 2, m, 2+m; k),$$

where $k = 1$ or $m-1$. We have to show that m is odd and $k = 1$.

Assume to the contrary that m is even or $k = m-1$. Moreover, let us choose m so that it is the smallest number for which this happens. A quick check with the computer algebra package MAGMA [4] shows that $m \geq 12$.

Define the binary relation $\sim$ on $V(\Gamma')$ by letting $u \sim v$ whenever $u = v$ or $|\Gamma'(u) \cap \Gamma'(v)| = 4$ for any $u, v \in V(\Gamma')$. Suppose that $u_0 \sim u_i$, $i \neq 0$. If u_i is adjacent with u_1 or u_{-1}, then $i = 2$ or -2, and

$$\big|\{v_0, v_2, v_m, v_{2+m}\} \cap \{v_i, v_{2+i}, v_{m+i}, v_{2+m+i}\}\big| = 3.$$

because $n = 2m \geq 24$. As $m \geq 12$, this cannot happen and the common neighbors of u_0 and u_i are the vertices v_0, v_2, v_m and v_{2+m}. This yields that $\{0, 2, m, 2+m\} + i = \{0, 2, m, 2+m\}$ holds in $\mathbb{Z}_n$. Using that $m \geq 12$, we find that $i = m$. Note that this implies that $u_i \sim u_j$ if and only if $i = j$ or $i = j + m$.

A similar argument shows that $v_i \sim v_j$ if and only if $i = j$ or $i = j + m$.

Now suppose that $u_0 \sim v_i$. Then both u_1, u_{-1} are adjacent with v_i, and it is not hard to show that $i = 1$ or $i = m+1$. This implies that $u_i \sim v_j$ if and only if $j - i = 1$ or $j - i = m+1$. All these show that $\sim$ is an equivalence relation whose classes are the sets

$$B_i := \{u_i, u_{i+m}, v_{i+1}, v_{i+1+m}\},\ i \in \mathbb{Z}_n.$$

Clearly, $\sim$ is invariant under $\mathrm{Aut}(\Gamma')$, so the classes above form a block system for $\mathrm{Aut}(\Gamma')$. This block system will be denoted by $\mathcal{B}$. Part of the graph with $k = m-1$ is shown in Figure 1.

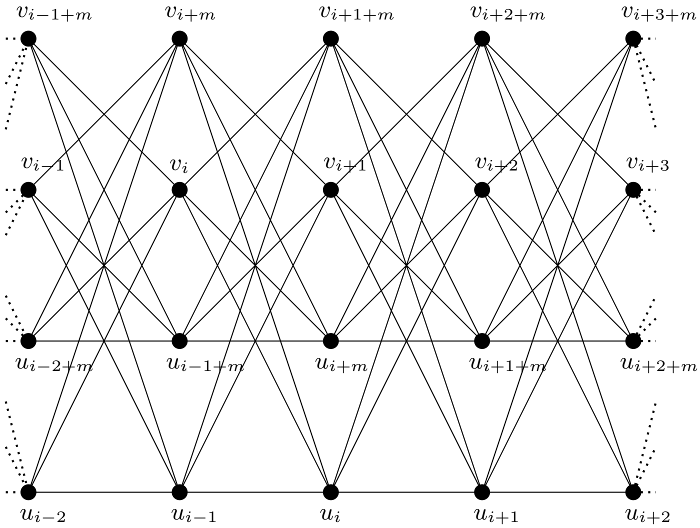

Figure 1: The Nest graph $\mathcal{N}(2m; 2, m, 2+m; m-1)$.

Let K be the kernel of the action of $\mathrm{Aut}(\Gamma')$ on $\mathcal{B}$. We prove next that K is faithful on every block. Suppose that $g \in K$ fixes pointwise the block B_i for some $i \in \mathbb{Z}_n$. Any pair of vertices in B_{i+1} is contained in a unique 4-cycle intersecting B_i at two vertices (see Figure 1). This means that g maps any pair of B_{i+1} to itself, implying that g fixes pointwise B_{i+1}. Repeating the argument, we conclude that g fixes pointwise each block B_i, i.e., g is the identity automorphism.

Let $n = 2m$, $C = \langle \rho \rangle$, where $\rho = (u_0, u_1, \ldots, u_{n-1})(v_0, v_1, \ldots, v_{n-1})$. Using the facts that K is faithful on every block B_i and the quotient graph $\Gamma/\mathcal{B}$ is an $n/2$-cycle whose automorphism group is isomorphic to the dihedral group $D_{n/2}$ of order n, we obtain the bound

$$|\mathrm{Aut}(\Gamma')| \le |K| \cdot n = 24n.$$

Thus $|C|^2 = n^2 \ge |\mathrm{Aut}(\Gamma')|$ because $n = 2m \ge 24$. By Theorem 2.10, C has a non-trivial core in $\mathrm{Aut}(\Gamma')$, let this core be denoted by N.

Assume first that $|N|$ is even. Then as C_2 is characteristic in N and $N \lhd \mathrm{Aut}(\Gamma')$, we obtain that $C_2 \lhd \mathrm{Aut}(\Gamma')$. Thus $\mathrm{Orb}(C_2, V(\Gamma))$ is a block system for $\mathrm{Aut}(\Gamma')$. But this is impossible because u_0 has one neighbour from the orbit $\{u_1, u_{1+m}\}$ and two from the orbit $\{v_0, v_m\}$.

Let $|N|$ be odd and choose an odd prime divisor p of $|N|$. It follows as above that $C_p \lhd \mathrm{Aut}(\Gamma')$. Clearly, p divides m.

Assume that $m = p$ or $2p$. If $m = p$, then by our initial assumptions, $k = p-1$. But this means that the edge $\{v_0, v_k\}$ is contained in a C_p-orbit, contradicting that $C_p \lhd \mathrm{Aut}(\Gamma')$ and Γ' is edge-transitive. If $m = 2p$, then u_0 has one neighbour from the C_p-orbit $\{u_{4i+1} : 0 \le i \le p-1\}$ and two from the C_p-orbit $\{v_{4i} : 0 \le i \le p-1\}$ (namely, v_0 and $v_{2+m} = v_{2+2p}$), which is a contradiction again.

Let $m > 2p$. By Lemma 3.4(3) and the remark after the lemma,

$$\Gamma'/C_p \cong \mathcal{N}(2m/p; f(2), f(m), f(2+m), f(k)),$$

where f is the homomorphism from $\mathbb{Z}_{2m}$ to $\mathbb{Z}_{2m/p}$ such that $f(1) = 1$. Since $m > 2p$ and p is odd, it follows that

$$f(2) = 2,\ f(m) = m/p \text{ and } f(2 + m) = 2 + m/p.$$

Furthermore, $f(k) = 1$ if $k = 1$ and $f(k) = m/p - 1$ if $k = m - 1$. By the minimality of m, we see that m/p is odd and $f(k) = 1$. This, however, contradicts that m is even or $k = m - 1$. □

Proof of Proposition 4.1. Let H and N be the setwise and the pointwise stabiliser, respectively, of the set $\{u_i : i \in \mathbb{Z}_n\}$ in $\mathrm{Aut}(\Gamma)$. Then $N \neq 1$ and $N \lhd H$. It follows that the N-orbits contained in $V := \{v_i : i \in \mathbb{Z}_n\}$ form a block system for the action of H on V, implying that $\mathrm{Orb}(N, V) = \mathrm{Orb}(C_d, V)$ for some $d > 1$. Since $N \le \mathrm{Aut}(\Gamma)_{u_0}$, it follows that every element in N maps $\{v_0, v_a, v_b, v_c\}$ to itself, and therefore, the latter set is a union of some orbits under N, hence some orbits under C_d as well. This yields that d is even, so that $n = 2m$, $\rho^m \in C_d$, and we may write w.l.o.g. that

$$a < m,\ b = m,\ c = a + m,\ \text{and } k < m.$$

Let η be the permutation of the vertex set acting as

$$u_i^\eta = u_i \text{ and } v_i^\eta = v_{i+m}\ (i \in \mathbb{Z}_n).$$

It is easy to check that $\eta \in \mathrm{Aut}(\Gamma)$.

Note that, by Lemma 3.1, Γ is arc-transitive, so $\mathrm{Aut}(\Gamma)_{u_0}$ is transitive on $\Gamma(u_0)$. Let $s = |\Gamma(v_0) \cap \Gamma(v_m)|$. It is easy to see that $s \ge 4$. Define the graph Δ as follows:

$$V(\Delta) = \Gamma(u_0) \text{ and } E(\Delta) = \{\{w, w'\} : |\Gamma(w) \cap \Gamma(w')| = s\}.$$

Note that Δ is vertex-transitive, in particular, it is regular.

Assume for the moment that u_1 and u_{-1} are adjacent in Δ. This means $|\Gamma(u_1) \cap \Gamma(u_{-1})| = s \ge 4$. Since $\Gamma(u_1) \cap \Gamma(u_{-1}) \cap \{u_i : i \in \mathbb{Z}_n\} = \{u_0\}$, we conclude that

$$|\{1, 1 + a, 1 + m, 1 + a + m\} \cap \{-1, -1 + a, -1 + m, -1 + a + m\}| \ge 3. \tag{4.1}$$

At least one of 1 and $1 + m$ is in the intersection. If it is 1, then $1 = -1 + a$ or $-1 + a + m$ because $n > 4$. As $a < m$, we find that $a = 2$. Similarly, if $1 + m$ is in the intersection, then $1 + m = -1 + a$ or $-1 + a + m$, and we find again that $a = 2$. Now, substituting $a = 2$ in (5.3), a contradiction arises because $n > 8$.

Therefore, u_1 and u_{-1} are not adjacent in Δ. Using also that $\eta \in \mathrm{Aut}(\Gamma)$, see above, we obtain that u_1 must be adjacent with v_0 or v_a.

Assume first that u_1 and v_0 are adjacent. Then $\Gamma(u_1) \cap \Gamma(v_0) = \{u_0, u_2, v_k, v_{-k}\}$, hence $2 \in \{0, n - a, m, n - a + m\}$ and $k \in \{1, 1 + a, 1 + m, 1 + a + m\}$. Since $a, k < m$ and $n > 4$, we find in turn that $a = m - 2$, and $k = 1$ or $m - 1$.

Now, if u_1 and v_a are adjacent, then $\Gamma(u_1) \cap \Gamma(v_a) = \{u_0, u_2, v_{a+k}, v_{a-k}\}$, whence $2 \in \{a, 0, a + m, m\}$ and $a + k \in \{1, 1 + a, 1 + m, 1 + a + m\}$. Since $a, k < m$ and $n > 4$, we find in turn that $a = 2$, and $k = 1$ or $m - 1$.

To sum up, $a = 2$ or $m - 2$ and $k = 1$ or $m - 1$, and so the proposition follows from Lemma 4.2. □

Corollary 4.3. *Let Γ be a G-edge-transitive Nest graph of order $2n$ such that $n > 8$ and*

$$C < G, \; C = \langle \rho \rangle, \text{ and } \rho = (u_0, u_1, \dots, u_{n-1})(v_0, v_1, \dots, v_{n-1}),$$

and suppose that $\mathrm{Orb}(C_{n/2}, V(\Gamma))$ is a block system for G. Then $C_{n/2} \lhd G$.

Proof. Let K be the kernel of the action of G on the block system $\mathrm{Orb}(C_{n/2}, V(\Gamma))$, and let K^* and $(C_{n/2})^*$ denote the image of K and $C_{n/2}$, respectively, induced by their action on $U := \{u_i : i \in \mathbb{Z}_n\}$. Since the subgraph of Γ induced by U is a cycle of length $n > 8$, it follows that $(C_{n/2})^*$ is characteristic in K^*. Therefore, if K is faithful on U, then $C_{n/2}$ is characteristic in K and as $K \lhd G$, we obtain that $C_{n/2} \lhd G$, as required.

If K is not faithful on U, then by Proposition 4.1, $n = 2m$, m is odd, and $\Gamma \cong \Gamma' := \mathcal{N}(2m; 2, m, 2+m; 1)$. Consider the group $\langle C, \varphi, \eta \rangle$, where φ and η are defined in Lemma 3.3. This is transitive on $V(\Gamma')$ and also contains the stabiliser of u_0 in $\mathrm{Aut}(\Gamma')$, therefore, $\mathrm{Aut}(\Gamma') = \langle C, \varphi, \eta \rangle$. A straightforward computation shows that $\varphi\rho^2\varphi = \rho^{-2}$ and $\eta\rho^2 = \rho^2\eta$, and hence $C_{n/2} \lhd \mathrm{Aut}(\Gamma')$. All these show that $C_{n/2} \lhd G$ holds in this case as well. □

5 Proof Theorem 1.3

Throughout this section we keep the following notation.

Hypothesis 5.1.

$\Gamma = \mathcal{N}(n; a, b, c; k)$ *is a Nest graph of order $2n$, $n \geq 4$,*

$C = \langle \rho \rangle$, *where* $\rho = (u_0, u_1, \dots, u_{n-1})(v_0, v_1, \dots, v_{n-1})$,

$G \leq \mathrm{Aut}(\Gamma)$ *such that $\rho \in G$, G acts transitively on $E(\Gamma)$, and* $\mathrm{core}_G(C) = 1$.

Instead of Theorem 1.3 we show the following slightly more stronger theorem. The proof will be given in the end of the section.

Theorem 5.2. *Assuming* Hypothesis 5.1, *Γ is isomorphic to one of the graphs: $\mathcal{N}(5; 1, 2, 3; 2)$, $\mathcal{N}(8; 1, 3, 4; 3)$, $\mathcal{N}(8; 1, 2, 5; 3)$ and $\mathcal{N}(12; 2, 4, 8; 5)$.*

We start with a computational result, which we retrieved from [12, Table 1] with the help of MAGMA [4]. Here we use the obvious facts that C is also core-free in $\mathrm{Aut}(\Gamma)$ and that $\mathrm{Aut}(\Gamma)$ is primitive whenever so is G.

Lemma 5.3. *Assuming* Hypothesis 5.1, *if $n \leq 50$, then the following hold.*

(1) *Γ is isomorphic to one of the graphs:*

$$\mathcal{N}(5; 1, 2, 3; 2), \; \mathcal{N}(8; 1, 3, 4; 3), \; \mathcal{N}(8; 1, 2, 5; 3) \text{ and } \mathcal{N}(12; 2, 4, 8; 5).$$

(2) *G is either primitive and $\Gamma \cong \mathcal{N}(5; 1, 2, 3; 2)$ or $\mathcal{N}(8; 1, 3, 4; 3)$; or for $N := \mathrm{soc}(\mathrm{Aut}(\Gamma))$, $N \cong \mathbb{Z}_2^2$ if $n = 8$ and $N \cong \mathbb{Z}_2^4$ if $n = 12$, and the N-orbits have length 4.*

The existence of a non-trivial non-cyclic block system is established next.

Lemma 5.4. *Assuming* Hypothesis 5.1, *suppose that $n > 8$. Then G admits a non-trivial non-cyclic block system.*

Proof. Observe that, if G is primitive, then Corollary 2.9 shows that $\Gamma \cong \mathcal{N}(5;1,2,3;2)$ or $\mathcal{N}(8;1,3,4;3)$. As $n > 8$, G is imprimitive.

Let $\mathcal{B}$ be a non-trivial block system with blocks of size d. If $\mathcal{B}$ is cyclic, then by Lemma 3.4(1) and Corollary 4.3, $C_d \lhd G$, where C_d is the subgroup of C of order d. This contradicts our assumption that $\mathrm{core}_G(C) = 1$, so $\mathcal{B}$ is non-cyclic. □

In the next two lemmas we study non-cyclic block systems with blocks of size 2.

Lemma 5.5. *Assuming* Hypothesis 5.1*, suppose that* $n > 50$ *and* $\mathcal{B}$ *is a non-cyclic block system for* G *with blocks of size* 2*. Then* $\Gamma/\mathcal{B}$ *has valency* 12.

Proof. Let K be the kernel of the action of G on $\mathcal{B}$, and for a subgroup $X \le G$, denote by $\bar{X}$ the image of X induced by its action on $\mathcal{B}$. For a block $B \in \mathcal{B}$, we write $B = \{u_B, v_B\}$, where $u_B \in \{u_i : i \in \mathbb{Z}_n\}$ and by $v_B \in \{v_i : i \in \mathbb{Z}_n\}$, and define the permutation τ of $V(\Gamma)$ as

$$\tau := \prod_{B \in \mathcal{B}} (u_B\, v_B). \tag{5.1}$$

Observe that τ commutes with any element of G.

Now define the graph Γ' as

$$V(\Gamma') := V(\Gamma) \text{ and } E(\Gamma') := \{\{u_0, u_1\}^x : x \in \langle G, \tau\rangle\} \tag{5.2}$$

Then $E(\Gamma) = \{\{u_0, u_1\}^x : x \in G\} \subseteq E(\Gamma')$. Also, $\langle \tau, G\rangle \le \mathrm{Aut}(\Gamma')$, hence Γ' is both vertex- and edge-transitive. Since τ commutes with every element of G, it follows that $E(\Gamma') = E(\Gamma) \cup E(\Gamma)^\tau$ and $E(\Gamma) = E(\Gamma)^\tau$ or $E(\Gamma) \cap E(\Gamma)^\tau = \emptyset$. Notice that

$$\Gamma/\mathcal{B} = \Gamma'/\mathcal{B},$$

hence we are done if show that $\Gamma'/\mathcal{B}$ has valency 12.

Denote by d and d' the valency of Γ' and $\Gamma'/\mathcal{B}$, respectively. Now, $d = |E(\Gamma')|/n = 6\frac{|E(\Gamma')|}{|E(\Gamma)|}$. This shows that $d = 6$ if $E(\Gamma) = E(\Gamma')$, and $d = 12$ otherwise.

Assume for the moment that $d = 6$, i.e., $E(\Gamma) = E(\Gamma')$. In this case $\tau \in K$, hence $\mathcal{B}$ is normal and Γ' is a normal r-cover of $\Gamma'/\mathcal{B}$ and $r = 1$ or $r = 2$.

If $r = 2$, then $d' = 3$. As $n > 50$, this is impossible due to Lemma 2.3. Here we use the facts that $\Gamma'/\mathcal{B}$ is edge-transitive and $\bar{C}$ is regular on $V(\Gamma'/\mathcal{B})$. It is well-known that an edge-transitive circulant graph is also arc-transitive. Thus $r = 1$ and $d' = 6$. As $n > 50$, Lemma 2.4 can be applied to $\Gamma'/\mathcal{B}$ and $\bar{C}$. This says that $\mathrm{Aut}(\Gamma'/\mathcal{B})$ has a normal subgroup N such that

(1) $N = \bar{C}$, or

(2) $n \equiv 4 \pmod 8$ and $N = \bar{C}_{n/4}$, or

(3) $N \cong \mathbb{Z}_3^\ell$ for $\ell \ge 2$ and $\bar{C}_3 \le N$.

In case (1), $N < \bar{G}$, hence $KC \lhd G$. The condition $r = 1$ yields that $K = \langle \tau \rangle$. Thus KC is abelian and $\langle x^2 : x \in KC\rangle = C_n$ if n is odd and $C_{n/2}$ if n is even. Using that the latter group is characteristic in KC and $KC \lhd G$, we obtain that $\mathrm{core}_G(C) \ne 1$, a contradiction.

In case (2), $N < \bar{G}$, hence $KC_{n/4} \lhd G$. Since KC is abelian, it follows that $C_{n/4}$ is characteristic in KC, implying that $C_{n/4} \lhd G$, a contradiction.

In case (3), $\bar{C}_3 \leq \bar{G} \cap N$. Since $\bar{G} \cap N \lhd \bar{G}$, it follows that G contains a normal subgroup M such that $M = \langle \tau \rangle \times S$, where $S \cong \mathbb{Z}_3^{\ell'}$ for some $\ell' \geq 1$. Thus S is normal in G, and we obtain that $\mathrm{Orb}(S, V(\Gamma))$ is a non-trivial cyclic block system for G. This contradicts Lemma 5.4, and we conclude that $d = 12$.

The graph Γ' is a normal r-cover of $\Gamma'/\mathcal{B}$, where $r = 1$ or $r = 2$, and we have $d' = d/r = 12/r$. If $r = 2$, then u_0 is adjacent with 6 vertices that are contained in the set $\{u_i : i \in \mathbb{Z}_n\}$. It follows from the definition of Γ' that this impossible. Thus $r = 1$, and so $d' = 12$. □

Lemma 5.6. *Assuming* Hypothesis 5.1, *suppose that $n > 50$ and $\mathcal{B}$ is a non-cyclic block system for G with blocks of size 2. Then there is a normal non-cyclic block system for G with blocks of size 4.*

Proof. Let K be the kernel of the action of G on $\mathcal{B}$, and for a subgroup $X \leq G$, denote by $\bar{X}$ the image of X induced by its action on $\mathcal{B}$. For the sake of simplicity we write $\bar{\Gamma}$ for $\Gamma/\mathcal{B}$.

By Lemma 5.5, $\bar{\Gamma}$ has valency 12. This implies that $K = 1$. As $\bar{C} \leq \mathrm{Aut}(\bar{\Gamma})$ and it is regular on $V(\bar{\Gamma})$, Theorem 2.1 can be applied to $\bar{\Gamma}$ and $\bar{C}$. As $n > 50$, $\bar{\Gamma}$ cannot be the complete graph. Also, if $\bar{C} \lhd \mathrm{Aut}(\bar{\Gamma})$, then $C \lhd G$ because $K = 1$. This is also impossible, hence $\bar{\Gamma}$ is in one of the families (c) and (d) of Theorem 2.1.

Case 1: $\bar{\Gamma}$ is in family (c).

In this case $\mathrm{Orb}(\bar{C}_d, V(\bar{\Gamma}))$ is a block system for $\mathrm{Aut}(\bar{\Gamma})$, hence for $\bar{G}$ as well, where $d \in \{2, 3, 4, 6\}$. Let N be the unique subgroup of G for which $\bar{N}$ is the kernel of the action of $\bar{G}$ on $\mathrm{Orb}(\bar{C}_d, V(\bar{\Gamma}))$. Note that $N \lhd G$ and $N \cong \bar{N}$ because $K = 1$. Let $\mathcal{B}' = \mathrm{Orb}(N, V(\Gamma))$. It follows that $\mathcal{B}'$ is non-cyclic and it has blocks of size $2d$.

Let $d = 2$. Then $\mathcal{B}'$ is normal with blocks of size 4, so the conclusion of the lemma holds.

Let $d = 3$. Then the Sylow 3-subgroup of $\bar{N}$ is normal in $\bar{G}$. It follows in turn that, the Sylow 3-subgroup of N is normal in G, the orbits of the latter subgroup form a non-trivial cyclic block system for G. This contradicts Lemma 5.4.

Let $d = 4$. Then $\bar{\Gamma}$ has valency 3. It follows from Lemma 2.3 that $n \leq 6$, but this is excluded.

Finally, let $d = 6$. Let τ be the permutation of $V(\Gamma)$ defined in (5.1) and Γ' be the graph defined in (5.2). Let Δ be the subgraph of Γ' induced by the set $u_0^N \cup u_1^N$. It is not hard to show that Δ is a bipartite graph, it has valency 6, and it is also edge-transitive. Moreover, if $B \in \mathcal{B}$ such that $B \subset u_1^N$, then

$$|\Delta(u_0) \cap B| = 1. \tag{5.3}$$

Since $\langle \tau \rangle \times C_6 \leq \mathrm{Aut}(\Delta)$, it follows that Δ is uniquely determined by $\Delta(u_0)$. It follows from the definition of Γ' that $|\Gamma'(u_0) \cap \{u_i : i \in \mathbb{Z}_n\}| = 4$. Therefore, replacing $u_0^N \cup u_1^N$ with $u_0^N \cup u_{n-1}^N$ if necessary, we may assume w.l.o.g. that $|\Delta(u_0) \cap \{u_i : i \in \mathbb{Z}_n\}| \leq 2$. This together with (5.3) show that there are 6 possibilities for Δ. Letting $B = \{u_0, v_b\}$ and $n = 6l$, the biparts of Δ are $\{u_{xl}, v_{xl+b} : x \in \mathbb{Z}_6\}$ and $\{u_{xl+1}, v_{xl+b+1} : x \in \mathbb{Z}_6\}$. Using (5.3), Δ can be described as follows. There exists $i \in \mathbb{Z}_6$ such that $\Delta(u_0) \cap \{u_i :$

$i \in \mathbb{Z}_n\} = \{u_1, u_{il+1}\}$. Then as $\tau \in \mathrm{Aut}(\Delta)$, $\Delta(v_0) \cap \{v_i : i \in \mathbb{Z}_n\} = \{v_{b+1}, v_{il+b+1}\}$, and for every $j \in \mathbb{Z}_6$,

$$\Delta(u_{jl}) = \{v_{xl+b+1} : x \in \mathbb{Z}_6, x \neq j, j+i\} \cup \{u_{jl+1}, u_{(j+i)l+1}\},$$
$$\Delta(v_{jl+b}) = \{u_{xl+b+1} : x \in \mathbb{Z}_6, x \neq i, j+i\} \cup \{v_{jl+b+1}, v_{(j+i)l+b+1}\}.$$

A computation with MAGMA [4] shows that none of these 6 graphs is edge-transitive.

Case 2: $\bar{\Gamma}$ is in family (d).

We finish the proof by showing this case does not occur. Theorem 2.1 shows that $\mathcal{B}_1 := \mathrm{Orb}(\bar{C}_d, V(\bar{\Gamma}))$ and $\mathcal{B}_2 := \mathrm{Orb}(\bar{C}_{n/d},, V(\bar{\Gamma}))$ are blocks for $\bar{G}$ for some divisor d of n such that $d \in \{4, 5, 7\}$ and $\gcd(d, n/d) = 1$. Furthermore,

$$\bar{C}_d \times \bar{C}_{n/d} < \bar{G} \leq \mathrm{Aut}(\Gamma') = G_1 \times G_2,$$

where $\bar{C}_d \leq G_1$, $G_1 \cong S_d$, $\bar{C}_{n/d} < G_2$ and $G_2 \cong \mathrm{Aut}(\Gamma'/\mathcal{B}_1)$.

Let $d = 4$. Then $\bar{\Gamma}/\mathcal{B}_1$ has valency 4. Using also that n/d is odd and that $n > 20$, it follows from Lemma 2.3 that $\bar{C}_{n/d} \lhd \bar{G}$, hence $C_{n/d} \lhd G$, a contradiction.

Let $d = 5$. Then $\bar{\Gamma}/\mathcal{B}_1$ has valency 3, hence $n \leq 30$ by Lemma 2.3, which is excluded.

Finally, let $d = 7$. Then $\bar{\Gamma}/\mathcal{B}_1$ is a cycle of length $n/7$, implying that $\bar{C}_{n/d} \lhd \bar{G}$, so $C_{n/d} \lhd G$, a contradiction. □

Before the proof of Theorem 5.2 we need two more lemmas dealing with non-cyclic block systems with blocks of size at least 4.

Lemma 5.7. *Assuming* Hypothesis 5.1*, suppose that* $n > 50$ *and* $\mathcal{B}$ *is a minimal non-cyclic block system for* G *with blocks of size at least* 4*, and let* $B \in \mathcal{B}$ *be any block. Then the permutation group of* B *induced by* $G_{\{B\}}$ *is an affine group.*

Proof. For a subgroup $X \leq G_{\{B\}}$, denote by X^* the image of X induced by its action on B. As B is minimal, $(G_{\{B\}})^*$ is a primitive permutation group. Also, $(C_{\{B\}})^*$ is a semiregular cyclic subgroup of $(G_{\{B\}})^*$ with 2 orbits, hence Theorem 2.5 can be applied to $(G_{\{B\}})^*$. This shows that $(G_{\{B\}})^*$ is either an affine group or it is one of the groups in the families (a) – (f) in part (2) of Theorem 2.5. Assume that the latter case occurs. We derive in three steps that this leads to a contradiction. Let K be the kernel of the action of G on $\mathcal{B}$.

Step 1: K acts faithfully on every block in $\mathcal{B}$.

Since $K \lhd G_{\{B\}}$, it follows that $K^* \lhd (G_{\{B\}})^*$. By Corollary 2.6, K^* is primitive and belongs to the same family as $(G_{\{B\}})^*$.

Assume to the contrary that K is not faithful on every block. Using the connectedness of Γ, it is easy to show that there are blocks B, B' in $\mathcal{B}$ with the following properties: The kernel of the action of K on B is non-trivial on B', and Γ has an edge $\{w, w'\}$ such that $w \in B$ and $w' \in B'$. Denote by N the latter kernel. Now as $N \lhd K$ and K is primitive on B', N is transitive on B'. Thus the orbit $(w')^N = B'$, and so w is adjacent with any vertex in B'. Since $\mathcal{B}$ is normal, it follows that the subgraph of Γ induced by $B \cup B'$ is isomorphic to the complete bipartite graph $K_{m,m}$, where $m = |B|$. On the other hand, $m \geq 6$, showing that $\Gamma \cong K_{6,6}$, a contradiction.

Denote by B and B' the blocks containing u_0 and u_1, respectively.

Step 2: The action of K on B is equivalent with its action on B'.

Assume to the contrary that the actions are inequivalent. Then K is a group described in one of the cases (i) and (ii) of Lemma 2.7(2). If K is described in Lemma 2.7(2)(i), then $|B| = 6$ or 12, and the stabilizer K_{u_0} acts transitively on the block B'. This implies that every vertex in B is adjacent with every vertex in B', hence $|B| = 6$ and $\Gamma \cong K_{6,6}$. This is impossible.

Let K be a group described in Lemma 2.7(2)(ii). Then K belongs to family (c) in Theorem 2.5(2) with $d \geq 4$, and the elements in B and B' correspond to the points and the hyperplanes of the projective geometry $\mathrm{PG}(d-1, q)$, respectively. The set B' splits into two K_{u_0}-orbits of lengths

$$(q^{d-1} - 1)/(q-1) \text{ and } q(q^{d-1} - 1)/(q-1).$$

The first orbit consists of the hyperplanes of $\mathrm{PG}(d-1, q)$ through the point represented by u_0, and the second orbit consists of the remaining hyperplanes. Clearly, the minimum of these numbers is bounded above by the valency of Γ, implying that $q^{d-1} - 1 \leq 6(q-1)$, and hence $q^{d-2} < 6$. This is impossible because $d \geq 4$.

Step 3: $\mathrm{core}_G(C) \neq 1$.

Since K acts equivalently on B and B', it follows that $K_{u_0} = K_v$ for some vertex $v \in B'$ (see [8, Lemma 1.6B]). Define the binary relation $\sim$ on $V(\Gamma)$ by letting $u \sim v$ if and only if $K_u = K_v$. It is not hard to show, using that $K \lhd G$, that $\sim$ is a G-congruence (see [8, Exercise 1.5.4]), and so there is a block for G containing u_0 and v. Also, as K is not regular, this block is non-trivial, and this shows that $v \neq u_1$.

By Lemma 2.7(1), K is 2-transitive on B', unless $|B'| = 10$, $K = A_5$ or S_5, and it has subdegrees $1, 3$ and 6.

Assume first that K is 2-transitive on B'. Then the orbit $u_1^{K_{u_0}} = u_1^{K_v} = |B'| - 1$ and each vertex in $u_1^{K_v}$ is adjacent with u_0. Hence u_0 has $|B'| - 1$ neighbours in B'. On the other hand, as $\mathcal{B}$ is normal, this number divides 6, so $|B| = |B'| = 4$, contradicting that $G^*_{\{B\}}$ is an almost simple group.

We are left with the case that $|B'| = 10$, $K = A_5$ or S_5, and it has subdegrees $1, 3$ and 6. Consequently, u_0 has 3 or 6 neighbours in B'.

If u_0 has 6 neighbours, then it is clear that $n = 10$, which is excluded.

Now assume that u_0 has 3 neighbours in B'. In this case Γ is a normal 3-cover of a cycle of length $n/5$. Since $\Gamma/\mathcal{B}$ is a cycle of length $n/5$, it follows that $|G| \leq |K| \cdot 2n/5 = 48n$. Using that $n > 50$, Theorem 2.10 shows that $\mathrm{core}_G(C) \neq 1$. □

Lemma 5.8. *Assuming* Hypothesis 5.1*, suppose that* $n > 50$ *and* $\mathcal{B}$ *is a minimal non-cyclic block system for* G *with blocks of size at least* 4*. Then the blocks have size* 4*.*

Proof. Let K be the kernel of the action of G on $\mathcal{B}$, and let $B \in \mathcal{B}$ be the block containing u_0. Denote by $(G_{\{B\}})^*$ the permutation group of B induced by $G_{\{B\}}$. By Lemma 5.7, $(G_{\{B\}})^*$ is an affine group, and thus it is one of the groups in the families (a) – (d) in part (1) of Theorem 2.5. In particular, $|B| \in \{4, 8, 16\}$. Assume to the contrary that $|B| > 4$. By Lemma 3.6, $\mathcal{B}$ is normal, hence Γ is a normal r-cover of $\Gamma/\mathcal{B}$ for $r \in \{1, 2, 3\}$.

Case 1: $r = 1$.

In this case K is regular on every block, in particular, $K \cong \mathbb{Z}_2^3$ or $\mathbb{Z}_2^4$. On the other hand, by Lemma 3.6(2), $C_{|B|/2} < K$, a contradiction.

Case 2: $r = 2$.

In this case $\Gamma/\mathcal{B}$ has valency 3. It follows from Lemma 2.3 that $n \leq 48$, but this is excluded.

Case 3: $r = 3$.

Then $\Gamma/\mathcal{B}$ is a cycle of length $2n/|B|$. This implies that the action of G_{u_0} on $\Gamma(u_0)$ admits a block system consisting of two blocks of size 3. Consequently, the restriction of G_{u_0} to $\Gamma(u_0)$ is a $\{2,3\}$-group. This together with the fact that Γ is connected yield that G_{u_0} is also a $\{2,3\}$-group. Now checking the stabilisers in part (1) of Theorem 2.5, we find that $|B| = 16$ and

$$(G_{\{B\}})^*_{u_0} \cong (\mathbb{Z}_3 \times \mathbb{Z}_3) : \mathbb{Z}_4 \text{ or } (S_3 \times S_3) : \mathbb{Z}_2 \tag{5.4}$$

Assume for the moment K is not faithful on B. Then there exist adjacent blocks B' and B'' such that the kernel of the action of K on B' is non-trivial on B''. Denote this kernel by L. The L-orbits contained in B'' have the same size, which is equal to 2^s for some $1 \leq s \leq 4$. On the other hand, for $w \in B'$, the set $\Gamma(w) \cap B''$ is L-invariant, implying that $|\Gamma(w) \cap B''|$ is equal to some power of 2, a contradiction. Thus K is faithful on B.

The group K contains a normal subgroup E such that $E \cong \mathbb{Z}_2^4$. Note that $\mathcal{B} = \mathrm{Orb}(E, V(\Gamma))$. Let P be the Sylow 3-subgroup of $G_{\{B\}}$. Since $\Gamma/\mathcal{B}$ is a cycle, it follows that $P \leq K$. This also shows that $P \cong \mathbb{Z}_3^2$. Also, $C_8 \leq K$, and in view of (5.4), we obtain that $|(G_{\{B\}})^* : K| \leq 2$ and if the index is equal to 2, then $(G_{\{B\}})^*_{u_0} \cong (S_3 \times S_3) : \mathbb{Z}_2$. A direct check by MAGMA [4] shows that in the latter case $(G_{\{B\}})^*$ has a unique subgroup of index 2 containing an element of order 8, which is also primitive. All these show that K is primitive on B.

Denote by Δ be the subgraph of Γ induced by $u_0^E \cup u_1^E$. Using that $E \cong \mathbb{Z}_2^4$ acting regularly on both u_0^E and u_1^E, it is not hard to show that Δ is the union of four 3-dimensional cube Q_3. If Δ_1 is a component of Δ, then $|V(\Delta_1) \cap B| = 4$ (note that $B = u_0^E$) and $V(\Delta_1) \cap B$ is a block for K. This, however, contradicts the fact that K is primitive on B. □

We are ready to settle Theorem 5.2, and therefore, Theorem 1.3 as well.

Proof of Theorem 5.2. In view of Lemma 5.3, we may assume that $n > 50$. It follows from Lemmas 5.4 – 5.8 that G admits a normal non-cyclic block system with blocks of size 4. Denote this block system by $\mathcal{B}$. Let K be the kernel of the action of G on $\mathcal{B}$, and for a subgroup $X \leq G$, denote by $\bar{X}$ the image of X induced by its action on $\mathcal{B}$. As $\mathcal{B}$ is normal, Γ is a normal r-cover of $\Gamma/\mathcal{B}$ for some $r \in \{1,2,3\}$. We exclude below all possibilities case-by-case.

Case 1: $r = 1$.

In this case $|K| = 4$ and $K \cap C = C_2$. If $K \cong \mathbb{Z}_4$, then C_2 is characteristic in K, and therefore, it is normal in G. This is impossible because $\mathrm{core}_G(C) = 1$, hence $K \cong \mathbb{Z}_2^2$

The graph $\Gamma/\mathcal{B}$ is edge-transitive, it has valency 6, and $\bar{C}$ is regular on $V(\Gamma/\mathcal{B})$. As $n > 50$, Lemma 2.4 can be applied to $\Gamma/\mathcal{B}$ and $\bar{C}$. It follows that $\mathrm{Aut}(\Gamma/\mathcal{B})$ has a normal subgroup N such that

(1) $N = \bar{C}$, or

(2) $n \equiv 4 \pmod 8$ and $N = \bar{C}_{n/4}$, or

(3) $N \cong \mathbb{Z}_3^\ell$ for $\ell \geq 2$ and $\bar{C}_3 \leq N$.

In case (1), we obtain that $KC \lhd G$, whereas in case (2), $KC_{n/4} \lhd G$. In either case, $|KC : C| = 2$, and therefore, for the derived subgroup $(KC)'$, $(KC)' \leq C$. Thus $(KC)' \leq \text{core}_G(C)$, and so $(KC)' = 1$, i.e., KC is an abelian group. Then we obtain that $C_{n/2} \lhd G$ in case (1), and $C_{n/4} \lhd G$ in case (2). None of these is possible because $\text{core}_G(C) = 1$.

In case (3), G contains a normal subgroup M such that $KC_3 \leq M$ and $M/K \cong \mathbb{Z}_3^{\ell'}$ for some $\ell' \geq 1$. Then M can be written as $M = KS$ where $C_3 \leq S$ and $S \cong \mathbb{Z}_3^{\ell'}$. As $C_2 \leq K$, we obtain that C_3 commutes with K, and so $C_3 \leq O_3(M)$, where $O_3(M)$ denotes the largest normal 3-subgroup of M. As $O_3(M)$ is characteristic in M, $O_3(M) \lhd G$. This yields that $\text{Orb}(O_3(M), V(\Gamma))$ is a non-trivial cyclic block system, a contradiction to Lemma 5.4.

Case 2: $r = 2$.

In this case $\Gamma/\mathcal{B}$ has valency 3, hence $n \leq 12$ by Lemma 2.3, which is excluded.

Case 3: $r = 3$.

The group K is faithful on every block of $\mathcal{B}$. This can be shown by copying the argument that has been used in Case 3 in the proof of Lemma 5.8. Since $\Gamma/\mathcal{B}$ is a cycle of length $n/2$, it follows that $|G| \leq |K| \cdot n = 24n$. Using that $n > 50$, Theorem 2.10 shows that $\text{core}_G(C) \neq 1$, a contradiction. □

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